Note on Differential Geometry and General Relativity

Kai Shi

November 28, 2023

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Chapter 1

Basics of GR

1.1 Differemorphism and coordinates transformation

1.1.1 Differemorphism invariance and isometry.

GR is differemorphism invariant means when applying differemorphism to the geometry terms of GR, the theory is invariant. That is

$$g_{ab}u^a v^b|_p = \phi^*(\phi_* g_{ab})u^a v^b|_p = \phi_* g_{ab}\phi_* u^a \phi_* v^b|_p = \phi_*(g_{ab}u^a v^b)|_q \tag{1.1}$$

Such an character ensures that within a GR theory, all differemorphisms of the metric form a equivalent class, or to say that the differemorphism is nothing but a gauge for given GR theory: there are infinite equivalent "viewpoints" for one to view the same structure of manifold, or more specifically, the inner product. The geometry is stage for vectors who are actors, differemorphism decides where you put the camera - $\phi_* g_{ab}$ which can be denoted as $\tilde{g}_{ab}$ is still the same stage, except viewed from different point.

However, when one denote $\phi_* g_{ab}$ as $\tilde{g}_{ab}$, they have no confidence to claim that $\phi_* g = g$, which is only true when ϕ^* is an isometry.

1.1.2 Mapping on manifold and coordinates transformation.

It is a fact that the tensor (we take a vector for simplicity) is invariant within a coordinates transformation,

$$v^a = v^\mu \frac{\partial}{\partial x^\mu}^a = v^\mu \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial}{\partial x'^\alpha}^a = v'^\alpha \frac{\partial}{\partial x'^\alpha}^a, \tag{1.2}$$

where $v^\mu = v^\mu(x^\mu)$ while $v'^\mu = v'^\mu(x'^\mu)$ which applies the new coordinate. It is known that there is a equivalence between coordinate transformation and active tensor transformation, depending on which perspective one would choose - the passive viewpoint or active viewpoint. I would justify that such a coordinate transformation always has an dual active tensor transformation, but in practice we can distinguish whether we are changing the coordinate or changing the coordinate system.

According to the effect of pushforward and pullback in active viewpoint, from the book Differential Geometry for Physicists, 2 coordinate systems are given to describe 2 points such as $\{x^\mu\}$ at p and $\{y^\mu\}$

at $\phi(p)$, then we would have

$$\phi_* \left(v^\mu(p) \frac{\partial}{\partial x^\mu}^a \Big|_p \right) \Big|_{\phi(p)} = v^\mu(p) \frac{\partial y^\alpha(\phi(p))}{\partial x^\mu(p)} \frac{\partial}{\partial y^\alpha(\phi(p))}^a \Big|_{\phi(p)}, \tag{1.3}$$

where on RHS everything is described with $\{y^\mu\}$ at $\phi(p)$, there for the function form of v^μ must be replaced as

$$v^\mu(x^\rho) \longmapsto v^\mu(y^\rho). \tag{1.4}$$

The 1.3 above is a push-forward described in active viewpoint, which essentially brings the whole tensor to a new point, effectively we copies the coefficient at the original point to a new point, and additionally change the coordinate-basis accordingly to describe it locally.

It looks very promising that the pushed forward vector in active viewpoint will change its form just like the ordinary coordinate transformation by comparing 1.2 and 1.3 ! But how does that come true?

A comparison of 2 push-forward definitions, in active viewpoint

Before the actual proof, one question comes naturally: does 1.4 comply with the definition of pushforward pullback proposed in Liang's book? We can compare it with the definition in Liang's book

$$(\phi_* v)(f)|_{\phi(p)} = v(\phi^* f)|_p. \tag{1.5}$$

Using the equivalent definition of $v(f)$, we may rewrite equation as

$$(\phi_* v)^a (df)_a|_{\phi(p)} = v^a d(\phi^* f)_a|_p, \tag{1.6}$$

due to the fact that the exterior derivative d commutes with ϕ^*,

$$d(\phi^* f)_a|_p = \phi^*(df)_a|_p, \tag{1.7}$$

such that we also know

$$v^a d(\phi^* f)_a|_p = v^a (\phi^* df)_a|_p, \tag{1.8}$$

consequently we can state

$$(\phi_* v)^a (df)_a|_{\phi(p)} = v^a (\phi^* df)_a|_p. \tag{1.9}$$

Additionally, $df_a\Big|_{\phi(p)}$ is originally described by y^μ as

$$df_a|_{\phi(p)} = \frac{\partial f}{\partial y^\mu} dy^\mu_a|_{\phi(p)}, \tag{1.10}$$

so if we want to describe it at p, like the right-hand-side, then a coordinate transformation is necessary, that is

$$\frac{\partial f}{\partial y^\mu} dy^\mu_a|_{\phi(p)} = \frac{\partial f}{\partial y^\mu} \frac{\partial y^\mu}{\partial x^\nu} dx^\nu_a|_p = \phi^* df_a|_p = d(\phi^* f)\Big|_p. \tag{1.11}$$

Substitute both sides of it into 1.6, ensuring the points complies the sides, we will see

$$(\phi_* v)^\mu(y) \frac{\partial f}{\partial y^\mu} = v^\mu(x) \frac{\partial f}{\partial y^\mu} \frac{\partial y^\mu}{\partial x^\nu}, \tag{1.12}$$

thus we have

$$(\phi_* v)^\mu(y) = v^\mu(x) \frac{\partial y^\mu}{\partial x^\nu}, \tag{1.13}$$

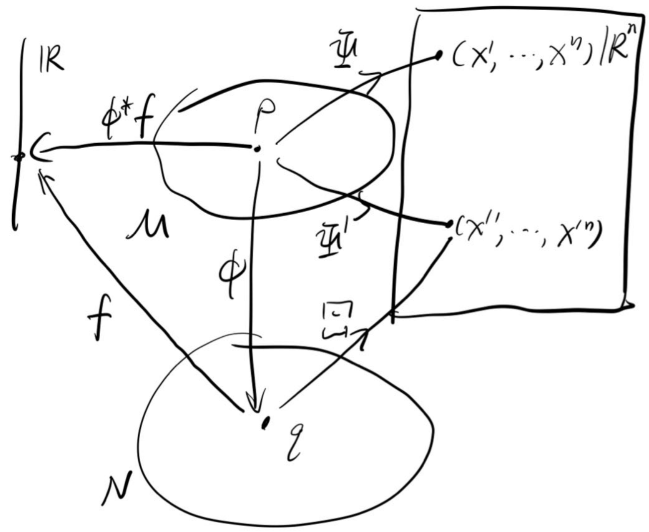

Figure 1.1: Induced coordinate system from a map between manifolds. Here Ψ is the original coordinate system at p, Ξ is the coordinate system at $q = \phi(p)$, while Ψ' marks a new induced coordinate system, such that for any point at p in $\mathcal{M}$, $\Xi(q) = \Psi(p)$. For simplicity, denoting y^μ for Ξ while x^μ for Ψ and x'^μ for Ψ'.

where the round bracket indicates in which coordinate system the component is described.

One last thing to mention is that the function or more specifically the relationship between v^μ and coordinates is the physical object that we need! So the form of the function, a.k.a

$$F_\nu(x^\rho) : x^\rho \longmapsto v^\mu(x^\rho)\frac{\partial y^\mu(x^\sigma)}{\partial x^\nu}, \tag{1.14}$$

is essential, as you can simply replace x with y according to the coordinate transformation in the final result - as y is fundamentally the local coordinate system to describe the new point. Then any replacement of the symbol does not affect the result any more, as the relationship is fixed concisely. This proof can be generalized to any kind of tensor.

Identification of the tensor component transformation in active viewpoint and passive viewpoint

In conclusion, utilizing $y^\mu(\phi(p))$ to induce a coordinate transformation $x'^\mu(p) = y^\mu(\phi(p))$ will give us the passive viewpoint result of a mapping, that is, a local transformation at original point, and equals the active viewpoint tensor transformation to the passive one,

$$\phi_*(v^\mu \frac{\partial}{\partial x^\mu}^a \Big|_p)\Big|_{\phi(p)} = v^\mu(p)\frac{\partial y^\alpha(\phi(p))}{\partial x^\mu(p)}\frac{\partial}{\partial y^\alpha(\phi(p))}^a |_{\phi(p)}= v^\mu(p)\frac{\partial x'^\alpha(p)}{\partial x^\mu(p)}\frac{\partial}{\partial x'^\alpha(p)}^a |_p = v'^\alpha \frac{\partial}{\partial x'^\alpha}^a |_p \tag{1.15}$$

The function $v^\mu(p)\frac{\partial y^\alpha(\phi(p))}{\partial x^\mu(p)}$ is *newnewold* and $v^\mu(p)\frac{\partial x'^\alpha(p)}{\partial x^\mu(p)}$ in (1.4) is *oldoldnew* in Liang's book. Refer to 1.1 for an explicit feeling.

To fully rewrite the coordinate transformation into active viewpoint, though this is tedious and totally

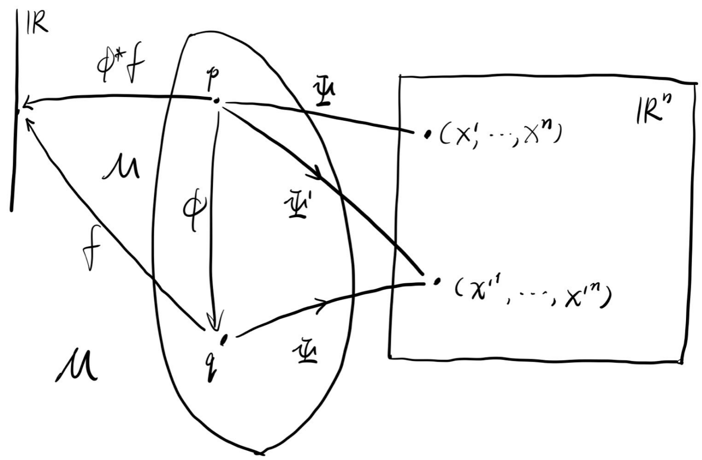

Figure 1.2: This is a simplified version of the mapping where only one manifold is included. To make it more general and clear in appearance we separate p and $\phi(p)$, if we make ϕ the identical mapping, then $\phi(p) = p$ — but 2 coordinate systems are preserved

unnecessary except justifying that we can treat coordinate transformation like how we do the push-forward using 1.3.

First of all, we define ϕ as a identical transformation at p such that

$$\phi(p) = p, \tag{1.16}$$

and we use $\{y^\mu\}$ to describe $\phi(p)$, then $y(\phi(p)) = y(p)$ and $y(p)$ simply induces itself as the new coordinate system $x'(p)$. Whereby we can rewrite the coordinates transformation into an active viewpoint language, where

$$v^a = v^\mu \frac{\partial}{\partial x^\mu}^a |_p = \phi^*\phi_*(v^\mu \frac{\partial}{\partial x^\mu}^a) |_p = \phi^*(v^\mu \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial}{\partial y^\alpha}^a) |_p = \phi^*(v'^\alpha \frac{\partial}{\partial y^\alpha}^a) |_p = v'^\alpha \frac{\partial x'^\nu}{\partial y^\alpha} \frac{\partial}{\partial x'^\nu}^a |_p = v'^\alpha \delta^\nu{}_\alpha \frac{\partial}{\partial x'^\nu}^a |_p, \tag{1.17}$$

where $y(p) = x'(p)$ is used, only if one uses the right corresponding coordinate system to describe v^μ and v'^μ in the concrete form. To avoid confusion in names, $\phi^*\phi_*$ will be called as unity mapping. This calculation can also be written in a more compactified form with the action of pull-back and push-forward as, where we will use the differential form for example,

$$\phi_*\phi^*(\omega_\mu dx^\mu_a) = \phi_*(\omega_\mu \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu) = \omega_\mu \frac{\partial x^\mu}{\partial x'^\nu} \frac{\partial x'^\nu}{\partial x'^\rho} dx'^\rho = \omega'_\nu dx'^\nu, \tag{1.18}$$

where $y^\mu(q) = x^\mu(p)$ is repetitively used. Therefore, any given coordinate transformation can be regarded as being functioned by a unity operator made of identical mapping ϕ, each is endowed with an coordinate system.

•By observing the calculation above, we can also understand why usually the pullback and pushforward mapping can be denoted by $\phi^\mu_{*\nu} = \frac{\partial y^\mu}{\partial x^\nu}$, and $\phi^{*\nu}{}_\mu = \frac{\partial x^\nu}{\partial y^\mu}$, where $\phi^{*\nu}{}_\mu \omega_\nu = \omega_\mu \frac{\partial x^\nu}{\partial y^\mu}$, $\phi^{*\nu}{}_\mu dx^\mu = \frac{dx^\mu}{dy^\nu} dy^\nu$.

Derive the mapping of a tensor from coordinate transformation

The calculation above implies that if we have any mapping between different points in a manifold, then a new local coordinate system can be induced utilizing the mapping. In this part it would be proven that if a local coordinate transformation is given, then how the tensor is mapped will be totally decided.

The proof is very easy: if we are given a coordinate transformation at p,

$$x^\mu \longmapsto x'^\mu, \tag{1.19}$$

the coordinate transformation shows us

$$v^\mu \frac{\partial}{\partial x^\mu}^a = v^\mu \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial}{\partial x'^\nu} = v'^\nu \frac{\partial}{\partial x'^\nu}, \tag{1.20}$$

then with the identification of $y^\mu(q) = x'^\mu(p)$ we have

$$v'^\nu \frac{\partial}{\partial x'^\nu} \xrightarrow{x' \to y} v^\nu \frac{\partial}{\partial y^\nu}. \tag{1.21}$$

We use the fact that "the image of curve's tangent equals to the tangent of curve's image here".

•An example.

Suppose there is a mapping that induces a coordinate transformation as

$$r' = r + \theta^2 \qquad \theta' = \theta \qquad \phi' = \phi, \tag{1.22}$$

and $v^a = r(\partial/\partial\theta)^a$. Then the coordinate transformation reforms the vector as

$$\begin{aligned} r\frac{\partial}{\partial \theta}^a &= r\frac{\partial r'}{\partial \theta}\frac{\partial}{\partial r'}^a + r\frac{\partial \theta'}{\partial \theta}\frac{\partial}{\partial \theta'}^a + r\frac{\partial \phi'}{\partial \theta}\frac{\partial}{\partial \phi'}^a \\ &= 2r\theta'\frac{\partial}{\partial r'}^a + r\frac{\partial}{\partial \theta'}^a \xrightarrow{x \to x'} (r' - \theta'^2)[2\theta'\frac{\partial}{\partial r'}^a + \frac{\partial}{\partial \theta'}^a] \end{aligned} \tag{1.23}$$

This corresponds to a description of v^a with Ψ' in Fig 1.2, but still at p point. However, by applying the equivalence of $y^\mu(\phi(p)) = x'^\mu(p)$, then we can regard it as a new tensor at some new point describe by "old" (which means the domain of x covers $\phi(p)$ such that $y^\mu(\phi(p)) = x^\mu(\phi(p))$). Then the new tensor described by old coordinate system is,

$$(r - \theta^2)[2\theta\frac{\partial}{\partial r}^a + \frac{\partial}{\partial \theta}^a]. \tag{1.24}$$

Basically the process can be understood within Fig 1.2 as

$$\Psi(p) \xrightarrow{\text{coordinate transformation}} \Psi'(p) \xrightarrow{\text{new tensor}} \Psi(q) \tag{1.25}$$

Proof of "the image of the tangent to a curve is the tangent to the image of a curve".

• Suppose we have a mapping such that $x'^\mu(p) = x^\mu(\phi(p))$, taken one coordinate curve of $\{x'^\mu\}$ as the object we would use to prove. Then by definition we have

$$\phi^* x^\mu(p) = x^\mu(\phi(p)) \tag{1.26}$$

where x^μ and ϕ separately give 3 curves

$$(x^\mu)^{-1} : x^\mu \longmapsto p \qquad \phi \circ (x^\mu)^{-1} : x^\mu \longmapsto \phi(p) \qquad (\phi^* x^\mu)^{-1} = \phi^{-1} \circ (x^\mu)^{-1} : x'^\mu \longmapsto p \tag{1.27}$$

$$\phi_*[\frac{\partial}{\partial x^\mu}\Big|_p](f)\Big|_{\phi(p)} = \frac{\partial}{\partial x^\mu}\Big|_p \phi^*(f)\Big|_p = \frac{\partial}{\partial x^\mu}\Big|_p f\Big|_{\phi(p)} = \frac{\partial}{\partial x^\mu} f(\phi \circ (x^\mu)^{-1} \circ x^\mu) \tag{1.28}$$

By comparison with the definition of the tangent to curve, suppose a curve C and its tangent T^a

$$T(f) = T^a \nabla_a f = \frac{\partial f(C(t))}{\partial t}, \tag{1.29}$$

we would know that $\frac{\partial}{\partial x^\mu}^a$ is the tangent to curve $\phi \circ (x^\mu)^{-1}$, which is exactly the image of curve $(x^\mu)^{-1}$. In conclusion, the image of tangent to a curve the tangent to image of the curve. By contracting the image of basis vector $\phi_* \partial^a_\mu$ and the local basis covector dy_a can also finish the proof, but won't be shown here.

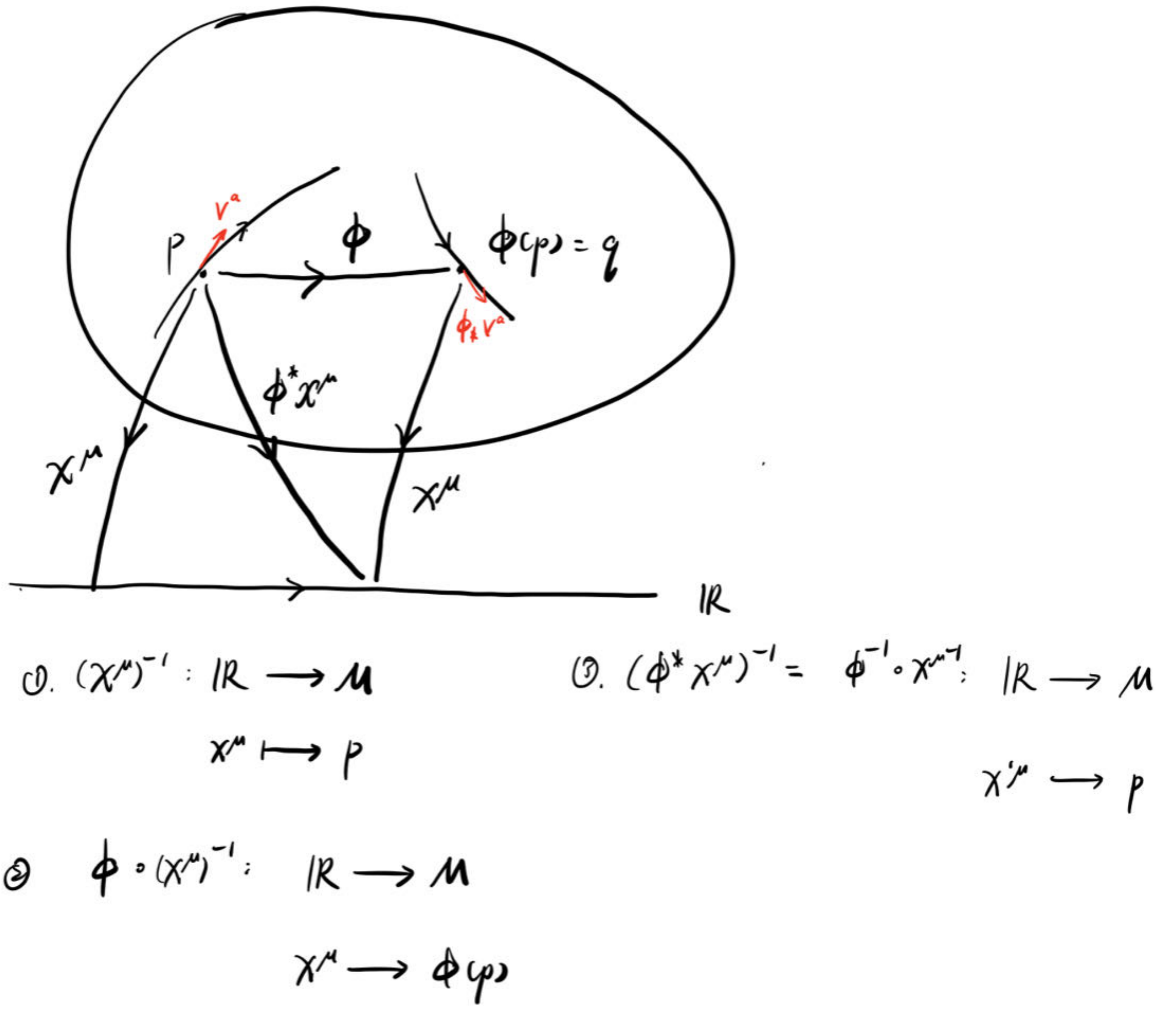

Figure 1.3: This figure show 3 different curves induced by a mapping *phi*, and the tangent to the original curve will be mapped as the tangent to the image of the curve.

Active and passive viewpoint are equally universal from the discussion above.