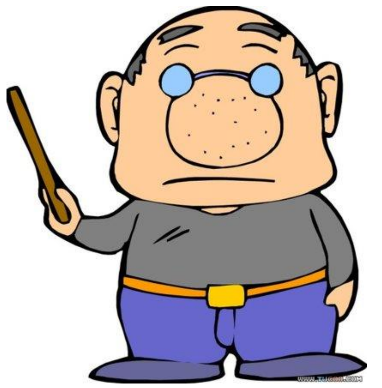
Laplace Transform



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The Laplace Transform (one of integral transform)

Purpose:

Initial O.D.E $\xrightarrow{\textit{Hard}}$ *Solution of Initial O.D.E*

$\downarrow$ *Integral Transform* $\qquad$ $\uparrow$ *Inverse Transform*

Algebra problem $\longrightarrow$ *Solution of algebra problem*

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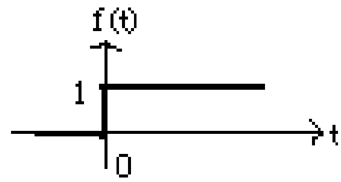
Definition of Laplace Transform

$$\mathcal{L}\{f(t)\}=\int_0^\infty e^{-st} f(t)dt=F(s)$$

Basic Laplace Transforms of Function

1. *Unit step*

$$f(t)=\begin{cases}0; t<0\\ 1; t\geq 0\end{cases}$$



$$\mathcal{L}\{f(t)\}=\frac{1}{s}$$

$$\mathcal{L}\{f(t)\}=\int_0^\infty e^{-st} f(t)dt=\int_0^\infty e^{-st}dt=-\frac{e^{-st}}{s}\Big|_0^\infty$$

$$=\frac{1}{s}\quad (s>0)$$

2. $f(t)=t^n \quad n=1,2,3,\ldots \quad t\geq 0$

$$\mathcal{L}\{f(t)\}=\int_0^\infty e^{-st}t^n dt=\int_0^\infty e^{-\tau}\left(\frac{\tau}{s}\right)^n\frac{d\tau}{s}$$

$$(Let\ st=\tau \quad dt=d\tau/s)$$

Note:
Gamma Function
$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$
$\Gamma(n+1)=n!$

$$\frac{1}{s^{n+1}}\int_0^\infty e^{-\tau}\tau^n d\tau=\frac{n!}{s^{n+1}}\quad (s>0)$$

3. $f(t) = e^{at} \quad t \geq 0$ $\quad \boxed{a : \text{constant}}$

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{at} e^{-st} dt = \int_0^{\infty} e^{(a-s)t} dt$$

$$= \lim_{k \to \infty} \left[\frac{1}{a-s} e^{(a-s)t} \right] \Big|_0^k$$

$$= \lim_{k \to \infty} \left[\frac{1}{a-s} e^{(a-s)k} - \frac{1}{a-s} \right]$$

$$= -\frac{1}{a-s} = \frac{1}{s-a}$$

$$(a - s < 0, \Rightarrow s > a)$$

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4. $f(t) = \sin at$

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} \sin at dt = \frac{e^{-st}}{s^2 + a^2} \left[-s \sin at - a \cos at \right] \Big|_0^{\infty}$$

$$= \frac{a}{s^2 + a^2}$$

5. $f(t) = \cos at$

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} \cos at dt = \frac{e^{-st}}{s^2 + a^2} \left[-s \cos at + a \sin at \right] \Big|_0^{\infty}$$

$$= \frac{s}{s^2 + a^2}$$

Inverse Laplace Transform

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}F(s)ds$$

$(a \text{ is a real number})$

$f(t) \to F(s)$	$F(s) \to f(t)$
$1 \to 1/s$	$1/s \to 1$
$t \to 1/s^2$	$1/s^2 \to t$
$t^n \to n!/s^{n+1}$	$1/s^{n+1} \to t^n/n!$
$e^{at} \to 1/(s-a)$	$1/(s-a) \to e^{at}$

$f(t) \to F(s)$	$F(s) \to f(t)$
$\sin at \to \dfrac{a}{s^2+a^2}$	$\dfrac{1}{s^2+a^2} \to \dfrac{1}{a}\sin at$
$\cos at \to \dfrac{s}{s^2+a^2}$	$\dfrac{s}{s^2+a^2} \to \cos at$

Existence of $\mathcal{L}\{f(t)\}$

(1) $f(t)$ *is piecewise continuous on*

$$t \in [0^+, k] \quad \text{for } k > 0$$

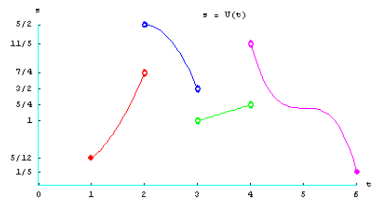
(2) *There are number* M & b

$$\Rightarrow |f(t)| \le Me^{bt} \quad \text{for } t > 0$$

Thus $\forall\ s > b \quad \mathcal{L}\{f(t)\} = F(s)$ *converges*

- *Definition of piecewise continuity f is piecewise continuous on* $[a,b]$ *if* There are finite points $a<t_1<t_2<\cdots<t_n<b$

- Such that f is continuous on each open interval (a,t_1) $(t_{j\text{-}1},t_j)$, *and* (t_n,b)



- *All of the following* limits *are finite* $\lim\limits_{x\to a^+} f(t),\quad \lim\limits_{x\to t_j^+} f(t),\quad \lim\limits_{x\to t_j^-} f(t),\quad \lim\limits_{x\to b^-} f(t)$

Origin: http://mathfaculty.fullerton.edu/mathews/c2003/FourierSeriesComplexMod.html

◎Basic Properties os Laplace Transform

1. Linearity

$$\mathcal{L}\{\alpha f(t)+\beta g(t)\}=\alpha F(s)+\beta G(s)$$

$$\textit{Where}\ \ \mathcal{L}\{\mathrm{f}(\mathrm{t})\}=F(s),\ \ \mathcal{L}\{\mathrm{g}(\mathrm{t})\}=G(s)$$

$$\alpha\ ,\beta\ \ \textit{are real numbers}$$

$$\textit{Note}:\mathcal{L}^{-1}\{\alpha F(s)+\beta G(s)\}=\alpha f(t)+\beta g(t)$$

Proof:

$$\mathcal{L}\{\alpha f(t)+\beta g(t)\}=\int_0^\infty e^{-st}\big(\alpha f(t)+\beta g(t)\big)dt$$
$$=\alpha\int_0^\infty e^{-st}f(t)dt+\beta\int_0^\infty e^{-st}g(t)dt$$
$$=\alpha F(s)+\beta G(s)$$

2. Derivative

$$\mathcal{L}\{f'(t)\}=sF(s)-f(0)$$

f' ***is piecewise continuous on*** $[\mathrm{o,k}]$ ***for*** $\mathrm{k>0}$

also $\lim_{k\to\infty}e^{-sk}f(k)=0$ ***if*** $\mathrm{s<0}$

Proof: *For k>0*

$$\int_0^\infty e^{-st}f'(t)dt=\lim_{k\to\infty}\int_0^k e^{-st}f'(t)dt$$
$$=\lim_{k\to\infty}\left[\left(e^{-st}f(t)\right)\Big|_0^k-\int_0^k -se^{-st}f(t)dt\right]$$
$$=\lim_{k\to\infty}\left[e^{-sk}f(k)-f(0)+s\int_0^k e^{-st}f(t)dt\right]$$
$$=-f(0)+s\int_0^\infty e^{-st}f(t)dt=sF(s)-f(0)$$

Note: $\mathcal{L}\{f^{(n)}(t)\}=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$

Note:

$$f, f', \cdots, f^{(n-1)} \text{ are continuous on } [0,\infty]$$

$$\text{and } f^{(n)}(t) \text{ is piecewise continuous on } [0,k]$$

$$\text{for } k>0$$

$$\text{also} \lim_{k\to\infty} e^{-st} f^{(j)}(k) = 0 \text{ for } s>0$$

$$\text{and } j=1,2,\cdots,n\text{-}1$$

EX: $y'' - 6y' + 9y = t^2 e^{3t}$; $y(0) = 2, y'(0) = 17$

1. $y_h = C_1 e^{3t} + C_2 t e^{3t}$
2. $y_p = u_1(t) e^{3t} + u_2(t) t e^{3t}$

$$\Rightarrow \begin{cases} u_1'(t)e^{3t} + u_2'(t)te^{3t} = 0 \\ 3u_1'(t)e^{3t} + u_2'(t)e^{3t} + 3u_2'(t)te^{3t} = t^2 e^{3t} \end{cases}$$

$$\Rightarrow u_1'(t) = -t^3, u_2'(t) = t^2$$

$$\Rightarrow u_1(t) = -\frac{1}{4}t^4, \qquad u_2(t) = \frac{1}{3}t^3$$

$$\Rightarrow y_p = \left(-\frac{1}{4} + \frac{1}{3}\right) t^4 e^{3t} = \frac{1}{12} t^4 e^{3t}$$

3. $y = y_h + y_p = C_1 e^{3t} + C_2 t e^{3t} + \frac{1}{12} t^4 e^{3t}$
4. I.C. $\Rightarrow C_1 = 2; C_2 = 11 \Rightarrow y(t) = 2e^{3t} + 11te^{3t} + \frac{1}{12} t^4 e^{3t}$

By Laplace Transform

$$\mathcal{L}\{y'' - 6y' + 9y\} = \mathcal{L}\{t^2 e^{3t}\}$$

$$\Rightarrow \left(s^2Y(s) - sy(0) - y'(0)\right) - 6\left(sY(s) - y(0)\right) + 9Y(s) = \frac{2}{(s-3)^3}$$

$$\Rightarrow (s^2 - 6s + 9)Y(s) - 2s - 17 + 12 = \frac{2}{(s-3)^3}$$

$$\Rightarrow (s-3)^2 Y(s) = \frac{2}{(s-3)^3} + (2s+5)$$

$$\Rightarrow \mathrm{Y(s)} = \frac{2}{(s-3)^5} + \frac{2s}{(s-3)^2} + \frac{5}{(s-3)^2}$$

$$= \frac{2}{(s-3)^5} + \frac{11}{(s-3)^2} + \frac{2}{s-3}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{12}t^4 e^{3t} + 11te^{3t} + 2e^{3t}$$

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3. First Shifting Theorem (Shifting in the S variable)

(S—Shift)

$$\mathcal{L}\left\{e^{at} f(t)\right\} = F(s-a) \qquad \text{for } s>a+b$$

Proof: $\mathcal{L}\left\{e^{at} f(t)\right\} = \int_0^\infty e^{at} \cdot e^{-st} f(t)\,dt$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt = F(s-a)$$

$$\forall s-a>b \quad \Rightarrow \quad s>a+b$$

Ex:

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\{e^{at} \cdot t^n\} = \frac{n!}{(s-a)^{n+1}}$$

$$\mathcal{L}\{\cos kt\} = \frac{s}{s^2 + k^2}$$

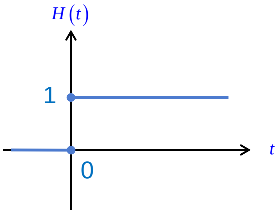
$$\mathcal{L}\{e^{at} \cos kt\} = \frac{s-a}{(s-a)^2 + k^2}$$

$$\mathcal{L}\{\sin kt\} = \frac{k}{s^2 + k^2}$$

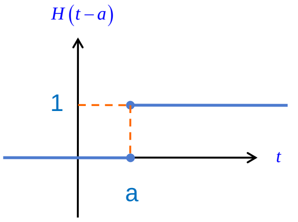
$$\mathcal{L}\{e^{at} \sin kt\} = \frac{k}{(s-a)^2 + k^2}$$

◎Heaviside Function

$$H(t) = \begin{cases} 0 \ ; \ if \ \ t < 0 \\ 1 \ ; \ if \ \ t \geq 0 \end{cases}$$



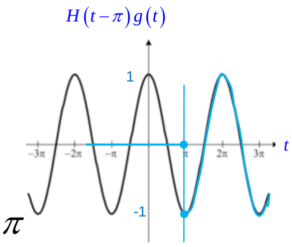
$$H(t-a) = \begin{cases} 0 \ ; \ if \ \ t < a \\ 1 \ ; \ if \ \ t \geq a \end{cases}$$



$$H(t-a)g(t) = \begin{cases} 0 \ ; \ if \ \ t < a \\ g(t) \ ; \ if \ \ t \geq a \end{cases}$$

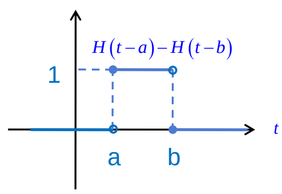
$Ex: \ g(t) = \cos t$

$$H(t\text{-}\pi)\,g(t) = \begin{cases} 0 \ ; \ if \ t<\pi \\ \cos t \ ; \ if \ t \geq \pi \end{cases}$$



◎*Pulse Function*

$$Pulse \ \ Function = H(t-a) - H(t-b) \ \ for \ \ a<b$$



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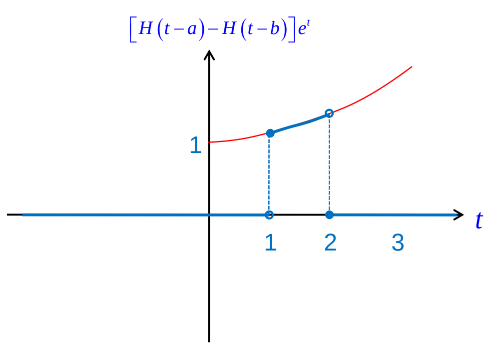
◎ $[H(t-a) - H(t-b)]e^t \quad a=1 \ , \ b=2$



4.The Second Shifting Theorem (Shifting in the t variable) (t-shift)

$$\mathcal{L}\{H(t-a)f(t-a)\}=e^{-as}F(s) \quad for \begin{matrix} s>b \\ a>0 \end{matrix}$$

Proof:

$$\mathcal{L}\{H(t-a)f(t-a)\}=\int_0^{\infty} e^{-st}H(t-a)f(t-a)dt$$

$$=\int_a^{\infty} e^{-st}f(t-a)dt$$

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Let $\omega=t-a$, $d\omega=dt$

then $\int_a^{\infty} e^{-st}f(t-a)dt=\int_0^{\infty} e^{-s(\omega+a)}f(\omega)d\omega$

$$=e^{-as}\int_0^{\infty} e^{-s\omega}f(\omega)d\omega=e^{-as}F(s)$$

EX: $H(t-1)\left[(t-1)^2\right]$

$$\mathcal{L}\left\{H(t-1)\left[(t-1)^2\right]\right\}$$
$$= e^{-as}\mathcal{L}\{t^2\}$$
$$= e^{-s}\cdot\frac{2}{s^3} = \frac{2e^{-s}}{s^3}$$

$f(t-1) = (t-1)^2$

$a = 1$

$f(t) = t^2$

EX: $\mathcal{L}\{H(t-1)\}$

$$= e^{-s}\mathcal{L}\{1\} = \frac{e^{-s}}{s}$$

$H(t-1)=1$ $\quad f(t-1)=1$

$f(t)=1$

EX:

$$g(t) = \begin{cases} 0 & ; \ 0 \le t < 2 \\ t^2+1 & ; \ t \ge 2 \end{cases} \qquad find \quad \mathcal{L}\{g(t)\} = ?$$

$$g(t) = H(t-2)(t^2+1) \Rightarrow \mathcal{L}\{g(t)\} = \mathcal{L}\left\{H(t-2)(t^2+1)\right\}$$
$$t^2+1 = (t-2)^2 + 4(t-2) + 5$$
$$\mathcal{L}\left\{H(t-2)\left[t^2+1\right]\right\} = \mathcal{L}\left\{H(t-2)\left[(t-2)^2+4(t-2)+5\right]\right\}$$
$$= \mathcal{L}\left\{H(t-2)\left[(t-2)^2\right]\right\} + 4\mathcal{L}\{H(t-2)[t-2]\} + 5\mathcal{L}\{H(t-2)\}$$
$$= e^{-2s}\frac{2}{s^3} + 4e^{-2s}\frac{1}{s^2} + 5e^{-2s}\frac{1}{s} = e^{-2s}\left[\frac{2}{s^3}+\frac{4}{s^2}+\frac{5}{s}\right]$$

5.Multiplication by $\frac{1}{S}$

If $\mathcal{L}\{f(t)\}=F(s) \quad s>b$

Then $\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\}=\frac{F(s)}{s} \quad$ for $s>max[0,b]$

Note: $\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\}=\int_0^t f(\tau)\,d\tau$

6. Differentiation with respect to S

If $\mathcal{L}\{f(t)\}=F(s)$ for $s>b$

Then $\mathcal{L}\{tf(t)\}=-\frac{dF(s)}{ds}$ for $s>b$

Note: $\mathcal{L}^{-1}\left\{\frac{dF(s)}{ds}\right\}=-tf(t)$

7. Initial-Value Theorem

Let f be continuous and f' be piecewise continuous on $0 \le t < t_0$ *for each finite* t_0 *and let f and f' be of exponential order as* $t \to \infty$

then $\lim_{s\to\infty}\left[sF(s)\right] = f(0)$

Proof: $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

f' is piecewise continuous & there are numbers M & b so that $\left|f'(t)\right| \le Me^{bt}$

$$\Rightarrow \lim_{s\to\infty}\int_0^\infty e^{-st} f'(t)\,dt = 0$$

$$\Rightarrow \lim_{s\to\infty}\left[sF(s) - f(0)\right] = 0$$

$$\Rightarrow \lim_{s\to\infty} sF(s) = f(0)$$

8. Final-Value Theorem

If $\lim_{t\to\infty} f(t)$ & $\lim_{s\to\infty} sF(s)$ *both exist , and*

$\mathcal{L}\{f(t)\} = F(s)$

Then $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Proof: $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

$$\Rightarrow \lim_{s\to 0}\int_0^\infty e^{-st} f'(t)\,dt = \lim_{s\to 0}\left[sF(s) - f(0)\right]$$

$$\Rightarrow \lim_{t\to\infty} f(t) - f(0) = \lim_{s\to 0} sF(s) - f(0)$$

$$\Rightarrow \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

◎Initial Value Problems Using Laplace Transform

For linear const-efficient O.D.E. with I.C.s

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = f(x)$$

$$y(0) = b_n \;, y'(0) = b_{n-1} \;, \cdots, y^{(n\text{-}1)}(0) = b_1$$

$Ex:$ $Solve\;\; y'\text{-}4y=1 \;\; ; \;\; y(0)=1$

$$\mathcal{L}\{y'-4y\}=\mathcal{L}\{1\} \qquad \mathcal{L}\{y\}=Y(s)$$

$$\Rightarrow \mathcal{L}\{y'\}-4\mathcal{L}\{y\}=\mathcal{L}\{1\}$$

$$\Rightarrow sY(s)-y(0)-4Y(s)=\frac{1}{s}\Rightarrow (s-4)Y(s)-1=\frac{1}{s}$$

$$\Rightarrow Y(s)=\frac{s+1}{s(s-4)}\Rightarrow y(t)=\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{s+1}{s(s-4)}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{-\frac{1}{4}}{s}+\frac{\frac{5}{4}}{s-4}\right\}=-\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+\frac{5}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}$$

$$=-\frac{1}{4}+\frac{5}{4}e^{4t}$$

$$Ex: \quad Solve \;\; y''+4y'+3y=e^{t} \;, y(0)=0 \;, y'(0)=2$$

Left-hand side

$$\mathcal{L}\{y''+4y'+3y\}$$
$$=[s^2Y(s)-sy(0)-y'(0)]+4[sY(s)-y(0)]+3Y(s)$$
$$=[s^2+4s+3]Y(s)-2$$

Right-hand side

$$\mathcal{L}\{e^t\}=\frac{1}{s-1}$$
$$\Rightarrow [s^2+4s+3]Y(s)-2=\frac{1}{s-1}$$
$$\Rightarrow Y(s)=\frac{2s-1}{(s-1)(s^2+4s+3)}$$

$$Y(s) = \frac{2s-1}{(s-1)(s^2+4s+3)} = \frac{2s-1}{(s-1)(s+3)(s+1)}$$

$$= \frac{1/8}{s-1} + \frac{-7/8}{s+3} + \frac{3/4}{s+1}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1/8}{s-1} + \frac{-7/8}{s+3} + \frac{3/4}{s+1}\right\}$$

$$= \frac{1}{8}e^{t} - \frac{7}{8}e^{-3t} + \frac{3}{4}e^{-t}$$

◎*Convolution:*

If f and g are defined on $[0, \infty]$, then the convolution $f * g$ is defined by

$$f * g = \int_0^t f(t-\tau)\, g(\tau)\, d\tau \qquad t \geq 0$$

◎*Convolution Theorem:*

If $f * g$ are defined, then

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$$

Proof:

Let $F(s)=\mathcal{L}\{f(t)\}, G(s)=\mathcal{L}\{g(t)\}$

$$F(s)G(s)=F(s)\int_0^\infty e^{-s\tau}g(\tau)d\tau=\int_0^\infty F(s)e^{-s\tau}g(\tau)d\tau$$

then

recall that

$$\mathcal{L}\{H(t-\tau)f(t-\tau)\}=e^{-s\tau}F(s)$$

$$\Rightarrow F(s)G(s)=\int_0^\infty \mathcal{L}\{H(t-\tau)f(t-\tau)\}g(\tau)d\tau$$

$$=\int_0^\infty \left\{\int_0^\infty e^{-st}H(t-\tau)f(t-\tau)dt\right\}g(\tau)d\tau$$

$$=\int_0^\infty\int_0^\infty e^{-st}H(t-\tau)f(t-\tau)dt\, g(\tau)d\tau$$

$$=\int_0^\infty\int_0^\infty e^{-st}g(\tau)H(t-\tau)f(t-\tau)dtd\tau$$

$$=\int_0^\infty\int_\tau^\infty e^{-st}g(\tau)f(t-\tau)dtd\tau$$

$$=\int_0^\infty\int_0^t e^{-st}g(\tau)f(t-\tau)d\tau dt$$

$$=\int_0^\infty e^{-st}\left\{\int_0^t g(\tau)f(t-\tau)d\tau\right\}dt$$

$$=\int_0^\infty e^{-st}\{f*g\}dt$$

$$=L\{f*g\}$$

$$Ex:\ \mathcal{L}^{-1}\left\{\frac{1}{s(s-4)^2}\right\}$$
$$=\mathcal{L}^{-1}\left\{\frac{1}{s}\times\frac{1}{(s-4)^2}\right\}=f*g$$
$$=\int_0^t f(t-\tau)g(\tau)d\tau=\boxed{\int_0^t 1\times\tau e^{4\tau}d\tau}$$

$$f*g=\frac{1}{16}-\frac{1}{16}e^{4t}+\frac{1}{4}te^{4t}$$

$$f(t)=\mathcal{L}^{-1}\{F(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}=1$$
$$g(t)=\mathcal{L}^{-1}\{G(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{(s-4)^2}\right\}=te^{4t}$$

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Ex: $\mathcal{L}\left\{\int_0^t e^{\tau}\sin(t-\tau)\,d\tau\right\}=?$

$$\mathcal{L}\left\{\int_0^t e^{\tau}\sin(t-\tau)\,d\tau\right\}=\mathcal{L}\{e^t*\sin(t)\}$$
$$=\mathcal{L}\{e^t\}\cdot\mathcal{L}\{\sin(t)\}=\frac{1}{(s-1)(s^2+1)}$$

Ex: $\mathcal{L}^{-1}\left\{\frac{1}{[s^2+4]^2}\right\}=?$

$$\mathcal{L}^{-1}\left\{\frac{1}{[s^2+4]^2}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\cdot\frac{1}{s^2+4}\right\}$$
$$=\frac{1}{2}\sin(2t)*\frac{1}{2}\sin(2t)=\frac{\sin(2t)-2t\cos(2t)}{16}$$

◎Nonhomogeneous O.D.E. solved by convolution

$$y'' + ay' + by = r(t)\ ,\ y(0) = k_0\ ,\ y'(0) = k_1$$

$$a\ ,\ b \ \text{ are constants}$$

① Take Laplace Transform on both sides

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + a\left[sY(s) - y(0)\right] + bY(s) = R(s)$$

$$\mathcal{L}\{y(t)\} = Y(s)\ ,\ \mathcal{L}\{r(t)\} = R(s)$$

$$\Rightarrow \left(s^2 + as + b\right)Y(s) = (s+a)y(0) + y'(0) + R(s)$$

$$\Rightarrow Y(s) = \frac{1}{s^2 + as + b}\left[(s+a)y(0) + y'(0)\right] + \frac{R(s)}{s^2 + as + b}$$

$$\text{Let } \ Q(s) = \frac{1}{s^2 + as + b} \Rightarrow Y(s) = Q(s)\left[(s+a)y(0) + y'(0)\right] + R(s)Q(s)]$$

$$y(t) = \mathcal{L}^{-1}\left\{Q(s)\left[(s+a)y(0) + y'(0)\right] + R(s)Q(s)\right\}$$

$$= \underbrace{\mathcal{L}^{-1}\left\{Q(s)\left[(s+a)y(0) + y'(0)\right]\right\}}_{y_h} + \underbrace{\mathcal{L}^{-1}\left\{R(s)Q(s)\right\}}_{y_p}$$

- Note:

$$If \;\; y(0)=0\,,\, y'(0)=0$$

$$Y(s)=Q(s)R(s)\Rightarrow Q(s)=\frac{Y(s)}{R(s)}$$

Output

Input

Transfer function

- Consider only y_p

$$y_p(t)=L^{-1}\{Q(s)R(s)\}=q(t)*r(t)$$
$$=\int_0^t q(t-\tau)r(\tau)d\tau$$

$$Ex: \;\; y''+3y'+2y=r(t)$$
$$y(0)=y'(0)=0$$

$$r(t)=\begin{cases}1 \;;\; 1<t<2\\ 0 \;;\; otherwise\end{cases}$$

Solution by Convolution

① *Take Laplace Transform*

$$(s^2+3s+2)Y(s)=R(s) \;\; ; \;\; R(s)=\mathcal{L}\{r(t)\}$$

$$\Rightarrow Q(s)=\frac{1}{s^2+3s+2} \qquad \therefore Y(s)=Q(s)R(s)$$

② $Q(s)=\dfrac{1}{s^2+3s+2}=\dfrac{1}{(s+1)(s+2)}=\dfrac{1}{s+1}-\dfrac{1}{s+2}$

$$\mathcal{L}^{-1}\{Q(s)\}=g(t)=\mathcal{L}^{-1}\left\{\frac{1}{s+1}-\frac{1}{s+2}\right\}=e^{-t}-e^{-2t}$$

③

$$\because Y(s)=Q(s)R(s)\quad \therefore y(t)=q(t)*r(t)$$

$$y(t)=\int_0^t q(t-\tau)r(\tau)d\tau$$

$$\because r(t)=\begin{cases}1\ ;\ 1<t<2\\ 0\ ;\ otherwise\end{cases}$$

$If\quad t\le 1\ ,\ then\quad r(t)=0\qquad \therefore y(t)=\int_0^t q(t-\tau)\cdot 0d\tau=0$

$If\quad 1<t<2\ ,\ then\ r(t)=1\qquad \because y(t)=\int_0^t q(t-\tau)\cdot r(\tau)d\tau$

$$y(t)=\int_1^t\left[e^{-(t-\tau)}-e^{-2(t-\tau)}\right]d\tau=\left[e^{-(t-\tau)}-\frac{1}{2}e^{-2(t-\tau)}\right]\Bigg|_1^t$$

$$=\left[e^0-\frac{1}{2}e^0\right]-\left[e^{-(t-1)}-\frac{1}{2}e^{-2(t-1)}\right]$$

$$=\frac{1}{2}-e^{-(t-1)}+\frac{1}{2}e^{-2(t-1)}$$

$If\ t\ge 2,\qquad then\ r(t)=0\qquad \therefore y(t)=\int_1^2[q(t-\tau)]d\tau$

$$y(t)=\left[e^{-(t-2)}-\tfrac{1}{2}e^{-2(t-2)}\right]-\left[e^{-(t-1)}-\tfrac{1}{2}e^{-2(t-1)}\right]$$

$$Ex: \quad Determine \quad f(t) = 2t^2 + \int_0^t f(t-\tau)e^{-\tau}d\tau$$

$Sol:$

$$Recognize \quad f(t) = 2t^2 + f(t) * e^{-t}$$

①Take Laplace Transform

$$F(s) = \frac{4}{s^3} + F(s)\cdot\frac{1}{s+1}$$

$$\Rightarrow F(s) = \frac{4}{s^3}\left(\frac{1}{1-\frac{1}{s+1}}\right) = \frac{4}{s^3}\left(\frac{s+1}{s}\right)$$

$$= \frac{4}{s^3} + \frac{4}{s^4}$$

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②Take Inverse Laplace Transform

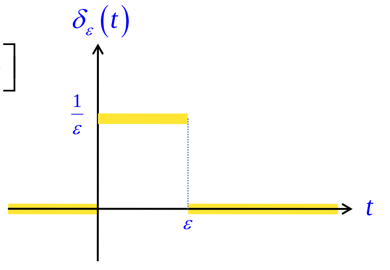
$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{4}{s^3} + \frac{4}{s^4}\right\}$$

$$= 2t^2 + \frac{2}{3}t^3$$

◎ *Unit Impulse & Dirac Delta Function*

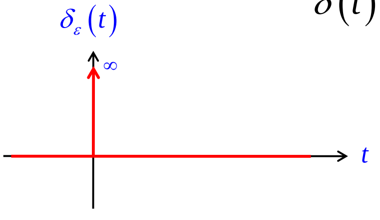
1. Impulse:

$$\delta_\varepsilon(t)=\frac{1}{\varepsilon}\left[H(t)-H(t-\varepsilon)\right]$$



2.Dirac Delta Function:

$$\delta(t)=\lim_{\varepsilon\to 0}\delta_\varepsilon(t)$$



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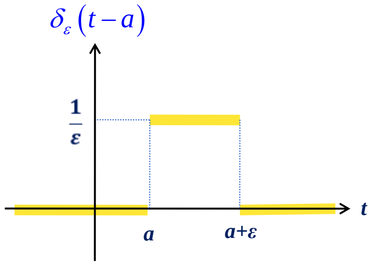
3. $$\delta_\varepsilon(t-a)=\frac{1}{\varepsilon}\left[H(t-a)-H(t-a-\varepsilon)\right]$$



$$\delta(t-a)=\lim_{\varepsilon\to 0}\delta_\varepsilon(t-a)$$

4. Laplace Transform

$$\mathcal{L}\{\delta_\varepsilon(t-a)\} = \frac{1}{\varepsilon}\left[\frac{1}{s}e^{-as} - \frac{1}{s}e^{-(a+\varepsilon)s}\right]$$

$$= \frac{e^{-as} - e^{-(a+\varepsilon)s}}{\varepsilon s} = \frac{e^{-as}(1-e^{-\varepsilon s})}{\varepsilon s}$$

$$Then\ \mathcal{L}\{\delta(t-a)\} = \lim_{\varepsilon \to 0} \frac{e^{-as}(1-e^{-\varepsilon s})}{\varepsilon s} = e^{-as}$$

1 Hospital`s rule

if $a=0$

$$L\{\delta(t)\} = 1$$

$$Ex:\ y''+2y'+2y=\delta(t-3)\ ;\ y(0)=y'(0)=0$$

Take Laplace Transform

$$s^2Y(s)+2sY(s)+2Y(s)=e^{-3s}$$

$$\Rightarrow Y(s)=\frac{e^{-3s}}{s^2+2s+2}=\frac{1}{(s+1)^2+1}e^{-3s}$$

We know $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}=\sin t$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}=e^{-t}\sin t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}e^{-3s}\right\}=H(t-3)e^{-(t-3)}\sin(t-3)$$

Therefore

$$y(t)=\mathcal{L}^{-1}\{Y(s)\}=H(t-3)e^{-(t-3)}\sin(t-3)$$

◎ *Linear O.D.E. with Polynomial coefficient*

$$\mathcal{L}\{t^n f(t)\}=(-1)^n\frac{d^n}{ds^n}F(s)$$

$$Ex:\ \ y''+2ty'-4y=1\ ;\ y(0)=y'(0)=0$$

Take Laplace Transform

$$\left[s^2Y(s)\right]+2\mathcal{L}\{ty'\}-4Y(s)=\frac{1}{s}$$

$$\mathcal{L}\{ty'\}=(-1)\frac{d}{ds}\mathcal{L}\{y'\}=(-1)\frac{d}{ds}\left[sY(s)\right]$$

$$=(-1)\left[Y(s)+sY'(s)\right]=-Y(s)-sY'(s)$$

$$\Rightarrow s^2Y(s)+2\left[-Y(s)-sY'(s)\right]-4Y(s)=\frac{1}{s}$$

$$\Rightarrow Y'(s)+\left(\frac{3}{s}-\frac{s}{2}\right)Y(s)=\frac{-1}{2s^2}$$

1^{st} $-$ *order O.D.E.*

find integrating factor $\sigma(s)=e^{\int\left(\frac{3}{s}-\frac{s}{2}\right)ds}=s^3e^{-\frac{s^2}{4}}$

$$\left(s^3e^{-\frac{s^2}{4}}\right)Y'(s)+\left(s^3e^{-\frac{s^2}{4}}\right)\left(\frac{3}{s}-\frac{s}{2}\right)Y(s)=s^3e^{-\frac{s^2}{4}}\left(\frac{-1}{2s^2}\right)$$

$$\Rightarrow \left[\left(s^3 e^{-\frac{s^2}{4}}\right) Y(s)\right]' = -\frac{1}{2} s e^{-\frac{s^2}{4}}$$

$$\Rightarrow s^3 e^{-\frac{s^2}{4}} Y(s) = \int -\frac{1}{2} s e^{-\frac{s^2}{4}} ds = e^{-\frac{s^2}{4}} + c$$

$$\Rightarrow Y(s) = \frac{1}{s^3} + \frac{c}{s^3} e^{-\frac{s^2}{4}}$$

$$\lim_{s \to \infty} Y(s) = y(0) = 0 \quad \Rightarrow \quad c=0 \quad \therefore Y(s) = \frac{1}{s^3}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2} t^2$$

◎Laplace Transform For system of O.D.E.

$$\begin{cases} x'' - 2x' + 3y' + 2y = 4 & x \to x(t) \\ 2y' - x' + 3y = 0 & y \to y(t) \end{cases}$$

$$x(0) = x'(0) = y(0) = 0$$

Find $x(t)$, $y(t)$

Solve:

①Take Laplace Transform

Let $\mathcal{L}\{x(t)\} = X(s)$, $\mathcal{L}\{y(t)\} = Y(s)$

② Find $X(s)$, $Y(s)$

$$X(s)=\frac{4s+6}{s^2(s+2)(s-1)},\quad Y(s)=\frac{2}{s(s+2)(s-1)}$$

③ Fraction Decomposition

④ Inverse Laplace Transform

$$x(t)=\mathcal{L}^{-1}\{X(s)\}=-\frac{7}{2}-3t+\frac{1}{6}e^{-2t}+\frac{10}{3}e^{t}$$

$$y(t)=\mathcal{L}^{-1}\{Y(s)\}=-1+\frac{1}{3}e^{-2t}+\frac{2}{3}e^{t}$$

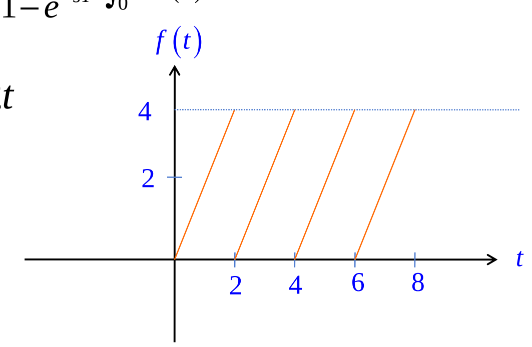
◎ *Laplace Transform of Periodic Function*

If f is periodic with period T on $0\le t<\infty$

and piecewise continuous on one period , then

$$\mathcal{L}\{f(t)\}=\frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}dt \quad for \quad s>0$$

$Ex:\ f(t)=2t$



$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2s}}\int_0^2 2te^{-st}dt$$

$$= \frac{1}{1-e^{-2s}}\frac{2\left[1-(1+2s)e^{-2s}\right]}{s^2}$$

$$= \frac{2}{s^2} - \frac{4}{s}\frac{e^{-2s}}{1-e^{-2s}}$$

Question: $\mathcal{L}^{-1}\left\{\frac{2}{s^2} - \frac{4}{s}\frac{e^{-2s}}{1-e^{-2s}}\right\} = ?$

Prove: $\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}dt$

Proof:

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

$$= \int_0^T f(t)e^{-st}dt + \int_T^{2T} f(t)e^{-st}dt + \cdots\cdots$$

$$= \sum_{n=0}^{\infty}\int_{nT}^{(n+1)T} f(t)e^{-st}dt$$

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$$Let \quad t=\tau+nT \quad , \quad dt=d\tau$$

$$\sum_{n=0}^{\infty}\int_{nT}^{(n+1)T} f(t)e^{-st}dt$$

$$=\sum_{n=0}^{\infty}\int_{0}^{T} f(\tau+nT)e^{-s(\tau+nT)}d\tau$$

$$=\sum_{n=0}^{\infty}e^{-snT}\int_{0}^{T} f(\tau+nT)e^{-s\tau}d\tau$$

$$=\sum_{n=0}^{\infty}e^{-snT}\int_{0}^{T} f(\tau)e^{-s\tau}d\tau$$

$$=\left(1+e^{-sT}+e^{-2sT}+\cdots\right)\int_{0}^{T} f(\tau)e^{-s\tau}d\tau$$

$$=\frac{1}{1-e^{-sT}}\int_{0}^{T} f(\tau)e^{-s\tau}d\tau$$

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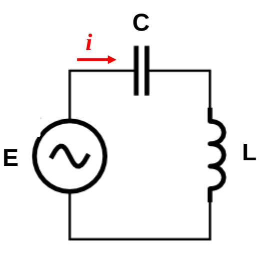
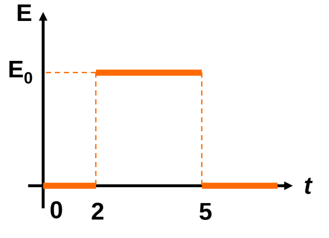
◎ **Application**





C, L are constants

The initial charge in capacitor is q_0

The initial current is zero

Find ***i*(*t*)**

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SOLUTION:

$$i(t) = \frac{dq(t)}{dt}$$

$$Lq''(t) + \frac{1}{C}q(t) = E(t); \ \ q(0) = q_0, \qquad q'(0) = 0$$

(1) Express E(t)

$$E(t) = E_0[H(t-2) - H(t-5)\ ]$$

(2) Take Laplace Transform

$$Lq''(t) + \frac{1}{C}q(t) = E_0[H(t-2) - H(t-5)\ ]$$

$$\Rightarrow L[s^2Q(s) - sq(0) - q'(0)] + \frac{1}{C}Q(s) = E_0\left\{\frac{e^{-2s}}{s} - \frac{e^{-5s}}{s}\right\}$$

$$\Rightarrow Q(s) = \frac{Lsq_0}{Ls^2 + \frac{1}{C}} + \frac{E_0[e^{-2s} - e^{-5s}]}{s\left\{Ls^2 + \frac{1}{C}\right\}}$$

$$Q(s) = \frac{Lsq_0}{Ls^2 + \frac{1}{C}} + \frac{E_0[e^{-2s} - e^{-5s}]}{s\left\{Ls^2 + \frac{1}{C}\right\}}$$

$$\Rightarrow Q(s) = \frac{sq_0}{s^2 + \frac{1}{LC}} + \frac{E_0}{Ls\left\{s^2 + \frac{1}{LC}\right\}}[e^{-2s} - e^{-5s}]$$

Let $w = \frac{1}{\sqrt{LC}}$

$$\Rightarrow Q(s) = \frac{sq_0}{s^2 + w^2} + \frac{E_0}{Ls\{s^2 + w^2\}}[e^{-2s} - e^{-5s}]$$

$$\mathcal{L}^{-1}\left\{\frac{sq_0}{s^2+w^2}\right\} = q_0\cos(wt)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s\{s^2+w^2\}}\right\} = 1 * \frac{\sin(wt)}{w} = \int_0^t \frac{\sin(w\tau)}{w}d\tau = \frac{1-\cos(wt)}{w^2}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s\{s^2+w^2\}}[e^{-2s}-e^{-5s}]\right\}$$

$$= \frac{1}{w^2}\{H(t-2)[1-cosw(t-2)]-H(t-5)[1-cosw(t-5)]\}$$

$$\Longrightarrow q(t) = \mathcal{L}^{-1}\{Q(s)\}$$

$$= q_0\cos(wt)$$

$$+\frac{E_0}{Lw^2}\{H(t-2)[1-cosw(t-2)]-H(t-5)[1$$

$$-cosw(t-5)]\}$$

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Appendix

Where the Laplace Transform Comes From?

$$\sum_{n=0}^{n=\infty} a_n x^n = A(x)$$

$$\Rightarrow \sum_{n=0}^{n=\infty} a(n) x^n = A(x)$$

$a(n) \xrightarrow{\text{associate with}} A(x)$

Ex: $1 \longrightarrow \frac{1}{1-x}$

$\frac{1}{n!} \longrightarrow e^x$

Discrete summation to Continuous Analog

$$\int_0^\infty a(t) x^t dt = A(x)$$

$x = e^{lnx}, x^t = e^{(lnx)t}, and\ 0 < x < 1\ (converge)$

Let $s = -lnx$ $\boxed{\Rightarrow s > 0}$

$$\Rightarrow \int_0^\infty f(t) e^{-st} dt = F(s)$$

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