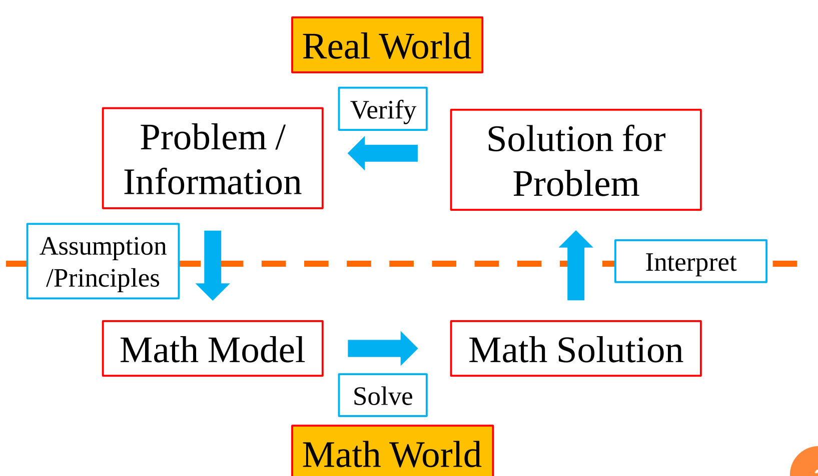
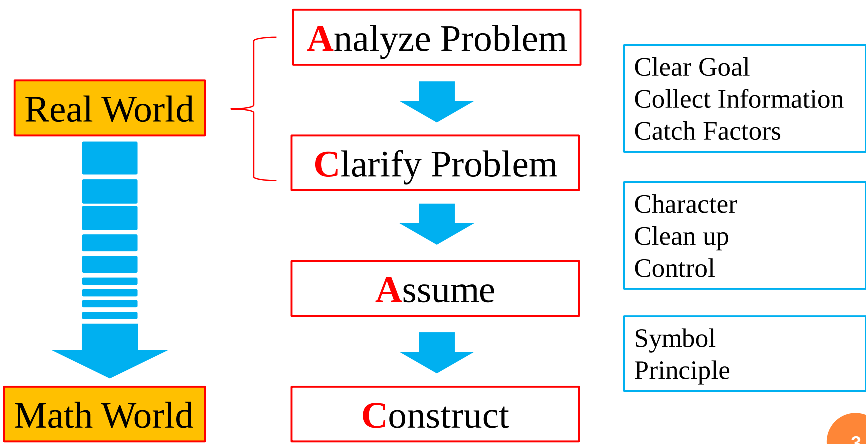
Introduction of Ordinary Differential Equation

Why We have to learn O.D.E.?



How to Build a Math Model?



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What is O.D.E.?

- In practice, it is hard to find the direct relationship between the variables, but it might be easier to find the relationship between the change of variables.
- For example:

A soup with an initial temperature of 150℃ was placed on a table in the environment temperature of 25 ℃ . If the temperature should be below 55 ℃ in order to drink it, then how long do you have to wait?

$$\Delta u = u(t+\Delta t) - u(t) = -k(u(t+\Delta t) - u_0) \times \Delta t$$

$$\Rightarrow \frac{du}{dt} = -k(u - u_0)$$

Most of Math Models are in the form of O.D.E.

1^{st} Order O.D.E.

$$O.D.E = Ordinary\ \ Differential\ \ Equation$$

常　　　微分　　　方程

$D.E$(微分方程)

方程式中描述未知函數與其導函數及自變數之關係
此方程稱微分方程

Ex：牛頓第二運動定律 $F = ma = m\dfrac{d^2x}{dt^2}$

◎ *Ordinary D.E. vs Partial D.E.*

O.D.E. ∶ *A D.E. contains ordinary derivative with respect to a single independent variable*

Ex: $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = 0, \quad \text{and} \quad \frac{dx}{dt} + \frac{dy}{dt} = 3x + 2y$

P.D.E. ∶ *A D.E. contains partial derivatives with respect to 2 or more independent variables*

Ex: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial t}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$

◎ Order

The order of D.E. is defined as the order of highest derivative

Ex: $3\frac{d^2x}{dt^2} + 10x = t$

$$3\frac{d^2x}{dt^2} + 10x^4 = t^5$$

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$$

◎ *Linear v.s. Non-Linear D.E.*

A nth-order O. D. E. is said to be linear in the variable y if the O.D.E. is expressed in the form

$$a_0(x)y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + a_2(x)y^{(n-2)}(x) + \cdots + a_n(x)y(x) = f(x)$$

where $a_0(x)$, $a_1(x)$, …, $a_n(x)$ are function of independent variable x alone

$$Ex: y'' + qy = 0$$

$$\left(x^2 + y^2\right)y' = 1$$

$$\theta'' + 3\sin\theta = 0$$

◎*Homogeneous*

If $f(x) = 0$, an n^{th} - order linear O.D.E. is homogeneous

If $f(x) \neq 0$, the linear O.D.E. is non-homogeneous

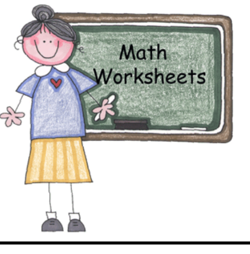
Ex:

$$m\frac{d^2x}{dt^2} + kx = F(t) \quad \text{(Mass-Spring System)}$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}(sin\theta) = 0 \quad \text{(Pandulum)}$$

◎ *Solution of D.E.*

A function is said to be a solution of a D.E. over a particular domain of the independent variable, if its substitution into the equation reduces that equation to an identity everywhere within that domain.



$$Ex: y''+y=2\sin x$$

$$y(x)=4\sin x - x\cos x$$

$$y''(x)=-4\sin x+2\sin x+x\cos x$$

$$y(x)=A\sin x+B\cos x - x\cos x$$

→ **General Solution** $of\ y''+y=2\ sinx$

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◎ Trivial solution

$$y''-2y'+y=0;\ y=xe^x$$

Left-hand side

$$y''-2y'+y=(xe^x+2e^x)-2(xe^x+e^x)+xe^x=0$$

$y=0$ is also the solution of the O.D.E., called **trivial solution.**

- **Families of Solutions**
  A solution containing an arbitrary constant represents a set $G(x, y) = 0$ of solutions is called a **one-parameter family of solutions.** A set $G(x, y, c_1, c_2, \ldots, c_n) = 0$ of solutions is called a ***n*-parameter family of solutions.**
- **Particular Solution**
  A solution free of arbitrary parameters. eg: $y = cx - x \cos x$ is a solution of $xy' - y = x^2 \sin x$ on $(-\infty, \infty)$, $y = x \cos x$ is a particular solution according to $c = 0$.

- **Singular Solution**
  A solution can not be obtained by particularly setting any parameters.

$$\frac{dy}{dx} = (y - x)^{2/3} + 1$$

$$Let\ u = y - x \ \Rightarrow \frac{du}{dx} = u^{\frac{2}{3}}$$

The general solution is

$$y - x = \frac{1}{27}(x + c)^3$$

However, $y = x$, *is also a solution.*

◎ *Differential Equation of 1st -Order*

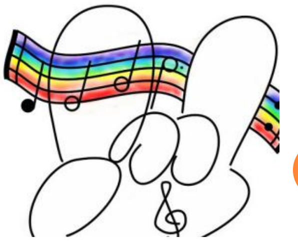
General First-Order O.D.E.

$$F\ (x, y, y') = 0$$

where x , y are independent & dependent variables , respectively .

4 subclasses are included in this chapter

1. *Separable equation*
2. *Exact equation*
3. *Linear equation*
4. *Others*



◎ Separable Equation

If $F\,(x, y, y') = 0$ *can be expressed as* $y' = X(x)\,Y(y)$

then we say the O.D.E. is separable

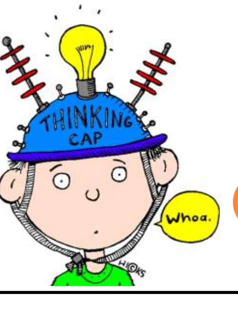
Note: 1. On the right side of D.E. , a function of x times a function of y

2. Only y′ is on the left side of D.E.

EX :

$$y' = xe^{x+2y} = [xe^{x}][e^{2y}]$$

$$y' = 3x - y$$



$$y' = X(x)Y(y) \Rightarrow \frac{dy}{dx} = X(x)Y(y)$$

$$\Rightarrow \frac{1}{Y(y)}dy = X(x)dx$$

$$\Rightarrow \int \frac{1}{Y(y)}dy = \int X(x)dx \longrightarrow$$ *General Solution of* $y' = X(x)Y(y)$

Ex

$$y' = -y^2$$

We want to obtain general solution $y(x)$

1. *Observe*: 1st – *order*, *Separable*

2. $\frac{dy}{dx} = -y^2 \Rightarrow \frac{dy}{dx} = [-1]\left[y^2\right]$

$$\frac{dy}{y^2} = (-1)\,dx \Rightarrow \int \frac{1}{y^2}dy = \int (-1)dx$$

$$\Rightarrow \frac{1}{y} + c_1 = x + c_2$$

$$\Rightarrow \frac{1}{y} = x + c_2 - c_1$$

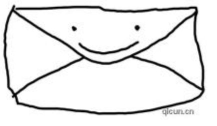
$$\Rightarrow \frac{1}{y} = x + c^* \quad (c_2 - c_1 = c^*)$$

$$\Rightarrow y = \frac{1}{x + c^*}$$

$$Ex: y' = \frac{4x}{1+2e^y} = [4x]\left[\frac{1}{1+2e^y}\right]$$

$$\frac{dy}{dx} = [4x]\left[\frac{1}{1+2e^y}\right]$$

$$\Rightarrow y + 2e^y = 2x^2 + c$$



◎ Losing Solution

$y' = y^2 - 4$

Solution: Rewrite this DE as

$$\frac{dy}{y^2-4} = dx \quad \text{or} \quad \left[\frac{\frac{1}{4}}{y-2} - \frac{\frac{1}{4}}{y+2}\right]dy = dx$$

$$\frac{1}{4}\ln|y-2| - \frac{1}{4}\ln|y+2| = x + c_1, \quad \frac{y-2}{y+2} = e^{4x+c_2}$$

$$y = 2\frac{1+ce^{4x}}{1-ce^{4x}}$$

If we rewrite the DE as $dy/dx = (y+2)(y-2)$, from the previous discussion, we have $y = \pm 2$ is a singular solution.

◎ **D.E. with Coefficients Homogeneous of Degree Zero :**

$$y' = f\left(\frac{y}{x}\right)$$

$$y' = \frac{y}{x} + 3\sqrt{\frac{x}{y}}$$

Ex: $1. Observe: 1^{st}-order, Non-Linear,$

$Homo\ of\ Degree\ Zero$

$$2. Let\ \frac{y}{x} = \upsilon \rightarrow dy = \upsilon dx + x d\upsilon$$

$$y' = \upsilon + 3\sqrt{\frac{1}{\upsilon}} \Rightarrow \frac{dy}{dx} = \upsilon + 3\sqrt{\frac{1}{\upsilon}}$$

$$\Rightarrow \frac{\upsilon dx + x d\upsilon}{dx} = \upsilon + 3\sqrt{\frac{1}{\upsilon}}$$

$$\Rightarrow \upsilon + x\frac{d\upsilon}{dx} = \upsilon + 3\sqrt{\frac{1}{\upsilon}}$$

$$\Rightarrow x\frac{d\upsilon}{dx} = 3\sqrt{\frac{1}{\upsilon}}$$

$$\Rightarrow \sqrt{\upsilon}\, d\upsilon = \frac{3}{x} dx$$

$$\Rightarrow \frac{2}{3}\upsilon^{\frac{3}{2}} = 3\ln x + c$$

$$\Rightarrow \frac{2}{3}\left(\frac{y}{x}\right)^{\frac{3}{2}} = 3\ln x + c$$

$$\Rightarrow y = x\left(\frac{9}{2}\ln x + \frac{3}{2}c\right)^{\frac{2}{3}}$$

$$= x\left(\frac{9}{2}\ln x + k\right)^{\frac{2}{3}}$$

◎ *Almost-Homogeneous Equations :*

$$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2} \qquad (a_1, a_2, b_1, b_2, c_1, c_2 \text{ are constants})$$

1. If $C_1, C_2 = 0$ ⟹ Homogeneous Equations

$$y' = \frac{a_1 x + b_1 y}{a_2 x + b_2 y} = \frac{a_1 + b_1 (\frac{y}{x})}{a_2 + b_2 (\frac{y}{x})} = f(\frac{y}{x})$$

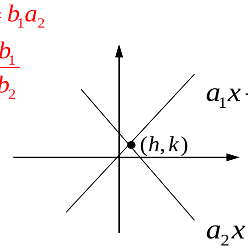
2. If C_1 or $C_2 \neq 0$

$$\begin{cases} a_1 b_2 - b_1 a_2 \neq 0 \\ a_1 b_2 - b_1 a_2 = 0 \end{cases}$$

(2-1) $a_1 b_2 - b_1 a_2 \neq 0$ $\Rightarrow a_1 b_2 \neq b_1 a_2$

$\Rightarrow \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

$$\begin{cases} a_1 x + b_1 y + c_1 = 0 \\ a_2 x + b_2 y + c_2 = 0 \end{cases}$$



⟹ Solve (x , y)=(h , k)

Let

$$x = u + h, y = v + k \Rightarrow dx = du, dy = dv$$

$$\Rightarrow a_1(u+h) + b_1(v+k) + c_1 \Rightarrow a_1 u + b_1 v + \underbrace{(a_1 h + b_1 k + c_1)}_{\Downarrow\ 0}$$

$$\frac{dv}{du} = \frac{a_1 u + b_1 v}{a_2 u + b_2 v}$$

$$\frac{dy}{dx} = y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2} \Rightarrow \frac{dv}{du} = \frac{a_1 u + b_1 v}{a_2 u + b_2 v}$$

(2-2) $a_1b_2 - b_1a_2 = 0$ & $a_1c_2 - a_2c_1 \neq 0 \Rightarrow$ *非重合之平行線*

$$\begin{cases} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \end{cases} \qquad \text{Let} \quad \begin{matrix} a_1x + b_1y = u \\ a_2x + b_2y = mu \end{matrix}$$

$$\Rightarrow y = \frac{u - a_1x}{b_1}, dy = \frac{du - a_1dx}{b_1}$$

$$y' = \frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} \Rightarrow (a_1x + b_1y + c_1)dx - (a_2x + b_2y + c_2)dy = 0$$

$$\Rightarrow (u + c_1)dx - (mu + c_2)\left(\frac{du - a_1dx}{b_1}\right) = 0$$

$$\Rightarrow b_1(u + c_1)dx - (mu + c_2)(du - a_1dx) = 0$$

$$\Rightarrow dx = \frac{mu + c_2}{b_1(u + c_1) + a_1(mu + c_2)}du$$

Ex : $y' = \dfrac{3x - y - 6}{x + y + 2}$

1.$Observe : 1^{st} - order, Almost - Home$

$$a_1b_2 - a_2b_1 \neq 0$$

2. $\begin{cases} 3x - y - 6 = 0 \\ x + y + 2 = 0 \end{cases} \Rightarrow (x, y) = (1, -3)$

3.$Let \ \ x = u + 1 \ , \ y = \upsilon - 3 \ \Rightarrow dx = du \ , \ dy = d\upsilon$

$$\therefore y' = \frac{dy}{dx} = \frac{d\upsilon}{du} = \frac{3u - \upsilon}{u + \upsilon}$$

4.$Let \dfrac{\upsilon}{u} = z \ \Rightarrow \ d\upsilon = zdu + udz$

Exact Equation

$$y' = -\frac{M(x,y)}{N(x,y)}$$

Test of exactness:

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Ex: $y' = \frac{3x+y}{1}$

Ex: $y' = 3x$

$$M(x,y) = 3x \qquad \frac{\partial M}{\partial y} = 0 \qquad F(x,y) \Rightarrow \begin{cases} \frac{\partial F}{\partial x} = M = 3x \\ \frac{\partial F}{\partial y} = N = -1 \end{cases}$$

$$N(x,y) = -1 \qquad \frac{\partial N}{\partial x} = 0$$

$$\frac{\partial F}{\partial x} = M = 3x \qquad \frac{\partial F}{\partial y} = N = -1$$

$$\Rightarrow F(x,y) = \frac{3}{2}x^2 + A(y) \qquad \frac{\partial F}{\partial y} = A'(y) = -1$$

$$\Rightarrow F(x,y) = \frac{3}{2}x^2 - y + c \qquad A(y) = -y + c$$

$$\Rightarrow F(x,y) = k \Rightarrow \frac{3}{2}x^2 - y = B = k - c$$

$y' = 3x$

1.Observe:

2.Separable:

◎ ***Integrating Factor***

If M&N fail to satisfy $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

the O.D.E. $M(x,y)dx + N(x,y)dy = 0$ *is not an exact equation.*

It may be possible to find a multplicative factor $\sigma(x,y)$*,so that*

$$\sigma(x,y)M(x,y)dx + \sigma(x,y)N(x,y)dy = 0$$

is an exact equation.

How do we find σ?

If we can find a σ(x,y) satisfying

$$\frac{\partial(\sigma M)}{\partial y}=\frac{\partial(\sigma N)}{\partial x}$$

them we call σ(x,y) an integrating factor of M(x,y)dx+N(x,y)dy=0

***Note**: σ is easy to find in some special cases*

From $\dfrac{\partial(\sigma M)}{\partial y}=\dfrac{\partial(\sigma N)}{\partial x}$

we can re-express it as

$$\frac{\partial\sigma}{\partial y}M+\sigma\frac{\partial M}{\partial y}=\frac{\partial\sigma}{\partial x}N+\sigma\frac{\partial N}{\partial x}$$

$$\Rightarrow N\frac{\partial\sigma}{\partial x}-M\frac{\partial\sigma}{\partial y}=\sigma\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$$

EX. $y' = \dfrac{1}{e^{-2y} - 3x}$

1.Observe ：1st-order , non-separable

2.Check Exactness :

$$dx + (3x - e^{-2y})dy = 0$$

$$\text{we know } \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

3. Integrating Factor σ

$$\left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\right) = \frac{0-3}{3x - e^{-2y}}$$ *(Not function of x only)*

$$\left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M}\right) = \frac{0-3}{-1} = 3$$ *(function of y only)*

$$\sigma = e^{\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} dy} = e^{3y}$$ *(function of y only)*

$$4.\ e^{3y}dx + e^{3y}(3x - e^{-2y})dy = 0$$

$$\Rightarrow \frac{\partial F}{\partial x} = e^{3y},\ \frac{\partial F}{\partial y} = e^{3y}(3x - e^{-2y})$$

$$\frac{\partial F}{\partial x} = e^{3y} \Rightarrow F(x,y) = xe^{3y} + A(y)$$

$$\frac{\partial F}{\partial y} = e^{3y}(3x - e^{-2y})$$

$$\Rightarrow \frac{\partial}{\partial y}(xe^{3y} + A(y)) = 3xe^{3y} + A'(y) = e^{3y}(3x - e^{-2y})$$

$$\Rightarrow A'(y) = -e^{y} \Rightarrow A(y) = -e^{y} + B$$

$$\Rightarrow F(x,y) = xe^{3y} - e^{y} + B$$

The general solution of the O.D.E is

$$F(x,y) = xe^{3y} - e^{y} + B = k \Rightarrow xe^{3y} - e^{y} = c \rightarrow c = k - B$$

◎ *First-order Linear Equation*

A general 1st –order Linear O.D.E

$$a_0(x)y' + a_1(x)y = f(x)$$

Dividing through by $a_0(x)$, we can represent the linear O.D.E. in the move concise form

$$y' + p(x)y = q(x)$$

we assume that $p(x)$ & $q(x)$ are continuous over the x interval of interest

Homogeneous case ∶ q(x)=0

$$y' + p(x)y = 0$$

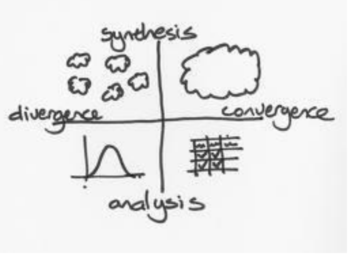
1. *Observe ∶ 1st-order ,separable*

$$y' = -p(x)y$$

2. *Solve*

$$\frac{dy}{dx} = -p(x)y \Rightarrow \frac{dy}{y} = -p(x)dx$$

$$\Rightarrow y = Ae^{\int -p(x)dx}$$



General Case : $q(x) \neq 0$

$$y' + p(x)y = q(x)$$

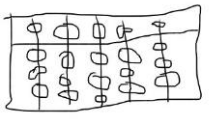
Method 1 : Integrating Factor

Method 2 : Variation of parameter

Method 1: Integrating Factor

$$\frac{dy}{dx}+p(x)y=q(x)\Rightarrow\left[p(x)y-q(x)\right]dx+dy=0$$

$$\therefore M(x,y)= \qquad N(x,y)=$$



$$\frac{\partial M}{\partial y}=p(x)\neq\frac{\partial N}{\partial x}=0$$

Check

$$\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=p(x) \qquad \text{(function of } x \text{ only)}$$

$$\therefore \sigma(x)=e^{\int p(x)dx}$$

$$F(x,y)=C \longleftarrow \text{General Solution}$$

$$\begin{cases}\dfrac{\partial F}{\partial x}=\sigma(x)\left[p(x)y-q(x)\right]=e^{\int p(x)dx}\left[p(x)y-q(x)\right]\\[2ex]\dfrac{\partial F}{\partial y}=\sigma(x)=e^{\int p(x)dx}\end{cases}$$

Note $\Rightarrow F(x,y)=ye^{\int p(x)dx}-\int q(x)e^{\int p(x)dx}dx+A(y)$

$$\frac{\partial F}{\partial x}=yp(x)e^{\int p(x)dx}-q(x)e^{\int p(x)dx}$$

$$\frac{\partial F}{\partial y} = e^{\int p(x)dx} \rightarrow F(x,y) = ye^{\int p(x)dx} + B(x)$$

$$\begin{cases} F(x,y) = ye^{\int p(x)dx} - \int q(x)e^{\int p(x)}dx + A(y) \\ F(x,y) = ye^{\int p(x)dx} + B(x) \end{cases}$$

$$\therefore A(y) = 0 \;,\; B(x) = -\int q(x)e^{\int p(x)dx}dx$$

General Solution: $ye^{\int p(x)dx} - \int q(x)e^{\int p(x)dx}dx = c$

i.e. $y\sigma(x) = \int q(x)\sigma(x)dx + c$

Method 2: Variation of Parameter

Step 1 ：Find the homogenous solution

$$y'+p(x)y=q(x) \quad \text{———} \quad (1)$$

Homogenous egn ： $y'+p(x)y=0$

Homogenous solution of egn (1)

Step 2 ： $y_h = Ae^{\int -p(x)dx}$

Let $y = A(x)e^{\int -p(x)dx}$

Note：Larange's idea is try to vary the parameter A
Thus , we seek a solution y(x) of the
non-homogenous equation in the form $y(x) = A(x)e^{-\int P(x)}$

Step 3 ：

$$y' + p(x)y = q(x)$$

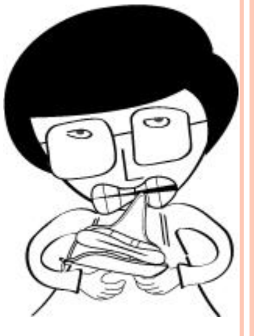
$$\left[A'(x)e^{-\int p(x)dx} + A(x)(-p(x))e^{-\int p(x)dx}\right]$$

$$+p(x)A(x)e^{-\int p(x)dx} = q(x)$$

$$\Rightarrow A'(x)e^{-\int p(x)dx} = q(x)$$

$$\Rightarrow A'(x) = q(x)e^{\int p(x)dx}$$

$$\Rightarrow A(x) = \int q(x)e^{\int p(x)dx}dx + c$$



The general solution of egn(1) is

$$y(x) = A(x)e^{-\int p(x)dx}$$

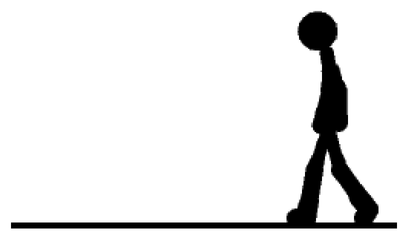
$$= e^{-\int p(x)dx}\left[\int q(x)e^{\int p(x)dx}dx + c\right]$$

$$\Rightarrow = y\sigma = \int q(x)\sigma(x)dx + c$$

Note：

$$y = e^{-\int p(x)dx}\left[\int\left[q(x)e^{\int p(x)dx}\right]dx + c\right]$$

$$= ce^{-\int p(x)dx} + e^{-\int p(x)dx}\int q(x)e^{\int p(x)dx}dx$$



◎ The Bernoulli Equation

$$y' + P(x)y = R(x)y^{\alpha}$$

1. Assume $v = y^{1-\alpha}$
2. Replace y with $v \Rightarrow \frac{dv}{dx} = (1-\alpha)y^{-\alpha}y'$

   $\Rightarrow \frac{1}{1-\alpha}v' + P(x)v = R(x)$

   This is a Linear O.D.E.
3. Follow the procedure of solving a linear O.D.E.

Convert a Non-linear O.D.E. to a Linear O.D.E.

◎ The Riccati Equation

$$y' = p(x)y^2 + Q(x)y + R(x)$$

(1) P(x)=0

$$y' = Q(x)y + R(x) \rightarrow Linear\ \ O.D.E.$$

(2) Otherwise: Let $y(x) = S(x) + \dfrac{1}{Z(x)}$

$$y'(x) = S'(x) + \frac{-Z'(x)}{Z^2(x)}$$

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$$\Rightarrow S'(x) + \frac{-Z'(x)}{Z^2(x)} = P(x)\left[S(x) + \frac{1}{Z(x)}\right]^2 + Q(x)\left[S(x) + \frac{1}{Z(x)}\right] + R(x)$$

$$\Rightarrow S'(x) - \frac{Z'(x)}{Z^2(x)} = \left[P(x)S^2(x) + Q(x)S(x) + R(x)\right]$$

$$+\left[P(x)\frac{1}{Z^2(x)} + 2P(x)S(x)\frac{1}{Z(x)} + Q(x)\frac{1}{Z(x)}\right]$$

$$If\ \ S'(x) = P(x)S^2(x) + Q(x)S(x) + R(x)$$

$$\Rightarrow Z'(x) + \left[2P(x)S(x) + Q(x)\right]Z(x) = -P(x)$$

1^{st} –order, Linear O.D.E.

$$Ex: y' = \frac{1}{x} y^2 + \frac{1}{x} y - \frac{2}{x}$$

1. Observe:Riccati Equation

2. Let $y(x) = S(x) + \dfrac{1}{Z(x)}$

   Inspection: $\ddot{S}(x) = 1$

   so that $S'(x) = \dfrac{1}{x} S^2(x) + \dfrac{1}{x} S(x) - \dfrac{2}{x}$

   Then let $y(x) = 1 + \dfrac{1}{Z(x)}$ $\quad y' = -\dfrac{Z'}{Z^2}$

$$\Rightarrow -\frac{Z'}{Z^2} = \frac{1}{x}\left(1 + \frac{1}{Z}\right)^2 + \frac{1}{x}\left(1 + \frac{1}{Z}\right) - \frac{2}{x}$$

$$\Rightarrow -\frac{Z'}{Z^2} = \frac{1}{x}\left(1 + \frac{2}{Z} + \frac{1}{Z^2}\right) + \frac{1}{x}\left(1 + \frac{1}{Z}\right) - \frac{2}{x}$$

$$= \frac{1}{x} + \frac{1}{x}\left(\frac{2}{Z} + \frac{1}{Z^2}\right) + \frac{1}{x} + \frac{1}{x}\frac{1}{Z} - \frac{2}{x}$$

$$= 3\frac{1}{x}\frac{1}{z} + \frac{1}{xZ^2}$$

$$\Rightarrow Z' + 3\left(\frac{1}{x}\right) Z + \frac{1}{x} = 0 \rightarrow 1^{st} - order\ \ Linear$$

Integrating Factor $\sigma = e^{\int \frac{3}{x}dx} = x^3$

$$x^3 Z' + 3x^2 Z = -x^2$$

$$\Rightarrow \left(x^3 Z\right)' = -x^2 \;\Rightarrow x^3 Z = \int\left(-x^2\right)dx$$

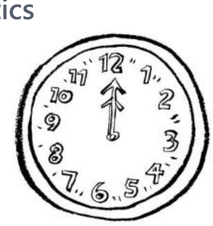
$$= -\frac{1}{3}x^3 + c$$

$$\Rightarrow Z(x) = -\frac{1}{3} + \frac{c}{x^3}$$

$$y(x) = 1 + \frac{1}{-\frac{1}{3}+\frac{c}{x^3}} = \frac{3c+2x^3}{3c-x^3}$$

$$= \frac{k+2x^3}{k-x^3} \quad (k=3c)$$

1st –order O.D.E.



(1)Separable　　$y' = X(x)Y(x)$

(2)Linear　　$y' + p(x)y = q(x)$

(3)Exact　　$y' = -\dfrac{M(x,y)}{N(x,y)}$

(4)Homogeneous　　$y' = f\left(\dfrac{y}{x}\right)$

(5)Almost-Homogeneous　　$y = \dfrac{a_1 x + b_1 y + c_1}{a_{2x} + b_2 y + c_2}$

(6)Bernoulli Equation $y' + P(x)y = R(x)y^{\alpha}$

(7)Riccati Equation $y' = P(x)y^2 + Q(x)y + R(x)$

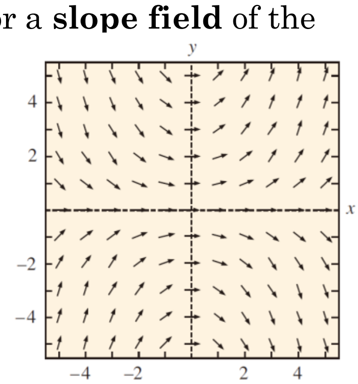
(8)Integrating Factor $\sigma(x, y)M(x, y)dx$

$+\sigma(x, y)N(x, y)dy = 0$

◎ Direction Field

- If we evaluate f over a rectangular grid of points, and draw a lineal element at each point (x, y) of the grid with slope $f(x, y)$, then the collection is called a **direction field** or a **slope field** of the following DE

$$dy/dx = f(x, y)$$



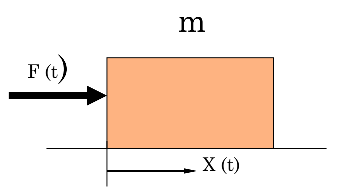
(a) Direction field for $dy/dx = 0.2xy$

◎ Initial Value Problem (I. V. P.)

the D.E. together with the initial conditions (I.C.) is called initial value problem.

➡ General solution $\xrightarrow{\text{I.C.}}$ Particular solution

E.X：



I.C. $x(0) = 0$

$\dot{x}(0) = v$

$m\ddot{x} = F(t) \xrightarrow{\text{I.C.}}$ Only one solution exists !

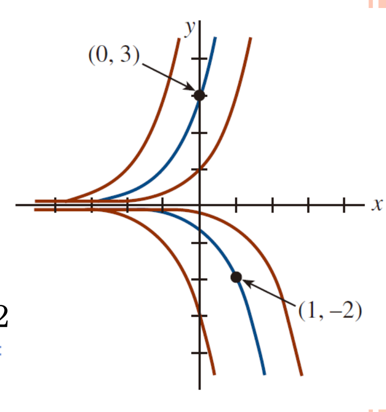
◎ Initial Value Problem (I. V. P.)

We know $y = ce^x$ is the solutions of $y' = y$ on $(-\infty, \infty)$. If $y(0) = 3$, then $3 = ce^0 = c$. Thus $y = 3e^x$ is a solution of this initial-value problem

$$y' = y, \quad y(0) = 3.$$

If we want a solution pass through (1, -2), that is $y(1) = -2$, $-2 = ce$, or $c = -2e^{-1}$. The function $y = -2e^{x-1}$ is a solution of the initial-value problem

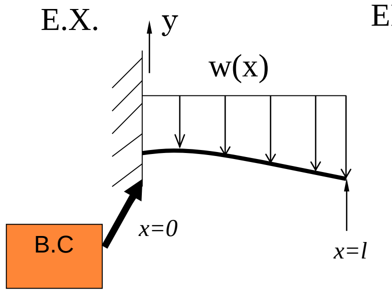
$$y' = y, \quad y(1) = -2.$$



◎ Boundary Value Problem (B.V.P)

D.E. ＋ Boundary condition (B.C.) $\longrightarrow$ B.V.P

E.X.

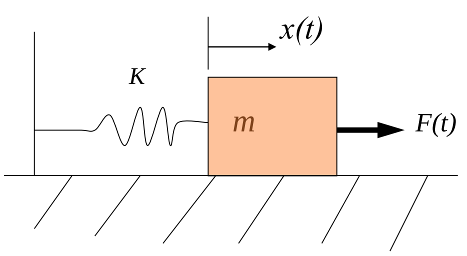


$$EI\, y''''(x) = -w(x)$$

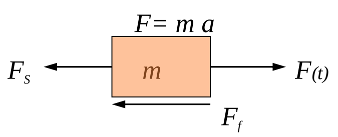
$$y(0) = 0 \quad y'(0) = 0$$
$$y''(L) = 0 \quad y'''(L) = 0$$

General solution $\xrightarrow{\text{B.C.}}$ Particular solution

◎ *Modeling* ：



Newton's Second Law of Motion



$$mx''(t) = F(t) - F_S - F_f$$
$$F_S = k\,x(t)$$
$$F_f = cx'(t)$$

$$mx'' = F(t) - kx(t) - cx'(t)$$

◎ Linear Model

- Newton's Law of Cooling /Warming

$$\frac{dT}{dt} = k\,(T - T_m)$$

Temperature of object

Ambient temperature

Time

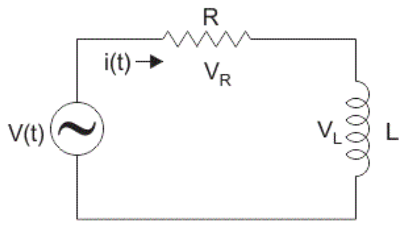
constant

Temperature of object

Example

- When a cake is removed from an oven, its temperature is measured at 300 °F, 3 minutes later, its temperature is 200°F. How long will it take from the cake to cool off to a room temperature of 70 °F?

Series Circuit (LR series circuit)



Kirchhoff's second law

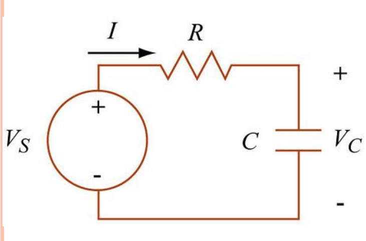
$$L\frac{di(t)}{dt}+Ri(t)=V(t)$$

Example: $V = 12\ \mathrm{V_{DC}}$, $L = ½$ henry, $R = 10\Omega$
Please find the current $i(t)$ if the initial current is zero

RC circuit

$$Ri(t)+\frac{1}{c}q(t)=V(t)$$

current　　charge



$$i(t)=\frac{dq(t)}{dt}$$

$$\Rightarrow R\frac{dq(t)}{dt}+\frac{1}{c}q(t)=V(t)$$

Carbon dating (c-14)

- The half-life of C-14 is close to 5730 years. A fossilized bone is found to contain 0.1 % of its original amount of C-14. Determine the age of the fossil.

Sol:

amount of C-14

$$\frac{dA}{dt} = -kA$$

Carbon dating (cont.)

- $A(t) = A_0 e^{kt} \quad \Rightarrow \frac{1}{2} A_0 = A_0 e^{k(5730)}$

$$\Rightarrow k = \frac{\ln\left(\frac{1}{2}\right)}{5730} = -0.00012$$

- $A(t) = A_0 e^{-0.00012\,t}$

$$From\ the\ final\ containing\ amount\ of\ C-14$$

$$0.001A_0 = A_0 e^{-0.00012\,t}$$

$$t = \frac{\ln(0.001)}{-0.00012} = 57103\ (years)$$

Chemical reaction

- Suppose that a_1 grams of chemical A are combined with b_1 grams of chemical B.
- If there are M parts of A and N parts of B formed in the compound and *X(t)* is the number of grams of chemical C.

$$\frac{M}{M+N}X \longrightarrow \text{chemical A used in grams}$$
$$\frac{N}{M+N}X \longrightarrow \text{chemical B used in grams}$$

- By the law of mass reaction, the rate of reaction satisfies

$$\frac{dX}{dt} = k\left(a_1 - \frac{N}{M+N}X\right)\left(b_1 - \frac{N}{M+N}X\right)$$

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example

- A compound C can be formed by the combination of chemical A & B. A resulting reaction between the two chemicals is such that for each gram of A, 4 grams of B is used. It is observed the 30 grams of the compound C is formed in 10 minutes. Determine the amount of C at time t if the rate of the reaction is proportional to the amount of A & B remaining.

- If initially there are 50 grams of A and 32 grams of B. How much of the compound C is present at 15 minutes?

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F-16 Landing with Parachute