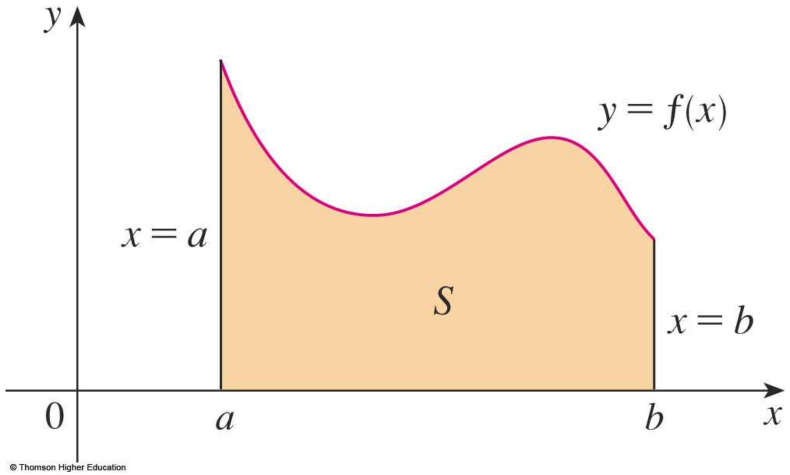
Integrals

Lecture Note 6

Sec. 5.1 – Sec. 5.5

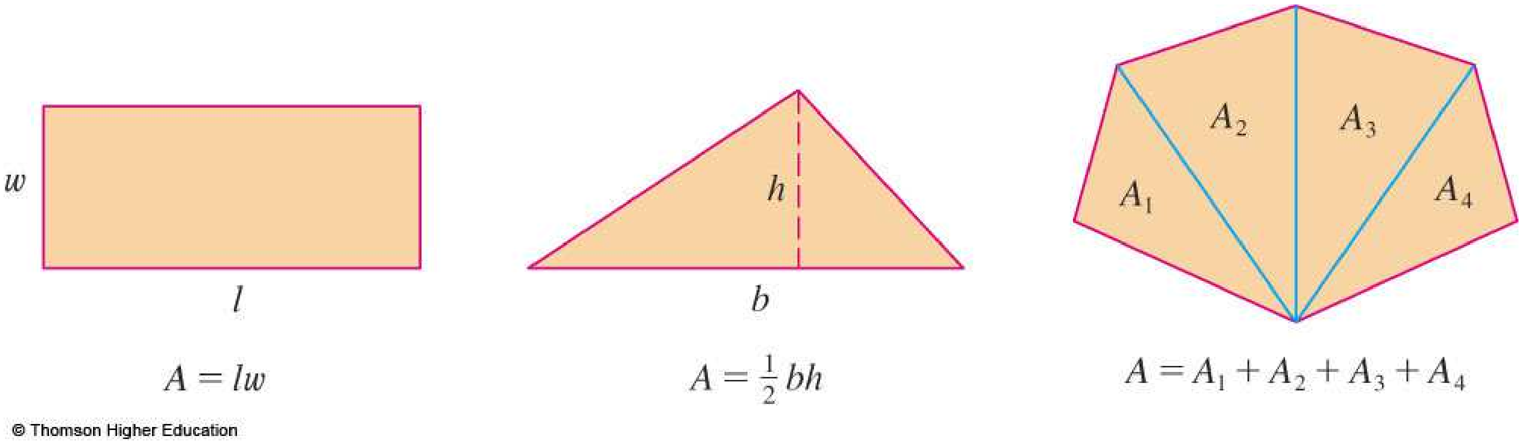
<u>Sec. 5.1</u> Areas and Distances

<u>The Area Problem</u>

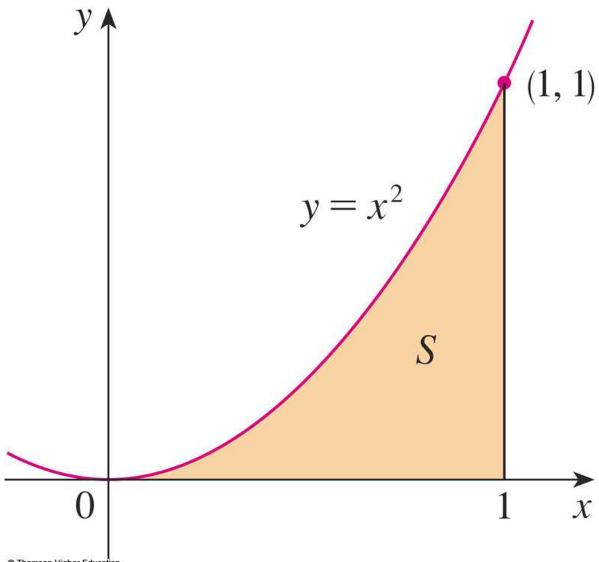


有沒有通用的方法，可以求在一個範圍內曲線下的面積 A？

面積的定義，與多邊形面積的求法：



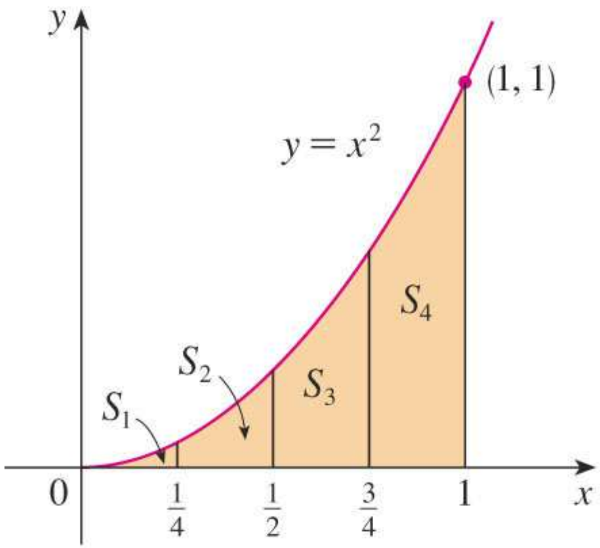
以上方法無法應用到求曲線下的的面積。



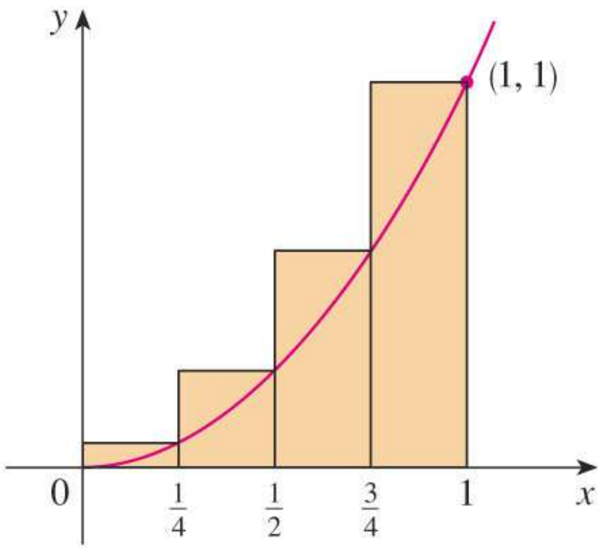
利用多個長方形近似曲線下的面積：

$$R_4 = \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 + \frac{1}{4}\cdot(1)^2 = 0.46875$$

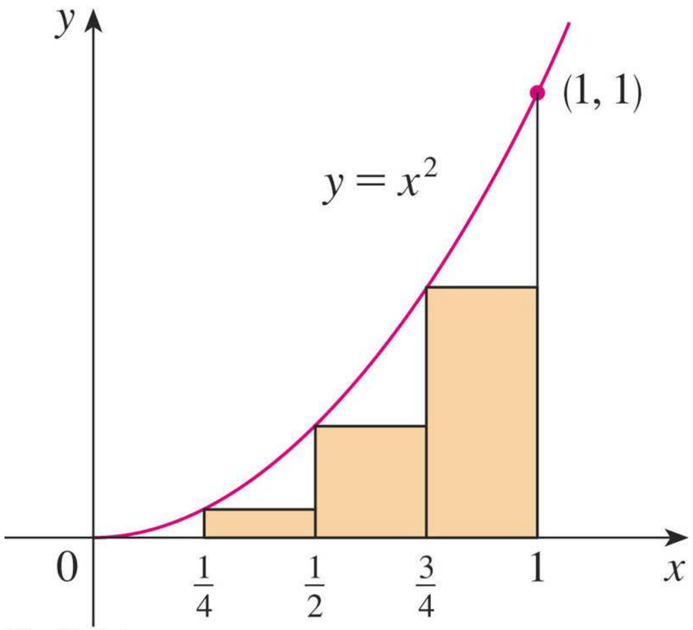
$$A < 0.46875$$



(a)

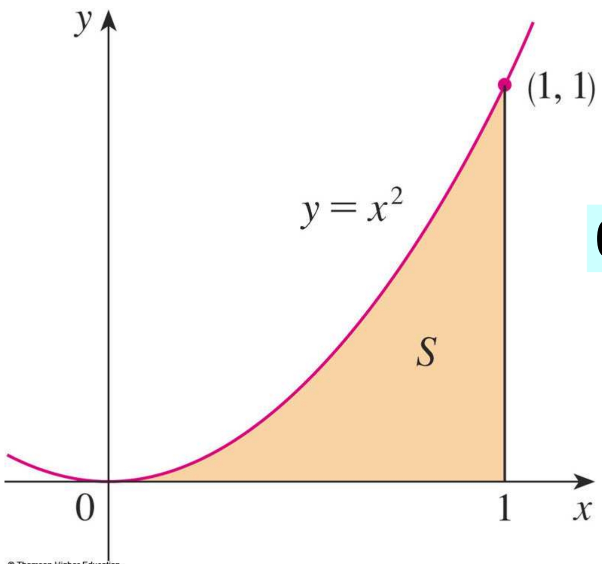


(b)



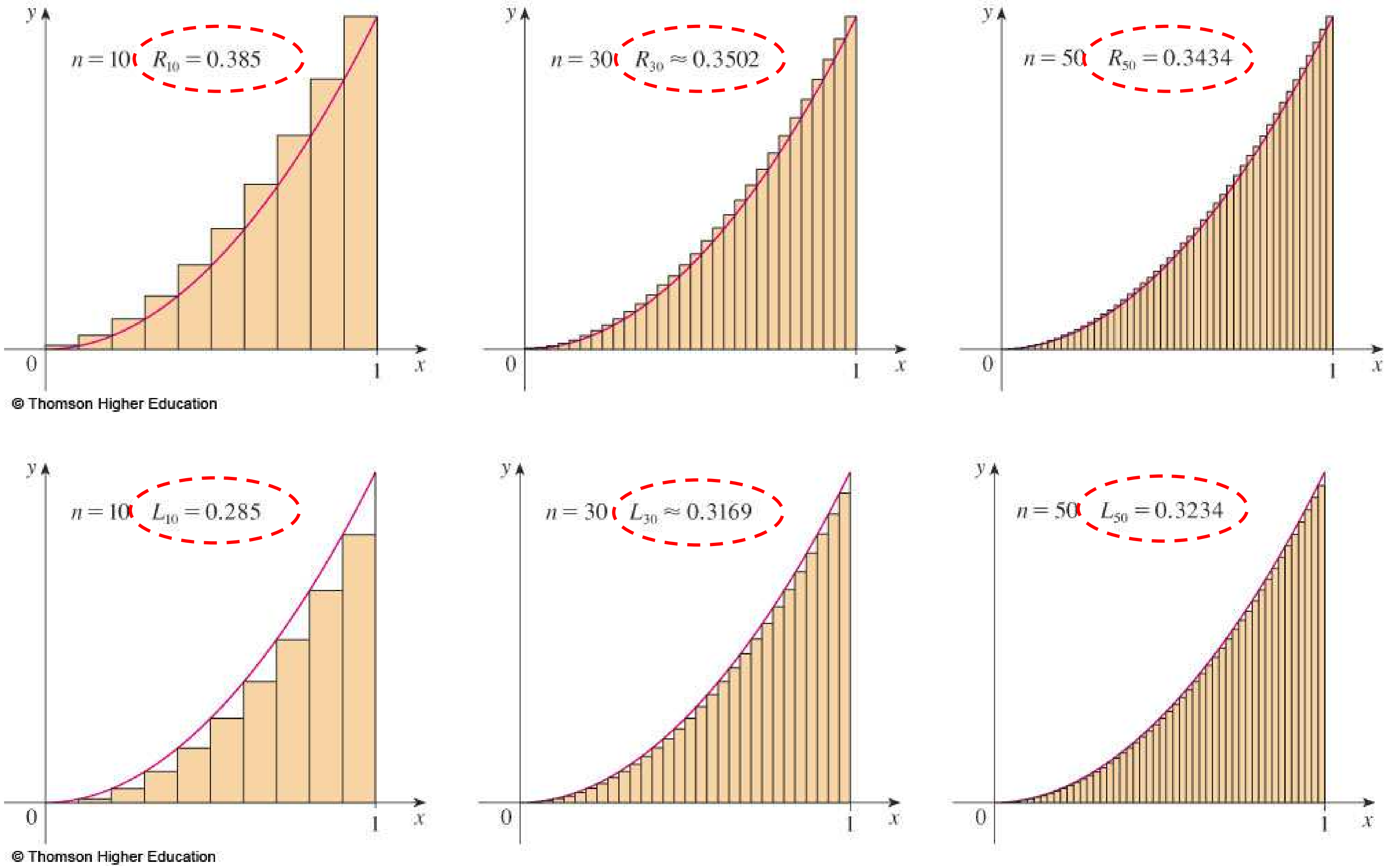
$$L_4 = \frac{1}{4}\cdot(0)^2 + \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 = 0.21875$$

$$A > 0.21875$$



$$0.21875 < A < 0.46875$$

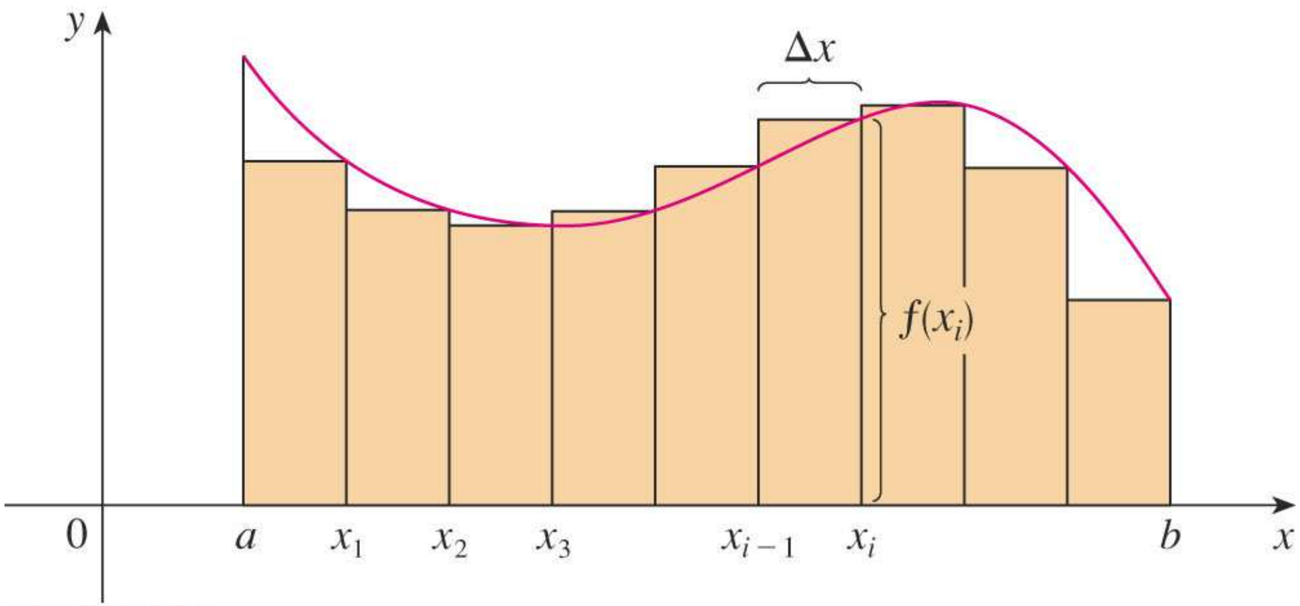
分割的越細，得到的面積越接近實際曲線下的面積。



DEFINITION **The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:**

$$A \equiv \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x \right]$$

取矩形右端點當高度值

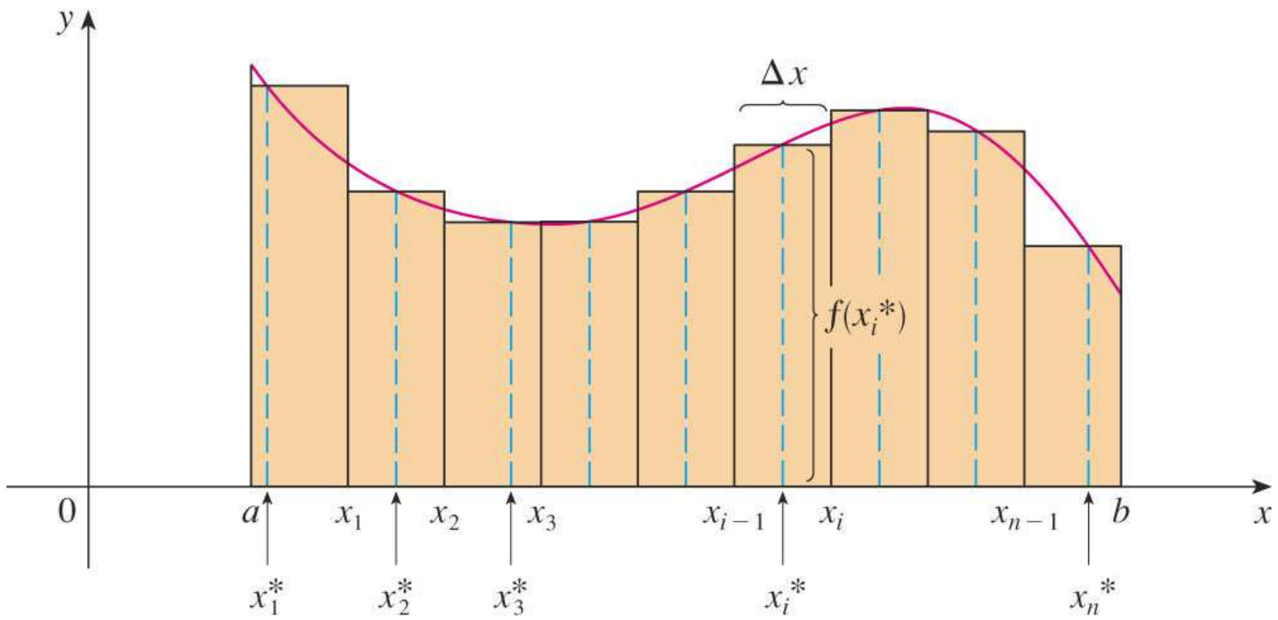


取矩形左端點當高度值：

$$A=\lim_{n\to\infty} L_n=\lim_{n\to\infty}\left[f(x_0)\Delta x+f(x_1)\Delta x+\cdots+f(x_{n-1})\Delta x\right]$$

取矩形中間任一點（Sample Point）當高度值：

$$A=\lim_{n\to\infty}\left[f(x_1^*)\Delta x+f(x_2^*)\Delta x+\cdots+f(x_n^*)\Delta x\right]$$



Sigma Notation

$$f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x$$

Use of sigma notation for area formula

右端點當高度值： $$A = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

左端點當高度值： $$A = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_{i-1})\Delta x$$

中間某點當高度值： $$A = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

Example

Let A be the area of the region that lies under the graph of $f(x)=x^2$ between $x=0$ and $x=1$. Find A.

(a)

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}$$

$$x_1 = \frac{1}{n}, \quad x_2 = \frac{2}{n}, \quad x_i = \frac{i}{n}, \quad x_n = \frac{n}{n} = 1$$

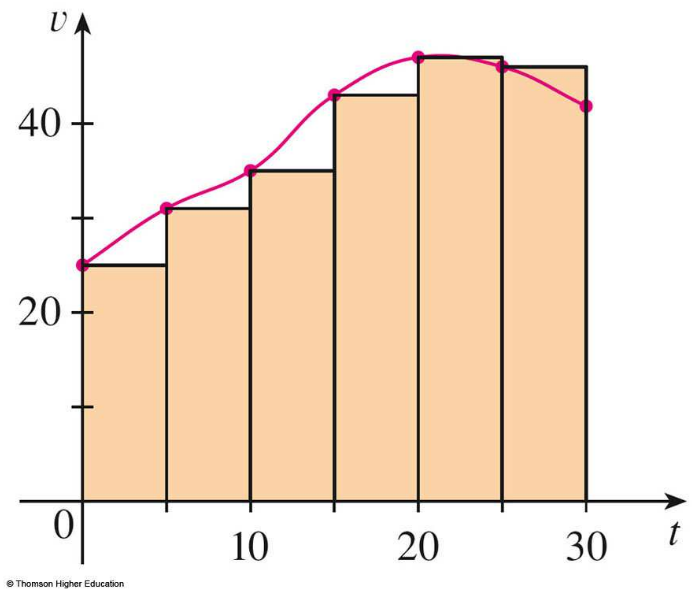
$$\begin{aligned} R_n &= f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x \\ &= x_1^2 \Delta x + x_2^2 \Delta x + \cdots + x_n^2 \Delta x \\ &= \left(\frac{1}{n}\right)^2 \frac{1}{n} + \left(\frac{2}{n}\right)^2 \frac{1}{n} + \cdots + (1)^2 \frac{1}{n} \\ &= \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + n^2\right) \end{aligned}$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\begin{aligned}
A &= \lim_{n\to\infty} R_n \\
&= \lim_{n\to\infty} \left[\frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \right] \\
&= \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} \\
&= \lim_{n\to\infty} \frac{1}{6} \cdot \frac{n+1}{n} \cdot \frac{2n+1}{n} \\
&= \frac{1}{3}
\end{aligned}$$

The Distance Problem

距離 = 速度 × 時間



左端點高度當速度值：

$$v(t_0)\Delta t + v(t_1)\Delta t + \cdots + v(t_{n-1})\Delta t = \sum_{i=1}^{n} v(t_{i-1})\Delta t$$

$$d = \lim_{n\to\infty} \sum_{i=1}^{n} v(t_{i-1})\Delta t$$

右端點高度當速度值：

$$d = \lim_{n\to\infty} \sum_{i=1}^{n} v(t_i)\Delta t$$

Sec. 5.2 The Definite Integral

DEFINITION OF A DEFINITE INTEGRAL **If f is a function defined for $a \leqq x \leqq b$, we divide the interval [a, b] into n subintervals of equal width $x=(b-a)/n$. We let $x_0(=a)$, x_1, x_2, …, $x_n(=b)$ be the endpoints of these subintervals and we let x_1*, x_2*, …, x_n* be any sample points in these subintervals, so x_i* lies in the i-th subinterval [x_{i-1}, x_i]. Then the definite integral of f from a to b is**

$$\int_a^b f(x)dx = \overbrace{\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x}^{\text{Riemann Sum}}$$

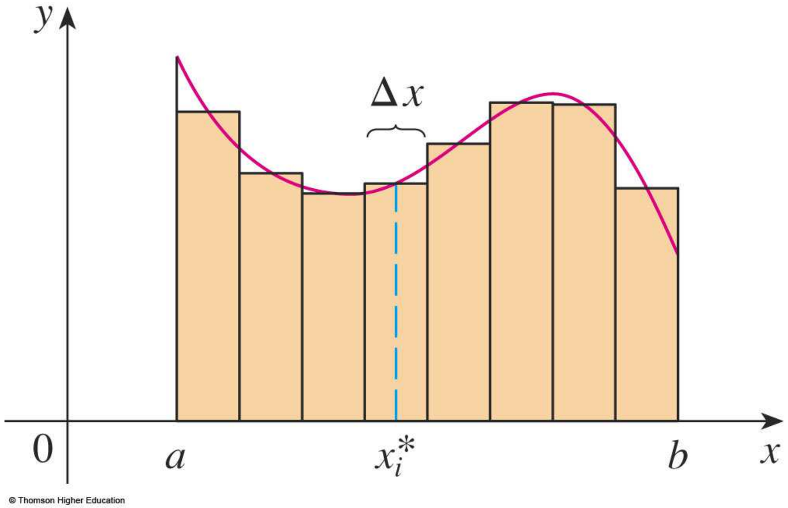
provided that this limit exists. If it does exist, we say that f is integrable on [a, b].

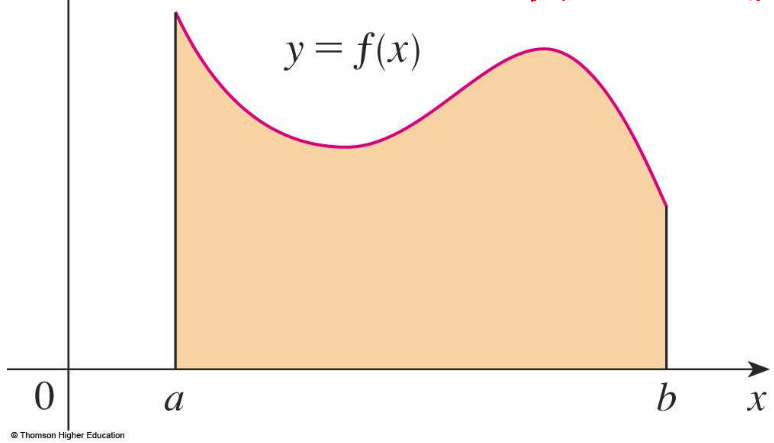
$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

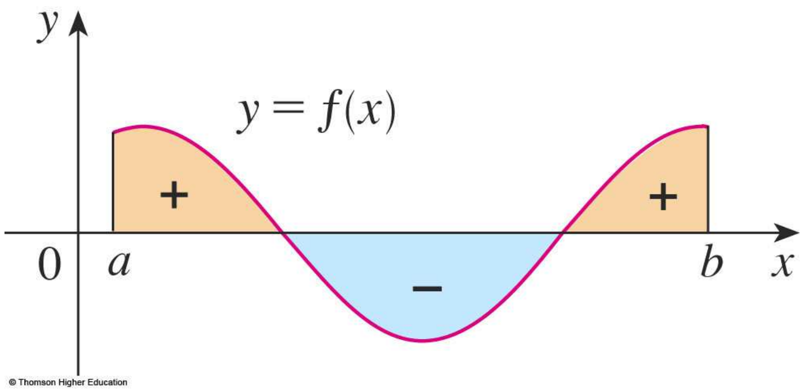
近似的面積

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

真正的面積







$$\int_a^b f(x)dx = A_1 - A_2$$

淨面積（Net Area）

Example 1

Express $\lim\limits_{n\to\infty}\sum_{i=1}^{n}(x_i^3 + x_i \sin x_i)\Delta x$ **as an integral on the interval [0, π]**

$$\lim_{n\to\infty}\sum_{i=1}^{n}(x_i^3 + x_i \sin x_i)\Delta x = \int_0^{\pi}(x^3 + x\sin x)dx$$

Evaluate Integrals

一些級數和的公式

$$\sum_{i=1}^{n} i = 1+2+\ldots+n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1^2+2^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

Page A36 for PROOF

$$\sum_{i=1}^{n} i^3 = 1^3+2^3+\ldots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

一些級數和的恆等式

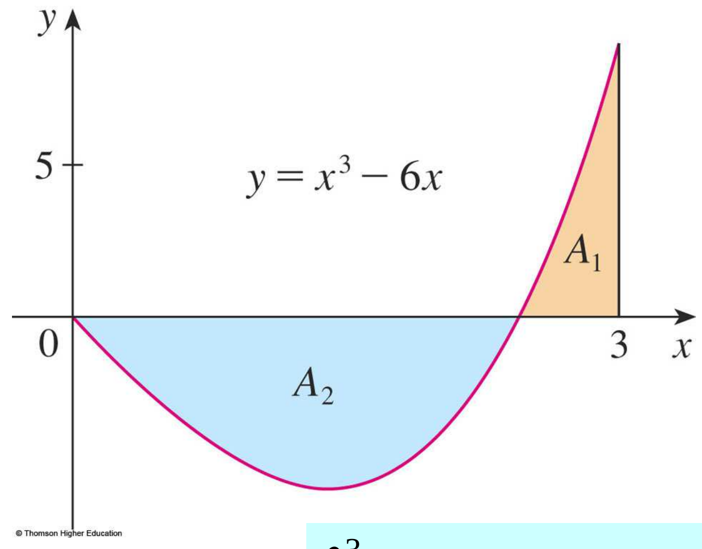
$$\sum_{i=1}^{n} c = nc$$

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$$

Example 2

Evaluate $\int_0^3 (x^3 - 6x)dx$



$$\int_0^3 (x^3 - 6x)dx = A_1 - A_2$$

$$\int_0^3 (x^3 - 6x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(\frac{3i}{n})\frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n}\left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right] = \lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n}\left[\frac{27}{n^3}i^3 - \frac{18}{n}i\right]$$

$$= \lim_{n\to\infty}\left[\frac{81}{n^4}\sum_{i=1}^{n} i^3 - \frac{54}{n^2}\sum_{i=1}^{n} i\right] = \lim_{n\to\infty}\left\{\frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{54}{n^2}\frac{n(n+1)}{2}\right\}$$

$$= \lim_{n\to\infty}\left[\frac{81}{4}\left(1+\frac{1}{n}\right)^2 - 27\left(1+\frac{1}{n}\right)\right] = \frac{81}{4} - 27 = -6.75$$

Example

$$\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\cdots+\frac{n}{n^2+n^2}\right]$$

$$\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\cdots+\frac{n}{n^2+n^2}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[\frac{n^2}{n^2+1^2}+\frac{n^2}{n^2+2^2}+\frac{n^2}{n^2+3^2}+\cdots+\frac{n^2}{n^2+n^2}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{1+\left(1/n\right)^2}+\frac{1}{n^2+\left(2/n\right)^2}+\frac{1}{n^2+\left(3/n\right)^2}+\cdots+\frac{1}{n^2+\left(1/n\right)^2}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+\left(i/n\right)^2}$$

$$=\int_0^1\frac{1}{1+x^2}dx$$

Properties of the Definite Integral

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

上下限交換差一個負號

$$\int_a^a f(x)dx = 0$$

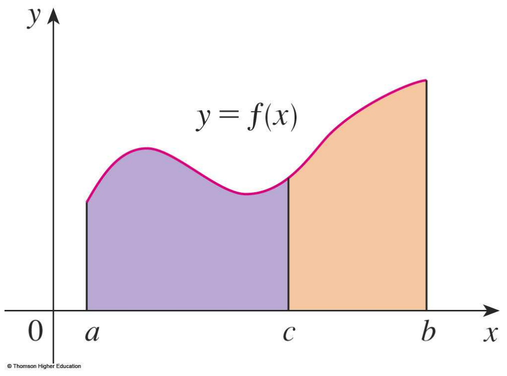
上下限相同，即面積為零

$$\int_a^b c\,dx = c(b-a), \quad \text{where } c \text{ is any constant}$$

$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$$



Prove that $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

$$
\begin{aligned}
\int_a^b [f(x) \pm g(x)]dx &= \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) \pm g(x_i)]\Delta x \\
&= \lim_{n\to\infty} \left[\sum_{i=1}^{n} f(x_i)\Delta x \pm \sum_{i=1}^{n} g(x_i)\Delta x \right] \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \pm \lim_{n\to\infty} \sum_{i=1}^{n} g(x_i)\Delta x \\
&= \int_a^b f(x)dx \pm \int_a^b g(x)dx
\end{aligned}
$$

Example 6

Evaluate $\int_0^1 (4+3x^2)dx$

$$\int_0^1 (4+3x^2)dx = \int_0^1 4dx + 3\int_0^1 x^2 dx$$

$$= 4(1-0) + 3\cdot\frac{1}{3}$$

$$= 5$$

Example 7

It is known that $\int_0^{10} f(x)dx = 17$ **and** $\int_0^{8} f(x)dx = 12$ **,**

find $\int_8^{10} f(x)dx.$

$$\int_0^{8} f(x)dx + \int_8^{10} f(x)dx = \int_0^{10} f(x)dx$$

$$\int_8^{10} f(x)dx = \int_0^{10} f(x)dx - \int_0^{8} f(x)dx = 17 - 12 = 5$$

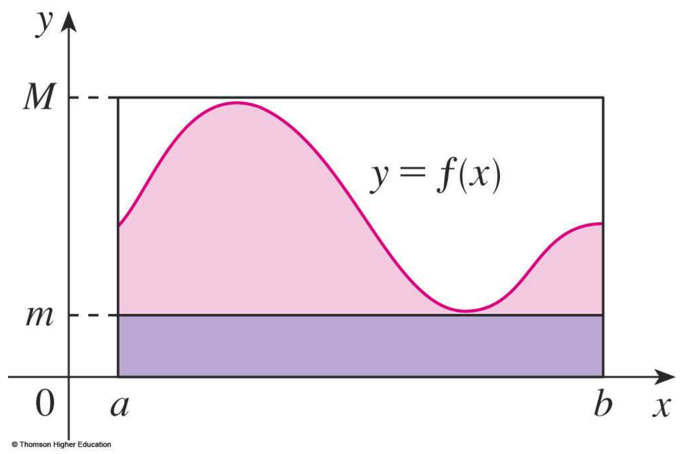
COMPARISON PROPERTIES OF THE INTEGRAL

If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0.$

If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx.$

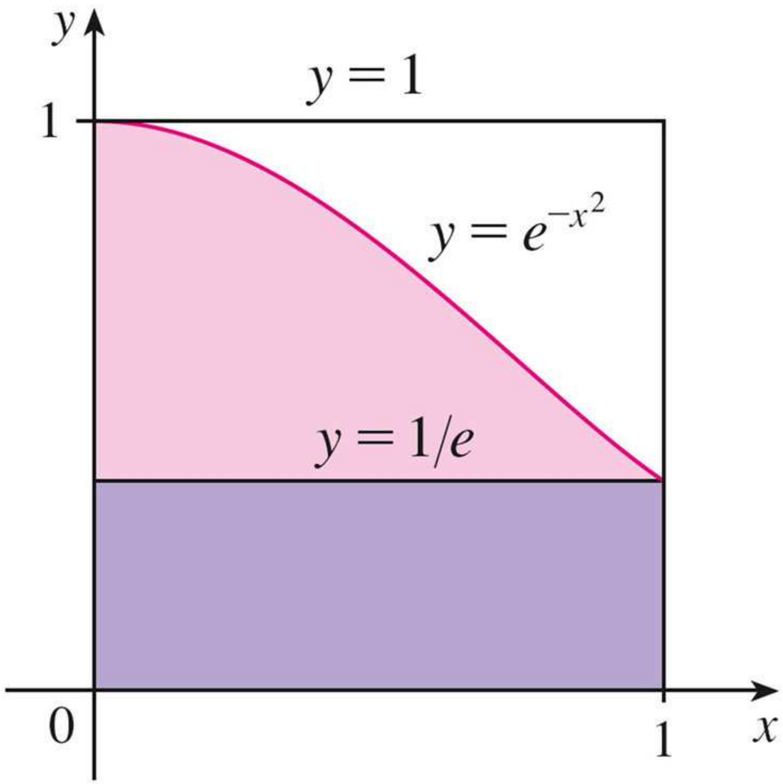
If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$



Example 8

Use the comparison property to estimate $\int_0^1 e^{-x^2}\,dx$

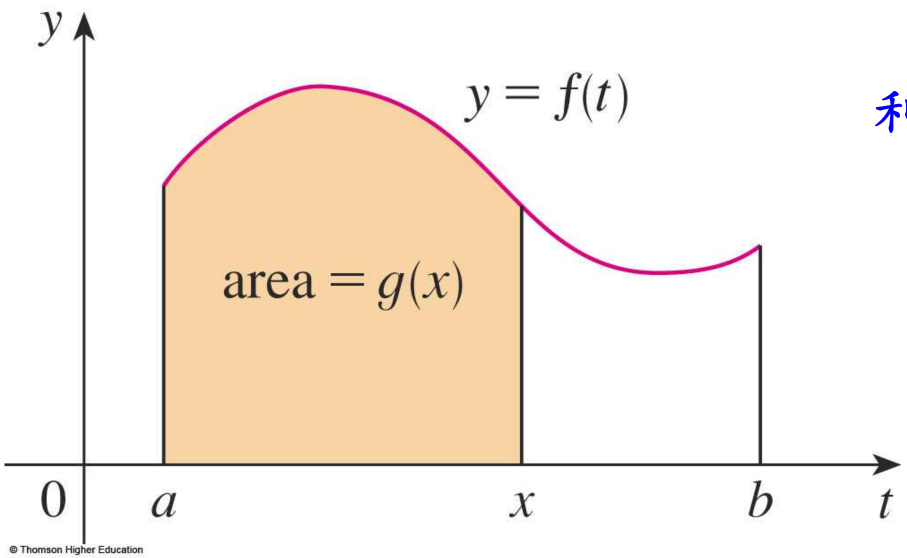


on $[0,1]$, $e^{-1} \le e^{-x^2} \le 1$

$$e^{-1}(1-0) \le \int_0^1 e^{-x^2}\,dx \le 1\cdot(1-0)$$

$$e^{-1} \le \int_0^1 e^{-x^2}\,dx \le 1$$

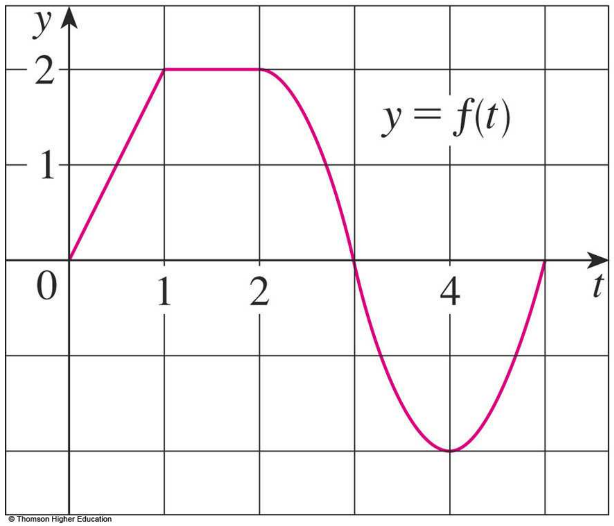
<u>Sec. 5.3</u> The Fundamental Theory of Calculus



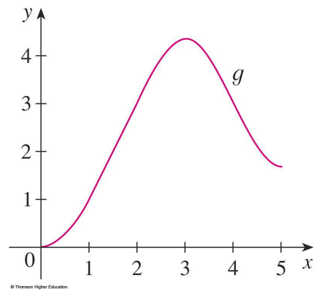
利用定積分為面積觀念，可以定義一函數

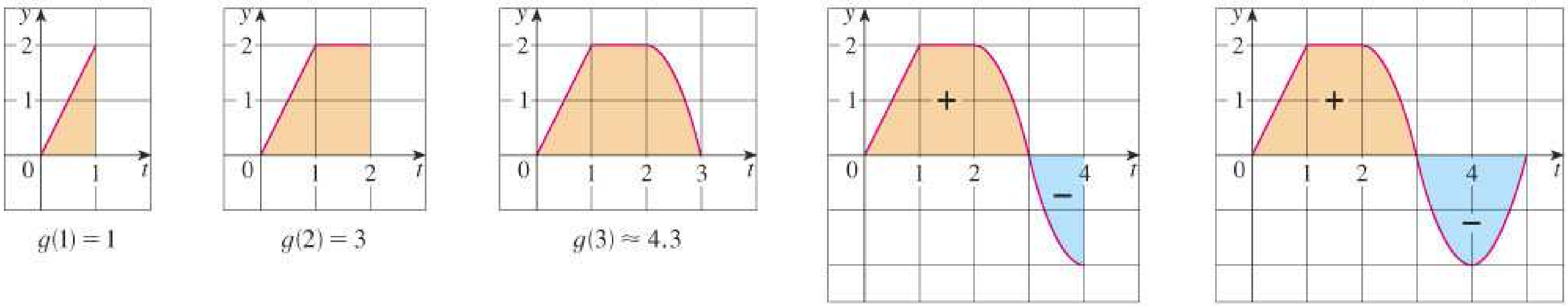
$$g(x)=\int_a^x f(t)dt$$

Example 1



$$g(x) = \int_{a=0}^{x} f(t)dt$$





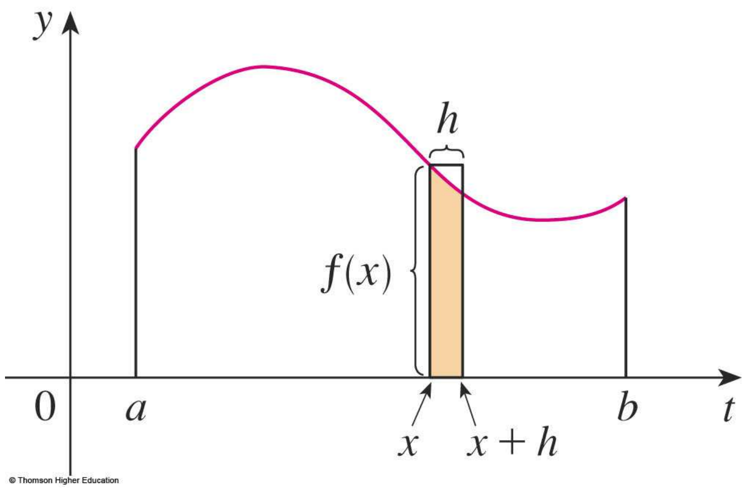
$g(1) = 1$ $g(2) = 3$ $g(3) \approx 4.3$ $g(4) \approx 3$ $g(5) \approx 1.7$

$g(x)=\int_a^x f(t)dt$ 是一個函數，可否對這一函數微分？如何微分？

$$g(x)=\int_a^x f(t)dt$$

$$g(x+h)=\int_a^{x+h} f(t)dt$$

$$g(x+h)-g(x)=\int_x^{x+h} f(t)dt \approx hf(x)$$



$$\frac{g(x+h)-g(x)}{h} \approx f(x) \quad \Rightarrow \quad g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=f(x)$$

微積分基本定理 I： $g'(x)=\frac{d}{dx}g(x)=\frac{d}{dx}\int_a^x f(t)dt=f(x)$

THE FUNDAMENTAL THEORY OF CALCULUS, PART I If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)dt \qquad a \le x \le b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

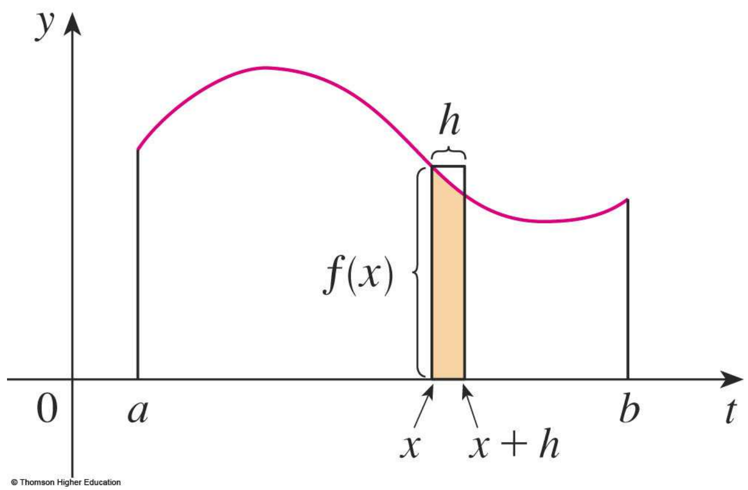
PROOF

$$g(x+h) - g(x) = \int_x^{x+h} f(t)dt$$

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)dt$$

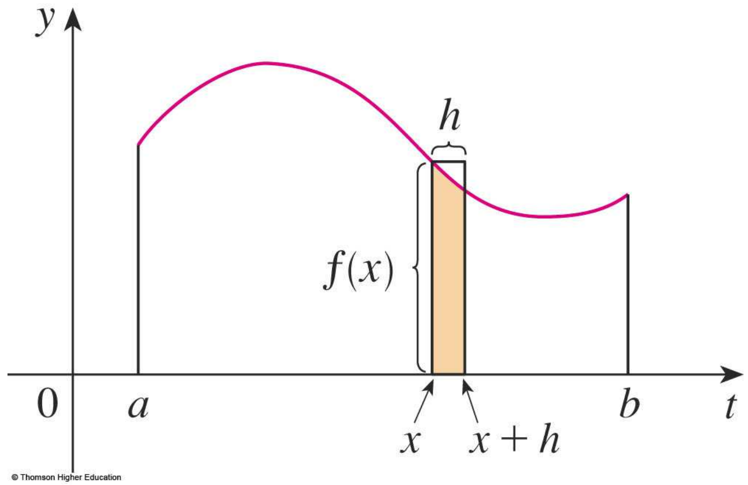
$$f(u) \le \frac{1}{h}\int_x^{x+h} f(t)dt \le f(v)$$

$$mh \le \int_x^{x+h} f(t)dt \le Mh$$



微小區間內的極小值　　　　微小區間內的極大值

$$f(u) \le \frac{g(x+h)-g(x)}{h} \le f(v)$$



連續條件

$$\lim_{h\to 0} f(u) = f(x)$$

$$\lim_{h\to 0} f(v) = f(x)$$

$$\Rightarrow \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = f(x)$$

夾擠定理

$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = f(x)$$

$$g'(x) = \frac{d}{dx} g(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$$

微積分基本定理 I

Example 2

Find the derivative of the function $g(x) = \int_0^x \sqrt{1+t^2}\,dt$

Sol:

$f(t) = \sqrt{1+t^2}$ is continuous

$g'(x) = \sqrt{1+x^2}$

Example 3

Find the derivative of the function $g(x) = \int_0^x \sin(\pi t^2/2)\,dt$

Sol:

$g'(x) = \sin(\pi x^2/2)$

Example 4

Find $\dfrac{d}{dx}\displaystyle\int_1^{x^4} \sec t\,dt$

Sol:

let $u = x^4$ 　　利用鍊鎖律

$$\begin{aligned}\frac{d}{dx}\int_1^{x^4} \sec t\,dt &= \frac{d}{dx}\int_1^{u} \sec t\,dt \\ &= \frac{d}{du}\left(\int_1^{u} \sec t\,dt\right)\frac{du}{dx} \\ &= \sec u\,\frac{du}{dx} \\ &= \sec\left(x^4\right)\cdot 4x^3\end{aligned}$$

THE FUNDAMENTAL THEORY OF CALCULUS, PART II If ***f*** is **continuous on [*a*, *b*], then**

$$\int_a^b f(x)dx = F(b) - F(a)$$

where *F* is any antiderivative of *f*, that is, a function such that *F'* (*x*) = *f*(*x*).

PROOF

$$\text{Let} \quad g(x) = \int_a^x f(t)dt \Rightarrow g'(x) = f(x)$$

g 為 f 的反導數

假設 F 為 f 的其他反導數，則 $F(x) = g(x) + C$

$$g(a) = \int_a^a f(t)dt = 0, \qquad g(b) = \int_a^b f(t)dt$$

$$\underbrace{F(b) - F(a)}_{\text{求函數 } f \text{ 之反導數，將定積分上下限代入相減}} = [g(b) + C] - [g(a) + C] = g(b) - g(a) = \underbrace{g(b) = \int_a^b f(t)dt}_{\text{函數 } f \text{ 之定積分}}$$

Example 5

Find $\int_1^3 e^x dx$

$$\int_1^3 e^x dx = \underbrace{F(3) - F(1)}_{= F(x)\Big]_1^3 = e^x\Big]_1^3} = e^3 - e$$

Example 6

Find the area under the parabola $y=x^2$ from 0 to 1.

$$A=\int_0^1 x^2dx=\frac{x^3}{3}\bigg]_0^1=\frac{1^3}{3}-\frac{0^3}{3}=\frac{1}{3}$$

Example 7

Find $\int_3^6 \frac{dx}{x}$

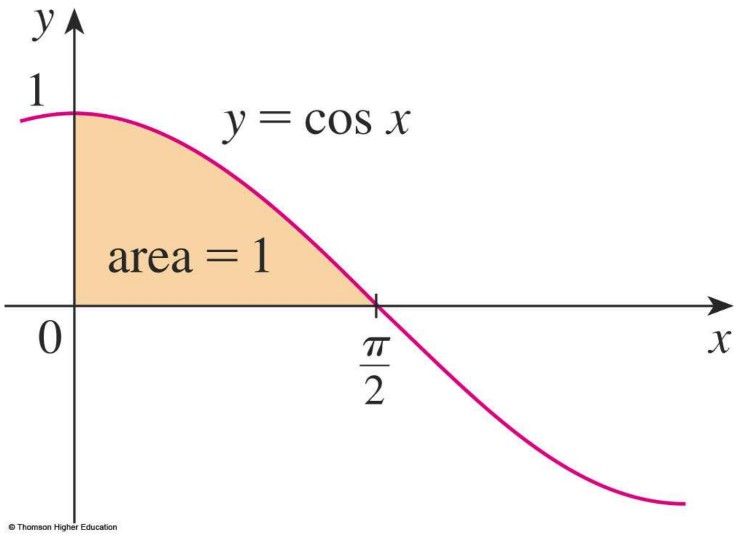
$$\int_3^6 \frac{dx}{x}=\ln x\big]_3^6=\ln 6-\ln 3=\ln 2$$

Example 8

Find the area under the cosine curve from 0 to b, where $0 \leqq b \leqq \pi/2$

$$A = \int_0^b \cos x dx = \sin x \Big]_0^b = \sin b - \sin 0 = \sin b$$

e.g. $\sin \frac{\pi}{2} = 1$



Example 9

What is wrong with the following calculation?

$$\int_{-1}^{3} \frac{1}{x^2} dx = \left.\frac{x^{-1}}{-1}\right]_{-1}^{3} = \frac{-1}{3} - 1 = \frac{-4}{3}$$

The function $f(x)=1/x^2$ is not continuous on [-1, 3], the FTC fails.

$$\int_{-1}^{3} \frac{1}{x^2} dx \quad \text{does not exist.}$$

Example

$$\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\cdots+\frac{n}{n^2+n^2}\right]=?$$

$$\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\cdots+\frac{n}{n^2+n^2}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+\left(i/n\right)^2}$$

$$=\int_0^1\frac{1}{1+x^2}dx$$

$$=\tan^{-1}x\Big]_0^1=\tan^{-1}1-0=\pi/4$$

Summary

THE FUNDAMENTAL THEORY OF CALCULUS **If *f* is continuous on [*a*, *b*]**

1. If $g(x)=\int_a^x f(t)dt$, then $g'(x)=f(x)$.

2. $\int_a^b f(x)dx=F(b)-F(a)$, **where $F'(x) = f(x)$.**

透過微積分基本定理，定積分和反導數的關連性才建立起來

Sec. 5.4 Indefinite Integrals and the Net Change Theorem

Indefinite Integrals

$$\int f(x)dx = F(x) \quad \text{means} \quad F'(x) = f(x)$$

Examples

$$\int x^2 dx = \frac{x^3}{3} + C$$

$$\int \sec^2 x dx = \tan x + C$$

TABLE OF INDEFINITE INTEGRALS

$$\int cf(x)dx = c\int f(x)dx$$

$$\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$$

$$\int kdx = kx + C$$

$$\int \frac{1}{x}dx = \ln|x| + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int \sin x dx = -\cos x + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \sec^2 x dx = \tan x + C$$

$$\int \csc^2 x dx = -\cot x + C$$

$$\int \sec x \tan x dx = \sec x + C$$

$$\int \csc x \cot x dx = -\csc x + C$$

$$\int \frac{1}{1+x^2}dx = \tan^{-1} x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C$$

Example 1

Find $\int (10x^4 - 2\sec^2 x)dx$

$$\int (10x^4 - 2\sec^2 x)dx = 10\int x^4 dx - 2\int \sec^2 x dx$$

$$= 10\frac{x^5}{5} - 2\tan x + C$$

$$= 2x^5 - 2\tan x + C$$

Example 2

Find $\int \frac{\cos\theta}{\sin^2\theta} d\theta$

$$\begin{aligned}\int \frac{\cos\theta}{\sin^2\theta} d\theta &= \int \left(\frac{1}{\sin\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right) d\theta \\ &= \int \csc\theta \cot\theta d\theta \\ &= -\csc\theta + C\end{aligned}$$

可利用 **Sec. 5-5** 的變數變換法積分

Example 3

Find $\int_0^3 (x^3 - 6x)dx$

$$\int_0^3 (x^3 - 6x)dx = \frac{x^4}{4} - 6\frac{x^2}{2}\Big]_0^3$$

$$= \left(\frac{1}{4}\cdot 3^4 - 3\cdot 3^2\right) - \left(\frac{1}{4}\cdot 0^4 - 3\cdot 0^2\right)$$

$$= -6.75$$

Example 4

Find $\int_0^2 (2x^3 - 6x + \frac{3}{1+x^2})dx$

$$\int_0^2 (2x^3 - 6x + \frac{3}{1+x^2})dx = 2\frac{x^4}{4} - 6\frac{x^2}{2} + 3\tan^{-1} x\Bigg]_0^2$$

$$= \frac{1}{2}x^4 - 3x^2 + 3\tan^{-1} x\Bigg]_0^2$$

$$= \left(\frac{1}{2}\cdot 2^4 - 3\cdot 2^2 + 3\tan^{-1} 2\right) - 0$$

$$= -4 + 3\tan^{-1} 2$$

Example 5

Find $\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2} dx$

$$\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2} dt = \int_1^9 (2 + t^{1/2} - t^{-2}) dt$$

$$= 2t + \frac{2}{3}t^{3/2} + \frac{1}{t}\Big]_1^9$$

$$= (2 \cdot 9 + \frac{2}{3}9^{3/2} + \frac{1}{9}) - (2 \cdot 1 + \frac{2}{3}1^{3/2} + \frac{1}{1})$$

$$= 32\frac{4}{9}$$

The Net Change（淨變化量）

If *f* is continuous on [*a*, *b*], then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where *F'* (*x*) = *f*(*x*).

微積分第二基本定理

$$\Rightarrow \quad \underbrace{\int_a^b \overbrace{F'(x)}^{F(x)\text{ 的變化率}} dx}_{\text{變化率的定積分}} = \underbrace{F(b) - F(a)}_{F(x)\text{ 在一段範圍（}a\sim b\text{）內的總改變量}} \quad \Rightarrow \quad \text{淨變化量}$$

THE NET CHANGE THEOREM **The integral of a rate of change is the net change:**

$$\int_a^b F'(x)dx = F(b) - F(a)$$

一個函數變化率的定積分 = 函數的淨變化量

$V(t)$ 代表蓄水池內水的體積

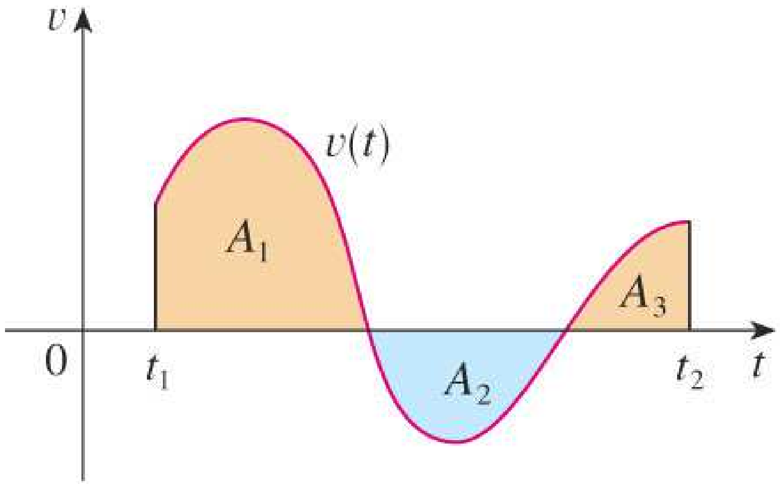
$$\int_{t_1}^{t_2} \underbrace{V'(t)}_{\text{水的體積量的變化率}}dt = \underbrace{V(t_2) - V(t_1)}_{\text{水的體積的變化量}}$$

$\dfrac{dn}{dt}$ 代表人口的成長率

$$\int_{t_1}^{t_2} \frac{dn}{dt}dt = \underbrace{n(t_2) - n(t_1)}_{\text{人口的增加量（變化量）}}$$

$v(t) = s'(t)$ 代表速度

$$\int_{t_1}^{t_2} v(t)dt = \underbrace{s(t_2) - s(t_1)}_{\text{位移量}}$$



$$\text{displacement} = \int_{t_1}^{t_2} v(t)\, dt = A_1 - A_2 + A_3$$

$$\text{distance} = \int_{t_1}^{t_2} |v(t)|\, dt = A_1 + A_2 + A_3$$

Example 6

A particle moves along a line so that its velocity at time t is $v(t)=t^2-t-6$ m/s.

(a) Find the displacement of the particle during $1 \leqq t \leqq 4$.

(b) Find the distance traveled during this time period.

(a)

$$s(4)-s(1)=\int_1^4 v(t)dt=\int_1^4 (t^2-t-6)dt=\left.\frac{t^3}{3}-\frac{t^2}{2}-6t\right]_1^4=-\frac{9}{2}$$

(b)

$$d=\int_1^4 |v(t)|dt=\int_1^3 -v(t)dt+\int_3^4 v(t)dt$$

$$=\int_1^3 -(t^2-t-6)dt+\int_3^4 (t^2-t-6)dt$$

$$=-\left.\left(\frac{t^3}{3}-\frac{t^2}{2}-6t\right)\right]_1^3+\left.\left(\frac{t^3}{3}-\frac{t^2}{2}-6t\right)\right]_3^4=\frac{61}{6}$$

<u>Sec. 5.5</u> The Substitution Rule

<u>Example</u>

$$\int 2x\sqrt{1+x^2}\,dx = ?$$

先觀察積分困難點

積分變數變換

$$\int 2x\sqrt{1+x^2}\,dx = \int \sqrt{1+x^2}\,(2x\,dx)$$

$$= \int \sqrt{u}\,du$$

$$= \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{3}(1+x^2)^{3/2} + C$$

$$u = 1+x^2$$

$$du = 2x\,dx$$

可微分檢查正確性

根據 **Chain Rule**：

$$\frac{d}{dx}F(g(x))=\frac{d}{dx}F(u)=\frac{d}{du}F(u)\frac{du}{dx}=F'(g(x))g'(x)$$

$u=g(x)$ ⇒ $du=g'(x)dx$

$$\int \underbrace{F'(g(x))}_{f(g(x))}g'(x)dx=F(g(x))+C$$

$$\int \underbrace{f(g(x))}\underbrace{g'(x)dx}_{du=g'(x)dx}=\int f(u)du$$

若此函數之反導數容易求得，積分則可求出。

THE SUBSTITUTION RULE **If $u=g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then**

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Example 1

Find $\int x^3 \cos(x^4+2)dx$

$u = x^4 + 2 \quad \Rightarrow \quad du = 4x^3 dx$

$$\int x^3 \cos(x^4+2)dx = \frac{1}{4}\int \cos(x^4+2)\left(4x^3 dx\right) = \frac{1}{4}\int \cos u\, du$$

$$= \frac{1}{4}\sin u + C = \frac{1}{4}\sin(x^4+2) + C$$

Example 2

Find $\int \sqrt{2x+1}dx$

$u = 2x+1$ ➡ $du = 2dx$

$$\int \sqrt{2x+1}dx = \int \sqrt{u}\frac{du}{2} = \frac{1}{2}\int u^{1/2}du = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(2x+1)^{3/2} + C$$

$u = \sqrt{2x+1}$ ➡ $du = \frac{1}{\sqrt{2x+1}}dx$ ➡ $dx = \sqrt{2x+1}du = udu$

$$\int \sqrt{2x+1}dx = \int u \cdot udu = \int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}(2x+1)^{3/2} + C$$

Example 3

Find $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\,dx$

$u = 1 - 4x^2$ ⇨ $du = -8x\,dx$

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx = \frac{-1}{8}\int \frac{1}{\sqrt{1-4x^2}}\left(-8x\,dx\right)$$

$$= \frac{-1}{8}\int \frac{1}{\sqrt{u}}\,du$$

$$= \frac{-1}{8}\left(2\sqrt{u}\right) + C$$

$$= \frac{-1}{4}\sqrt{1-4x^2} + C$$

Example 4

Find $\int e^{5x} dx$

$u = 5x$ $\Rightarrow$ $du = 5dx$

$$\int e^{5x} dx = \frac{1}{5}\int e^{u} du = \frac{1}{5}e^{u} + C = \frac{1}{5}e^{5x} + C$$

Example 5

Find $\int \sqrt{1+x^2}\, x^5 dx$

$u = 1 + x^2 \Rightarrow du = 2x dx \qquad x^2 = u - 1$

$$\int \sqrt{1+x^2}\, x^5 dx = \frac{1}{2}\int \sqrt{1+x^2}\, x^4 \left(2x dx\right) = \frac{1}{2}\int \sqrt{u}\left(u-1\right)^2 du$$

$$= \frac{1}{2}\int \sqrt{u}\left(u^2 - 2u + 1\right) du = \frac{1}{2}\int \left(u^{5/2} - 2u^{3/2} + u^{1/2}\right) du$$

$$= \frac{1}{2}\left(\frac{2}{7}u^{7/2} - 2\cdot\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right) + C$$

$$= \frac{1}{7}u^{7/2} - \frac{2}{5}u^{5/2} + \frac{1}{3}u^{3/2} + C$$

$$= \frac{1}{7}\left(1+x^2\right)^{7/2} - \frac{2}{5}\left(1+x^2\right)^{5/2} + \frac{1}{3}\left(1+x^2\right)^{3/2} + C$$

Example 6

Find $\int \tan x dx$

$$\int \tan x dx = \ln\left|\sec x\right| + C$$

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx$$

$u = \cos x$ ⇒ $du = -\sin x dx$

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{1}{u} du$$

$$= -\ln\left|u\right| + C = -\ln\left|\cos x\right| + C = \ln\left|\sec x\right| + C$$

THE SUBSTITUTION RULE FOR DEFINITE INTEGRAL If $g'(x)$ is continuous on [a, b] and f is continuous on the range of $u=g(x)$, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

PROOF

$$\int_a^b f(g(x))g'(x)dx = F(g(x))\Big]_a^b = F(g(b)) - F(g(a))$$

$$\int_{g(a)}^{g(b)} f(u)du = F(u)\Big]_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$$

Example 7

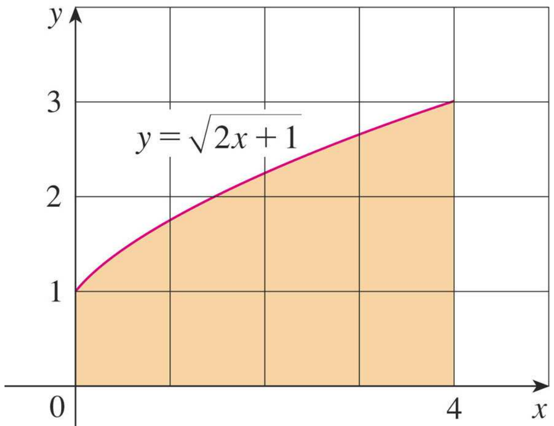
Find $\int_0^4 \sqrt{2x+1}\,dx$

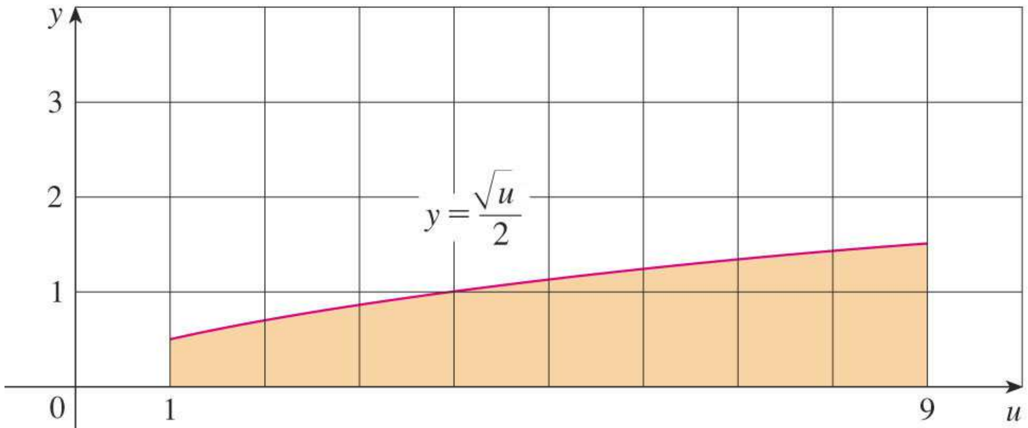
$u = 2x+1$ ⇒ $du = 2dx$

$u(0) = 1$

$u(4) = 9$

$$\int_0^4 \sqrt{2x+1}\,dx = \frac{1}{2}\int_{u(0)}^{u(4)} u^{1/2}du = \frac{1}{2}\int_1^9 u^{1/2}du = \frac{1}{3}u^{3/2}\Big]_1^9 = \frac{1}{3}(9^{3/2} - 1^{3/2}) = \frac{26}{3}$$





Example 8

Find $\displaystyle\int_1^2 \frac{dx}{(3-5x)^2}$

$u = 3-5x \quad \Rightarrow \quad du = -5dx$

$$u(1) = -2$$
$$u(2) = -7$$

$$\int_1^2 \frac{dx}{(3-5x)^2} = \frac{-1}{5}\int_{-2}^{7} \frac{du}{u^2} = \frac{1}{5}u^{-1}\Big]_{-2}^{-7} = \frac{1}{5}\left(\frac{1}{-7} - \frac{1}{-2}\right) = \frac{1}{14}$$

Example 9

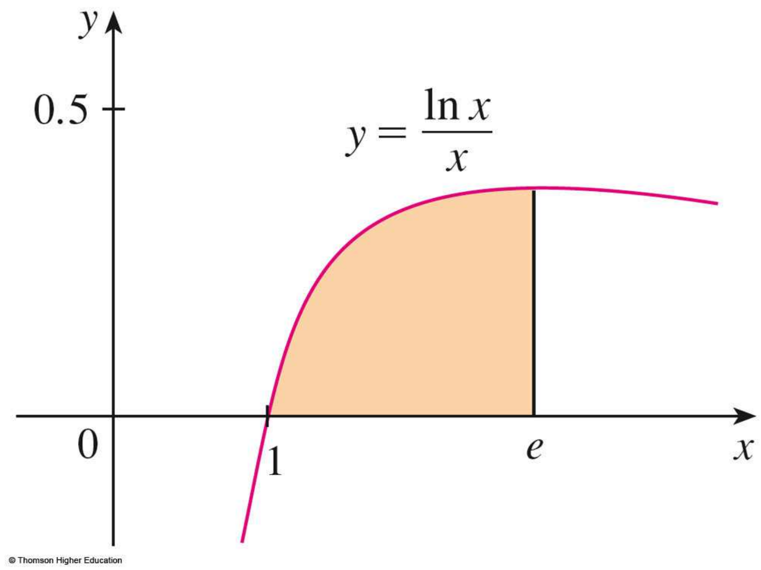
Find $\int_1^e \frac{\ln x}{x}\,dx$

$u = \ln x$ ⇨ $du = \frac{1}{x}\,dx$

$u(1) = 0$

$u(e) = 1$

$$\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 u\,du = \frac{1}{2}u^2\Big]_0^1 = \frac{1}{2}(1-0) = \frac{1}{2}$$

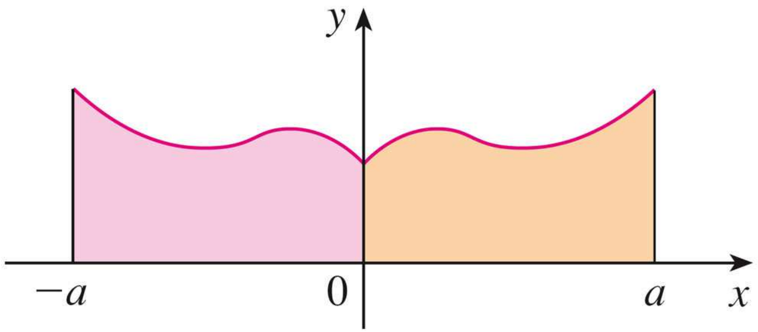


<u>INTEGRAL OF SYMMETRY FUNCTION</u> **Suppose *f* is continuous on [-a, a].**

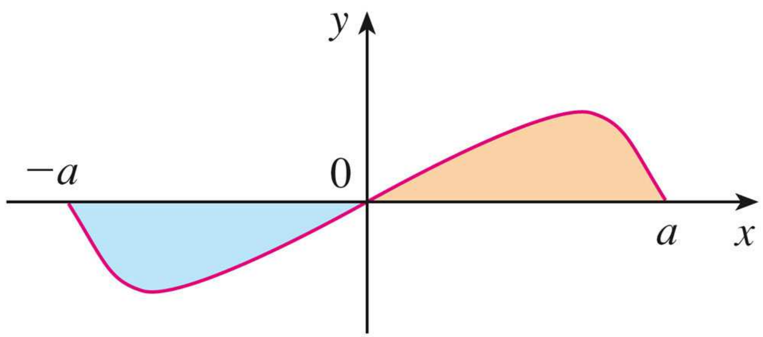
(a) If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$

(b) If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)dx = 0.$

<u>PROOF</u>



(a) f even, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$



(b) f odd, $\int_{-a}^{a} f(x)\,dx = 0$

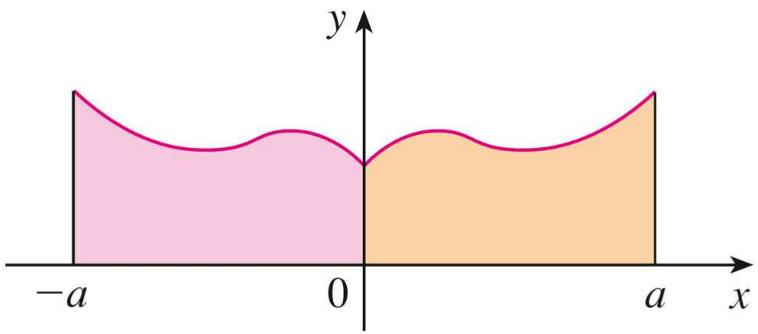
If f is even, then

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$$

$$= -\int_{0}^{-a} f(x)dx + \int_{0}^{a} f(x)dx$$

$$= -\int_{0}^{a} f(-u)(-du) + \int_{0}^{a} f(x)dx \qquad (\text{let } u = -x)$$

$$= \int_{0}^{a} f(u)(du) + \int_{0}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$



(a) f even, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

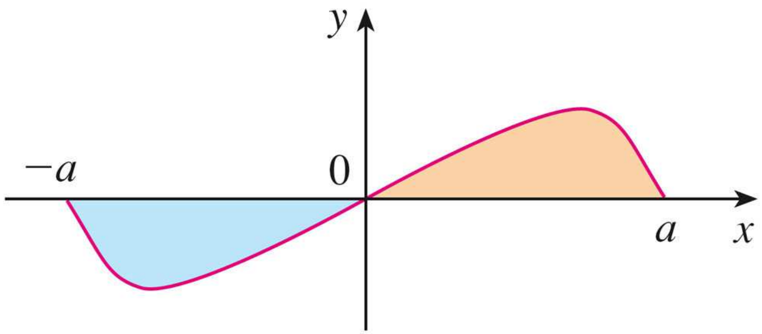
If f is odd, then

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$$

$$= -\int_{0}^{-a} f(x)dx + \int_{0}^{a} f(x)dx$$

$$= -\int_{0}^{a} f(-u)\left(-du\right) + \int_{0}^{a} f(x)dx \qquad \left(\text{let } u = -x\right)$$

$$= -\int_{0}^{a} f(u)\left(du\right) + \int_{0}^{a} f(x)dx = 0.$$



(b) f odd, $\int_{-a}^{a} f(x)\,dx = 0$

Example 10

Find $\int_{-2}^{2}(x^{6}+1)dx$

$$f(x)=x^{6}+1 \quad \Rightarrow \quad f(-x)=f(x)$$

$$\int_{-2}^{2}(x^{6}+1)dx=2\int_{0}^{2}(x^{6}+1)dx=2\left(\frac{1}{7}x^{7}+x\right)\Bigg]_{0}^{2}=2\left(\frac{128}{7}+2\right)=\frac{284}{7}$$

Example 11

Find $\int_{-1}^{1} \frac{\tan x}{1+x^2+x^4}\,dx$

$$f(x)=\frac{\tan x}{1+x^2+x^4} \quad \Rightarrow \quad f(-x)=-f(x)$$

$$\int_{-1}^{1} \frac{\tan x}{1+x^2+x^4}\,dx=0$$