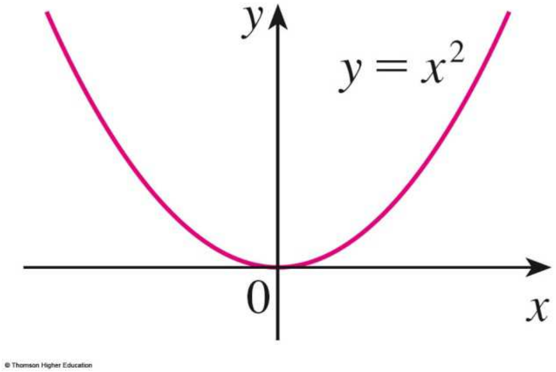
Applications of Differentiation

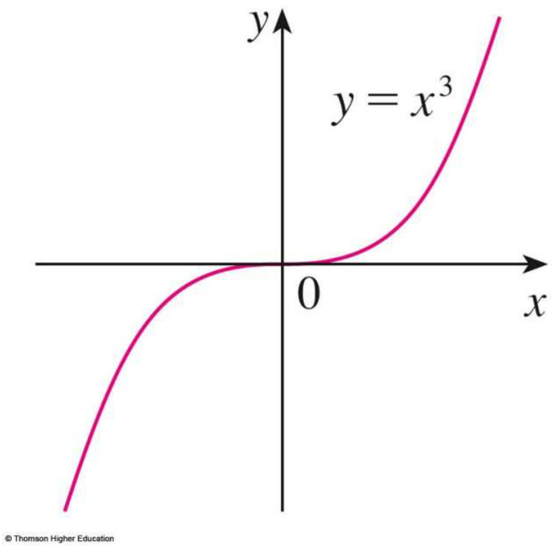
Lecture Note 5

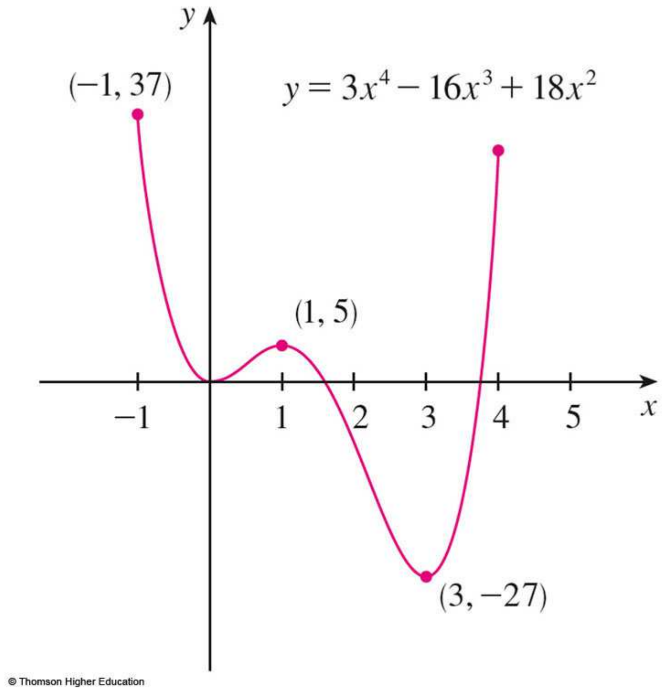
Sec. 4.1 – Sec. 4.9

Sec. 4.1 Maximum and Minimum Values

函數可能代表面積、 體積、 數量、 能量、成本…等

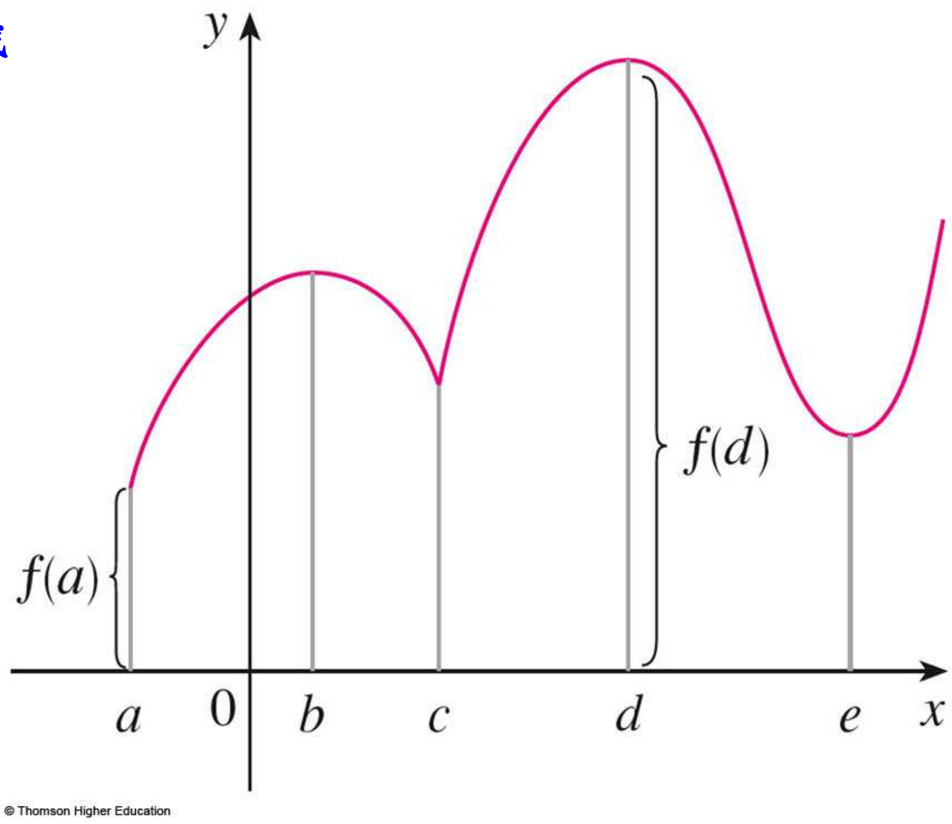






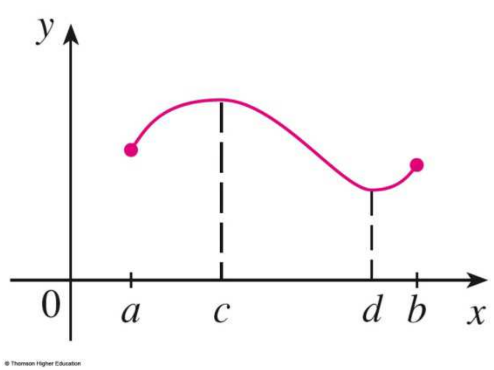
- **Absolute maximum (global maximum) and maximum value**
- **Absolute minimum (global minimum) and minimum value**
- **Local maximum (relative maximum)**
- **Local minimum (relative minimum)**

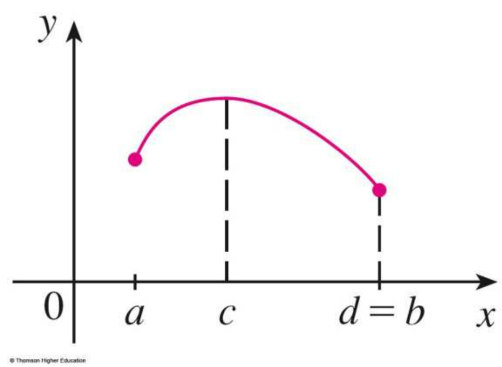
極值的定義

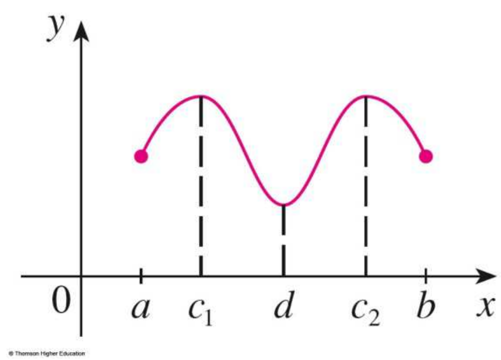


<u>THE EXYREME VALUE THEEOREM</u>（極值定理）**If *f* is continuous on a closed interval [*a*, *b*], then *f* attains an absolute maximum value *f*(*c*) and an absolute minimum value *f*(*d*) at some number *c* and *d* in [*a*, *b*].**

某些函數有極值，某些函數不存在極值

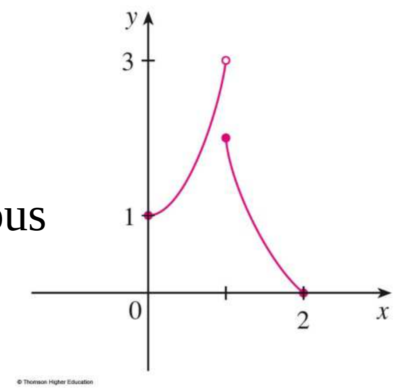


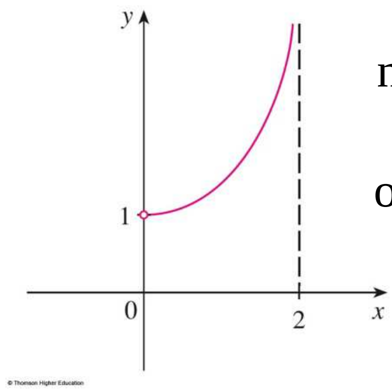




no max.
has min.

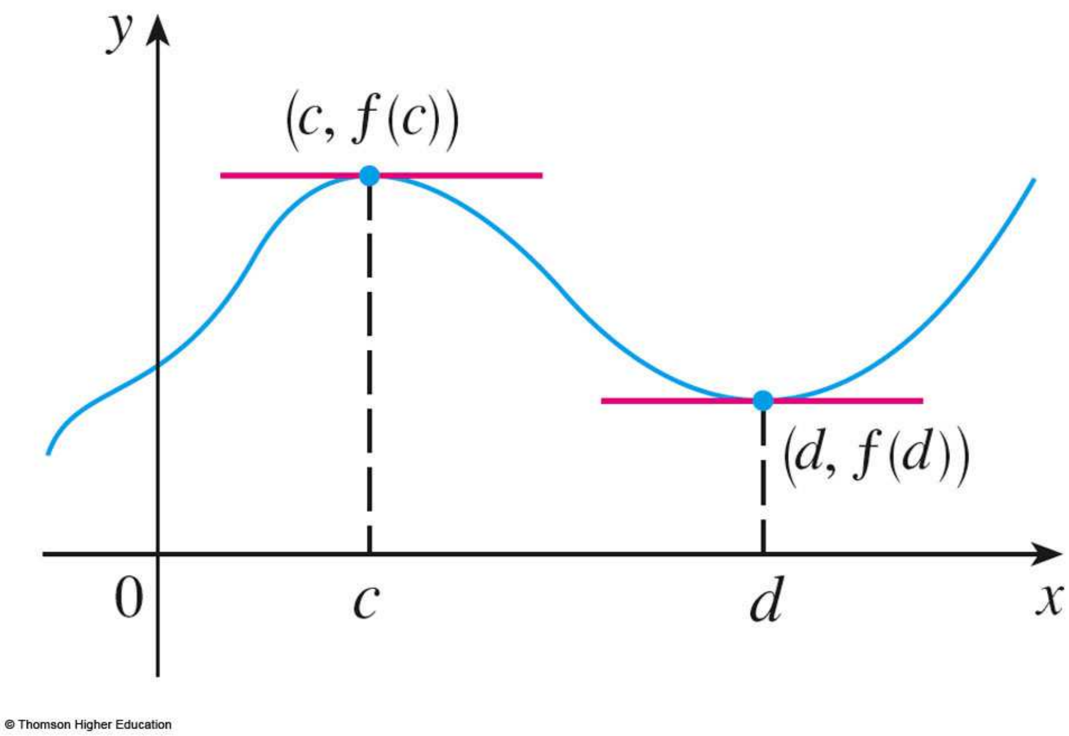
not continuous





no max. or min.

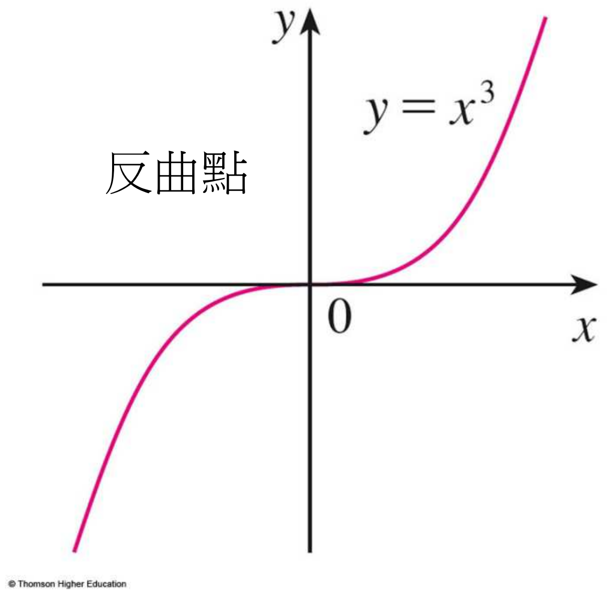
open interval

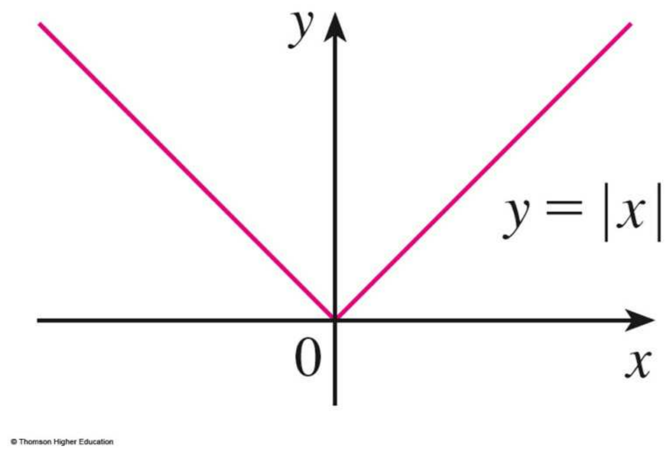


FERMAT'S THEOREM **(費瑪定理) If *f* has a local maximum or minimum at *c* , and if $f'(c)$ exists, then $f'(c) = 0$.**

Local maximum or minimum at c, and if $f'(c)$ exists $\begin{array}{c}\Rightarrow \\ \not\Leftarrow\end{array}$ **$f'(c) = 0$.**

反例：



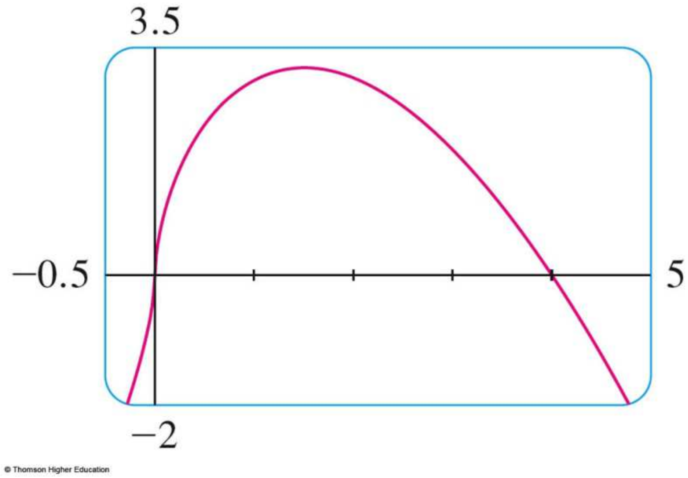


A <u>critical number</u> of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Example 7（找臨界點）

Find the critical numbers of function $f(x)=x^{3/5}(4-x)$

$$f'(x)=\frac{3}{5}x^{-2/5}(4-x)+x^{3/5}(-1)$$
$$=\frac{3(4-x)}{5x^{2/5}}-x^{3/5}$$
$$=\frac{3(4-x)-5x}{5x^{2/5}}=\frac{12-8x}{5x^{2/5}}$$



$$f'(x)=0 \quad \Rightarrow \quad x=\frac{3}{2}$$

$f'(x)$ does not exist when $x=0$

critical numbers: $x=\frac{3}{2}$ and $x=0$

Example 8（求絕對極值）

Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 1 \qquad -\frac{1}{2} \le x \le 4$$

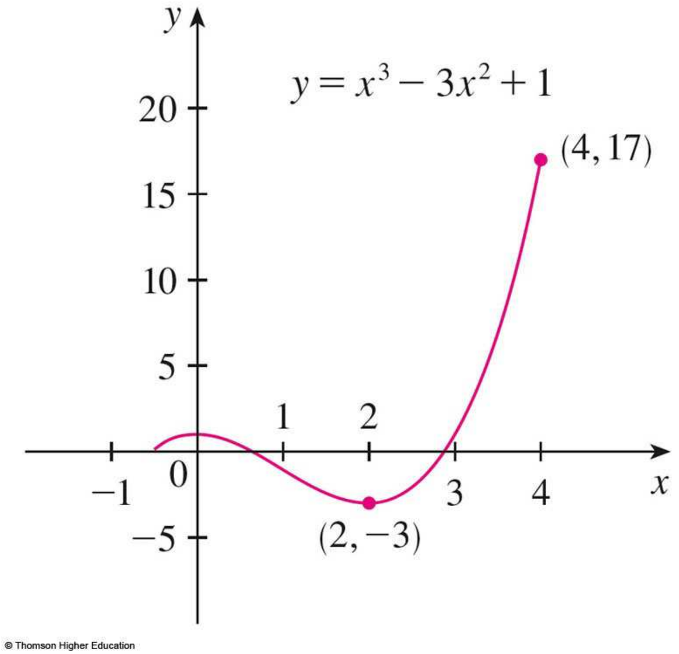
$$f'(x) = 3x^2 - 6x = 3x(x-2), \qquad f'(x) = 0 \quad \Rightarrow \quad x = 0 \text{ or } 2$$

$$f(0) = 1 \qquad f(2) = -3$$

$$f\left(-\frac{1}{2}\right) = \frac{1}{8} \qquad f(4) = 17$$

maximum value: $f(4) = 17$

minimum value: $f(2) = -3$



THE CLOSED INTERVAL METHOD **To find the absolute maximum and minimum values of a continuous function *f* on a closed interval [*a*, *b*]:**

1. **Find the values of *f* at the critical numbers of *f* in (*a*, *b*).**
2. **Find the values of *f* at the endpoints of the interval.**
3. **The largest of the values from Step 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.**

Example 9

Find the absolute maximum and minimum values of the function

$$f(x) = x - 2\sin x \qquad 0 \le x \le 2\pi$$

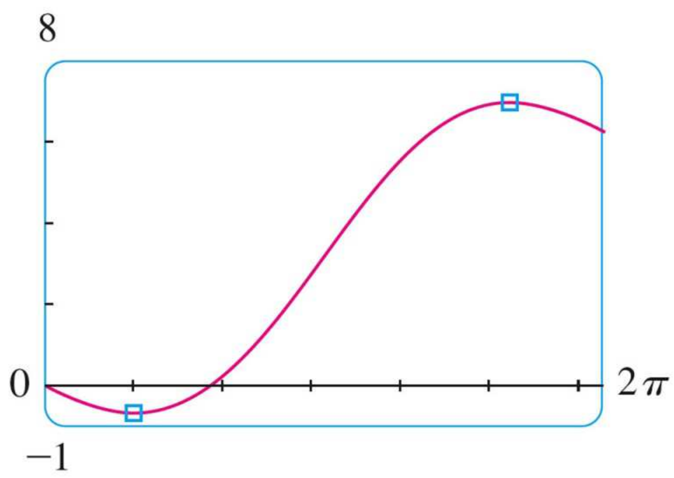
$$f'(x) = 1 - 2\cos x, \qquad f'(x) = 0 \;\Rightarrow\; \cos x = \frac{1}{2} \;\Rightarrow\; x = \pi/3 \text{ or } 5\pi/3$$

$$f(\pi/3) = \frac{\pi}{3} - 2\sin\frac{\pi}{3} = \frac{\pi}{3} - \sqrt{3} \qquad \text{minimum}$$

$$f(5\pi/3) = \frac{5\pi}{3} - 2\sin\frac{5\pi}{3} = \frac{5\pi}{3} + \sqrt{3} \qquad \text{maximum}$$

$$f(0) = 0$$

$$f(2\pi) = 2\pi - 2\sin(2\pi) = 2\pi$$



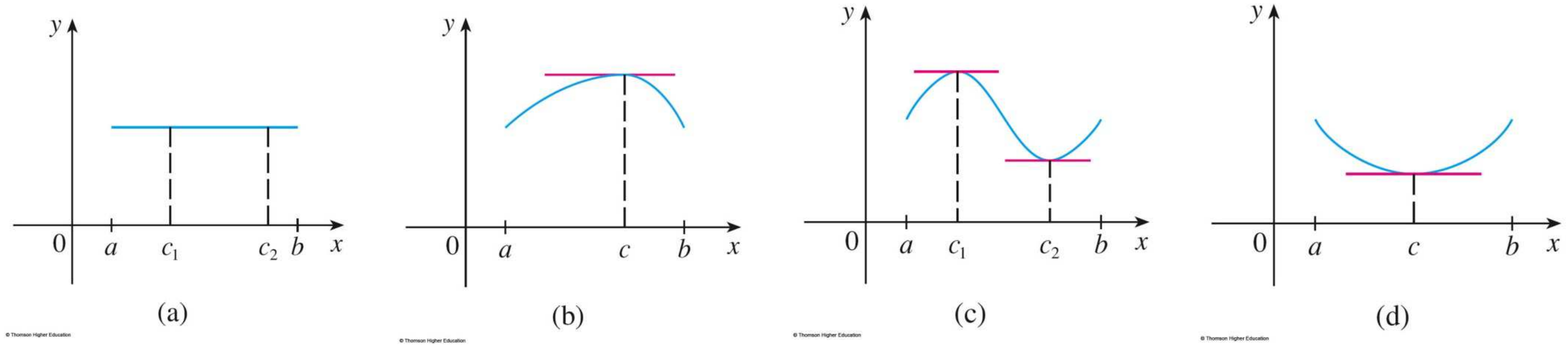
Sec. 4.2 The Mean Value Theory（均值定理）

ROLLE'S THEOREM **Let f be a function that satisfies the following three hypotheses:**

1. **f is continuous on the closed interval $[a, b]$**
2. **f is differentiable on the open interval (a, b)**
3. **$f(a)=f(b)$**

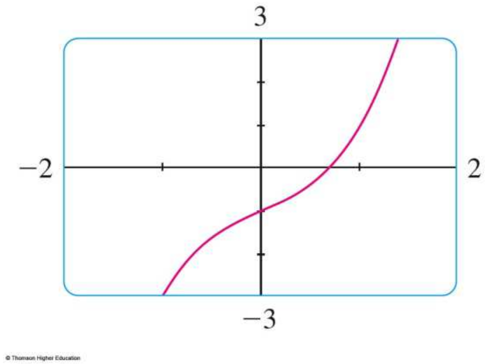
Then there is a number c in (a, b) such that $f'(c)=0$

和極值定理雷同，但條件較嚴格，不過得到的結果可更明確以數學式表示。



Example 2 (Rolle's 定理之應用)

Prove that the equation $x^3+x-1=0$ has exactly one real root.



Solution

Let $f(x) = x^3 + x - 1$

Then $f(0) = -1 < 0$ and $f(1) = 1 > 0$

By **Intermediate Value Theorem**, there is a number c between 0 and 1 such that $f(c)=0$. Thus the given equation has a root, at least.

Suppose that the equation had two roots a and b. Then $f(a)=f(b)=0$, and f is differentiable on (a, b) since it is a polynomial. Thus, by **Rolle's theorem**, there is a number c between a and b such that $f'(c)=0$. But

$$f'(x) = 3x^2 + 1 \geq 1$$

This gives a contradiction to Rolle's theorem. Therefore the equation can't have two roots.

Proof of Rolle's Theorem There are three cases:

CASE I ■ $f(x) = k$, **a constant**

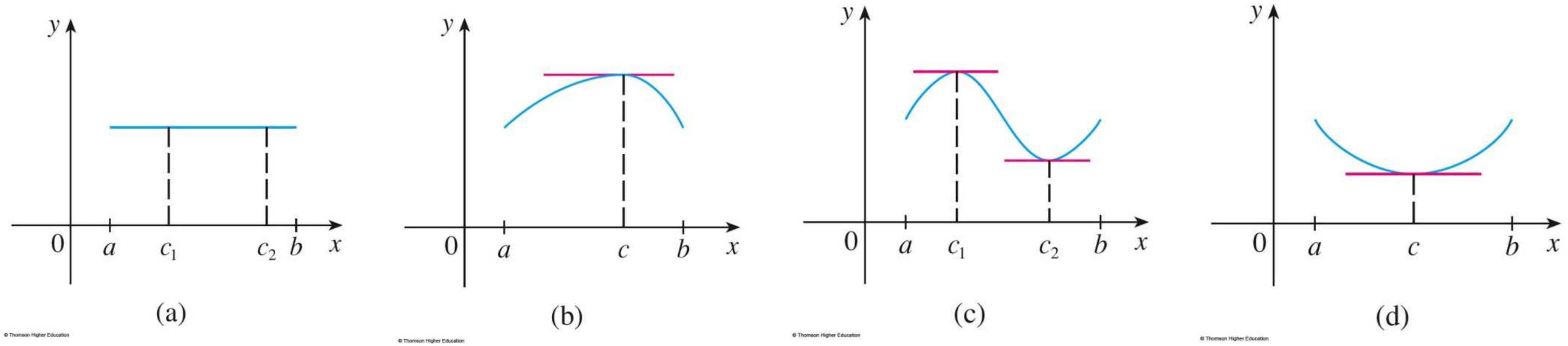
Then $f'(x) = 0$, so the number c can be taken to be *any* number in (a, b).

CASE II ■ $f(x) > f(a)$ **for some** x **in** (a, b) [as in Figure 1(b) or (c)]

By the Extreme Value Theorem (which we can apply by hypothesis 1), f has a maximum value somewhere in $[a, b]$. Since $f(a) = f(b)$, it must attain this maximum value at a number c in the open interval (a, b). Then f has a *local* maximum at c and, by hypothesis 2, f is differentiable at c. Therefore $f'(c) = 0$ by Fermat's Theorem.

CASE III ■ $f(x) < f(a)$ **for some** x **in** (a, b) [as in Figure 1(c) or (d)]

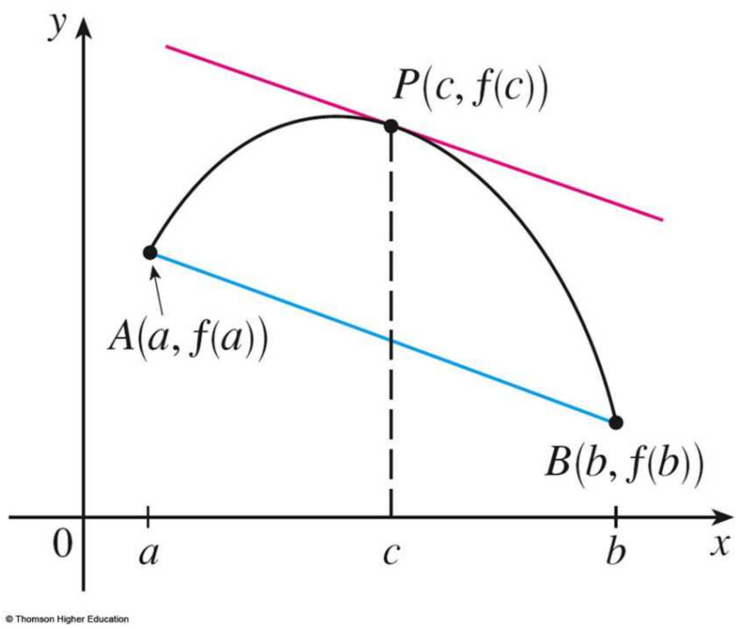
By the Extreme Value Theorem, f has a minimum value in $[a, b]$ and, since $f(a) = f(b)$, it attains this minimum value at a number c in (a, b). Again $f'(c) = 0$ by Fermat's Theorem. □

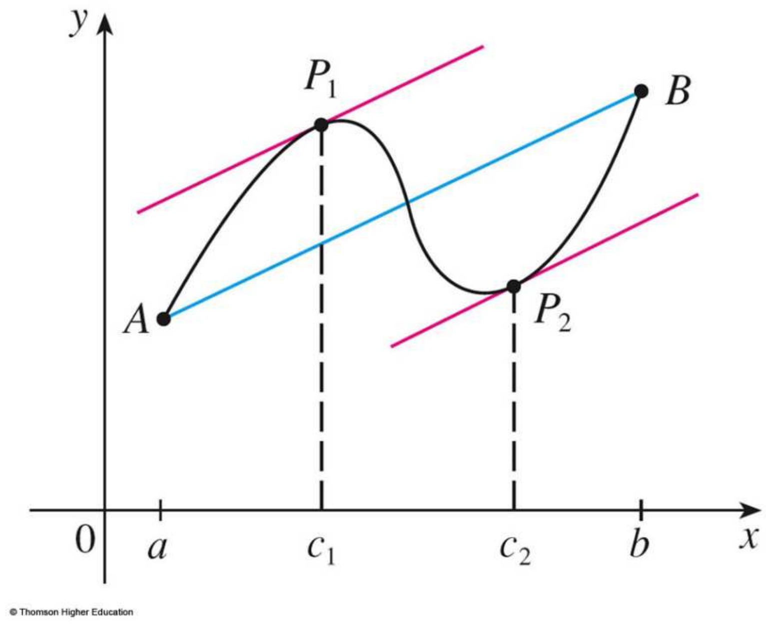


(a) (b) (c) (d)

THE MEAN VALUE THEOREM (均值定理)

透過觀察一些曲線：





$$m_{AB} = \frac{f(b) - f(a)}{b - a}$$

at the point $P(c, f(c))$, $\quad f'(c) = m_{AB}$

THE MEAN VALUE THEOREM **Let *f* be a function that satisfies the following hypotheses:**

1. ***f* is continuous on the closed interval [*a*, *b*]**
2. ***f* is differentiable on the open interval (*a*, *b*)**

Then there is a number *c* in (*a*, *b*) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or equivalent

$$f(b) - f(a) = f'(c)(b - a)$$

Example 3

Consider the equation $f(x)=x^3-x$, $a=0$ and $b=2$. Verify the Mean Value Theorem.

Solution

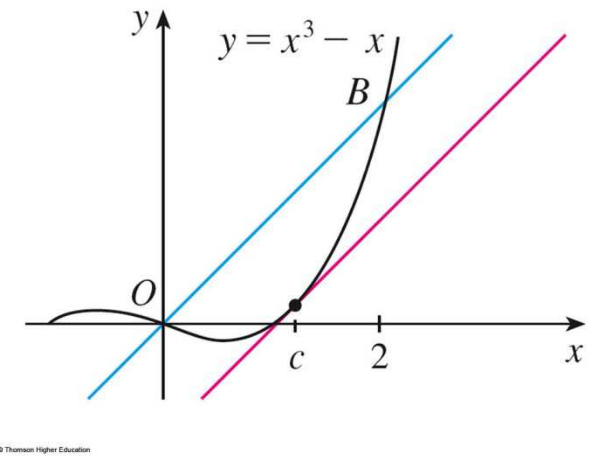
Since f is a polynomial, it is continuous and differentiable for all x. Therefore by the Mean Value Theorem, there is a number c in (0, 2) such that

$$f(2)-f(0)=f'(c)(2-0)$$

$$6-0=\left(3c^2-1\right)(2-0)$$

$$\Rightarrow \quad c^2=\frac{4}{3}, \quad c=\pm\frac{2}{\sqrt{3}}$$

But c must lie in (0, 2), so $c=\dfrac{2}{\sqrt{3}}$



Example 4

If an object moves in a straight line with position function *s*=*f*(*t*), then the average velocity between *t*=*a* and *t*=*b* is

$$\upsilon_{average} = \frac{f(b) - f(a)}{b - a}$$

and the velocity at *t*=*c* is *f*'(*c*). Thus the Mean Value Theorem tells us that at some time *t*=*c* between a and b the instantaneous velocity *f*'(*c*) is equal to that average velocity.

For example, if a car traveled 180km in 2 hours, then the speedometer must have read 90km/h at least once.

Proof of the Mean Value Theorem

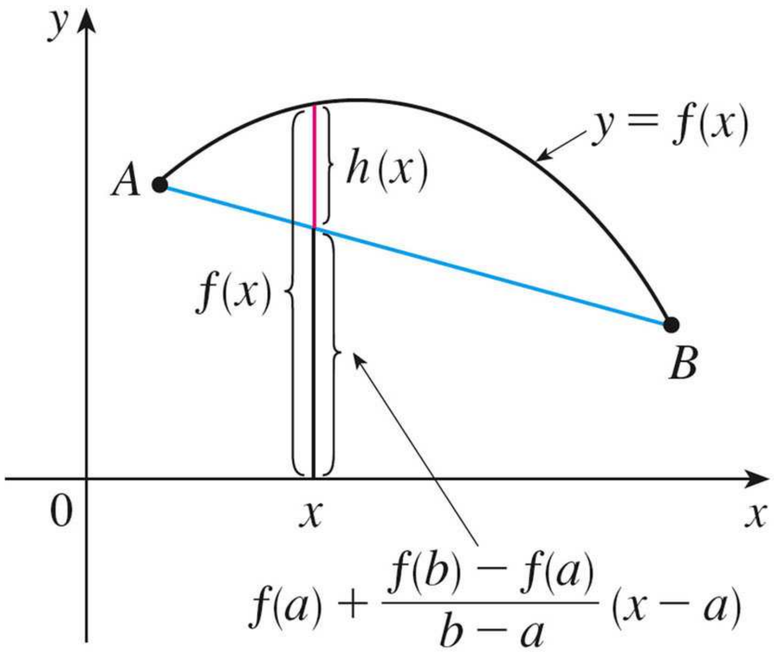
割線 *AB* 的斜率(平均變化率)爲：

$$m_{AB} = \frac{f(b) - f(a)}{b - a}$$

割線 *AB* 的直線方程式爲：

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$



參考左圖，函數和割線的差值 *h*(*x*)：

$$h(x) = f(x) - y$$
$$= f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

Proof of the Mean Value Theorem （續一）

檢查 $h(x)$ 是否滿足 Rolle's Theorem 的三個假設：

1. 因 $h(x)$ 為連續函數 f 和一階多項式之和，所以 $h(x)$ 在 $[a, b]$ 是連續的。
2. 因 f 和一階多項式在 (a, b) 均可微，所以 $h(x)$ 在 (a, b) 是可微的，即

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

3. 因

$$h(a) = f(a) - f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = 0$$

$$h(b) = f(b) - f(a) - \frac{f(b) - f(a)}{b - a}(b - a) = 0$$

所以

$$h(a) = h(b)$$

Proof of the Mean Value Theorem （續二）

根據 Rolle’s Theorem，在 (a, b) 區間中存在一個數字 c 使得 $h'(c)=0$：

$$0 = h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

即

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

證明完畢

Example 5

Suppose that $f(0)=-3$ and $f'(x) \leqq 5$ for all values of x. How large can $f(2)$ possibly be?

Solution

If f is differentiable (and therefore continuous) for all x. Then apply the Mean Value Theorem on the interval [0, 2], there is number c such that

$$f(b)-f(a)=f'(c)(b-a)$$

$$f(2)-f(0)=f'(c)(2-0) \quad \Rightarrow \quad f(2)=-3+2f'(c)$$

Since $f'(x) \leqq 5$ for all values of x,

$$f(2)=-3+2f'(c)\leq -3+10=7$$

The largest possible value for $f(2)$ is 7.

THEOREM **If $f'(x)=0$ for all x in an interval (a, b), then f is constant on (a, b).**

Proof

Let x_1 and x_2 be any two numbers in (a, b) with $x_1<x_2$. Then f must be differentiable on (a, b). By applying the Mean Value Theorem

$$f(x_2)-f(x_1)=f'(c)(x_2-x_1)=0 \quad \left(\because\ f'(c)=0\right)$$

That is

$$f(x_2)-f(x_1)=0 \quad \text{or} \quad f(x_2)=f(x_1)$$

This means that f is constant on (a, b).

COROLLARY **If $f'(x)=g'(x)$ for all x in an interval (a, b), then $f-g$ is constant on (a, b); that is, $f(x)=g(x)+c$ where c is a constant.**

Proof

Let

$$F(x)=f(x)-g(x)$$

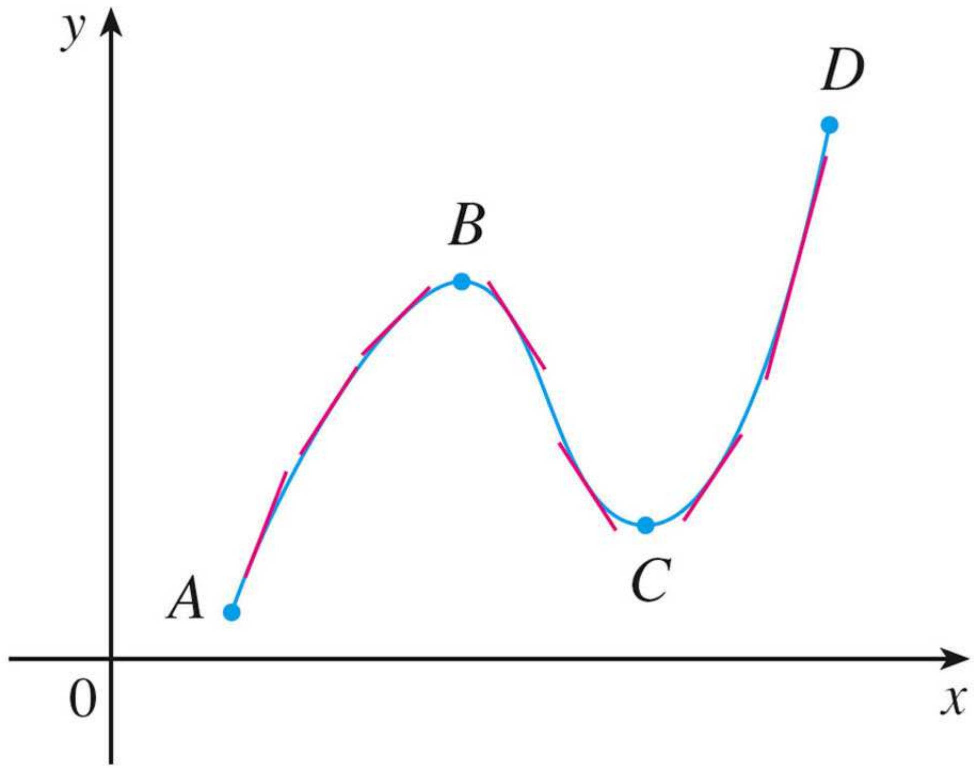
Then

$$F'(x)=f'(x)-g'(x)=0$$

For all x in (a, b). Thus by the last theorem, F is a constant; that f-g is constant

Sec. 4.3 How Derivatives Affect the Shapes of a Graph

What dose *f* ' say about *f*



INCREASING/DECREASING TEST

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

PROOF

(a) Let x_1 and x_2 be any two numbers in the interval with $x_1 < x_2$. According to the definition of an increasing function (page 20) we have to show that $f(x_1) < f(x_2)$.

Because we are given that $f'(x) > 0$, we know that f is differentiable on $[x_1, x_2]$. So, by the Mean Value Theorem there is a number c between x_1 and x_2 such that

$$\boxed{1} \qquad f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Now $f'(c) > 0$ by assumption and $x_2 - x_1 > 0$ because $x_1 < x_2$. Thus the right side of Equation 1 is positive, and so

$$f(x_2) - f(x_1) > 0 \qquad \text{or} \qquad f(x_1) < f(x_2)$$

This shows that f is increasing.

Part (b) is proved similarly. □

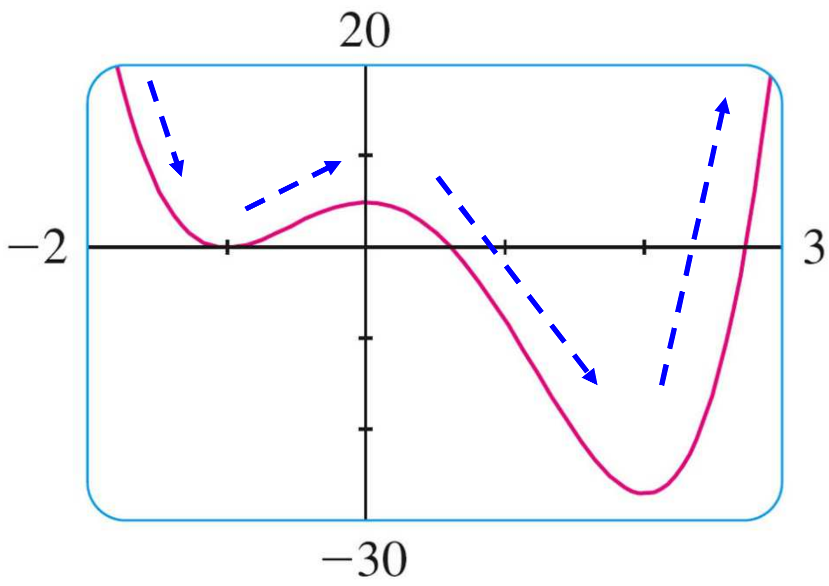
V EXAMPLE 1 Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

SOLUTION
$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x - 2)(x + 1)$$

To use the I/D Test we have to know where $f'(x) > 0$ and where $f'(x) < 0$. This depends on the signs of the three factors of $f'(x)$, namely, $12x$, $x - 2$, and $x + 1$. We divide the real line into intervals whose endpoints are the critical numbers -1, 0, and 2 and arrange our work in a chart. A plus sign indicates that the given expression is positive, and a minus sign indicates that it is negative. The last column of the chart gives the conclusion based on the I/D Test. For instance, $f'(x) < 0$ for $0 < x < 2$, so f is decreasing on $(0, 2)$. (It would also be true to say that f is decreasing on the closed interval $[0, 2]$.)

Interval	$12x$	$x - 2$	$x + 1$	$f'(x)$	f
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$-$	$-$	$+$	$+$	increasing on $(-1, 0)$
$0 < x < 2$	$+$	$-$	$+$	$-$	decreasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	$+$	increasing on $(2, \infty)$

The graph of f shown in Figure 2 confirms the information in the chart. □

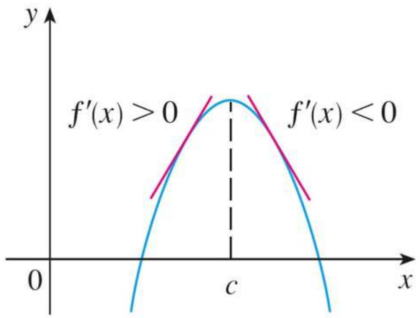

20
−2
3
−30

THE FIRST DERIVATIVE TEST Suppose that c is a critical number of a continuous function f.

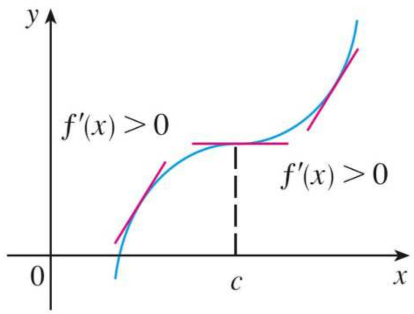
(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

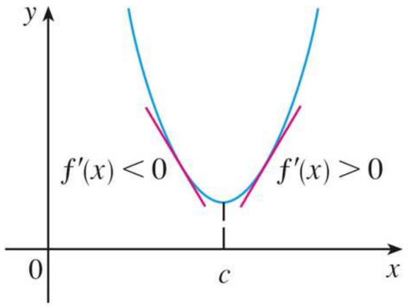
(c) If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.



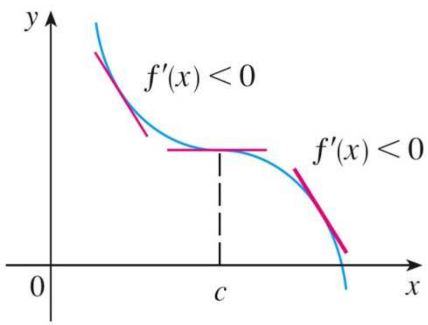
(a) Local maximum



(c) No maximum or minimum



(b) Local minimum



(d) No maximum or minimum

EXAMPLE 2 Find the local minimum and maximum values of the function f in Example 1.

SOLUTION From the chart in the solution to Example 1 we see that $f'(x)$ changes from negative to positive at -1, so $f(-1) = 0$ is a local minimum value by the First Derivative Test. Similarly, f' changes from negative to positive at 2, so $f(2) = -27$ is also a local minimum value. As previously noted, $f(0) = 5$ is a local maximum value because $f'(x)$ changes from positive to negative at 0.

EXAMPLE 3 Find the local maximum and minimum values of the function

$$g(x) = x + 2\sin x \qquad 0 \leqslant x \leqslant 2\pi$$

SOLUTION To find the critical numbers of g, we differentiate:

$$g'(x) = 1 + 2\cos x$$

So $g'(x) = 0$ when $\cos x = -\frac{1}{2}$. The solutions of this equation are $2\pi/3$ and $4\pi/3$. Because g is differentiable everywhere, the only critical numbers are $2\pi/3$ and $4\pi/3$ and so we analyze g in the following table.

Interval	$g'(x) = 1 + 2\cos x$	g
$0 < x < 2\pi/3$	$+$ 極大	increasing on $(0, 2\pi/3)$
$2\pi/3 < x < 4\pi/3$	$-$	decreasing on $(2\pi/3, 4\pi/3)$
$4\pi/3 < x < 2\pi$	$+$ 極小	increasing on $(4\pi/3, 2\pi)$

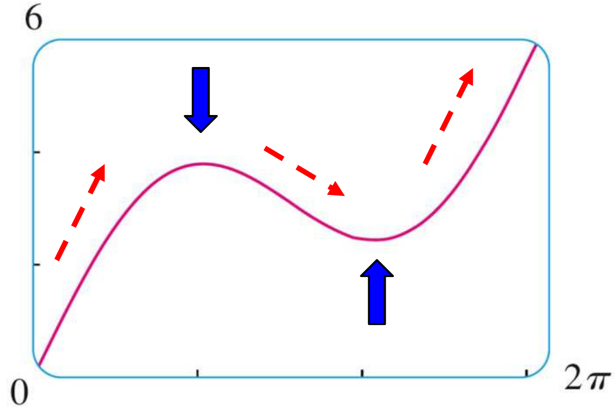
Because $g'(x)$ changes from positive to negative at $2\pi/3$, the First Derivative Test tells us that there is a local maximum at $2\pi/3$ and the local maximum value is

$$g(2\pi/3) = \frac{2\pi}{3} + 2\sin\frac{2\pi}{3} = \frac{2\pi}{3} + 2\left(\frac{\sqrt{3}}{2}\right) = \frac{2\pi}{3} + \sqrt{3} \approx 3.83$$

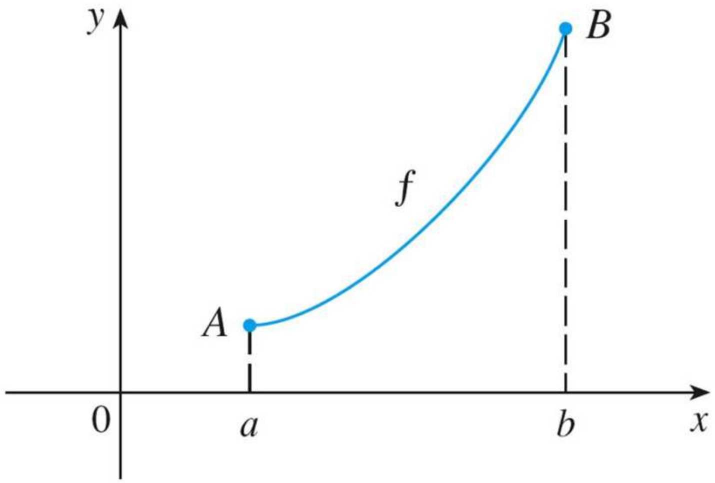
Likewise, $g'(x)$ changes from negative to positive at $4\pi/3$ and so

$$g(4\pi/3) = \frac{4\pi}{3} + 2\sin\frac{4\pi}{3} = \frac{4\pi}{3} + 2\left(-\frac{\sqrt{3}}{2}\right) = \frac{4\pi}{3} - \sqrt{3} \approx 2.46$$

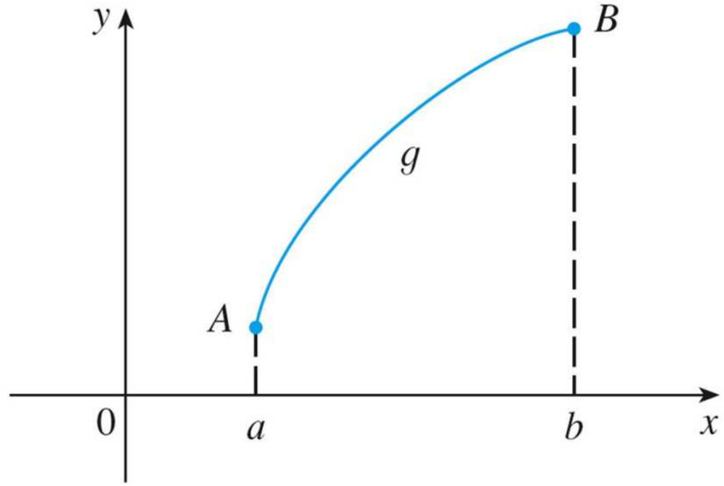
is a local minimum value. The graph of g in Figure 4 supports our conclusion. □



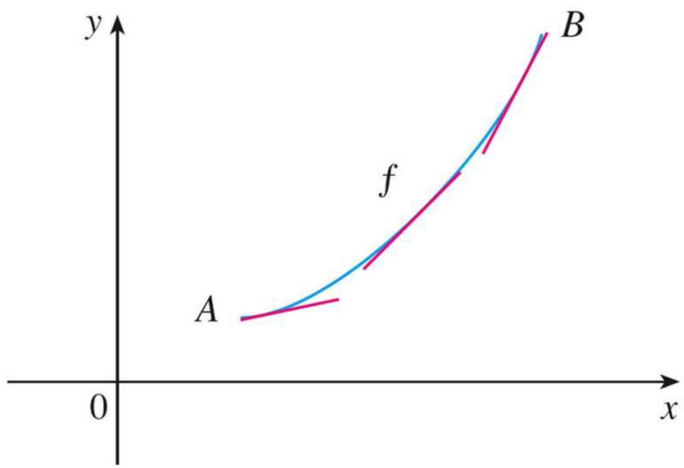
What dose f'' say about f



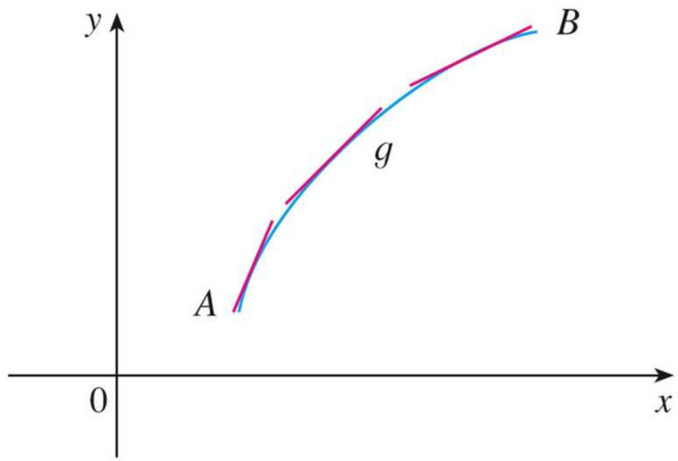
(a)



(b)

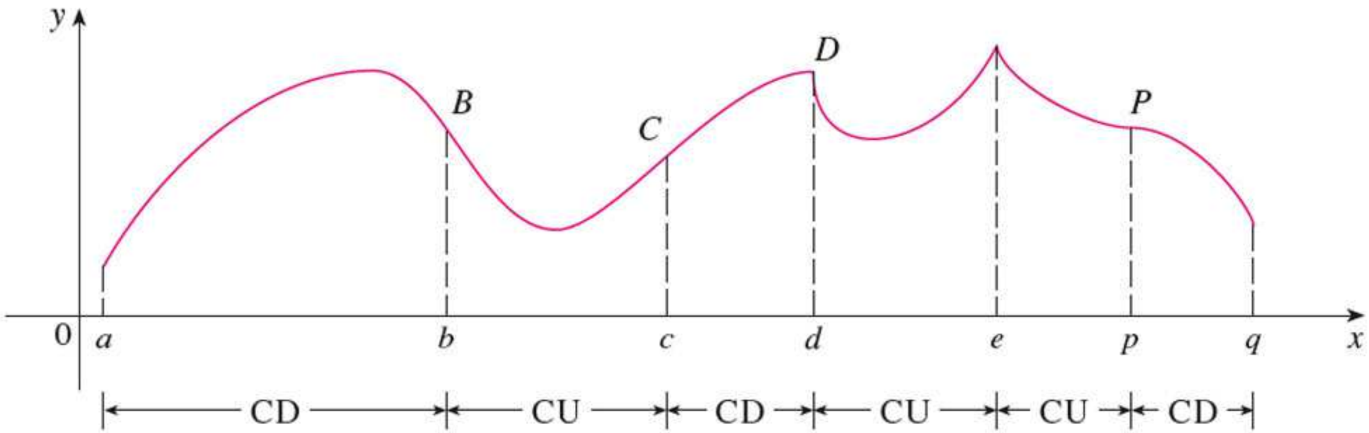


(a) Concave upward



(b) Concave downward

DEFINITION If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.



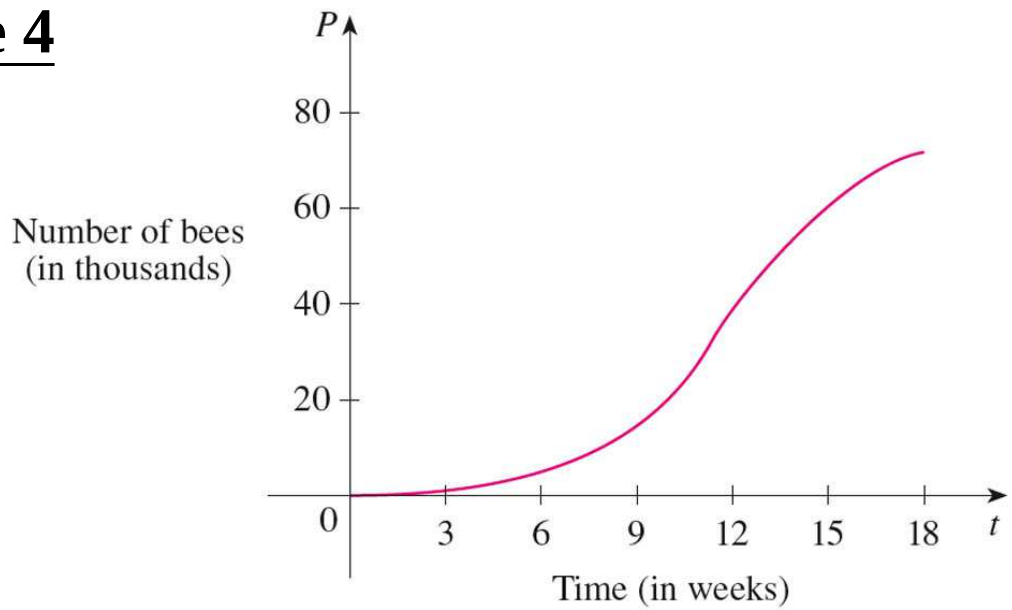
CONCAVITY TEST

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

DEFINITION A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

Example 4



EXAMPLE 5 Sketch a possible graph of a function f that satisfies the following conditions:

(i) $f(0) = 0, \quad f(2) = 3, \quad f(4) = 6, \quad f'(0) = f'(4) = 0$

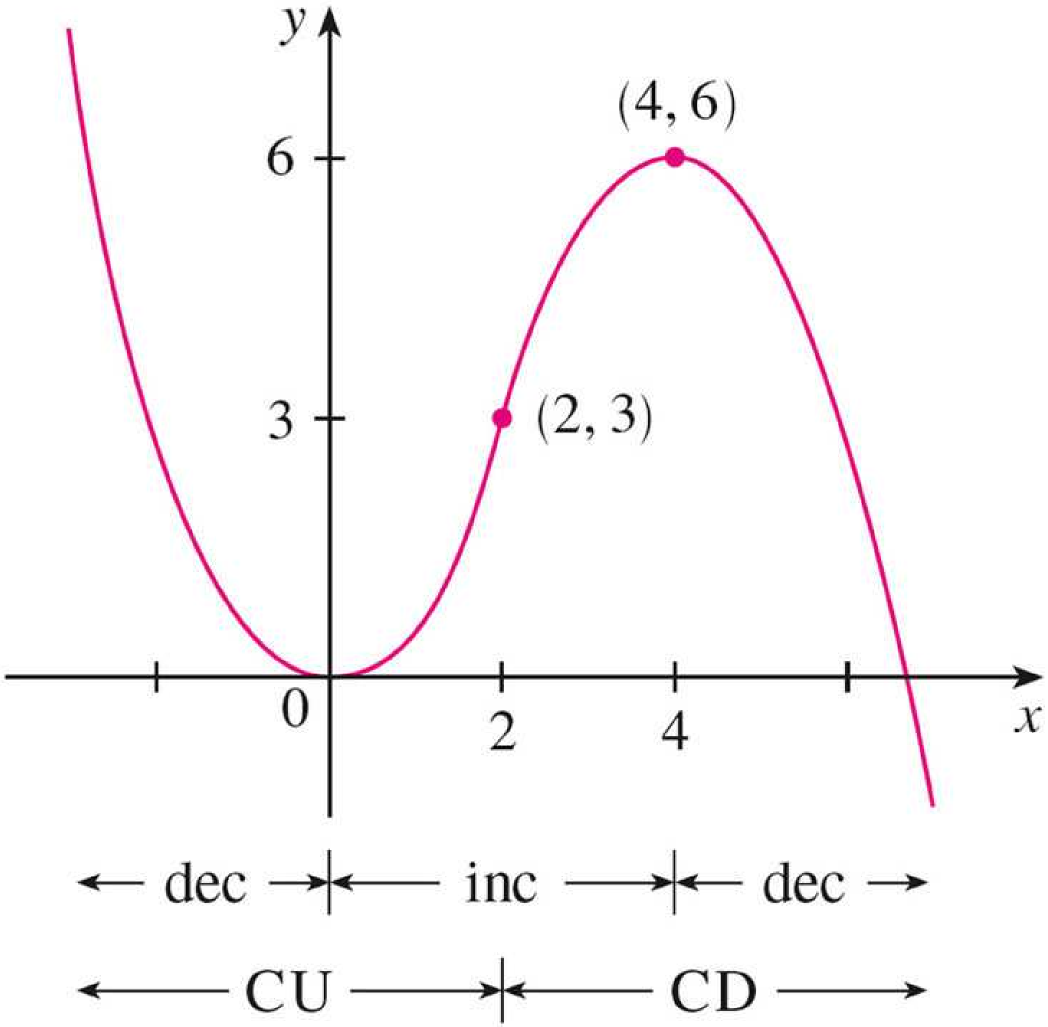
(ii) $f'(x) > 0$ for $0 < x < 4$, $\quad f'(x) < 0$ for $x < 0$ and for $x > 4$

(iii) $f''(x) > 0$ for $x < 2$, $\quad f''(x) < 0$ for $x > 2$

SOLUTION Condition (i) tells us that the graph has horizontal tangents at the points $(0, 0)$ and $(4, 6)$. Condition (ii) says that f is increasing on the interval $(0, 4)$ and decreasing on the intervals $(-\infty, 0)$ and $(4, \infty)$. It follows from the I/D Test that $f(0) = 0$ is a local minimum and $f(4) = 6$ is a local maximum.

Condition (iii) says that the graph is concave upward on the interval $(-\infty, 2)$ and concave downward on $(2, \infty)$. Because the curve changes from concave upward to concave downward when $x = 2$, the point $(2, 3)$ is an inflection point.

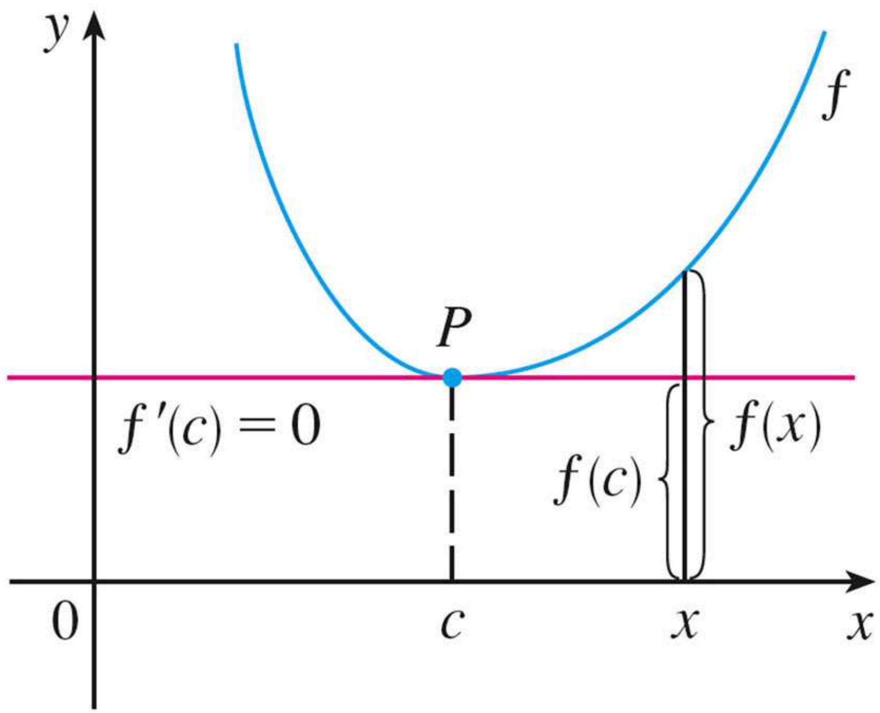
We use this information to sketch the graph of f in Figure 9. Notice that we made the curve bend upward when $x < 2$ and bend downward when $x > 2$. □


y
(4, 6)
6
3
(2, 3)
0
2
4
x
dec
inc
dec
CU
CD

THE SECOND DERIVATIVE TEST Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

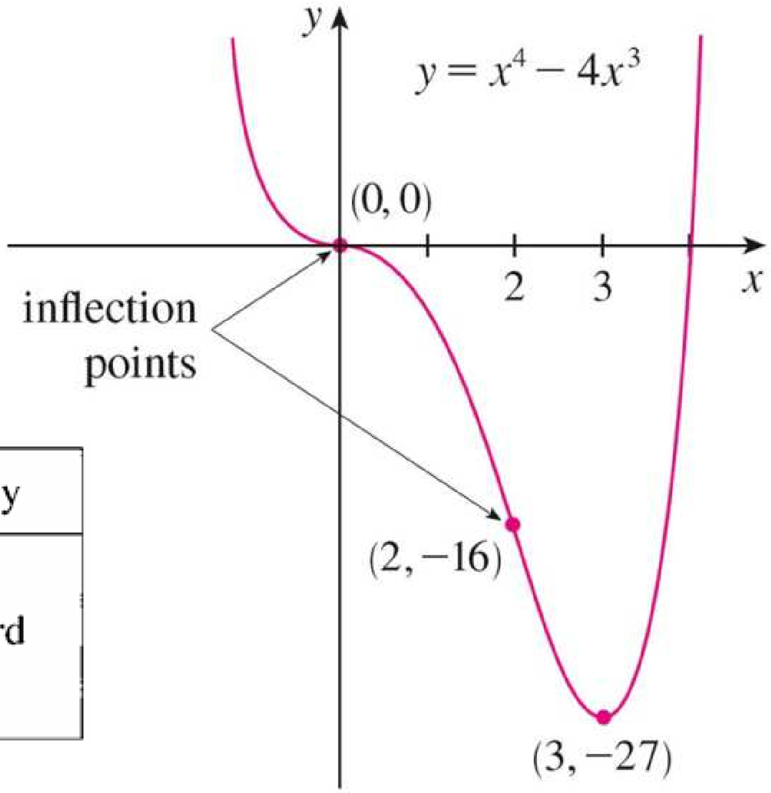


EXAMPLE 6 Discuss the curve $y = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve.

SOLUTION If $f(x) = x^4 - 4x^3$, then

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$$

$$f''(x) = 12x^2 - 24x = 12x(x - 2)$$

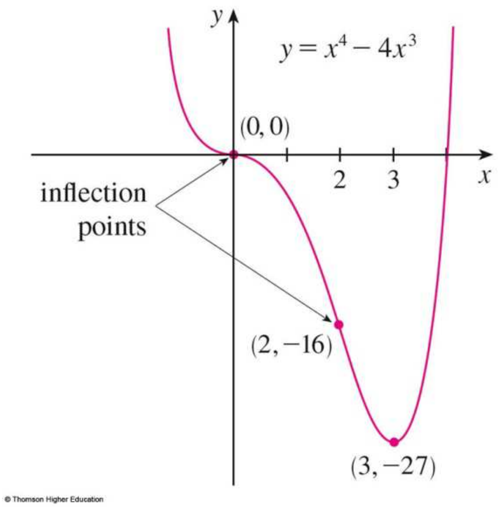


Interval	$f''(x) = 12x(x - 2)$	Concavity
$(-\infty, 0)$	$+$	upward
$(0, 2)$	$-$	downward
$(2, \infty)$	$+$	upward

To find the critical numbers we set $f'(x) = 0$ and obtain $x = 0$ and $x = 3$. To use the Second Derivative Test we evaluate f'' at these critical numbers:

$$f''(0) = 0 \qquad f''(3) = 36 > 0$$

Since $f'(3) = 0$ and $f''(3) > 0$, $f(3) = -27$ is a local minimum. Since $f''(0) = 0$, the Second Derivative Test gives no information about the critical number 0. But since $f'(x) < 0$ for $x < 0$ and also for $0 < x < 3$, the First Derivative Test tells us that f does not have a local maximum or minimum at 0. [In fact, the expression for $f'(x)$ shows that f decreases to the left of 3 and increases to the right of 3.]



Since $f''(x) = 0$ when $x = 0$ or 2, we divide the real line into intervals with these numbers as endpoints and complete the following chart.

Interval	$f''(x) = 12x(x - 2)$	Concavity
$(-\infty, 0)$	+	upward
$(0, 2)$	−	downward
$(2, \infty)$	+	upward

The point $(0, 0)$ is an inflection point since the curve changes from concave upward to concave downward there. Also, $(2, -16)$ is an inflection point since the curve changes from concave downward to concave upward there.

Using the local minimum, the intervals of concavity, and the inflection points, we sketch the curve in Figure 11. ☐

NOTE The Second Derivative Test is inconclusive when $f''(c) = 0$. In other words, at such a point there might be a maximum, there might be a minimum, or there might be neither (as in Example 6). This test also fails when $f''(c)$ does not exist. In such cases the First Derivative Test must be used. In fact, even when both tests apply, the First Derivative Test is often the easier one to use.

EXAMPLE 7 Sketch the graph of the function $f(x) = x^{2/3}(6 - x)^{1/3}$.

SOLUTION You can use the differentiation rules to check that the first two derivatives are

$$f'(x) = \frac{4 - x}{x^{1/3}(6 - x)^{2/3}} \qquad f''(x) = \frac{-8}{x^{4/3}(6 - x)^{5/3}}$$

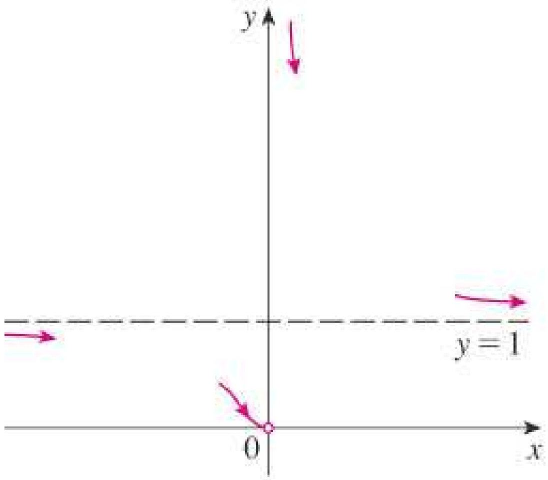
Since $f'(x) = 0$ when $x = 4$ and $f'(x)$ does not exist when $x = 0$ or $x = 6$, the critical numbers are 0, 4, and 6.

Interval	$4 - x$	$x^{1/3}$	$(6 - x)^{2/3}$	$f'(x)$	f
$x < 0$	+	−	+	−	decreasing on $(-\infty, 0)$
$0 < x < 4$	+	+	+	+	increasing on $(0, 4)$
$4 < x < 6$	−	+	+	−	decreasing on $(4, 6)$
$x > 6$	−	+	+	−	decreasing on $(6, \infty)$

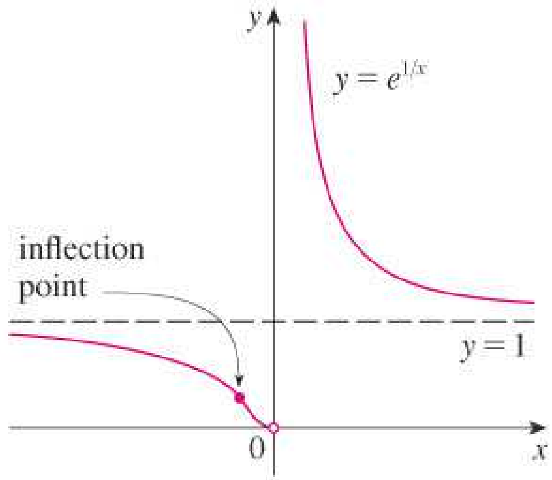
To find the local extreme values we use the First Derivative Test. Since f' changes from negative to positive at 0, $f(0) = 0$ is a local minimum. Since f' changes from positive to negative at 4, $f(4) = 2^{5/3}$ is a local maximum. The sign of f' does not chang

EXAMPLE 8 Use the first and second derivatives of $f(x) = e^{1/x}$, together with asymptotes, to sketch its graph.

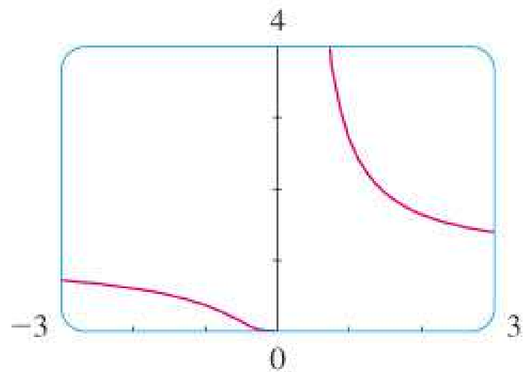
SOLUTION Notice that the domain of f is $\{x \mid x \neq 0\}$, so we check for vertical asymptotes by computing the left and right limits as $x \to 0$. As $x \to 0^+$, we know that $t = 1/x \to \infty$,



(a) Preliminary sketch



(b) Finished sketch



(c) Computer confirmation

so

$$\boxed{\lim_{x\to 0^+} e^{1/x} = \lim_{t\to\infty} e^t = \infty}$$

and this shows that $x = 0$ is a vertical asymptote. As $x \to 0^-$, we have $t = 1/x \to -\infty$, so

$$\boxed{\lim_{x\to 0^-} e^{1/x} = \lim_{t\to -\infty} e^t = 0}$$

As $x \to \pm\infty$, we have $1/x \to 0$ and so

$$\boxed{\lim_{x\to\pm\infty} e^{1/x} = e^0 = 1}$$

This shows that $y = 1$ is a horizontal asymptote.

Now let's compute the derivative. The Chain Rule gives

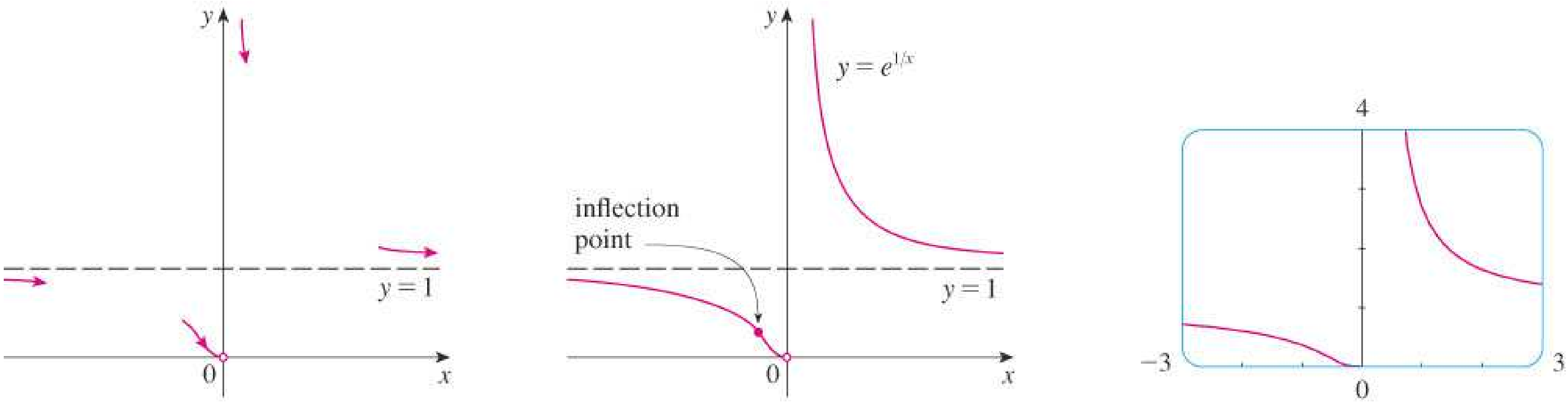
$$\boxed{f'(x) = -\frac{e^{1/x}}{x^2}}$$

Since $e^{1/x} > 0$ and $x^2 > 0$ for all $x \neq 0$, we have $f'(x) < 0$ for all $x \neq 0$. Thus f is decreasing on $(-\infty, 0)$ and on $(0, \infty)$. There is no critical number, so the function has no maximum or minimum. The second derivative is

$$\boxed{f''(x) = -\frac{x^2 e^{1/x}(-1/x^2) - e^{1/x}(2x)}{x^4} = \frac{e^{1/x}(2x + 1)}{x^4}}$$

Since $e^{1/x} > 0$ and $x^4 > 0$, we have $f''(x) > 0$ when $x > -\frac{1}{2}$ $(x \neq 0)$ and $f''(x) < 0$ when $x < -\frac{1}{2}$. So the curve is concave downward on $\left(-\infty, -\frac{1}{2}\right)$ and concave upward on $\left(-\frac{1}{2}, 0\right)$ and on $(0, \infty)$. The inflection point is $\left(-\frac{1}{2}, e^{-2}\right)$.

To sketch the graph of f we first draw the horizontal asymptote $y = 1$ (as a dashed line), together with the parts of the curve near the asymptotes in a preliminary sketch [Figure 13(a)]. These parts reflect the information concerning limits and the fact that f is decreasing on both $(-\infty, 0)$ and $(0, \infty)$. Notice that we have indicated that $f(x) \to 0$ as $x \to 0^-$ even though $f(0)$ does not exist. In Figure 13(b) we finish the sketch by incorporating the information concerning concavity and the inflection point. In Figure 13(c) we check our work with a graphing device.



(a) Preliminary sketch (b) Finished sketch (c) Computer confirmation

<u>Sec. 4.4</u> Indeterminate Forms and L'Hospital's Rule

$$\lim_{x \to 1} F(x) = \lim_{x \to 1} \frac{\ln x}{x - 1}$$

Although *F* is not defined when *x*=1, we need to know how *F* behaviors near 1 (i.e., the value of the limit)

Indeterminate form of type 0/0

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

Both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$

<u>L’HOSPITAL’S RULE</u> **Suppose *f* and *g* are differentiable and *g’*(*x*) $\neq$ 0 on an open interval *I* that contains *a* (except possible at *a*). Suppose that**

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0$$

or that

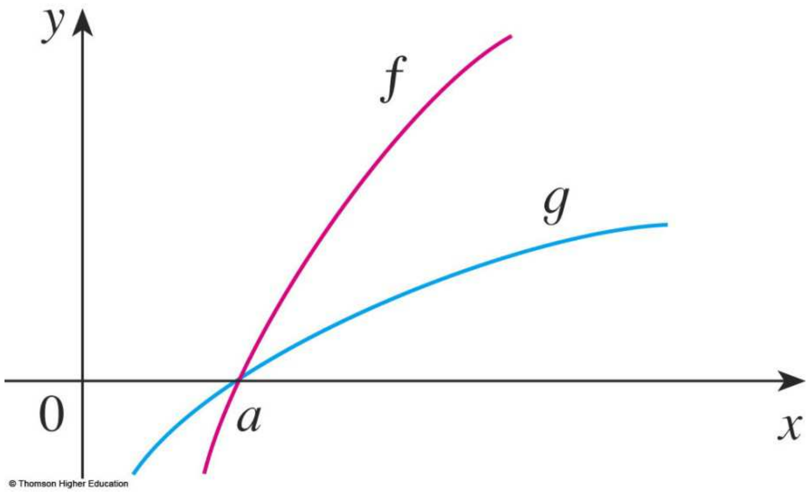
$$\lim_{x\to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to a} g(x) = \pm\infty$$

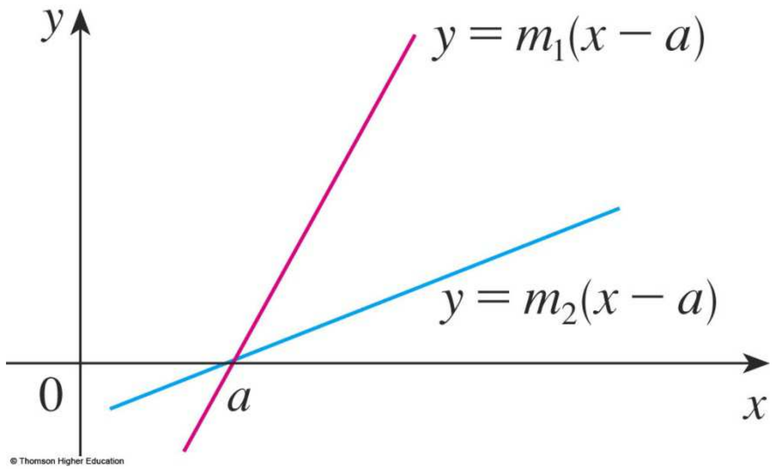
(In other words, we have an indeterminate form of type 0/0 or ∞/∞.) Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or - ∞).

羅必達法則的概念（線性化概念 的應用）





$$\frac{m_1(x-a)}{m_2(x-a)}=\frac{m_1}{m_2}$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

若兩個函數在趨近於**a**時，趨近於無窮大，函數比的極限又如何解釋？

NOTE 1 L'Hospital's Rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the given conditions are satisfied. It is especially important to verify the conditions regarding the limits of f and g before using l'Hospital's Rule.

NOTE 2 L'Hospital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity; that is, "$x \to a$" can be replaced by any of the symbols $x \to a^+$, $x \to a^-$, $x \to \infty$, or $x \to -\infty$.

NOTE 3 For the special case in which $f(a) = g(a) = 0$, f' and g' are continuous, and $g'(a) \neq 0$, it is easy to see why l'Hospital's Rule is true. In fact, using the alternative form of the definition of a derivative, we have

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \frac{\displaystyle\lim_{x \to a} \frac{f(x) - f(a)}{x - a}}{\displaystyle\lim_{x \to a} \frac{g(x) - g(a)}{x - a}} = \lim_{x \to a} \frac{\dfrac{f(x) - f(a)}{x - a}}{\dfrac{g(x) - g(a)}{x - a}}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \to a} \frac{f(x)}{g(x)}$$

It is more difficult to prove the general version of l'Hospital's Rule. See Appendix F.

Example 1

Find $\lim_{x \to 1} \frac{\ln x}{x-1}$

Since $\lim_{x \to 1} \ln x = \ln 1 = 0$ **and** $\lim_{x \to 1} (x-1) = 0$

we can apply l'Hospital's rule

$$\lim_{x \to 1} \frac{\ln x}{x-1} = \lim_{x \to 1} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x-1)} = \lim_{x \to 1} \frac{\frac{1}{x}}{1} = 1$$

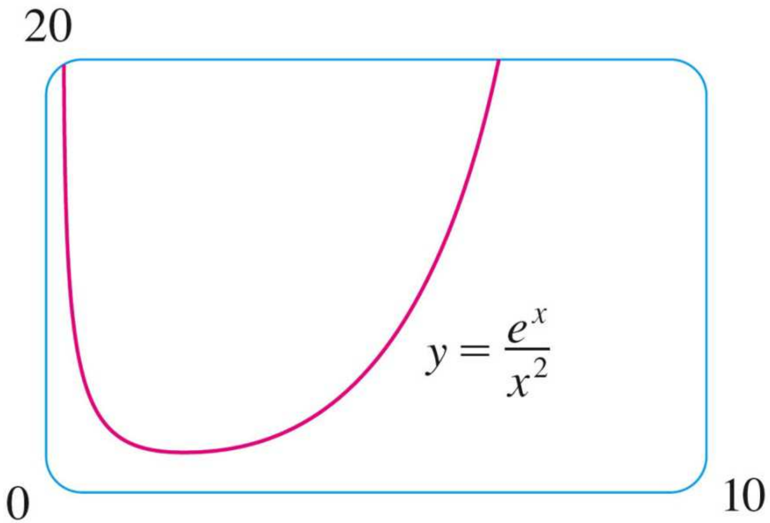
EXAMPLE 2 Calculate $\lim_{x \to \infty} \frac{e^x}{x^2}$.

SOLUTION We have $\lim_{x \to \infty} e^x = \infty$ and $\lim_{x \to \infty} x^2 = \infty$, so l'Hospital's Rule gives

$$\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x^2)} = \lim_{x \to \infty} \frac{e^x}{2x}$$

Since $e^x \to \infty$ and $2x \to \infty$ as $x \to \infty$, the limit on the right side is also indeterminate, but a second application of l'Hospital's Rule gives

$$\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{e^x}{2x} = \lim_{x \to \infty} \frac{e^x}{2} = \infty$$ □



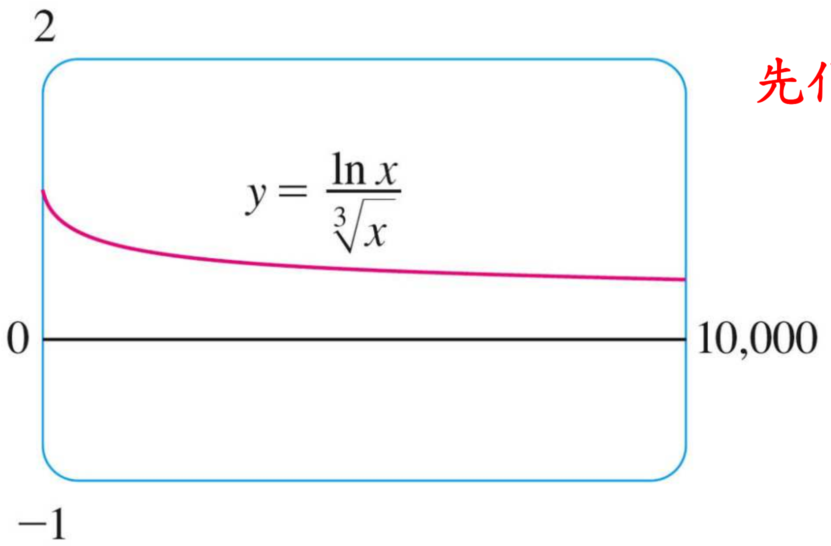
EXAMPLE 3 Calculate $\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}}$.

SOLUTION Since $\ln x \to \infty$ and $\sqrt[3]{x} \to \infty$ as $x \to \infty$, l'Hospital's Rule applies:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{3}x^{-2/3}}$$

Notice that the limit on the right side is now indeterminate of type $\frac{0}{0}$. But instead of applying l'Hospital's Rule a second time as we did in Example 2, we simplify the expression and see that a second application is unnecessary:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{3}x^{-2/3}} = \lim_{x\to\infty} \frac{3}{\sqrt[3]{x}} = 0 \qquad \square$$



先化簡，若仍為不定型再使用羅必達法則

EXAMPLE 4 Find $\lim_{x \to 0} \frac{\tan x - x}{x^3}$. (See Exercise 38 in Section 2.2.)

SOLUTION Noting that both $\tan x - x \to 0$ and $x^3 \to 0$ as $x \to 0$, we use l'Hospital's Rule:

$$\lim_{x \to 0} \frac{\tan x - x}{x^3} = \lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2}$$

Since the limit on the right side is still indeterminate of type $\frac{0}{0}$, we apply l'Hospital's Rule again:

$$\lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x \to 0} \frac{2 \sec^2 x \tan x}{6x}$$

Because $\lim_{x \to 0} \sec^2 x = 1$, we simplify the calculation by writing

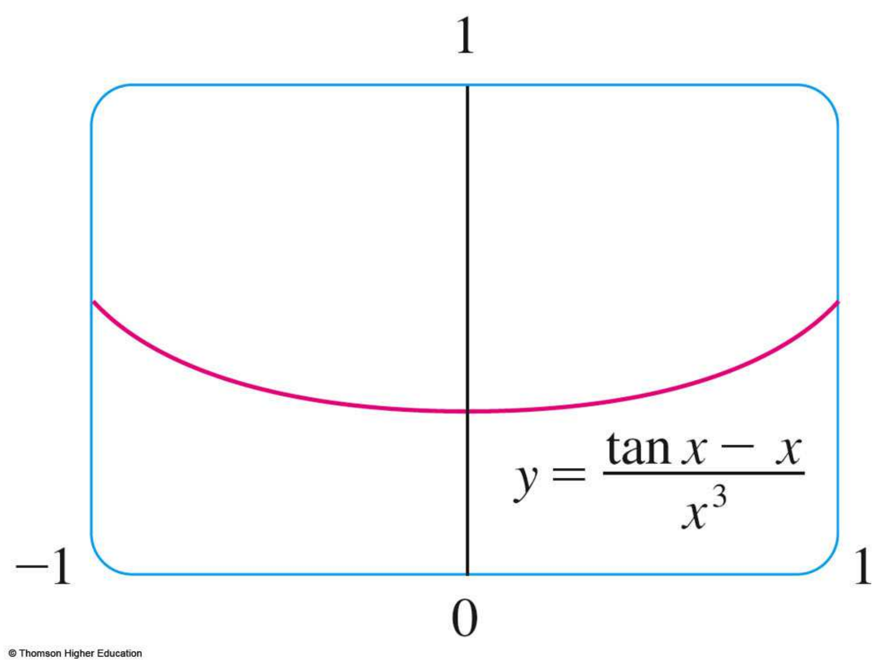
$$\lim_{x \to 0} \frac{2 \sec^2 x \tan x}{6x} = \frac{1}{3} \lim_{x \to 0} \sec^2 x \lim_{x \to 0} \frac{\tan x}{x} = \frac{1}{3} \lim_{x \to 0} \frac{\tan x}{x}$$

We can evaluate this last limit either by using l'Hospital's Rule a third time or by writing $\tan x$ as $(\sin x)/(\cos x)$ and making use of our knowledge of trigonometric limits.

Putting together all the steps, we get

$$\lim_{x\to 0}\frac{\tan x - x}{x^3} = \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2} = \lim_{x\to 0}\frac{2\sec^2 x\tan x}{6x}$$

$$= \frac{1}{3}\lim_{x\to 0}\frac{\tan x}{x} = \frac{1}{3}\lim_{x\to 0}\frac{\sec^2 x}{1} = \frac{1}{3}$$



EXAMPLE 5 Find $\lim_{x \to \pi^-} \dfrac{\sin x}{1 - \cos x}$.

SOLUTION If we blindly attempted to use l'Hospital's Rule, we would get

$$\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x} = \lim_{x \to \pi^-} \frac{\cos x}{\sin x} = -\infty$$

This is **wrong!** Although the numerator $\sin x \to 0$ as $x \to \pi^-$, notice that the denominator $(1 - \cos x)$ does not approach 0, so l'Hospital's Rule can't be applied here.

The required limit is, in fact, easy to find because the function is continuous at π an the denominator is nonzero there:

$$\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x} = \frac{\sin \pi}{1 - \cos \pi} = \frac{0}{1 - (-1)} = 0$$

INDETERMINATE PRODUCTS

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ (or $-\infty$), then it isn't clear what the value $\lim_{x\to a} f(x)g(x)$, if any, will be. There is a struggle between f and g. If f wins, the ans will be 0; if g wins, the answer will be ∞ (or $-\infty$). Or there may be a compromise wh the answer is a finite nonzero number. This kind of limit is called an **indeterminate fo of type $\mathbf{0 \cdot \infty}$**. We can deal with it by writing the product fg as a quotient:

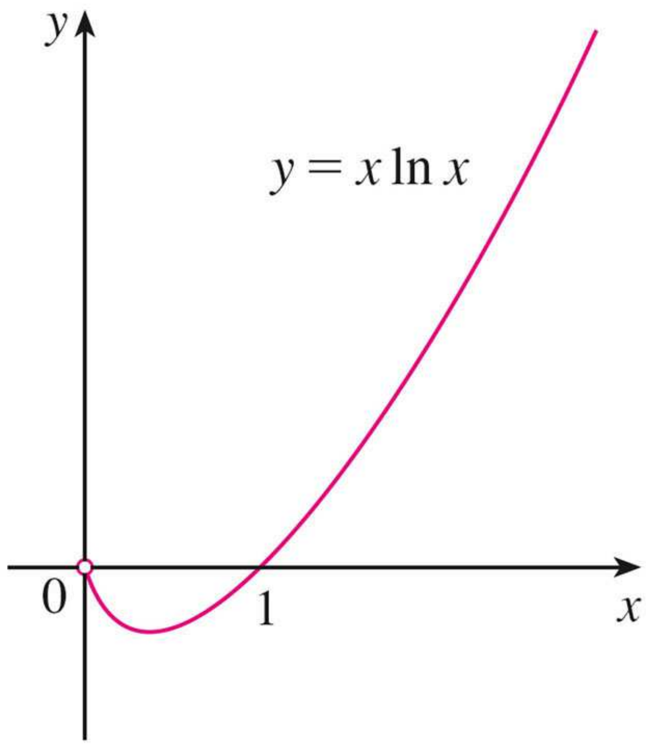
$$\boxed{fg = \frac{f}{1/g} \qquad \text{or} \qquad fg = \frac{g}{1/f}}$$

This converts the given limit into an indeterminate form of type $\frac{0}{0}$ or ∞/∞ so that we use l'Hospital's Rule.

EXAMPLE 6 Evaluate $\lim_{x \to 0^+} x \ln x$.

SOLUTION The given limit is indeterminate because, as $x \to 0^+$, the first factor (x) approaches 0 while the second factor ($\ln x$) approaches $-\infty$. Writing $x = 1/(1/x)$, we have $1/x \to \infty$ as $x \to 0^+$, so l'Hospital's Rule gives

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0$$



If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit

$$\lim_{x\to a} [f(x) - g(x)]$$

is called an **indeterminate form of type $\infty - \infty$**. Again there is a contest between f and g. Will the answer be ∞ (f wins) or will it be $-\infty$ (g wins) or will they compromise on a finite number? To find out, we try to convert the difference into a quotient (for instance, by using a common denominator, or rationalization, or factoring out a common factor) so that we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.

EXAMPLE 8 Compute $\lim_{x \to (\pi/2)^-} (\sec x - \tan x)$.

SOLUTION First notice that $\sec x \to \infty$ and $\tan x \to \infty$ as $x \to (\pi/2)^-$, so the limit is indeterminate. Here we use a common denominator:

$$\lim_{x \to (\pi/2)^-} (\sec x - \tan x) = \lim_{x \to (\pi/2)^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right)$$

$$= \lim_{x \to (\pi/2)^-} \frac{1 - \sin x}{\cos x} = \lim_{x \to (\pi/2)^-} \frac{-\cos x}{-\sin x} = 0$$

Note that the use of l'Hospital's Rule is justified because $1 - \sin x \to 0$ and $\cos x \to 0$ as $x \to (\pi/2)^-$.

Several indeterminate forms arise from the limit

$$\boxed{\lim_{x \to a} [f(x)]^{g(x)}}$$

1. $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ type 0^0
2. $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = 0$ type ∞^0
3. $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \pm\infty$ type 1^∞

Each of these three cases can be treated either by taking the natural logarithm:

$$\underline{\text{let} \quad y = [f(x)]^{g(x)}, \quad \text{then} \quad \ln y = g(x) \ln f(x)}$$

or by writing the function as an exponential:

$$\underline{[f(x)]^{g(x)} = e^{g(x) \ln f(x)}}$$

(Recall that both of these methods were used in differentiating such functions.) In either method we are led to the indeterminate product $g(x) \ln f(x)$, which is of type $0 \cdot \infty$.

EXAMPLE 9 Calculate $\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x}$.

SOLUTION First notice that as $x \to 0^+$, we have $1 + \sin 4x \to 1$ and $\cot x \to \infty$, so the given limit is indeterminate. Let

$$y = (1 + \sin 4x)^{\cot x}$$

Then

$$\ln y = \ln[(1 + \sin 4x)^{\cot x}] = \cot x \ln(1 + \sin 4x)$$

so l'Hospital's Rule gives

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \frac{\ln(1 + \sin 4x)}{\tan x} = \lim_{x \to 0^+} \frac{\dfrac{4 \cos 4x}{1 + \sin 4x}}{\sec^2 x} = 4$$

So far we have computed the limit of $\ln y$, but what we want is the limit of y. To find t we use the fact that $y = e^{\ln y}$:

$$\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x} = \lim_{x \to 0^+} y = \lim_{x \to 0^+} e^{\ln y} = e^4$$

EXAMPLE 10 Find $\lim_{x \to 0^+} x^x$.

SOLUTION Notice that this limit is indeterminate since $0^x = 0$ for any $x > 0$ but $x^0 = 1$ for any $x \neq 0$. We could proceed as in Example 9 or by writing the function as an exponential:

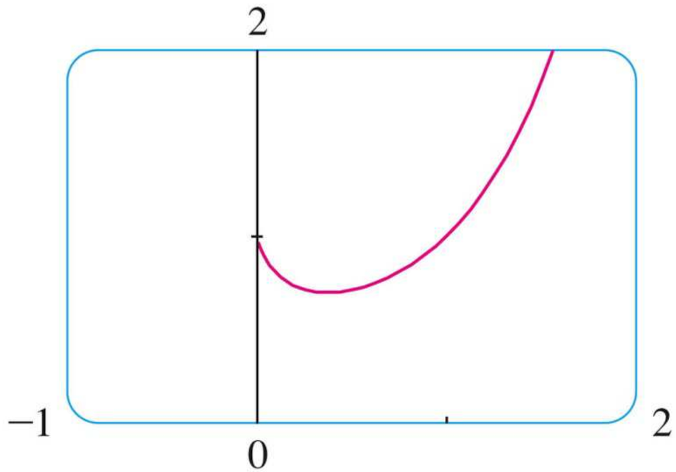
$$x^x = (e^{\ln x})^x = e^{x \ln x}$$

In Example 6 we used l'Hospital's Rule to show that

$$\lim_{x \to 0^+} x \ln x = 0$$

Therefore

$$\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{x \ln x} = e^0 = 1 \qquad \square$$



PROOF OF L'HOSPITAL'S RULE We are assuming that $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$. Let

$$L = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

We must show that $\lim_{x \to a} f(x)/g(x) = L$. Define

$$F(x) = \begin{cases} f(x) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases} \qquad G(x) = \begin{cases} g(x) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases}$$

Then F is continuous on I since f is continuous on $\{x \in I \mid x \neq a\}$ and

$$\lim_{x \to a} F(x) = \lim_{x \to a} f(x) = 0 = F(a)$$

Likewise, G is continuous on I. Let $x \in I$ and $x > a$. Then F and G are continuous on $[a, x]$ and differentiable on (a, x) and $G' \neq 0$ there (since $F' = f'$ and $G' = g'$). Therefore, by Cauchy's Mean Value Theorem, there is a number y such that $a < y < x$ and

$$\frac{F'(y)}{G'(y)} = \frac{F(x) - F(a)}{G(x) - G(a)} = \frac{F(x)}{G(x)}$$

Here we have used the fact that, by definition, $F(a) = 0$ and $G(a) = 0$. Now, if we let $x \to a^+$, then $y \to a^+$ (since $a < y < x$), so

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{F(x)}{G(x)} = \lim_{y\to a^+} \frac{F'(y)}{G'(y)} = \lim_{y\to a^+} \frac{f'(y)}{g'(y)} = L$$

A similar argument shows that the left-hand limit is also L. Therefore

$$\lim_{x\to a} \frac{f(x)}{g(x)} = L$$

This proves l'Hospital's Rule for the case where a is finite.

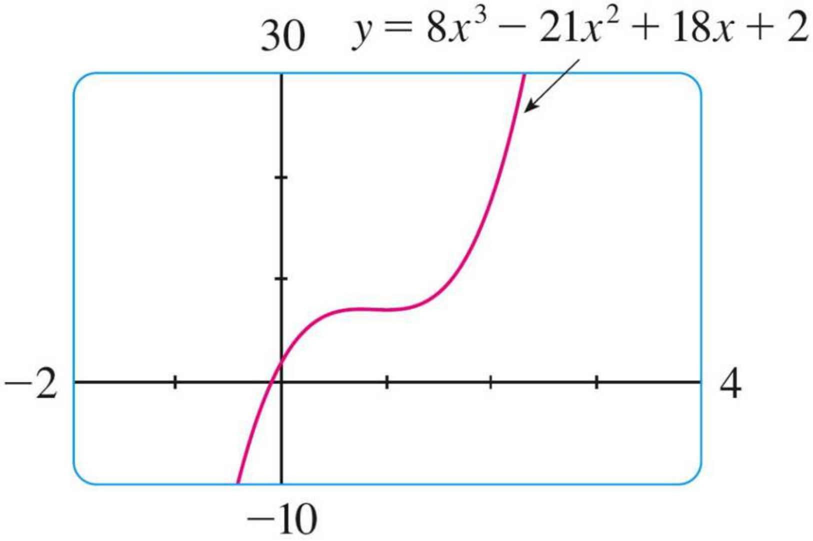
If a is infinite, we let $t = 1/x$. Then $t \to 0^+$ as $x \to \infty$, so we have

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{t\to 0^+} \frac{f(1/t)}{g(1/t)}$$

$$= \lim_{t\to 0^+} \frac{f'(1/t)(-1/t^2)}{g'(1/t)(-1/t^2)} \qquad \text{(by l'Hospital's Rule for finite } a\text{)}$$

$$= \lim_{t\to 0^+} \frac{f'(1/t)}{g'(1/t)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

Sec. 4.5 Summary of Curve Sketching



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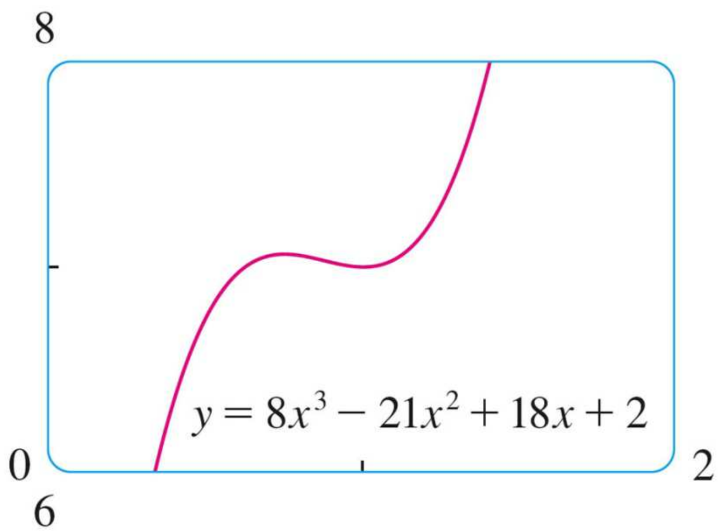

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有極大值在 x=0.75，極小值在 x=1

Guidelines for Sketching a Curve

A. Domain

B. Intercepts: (i) *x*-intercepts; (ii) *y*-intercepts

C. Symmetry: (i) even function; (ii) odd function; (iii) periodic function

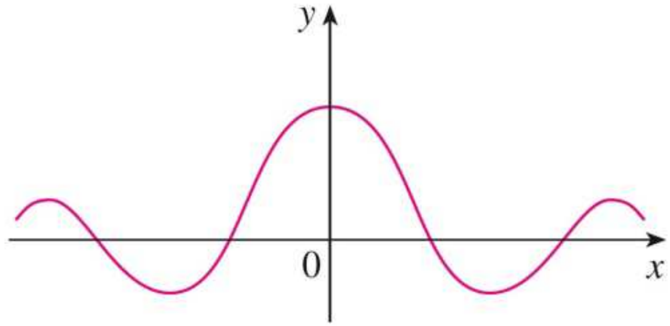
D. Asymptotes: (i) horizontal, (ii) vertical, (iii) slant asymptotes

E. Intervals of Increase or Decrease

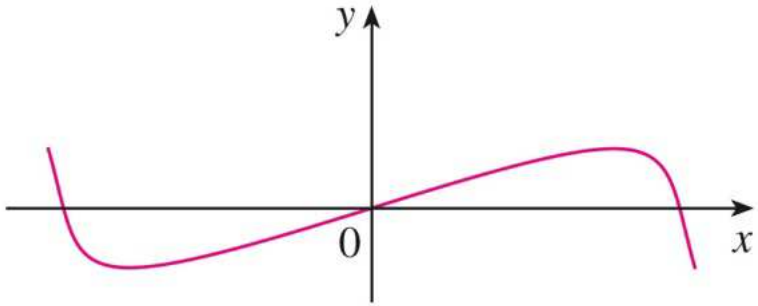
F. Local Max. or Mini. Values

G. Concavity and Points of Inflection

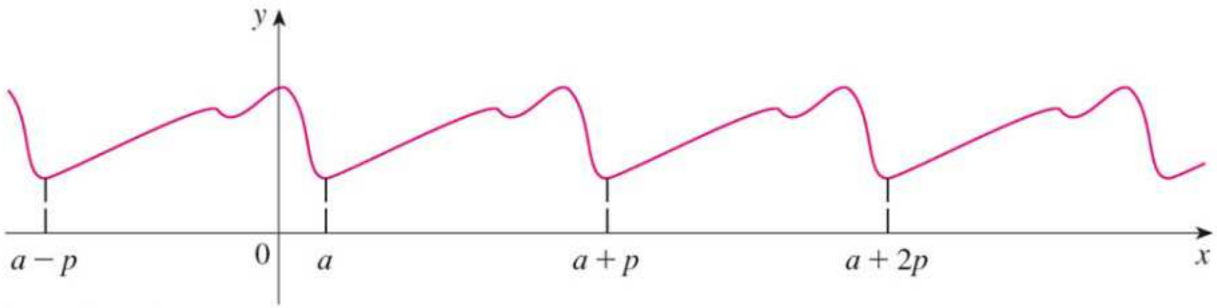
H. Sketch the Curve



(a) Even function: reflectional symmetry



(b) Odd function: rotational symmetry



$\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$

$y = L$ is a horizontal asymptote of $y = f(x)$

$\lim\limits_{x\to a+} f(x) = \infty$ $\lim\limits_{x\to a-} f(x) = \infty$

$\lim\limits_{x\to a+} f(x) = -\infty$ $\lim\limits_{x\to a-} f(x) = -\infty$

$x = a$ is a vertical asymptote of $y = f(x)$

Example 1

Use the guidelines to sketch the curve $y = f(x) = \dfrac{2x^2}{x^2-1}$

A. Domain $\{x \mid x^2 - 1 \neq 0\} = \{x \mid x \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

B. Intercepts $(x, y) = (0, 0)$

C. Symmetry $f(-x) = \dfrac{2(-x)^2}{(-x)^2 - 1} = \dfrac{2x^2}{x^2-1} = f(x) \quad \Rightarrow \quad \text{even function}$

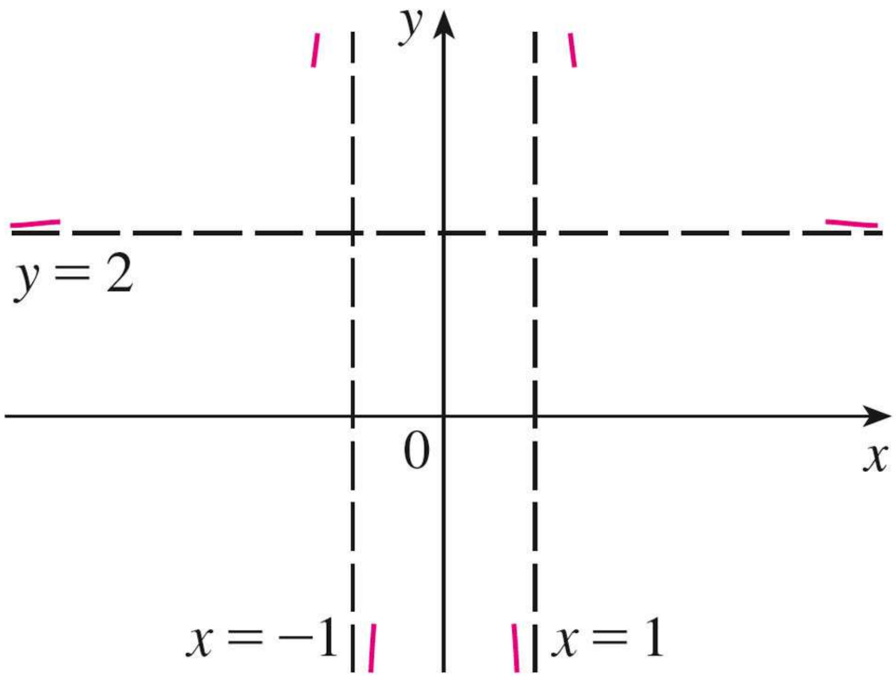
D. Asymptote

$$\lim_{x\to\pm\infty} \frac{2x^2}{x^2-1} = \lim_{x\to\pm\infty} \frac{2}{1-1/x^2} = 2 \quad \Rightarrow \quad y = 2 \text{ is a horizontal asymptote}$$

$$\lim_{x \to 1+} \frac{2x^2}{x^2 - 1} = \infty \qquad \lim_{x \to 1-} \frac{2x^2}{x^2 - 1} = -\infty$$

$$\lim_{x \to -1+} \frac{2x^2}{x^2 - 1} = -\infty \qquad \lim_{x \to -1-} \frac{2x^2}{x^2 - 1} = \infty$$

$x = 1$ and $x = -1$ are vertical asymptotes



E. Increase or Decrease

$$f'(x) = \frac{4x\left(x^2-1\right) - 2x^2 \cdot 2x}{\left(x^2-1\right)^2} = \frac{-4x}{\left(x^2-1\right)^2}$$

$f'(x) > 0$ when $x < 0\left(x \neq -1\right)$ and $f'(x) < 0$ when $x > 0\left(x \neq 1\right)$

Increasing: $(-\infty, -1)$ and $(-1, 0)$

Decreasing: $(0,1)$ and $(1, \infty)$

F. Max. or Mini

$$f'(x) = \frac{-4x}{\left(x^2-1\right)^2} = 0 \quad \Rightarrow \quad x = 0$$

$f(0) = 0$

G. Concavity

$$f''(x)=\frac{-4\left(x^2-1\right)^2+4x\cdot 2\left(x^2-1\right)\cdot 2x}{\left(x^2-1\right)^4}=\frac{12x^2+4}{\left(x^2-1\right)^3}$$

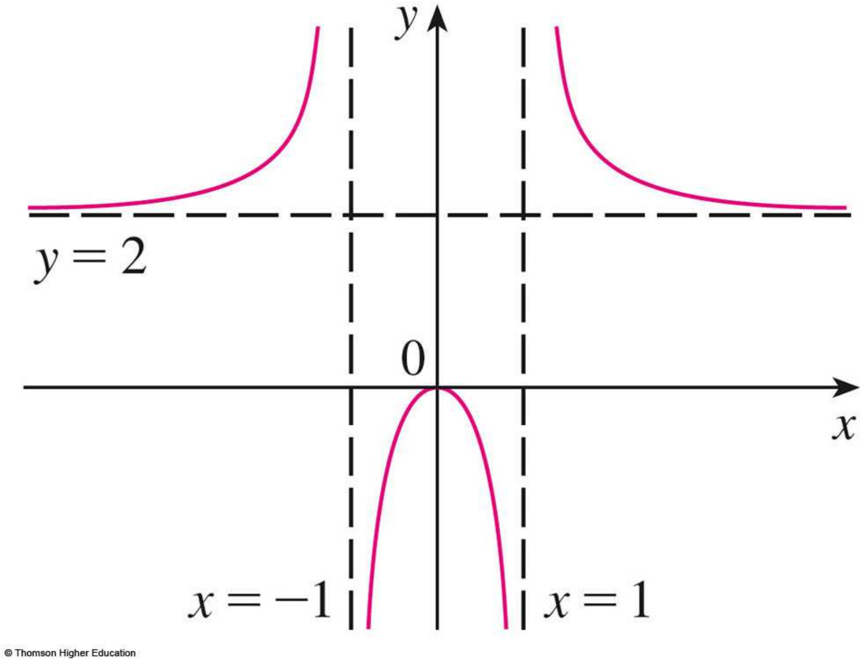
$\because 12x^2+4>0$ for all x, then

$$f''(x)>0 \quad \Leftrightarrow \quad x^2-1>0 \quad \Leftrightarrow \quad |x|>1$$

$$f''(x)<0 \quad \Leftrightarrow \quad x^2-1<0 \quad \Leftrightarrow \quad |x|<1$$

Concave upward: $(-\infty,-1)$ and $(1,\infty)$

Concave downward: $(-1,1)$



Example 2

Sketch the curve $f(x) = \dfrac{x^2}{\sqrt{x+1}}$

A. Domain $\{x \mid x+1>0\} = \{x > -1\} = (-1, \infty)$

B. Intercepts $(x, y) = (0, 0)$

C. Symmetry $f(-x) = \dfrac{(-x)^2}{\sqrt{-x+1}} = \dfrac{x^2}{\sqrt{-x+1}} \quad (\text{none})$

D. Asymptote $\lim\limits_{x\to\infty} \dfrac{x^2}{\sqrt{x+1}} = \infty \quad \Rightarrow \quad$ no horizontal asymptote

$\lim\limits_{x\to -1+} \dfrac{x^2}{\sqrt{x+1}} = \infty \quad \Rightarrow \quad x = 1$ is a vertical asymptote

E. Increase or Decrease

$$f'(x)=\frac{2x\sqrt{x+1}-x^2\cdot 1/2\sqrt{x+1}}{x+1}=\frac{x(3x+4)}{2(x+1)^{3/2}}$$

$f'(x)=0$ when $x=0$,

$f'(x)<0$ when $-1<x<0$,

$f'(x)>0$ when $x>0$,

Increasing: $(0,\infty)$

Decreasing: $(-1,0)$

F. Max. or Mini

$$f'(x)=\frac{x(x+4)}{2(x+1)^{3/2}}=0 \quad\Rightarrow\quad x=0$$

f' changes from $-$ to $+$, and $f(0)=0$ (mini.)

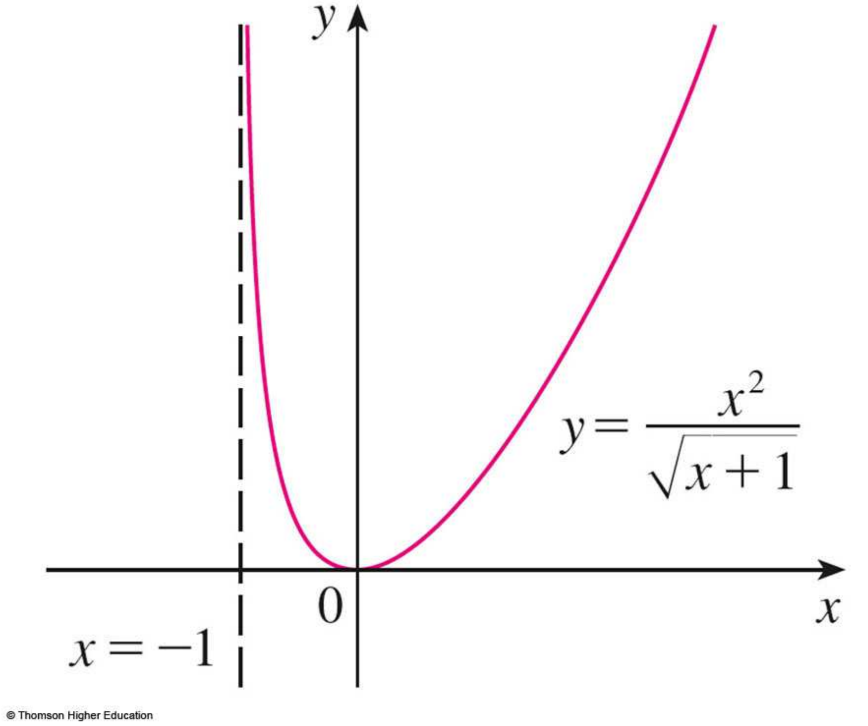
G. Concavity

$$f''(x)=\frac{2(x+1)^{3/2}(6x+4)-(3x^2+4x)\cdot 3(x+1)^{1/2}}{4(x+1)^3}=\frac{3x^2+8x+8}{4(x+1)^{5/2}}$$

$3x^2+8x+8>0$ for all x, then

$f''(x)>0$ for all x

Concave upward: $(-1,\infty)$



Example 4

Sketch the curve $y = f(x) = \dfrac{\cos x}{2+\sin x}$

A. Domain $x \in R$

B. Intercepts $f(0) = \dfrac{1}{2}.\quad y = 0$ when $\cos x = 0,\quad x = \dfrac{(2n+1)\pi}{2},\quad n = 0, \pm 1, \pm 2, \ldots$

C. Symmetry $f(-x) = \dfrac{\cos(-x)}{2+\sin(-x)} = \dfrac{\cos x}{2-\sin x}$ (neither even nor odd),

but $f(x+2n\pi) = \dfrac{\cos(x+2n\pi)}{2+\sin(x+2n\pi)} = \dfrac{\cos x}{2+\sin x} = f(x)$ is periodic

D. Asymptote: None

E. Increase or Decrease

$$f'(x) = \frac{(2+\sin x)(-\sin x) - \cos x(\cos x)}{(2+\sin x)^2} = -\frac{2\sin x + 1}{(2+\sin x)^2}$$

$$f'(x) > 0 \text{ when } 2\sin x + 1 < 0 \quad \Leftrightarrow \quad \sin x < -\frac{1}{2} \quad \Leftrightarrow \quad \frac{7\pi}{6} < x < \frac{11\pi}{6},$$

$$f'(x) < 0 \text{ when } 2\sin x + 1 > 0, \quad \Leftrightarrow \quad \sin x > -\frac{1}{2} \quad \Leftrightarrow \quad 0 < x < \frac{7\pi}{6}, \frac{11\pi}{6} < x < 2\pi.$$

Increasing: $(7\pi/6, 11\pi/6)$

Decreasing: $(0, 7\pi/6)$, $(11\pi/6, 2\pi)$

F. Max. or Mini

$$f'(x) = -\frac{2\sin x + 1}{(2+\sin x)^2} = 0 \quad \Rightarrow \quad x = \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$f\left(\frac{7\pi}{6}\right) = \frac{-1}{\sqrt{3}} \text{ (mini.)}$$

$$f\left(\frac{11\pi}{6}\right) = \frac{1}{\sqrt{3}} \text{ (max.)}$$

G. Concavity

$$f''(x) = -\frac{2\cos x(1-\sin x)}{(2+\sin x)^3}$$

$(2+\sin x)^3 > 0$ and $1-\sin x \geq 0$ for all x, then

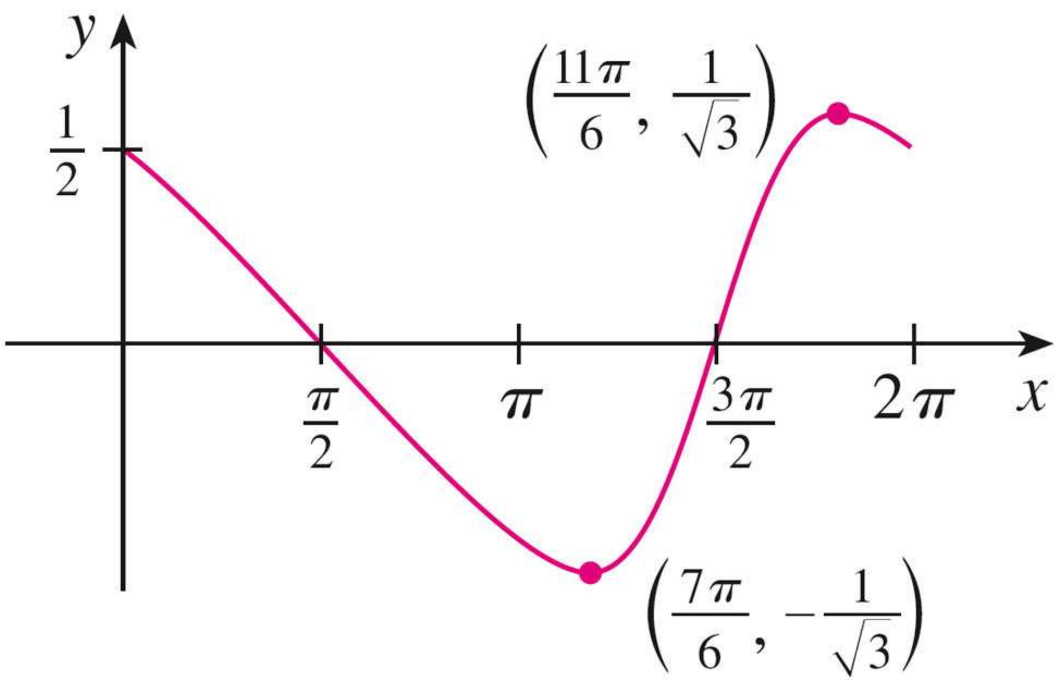
$$f''(x) > 0 \text{ when } \cos x < 0 \quad \Leftrightarrow \quad \frac{\pi}{2} < x < \frac{3\pi}{2}$$

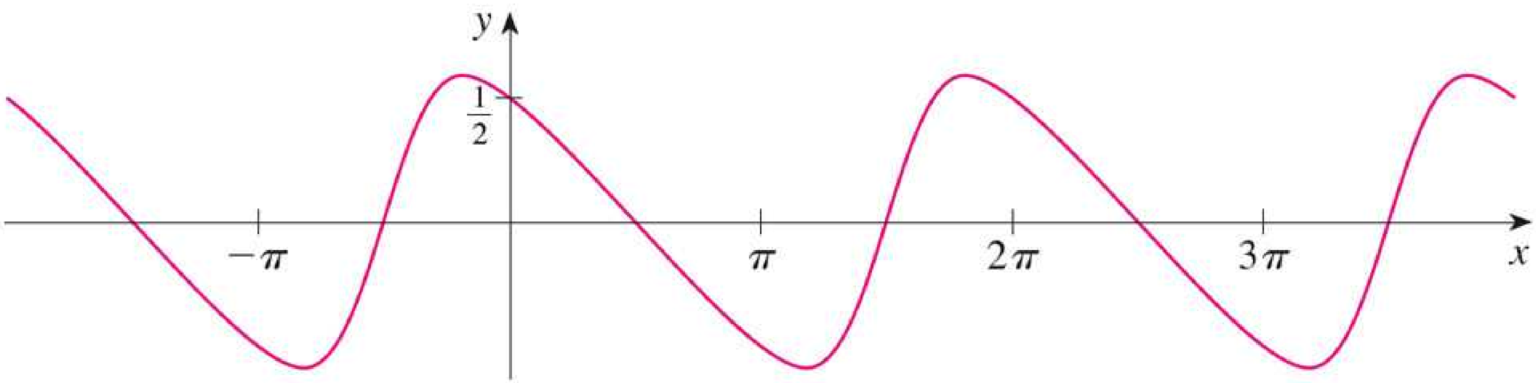
$$f''(x) < 0 \text{ when } \cos x > 0 \quad \Leftrightarrow \quad 0 < x < \frac{\pi}{2} \text{ and } \frac{3\pi}{2} < x < 2\pi.$$

Concave upward: $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Concave downward: $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$

$$y = f(x) = \frac{\cos x}{2 + \sin x}$$





SLANT ASYMPTOTES

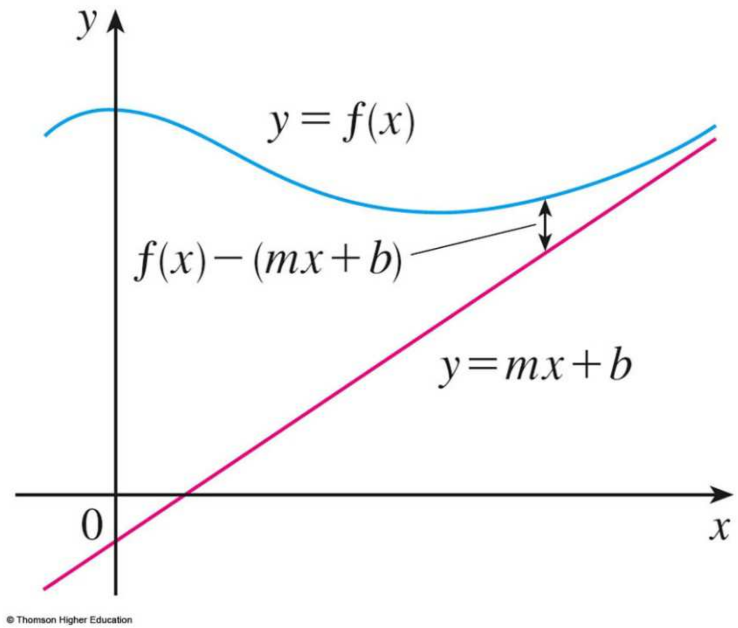
當漸近線為一斜直線時，稱為斜漸近線。若

$$\lim_{x\to\infty}\left[f(x)-(mx+b)\right]=0$$

如何找出此關係式？

函數為假分式或帶分式時使用

則 $\boxed{y=mx+b}$ 為函數的一條斜漸近線



Example 6

Sketch the curve $y = f(x) = \dfrac{x^3}{x^2+1}$

A. Domain $x \in R$

B. Intercepts $(x, y) = (0, 0)$

C. Symmetry $f(-x) = \dfrac{(-x)^3}{(-x)^2+1} = \dfrac{-x^3}{x^2+1} = -f(x) \quad (\text{odd function})$

D. Asymptote: No vertical and horizontal asymptotes, but

$$f(x) = \frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}$$

$$f(x) - x = -\frac{x}{x^2+1} = -\frac{\frac{1}{x}}{1+\frac{1}{x^2}} \to 0 \text{ as } x \to \pm\infty$$

$y = x$ is a slant asymptote.

E. Increase or Decrease

$$f'(x)=\frac{3x^2\left(x^2+1\right)-x^3\cdot 2x}{\left(x^2+1\right)^2}=\frac{x^2\left(x^2+3\right)}{\left(x^2+1\right)^2}$$

$f'(x)>0$ for all x (except 0), f is increasing on $(-\infty,\infty)$.

F. Max. or Mini

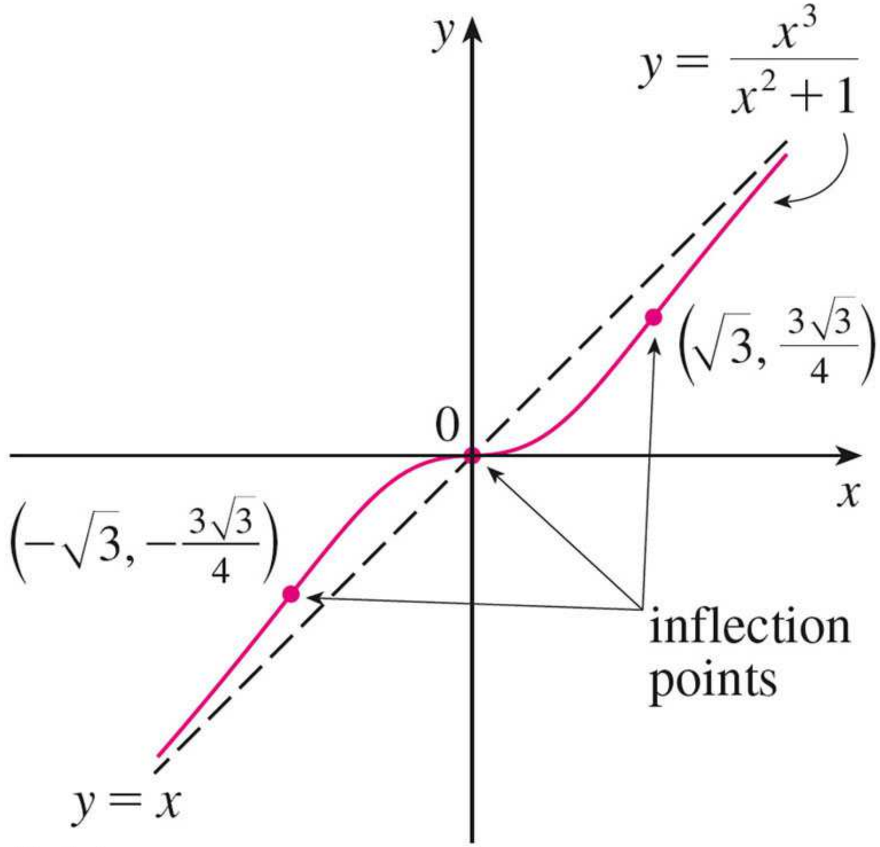
$f'(0)=0$, but f' does not change sign, so there is no max. or mini.

G. Concavity

$$f''(x)=\frac{\left(4x^3+6x\right)\left(x^2+1\right)^2-\left(x^4+3x^2\right)\cdot 2\left(x^2+1\right)\cdot 2x}{\left(x^2+1\right)^4}=\frac{2x\left(3-x^2\right)}{\left(x^2+1\right)^3}$$

$f''(x)=0$ when $x=0$ or $x=\pm\sqrt{3}$.

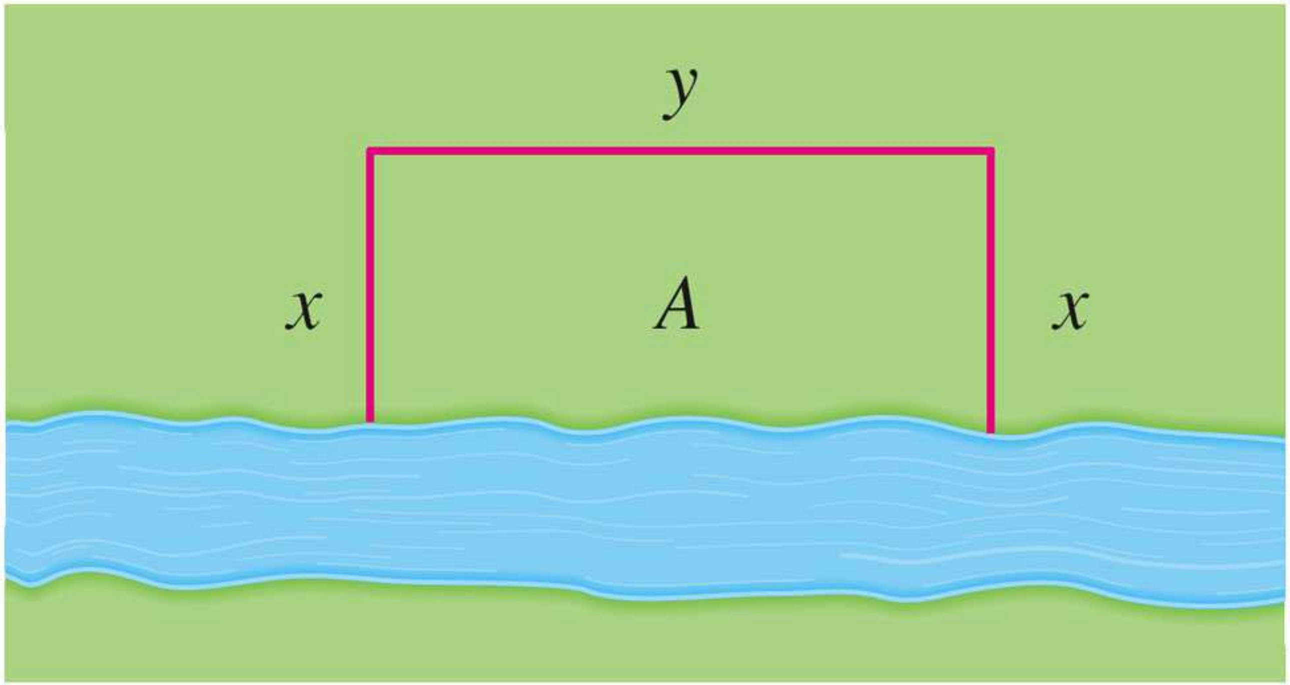
The points of inflection are $\left(-\sqrt{3},-\frac{3\sqrt{3}}{4}\right)$, $(0,0)$, and $\left(\sqrt{3},\frac{3\sqrt{3}}{4}\right)$.



y
$y = \frac{x^3}{x^2+1}$
$\left(\sqrt{3}, \frac{3\sqrt{3}}{4}\right)$
0
x
$\left(-\sqrt{3}, -\frac{3\sqrt{3}}{4}\right)$
inflection points
$y = x$

<u>Sec. 4.7</u> Optimization Problems

EXAMPLE 1 A farmer has 1200 m of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?



SOLUTION In order to get a feeling for what is happening in this problem, let's experiment with some special cases. Figure 1 shows three possible ways of laying out the 1200 m of fencing.

We see that when we try shallow, wide fields or deep, narrow fields, we get relatively small areas. It seems plausible that there is some intermediate configuration that produces the largest area.

Figure 2 illustrates the general case. We wish to maximize the area A of the rectangle. Let x and y be the depth and width of the rectangle (in meters). Then we express A in terms of x and y:

$$A = xy$$

We want to express A as a function of just one variable, so we eliminate y by expressing it in terms of x. To do this we use the given information that the total length of the fencing is 1200 m. Thus

$$2x + y = 1200$$

From this equation we have $y = 1200 - 2x$, which gives

$$A = x(1200 - 2x) = 1200x - 2x^2$$

Note that $x \geqslant 0$ and $x \leqslant 600$ (otherwise $A < 0$). So the function that we wish to maximize is

$$A(x) = 1200x - 2x^2 \qquad 0 \leqslant x \leqslant 600$$

The derivative is $A'(x) = 1200 - 4x$, so to find the critical numbers we solve the equation

$$1200 - 4x = 0$$

which gives $x = 300$. The maximum value of A must occur either at this critical number or at an endpoint of the interval. Since $A(0) = 0$, $A(300) = 180{,}000$, and $A(600) = 0$, the Closed Interval Method gives the maximum value as $A(300) = 180{,}000$.

[Alternatively, we could have observed that $A''(x) = -4 < 0$ for all x, so A is always concave downward and the local maximum at $x = 300$ must be an absolute maximum.]

Thus the rectangular field should be 300 m deep and 600 m wide.

STEPS IN SOLVING OPTIMIZATION PROBLEMS

1. **Understand the Problem** The first step is to read the problem carefully until it is clearly understood. Ask yourself: What is the unknown? What are the given quantities? What are the given conditions?

2. **Draw a Diagram** In most problems it is useful to draw a diagram and identify the given and required quantities on the diagram.

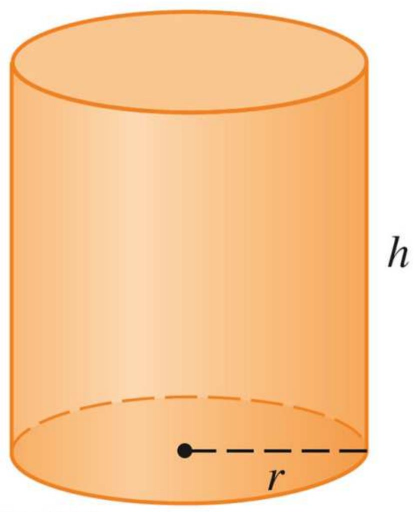
3. **Introduce Notation** Assign a symbol to the quantity that is to be maximized or minimized (let's call it Q for now). Also select symbols ($a, b, c, \ldots, x, y$) for other unknown quantities and label the diagram with these symbols. It may help to use initials as suggestive symbols—for example, A for area, h for height, t for time.

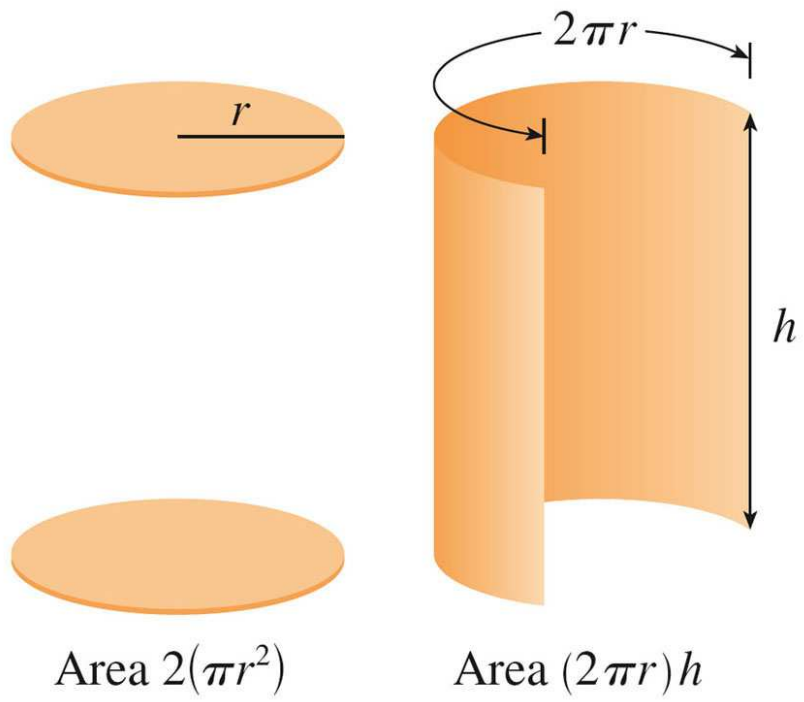
4. Express Q in terms of some of the other symbols from Step 3.

5. If Q has been expressed as a function of more than one variable in Step 4, use the given information to find relationships (in the form of equations) among these variables. Then use these equations to eliminate all but one of the variables in the expression for Q. Thus Q will be expressed as a function of *one* variable x, say, $Q = f(x)$. Write the domain of this function.

6. Use the methods of Sections 4.1 and 4.3 to find the *absolute* maximum or minimum value of f. In particular, if the domain of f is a closed interval, then the Closed Interval Method in Section 4.1 can be used.

EXAMPLE 2 A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.





SOLUTION Draw the diagram as in Figure 3, where r is the radius and h the height (both in centimeters). In order to minimize the cost of the metal, we minimize the total surface area of the cylinder (top, bottom, and sides). From Figure 4 we see that the sides are made from a rectangular sheet with dimensions $2\pi r$ and h. So the surface area is

$$A = 2\pi r^2 + 2\pi rh$$

To eliminate h we use the fact that the volume is given as 1 L, which we take to be 1000 cm^3. Thus

$$\pi r^2 h = 1000$$

which gives $h = 1000/(\pi r^2)$. Substitution of this into the expression for A gives

$$A = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}$$

Therefore the function that we want to minimize is

$$A(r) = 2\pi r^2 + \frac{2000}{r} \qquad r > 0$$

To find the critical numbers, we differentiate:

$$A'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4(\pi r^3 - 500)}{r^2}$$

Then $A'(r) = 0$ when $\pi r^3 = 500$, so the only critical number is $r = \sqrt[3]{500/\pi}$.

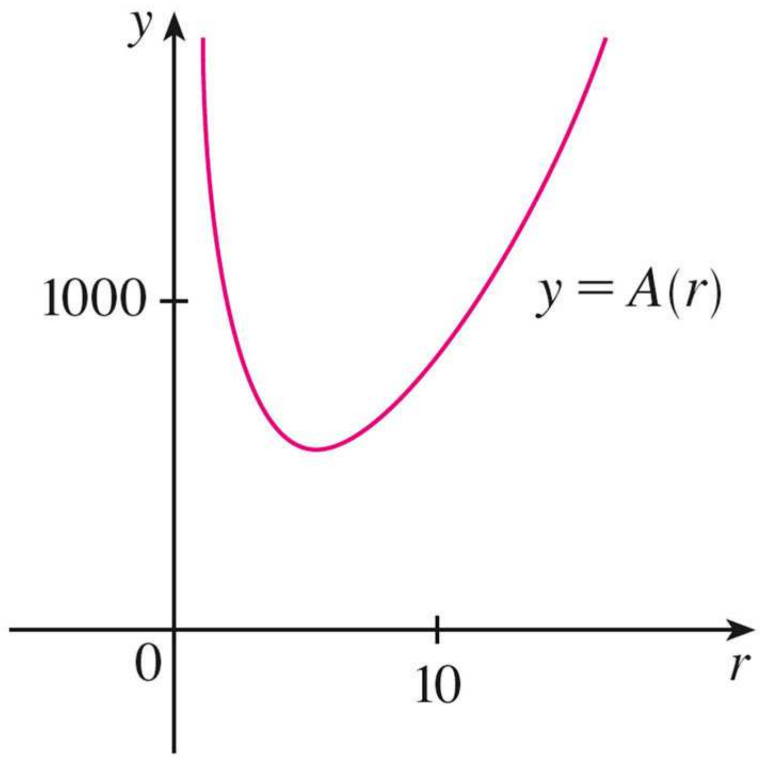
Since the domain of A is $(0, \infty)$, we can't use the argument of Example 1 concerning endpoints. But we can observe that $A'(r) < 0$ for $r < \sqrt[3]{500/\pi}$ and $A'(r) > 0$ for $r > \sqrt[3]{500/\pi}$, so A is decreasing for *all* r to the left of the critical number and increasing for *all* r to the right. Thus $r = \sqrt[3]{500/\pi}$ must give rise to an *absolute* minimum.

[Alternatively, we could argue that $A(r) \to \infty$ as $r \to 0^+$ and $A(r) \to \infty$ as $r \to \infty$, so there must be a minimum value of $A(r)$, which must occur at the critical number. See Figure 5.]

The value of h corresponding to $r = \sqrt[3]{500/\pi}$ is

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi(500/\pi)^{2/3}} = 2\sqrt[3]{\frac{500}{\pi}} = 2r$$

Thus, to minimize the cost of the can, the radius should be $\sqrt[3]{500/\pi}$ cm and the height should be equal to twice the radius, namely, the diameter. □


y
1000
0
10
r
y = A(r)

NOTE 2 An alternative method for solving optimization problems is to use implicit differentiation. Let's look at Example 2 again to illustrate the method. We work with the same equations

$$A = 2\pi r^2 + 2\pi r h \qquad \pi r^2 h = 100$$

but instead of eliminating h, we differentiate both equations implicitly with respect to r:

$$A' = 4\pi r + 2\pi h + 2\pi r h' \qquad 2\pi r h + \pi r^2 h' = 0$$

The minimum occurs at a critical number, so we set $A' = 0$, simplify, and arrive at the equations

$$2r + h + rh' = 0 \qquad 2h + rh' = 0$$

and subtraction gives $2r - h = 0$, or $h = 2r$.

FIRST DERIVATIVE TEST FOR ABSOLUTE EXTREME VALUES Suppose that c is a critical number of a continuous function f defined on an interval.

(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.

(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

EXAMPLE 3 Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

SOLUTION The distance between the point $(1, 4)$ and the point (x, y) is

$$d = \sqrt{(x - 1)^2 + (y - 4)^2}$$

(See Figure 6.) But if (x, y) lies on the parabola, then $x = \frac{1}{2}y^2$, so the expression for d becomes

$$d = \sqrt{\left(\tfrac{1}{2}y^2 - 1\right)^2 + (y - 4)^2}$$

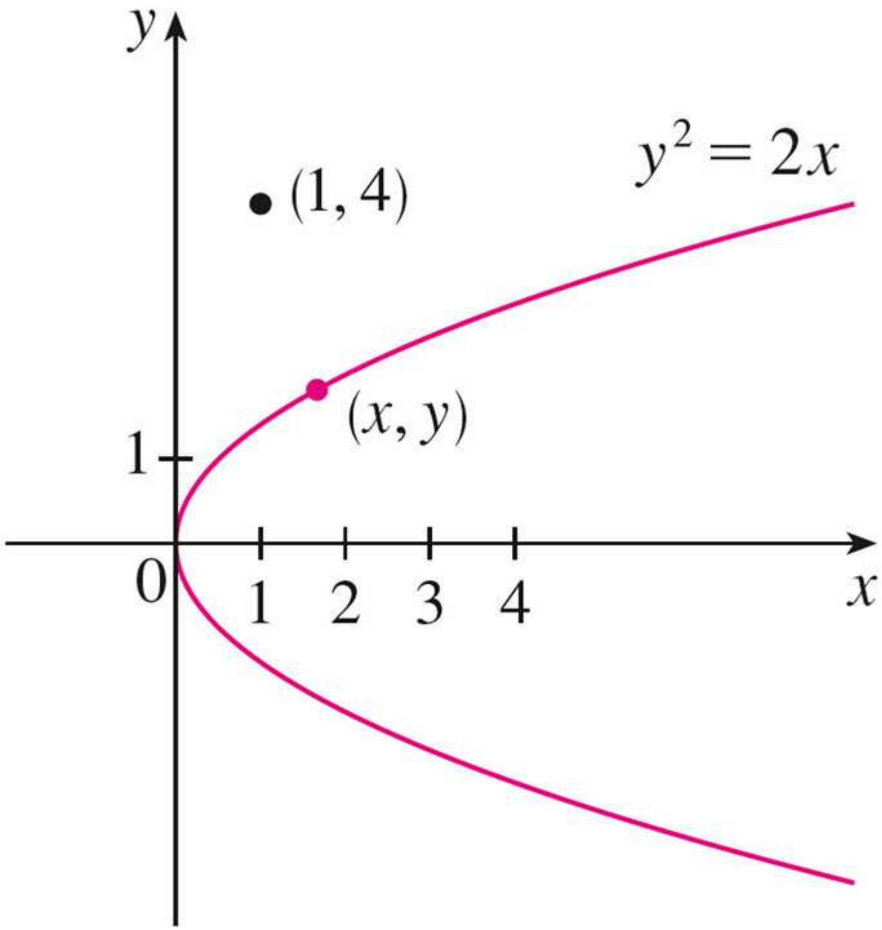
(Alternatively, we could have substituted $y = \sqrt{2x}$ to get d in terms of x alone.) Instead of minimizing d, we minimize its square:

$$d^2 = f(y) = \left(\tfrac{1}{2}y^2 - 1\right)^2 + (y - 4)^2$$

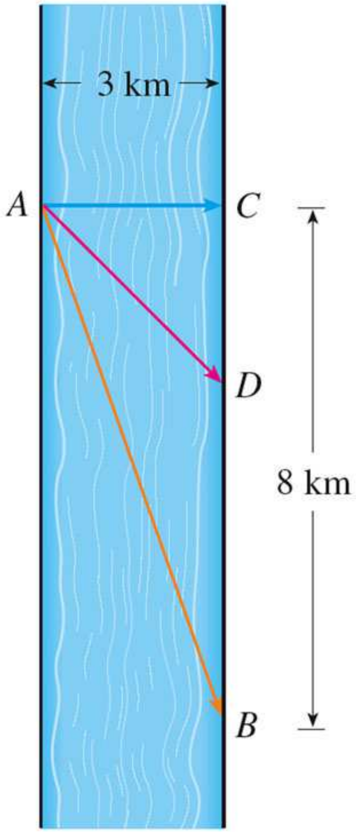
(You should convince yourself that the minimum of d occurs at the same point as the minimum of d^2, but d^2 is easier to work with.) Differentiating, we obtain

$$f'(y) = 2\left(\tfrac{1}{2}y^2 - 1\right)y + 2(y - 4) = y^3 - 8$$

so $f'(y) = 0$ when $y = 2$. Observe that $f'(y) < 0$ when $y < 2$ and $f'(y) > 0$ when $y > 2$, so by the First Derivative Test for Absolute Extreme Values, the absolute minimum occurs when $y = 2$. (Or we could simply say that because of the geometric nature of the problem, it's obvious that there is a closest point but not a farthest point.) The corresponding value of x is $x = \frac{1}{2}y^2 = 2$. Thus the point on $y^2 = 2x$ closest to $(1, 4)$ is $(2, 2)$. □


y
$y^2 = 2x$
(1, 4)
(x, y)
1
0
1 2 3 4
x

EXAMPLE 4 A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible (see Figure 7). He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point D between C and B and then run to B. If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible? (We assume that the speed of the water is negligible compared with the speed at which the man rows.)



SOLUTION If we let x be the distance from C to D, then the running distance is $|DB| = 8 - x$ and the Pythagorean Theorem gives the rowing distance as $|AD| = \sqrt{x^2 + 9}$. We use the equation

$$\text{time} = \frac{\text{distance}}{\text{rate}}$$

Then the rowing time is $\sqrt{x^2 + 9}/6$ and the running time is $(8 - x)/8$, so the total time T as a function of x is

$$T(x) = \frac{\sqrt{x^2 + 9}}{6} + \frac{8 - x}{8}$$

The domain of this function T is $[0, 8]$. Notice that if $x = 0$, he rows to C and if $x = 8$, he rows directly to B. The derivative of T is

$$T'(x) = \frac{x}{6\sqrt{x^2 + 9}} - \frac{1}{8}$$

Thus, using the fact that $x \geqslant 0$, we have

$$T'(x) = 0 \iff \frac{x}{6\sqrt{x^2 + 9}} = \frac{1}{8} \iff 4x = 3\sqrt{x^2 + 9}$$

$$\iff 16x^2 = 9(x^2 + 9) \iff 7x^2 = 81$$

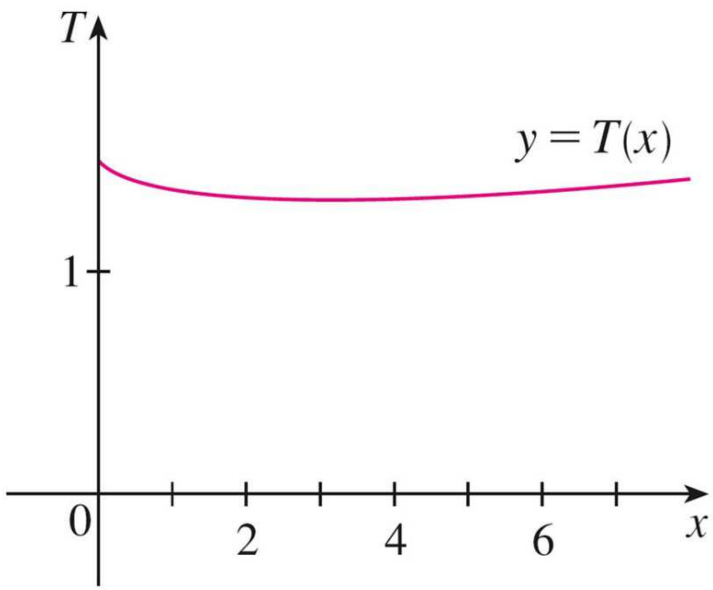
$$\iff x = \frac{9}{\sqrt{7}}$$

The only critical number is $x = 9/\sqrt{7}$. To see whether the minimum occurs at this criti cal number or at an endpoint of the domain [0, 8], we evaluate T at all three points:

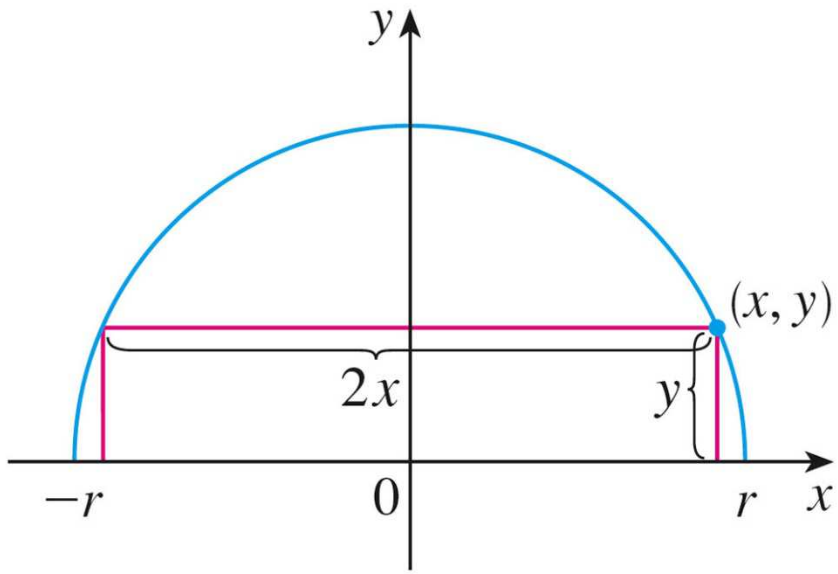
$$T(0) = 1.5 \qquad T\left(\frac{9}{\sqrt{7}}\right) = 1 + \frac{\sqrt{7}}{8} \approx 1.33 \qquad T(8) = \frac{\sqrt{73}}{6} \approx 1.42$$

Since the smallest of these values of T occurs when $x = 9/\sqrt{7}$, the absolute minimum value of T must occur there. Figure 8 illustrates this calculation by showing the graph of T.

Thus the man should land the boat at a point $9/\sqrt{7}$ km ($\approx$3.4 km) downstream fron his starting point.



V EXAMPLE 5 Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.



SOLUTION 1 Let's take the semicircle to be the upper half of the circle $x^2 + y^2 = r^2$ with center the origin. Then the word *inscribed* means that the rectangle has two vertices on the semicircle and two vertices on the x-axis as shown in Figure 9.

Let (x, y) be the vertex that lies in the first quadrant. Then the rectangle has sides of lengths $2x$ and y, so its area is

$$A = 2xy$$

To eliminate y we use the fact that (x, y) lies on the circle $x^2 + y^2 = r^2$ and so $y = \sqrt{r^2 - x^2}$. Thus

$$A = 2x\sqrt{r^2 - x^2}$$

The domain of this function is $0 \leqslant x \leqslant r$. Its derivative is

$$A' = 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}} = \frac{2(r^2 - 2x^2)}{\sqrt{r^2 - x^2}} .$$

which is 0 when $2x^2 = r^2$, that is, $x = r/\sqrt{2}$ (since $x \geqslant 0$). This value of x gives a maximum value of A since $A(0) = 0$ and $A(r) = 0$. Therefore the area of the largest inscribed rectangle is

$$A\left(\frac{r}{\sqrt{2}}\right) = 2\frac{r}{\sqrt{2}}\sqrt{r^2 - \frac{r^2}{2}} = r^2$$

SOLUTION 2 A simpler solution is possible if we think of using an angle as a variable. Let θ be the angle shown in Figure 10. Then the area of the rectangle is

$$A(\theta) = (2r\cos\theta)(r\sin\theta) = r^2(2\sin\theta\,\cos\theta) = r^2\sin 2\theta$$

We know that $\sin 2\theta$ has a maximum value of 1 and it occurs when $2\theta = \pi/2$. So $A(\theta)$ has a maximum value of r^2 and it occurs when $\theta = \pi/4$.

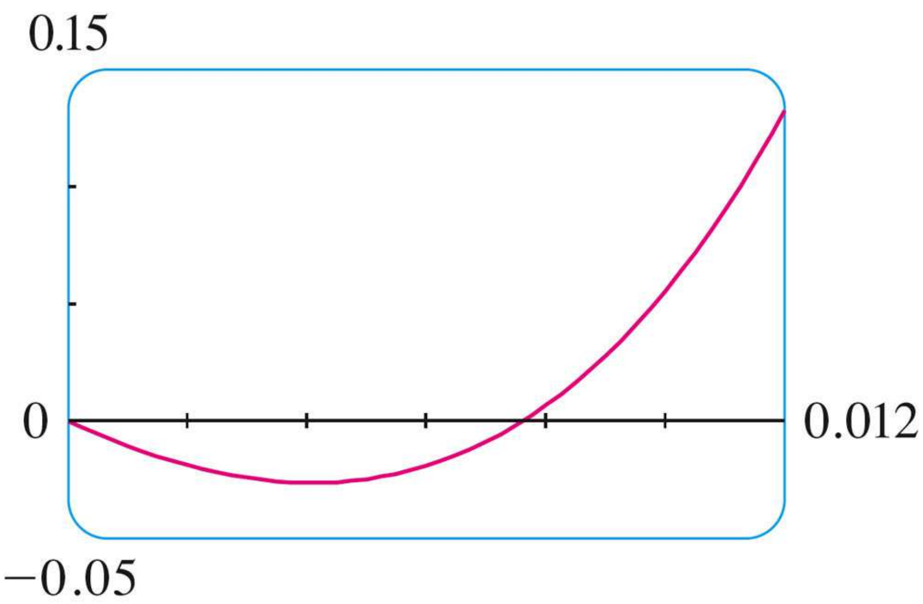
Notice that this trigonometric solution doesn't involve differentiation. In fact, we didn't need to use calculus at all. □

<u>Sec. 4.8</u> Newton's Method（略）

Suppose that a car dealer offers to sell you a car for \$18,000 or for payments of \$375 per month for five years. You would like to know what monthly interest rate the dealer is, in effect, charging you. To find the answer, you have to solve the equation

$$\boxed{1} \qquad 48x(1+x)^{60} - (1+x)^{60} + 1 = 0$$

(The details are explained in Exercise 39.) How would you solve such an equation?

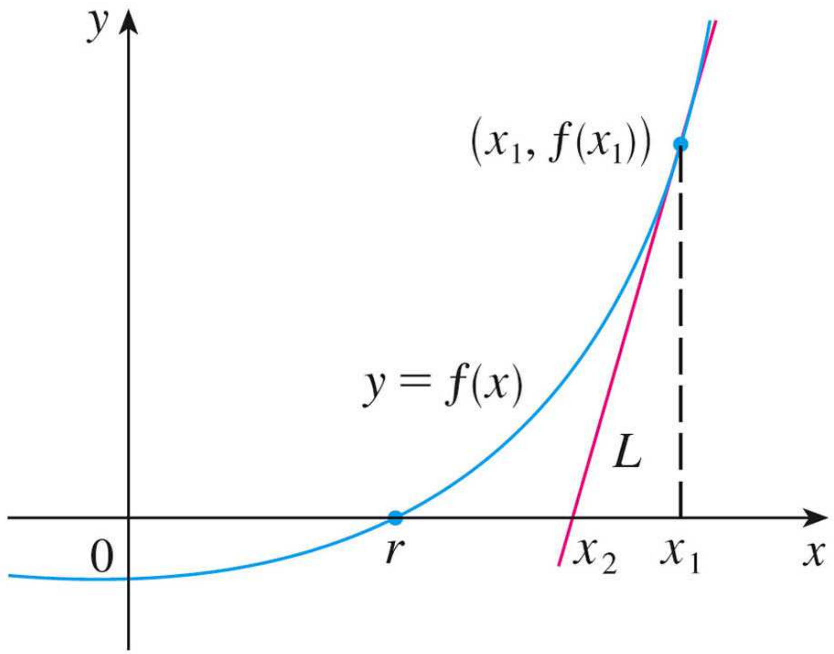


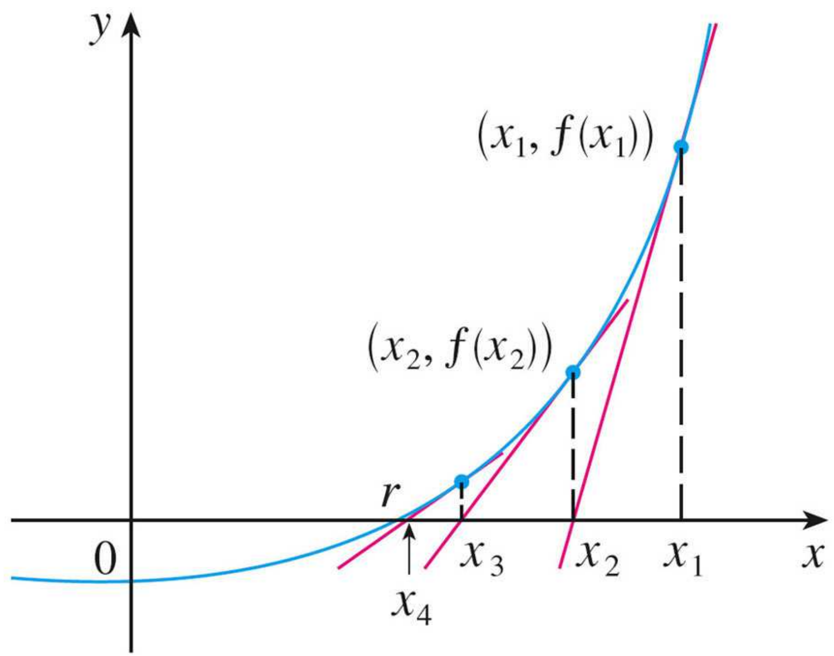
To find a formula for x_2 in terms of x_1 we use the fact that the slope of L is $f'(x_1)$, so its equation is

$$y - f(x_1) = f'(x_1)(x - x_1)$$

Since the x-intercept of L is x_2, we set $y = 0$ and obtain

$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$





If $f'(x_1) \neq 0$, we can solve this equation for x_2:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

We use x_2 as a second approximation to r.

Next we repeat this procedure with x_1 replaced by x_2, using the tangent line $(x_2, f(x_2))$. This gives a third approximation:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

If we keep repeating this process, we obtain a sequence of approximations x_1, x_2, x_3, x_4, as shown in Figure 3. In general, if the nth approximation is x_n and $f'(x_n) \neq 0$, then next approximation is given by

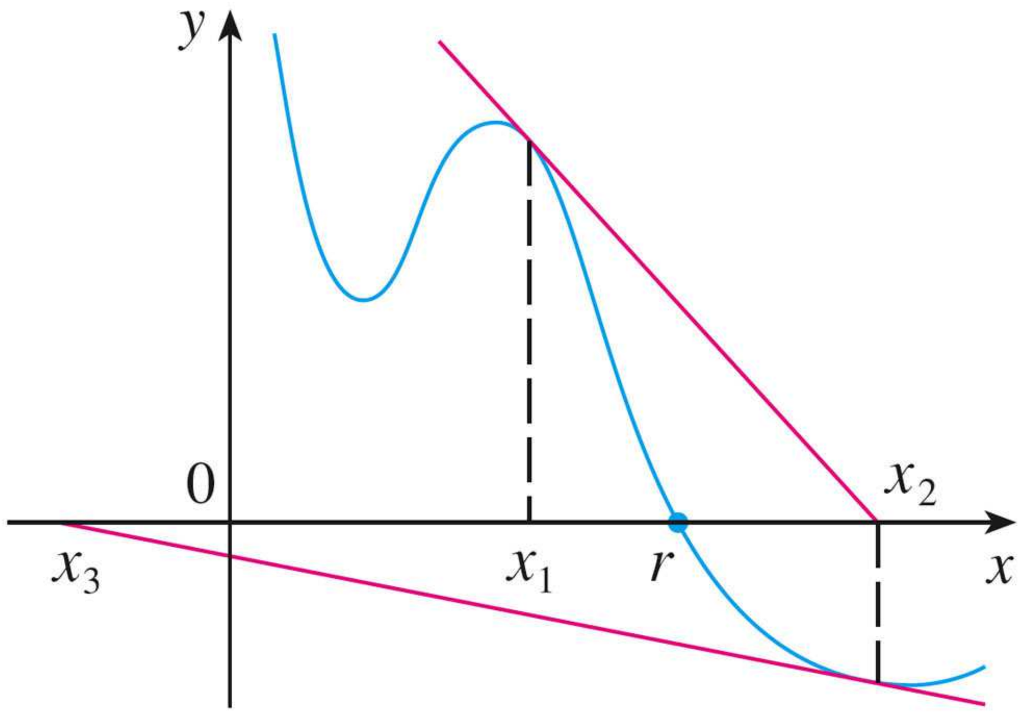
2

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

If the numbers x_n become closer and closer to r as n becomes large, then we say t the sequence *converges* to r and we write

$$\lim_{n \to \infty} x_n = r$$

若起始猜測點選擇的不好，找根可能會有困難。



Although the sequence of successive approximations converges to the desired root f functions of the type illustrated in Figure 3, in certain circumstances the sequence may n converge. For example, consider the situation shown in Figure 4. You can see that x_2 is worse approximation than x_1. This is likely to be the case when $f'(x_1)$ is close to 0. It mig even happen that an approximation (such as x_3 in Figure 4) falls outside the domain of Then Newton's method fails and a better initial approximation x_1 should be chosen. S Exercises 29–32 for specific examples in which Newton's method works very slowly does not work at all.

EXAMPLE 1 Starting with $x_1 = 2$, find the third approximation x_3 to the root of the equation $x^3 - 2x - 5 = 0$.

SOLUTION We apply Newton's method with

$$f(x) = x^3 - 2x - 5 \qquad \text{and} \qquad f'(x) = 3x^2 - 2$$

Newton himself used this equation to illustrate his method and he chose $x_1 = 2$ after some experimentation because $f(1) = -6$, $f(2) = -1$, and $f(3) = 16$. Equation 2 becomes

$$x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$$

With $n = 1$ we have

$$x_2 = x_1 - \frac{x_1^3 - 2x_1 - 5}{3x_1^2 - 2}$$

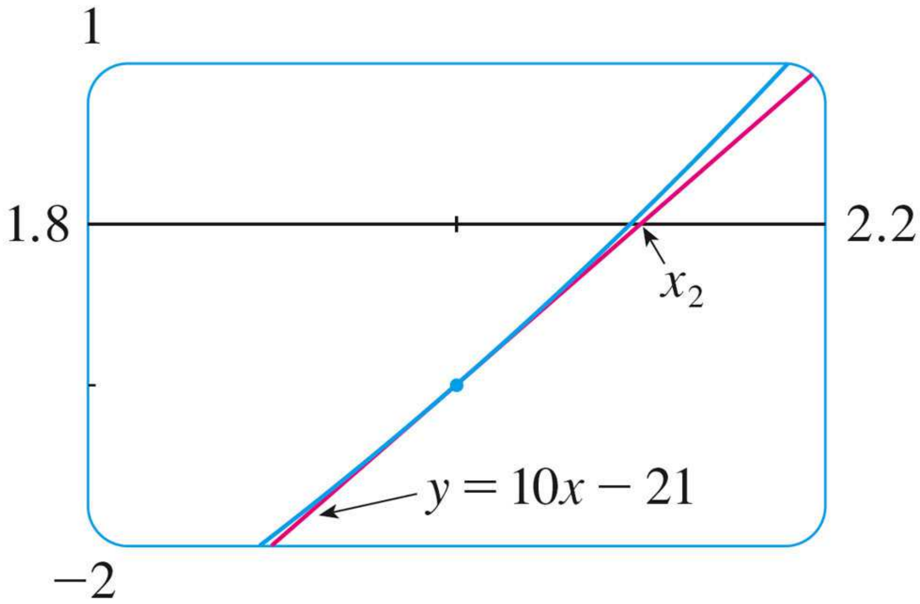
$$= 2 - \frac{2^3 - 2(2) - 5}{3(2)^2 - 2} = 2.1$$

Then with $n = 2$ we obtain

$$x_3 = x_2 - \frac{x_2^3 - 2x_2 - 5}{3x_2^2 - 2}$$

$$= 2.1 - \frac{(2.1)^3 - 2(2.1) - 5}{3(2.1)^2 - 2} \approx 2.0946$$

It turns out that this third approximation $x_3 \approx 2.0946$ is accurate to four decimal places.



V EXAMPLE 2 Use Newton's method to find $\sqrt[6]{2}$ correct to eight decimal places.

SOLUTION First we observe that finding $\sqrt[6]{2}$ is equivalent to finding the positive root of the equation

$$x^6 - 2 = 0$$

so we take $f(x) = x^6 - 2$. Then $f'(x) = 6x^5$ and Formula 2 (Newton's method) becomes

$$x_{n+1} = x_n - \frac{x_n^6 - 2}{6x_n^5}$$

If we choose $x_1 = 1$ as the initial approximation, then we obtain

$$x_2 \approx 1.16666667$$

$$x_3 \approx 1.12644368$$

$$x_4 \approx 1.12249707$$

$$x_5 \approx 1.12246205$$

$$x_6 \approx 1.12246205$$

Since x_5 and x_6 agree to eight decimal places, we conclude that

$$\sqrt[6]{2} \approx 1.12246205$$

to eight decimal places. □

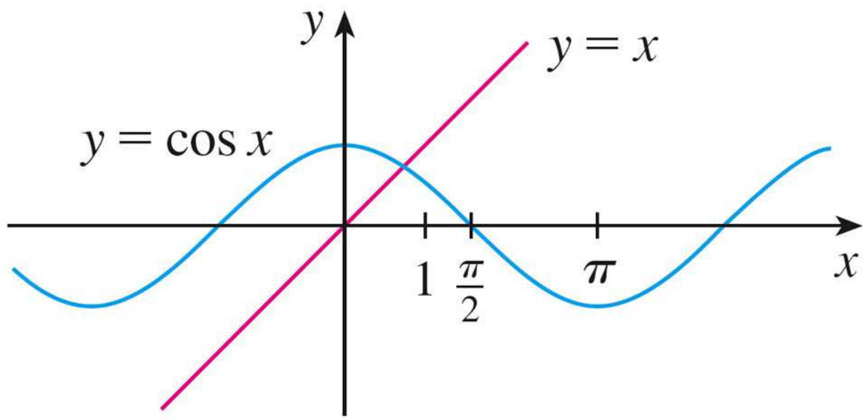
EXAMPLE 3 Find, correct to six decimal places, the root of the equation $\cos x = x$.

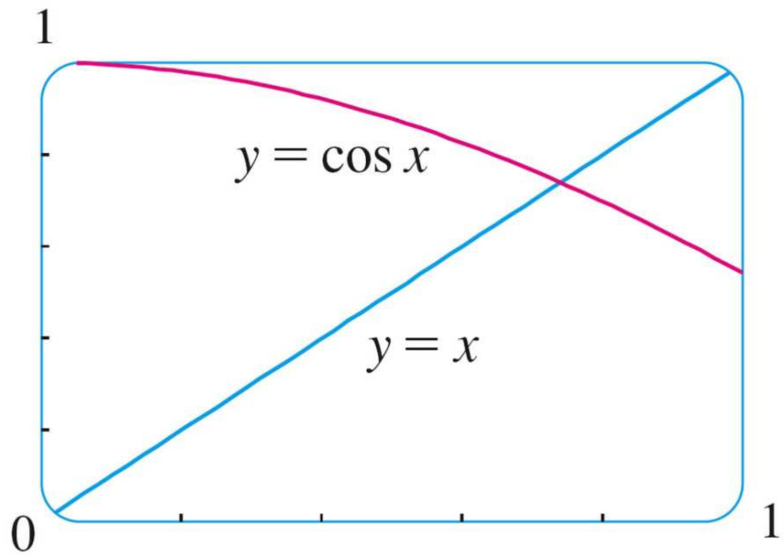
SOLUTION We first rewrite the equation in standard form:

$$\cos x - x = 0$$

Therefore we let $f(x) = \cos x - x$. Then $f'(x) = -\sin x - 1$, so Formula 2 becomes

$$x_{n+1} = x_n - \frac{\cos x_n - x_n}{-\sin x_n - 1} = x_n + \frac{\cos x_n - x_n}{\sin x_n + 1}$$





In order to guess a suitable value for x_1 we sketch the graphs of $y = \cos x$ and $y = x$ in Figure 6. It appears that they intersect at a point whose x-coordinate is somewhat less than 1, so let's take $x_1 = 1$ as a convenient first approximation. Then, remembering to put our calculator in radian mode, we get

$$x_2 \approx 0.75036387$$

$$x_3 \approx 0.73911289$$

$$x_4 \approx 0.73908513$$

$$x_5 \approx 0.73908513$$

Since x_4 and x_5 agree to six decimal places (eight, in fact), we conclude that the root of the equation, correct to six decimal places, is 0.739085.

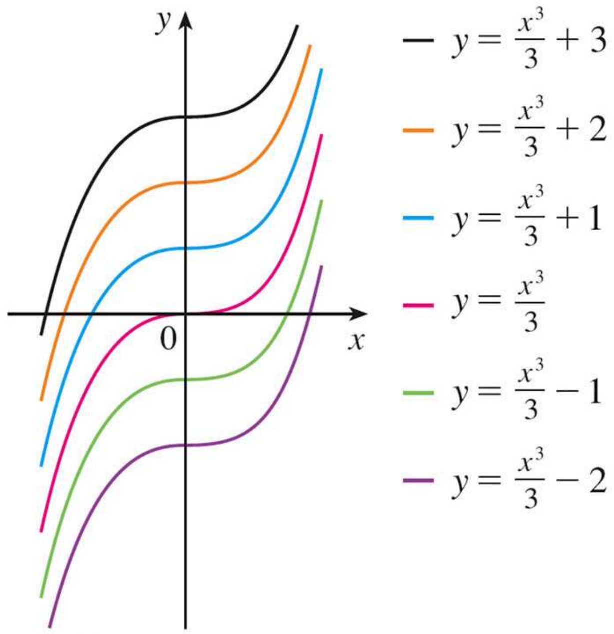
Sec. 4.9 Antiderivatives

DEFINITION **A function F is called an antiderivative of f on an interval I if $F'(x)=f(x)$ for all x in I.**

1 THEOREM If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.



EXAMPLE 1 Find the most general antiderivative of each of the following functions.
(a) $f(x) = \sin x$ (b) $f(x) = x^n, \quad n \geqslant 0$ (c) $f(x) = x^{-3}$

SOLUTION
(a) If $F(x) = -\cos x$, then $F'(x) = \sin x$, so an antiderivative of $\sin x$ is $-\cos x$. By Theorem 1, the most general antiderivative is $G(x) = -\cos x + C$.

(b) We use the Power Rule to discover an antiderivative of x^n:

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{(n+1)x^n}{n+1} = x^n$$

Thus the general antiderivative of $f(x) = x^n$ is

$$F(x) = \frac{x^{n+1}}{n+1} + C$$

This is valid for $n \geqslant 0$ because then $f(x) = x^n$ is defined on an interval.

(c) If we put $n = -3$ in part (b) we get the particular antiderivative $F(x) = x^{-2}/(-2)$ by the same calculation. But notice that $f(x) = x^{-3}$ is not defined at $x = 0$. Thus Theorem 1 tells us only that the general antiderivative of f is $x^{-2}/(-2) + C$ on any interval that does not contain 0. So the general antiderivative of $f(x) = 1/x^3$ is

$$F(x) = \begin{cases} -\dfrac{1}{2x^2} + C_1 & \text{if } x > 0 \\[2ex] -\dfrac{1}{2x^2} + C_2 & \text{if } x < 0 \end{cases}$$

□

EXAMPLE 2 Find all functions g such that

$$g'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}\ .$$

SOLUTION We first rewrite the given function as follows:

$$g'(x) = 4\sin x + \frac{2x^5}{x} - \frac{\sqrt{x}}{x} = 4\sin x + 2x^4 - \frac{1}{\sqrt{x}}$$

Thus we want to find an antiderivative of

$$g'(x) = 4\sin x + 2x^4 - x^{-1/2}$$

Using the formulas in Table 2 together with Theorem 1, we obtain

$$\begin{aligned} g(x) &= 4(-\cos x) + 2\frac{x^5}{5} - \frac{x^{1/2}}{\frac{1}{2}} + C \\ &= -4\cos x + \tfrac{2}{5}x^5 - 2\sqrt{x} + C \end{aligned}$$

EXAMPLE 3 Find f if $f'(x) = x\sqrt{x}$ and $f(1) = 2$.

SOLUTION The general antiderivative of

$$f'(x) = x^{3/2}$$

is

$$f(x) = \frac{x^{5/2}}{\frac{5}{2}} + C = \tfrac{2}{5}x^{5/2} + C$$

To determine C we use the fact that $f(1) = 2$:

$$f(1) = \tfrac{2}{5} + C = 2$$

Solving for C, we get $C = 2 - \frac{2}{5} = \frac{8}{5}$, so the particular solution is

$$f(x) = \frac{2x^{5/2} + 8}{5}$$

EXAMPLE 4 Find f if $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$, and $f(1) = 1$.

SOLUTION The general antiderivative of $f''(x) = 12x^2 + 6x - 4$ is

$$f'(x) = 12\frac{x^3}{3} + 6\frac{x^2}{2} - 4x + C = 4x^3 + 3x^2 - 4x + C$$

Using the antidifferentiation rules once more, we find that

$$f(x) = 4\frac{x^4}{4} + 3\frac{x^3}{3} - 4\frac{x^2}{2} + Cx + D = x^4 + x^3 - 2x^2 + Cx + D$$

To determine C and D we use the given conditions that $f(0) = 4$ and $f(1) = 1$. Since $f(0) = 0 + D = 4$, we have $D = 4$. Since

$$f(1) = 1 + 1 - 2 + C + 4 = 1$$

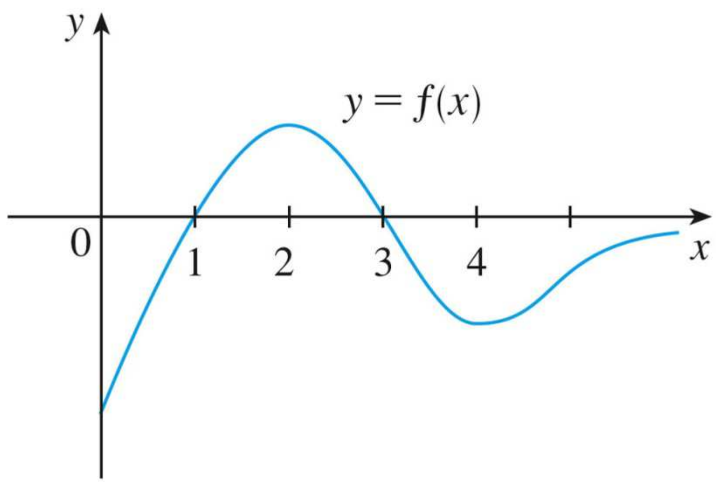
we have $C = -3$. Therefore the required function is

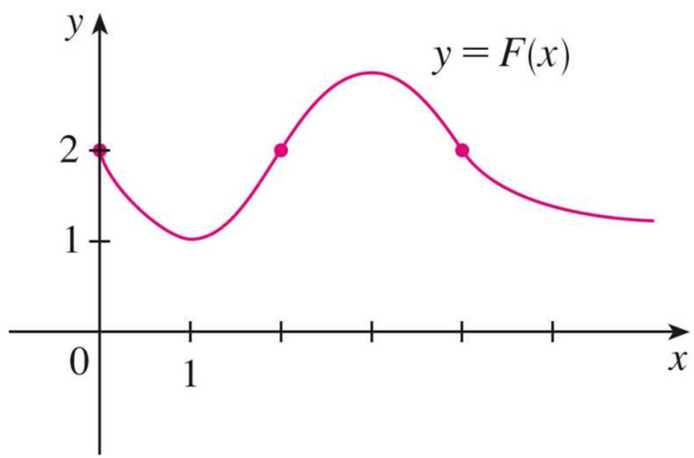
$$f(x) = x^4 + x^3 - 2x^2 - 3x + 4$$

□

EXAMPLE 5 The graph of a function f is given in Figure 2. Make a rough sketch of an antiderivative F, given that $F(0) = 2$.

SOLUTION We are guided by the fact that the slope of $y = F(x)$ is $f(x)$. We start at the point $(0, 2)$ and draw F as an initially decreasing function since $f(x)$ is negative when $0 < x < 1$. Notice that $f(1) = f(3) = 0$, so F has horizontal tangents when $x = 1$ and $x = 3$. For $1 < x < 3$, $f(x)$ is positive and so F is increasing. We see that F has a local minimum when $x = 1$ and a local maximum when $x = 3$. For $x > 3$, $f(x)$ is negative and so F is decreasing on $(3, \infty)$. Since $f(x) \to 0$ as $x \to \infty$, the graph of F becomes flatter as $x \to \infty$. Also notice that $F''(x) = f'(x)$ changes from positive to negative at $x = 2$ and from negative to positive at $x = 4$, so F has inflection points when $x = 2$ and $x = 4$. We use this information to sketch the graph of the antiderivative in Figure 3. ☐





EXAMPLE 6 A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

SOLUTION Since $v'(t) = a(t) = 6t + 4$, antidifferentiation gives

$$v(t) = 6\frac{t^2}{2} + 4t + C = 3t^2 + 4t + C$$

Note that $v(0) = C$. But we are given that $v(0) = -6$, so $C = -6$ and

$$v(t) = 3t^2 + 4t - 6$$

Since $v(t) = s'(t)$, s is the antiderivative of v:

$$s(t) = 3\frac{t^3}{3} + 4\frac{t^2}{2} - 6t + D = t^3 + 2t^2 - 6t + D$$

This gives $s(0) = D$. We are given that $s(0) = 9$, so $D = 9$ and the required position function is

$$s(t) = t^3 + 2t^2 - 6t + 9$$

EXAMPLE 7 A ball is thrown upward with a speed of 15 m/s from the edge of a cliff 140 m above the ground. Find its height above the ground t seconds later. When does it reach its maximum height? When does it hit the ground?

SOLUTION The motion is vertical and we choose the positive direction to be upward. At time t the distance above the ground is $s(t)$ and the velocity $v(t)$ is decreasing. Therefore the acceleration must be negative and we have

$$a(t) = \frac{dv}{dt} = -9.8$$

Taking antiderivatives, we have

$$v(t) = -9.8t + C$$

To determine C we use the given information that $v(0) = 15$. This gives $15 = 0 + C$, so

$$v(t) = -9.8t + 15$$

The maximum height is reached when $v(t) = 0$, that is, after $15/9.8 \approx 1.53$ s. Since $s'(t) = v(t)$, we antidifferentiate again and obtain

$$s(t) = -4.9t^2 + 15t + D$$

Using the fact that $s(0) = 140$, we have $140 = 0 + D$ and so

$$s(t) = -4.9t^2 + 15t + 140$$

The expression for $s(t)$ is valid until the ball hits the ground. This happens when $s(t) = 0$, that is, when

$$-4.9t^2 - 15t - 140 = 0$$

Using the quadratic formula to solve this equation, we get

$$t = \frac{15 \pm \sqrt{2969}}{9.8}$$

We reject the solution with the minus sign since it gives a negative value for t. Therefore the ball hits the ground after

$$\frac{15 + \sqrt{2969}}{9.8} \approx 7.1 \text{ s} \qquad \square$$