Differentiation Rules

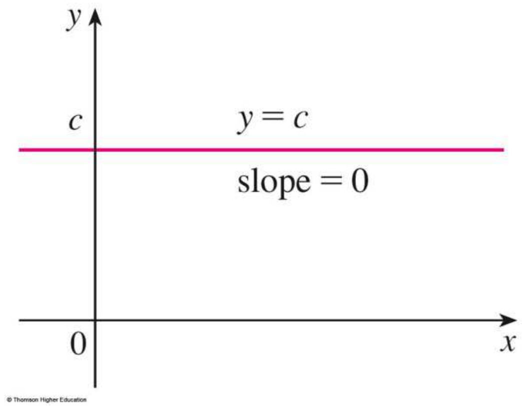
Lecture Note 4

Sec. 3.1 – Sec. 3.10

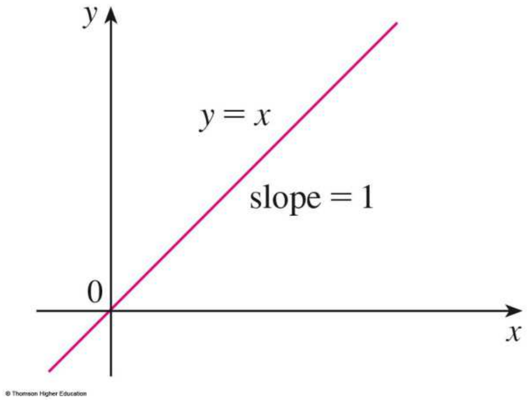
Sec. 3.1 Derivatives of Polynomials and Exponential Function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{d}{dx}(c) = \lim_{h \to 0} \frac{c-c}{h} = \lim_{h \to 0} 0 = 0$$



$$\frac{d}{dx}(x) = \lim_{h \to 0} \frac{(x+h) - x}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$



Rule: $$\boxed{\frac{d}{dx}x^n = nx^{n-1}, \quad n \text{ is a positive integer}}$$

Proof

$$\begin{aligned}
f'(a) &= \lim_{x\to a}\frac{x^n - a^n}{x-a} = \lim_{x\to a}\frac{(x-a)\left(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}\right)}{x-a} \\
&= \lim_{x\to a}\left(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}\right) \\
&= a^{n-1} + a^{n-2}a + \cdots + aa^{n-2} + a^{n-1} \\
&= na^{n-1}
\end{aligned}$$

$$f'(x) = nx^{n-1}$$

Generally, *n* can be any real number. (See Sec 3.6)

Example 1

(a) $f(x) = x^6, \quad \text{then} \quad f'(x) = 6x^5.$

(b) $y = x^{1000}, \quad \text{then} \quad y' = 1000x^{999}.$

(c) $y = t^4, \quad \text{then} \quad \dfrac{dy}{dt} = 4t^3.$

(d) $\dfrac{d}{dr}\left(r^3\right) = 3r^2$

Example 2

(a) $f(x) = \dfrac{1}{x^2}$

$$f'(x) = \frac{d}{dx}\left(x^{-2}\right) = -2x^{-2-1} = -2x^{-3} = -\frac{2}{x^3}$$

(b) $y = \sqrt[3]{x^2}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt[3]{x^2}\right) = \frac{d}{dx}\left(x^{2/3}\right) = \frac{2}{3}x^{(2/3)-1} = \frac{2}{3}x^{-1/3}$$

Example 3

Find equations of the tangent line and normal line to the curve $y = x\sqrt{x}$ at the point (1, 1). Illustrate by graphing the curve and these lines.

$$y = x\sqrt{x} = x^{3/2}$$

$$y' = \frac{3}{2}x^{1/2} = \frac{3}{2}\sqrt{x}$$

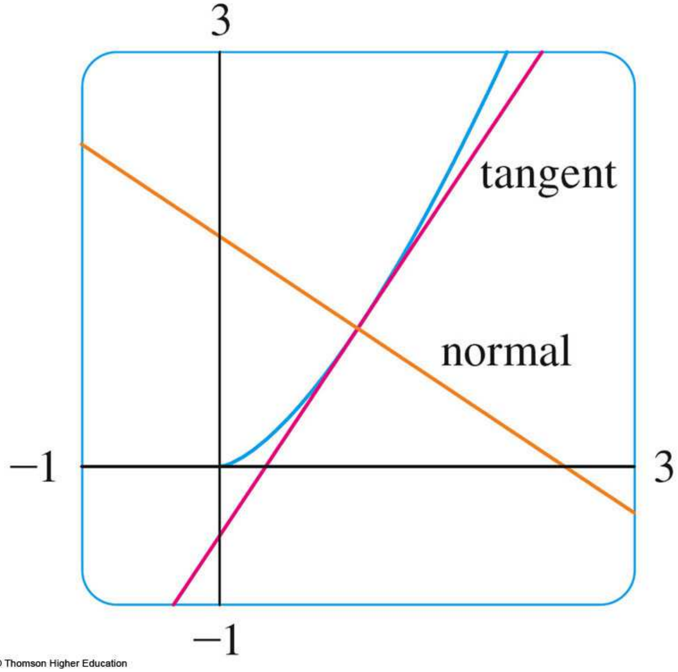
Tangent line:

$$m = y'(1) = \frac{3}{2}$$

$$y - 1 = \frac{3}{2}(x-1) \quad \Rightarrow \quad y = \frac{3}{2}x - \frac{1}{2}$$

Normal line: $m = -\dfrac{2}{3}$

$$y - 1 = -\frac{2}{3}(x-1) \quad \Rightarrow \quad y = -\frac{2}{3}x + \frac{5}{3}$$



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Example

Find the equations of tangent line and normal line to the curve $y=\dfrac{\sqrt{x}}{1+x^2}$ at the point $\left(1,\dfrac{1}{2}\right)$.

$$y'=\frac{\left(1+x^2\right)\dfrac{d}{dx}\left(\sqrt{x}\right)-\sqrt{x}\,\dfrac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2}$$

$$=\frac{\left(1+x^2\right)\dfrac{1}{2\sqrt{x}}-\sqrt{x}\left(2x\right)}{\left(1+x^2\right)^2}$$

$$=\frac{\left(1+x^2\right)-4x^2}{2\sqrt{x}\left(1+x^2\right)^2}=\frac{1-3x^2}{2\sqrt{x}\left(1+x^2\right)^2}$$

$$y'(1)=\frac{1-3(1)^2}{2\sqrt{1}\left(1+1^2\right)^2}=-\frac{1}{4}$$

tangent line:

$$y-\frac{1}{2}=-\frac{1}{4}(x-1)$$

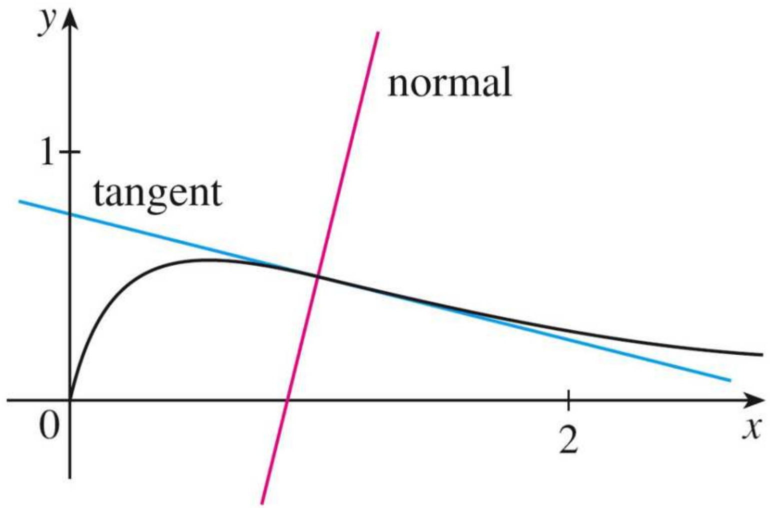
$$\Rightarrow y=-\frac{1}{4}x+\frac{3}{4}$$

Slop of normal line at $\left(1, \frac{1}{2}\right)$ is 4.

normal line:

$$y - \frac{1}{2} = 4(x-1)$$

$$\Rightarrow y = 4x - \frac{7}{2}$$



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Rule: $\boxed{\dfrac{d}{dx}\left[cf(x)\right] = c\dfrac{d}{dx} f(x), \quad c \text{ is a constant}}$

Proof

$$\frac{d}{dx}\left[cf(x)\right] = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} = \lim_{h\to 0} c\left[\frac{f(x+h)-f(x)}{h}\right]$$

$$= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= cf'(x)$$

Example 4

$$\frac{d}{dx}\left(3x^4\right) = 3\frac{d}{dx}\left(x^4\right) = 3\left(4x^3\right) = 12x^3$$

Rule: $$\boxed{\frac{d}{dx}\left[f(x)\pm g(x)\right]=\frac{d}{dx}f(x)\pm\frac{d}{dx}g(x)}$$

Proof

$$\begin{aligned}
\frac{d}{dx}\left[f(x)+g(x)\right]&=\lim_{h\to 0}\frac{\left[f(x+h)+g(x+h)\right]-\left[f(x)+g(x)\right]}{h}\\
&=\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right]\\
&=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&=f'(x)+g'(x)
\end{aligned}$$

Example 4

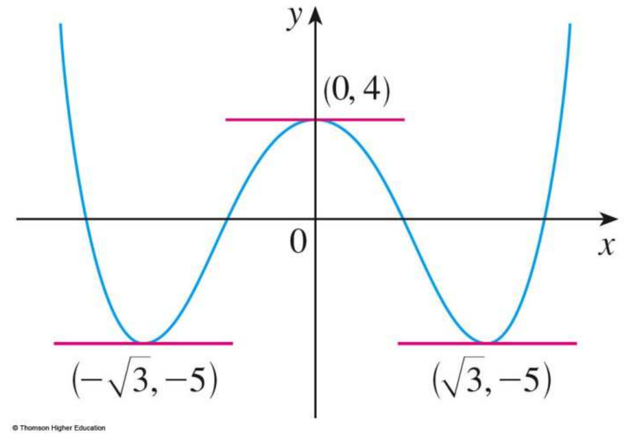
$$\frac{d}{dx}\left(x^8+12x^5-4x^4+10x^3-6x+5\right)=8x^7+60x^4-16x^3+30x^2-6$$

Example 6

Find the points on the curve $y = x^4 - 6x^2 + 4$ **where the tangent line is horizontal.**

$$y' = 4x^3 - 12x = 4x\left(x^2 - 3\right)$$

For tangent line is horizontal:



$$y' = 4x\left(x^2 - 3\right) = 0$$

$$x = 0$$

$$x = \pm\sqrt{3}$$

The points are:

$$(0,4), \quad \left(\sqrt{3}, -5\right), \quad \left(-\sqrt{3}, -5\right)$$

EXPONENTIAL FUNCTION

$$f(x) = a^x$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x a^h - a^x}{h} = \lim_{h \to 0} \frac{a^x\left(a^h - 1\right)}{h}$$

$$= a^x \lim_{h \to 0} \frac{a^h - 1}{h}$$

$$\because \quad \lim_{h \to 0} \frac{a^h - 1}{h} = f'(0) \quad \Rightarrow \quad \boxed{f'(x) = f'(0)\, a^x}$$

The rate of change of any exponential function is proportional to the function itself.

h	$\dfrac{2^h - 1}{h}$	$\dfrac{3^h - 1}{h}$
0.1	0.7177	1.1612
0.01	0.6956	1.1047
0.001	0.6934	1.0992
0.0001	0.6932	1.0987

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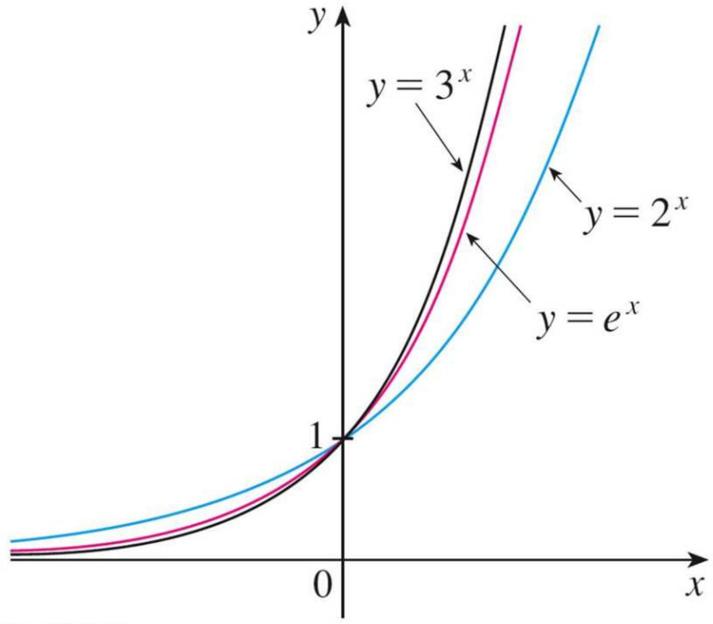

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DEFINITION OF THE NUMBER *e*

***e* is the number such that** $\lim_{h \to 0} \dfrac{e^h - 1}{h} = 1$

$$\boxed{f(x) = e^x \quad \Rightarrow \quad f'(0) = 1}$$

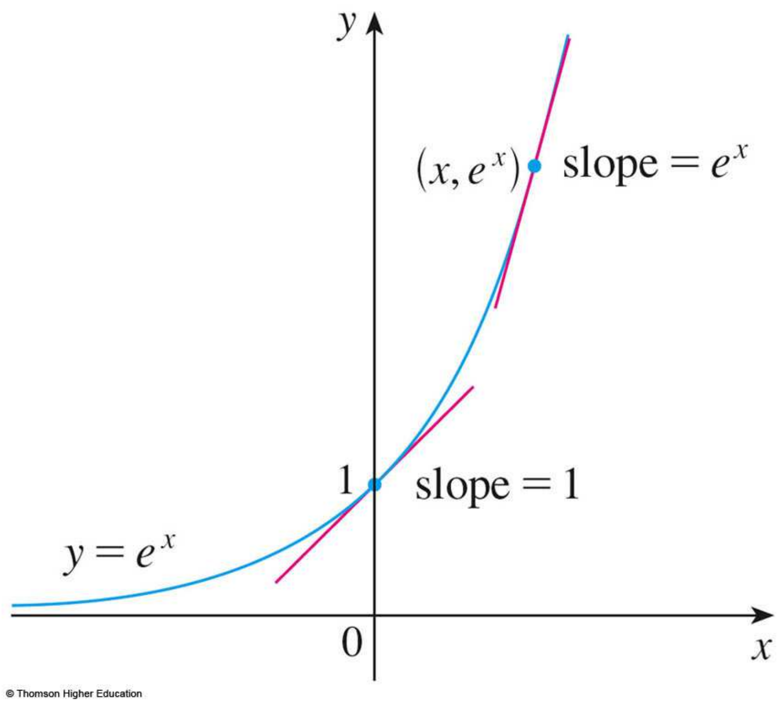
DERIVATIVE OF THE NATURAL EXPONENTIAL FUNCTION

$$\frac{d}{dx}\left(e^x\right)=e^x$$

$$f(x)=e^x$$

$$\because \quad \lim_{h\to 0}\frac{e^h-1}{h}=f'(0)=1$$

$$f'(x)=f'(0)e^x=e^x$$

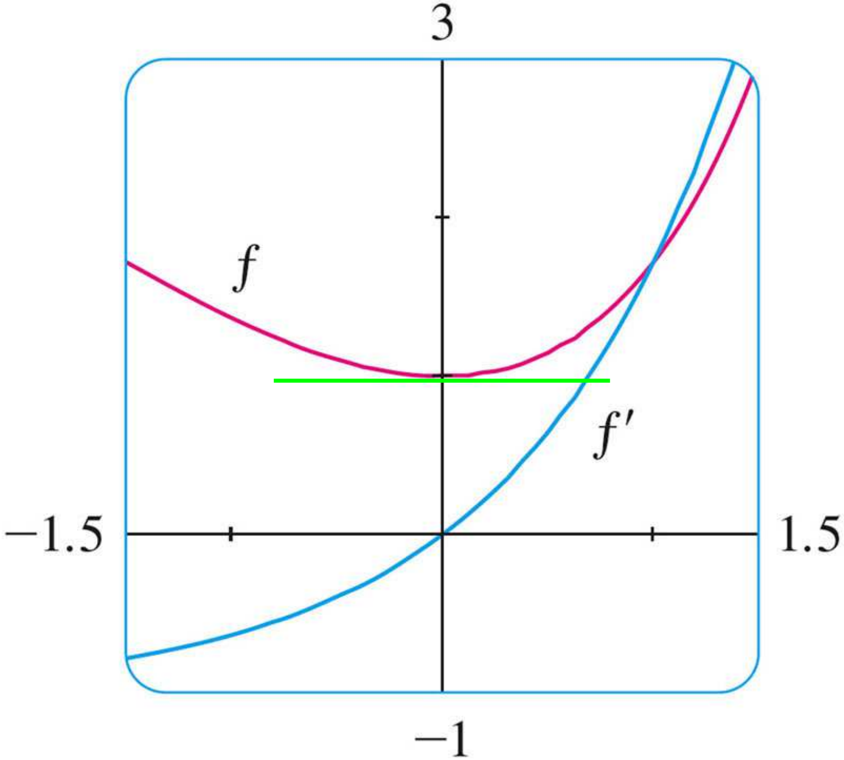


Example 8

If $f(x) = e^x - x$ **, find** f' **and** f''**, and compare the graph of** f **and** f' **.**

$$f'(x) = \frac{d}{dx}\left(e^x - x\right) = e^x - 1$$

$$f''(x) = \frac{d}{dx}\left(e^x - 1\right) = e^x$$



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Example 9

At what point on the curve $y = e^x$ **is the tangent line parallel to the line** $y = 2x$ **?**

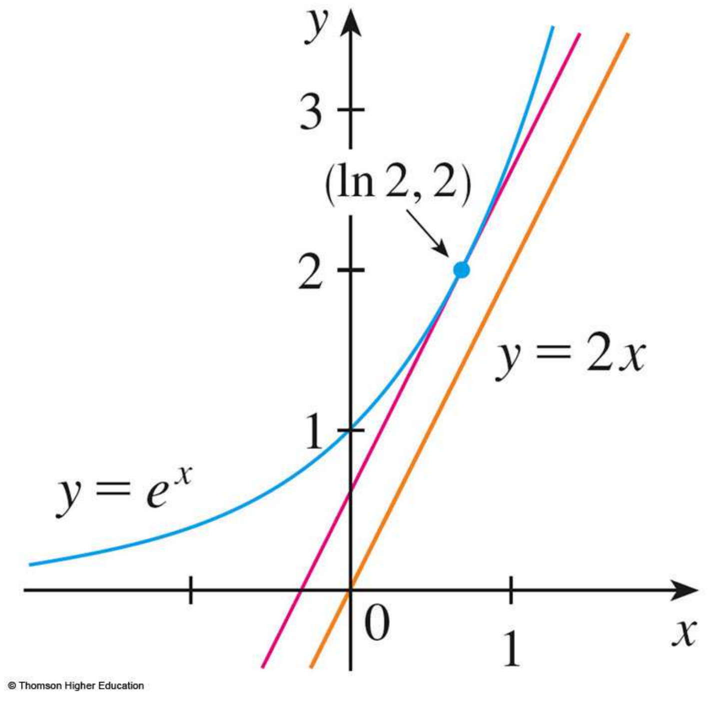
$$y = e^x \quad \Rightarrow \quad y' = e^x$$

$$y = 2x \quad \Rightarrow \quad y' = 2$$

$$e^a = 2 \quad \Rightarrow \quad a = \ln 2$$

The point is

$$\left(a, e^a\right) = \left(\ln 2, 2\right)$$



Sec. 3.2 The Product and Quotient Rules

Product rule:

$$\boxed{\frac{d}{dx}\left[f(x)g(x)\right] = f(x)\frac{d}{dx}\left[g(x)\right] + g(x)\frac{d}{dx}\left[f(x)\right]}$$

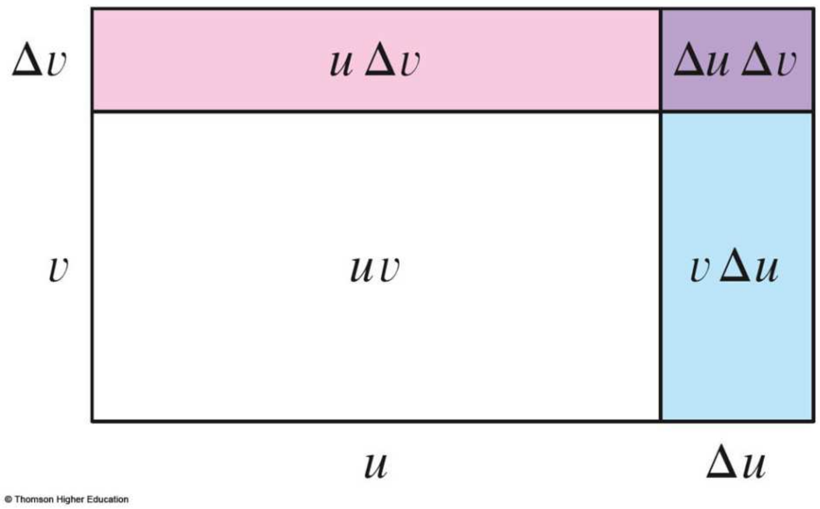
Quotient rule:

$$\boxed{\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}\left[f(x)\right] - f(x)\frac{d}{dx}\left[g(x)\right]}{\left[g(x)\right]^2}}$$

$$\frac{d}{dx}\left[f(x)g(x)\right]=?$$

$$u=f(x) \qquad \upsilon=g(x)$$

假想兩個函數分別代表一個長方形的兩個邊長



$$\Delta u=f(x+\Delta x)-f(x)$$

$$\Delta\upsilon=g(x+\Delta x)-g(x)$$

$$u(x+\Delta x)=u(x)+\Delta u$$

$$\upsilon(x+\Delta x)=\upsilon(x)+\Delta\upsilon$$

$$\begin{aligned}\Delta(uv)&=u(x+\Delta x)\upsilon(x+\Delta x)-u(x)\upsilon(x)\\&=(u+\Delta u)(\upsilon+\Delta\upsilon)-u(x)\upsilon(x)\\&=u\Delta\upsilon+\upsilon\Delta u+\Delta u\Delta\upsilon\end{aligned}$$

高階項

$$
\begin{aligned}
\frac{d}{dx}(uv) &= \lim_{\Delta x \to 0} \frac{\Delta(u\upsilon)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left(u\frac{\Delta \upsilon}{\Delta x} + \upsilon\frac{\Delta u}{\Delta x} + \Delta u \frac{\Delta \upsilon}{\Delta x} \right) \\
&= u \lim_{\Delta x \to 0} \frac{\Delta \upsilon}{\Delta x} + \upsilon \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \left(\lim_{\Delta x \to 0} \Delta u \right) \left(\lim_{\Delta x \to 0} \frac{\Delta \upsilon}{\Delta x} \right) \\
&= u\frac{d\upsilon}{dx} + \upsilon\frac{du}{dx} + 0\frac{d\upsilon}{dx}
\end{aligned}
$$

因此微分高階項可直接忽略

Rule: $$\boxed{\frac{d}{dx}\left[f(x)g(x)\right]=f(x)\frac{d}{dx}\left[g(x)\right]+g(x)\frac{d}{dx}\left[f(x)\right]}$$

Proof

$$\frac{d}{dx}\left[f(x)g(x)\right]=\lim_{h\to 0}\frac{\left[f(x+h)g(x+h)\right]-\left[f(x)g(x)\right]}{h}$$

$$=\lim_{h\to 0}\left[\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}\right]$$

$$=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)}{h}+\lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x)}{h}$$

$$=\lim_{h\to 0}f(x+h)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}+\lim_{h\to 0}g(x)\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=f(x)g'(x)+g(x)f'(x)$$

Example 1

(a) $f(x) = xe^x$

$$f'(x) = \frac{d}{dx}\left(xe^x\right) = x\frac{d}{dx}\left(e^x\right) + e^x\frac{d}{dx}(x)$$
$$= xe^x + e^x = e^x(x+1)$$

(b) $f^{(n)}(x) = ?$

$$f''(x) = e^x(x+1) + e^x = e^x(x+2)$$
$$f'''(x) = e^x(x+2) + e^x = e^x(x+3)$$
$$\vdots$$
$$f^{(n)}(x) = e^x(x+n)$$

Example 2

$$f(t) = \sqrt{t}\,(a+bt)$$

$$\begin{aligned} f'(t) &= \sqrt{t}\,\frac{d}{dx}(a+bt) + (a+bt)\frac{d}{dx}\left(\sqrt{t}\right) \\ &= \sqrt{t}\cdot b + (a+bt)\cdot\frac{1}{2}t^{-1/2} \\ &= b\sqrt{t} + \frac{a+bt}{2\sqrt{t}} = \frac{a+3bt}{2\sqrt{t}} \end{aligned}$$

Example 3

If $f(x)=\sqrt{x}g(x)$ **, where** $g(4)=2$ **and** $g'(4)=3$ **, find** $f'(4)$**.**

$$f'(x)=\frac{d}{dx}\left[\sqrt{x}g(x)\right]$$

$$=\sqrt{x}g'(x)+g(x)\cdot\frac{1}{2}x^{-1/2}$$

$$=\sqrt{x}g'(x)+\frac{g(x)}{2\sqrt{x}}$$

$$f'(4)=\sqrt{4}g'(4)+\frac{g(4)}{2\sqrt{4}}=2\cdot 3+\frac{2}{2\cdot 2}=6.5$$

Rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\frac{d}{dx}\left[f(x)\right]-f(x)\frac{d}{dx}\left[g(x)\right]}{\left[g(x)\right]^2}$$

Proof

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h}$$

$$=\lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x+h)}{hg(x+h)g(x)}$$

$$=\lim_{h\to 0}\frac{\frac{f(x+h)g(x)-f(x)g(x)}{h}-\frac{f(x)g(x+h)-f(x)g(x)}{h}}{g(x+h)g(x)}$$

$$=\frac{\lim_{h\to 0}g(x)\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}-\lim_{h\to 0}f(x)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}}{\lim_{h\to 0}g(x+h)\cdot\lim_{h\to 0}g(x)}$$

$$=\frac{g(x)f'(x)-f(x)g'(x)}{\left[g(x)\right]^2}$$

Example 4

$$y=\frac{x^2+x-2}{x^3+6},\quad y'=?$$

$$y'=\frac{\left(x^3+6\right)\frac{d}{dx}\left(x^2+x-2\right)-\left(x^2+x-2\right)\frac{d}{dx}\left(x^3+6\right)}{\left(x^3+6\right)^2}$$

$$=\frac{\left(x^3+6\right)(2x+1)-\left(x^2+x-2\right)\left(3x^2\right)}{\left(x^3+6\right)^2}$$

$$=\frac{\left(2x^4+x^3+12x+6\right)-\left(3x^4+3x^3-6x^2\right)}{\left(x^3+6\right)^2}$$

$$=\frac{-x^4-2x^3+6x^2+12x+6}{\left(x^3+6\right)^2}$$

Example 5

Find an equations of the tangent line to the curve $y = e^x / (1 + x^2)$ **at the point (1, ½ *e*).**

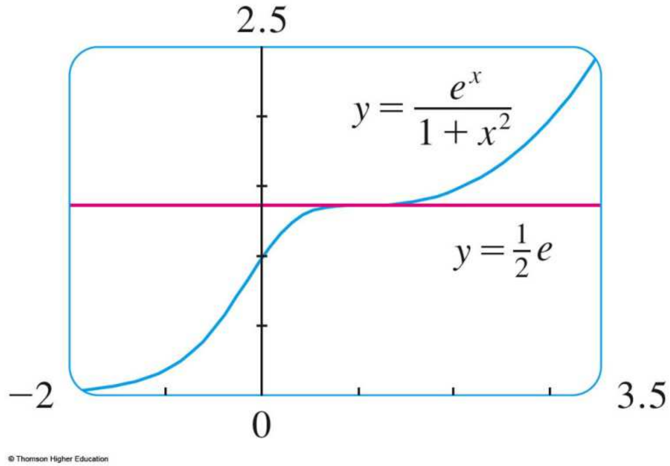
$$y' = \frac{(1+x^2)\frac{d}{dx}(e^x) - (e^x)\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$= \frac{(1+x^2)(e^x) - (e^x)(2x)}{(1+x^2)^2} = \frac{e^x(1-2x+x^2)}{(1+x^2)^2} = \frac{e^x(1-x)^2}{(1+x^2)^2}$$

$$y'(1) = \frac{e^x(1-1)^2}{(1+1^2)^2} = 0$$

The tangent line

$$y = \frac{1}{2}e$$



Sec. 3.3 Derivatives of Trigonometric Functions

Two important limits

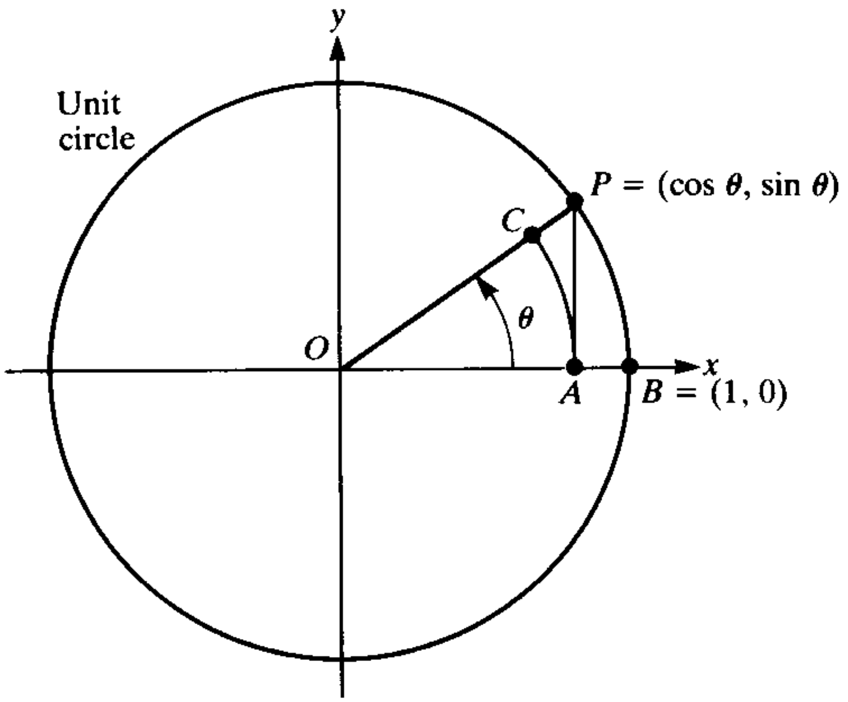
1. $$\boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1}$$

2. $$\boxed{\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} = 0}$$ OR $$\boxed{\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0}$$

這兩個極限的含意為何？

這兩個極限會用來證明三角函數的微分公式

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$



Proof

$$\text{Area of sector } OAC = \frac{\theta \cos^2 \theta}{2}$$

$$\text{Area of triangle } OAP = \frac{\cos\theta \cdot \sin\theta}{2}$$

$$\text{Area of sector } OBP = \frac{\theta \cdot 1^2}{2} = \frac{\theta}{2}$$

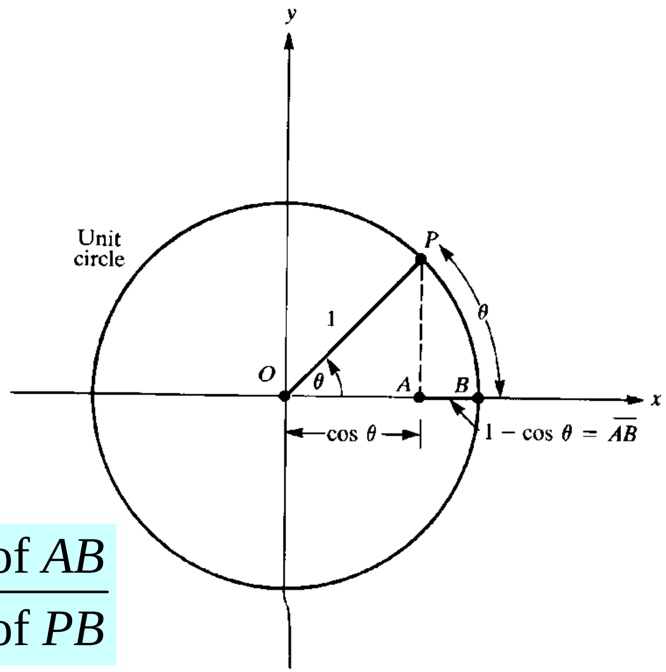
$$\text{Area of sector } OAC < \text{Area of triangle } OAP < \text{Area of sector } OBP$$

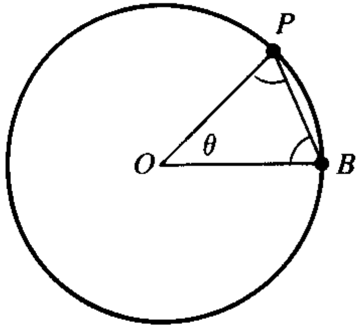
$$\frac{\theta \cos^2 \theta}{2} < \frac{\cos\theta \cdot \sin\theta}{2} < \frac{\theta}{2} \quad \Rightarrow \quad \cos\theta < \frac{\sin\theta}{\theta} < \frac{1}{\cos\theta}$$

$$\lim_{\theta \to 0} \cos\theta = 1, \qquad \lim_{\theta \to 0} \frac{1}{\cos\theta} = 1$$

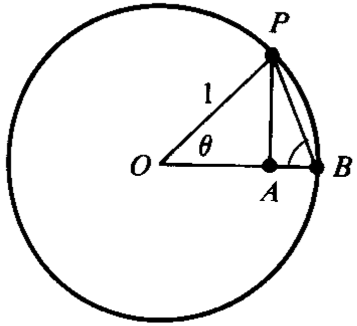
$$\text{Squeeze theorem:} \quad \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}=0$$





$OPB = OBP$



Angle $ABP = \frac{\pi-\theta}{2}$

Proof

$$\frac{1-\cos\theta}{\theta}=\frac{\text{Length of } AB}{\text{Length of arc } PB}<\frac{\text{Length of } AB}{\text{Length of } PB}$$

$$\frac{\text{Length of } AB}{\text{Length of } PB}=\cos\left(\frac{\pi-\theta}{2}\right)$$

$$0<\frac{1-\cos\theta}{\theta}<\cos\left(\frac{\pi-\theta}{2}\right)$$

$$\lim_{\theta\to 0}0=0,\qquad \lim_{\theta\to 0}\cos\left(\frac{\pi-\theta}{2}\right)=\cos\frac{\pi}{2}=0$$

Squeeze theorem: $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}=0$

Example

$$\lim_{x\to 0}\frac{\tan x}{x} = ?$$

$$\begin{aligned}
\lim_{x\to 0}\frac{\tan x}{x} &= \lim_{x\to 0}\frac{\sin x/\cos x}{x} \\
&= \lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\frac{1}{\cos x}\right) \\
&= \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{\cos x} \\
&= 1\cdot\frac{1}{1} = 1
\end{aligned}$$

此極限的結果有何含意？

Example

$$\lim_{x\to 0}\frac{\sin 5x}{\sin 2x}=?$$

$$\begin{aligned}\frac{\sin 5x}{\sin 2x}&=\frac{\sin 5x}{5x}\cdot\frac{2x}{\sin 2x}\cdot\frac{5x}{2x}\\&=\frac{\sin 5x}{5x}\cdot\frac{2x}{\sin 2x}\cdot\frac{5}{2}\\&=\frac{5}{2}\cdot\frac{\sin 5x}{5x}\cdot\frac{1}{\dfrac{\sin 2x}{2x}}\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 5x}{\sin 2x}&=\frac{5}{2}\cdot\lim_{x\to 0}\frac{\sin 5x}{5x}\cdot\frac{1}{\dfrac{\sin 2x}{2x}}\\&=\frac{5}{2}\cdot\lim_{x\to 0}\frac{\sin 5x}{5x}\cdot\frac{1}{\lim_{x\to 0}\dfrac{\sin 2x}{2x}}\\&=\frac{5}{2}\cdot 1\cdot\frac{1}{1}=\frac{5}{2}\end{aligned}$$

DERIVATIVES OF TRIGONMETRIC FUNCTIONS

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sin x) = \cos x$$

Proof

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h} \\
&= \lim_{h\to 0}\left[\frac{\sin x\cos h-\sin x}{h}+\frac{\cos x\sin h}{h}\right] \\
&= \lim_{h\to 0}\left[\sin x\cdot\frac{\cos h-1}{h}+\cos x\cdot\frac{\sin h}{h}\right] \\
&= \sin x\cdot\lim_{h\to 0}\frac{\cos h-1}{h}+\cos x\cdot\lim_{h\to 0}\frac{\sin h}{h} \\
&= \sin x\cdot 0+\cos x\cdot 1=\cos x
\end{aligned}$$

$$\frac{d}{dx}\left(\sec x\right)=\sec x\tan x$$

Proof

$$\begin{aligned}\frac{d}{dx}\left(\sec x\right)&=\frac{d}{dx}\left(\frac{1}{\cos x}\right)\\&=\frac{-\frac{d}{dx}\left(\cos x\right)}{\cos^2 x}\\&=\frac{\sin x}{\cos^2 x}\\&=\frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}\\&=\sec x\cdot\tan x\end{aligned}$$

Example 2

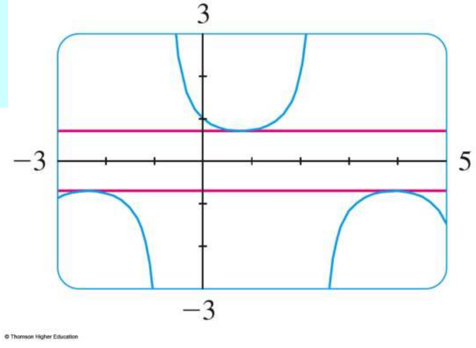
Differentiate $f(x)=\dfrac{\sec x}{1+\tan x}$. For what value of x does the graph of f have horizontal tangent?

$$f'(x)=\frac{(1+\tan x)(\sec x)'-(\sec x)(1+\tan x)'}{(1+\tan x)^2}$$

$$=\frac{(1+\tan x)(\sec x\tan x)-(\sec x)\left(\sec^2 x\right)}{(1+\tan x)^2}$$

$$=\frac{\sec x\left(\tan x+\tan^2 x-\sec^2 x\right)}{(1+\tan x)^2}=\frac{\sec x(\tan x-1)}{(1+\tan x)^2}$$

$$f'(x)=0 \quad \text{when} \quad \tan x=1. \quad \Rightarrow \quad x=n\pi+\frac{\pi}{4}$$



<u>Sec. 3.4</u> The Chain Rule

※用於合成函數的微分

<u>Example 1</u>

$$F(x)=\sqrt{x^2+1}, \qquad \frac{d}{dx}F(x)=?$$

Let $y=f(u)=\sqrt{u}, \qquad u=g(x)=x^2+1$

Then $F(x)=f(g(x))=(f\circ g)(x)$

$$\frac{d}{dx}F(x)=\frac{dy}{dx}\overset{?}{=}\frac{dy}{du}\frac{du}{dx}=\left(\frac{1}{2\sqrt{u}}\right)(2x)=\frac{x}{\sqrt{u}}=\frac{x}{\sqrt{x^2+1}}$$

THE CHAIN RULE If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F=f \circ g$ defined by $F=f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y=f(u)$ and $u=g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

連鎖律

鍊微法則

Example 2

$$\frac{d}{dx}\left[\sin\left(x^2\right)\right] = ?$$

Let $u(x) = x^2, \quad y = \sin\left(x^2\right) = \sin u$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{d(\sin u)}{du}\frac{d\left(x^2\right)}{dx} = (\cos u)(2x) = 2x\cos\left(x^2\right)$$

$$\frac{d}{dx}\left(\sin^2 x\right) = ?$$

Let $u(x) = \sin x, \quad y = \sin^2 x = u^2$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{d\left(u^2\right)}{du}\frac{d(\sin x)}{dx} = (2u)(\cos x) = 2\sin x \cdot \cos x$$

Example 3

$$y = \left(x^3 - 1\right)^{100}$$

Let $u(x) = x^3 - 1, \quad y = u^{100}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= \frac{d}{du}\left(u^{100}\right)\frac{d}{dx}\left(x^3 - 1\right) \\ &= 100u^{99}\left(3x^2\right) \\ &= 300x^2\left(x^3 - 1\right)^{99}\end{aligned}$$

THE POWER RULE COMBINED WITH THE CHAIN RULE **If *n* is any real number and *u*=*g*(*x*) is differentiable, then**

$$\frac{d}{dx}\left(u^{n}\right)=nu^{n-1}\frac{du}{dx}$$

Alternatively

$$\frac{d}{dx}\left[g\left(x\right)\right]^{n}=n\left[g\left(x\right)\right]^{n-1}\cdot g'\left(x\right)$$

Example 4

$$f(x)=\frac{1}{\sqrt[3]{x^2+x+1}}$$

Let $u(x)=x^2+x+1, \quad y=\dfrac{1}{\sqrt[3]{u}}$

$$\begin{aligned}\frac{dy}{dx}&=\frac{dy}{du}\frac{du}{dx}\\&=\frac{d}{du}\left(u^{-1/3}\right)\frac{d}{dx}\left(x^2+x+1\right)\\&=\left(-\frac{1}{3}u^{-4/3}\right)(2x+1)\\&=-\frac{1}{3}\left(x^2+x+1\right)^{-4/3}(2x+1)\end{aligned}$$

Example 5

$$g(t) = \left(\frac{t-2}{2t+1}\right)^9$$

$$g'(t) = 9\left(\frac{t-2}{2t+1}\right)^8 \frac{d}{dt}\left(\frac{t-2}{2t+1}\right)$$

$$= 9\left(\frac{t-2}{2t+1}\right)^8 \frac{(2t+1)\cdot 1 - 2(t-2)}{(2t+1)^2}$$

$$= \frac{45(t-2)^8}{(2t+1)^{10}}$$

Example 6

$$y = (2x+1)^5 (x^3 - x + 1)^4$$

$$\begin{aligned}
\frac{dy}{dx} &= (2x+1)^5 \frac{d}{dx}(x^3 - x + 1)^4 + (x^3 - x + 1)^4 \frac{d}{dx}(2x+1)^5 \\
&= (2x+1)^5 \cdot 4(x^3 - x + 1)^3 \frac{d}{dx}(x^3 - x + 1) + (x^3 - x + 1)^4 \cdot 5(2x+1)^4 \frac{d}{dx}(2x+1) \\
&= 4(2x+1)^5 (x^3 - x + 1)^3 (3x^2 - 1) + 10(x^3 - x + 1)^4 (2x+1)^4 \\
&= 2(2x+1)^4 (x^3 - x + 1)^3 (17x^3 + 6x^2 - 9x + 3)
\end{aligned}$$

Derivative of an exponential function with any base *a*>0:

$$a^x = \left(e^{\ln a}\right)^x = e^{(\ln a)x}$$

$$\begin{aligned}\frac{d}{dx}\left(a^x\right) &= \frac{d}{dx}\left(e^{(\ln a)x}\right)\\ &= e^{(\ln a)x}\frac{d}{dx}\left((\ln a)x\right)\\ &= e^{(\ln a)x}(\ln a)\\ &= a^x \ln a\end{aligned}$$

$$\boxed{\frac{d}{dx}\left(a^x\right) = a^x \ln a}$$

Recall the relations:

$$\log_b\left(b^x\right) = x \quad \text{for every } x \in \mathrm{R}$$

$$b^{\log_b x} = x \quad \text{for every } x > 0$$

Example 8

$$f(x) = \sin\left(\cos\left(\tan x\right)\right)$$

$$\begin{aligned}
f'(x) &= \cos\left(\cos\left(\tan x\right)\right)\frac{d}{dx}\cos\left(\tan x\right) \\
&= \cos\left(\cos\left(\tan x\right)\right)\left(-\sin\left(\tan x\right)\right)\frac{d}{dx}\left(\tan x\right) \\
&= \cos\left(\cos\left(\tan x\right)\right)\left(-\sin\left(\tan x\right)\right)\sec^2 x \\
&= -\cos\left(\cos\left(\tan x\right)\right)\sin\left(\tan x\right)\sec^2 x
\end{aligned}$$

Example 9

$$y = e^{\sec 3\theta}$$

$$\begin{aligned}\frac{dy}{d\theta} &= e^{\sec 3\theta}\frac{d}{d\theta}\left(\sec 3\theta\right)\\ &= e^{\sec 3\theta}\sec 3\theta \tan 3\theta \frac{d}{d\theta}\left(3\theta\right)\\ &= 3e^{\sec 3\theta}\sec 3\theta \tan 3\theta\end{aligned}$$

Proof of the Chain Rule

Assuming $g'(x) \neq 0$.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$ $\Delta u \neq 0$ if Δx is small enough

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$ since both limits exist

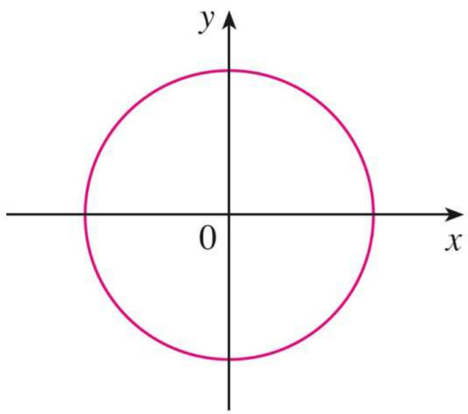
$$= \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$ since $\Delta u \to 0$ as $\Delta x \to 0$

$$= \frac{dy}{du} \cdot \frac{du}{dx}$$ definition of derivative

<u>Sec. 3.5</u> Implicit Differentiation 隱微分

Some functions are defined implicitly by a relation between *x* and *y* such as

$$x^2 + y^2 = 25$$ 此方程式包含兩個函數



(a) $x^2 + y^2 = 25$

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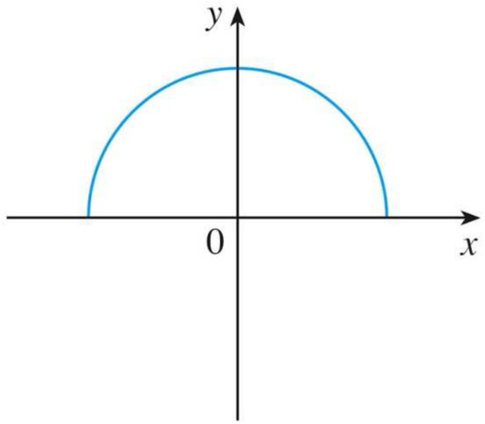

(b) $f(x) = \sqrt{25 - x^2}$

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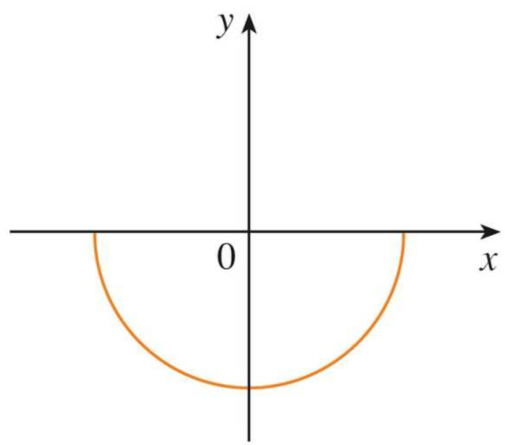

(c) $g(x) = -\sqrt{25 - x^2}$

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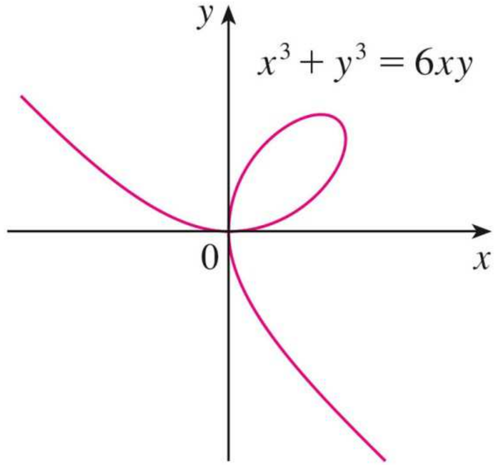
$$x^3 + y^3 = 6xy$$

$$x^3 + \left[f(x) \right]^3 = 6xf(x)$$



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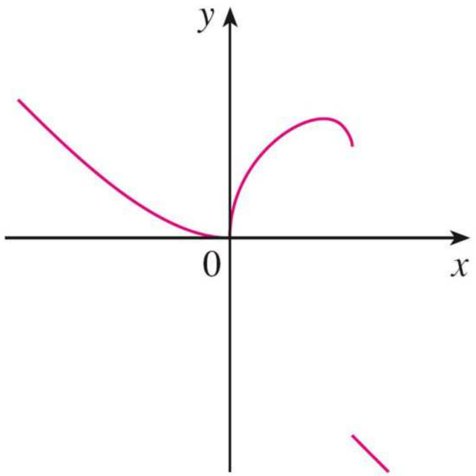

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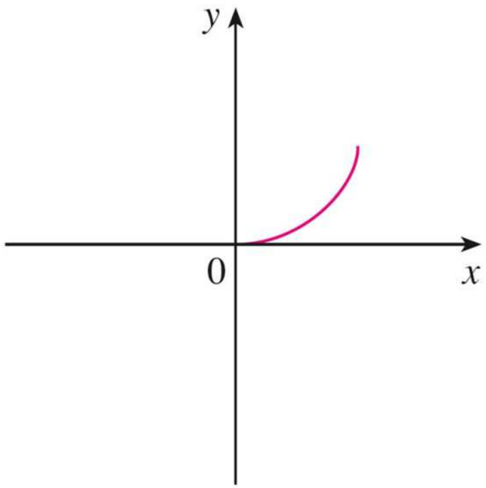

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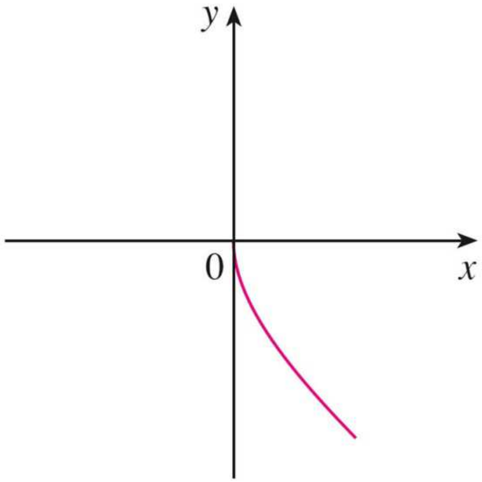

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Example 1

$$(\mathrm{a})\ x^2 + y^2 = 25, \quad \frac{dy}{dx} = ?$$

(b) Find the tangent to the circle $x^2 + y^2 = 25$ at the point $(3,4)$.

SOLUTION I

(a)

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(25)$$

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}$$

(b)

at the point $(3,4)$, $\quad \dfrac{dy}{dx} = -\dfrac{x}{y} = -\dfrac{3}{4}$

equation of the tangent line:

$$y - 4 = -\frac{3}{4}(x - 3) \quad \text{or} \quad 3x + 4y = 25$$

SOLUTION II

(a) $x^2 + y^2 = 25 \quad \Rightarrow \quad y = \pm\sqrt{25 - x^2}$

the point $(3,4)$ is on $y = \sqrt{25 - x^2}$

$$\begin{aligned} y' &= \frac{dy}{dx} \\ &= \frac{1}{2}\left(25 - x^2\right)^{-1/2} \frac{d}{dx}\left(25 - x^2\right) \\ &= \frac{1}{2}\left(25 - x^2\right)^{-1/2} (-2x) \\ &= -\frac{x}{\sqrt{25 - x^2}} \end{aligned}$$

(b)

at the point $(3, 4)$, $\quad x = 3$

$$\frac{dy}{dx} = -\frac{x}{\sqrt{25 - x^2}} = -\frac{3}{4}$$

equation of the tangent line:

$$y - 4 = -\frac{3}{4}(x - 3) \quad \text{or} \quad 3x + 4y = 25$$

Example 2

(a) $x^3 + y^3 = 6xy, \quad \dfrac{dy}{dx} = ?$

(b) Find the tangent to the curve $x^3 + y^3 = 6xy$ at the point $(3,3)$.

(c) At what points in the first quadrant is the tangent line horizontal?

(a)

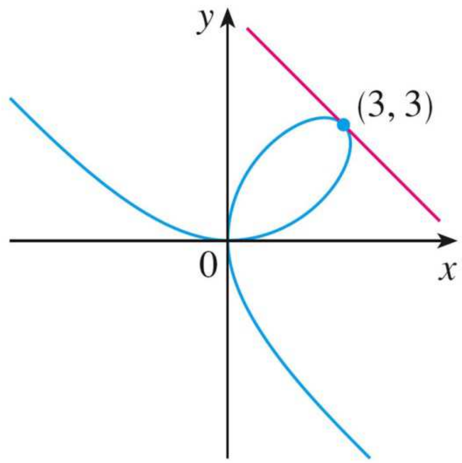
$$3x^2 + 3y^2 y' = 6y + 6xy'$$
$$x^2 + y^2 y' = 2y + 2xy'$$

$$y' = \frac{2y - x^2}{y^2 - 2x}$$

(b)

at the point $(3,3)$,

$$y' = \frac{2\cdot 3 - 3^2}{3^2 - 2\cdot 3} = -1$$



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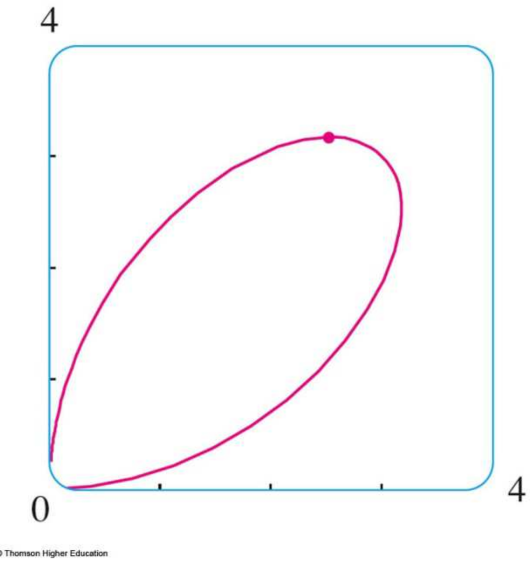
equation of the tangent line:

$$y - 3 = -(x - 3) \quad \text{or} \quad x + y = 6$$

(c)

The tangent line is horizontal if $y'=0$:

$2y-x^2=0$ $\left(\text{provided that } y^2-2x\neq 0\right)$



Substitute $2y-x^2=0$ into the equation:

$$x^3+\left(\frac{1}{2}x^2\right)^3=6x\left(\frac{1}{2}x^2\right)$$

$$8x^3+x^6=24x^3$$

$$x^6=16x^3$$

$x^3\left(x^3-16\right)=0,\quad x=16^{1/3},\quad x=0$ (not in the 1st quadrant)

$$x=16^{1/3}=2^{4/3},\quad y=\frac{1}{2}\left(2^{8/3}\right)=2^{5/3}$$

The tangent line is horizontal at $\left(2^{4/3},2^{5/3}\right)$

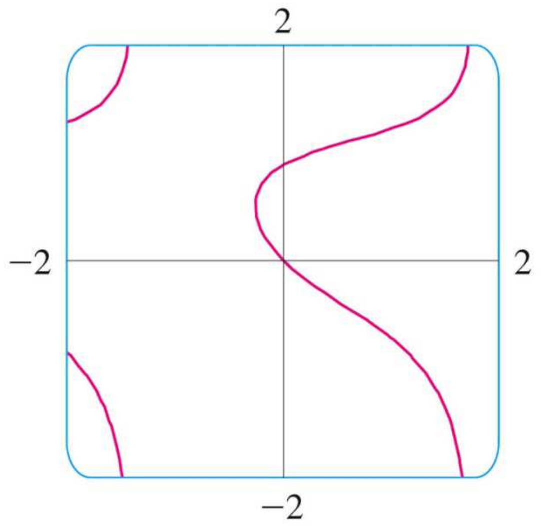
Example 3

$$\sin(x+y) = y^2 \cos x, \quad y' = ?$$

$$(1+y')\cos(x+y) = 2yy'\cos x - y^2 \sin x$$

$$\cos(x+y) + y^2 \sin x = 2yy'\cos x - y'\cos(x+y)$$

$$y' = \frac{\cos(x+y) + y^2 \sin x}{2y\cos x - \cos(x+y)}$$



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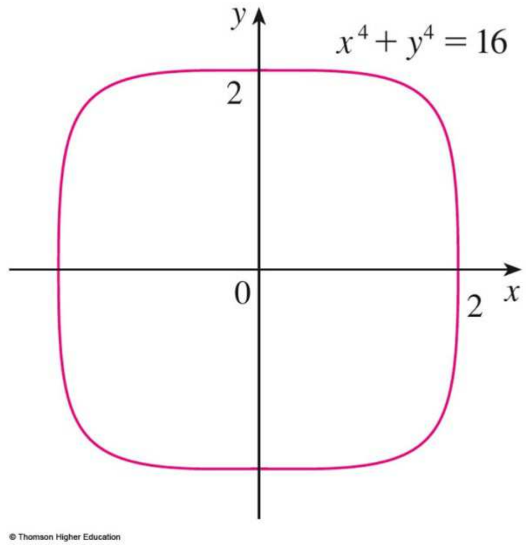
Example 4



$$x^4 + y^4 = 16, \quad y'' = \frac{d^2 y}{dx^2} = ?$$

$$4x^3 + 4y^3 y' = 0 \quad \Rightarrow \quad y' = -\frac{x^3}{y^3}$$

$$12x^2 + 12y^2 y' y' + 4y^3 y'' = 0 \quad \Rightarrow \quad y'' = -\frac{3\left(x^2 + y^2 y' y'\right)}{y^3}$$

$$y'' = -\frac{3\left(x^2 + y^2\left(-\frac{x^3}{y^3}\right)^2\right)}{y^3} = -\frac{3\left(x^2 + \frac{x^6}{y^4}\right)}{y^3} = -\frac{3x^2\left(y^4 + x^4\right)}{y^7} = -\frac{48x^2}{y^7}$$

DERIVATIVE OF INVERSE TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$$

$$\boxed{\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}}$$

The most important one.

$$\frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$$

DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

$$y = \sin^{-1} x \quad \Rightarrow \quad \sin y = x \ \text{ and } \ -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

Differentiate sin*y*=*x* implicitly w.r.t. *x*

$$\cos y \frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\cos y}$$

$$\cos y \ge 0 \ \text{ since } \ -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

$$\boxed{\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}}$$

$$y = \tan^{-1} x \quad \Rightarrow \quad \tan y = x \;\; \text{and} \;\; -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

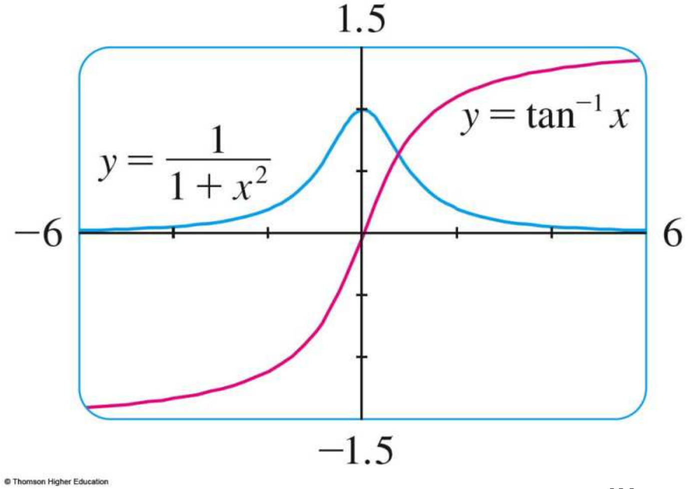
Differentiate tan*y*=*x* implicitly w.r.t. *x*

$$\sec^2 y \frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$\sec^2 y \ge 0 \;\; \text{since} \;\; -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

$$\boxed{\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1 + x^2}}$$



Example 5

(a) $y=\dfrac{1}{\sin^{-1}x}, \quad \dfrac{dy}{dx}=?$

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sin^{-1}x\right)^{-1}=-\left(\sin^{-1}x\right)^{-2}\frac{d}{dx}\left(\sin^{-1}x\right)$$

$$=-\left(\sin^{-1}x\right)^{-2}\frac{1}{\sqrt{1-x^2}}$$

$$=-\frac{1}{\left(\sin^{-1}x\right)^{2}\sqrt{1-x^2}}$$

(b) $y=x\tan^{-1}\sqrt{x}, \quad \dfrac{dy}{dx}=?$

$$\frac{dy}{dx}=\frac{d}{dx}\left(x\tan^{-1}\sqrt{x}\right)=x\frac{1}{1+x}\left(\frac{1}{2}x^{-1/2}\right)+\tan^{-1}\sqrt{x}$$

$$=\frac{\sqrt{x}}{2(1+x)}+\tan^{-1}\sqrt{x}$$

Sec. 3.6 Derivative of Logarithmic Functions

1 $$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

PROOF Let $y = \ln x$. Then

$$e^y = x$$

Differentiating this equation implicitly with respect to x, we get

$$e^y \frac{dy}{dx} = 1$$

and so $$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

EXAMPLE 1 Differentiate $y = \ln(x^3 + 1)$.

SOLUTION To use the Chain Rule, we let $u = x^3 + 1$. Then $y = \ln u$, so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}\frac{du}{dx} = \frac{1}{x^3 + 1}(3x^2) = \frac{3x^2}{x^3 + 1}$$

EXAMPLE 2 Find $\dfrac{d}{dx}\ln(\sin x)$.

SOLUTION Using (2), we have

$$\frac{d}{dx}\ln(\sin x) = \frac{1}{\sin x}\frac{d}{dx}(\sin x) = \frac{1}{\sin x}\cos x = \cot x$$

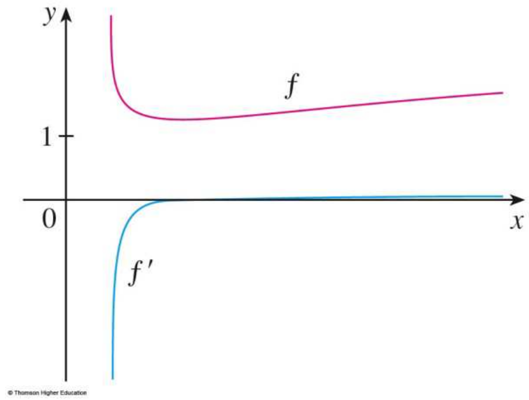
EXAMPLE 3 Differentiate $f(x) = \sqrt{\ln x}$.

SOLUTION This time the logarithm is the inner function, so the Chain Rule gives

$$f'(x) = \tfrac{1}{2}(\ln x)^{-1/2}\frac{d}{dx}(\ln x) = \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$$

$$\boxed{2} \qquad \boxed{\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}} \quad \text{or} \quad \boxed{\frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}}$$

EXAMPLE 4 Find $\dfrac{d}{dx}\ln\dfrac{x+1}{\sqrt{x-2}}$.



SOLUTION 1

$$\frac{d}{dx}\ln\frac{x+1}{\sqrt{x-2}} = \frac{1}{\dfrac{x+1}{\sqrt{x-2}}}\,\frac{d}{dx}\frac{x+1}{\sqrt{x-2}}$$

$$= \frac{\sqrt{x-2}}{x+1}\,\frac{\sqrt{x-2}\cdot 1 - (x+1)\left(\frac{1}{2}\right)(x-2)^{-1/2}}{x-2}$$

$$= \frac{x-2-\frac{1}{2}(x+1)}{(x+1)(x-2)} = \frac{x-5}{2(x+1)(x-2)}$$

SOLUTION 2 If we first simplify the given function using the laws of logarithms, then the differentiation becomes easier:

$$\frac{d}{dx}\ln\frac{x+1}{\sqrt{x-2}} = \frac{d}{dx}\left[\ln(x+1) - \tfrac{1}{2}\ln(x-2)\right] = \frac{1}{x+1} - \frac{1}{2}\left(\frac{1}{x-2}\right)$$

EXAMPLE 7 Find $f'(x)$ if $f(x) = \ln|x|$.

SOLUTION Since

$$f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

it follows that

$$f'(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\ \dfrac{1}{-x}(-1) = \dfrac{1}{x} & \text{if } x < 0 \end{cases}$$

Thus $f'(x) = 1/x$ for all $x \neq 0$.

$$\boxed{6} \qquad \boxed{\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}}$$

$$\log_a x = \frac{\ln x}{\ln a}$$

Since $\ln a$ is a constant, we can differentiate as follows:

$$\frac{d}{dx}(\log_a x) = \frac{d}{dx}\frac{\ln x}{\ln a} = \frac{1}{\ln a}\frac{d}{dx}(\ln x) = \frac{1}{x \ln a}$$

EXAMPLE 12 Using Formula 6 and the Chain Rule, we get

$$\frac{d}{dx}\log_{10}(2 + \sin x) = \frac{1}{(2 + \sin x)\ln 10}\frac{d}{dx}(2 + \sin x) = \frac{\cos x}{(2 + \sin x)\ln 10}$$

LOGARITHMIC DIFFERENTIATION

當有一個函數是由很多種函數（冪函數、指數函數、根式函數、三角函數 …等）乘除在一起所組成的函數，可利用對數微分法求其微分。

STEPS IN LOGARITHMIC DIFFERENTIATION

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the properties of logarithms to simplify.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'.

對數微分法求微分的範例：

EXAMPLE 15 Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

SOLUTION We take logarithms of both sides of the equation and use the properties of logarithms to simplify:

$$\ln y = \tfrac{3}{4}\ln x + \tfrac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1} - 5\cdot\frac{3}{3x+2}$$

Solving for dy/dx, we get

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

Because we have an explicit expression for y, we can substitute and write

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

□

利用對數微分公式，證明任意次方冪函數的微分公式

THE POWER RULE If n is any real number and $f(x) = x^n$, then

$$f'(x) = nx^{n-1}$$

PROOF Let $y = x^n$ and use logarithmic differentiation:

$$\ln|y| = \ln|x|^n = n\ln|x| \qquad x \neq 0$$

Therefore $$\frac{y'}{y} = \frac{n}{x}$$

Hence $$y' = n\frac{y}{x} = n\frac{x^n}{x} = nx^{n-1}$$

⊘ You should distinguish carefully between the Power Rule $[(d/dx)\ x^n = nx^{n-1}]$, w the base is variable and the exponent is constant, and the rule for differentiating expo tial functions $[(d/dx)\ a^x = a^x \ln a]$, where the base is constant and the exponent is vari

In general there are four cases for exponents and bases:

1. $\dfrac{d}{dx}(a^b) = 0$ $\quad$ (a and b are constants)

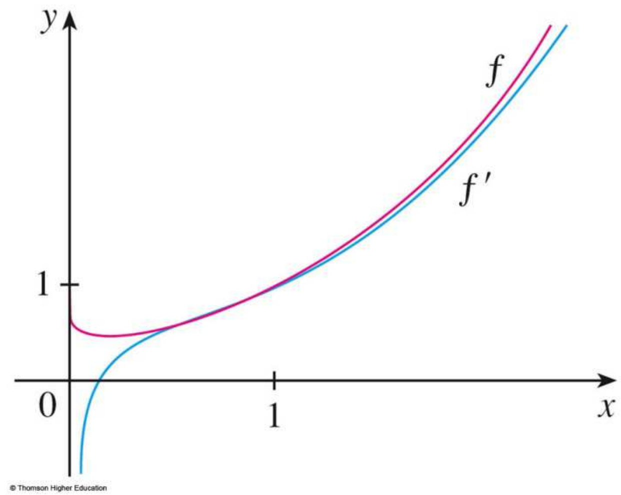
2. $\dfrac{d}{dx}[f(x)]^b = b[f(x)]^{b-1}f'(x)$ 冪函數

3. $\dfrac{d}{dx}[a^{g(x)}] = a^{g(x)}(\ln a)g'(x)$ 指數函數

4. To find $(d/dx)[f(x)]^{g(x)}$, logarithmic differentiation can be used, as in the next example.

※冪函數和指數函數的合成函數

EXAMPLE 16 Differentiate $y = x^{\sqrt{x}}$.



SOLUTION 1 Using logarithmic differentiation, we have

方法一：寫成對數

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x} \ln x$$

$$\frac{y'}{y} = \sqrt{x} \cdot \frac{1}{x} + (\ln x)\frac{1}{2\sqrt{x}}$$

$$y' = y\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right)$$

SOLUTION 2 Another method is to write $x^{\sqrt{x}} = (e^{\ln x})^{\sqrt{x}}$:

方法二：寫成指數

$$\frac{d}{dx}\left(x^{\sqrt{x}}\right) = \frac{d}{dx}\left(e^{\sqrt{x} \ln x}\right) = e^{\sqrt{x} \ln x} \frac{d}{dx}\left(\sqrt{x} \ln x\right)$$

$$= x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right) \qquad \text{(as in Solution 1)}$$

We have shown that if $f(x) = \ln x$, then $f'(x) = 1/x$. Thus $f'(1) = 1$. We now use this fa
to express the number e as a limit.

From the definition of a derivative as a limit, we have

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{x\to 0} \frac{f(1+x)-f(1)}{x}$$

$$= \lim_{x\to 0} \frac{\ln(1+x)-\ln 1}{x} = \lim_{x\to 0} \frac{1}{x}\ln(1+x)$$

$$= \lim_{x\to 0} \ln(1+x)^{1/x}$$

Because $f'(1) = 1$, we have

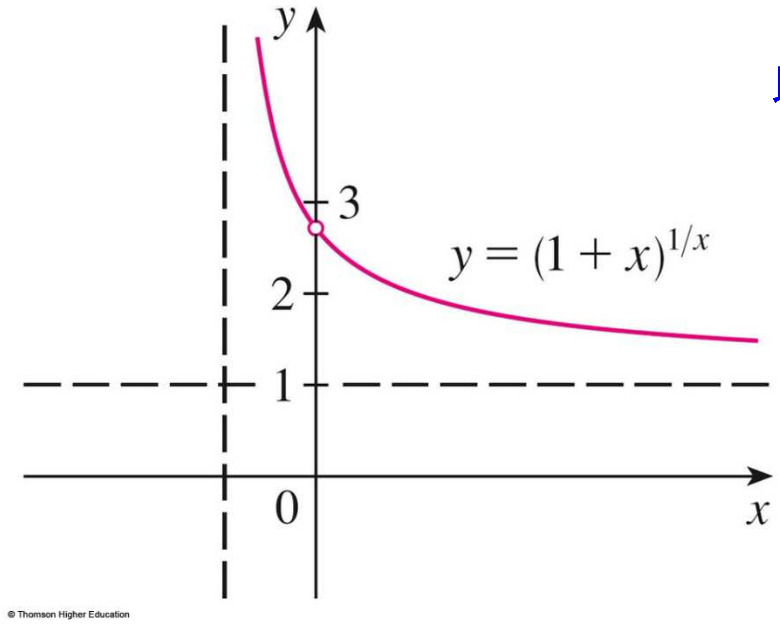
$$\lim_{x\to 0} \ln(1+x)^{1/x} = 1$$

Then, by Theorem 2.5.8 and the continuity of the exponential function, we have

$$\boxed{e = e^1 = e^{\lim_{x\to 0}\ln(1+x)^{1/x}} = \lim_{x\to 0} e^{\ln(1+x)^{1/x}} = \lim_{x\to 0}(1+x)^{1/x}}$$

8

$$e = \lim_{x \to 0} (1 + x)^{1/x}$$



此函數在 **x=0** 沒有定義

If we put $n = 1/x$ in Formula 8, then $n \to \infty$ as $x \to 0^+$ and so an alternative expression for e is

9

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

<u>Sec. 3.9</u> Related Rates

當<u>一個函數</u>的變化率會影響<u>另一個函數</u>的變化率時，如何計算？

$$y = y\left(x\left(t\right)\right)$$

合成函數型式

$$\frac{d}{dt}y\left(x\left(t\right)\right) = y'\left(t\right) = F\left(x\left(t\right), x'\left(t\right)\right)$$

$$f\left(x\left(t\right), y\left(t\right)\right) = 0$$

隱函數型式

$$\frac{df}{dt} = F\left(x, y, x'\left(t\right), y'\left(t\right)\right) = 0$$

These equations can be used to compute the rate of change of *y* in terms of the rate of change of *x*, or *vice versa*.

Example 1

Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm³/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

$$V(r)=\frac{4}{3}\pi r^3$$ 其中 $$r=r(t)$$ 合成函數

$$\frac{dV}{dt}=\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)=4\pi r^2\frac{dr}{dt} \quad \Rightarrow \quad \frac{dr}{dt}=\frac{1}{4\pi r^2}\frac{dV}{dt}$$

when $\frac{dV}{dt}=100\ \text{cm}^3/\text{s}$ and $r=25$ cm,

$$\frac{dr}{dt}=\frac{1}{4\pi r^2}\frac{dV}{dt}=\frac{1}{4\pi(25)^2}\times 100=0.0127\ \text{cm}/\text{s}$$

Example 2

A ladder 5 m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 m/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 3 m from the wall?

$$x = x(t)$$
$$y = y(t)$$

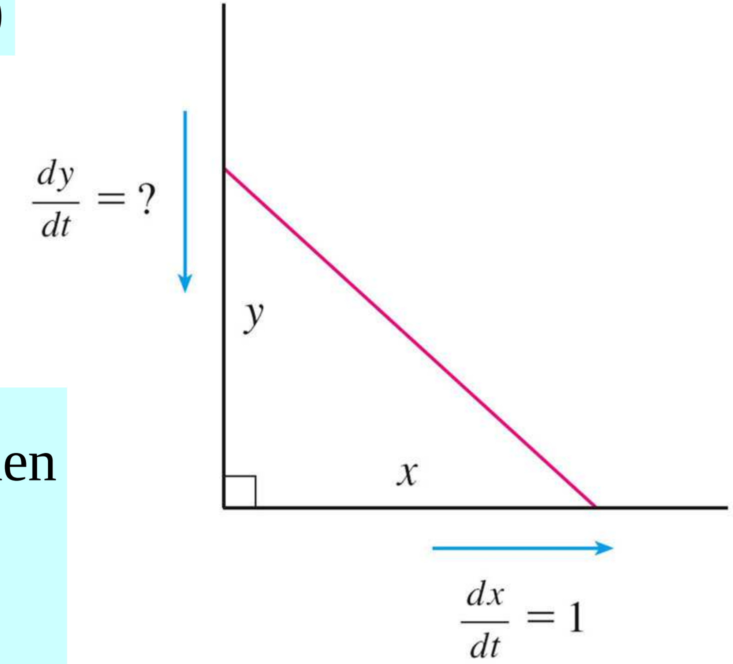
$$x^2 + y^2 - 25 = 0$$

隱函數

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \quad \Rightarrow \quad \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

when $x = 3$ m, $y = 4$ m. If $\frac{dx}{dt} = 1$ m/s, then

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{3}{4}\times 1 = -\frac{3}{4} \text{ m/s}$$



Education

Example 3

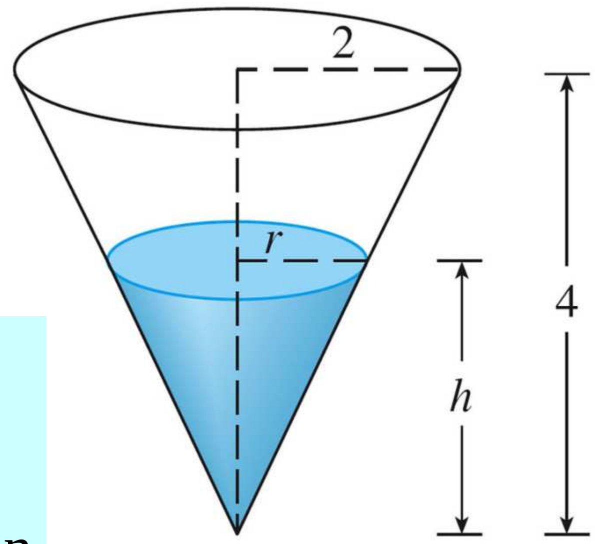
A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m³/min, find the rate at which the water level is rising when the water is 3 m deep?

$$V = \frac{1}{3}\pi r^2 h, \quad r = \frac{h}{2} \quad \Rightarrow \quad V = \frac{\pi}{12} h^3$$

$$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt} \quad \Rightarrow \quad \frac{dh}{dt} = \frac{4}{\pi h^2} \frac{dV}{dt}$$

when $h = 3$ m and $\frac{dV}{dt} = 2 \ \text{m}^3/\text{min}$, then

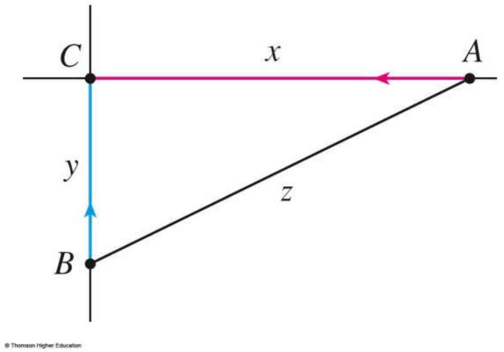
$$\frac{dh}{dt} = \frac{4}{\pi h^2} \frac{dV}{dt} = \frac{4}{\pi (3)^2} \times 2 = \frac{8}{9\pi} \approx 0.28 \ \text{m}/\text{min}$$



Example 4

Car A is traveling west at 90 km/h and car B is traveling north at 100 km/h. Both are headed for the intersection of the two road. At what rate are the cars approaching each other when car A is 60 m and car B is 80 m from the intersection?

$$z^2 = x^2 + y^2$$



$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \quad \Rightarrow \quad \frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt} \right)$$

when $x = 0.06$ km and $y = 0.08$ km, $z = 0.1$ km.

If $\dfrac{dx}{dt} = -90\ \text{km/h}$, and $\dfrac{dy}{dt} = -100\ \text{km/h}$, then

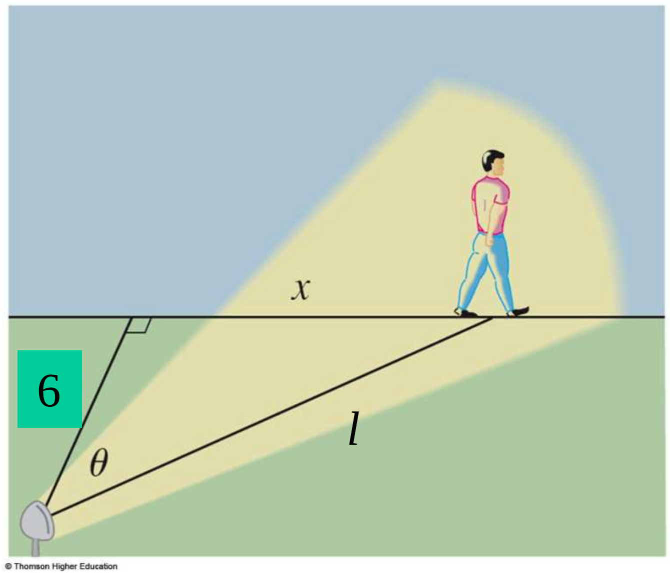
$$\frac{dz}{dt} = \frac{1}{0.1}\left(0.06(-90) + 0.08(-100)\right) = -134\ \text{km/h}$$

Example

A man walks along a straight path at a speed of 1.5 m/s. A searchlight is located on the ground 6 m from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 8 m from the point on the path closest to the searchlight?

$$\frac{x}{6} = \tan\theta \quad \Rightarrow \quad x = 6\tan\theta$$

$$\frac{dx}{dt} = 6\sec^2\theta\frac{d\theta}{dt} \quad \Rightarrow \quad \frac{d\theta}{dt} = \frac{1}{6}\cos^2\theta\frac{dx}{dt}$$

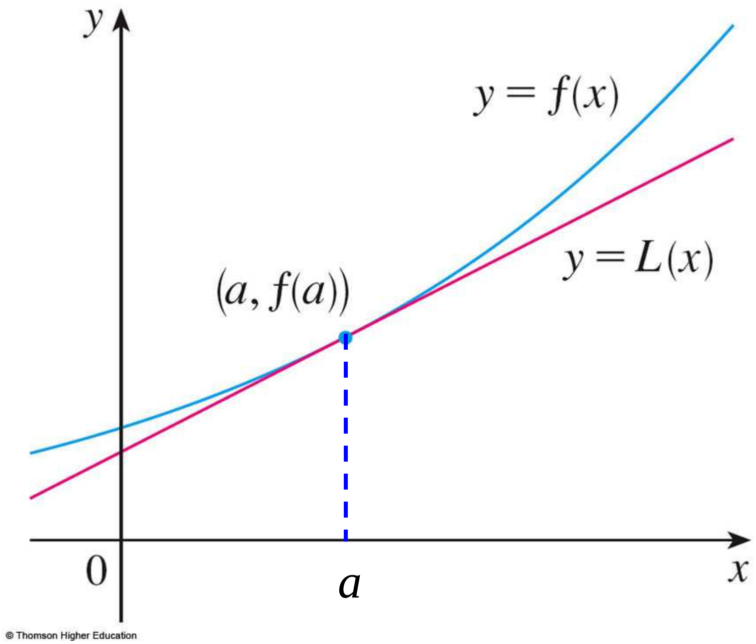


when $x = 8$ m, $l = 10$ m. So $\cos\theta = \frac{3}{5}$.

If $\frac{dx}{dt} = 1.5\ \text{m/s}$, then

$$\frac{d\theta}{dt} = \frac{1}{6}\cos^2\theta\frac{dx}{dt} = \frac{1}{6}\left(\frac{3}{5}\right)^2(1.5) = 0.09\ \text{rad/s}$$

Sec. 3.10 Linear Approximation and Differentials



Use the tangent line at $(a, f(a))$ as an approximation to the curve $f(x)$ when x is near a.

Equation of the tangent line:

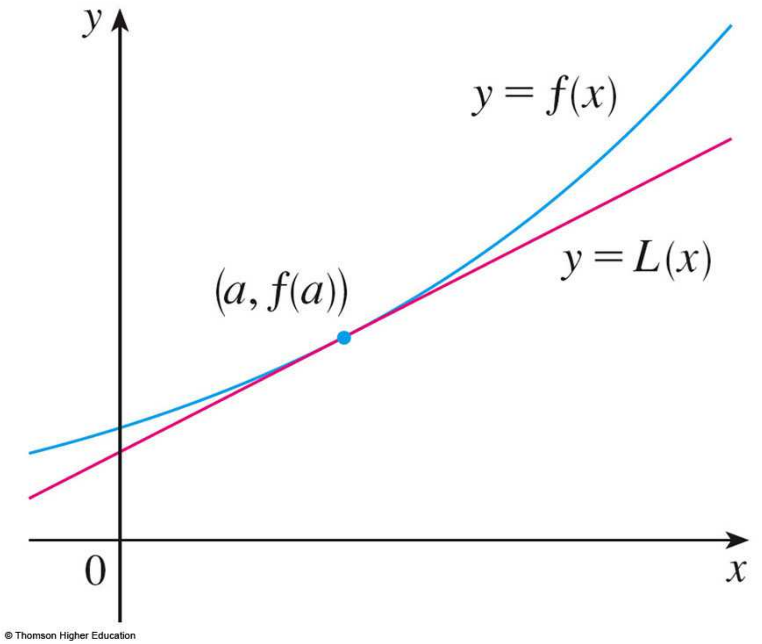
$$y = f(a) + f'(a)(x-a)$$

The approximation:

$$f(x) \approx f(a) + f'(a)(x-a)$$

Linear approximation or **tangent line approximation** of f at a.

Linearization



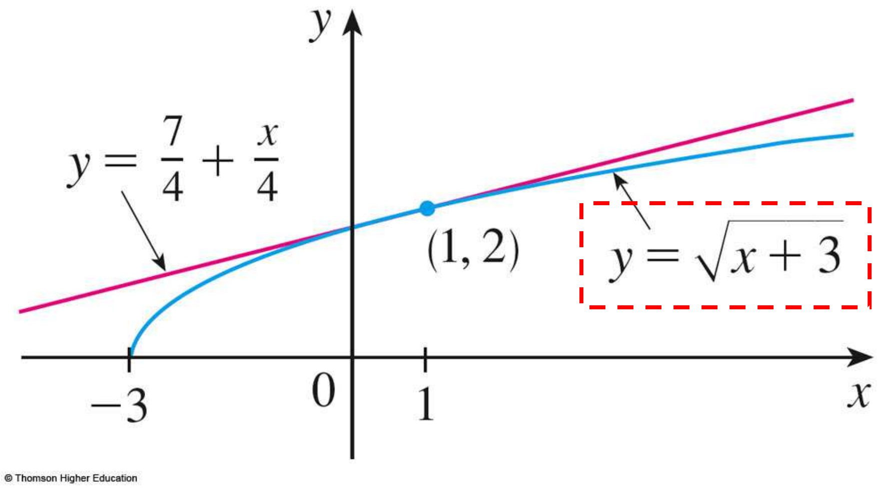
Linearization of *f* at *a*:

$$L(x) = f(a) + f'(a)(x - a)$$

Example 1

Find the linearization of the function f below at a=1 and use it to approximate the number $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximation overestimates or underestimates?

$$f(x)=\sqrt{x+3}$$



$$f'(x)=\frac{1}{2\sqrt{x+3}}$$

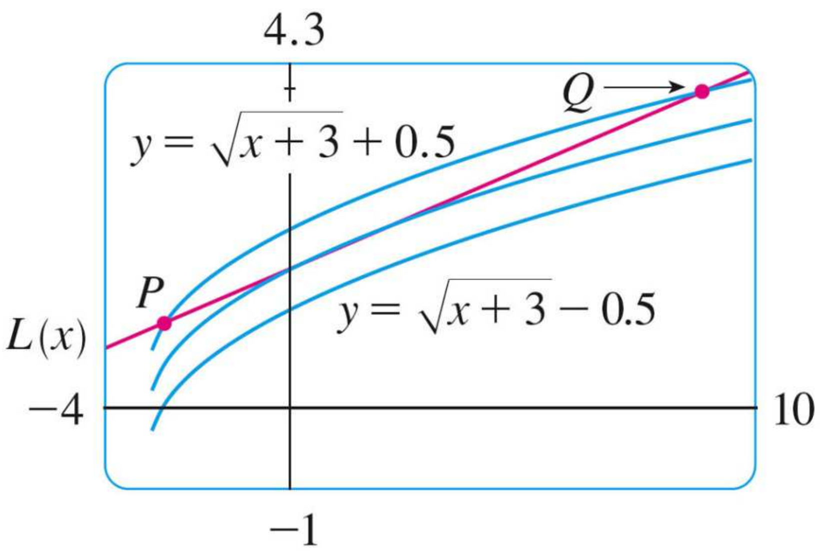
$$f(1)=2 \text{ and } f'(1)=\frac{1}{2\sqrt{1+3}}=\frac{1}{4}$$

$$\text{Linearization: } L(x)=f(1)+f'(1)(x-1)=2+\frac{1}{4}(x-1)=\frac{7}{4}+\frac{x}{4}$$

$$\Rightarrow \quad \sqrt{x+3}\approx\frac{7}{4}+\frac{x}{4} \quad (\text{when } x \text{ is near } 1)$$

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4} \quad (\text{when } x \text{ is near } 1)$$

$$\sqrt{3.98} \approx \frac{7}{4} + \frac{0.98}{4} = 1.995 \qquad \sqrt{4.05} \approx \frac{7}{4} + \frac{1.05}{4} = 2.0125$$



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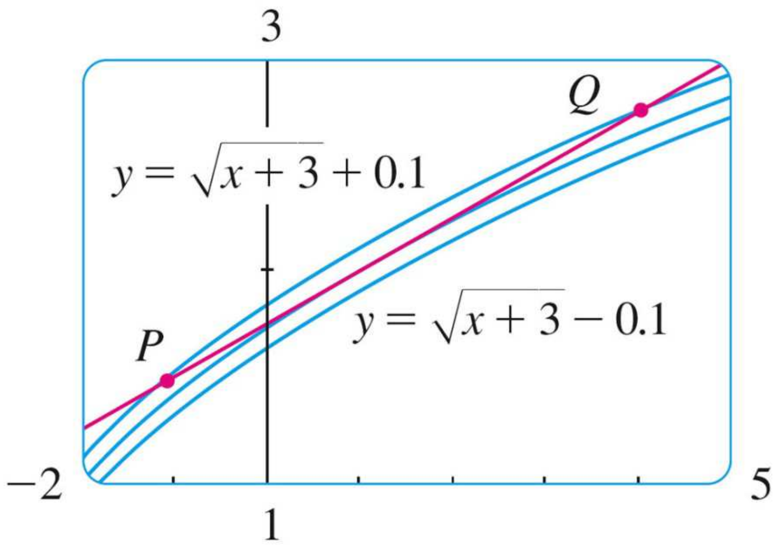

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Problem of Pendulum

$$F_r = ma_r: \quad T - mg\cos\theta = ml\dot{\theta}^2$$
$$F_\theta = ma_\theta: \quad -mg\sin\theta = ml\ddot{\theta}$$

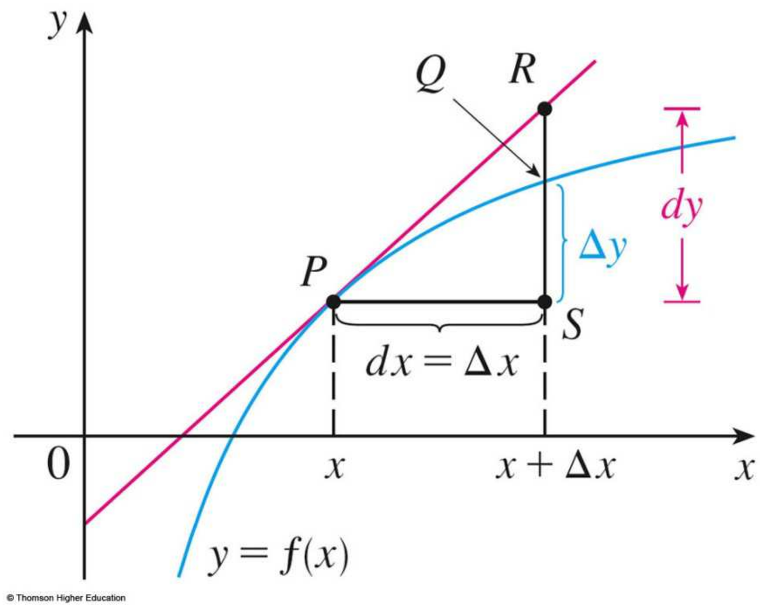
$$\cos\theta \approx \cos 0 + \left(\cos\theta\right)'\Big|_{\theta=0}\left(\theta - 0\right) = 1$$
$$\sin\theta \approx \sin 0 + \left(\sin\theta\right)'\Big|_{\theta=0}\left(\theta - 0\right) = \theta$$

Linearization as $\theta \to 0$:

$$\begin{cases} T - mg = ml\dot{\theta}^2 \\ -mg\theta = ml\ddot{\theta} \end{cases}$$

Differential and Difference

微分　　　　差分



Relation between the differentials dy and dx:

$$dy = f'(x)\,dx$$

Difference of y as x increases to $x+\Delta x$:

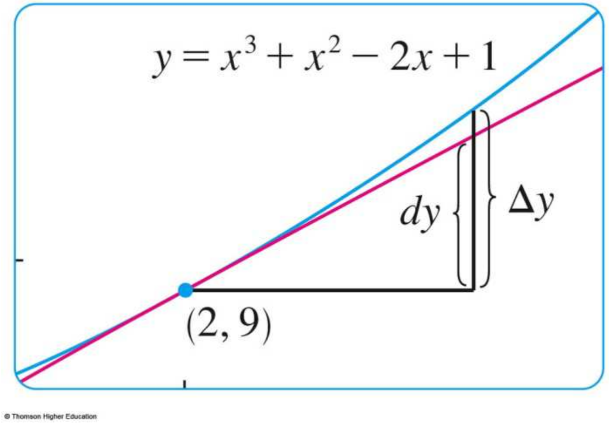
$$\Delta y = f(x+\Delta x) - f(x)$$

當 x 的增量很小時，可以用微分來替代差分

工程上的計算，可允許誤差值存在

Example 3

Compare the values of Δy and dy if $y=f(x)=x^3+x^2-2x+1$ and x changes (a) from 2 to 2.05 and (b) from 2 to 2.01.



(a)

$$\Delta y = f(2.05) - f(2) = 0.717625$$

$$dy = f'(x)\,dx = \left(3x^2 + 2x - 2\right)dx$$

When $x = 2$ and $dx = \Delta x = 0.05$, $dy = \left[3(2)^2 + 2(2) - 2\right]0.05 = 0.7$

(b)

$$\Delta y = f(2.01) - f(2) = 0.140701$$

When $x = 2$ and $dx = \Delta x = 0.01$, $dy = \left[3(2)^2 + 2(2) - 2\right]0.01 = 0.14$

Example 4

The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?

$$V = \frac{4}{3}\pi r^3, \qquad dV = 4\pi r^2 dr$$

When $r = 21$ and $dr = \Delta r = 0.05$, $dV = 4\pi \left(21\right)^2 0.05 \approx 277 \ \text{ cm}^3$

Relative errors: $\dfrac{\Delta V}{V} \approx \dfrac{dV}{V} = \dfrac{4\pi r^2 dr}{\frac{4}{3}\pi r^3} = 3\dfrac{dr}{r}$

$$\frac{dr}{r} = \frac{0.05}{21} \approx 0.0024 \quad \Rightarrow \quad \frac{dV}{V} \approx 3 \times 0.0024 = 0.0072$$

Sec. 3.11 Hyperbolic Functions and Their Derivatives

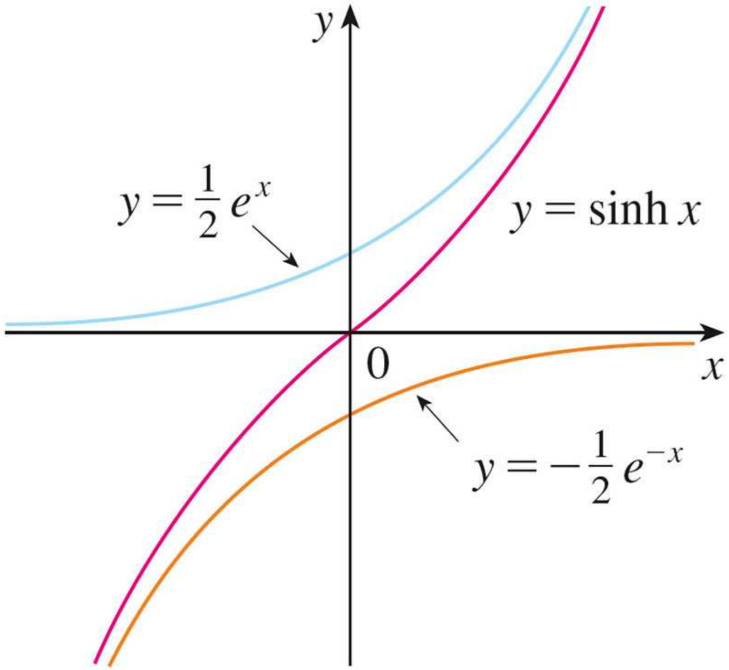
（略）

DEFINITION OF THE HYPERBOLIC FUNCTIONS

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$



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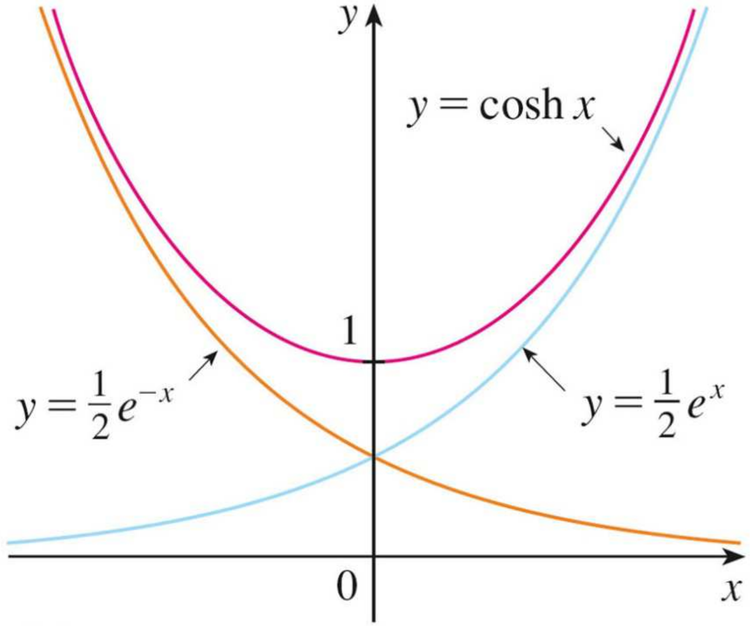

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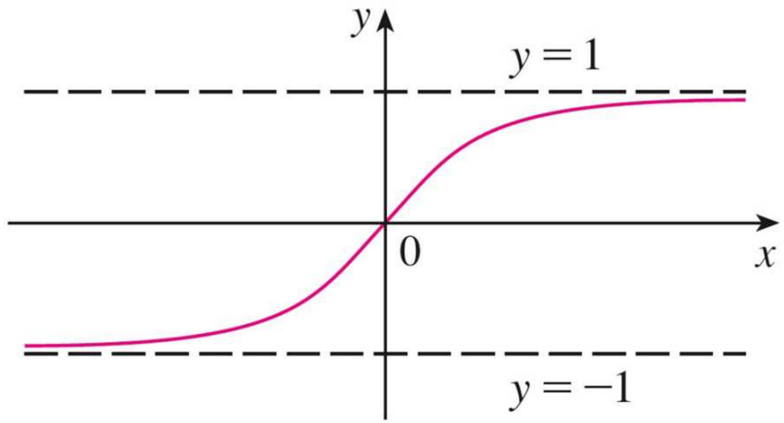

$y = \tanh x$

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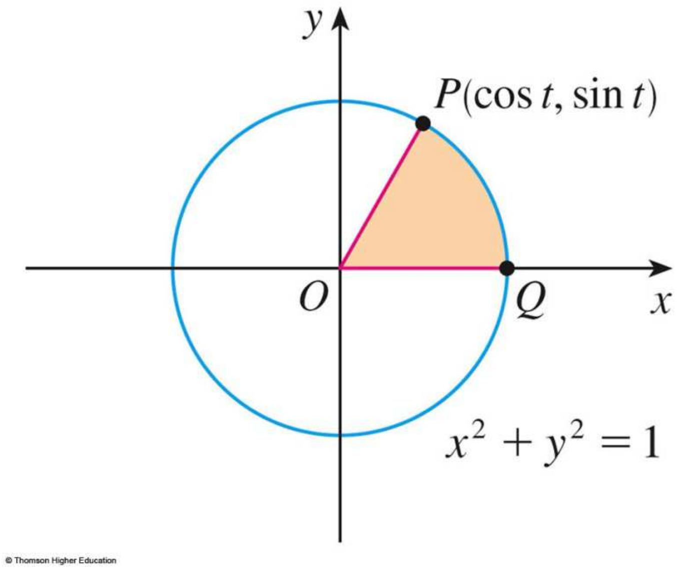
HYPERBOLIC IDENTITIES

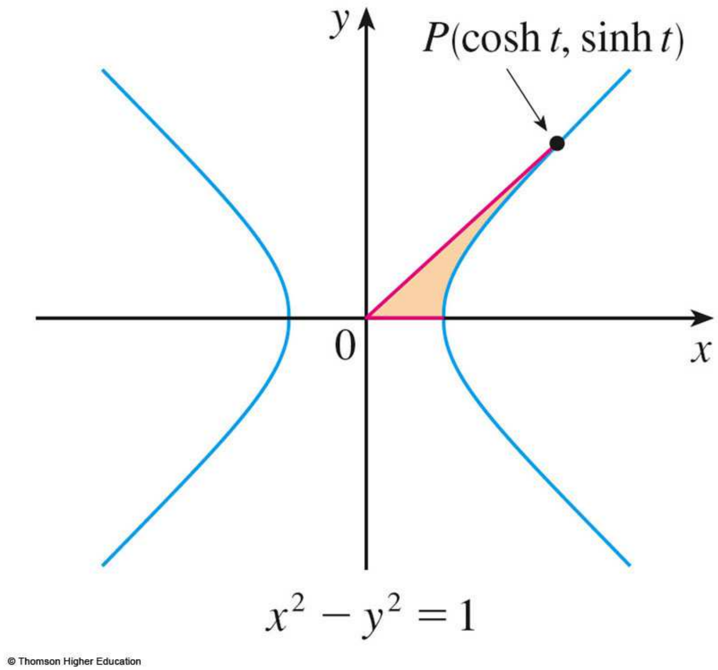
$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$





$$\sin^2 t + \cos^2 t = 1$$

$$\cosh^2 t - \sinh^2 t = 1$$

t 代表角度

t 代表兩倍陰影面積

EXAMPLE 1 Prove (a) $\cosh^2 x - \sinh^2 x = 1$ and (b) $1 - \tanh^2 x = \operatorname{sech}^2 x$.

SOLUTION

(a)
$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$
$$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{4}{4} = 1$$

(b) We start with the identity proved in part (a):

$$\cosh^2 x - \sinh^2 x = 1$$

If we divide both sides by $\cosh^2 x$, we get

$$1 - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

or
$$1 - \tanh^2 x = \operatorname{sech}^2 x$$
□

1 DERIVATIVES OF HYPERBOLIC FUNCTIONS

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

V EXAMPLE 2 Any of these differentiation rules can be combined with the Chain Rule. For instance,

$$\frac{d}{dx}\left(\cosh \sqrt{x}\right) = \sinh \sqrt{x} \cdot \frac{d}{dx}\sqrt{x} = \frac{\sinh \sqrt{x}}{2\sqrt{x}}$$

$\boxed{2}$

$$y = \sinh^{-1}x \iff \sinh y = x$$

$$y = \cosh^{-1}x \iff \cosh y = x \quad \text{and} \quad y \geqslant 0$$

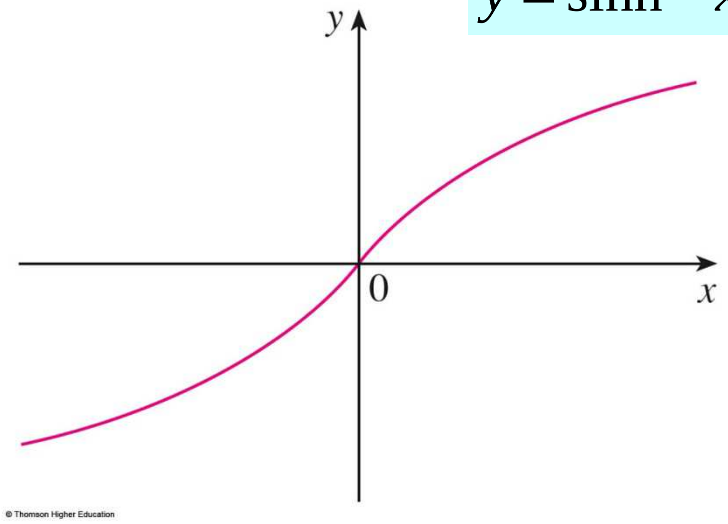
$$y = \tanh^{-1}x \iff \tanh y = x$$

$\boxed{3}$ $$\sinh^{-1}x = \ln\left(x + \sqrt{x^2 + 1}\right) \qquad x \in \mathbb{R}$$

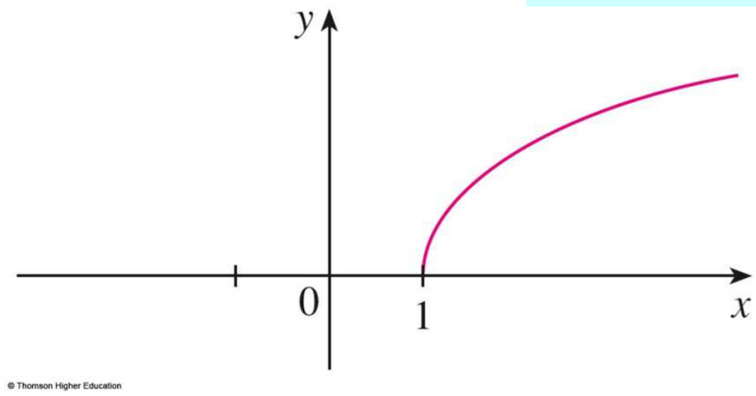
$\boxed{4}$ $$\cosh^{-1}x = \ln\left(x + \sqrt{x^2 - 1}\right) \qquad x \geqslant 1$$

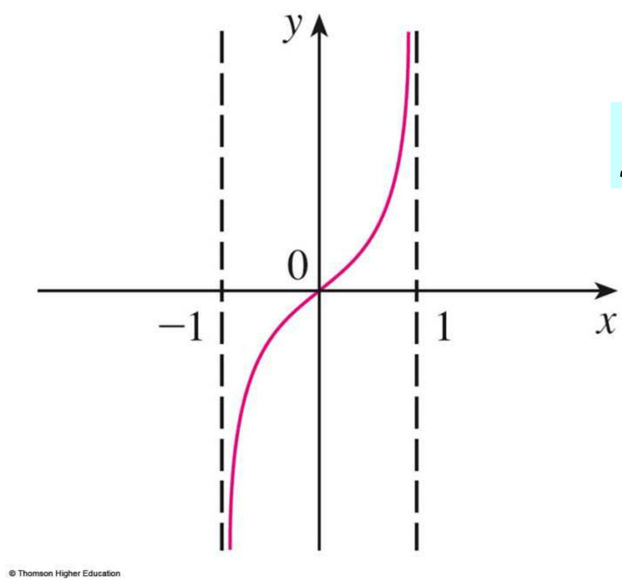
$\boxed{5}$ $$\tanh^{-1}x = \tfrac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right) \qquad -1 < x < 1$$

$y = \sinh^{-1} x$



$y = \cosh^{-1} x$





$y = \tanh^{-1} x$

EXAMPLE 3 Show that $\sinh^{-1}x = \ln\left(x + \sqrt{x^2 + 1}\right)$.

SOLUTION Let $y = \sinh^{-1}x$. Then

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

so

$$e^y - 2x - e^{-y} = 0$$

or, multiplying by e^y,

$$e^{2y} - 2xe^y - 1 = 0$$

This is really a quadratic equation in e^y:

$$(e^y)^2 - 2x(e^y) - 1 = 0$$

Solving by the quadratic formula, we get

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}$$

Note that $e^y > 0$, but $x - \sqrt{x^2 + 1} < 0$ (because $x < \sqrt{x^2 + 1}$). Thus the minus sign is inadmissible and we have

$$e^y = x + \sqrt{x^2 + 1}$$

Therefore

$$y = \ln(e^y) = \ln\left(x + \sqrt{x^2 + 1}\right)$$

(See Exercise 25 for another method.) □

6 DERIVATIVES OF INVERSE HYPERBOLIC FUNCTIONS

$$\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}} \qquad \frac{d}{dx}(\operatorname{csch}^{-1}x) = -\frac{1}{|x|\sqrt{x^2+1}}$$

$$\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}} \qquad \frac{d}{dx}(\operatorname{sech}^{-1}x) = -\frac{1}{x\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2} \qquad \frac{d}{dx}(\coth^{-1}x) = \frac{1}{1-x^2}$$

V EXAMPLE 4 Prove that $\dfrac{d}{dx}(\sinh^{-1}x) = \dfrac{1}{\sqrt{1+x^2}}$.

SOLUTION 1 Let $y = \sinh^{-1}x$. Then $\sinh y = x$. If we differentiate this equation implicitly with respect to x, we get

$$\cosh y \frac{dy}{dx} = 1$$

Since $\cosh^2 y - \sinh^2 y = 1$ and $\cosh y \geqslant 0$, we have $\cosh y = \sqrt{1 + \sinh^2 y}$, so

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}$$

SOLUTION 2 From Equation 3 (proved in Example 3), we have

$$\begin{aligned}
\frac{d}{dx}(\sinh^{-1}x) &= \frac{d}{dx}\ln\left(x + \sqrt{x^2 + 1}\right) \\
&= \frac{1}{x + \sqrt{x^2 + 1}}\,\frac{d}{dx}\left(x + \sqrt{x^2 + 1}\right) \\
&= \frac{1}{x + \sqrt{x^2 + 1}}\left(1 + \frac{x}{\sqrt{x^2 + 1}}\right) \\
&= \frac{\sqrt{x^2 + 1} + x}{\left(x + \sqrt{x^2 + 1}\right)\sqrt{x^2 + 1}} \\
&= \frac{1}{\sqrt{x^2 + 1}}
\end{aligned}$$

EXAMPLE 5 Find $\dfrac{d}{dx}[\tanh^{-1}(\sin x)]$.

SOLUTION Using Table 6 and the Chain Rule, we have

$$\frac{d}{dx}[\tanh^{-1}(\sin x)] = \frac{1}{1-(\sin x)^2}\frac{d}{dx}(\sin x)$$

$$= \frac{1}{1-\sin^2 x}\cos x = \frac{\cos x}{\cos^2 x} = \sec x$$