2ND / HIGHER–ORDER O.D.E.

1

◎ *Linear O.D.E of 2^{nd} – order & Higher*

Review ：

1^{st} – order Linear ：$y' + p(x)y = q(x)$

$$y\sigma = \int \sigma \times q(x)dx + c$$

Non – Linear ：$y' = f(x,y)$

(1). Separable (2). Exact (3). Bernoulli (4).Riccati

2

2^{nd} – order / higher order

Linear ：

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \dots\dots\dots + a_{n-1}(x)y' + a_n(x)y = f(x)$$

(1) <u>*Constant Coefficient*</u> , <u>*Homogeneous*</u>

$a_j(x)$ are const $f(x) = 0$

(2) *Const , coeff , Non- Homogeneous*

(3) *Non-const , coeff*

◎ ***Theory of solution of Higher order Linear O.D.E***

Note ： *We consider 2^{nd}–order Linear O.D.E here but it can be applied to higher order Linear O.D.E as well*

ex ：

$$y'' - 12x = 0 \rightarrow y'' = 12x$$

$$\Rightarrow y' = \int 12x dx = 6x^2 + c$$

$$\Rightarrow y = \int (6x^2 + c) dx = 2x^3 + cx + k$$

Suppose we want a solution satisfying I.C

$$y(0) = 3 \rightarrow k = 3$$
$$y(x) = 2x^3 + cx + 3$$

Suppose we also specify another I.C

$$y'(0) = -1 \Rightarrow c = -1$$
$$y(x) = 2x^3 + x + 3$$

Summary ：

(1) The general solution of the 2nd–order Linear O.D.E involved 2 arbitrary constants

(2) Initial conditions are needed is order to specify a particular solution one specifies a point lying on the solution curve & the other specifies the slope at that point

◎ ***Theorem*** ：

Let p , q and f be continuous on an open interval I

Let x_0 be in I & Let A , B be any real numbers ,
then the initial –value problem

$$y'' + p(x)y' + q(x)y = f(x)$$

I.C

$$y(x_0) = A \quad , \quad y'(x_0) = B$$

has a unique solution defined for all x in I

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Note ： *It can be applied to n^{th}-order Linear O.D.E*

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \dots\dots\dots + a_{n-1}(x)y' + a_n(x)y = f(x)$$

I.C.

$$y(x_0) = A_0 \ , \ y'(x_0) = A_1 \ , \dots\dots\dots y^{(n-1)}(x_0) = A_{n-1}$$

The O.D.E has a unique solution defined
for all x in I

◎ ***Homogeneous Equation*** ：

$$y'' + p(x)y' + q(x)y = 0$$

Theorem ： *general solution of* $y'' + p(x)y' + q(x)y = 0$

Let y_1 & y_2 *be solutions of* $y'' + p(x)y' + q(x)y = 0$ *on an interval* I. *Then any linear combination of these solution is also a solution*

$$\left.\begin{matrix} y_1 \text{ (solution)} \\ y_2 \text{ (solution)} \end{matrix}\right] \longrightarrow c_1y_1 + c_2y_2 \text{ (solution)}$$

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Proof ：*Let* c_1, c_2 *be real numbers* $y(x) = c_1y_1 + c_2y_2$, *substitute into the O.D.E*

we obtain

$$\begin{aligned}&[c_1y_1 + c_2y_2]'' + p(x)[c_1y_1 + c_2y_2]' + q(x)[c_1y_1 + c_2y_2] \\ &= c_1[y''_1 + p(x)y'_1 + q(x)y_1] + c_2[y''_2 + p(x)y'_2 + q(x)y_2] \\ &= 0\end{aligned}$$

Theorem ∶ *Let p , q be continuous on an interval I the 2nd–order linear Homogeneous O.D.E. admits exact 2 linear independent solution. If y_1 & y_2 are such a set of L.I solutions on I , then the general solution of $y'' + p(x)y' + q(x)y = 0$ is*

$y(x) = c_1 y_1 + c_2 y_2$ *where c_1, c_2 are arbitrary const .*

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Definition ∶ *Linear dependence &*
Linear independence

Two functions f & g are linear dependent on an open interval I if for some constant c , either f(x) =c g(x) for all x in I , or g(x)=c f(x) for all x in I
If f & g are not linear dependent on I, then they are said to be linear independent

Ex ∶ [x , $2x$] is linear dependent

[e^x , e^{-x}, sinhx] is linear dependent

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$$

$$\Rightarrow e^x = 2\sinh x + e^{-x}$$

$$\Rightarrow e^{-x} = e^x - 2\sinh x$$

8 A set of functions $f_1(x)$, $f_2(x)$, $f_3(x)$,…,$f_n(x)$ is said to be linearly dependent on an interval I if there exist constants C_1, C_2, C_3,…,C_n not all zeros, such that

$$C_1 f_1(x) + C_2 f_2(x) + C_3 f_3(x) + \ldots + C_n f_n(x) = 0$$

for every x in the interval I.

If the set of functions is not linearly dependent on the interval, it is said to be linearly independent.

◎ ***Wronskian of solution*** ：

Let y_1 , y_2 *be solutions of* $y'' + p(x)y' + q(x)y = 0$

$$w(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x)$$

is called the wronskian of solutions

ex $y'' + y = 0$ $\qquad y_1 = \cos x \qquad y_2 = \sin x$

$$w(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

Theorem ：*Wronskian Test*

Let y_1 & y_2 *be solutions of* $y'' + p(x)y' + q(x)y = 0$ *on an open interval I , then*

1. *Either* $W(x)=0$ *for all x in I or* $W(x) \neq 0$ *for all x in I*

2. y_1 & y_2 *are linear independent on I if and only if* $W(x) \neq 0$ *in I*

so,

$$w(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

Ex ：

$$y'' + xy = 0$$

$$y_1(x) = 1 - \frac{1}{6}x^3 + \frac{1}{180}x^6 - \frac{1}{12960}x^9 + \ldots\ldots\ldots\ldots\ldots$$

$$y_2(x) = x - \frac{1}{12}x^4 + \frac{1}{504}x^7 - \frac{1}{45360}x^{10} + \ldots\ldots\ldots\ldots$$

$W(x)$ (y_1, y_2) would be difficult to evaluate , but at $x = 0$, we easily obtain

$$W(0) = y_1(0)y_2'(0) - y_1'(0)y_2(0) = (1)(1) - (0)(0)$$

$$= 1$$

Non vanishing of the wronskian at this point is enough to conclude linear independence of these solutions.

Theorem ：

Let y_1 & y_2 be linear independent solutions of $y'' + p(x)y' + q(x)y = 0$ on an open interval I . Then , every solution of this O.D.E on I is a linear combination of y_1 & y_2

Higher Order Linear Homogeneous O.D.E. -- Constant Coefficient

◎ *General solution of Higher-order homogeneous Linear O.D.E.*

Let $y_1 \,\&\, y_2$ *be solution of* $y'' + p(x)y' + q(x)y = 0$ *on an open interval* I

(1) $y_1 \,\&\, y_2$ *form a fundamental set of solution in* I *if* $y_1 \,\&\, y_2$ *are linear independent in* I

(2) *when* $y_1 \,\&\, y_2$ *form a fundamental set of solution, we call* $c_1y_1 + c_2y_2$, *with* $c_1 \,\&\, c_2$ *arbitrary constants, the genearal solution of the O.D.E. in* I

Proof： $Let\ \phi\ be\ any\ solution\ of\ \ y''+p(x)y'+q(x)y=0$
$on\ I.\ We\ want\ to\ show\ that\ \phi=c_1y_1+c_2y_2$
$is\ the\ unique\ solution\ on\ I\ of\ the\ initial-value$
$problem\ \ [I.C.\ y(x_0)=A\quad y'(x_0)=B]$

$$\phi(x_0)=c_1y_1(x_0)+c_2y_2(x_0)=A$$
$$\phi'(x_0)=c_1y_1'(x_0)+c_2y_2'(x_0)=B$$

$Assume\ \omega(x_0)\neq 0\ ,We\ \ find\ \ that$

$$c_1=\frac{Ay_2'(x_0)-By_2(x_0)}{\omega(x_0)}\qquad c_2=\frac{By_1(x_0)-Ay_1'(x_0)}{\omega(x_0)}$$

Summary：

(1) $\phi(x)=c_1y_1(x)+c_2y_2(x)$ is a
solution of the initial value problem

(2) $c_1\ \&\ c_2$ are unique with the I.C.
$\phi(x)$ is the unique solution of the
initial value problem

Review : $y'' + p(x)y' + q(x)y = 0$

$\Downarrow$

General solution

$\Downarrow$

$y(x) = c_1 y_1(x) + c_2 y_2(x)$

$\Downarrow$

Need to be L.I. (Basis)

$\Downarrow$

Wronskian Test to determine L.I./L.D.

I.C. $y(x_0) = A$

$y'(x_0) = B$

$\Downarrow$

Particular solution (unique)

◎ $y'' + p(x)y' + q(x)y = f(x)$

Let y_1 & y_2 be a fundamental set of solutions of $y'' + p(x)y' + q(x)y = 0$ on an open interval I

Let y_p be any solution of Non-homo. eqn. on I. Then for any solution of the Non-homo. eqn., φ there exists numbers c_1 & c_2 such that $\varphi = c_1 y_1 + c_2 y_2 + y_p = y_h + y_p$

Proof： $\varphi \,\&\, y_p$ are both solutions of

$$y'' + p(x)y' + q(x)y = f(x)$$

$$\Rightarrow \varphi'' + p(x)\varphi' + q(x)\varphi = f(x) ------(1)$$

$$y_p'' + p(x)y_p' + q(x)y_p = f(x) ----(2)$$

$(1)-(2)$

$$(\varphi - y_p)'' + p(x)(\varphi - y_p)' + q(x)(\varphi - y_p) = 0$$

$\therefore (\varphi - y_p)$ is a solution of $y'' + p(x)y' + q(x)y = 0$

$$\Rightarrow \varphi - y_p = c_1y_1 + c_2y_2$$

$$\Rightarrow \varphi = c_1y_1 + c_2y_2 + y_p$$

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◎ *The Constant coefficient Homogeneous Linear Equation*

$$y^{(n)} + a_1y^{(n-1)} + a_2y^{(n-2)} + \cdots + a_{n-1}y' + a_ny = 0$$

where a_j are constants, $j = 1,2,\ldots,n$

Consider the 2^{nd} *– order O.D.E.*

$y'' + Ay' + By = 0 \quad A, B$ *are constants*

Look for solutions $y(x) = e^{\lambda x}$

Substitute $e^{\lambda x}$ *into equation*

$$\lambda^2 e^{\lambda x} + A\lambda e^{\lambda x} + Be^{\lambda x} = 0 \Rightarrow (\lambda^2 + A\lambda + B)e^{\lambda x} = 0$$

This can only be true if $\quad \lambda^2 + A\lambda + B = 0$

It is roots are $\quad \lambda_{1,2} = \dfrac{-A \pm \sqrt{A^2 - 4B}}{2}$

Case 1： $A^2 - 4B > 0 \Rightarrow \lambda_1, \lambda_2$ *are real numbers*

distinct roots

$$\lambda_1 = \frac{-A + \sqrt{A^2 - 4B}}{2} \quad , \quad \lambda_2 = \frac{-A - \sqrt{A^2 - 4B}}{2}$$

The solution are

$$y_1(x) = e^{\lambda_1 x} \quad , \quad y_2(x) = e^{\lambda_2 x}$$

The general solution is

$$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$Ex:\ y'' - y' - 6y = 0$

$characteristic\ equation\ \lambda^2 - \lambda - 6 = 0$

$\lambda_1 = -2\ ,\ \lambda_2 = 3$

$General\ Solution$

$y(x) = c_1 e^{-2x} + c_2 e^{3x}$

$\text{Case2}: A^2 - 4B = 0 \Rightarrow \lambda_1, \lambda_2\ are\ real\ numbers$

$repeated\ root$

$$\lambda_1, \lambda_2 = \frac{-A}{2} \Rightarrow y_1(x) = e^{-\frac{A}{2}x}$$

$How\ to\ find\ the\ second\ solution?$

$\Rightarrow Reduction\ of\ order$

$Try\ y_2(x) = u(x)e^{-\frac{A}{2}x}, substitute\ into\ the\ O.D.E.$

$$y_2'(x) = u'(x)e^{-\frac{A}{2}x} - \frac{A}{2}u(x)e^{-\frac{A}{2}x}$$

$$y_2''(x) = \frac{A^2}{4}u(x)e^{-\frac{A}{2}x} - Au'(x)e^{-\frac{A}{2}x} + u''(x)e^{-\frac{A}{2}x}$$

$$\Rightarrow u''(x) + (B - \frac{A^2}{4})u = 0 \Rightarrow u''(x) = 0$$

$$We\ choose\ u(x) = x \Rightarrow y_2(x) = xe^{-\frac{A}{2}x}$$

Since y_1 & y_2 *are linear independent*

they form a fundamental set of solutions

The general solution is

$$y(x) = c_1 e^{-\frac{A}{2}x} + c_2 x e^{-\frac{A}{2}x}$$

$$Ex:\ y'' - 6y' + 9y = 0$$

Characteristic equation

$$\lambda^2 - 6\lambda + 9 = 0 \Rightarrow \lambda_{1,2} = 3$$

The general solution is

$$y(x) = c_1 e^{3x} + c_2 x e^{3x}$$

*Case*3: $A^2 - 4B < 0 \Rightarrow \lambda_1, \lambda_2$ *are Complex Roots*

$$\lambda_1, \lambda_2 = \frac{-A \pm \sqrt{4B - A^2}\,i}{2} = p \pm iq\ ,\ p = -\frac{A}{2}\ ,\ q = \frac{\sqrt{4B - A^2}}{2}$$

This yields 2 solutions:

$$y_1(x) = e^{(p+iq)x} \qquad y_2(x) = e^{(p-iq)x}$$

Therefore, the general solution is

$$y(x) = c_1 e^{(p+iq)x} + c_2 e^{(p-iq)x}$$

$$= e^{px}\left[c_1 e^{iqx} + c_2 e^{-iqx}\right]$$

An Alternative form for the complex roots case

Euler's formula

$$e^{ix} = \cos x + i\sin x$$

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$$e^{-ix} = \cos x - i\sin x$$

$$y_1(x) \to e^{(p+iq)x} = e^{px}\left[\cos(qx) + i\sin(qx)\right]$$

$$y_2(x) \to e^{(p-iq)x} = e^{px}\left[\cos(qx) - i\sin(qx)\right]$$

The general solution

$$y(x) = e^{px}\left[c_1 e^{iqx} + c_2 e^{-iqx}\right]$$

If we choose $c_1 = c_2 = \dfrac{1}{2}$, $y_3(x) = e^{px}\cos(qx)$

$$c_1 = \frac{1}{2i}, \; c_2 = \frac{-1}{2i} \Rightarrow y_4(x) = e^{px}\sin(qx)$$

$$2\ \ fundamental\ \ set.\ \ \ Linear\ \ independent$$

$$y_3 = e^{px}\cos(qx) \qquad y_4 = e^{px}\sin(qx)$$

$$General\ \ solution$$

$$y(x) = e^{px}\left[c_3\cos(qx) + c_4\sin(qx)\right]$$

$$c_3, c_4\ \ are\ \ arbitrary\ \ constants$$

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Higher Order Linear Homogeneous O.D.E. -- Non Const. Coeff.

◎**Non-constant Coefficient Homogeneous Eqn.**

Euler's Equation

(Cauchy-Euler Equation)

(Equi-dimensional Equation)

Euler's Equation of 2^{nd}-order

$$y'' + \frac{1}{x}Ay' + \frac{1}{x^2}By = 0 \quad for\ x > 0$$

A, B are constants

- $Let\ x = e^t \quad y(x) = y(e^t) = Y(t)$

$$y'(x) = \frac{dY}{dt}\frac{dt}{dx} = Y'(t)\frac{1}{x}$$
$$\Rightarrow Y'(t) = \frac{dY}{dt} = xy'(x)$$

$$y''(x) = \frac{d}{dx}y'(x) = \frac{d}{dx}\left\{\frac{1}{x}Y'(t)\right\}$$
$$= -\frac{1}{x^2}Y'(t) + \frac{1}{x}\frac{dY'}{dt}\frac{dt}{dx} = -\frac{1}{x^2}Y'(t) + \frac{1}{x}Y''(t)\frac{1}{x}$$
$$= \frac{1}{x^2}\{Y''(t) - Y'(t)\}$$
$$\Rightarrow x^2y''(x) = Y''(t) - Y'(t)$$

The other form of Euler's Equation

$$x^2 y'' + Axy' + By = 0$$

$$\Rightarrow \{Y''(t) - Y'(t)\} + AY'(t) + BY(t) = 0$$

$$\Rightarrow Y''(t) + (A-1)Y'(t) + BY(t) = 0$$

const coeff homo linear O.D.E

Next : (1) solve the const coeff homo linear O.D.E

(2) let $t = \ln x$ to obtain $y(x)$

Ex : $x^2 y'' + 2xy' - 6y = 0$

(1) Observe : Euler's Equation ,2^{nd} - order

(2) Let $x = e^t$, $t = \ln x$

$$y(x) = y(e^t) = Y(t)$$

$$xy'(x) = Y'(t) \quad , x^2 y''(x) = Y'' - Y'(t)$$

(3) Substitute into the O.D.E

$$\{Y''(t) - Y'(t)\} + 2(Y'(t)) - 6Y(t) = 0$$

$$\Rightarrow Y''(t) + Y'(t) - 6Y(t) = 0$$

$$Y''(t) + Y'(t) - 6Y(t) = 0$$

$$\text{characteristic equation}: \lambda^2 + \lambda - 6 = 0$$

$$\lambda_1 = 2 \qquad \lambda_2 = -3$$

$$\text{The general solution } Y(t) = c_1 e^{2t} + c_2 e^{-3t}$$

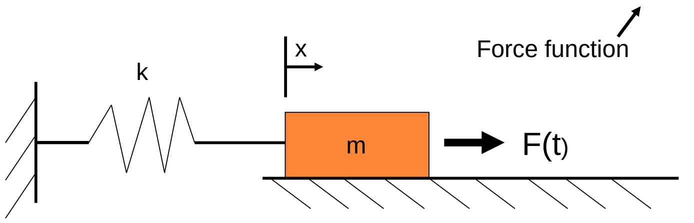
$$\text{Let } t = \ln x$$

$$y(x) = c_1 e^{2\ln x} + c_2 e^{-3\ln x} = c_1 x^2 + c_2 x^{-3}$$

Higher Order Linear Non-Homogeneous O.D.E.

Non – Homogenos Equation

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots a_{n-1}(x)y' + a_n(x)y = f(x)$$



$$mx(t)'' + kx(t) = F(t)$$

Remind: General Solution of the non-homogeneous O.D.E.

$$y(x) = y_h(x) + y_p(x)$$

Method1 : The method of Variation of Parameter

$$y'' + p(x)y' + q(x)y = f(x)$$

suppose *we can find a fundamental set of solutions* $y_1 \,\&\, y_2$ *for the* homogeneous equation

Let $y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$

$$y_p'(x) = c_1'y_1 + c_1y_1' + c_2'y_2 + c_2y_2'$$

suppose $c_1'y_1 + c_2'y_2 = 0 \rightarrow (1)$

$$y_p' = c_1 y_1' + c_2 y_2'$$

$$y_p'' = c_1' y_1' + c_1 y_1'' + c_2' y_2' + c_2 y_2''$$

Substitute into O.D.E

$$\left[c_1' y_1' + c_1 y_1'' + c_2' y_2' + c_2 y_2''\right] + p(x)\left[c_1 y_1' + c_2 y_2'\right]$$

$$+ q(x)\left[c_1 y_1 + c_2 y_2\right] = f(x)$$

$$\Rightarrow c_1' y_1' + c_2' y_2' = f(x) \rightarrow (2)$$

$$Form\ (1)\ (2)\ \begin{cases} c_1' y_1 + c_2' y_2 = 0 \\ c_1' y_1' + c_2' y_2' = f(x) \end{cases}$$

$$c_1' = -\frac{y_2 f(x)}{w}, \qquad c_2' = \frac{y_1 f(x)}{w}$$

w : *wronskian of* y_1 & y_2

$\Rightarrow$ *Integrate these equations to obtain* c_1 & c_2

$\Rightarrow y_p$ *is found*

High Order Linear O.D.E

(1) Constant Coefficient Linear Homogeneous

(2) Non - constant Coefficient Linear Homogeneous

(Euler's Equation)

(3) Non - Homogeneous Equation

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \dots\dots\dots + a_{n-1}(x)y' + a_n(x)y = f(x)$$

Method 1 *: Variation of parameters*

$$y_p = c_1(x)y_1(x) + c_2(x)y_2(x)$$

$$c_1' = -\frac{y_2 f(x)}{w}, \quad c_2' = \frac{y_1 f(x)}{w}$$

w : wronskian of y_1 & y_2

$$ex: \; y'' - \frac{4}{x}y' + \frac{4}{x^2}y = x^2 + 1 \qquad for \quad x > 0$$

$$Homogenous \;\; Equation \qquad y'' - \frac{4}{x}y' + \frac{4}{x^2}y = 0$$

$$\rightarrow Euler's \;\; Equation \qquad y_1(x) = x, \; y_2(x) = x^4$$

$$y_h = c_1 x + c_2 x^4$$

$$wronskian: \; w(x) = \begin{vmatrix} x & x^4 \\ 1 & 4x^3 \end{vmatrix} = 3x^4$$

$$c_1'(x) = \frac{-y_2 f(x)}{w} = \frac{-x^4(x^2+1)}{3x^4} = -\frac{1}{3}(x^2+1)$$

$$c_1(x) = -\frac{1}{9}x^3 - \frac{1}{3}x$$

$$c_2'(x) = \frac{y_1 f(x)}{w} = \frac{x(x^2+1)}{3x^4} = \frac{1}{3}(\frac{1}{x} + \frac{1}{x^3})$$

$$c_2(x) = \frac{1}{3}\ln|x| - \frac{1}{6x^2}$$

$$\Rightarrow y_p = c_1(x)y_1 + c_2(x)y_2$$

$$= (-\frac{1}{9}x^3 - \frac{1}{3}x)x + (\frac{1}{3}\ln|x| - \frac{1}{6x^2})x^4$$

General Solution is

$$y(x) = y_h + y_p$$

$$= c_1x + c_2x^4 + (-\frac{1}{9}x^3 - \frac{1}{3}x)x + (\frac{1}{3}\ln|x| - \frac{1}{6x^2})x^4$$

$$= c_1x + c_2x^4 - \frac{1}{9}x^4 - \frac{1}{2}x^2 + \frac{1}{3}x^4\ln|x|$$

◎ The Method of Undetermined Coefficients

(only if p(x) & g(x) are constants)

y" +Ay' +By= f(x)

Guess the general form of y_p from f(x)

$f(x) = f_1(x) + \ldots + f_k(x)$

Note : Repeated differentiation of each $f_j(x)$ term produces only a finite number of linear independent terms

$$ex: f(x) = 2xe^{-x}$$

$$\rightarrow \left\{2xe^{-x}, 2e^{-x} - 2xe^{-x}, -4e^{-x} + 2xe^{-x}, \ldots\ldots\ldots\right\}$$

only contains 2 L.I. function $\left\{e^{-x}, xe^{-x}\right\}$

$$f(x) = \frac{1}{x}$$

$$\rightarrow \left\{\frac{1}{x}, -\frac{1}{x^2}, \frac{1}{x^3}, \frac{-6}{x^4} \ldots\ldots\ldots\ldots\right\}$$

infinite number of L.I. terms $\left\{\frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}, \ldots\ldots\ldots\ldots\right\}$

Special forms

$f(x)$	y_p
k	c
e^{ax}	ce^{ax}
$\cos ax/\sin ax$	$A\cos ax+B\sin ax$
x^n	$A_nx^n+A_{n-1}x^{n-1}+\ldots\ldots+A_1x+A_0$

$ex: y'' - 4y = 8x^2 - 2x$

$Aussume \quad y_p(x) = ax^2 + bx + c \ , \ y'_p = 2ax + b \ , \ y''_p = 2a$

Substitute into O.D.E

$2a - 4(ax^2+bx+c) = 8x^2 - 2x$

$\Rightarrow (-4a-8)x^2 + (-4b+2)x + (2a-4c) = 0$

$\Rightarrow a = -2 \qquad b = \frac{1}{2} \qquad c = -1$

$\therefore y_p = -2x^2 + \frac{1}{2}x - 1$

$Homogeneous \quad Part: \ y_h = c_1e^{2x} + c_2e^{-2x}$

$General \quad Solution:$

$y = y_h + y_p = c_1e^{2x} + c_2e^{-2x} - 2x^2 + \frac{1}{2}x - 1$

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Ex: $y'' - 6y' + 9y = 5e^{3x}$

$y_h = c_1e^{3x} + c_2xe^{3x} \qquad y_p = Ax^2e^{3x} \quad (A = \frac{5}{2})$

◎ The principle of superposition

Ex: $y'' + 4y = x + 2e^{-2x}$

$y'' + 4y = x \rightarrow \ y_p = 1/4\ x$

$y'' + 4y = 2e^{-2x} \ \rightarrow y_p = 1/4\ e^{-2x}$

$\therefore y_p = 1/4(x + e^{-2x})$

General solution is

$y(x) = c_1\cos(2x) + c_2\sin(2x) + 1/4(x + e^{-2x})$

3.1 Preliminary Theory: Linear Equ.

- **Initial-value Problem**

  An *n*th-order initial problem is

  Solve:

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

  Subject to:

$$y(x_0)=y_0\ ,\ y'(x_0)=y_1\ ,\cdots,\ y^{(n-1)}(x_0)=y_{n-1}$$

  with n initial conditions.

Theorem 3.1.1 Existence of a Unique Solution

Let $a_n(x)$, $a_{n-1}(x)$, …, $a_0(x)$, and $g(x)$ be continuous on I, $a_n(x) \neq 0$ for all x on I. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of (1) exists on the interval and is unique.

Unique Solution of an IVP

- The IVP

$$3y''' + 5y'' + y' + 7y = 0,\ y(1) = 0,\ y'(1) = 0,\ y''(1) = 0$$

possesses the trivial solution $y = 0$. Since this DE with constant coefficients, from Theorem 3.1.1, hence $y = 0$ is the only one solution on any interval containing $x = 1$.

Unique Solution of an IVP

- Please verify $y = 3e^{2x} + e^{-2x} - 3x$, is a solution of

$$y'' - 4y = 12x,\ y(0) = 4,\ y'(0) = 1$$

- This DE is linear and the coefficients and $g(x)$ are all continuous, and $a_2(x) \neq 0$ on any I containing $x = 0$. This DE has an unique solution on I.

BOUNDARY-VALUE PROBLEM

Solve: $$a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

Subject to:

$$y(a)=y_0\ ,\ y(b)=y_1$$

is called a **boundary-value problem (BVP).**

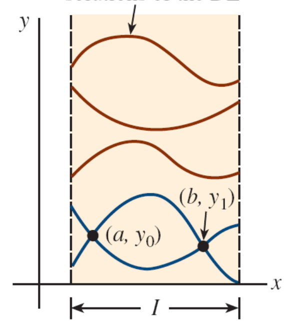

FIGURE 3.1.1 Colored curves are solutions of a BVP

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Applications of Higher Order Linear Homogeneous O.D.E.

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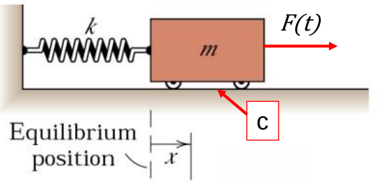
◎ APPLICATION OF 2ND –ORDER ODE

- Free Oscillation



$$mx'' + cx' + kx = F(t)$$

$$x' = \frac{dx}{dt}, \qquad x'' = \frac{d^2x}{dt^2}$$

Initial displacement
$x(0) = x_0$

Initial velocity
$x'(0) = x_0'$

c: damping coefficient
k: spring stiffness

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Figure source: https://open.usq.edu.au/pluginfile.php/77992/mod_resource/content/3/mec3403/vibration-1/vibration-1.htm

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- Free Oscillation => F(t) = 0 (unforced)

O. D. E. => $mx'' + cx' + kx = 0$

Let $x(t) = e^{\lambda t}$

⇒ characteristic equation

⇒ $m\lambda^2 + c\lambda + k = 0$

⇒ Roots: $\lambda = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$

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CASE 1: C = 0 (NO DAMPING)

Friction is small enough to be neglected

- $mx'' + kx = 0,\ \lambda = \pm i\sqrt{\frac{k}{m}}$

$$\Rightarrow x(t) = c_1 e^{iwt} + c_2 e^{-iwt}, w = \sqrt{\frac{k}{m}}$$

$$= A\cos(wt) + B\sin(wt),$$

- Let $A = E\ sin\phi,\ B = E\ cos\phi$

$$x(t) = E(sin\phi\cos(wt) + cos\phi\sin(wt))$$

$$= E\sin(wt + \phi)$$

where $E = \sqrt{A^2 + B^2}$ (amplitude)

$\phi = tan^{-1}\frac{A}{B}$ (phase angle)

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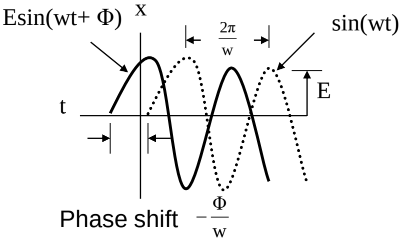


$$x(t) = E\sin(wt + \Phi)$$

$$\Phi = \tan^{-1}\left(\frac{x_0}{x_0'/w}\right)$$

$$= \tan^{-1}\left(\frac{x_0 w}{x_0'}\right)$$

But it is easier to apply the I.C. to x(t) =A coswt +B sinwt

$x(0) = x_0 = A \quad x'(0) = x'_0 = \ W\ B \Rightarrow x(t) = x_0\cos wt + \frac{x_0'}{w}\sin wt$

$$E = \sqrt{A^2 + B^2} = \sqrt{(x_0)^2 + \left(\frac{x_0'}{w}\right)^2}$$

◎ Mathematic ←→ Physics

$$w = \sqrt{\frac{k}{m}}$$

k ↑ , w ↑

m ↑, w ↓

$$E = \sqrt{(x_0)^2 + (\frac{x_0'}{w})^2}$$

x_0 ↑ , E ↑

Case 2 : c > 0

$$Critical \quad Damping \qquad C_{cr} = \sqrt{4mk}$$

$$\lambda = \frac{-c}{2m} \Rightarrow x(t) = (A+Bt)e^{-\frac{c}{2m}t}$$

$$Althoug(A+Bt) \quad growsunboundedly$$

Although (A+Bt) grows unboundedly as t increases
the exponential function decays more powerfully

◎ Underdamped ($C < C_{cr}$)

$$\lambda = \frac{1}{2m}(-c \pm \sqrt{c^2 - c_{cr}^2}) = \frac{1}{2m}(-c \pm i\sqrt{c_{cr}^2 - c^2})$$

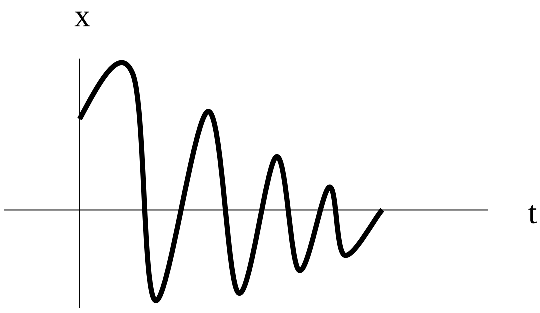
$$= \frac{-c}{2m} \pm i\sqrt{w^2 - (\frac{c}{2m})^2}$$

$$Greneral \quad Solution:$$

$$x(t) = e^{-\frac{c}{2m}t}\left[A\cos(\sqrt{\omega^2 - (\frac{c}{2m})^2}t) + B\sin(\sqrt{\omega^2 - (\frac{c}{2m})^2}t)\right]$$

(1) As $t \to \infty$, the oscillation will "damp out" because of the $e^{-2/2m\,t}$ factor

(2) The frequency is reduced from natural frequency W to $\sqrt{w^2 - (\frac{c}{2m})^2}$



◎ *Overdampecd ($C > C_{cr}$)*

$\lambda_{1,2}$ are both real & negative $\lambda = \dfrac{-c \pm \sqrt{c^2 - 4mk}}{2m}$

$$x(t) = e^{-\frac{c}{2m}t}\left[A\cosh\left(\sqrt{(\frac{c}{2m})^2 - w^2}\,t\right) + B\sinh\left(\sqrt{(\frac{c}{2m})^2 - w^2}\,t\right)\right]$$

2. Force Oscillation

$$mx'' + cx' + kx = f(t) \qquad f(t) = f_0 \cos \Omega t$$

◎ *Undamped case c = 0*

$$mx'' + kx = f_0 \cos \Omega t$$

The homogeneous solution is

x(t) =A coswt+ B sinwt

$$w=\sqrt{\frac{k}{m}},\ (natural\quad frequency)$$

by the method of undetermined coefficient

$$x_p(t)=C\cos\Omega t+D\sin\Omega t$$

$$For\quad Non-resonant\quad ocillation\quad (w\neq\Omega)$$

$$x_p'=-\Omega C\cos\Omega t+\Omega D\sin\Omega t$$

$$x_p''=-\Omega^2 C\cos\Omega t-\Omega^2 D\sin\Omega t$$

$$\Rightarrow(-\Omega^2 C\cos\Omega t-\Omega^2 D\sin\Omega t)+w^2(C\cos\Omega t+D\sin\Omega t)$$

$$=\frac{f_0}{m}\cos\Omega t$$

$$\Rightarrow(w^2-\Omega^2)C\cos\Omega t+(w^2-\Omega^2)D\sin\Omega t=\frac{f_0}{m}\cos\Omega t$$

$$\because w\neq\Omega\qquad c=\frac{f_0/m}{w^2-\Omega^2}\quad,\quad D=0\Rightarrow x_p=\frac{f_0/m}{w^2-\Omega^2}\cos\Omega t$$

The general solution is

$$x(t)=x_h+x_p$$

$$=A\ \cos wt+\ B\ \sin wt\ +\frac{f_0/m}{w^2-\Omega^2}\cos\Omega t$$

$$=E\sin(wt+\phi)+\frac{f_0/m}{w^2-\Omega^2}\cos\Omega t$$

Note : (1) f_0 $\&\,\Omega$ *a re controllable parameters*

(2) The force response is at the same frequency as the forcing function Ω

Resonant Oscillation $(w=\Omega)$

$$x_p(t)=t(C\cos wt+D\sin wt)$$

$$\Rightarrow C=0, D=f_0/2mw \qquad \therefore x_p=\frac{f_0}{2mw}t\sin wt$$

1. The response is not a harmonic oscillation ,but a harmonic function times t

2. The magnitude dose not grow unboundedly in a real application

3 . Resonant case is sometimes welcome & sometimes unwelcome

$Beats\ (\Omega \to \omega)$

$(One\ can\ never\ get\ \Omega - exactly\ equal\ w$

$It\ is,\ therefore,\ of\ interest\ to\ look\ at\ the$

$solution\ x(t)\ as\ \Omega\ approaches\ w)$

$Use\ simple\ I.C.:\ x(0)=0\ ,\ \ x'(0)=0$

$$x(t) = -\frac{f_0/m}{w^2-\Omega^2}(\cos wt - \cos \Omega t)$$

$$= \frac{2f_0/m}{w^2-\Omega^2}\sin(\frac{w+\Omega}{2}t)\sin(\frac{w-\Omega}{2}t)$$

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$Damped\ case\ (c>0)$

$$mx'' + cx' + kx = f_0 \cos \Omega t$$

$Homogenous\ solution$

$$x(t) = \begin{cases} e^{-\frac{c}{2m}t}\left[A\cos(\sqrt{w^2-(\frac{c}{2m})^2})t + B\sin(\sqrt{w^2-(\frac{c}{2m})^2})t\right] & (c<c_{cr}) \\ e^{-\frac{c}{2m}t}(A+Bt) \quad (c=c_{cr}) & \\ e^{-\frac{c}{2m}t}\left[A\cosh\sqrt{(\frac{c}{2m})^2+w^2}\ t + B\sinh\sqrt{(\frac{c}{2m})^2+w^2}\ t\right] & (c>c_{cr}) \\ c_{cr}=? \qquad w=? & \end{cases}$$

$$\text{Assume } x_p = C\cos\Omega t + D\sin\Omega t$$

$$\Rightarrow x_p(t) = \frac{\frac{f_0}{m}(w^2-\Omega^2)}{(w^2-\Omega^2)^2 + (\frac{c\Omega}{m})^2}\cos\Omega t$$

$$+\frac{\frac{f_0 c\Omega}{m^2}}{(\omega^2-\Omega^2)^2 + (\frac{c\Omega}{m})^2}\sin\Omega t$$

$$\Rightarrow x_p(t) = E\cos(\Omega t + \phi)$$

$$E = \frac{\frac{f_0}{m}}{\sqrt{(w^2-\Omega^2)^2 + (\frac{c\Omega}{m})^2}} \qquad \phi = \tan^{-1}\frac{\frac{c\Omega}{m}}{\Omega^2 - w^2}$$

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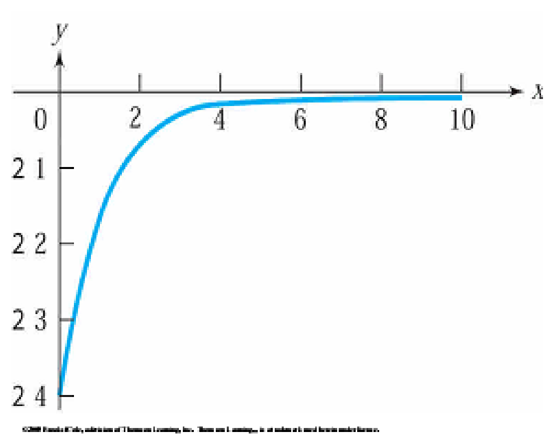


Figure 2.5 *An example of overdamped motion, no driving force.*

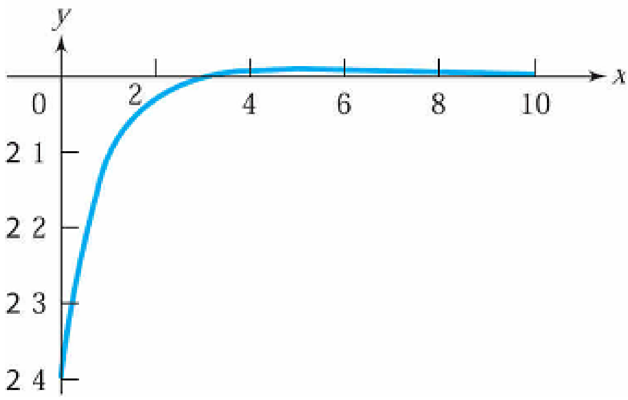


Figure 2.6 *An example of critical damped motion, no driving force.*

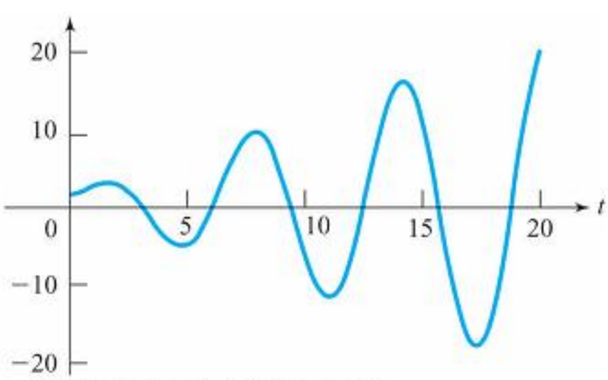


Figure 2.11 *Resonance*

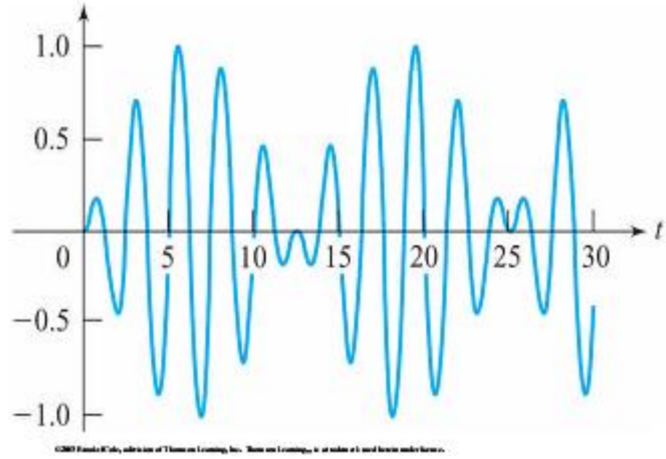
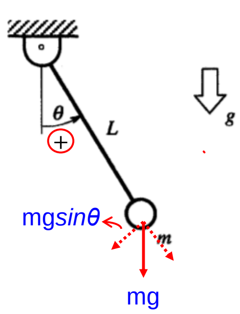


Figure 2.12 ***Beats***

(1)



Equation of Motion

$$-mL\theta'' = mgsin\theta$$
$$\Rightarrow mL\theta'' + mgsin\theta = 0$$

- If $\|\theta\| \ll 1 \;\Rightarrow sin\theta \approx \theta \Rightarrow mL\theta'' + mg\theta = 0$
  $$\Rightarrow \theta'' + \frac{g}{L}\theta = 0$$
- Damping due to friction or air resistance
  $$\theta'' + \epsilon\theta + \frac{g}{L}\theta = 0$$

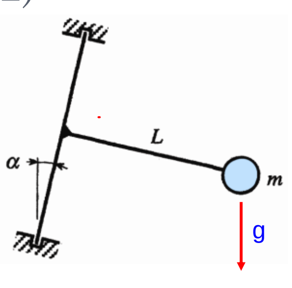
If a pendulum mechanism converts each oscillation to one second of recorded time. How does the clock maintain its accuracy even when its amplitude of oscillation has diminished to a small fraction of its initial value

$\Longrightarrow$ If $\epsilon \ll 1$, then $\theta'' + \epsilon\theta + \frac{g}{L}\theta = 0$ is underdamped

Solution:

$$\theta(t) = e^{\frac{-\epsilon t}{2}}\left[A\cos\left(\sqrt{\frac{g}{L} - \left(\frac{\epsilon}{2}\right)^2}\,t\right) + B\ \sin\left(\sqrt{\frac{g}{L} - \left(\frac{\epsilon}{2}\right)^2}\,t\right)\right]$$

(2)



- The mass of rod is negligible compared with m
- The rod is welded at the right angle to the other
- θ is the angle of rotation of the pendulum w.r.t. the equilibrium position where m is at the lowest point
- The system rotates without friction

1. Please derive the equation of motion.
2. What is the frequency of small amplitude oscillation?

- Potential Energy $= mgh = mgL\,(1-\cos\theta)sin\alpha$

  Kinetic Energy $= \frac{1}{2}mv^2 = \frac{1}{2}m(L\theta')^2$

  $\Rightarrow P.E.+K.E.= constant$

  $\Rightarrow mgL\,(1-\cos\theta)sin\alpha + \frac{1}{2}m(L\theta')^2 = constant$

- Differentiate with respect to t

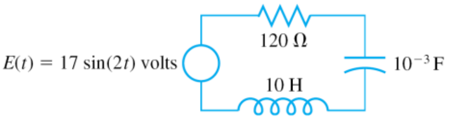
$$\frac{d}{dt}\left[mgL\,(1-\cos\theta)sin\alpha+\frac{1}{2}m(L\theta')^2\right]=0$$

$$\Rightarrow mgL\theta'\sin\theta\; sin\alpha + mL^2\theta''\theta' = 0$$

$$\Rightarrow\ \theta'' + \frac{g}{L}sin\alpha\; sin\theta = 0$$

- If $\theta \ll 1,\ \Rightarrow \theta'' + \frac{g}{L}sin\alpha\,(\theta) = 0$

(3) RLC SERIES CIRCUIT



- $E(t) = Li'(t) + Ri(t) + \frac{1}{C}q(t)$

i: current, q: charge

$\Rightarrow q'(t) = i(t)$

$\Rightarrow E(t) = Lq''(t) + Rq'(t) + \frac{1}{C}q(t)$

$\Rightarrow 17\sin(2t) = 10q''(t) + 120q'(t) + 1000q(t)$

Initial Conditions:

$$q(0) = \frac{1}{2000}(coulumb), \qquad q'(0) = 0$$

$\Rightarrow q(t) =$

$$\frac{1}{1500}e^{-6t}[7\cos(8t) - \sin(8t)] + \frac{1}{240}[4\sin(2t) - \cos(2t)]$$

$\Rightarrow q'^{(t)} = i(t)$

$$\approx \underbrace{\frac{1}{30}e^{-6t}[-7\,sin(8t) + \cos(8t)]}_{\text{暫態響應}} + \underbrace{\frac{1}{120}[4\cos(2t) + sin(2t)]}_{\text{穩態響應}}$$

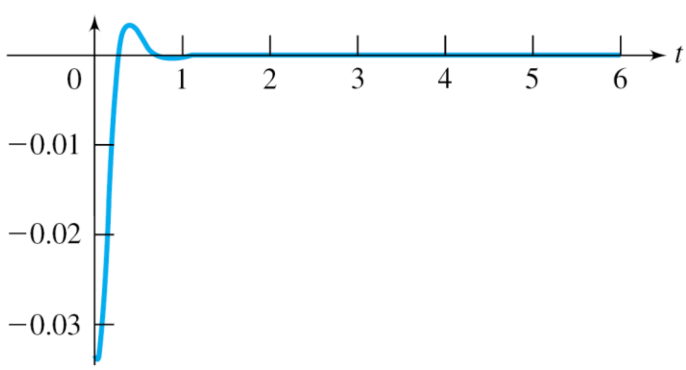


FIGURE 2.14 *Transient part of the current for the circuit of Figure 2.13.*

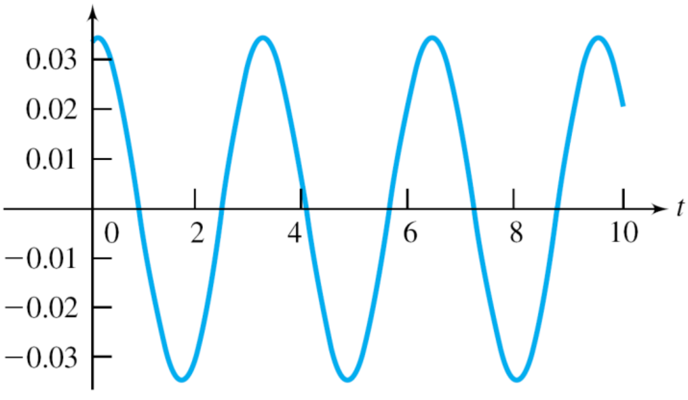


FIGURE 2.15 *Steady-state part of the current for the circuit of Figure 2.13.*

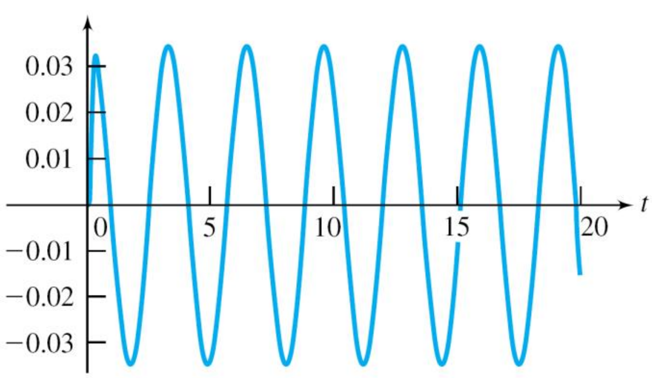


FIGURE 2.16 *Current function for the circuit of Figure 2.13.*