Holographic Renormalization

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December 8, 2023

Red is for important piece. Green is for ambiguity and unknown. Blue is for definition. Einstein summation convention is used otherwise clarification will be given.

1 Review of AdS

suppose there is a 5 dimensional flat space $(\mathbb{R}^3, \tilde{\eta}_{ab})$ with signature {-1,-1,+1,+1,+1}, then its line element is

$$ds^2 = -\mathrm{d}T^2 - \mathrm{d}W^2 + \mathrm{d}X^2 + \mathrm{d}Y^2 + \mathrm{d}Z^2 \tag{1}$$

suppose there is a embedding hypersurface which is constrained by the following equation:

$$-T^2 - W^2 + X^2 + Y^2 + Z^2 = -\ell^2 \tag{2}$$

And the induced metric on the hyper surface is the AdS_4 metric, where ℓ is the AdS radius. By moving terms in the above equation, we find that

$$X^2 + Y^2 + Z^2 + \ell^2 = T^2 + W^2 \tag{3}$$

which means for any given array of (X, Y, Z), a circle of (T, W) which satisfies $T^2 + W^2 = X^2 + Y^2 + Z^2 + \ell^2$ will be specified. This implies that the topology of the AdS is $\mathbb{S}^1 \times \mathbb{R}^3$, which is distinct from the usual Minkowski occupies $\mathbb{R}^1 \times \mathbb{S}^3$ topology. Now let's introduce 3 different kinds of usual coordinate transformation of AdS_4.

- Static spherically symmetric coordinate system {t,r,θ,φ},$0 < t < 2\pi l, 0 < r < \infty, 0 < \theta < \pi, 0 < \varphi < 2\pi$, such a set of coordinates fulfill the constraints such that they describe a well-defined hyperspace, and the induced metric on the surface is done by implementation of coordinates into the original metric – basically nothing but a pull-back of the embedding mapping.

$$\begin{aligned} &T = (l^2 + r^2)^{1/2} \sin(l^{-1}t), \quad W = (l^2 + r^2)^{1/2} \cos(l^{-1}t), \\ &X = r \sin\theta \cos\varphi, \quad Y = r \sin\theta \sin\varphi, \quad Z = r \cos\theta \end{aligned} \tag{4}$$

and the induced metric becomes

$$(ds^*)^2 = -(1 + l^{-2}r^2)dt^2 + (1 + l^{-2}r^2)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{5}$$

This coordinate system covers the spacetime except the coordinate singularity $r = l$. And it would be easy to check that the upper metric satisfies

$$G_{ab} + \Lambda g_{ab} = 0 \tag{6}$$

where $\Lambda = 1/4R = -3/\ell^2$.

- A second coordinate system $\{\hat{t}, \chi, \theta, \varphi\}$, $0 < \hat{t} < 2\pi l, 0 < \chi < \infty, 0 < \theta < \pi, 0 < \varphi < 2\pi$, which is defined as

$$\begin{gathered} T = l\sin(l^{-1}\hat{t}), \quad X = l\cos(l^{-1}\hat{t})\sinh\chi\sin\theta\cos\varphi, \quad Y = l\cos(l^{-1}\hat{t})\sinh\chi\sin\theta\sin\varphi, \\ Z = l\cos(l^{-1})\hat{t}\sinh\chi\cos\theta, \quad W = l\cos(l^{-1}\hat{t})\cosh\chi \end{gathered} \tag{7}$$

This coordinates gives the following metric

$$(ds^*)^2 = -d\hat{t}^2 + l^2\cos^2(l^{-1}\hat{t})[(d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)] \tag{8}$$

which gives us a $k = -1$ RW universe except $l^{-1}\hat{t} = \pm\pi/2$.

- The third coordinate system $\{\tilde{t}, \psi, \theta, \varphi\}$, $0 < \tilde{t} < 2\pi l, 0 < \psi < \infty, 0 < \theta < \pi, 0 < \varphi < 2\pi$, where

$$\begin{gathered} T = l\sin(l^{-1}\hat{t})\cosh\psi, \quad W = l\cos(l^{-1}\hat{t})\cos\psi, \\ X = l\sinh\psi\sin\theta\cos\varphi, \quad Y = l\sin\psi\sin\theta\cos\varphi, \quad Z = l\sinh\psi\cos\theta \end{gathered} \tag{9}$$

which gives the metric

$$(ds^*)^2 = -\cosh^2\psi d\tilde{t}^2 + l^2(d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta d\varphi^2)) \tag{10}$$

Through a coordinate transformation

$$\begin{gathered} t' = l^{-1}\tilde{t}, \quad \psi' = \arctan(\sin\psi), \quad (0 < \psi' < \frac{\pi}{2}) \\ a.k.a d\psi' = d\psi/\cosh\psi, \quad \cosh\psi\cos\psi' = 1, \quad \sinh\psi = \cosh\psi\sinh\psi' \end{gathered} \tag{11}$$

Finally we find the conformal format of the metric,

$$(ds^*)^2 = l^2\cosh^2\psi[-dt'^2 + d\psi'^2 + \sin\psi'^2(d\theta^2 + \sin^2\theta d\varphi^2)], \tag{12}$$

consequently we find that the spatial boundary of the AdS is timelike and lives on $\psi' = \pi/2$ while for the Minkowski space-time $\psi' \in (-\pi/2, \pi/2)$ which means AdS only covers half of the Minkowski conformally.

- the fourth coordinate system worth mentioning is the Poincare coordinate system $\{t, r, x, y\}$, with which by functioning the coordinate transformation 13 would give us the metric 14

$$\begin{aligned} &T = \frac{lr}{2}(x^2 + y^2 - t^2 + \frac{1}{r^2} + 1), \qquad Z = \frac{lr}{2}(x^2 + y^2 - t^2 + \frac{1}{r^2} - 1), \\ &W = lrt, \quad X = lrx, \quad Y = lry, \end{aligned} \tag{13}$$

$$(ds^*)^2 = l^2(\frac{1}{r^2}dr^2 + r^2(dx^2 + dy^2 - dt^2)). \tag{14}$$

This coordinate system is exceptionally important in AdS/CFT correspondence, and the boundary lives on $r = \infty$.

- The fifth coordinate transformation is based on the fourth one, with which we do the transformation

$$z = \frac{1}{r}, \tag{15}$$

to grab the new metric

$$(ds^*)^2 = \frac{l^2}{z^2}(-dt^2 + dx^2 + dy^2 + dz^2). \tag{16}$$

Correspondingly, the boundary lies on where $z = 0$.

2 AdS/CFT & Holographic Renormalization

To discuss the AdS/CFT, at the very beginning we choose a coordinate system for our problem,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = N(r)^2 dr^2 + r^2\gamma_{ij}dx^i dx^j, \tag{17}$$

with the conformal boundary lives on $r \to \infty$ and $N(r) \to 1/r$ near the boundary[5]. Suppose we have an AdS_{d+1}/CFT_d holography in pure AdS, its dictionary is usually merely defined with respect to the leading order of expansion coefficients of the asymptotic solution of a bulk equation of motion, for which the dictionary states

$$\begin{aligned} \mathcal{Z}_{CFT_d}[\gamma_{ij}, J] &= \lim_{r_c\to\infty} \mathcal{Z}_{AdS_{d+1}}[g^0_{ij} = r_c^2\gamma_{ij}, f(r_c)], \\ f(r_c) &= (\frac{1}{r_c})^\Delta \sum_0^\infty f_{(k)}(\frac{1}{r_c})^k, \quad J = f_{(0)}, \end{aligned} \tag{18}$$

where g^0_{ij} is the induced metric on boundary denoted by $r_c \to \infty$ while γ stands for the conformal boundary metric. Additionally, f can be any kind of field provided that it corresponds to the appropriate boundary source, and for simplicity all dependent coordinates except $1/r$ are omitted. However, usually we prefer defining $z = \frac{1}{r}$ as new coordinate to suppress the divergence of r_c on conformal boundary. Then in this more convenient coordinate system the metric is represented by

$$ds^2 = \frac{1}{z^2}(N(z)^2 dz^2 + \gamma_{ij}dx^i dx^j). \tag{19}$$

Meanwhile, the dictionary takes the form of

$$\begin{aligned} \mathcal{Z}_{CFT_d}[\gamma_{ij}, J] &= \lim_{z\to 0} \mathcal{Z}_{AdS_{d+1}}[g^0_{ij} = z^{-2}\gamma_{ij}, f(z)], \\ f(z) &= z^\Delta \sum_0^\infty f_{(k)} z^k, \quad J = f_{(0)}. \end{aligned} \tag{20}$$

It also worth noting that the conformal boundary now lives on $z = 0$ surface.

The one point boundary correlerator is defined as

$$\langle O \rangle = \frac{1}{\sqrt{\gamma}\mathcal{Z}_{CFT}} \frac{\delta \mathcal{Z}_{CFT}[J]}{\delta J} = \frac{1}{\sqrt{\gamma}} \frac{\delta \log(\mathcal{Z}_{CFT}[J])}{\delta J} = \frac{1}{\sqrt{\gamma}} \frac{\delta \log(\mathcal{Z}_{E,bulk}[f])}{\delta f_{(0)}}\Big|_{z\to 0}, \tag{21}$$

where Z_E is the Euclidean action by doing wick rotation as $\tau = it$ over the action whereby a higher point of correlerator is defined by

$$\langle O_1 ... O_n \rangle = (\frac{\delta}{\sqrt{\gamma}\delta J})_1 ... (\frac{\delta}{\sqrt{\gamma}\delta J})_n \log(\mathcal{Z}_{CFT}[J]). \tag{22}$$

Then using the dictionary we can replace the generating functional of CFT with bulk generating functional $\mathcal{Z}_{E,bulk} = \exp(iS_{\text{bulk}}) = \exp(-S_E)$

$$\begin{aligned} \langle O_1 ... O_n \rangle &= (\frac{\delta}{\sqrt{\gamma}\delta f_{(0)}})_1 ... (\frac{\delta}{\sqrt{\gamma}\delta f_{(0)}})_n \log(\mathcal{Z}_{E,bulk}[f])\Big|_{z\to 0} \\ &= (\frac{\delta}{\sqrt{\gamma}\delta f_{(0)}})_1 ... (\frac{\delta}{\sqrt{\gamma}\delta f_{(0)}})_n \log(\exp(-S_E[f]))\Big|_{z\to 0} \\ &= -(\frac{\delta}{\sqrt{\gamma}\delta f_{(0)}})_1 ... (\frac{\delta}{\sqrt{\gamma}\delta f_{(0)}})_n S_E[f]\Big|_{z\to 0}. \end{aligned} \tag{23}$$

Generally speaking, due to the appearance of the inverse of conformal factor of the AdS spacetime $\frac{1}{z}$ (or at least asymptotic AdS spacetime) in the bulk action, the action S_E becomes highly prone to divergence, or in a more explicit sense one may find that volume element carries terms like $(\frac{1}{z})^{\mathscr{F}(d,\Delta)}$ with $\mathscr{F}(d, \Delta) > 0$. Luckily, in some preferred cases the divergence terms can be canceled out totally, such that the renormalized correlerator is finite.

3 Holographic renormalization in $AdS_{d+1}/CFT_d(d>2)$-scalar case

3.1 Quantization and constraints

We take the one scalar field in AdS_{d+1} as example to do the holographic renormalization, and clarify more AdS/CFT basics within the process. First of all, in this case the AdS_4 metric is

$$ds^2 = \frac{\ell^2}{z^2}(dz^2 + \eta_{ij}dx^i dx^j), \tag{24}$$

where η is the d-dimensional Minkowski metric and AdS radius is always taken to be 1 for simplicity. From the metric we can read that the boundary is conformally flat. To reveal more critical points within AdS/CFT without bringing about extra confusion, for our discussion we take a free real scalar field without any potential whose action is given by

$$S = -\frac{1}{2}\int d^{d+1}x\sqrt{-g}(\nabla_a\phi\nabla^a\phi + m^2\phi^2). \tag{25}$$

Then the application of variation principle yields the equation of motion (Klein Gordon equation), as

$$(\Box - m^2)\phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\phi) - m^2\phi = 0. \tag{26}$$

And the substitution of the determinant leads to

$$z^2\partial_z^2\phi + (1-d)z\partial_z\phi + z^2\partial_{\vec{x}}^2\phi - m^2\phi = 0. \tag{27}$$

Now we start to solve the equation of motion asymptotically, because this is more generic as it only requires the spacetime to be asymptotically AdS. Near the boundary $z \to 0$ we do the expansion of ϕ as

$$\phi = z^\Delta(\phi_0(x) + z\phi_1(x) + z^2\phi_2(x) + ...), \tag{28}$$

with which the asymptotic equation of motion comes out as

$$0 = z^\Delta\left[\Delta(\Delta-1)\phi_0(x) + (1-d)\Delta\phi_0(x) - m^2\phi_0(x) + z^2\partial_{\vec{x}}^2\phi_0(x)\right] + \mathcal{O}(z^{\Delta+1}). \tag{29}$$

It worth mentioning that due to the flatness of the boundary, then the translation symmetry ensure us to do Fourier transformation of $\phi(x)$ with the invariant modes $e^{-ik_\mu x^\mu}$. The $z^2\partial_{\vec{x}}^2\phi_0(x)$ is only high order term that not absorbed into $\mathcal{O}(z^{\Delta+1})$ because it will be used to deduce the recurrence relation of the expansion coefficient. To fulfill the equation of motion, first of all we can impose the condition to make the least order term to be 0, specifically resulting in the parameter part before $\phi_{(0)}$ to be 0 such that the freedom of ϕ_0 is not intervened - or ϕ will be too trivial. Imposing such a condition produces

$$(\Delta^2 - \Delta d - m^2) = 0, \tag{30}$$

which yields

$$\Delta_\pm = \frac{d \pm \sqrt{d^2+4m^2}}{2}. \tag{31}$$

This implies that there are 2 series within the asymptotic expansion of ϕ, leading by $\psi^+ z^{\Delta^+}$ and $\psi^- z^{\Delta^-}$, so the asymptotic expansion of ϕ is

$$\phi = z^{\Delta_+}\sum_{n=0}\psi_n^+ z^n + z^{\Delta_-}\sum_{n=0}\psi_n^- z^n. \tag{32}$$

Remarkably, the positiveness of $\sqrt{d^2+4m^2}$ implies that the mass in AdS can be negative but not too negative. (BF bound, the breakthrough will leads to instability) Then we can discuss the recurrence relation and choice of source in AdS/CFT holography.

First, if we carry out expansion of ϕ to higher orders, we will find the following recurrence relation between expansion coefficients of different orders,

$$\begin{aligned} -\frac{\partial_{\vec{x}}^2 \phi_{2n-2}}{2n(2\Delta+2n-d)} &= \phi_{2n}, \\ -\frac{\partial_{\vec{x}}^2 \phi_{2n-1}}{2n(2\Delta+2n+1-d)} &= \phi_{2n+1}. \end{aligned} \tag{33}$$

It worth mentioning that the second recurrence relationship implies that all odd order expansion vanish, as a result of that ϕ_1 must be 0 - the reason why this happens is that Δ is already taken to be the solution to 30, so that the coefficient of subleading term $z^{\Delta} P(\Delta, m)\phi_1 z$ does not vanish unless $\phi_1 = 0$.

It also causes trouble that sometimes denominator vanishes when terms within 2 series leaded by ψ^+ and ψ^- degenerate, or in other words own same power. To deal with this, one can add a multiplier log z to the degenerate term of one series and cut off that branch, which from the perspective of PDE includes a new orthogonal basis. For example, by denoting the Δ_+ as Δ and correspondingly Δ_- as $d-\Delta$, then

$$\phi = z^{d-\Delta}(\psi_0^- + \psi_2^- z^2 + \psi_4^- z^4 + ... + \psi_{2n^*}^- z^{2n^*} \log z) + z^{\Delta}(\psi_0^+ + \psi_2^+ z^2 + \psi_4^+ z^4 + ...) \tag{34}$$

The additional log term will modify the recurrence relationship of the degenerate term into

$$\psi_{2n^*}^- = \frac{\partial_x^2 \psi_{2(n^*-1)}^-}{2\Delta - 4n^* - d} \tag{35}$$

such that the denominator vanishing is avoided.

Recalling AdS/CFT dictionary, one will find that at least 2 duality are possible when choosing the correspondence between "source" and "leading order coefficient", ψ^- or ψ^+, that are initially independent. However, by imposing either ψ^+ or ψ^- as one boundary condition at $z=0$, and additionally another "regular" or "ingoing" boundary condition, based on interests and signature of space time, at $z \to \infty$, then due to the fact that one order-2 PDE requires only 2 independent boundary conditions to solve, ψ^- and ψ^+ will be no longer irrelevant. In AdS/CFT language, the independent boundary condition is taken to be "*source*" and the dependent one is regarded as "VEV", accordingly we are able to write down $\psi^+[\psi^-]$ or $\psi^-[\psi^+]$. Usually, treating ψ^- as source and ψ^+ as VEV is called standard quantization. Conversely, when they are exchanged, it is referred to as alternative quantization. Remarkably, the universality of 2 quantization method is not same. To see this, we can investigate the inner product of VEV, which works as basis in Hilbert space of a QM. The inner product is defined with the Klein-Gordon(symplectic) current

$$j^a(\psi, \varphi) = -i(\bar{\psi}\nabla^a \varphi - \varphi \nabla^a \bar{\psi}), \tag{36}$$

where ψ and φ are solutions to Klein-Gordon equation, by

$$(\psi, \varphi) = -\int_{\Sigma} j^a n_a = \int_{\Sigma} dz d^{d-1}x \; (\frac{1}{z})^{d-1} i(\bar{\psi}\nabla_t \varphi - \varphi \nabla_t \bar{\psi}). \tag{37}$$

The inner product of any ψ, φ appears to be like

$$\int_{\Sigma} dz d^{d-1}x \; z^{2\Delta_{+(-)}-(d-1)} P(\psi^{+(-)}(x), \varphi^{+(-)}(x), z) \tag{38}$$

by substituting the expansion of ψ and φ into the inner product, whose convergence relies on the leading part. $z^{2\Delta-(d-1)}$. Another point worth a highlight is since we tend not to look into the black hole, so the divergence of integral only happens near $z \to 0$. To avoid this from happening, we must require that

$$2\Delta_{+(-)} - (d-1) > -1 \Rightarrow \Delta_{+(-)} > \frac{d}{2} - 1. \tag{39}$$

Therefore, the standard quantization always works, but alternative one only works for those solutions with a $\Delta_- > \frac{d}{2} - 1$. So far as the inner product is well defined, the Hilbert space is done too, which can be delicately (choosing the positive frequency solutions) uplifted to a Fock space in which the QFT live. Finally, the field operator in bulk defined is,

$$\hat{\varphi}(z,x) = \sum_K a_K^\dagger \bar{\varphi}_K(z,x) + a_K \varphi_K(z,x). \tag{40}$$

where $K = -k^a k_a$.

3.2 Regularization and counter terms

As shown above, the one point correlation function goes divergent easily, which necessarily requires renormalization to obtain finite result. Before the normalization, we need to know the behavior of divergence one point correlation function.

Under Euclidean signature, the bulk generating functional reads

$$Z_{E,bulk}(\phi) = \exp(-S_E) = \exp(-\frac{1}{2}\int d^{d+1}x\sqrt{g}(\nabla_a\phi\nabla^a\phi + m^2\phi^2)), \tag{41}$$

which produces the one point correlation function, with AdS/CFT dictionary, generally as

$$\begin{aligned}\langle O\rangle_J &= \frac{1}{\sqrt{\gamma}}\frac{\delta \log(\mathcal{Z}_{CFT}[J])}{\delta J} \\ &= \frac{1}{\sqrt{\gamma}}\frac{\delta \log Z_E}{\delta f_{(0)}}\Big|_{z\to 0},\end{aligned} \tag{42}$$

or more specifically with standard quantization it gives,

$$\langle O\rangle_{\psi_0^-} = \frac{1}{\sqrt{\gamma}}\frac{\delta \log Z_E(\phi)}{\delta\psi_0^-}\Big|_{z\to 0} = -\frac{1}{\sqrt{\gamma}}\frac{\delta S_E(\phi)}{\delta\psi_0^-}\Big|_{z\to 0} \tag{43}$$

The on shell condition kills all other terms except the boundary terms, and since the holography dictionary asks for a restriction to $z=0$ surface, the left action we need to take care of is

$$S_E|_{boundary} = \frac{1}{2}\int_{z\to 0} d^d x (\frac{1}{z})^{d-1}\phi\partial_z\phi. \tag{44}$$

From the bulk perspective, the term diverges due to the vastness of integral region – and in terminology it is called as IR divergence, while from boundary it stands for UV divergence.(I don't quite understand.) The asymptotic expansion of ϕ renders the $\phi\partial_z\phi$ into

$$\phi\partial_z\phi = z^{2d-2\Delta-1}[(d-\Delta){\psi_0^-}^2+...]+z^{d-1}[\Delta\psi_0^-\psi_0^+ +...]+z^{d-1}[(d-\Delta)\psi_0^-\psi_0^+ +...]+z^{2\Delta-1}(\Delta{\psi_0^+}^2+...), \tag{45}$$

where a simple power counting exclude the possibilities for the last 3 series to go divergent - except for a $\log z$ term - after recalling that Δ stands for $\Delta^+ > \frac{d}{2}$. Dealing with the first series leads to

$$\int d^d x(\frac{1}{z})^{d-1} z^{2d-2\Delta-1}[(d-\Delta){\psi_0^-}^2 + (2d-2\Delta+1)z^2\psi_0^-\psi_2^- + ...], \tag{46}$$

where only finite terms whose power $2d-2\Delta-1+2n$ fulfill $2d-2\Delta-1+2n-(d-1)<0$ are not preferable. Equivalently, that is $n\leq\frac{2\Delta-d}{2}$, where $2n$ is the power in z after $2d-2\Delta-1$ is subtracted. To eliminate the divergence terms, we need to subtract the counters like

$$S_{counter}=\frac{1}{2}\int d^dx(\frac{1}{z})^d\frac{d-\Delta}{2}\phi^2+b\phi\Box\phi+c\phi\Box^2\phi+...,\tag{47}$$

where b,c are some other constants obtained from order-by-order comparison. However, the coefficient of the leading order is specifically given as it plays the critical role here, and one may find that leading order term is $\frac{1}{2}(d-\Delta)z^{d-2\Delta}$, which will exactly cancel the first term in 46.

3.3 Renormalized 1 point correlation function

As the counter term is now found, we can cancel all divergence with it before taking regularization parameter to the limit, that is we define a subtracted action firstly as

$$S_{sub}=S_E+S_{counter}.\tag{48}$$

Note that the regularization parameter is gone after the renormalization, consequently the renormalized operator comes out as

$$\langle O\rangle_{\psi_0^-}=-\frac{1}{\sqrt{\gamma}}\frac{\delta S_{ren}(\phi)}{\delta\psi_0^-}=-\lim_{\epsilon\to 0}\frac{1}{\sqrt{\gamma}}\frac{\delta S_{sub}(\phi)}{\delta\psi_0^-}\Big|_{\epsilon=z}.\tag{49}$$

As a result of choosing standard quantization, we can impose the relationship (with λ to indicate who move together),

$$\lim_{z\to 0}\phi(\lambda)=\psi_0^-(\lambda)z^{d-\Delta}+\mathcal{O}(z^{d-\Delta+1})+G[\psi_0^+],\tag{50}$$

while keep the ψ_0^+ series free, with which the one-point correlation function is modified as

$$\begin{aligned}&-\lim_{z\to 0}\frac{1}{\sqrt{\gamma}}\frac{\delta S_{sub}(\phi)}{\delta\psi_0^-}\\&=-\lim_{z\to 0}\frac{1}{\sqrt{\gamma}}\frac{\delta\phi}{\delta\psi_0^-}\frac{\delta S_{sub}(\phi)}{\delta\phi}\\&=-\lim_{z\to 0}\frac{1}{z^{\Delta-d}\sqrt{\gamma}}\frac{\delta S_{sub}(\phi)}{\delta\phi}\\&=-\lim_{z\to 0}\frac{1}{z^{\Delta}\sqrt{g^0}}\frac{\delta S_{sub}(\phi)}{\delta\phi},\end{aligned}\tag{51}$$

where g^0 is the effectively induced metric on asymptotic boundary, which does not really exist in physical space-time. Now we calculate the variation of the original action and the counter term separately.

On one hand, for the variation of the original action, an explicit calculation gives

$$\begin{aligned}&\frac{1}{z^{\Delta}\sqrt{g^0}}\frac{\delta S_E}{\delta\phi}=-\frac{1}{z^{\Delta-1}}\partial_z\phi\\&=-\frac{1}{z^{\Delta-1}}\partial_z\left\{z^{d-\Delta}(\psi_0^-+\psi_2^-z^2+...)+z^{\Delta}(\psi_0^++\psi_2^++...)\right\}\\&=-z^{d-2\Delta}\left[(d-\Delta)\psi_0^-+(d-\Delta+2)\psi_2^-z^2+...\right]-\left[\Delta\psi_0^++(\Delta+2)\psi_2^+z^2+...\right],\end{aligned}\tag{52}$$

where obviously only finite divergent terms are require a cancel-out as adding power in z keeps suppressing the divergence. On the other hand, the variation over the counter term gives

$$\begin{aligned}&\frac{1}{z^{\Delta}\sqrt{g^0}}\frac{\delta S_{counter}}{\delta\phi}=\lim_{z\to 0}\frac{1}{z^{\Delta}}(d-\Delta)\phi\\&=\lim_{z\to 0}(d-\Delta)z^{d-2\Delta}(\psi_0^-+\psi_2^-z^2+...)+(d-\Delta)(\psi_0^++\psi_2^+z^2+...).\end{aligned}\tag{53}$$

A brief observation indicates that we are fortunate enough to cancel the prime divergence now by adding 52 and 53 up! By taking the regularization parameter to limit on the subtracted action, the renormalized one point correlation is attained by

$$\begin{aligned}\langle O\rangle_{\psi_0^-,ren} =& -\lim_{z\to 0}\left(\frac{1}{z^\Delta\sqrt{g^0}}\frac{\delta S_{sub}}{\delta\phi}\right)\\ =& -\lim_{z\to 0}\left(\frac{\delta S_E}{\delta\phi}+\frac{\delta S_{counter}}{\delta\phi}\right)\\ =& -\lim_{z\to 0}\left(-\frac{1}{z^{\Delta-1}}\partial_z\phi+\frac{1}{z^\Delta}(d-\Delta)\phi\right)\\ =&(2\Delta-d)\psi_0^+ + f[\psi_0^-]+...,\end{aligned} \tag{54}$$

where ... terms denotes some terms yet to be canceled like terms $\sim z^{d-2\Delta+2}$, while $f[\psi_0^-]$ is shorthand for finite terms who are only related to the source ψ_0^- directly or utilizing the recurred relation, and the others are all suppressed to 0 through the taking of limit. The rest divergent terms are not a problem, as long as we introduce more counter terms to vanish them, just like what we have done before.

One last thing worth saying is that the alternative quantization can also give us the renormalized one point correlation function, and the result reads

$$\langle O\rangle_{\psi_0^+} = (2\Delta_- - d)\psi_0^- + f[\psi_0^+], \tag{55}$$

but the details are not going to be shown here, where a Legendre transformation will be applied.

4 Holographic renormalization in AdS_4-Maxwell-field case

The fact is that when it comes to the discussion of Maxwell field in AdS_4, holographic renormalization is not needed. Suppose the theory is described by the following action,

$$S = -\int \frac{1}{4}F_{ab}F^{ab}\epsilon = -\int dz d^d x\sqrt{-g}\partial_\mu A_\nu \partial^\mu A^\nu \tag{56}$$

in AdS_4 spacetime depicted with Poincare coordinates,

$$ds^2 = \frac{\ell^2}{z^2}(-dt^2+dz^2+dx^2+dy^2). \tag{57}$$

From the discussion of scalar field, it is critical for us to specify *source* and VEV by solving the equation of motion asymptotically. In this case we use Lorenzian gauge, and only investigate a sub-space of the solutions where A_z is imposed to be 0. The equation of motion is given by

$$\nabla_a F^{ab} = \nabla_a\nabla^a A^b - R_a{}^b A^a = 0, \tag{58}$$

whose concrete version in the coordinate system is

$$\begin{cases} z^4(\partial_z^2 A_t + \partial_{\vec{x}}^2 A_t) = 0\\ z^3(\partial_t A_t - \partial_x A_x - \partial_y A_y) = 0\\ z^4(\partial_z^2 A_x + \partial_{\vec{x}}^2 A_x) = 0\\ z^4(\partial_z^2 A_y + \partial_{\vec{x}}^2 A_y) = 0\end{cases}. \tag{59}$$

What we care is the leading order coefficient of asymptotic expansion of A_μ in z for which act as the source in AdS/CFT correspondence, thus we apply the following asymptotic expansion on vector field at $z=0$ as

$$A_\mu = z^\Delta \sum_{n=0} A_{(n)\mu}(x) z^n, \mu \neq z, \tag{60}$$

whose substitution into the equation of motion yields

$$\begin{cases} \Delta(\Delta-1)A_{(0)\mu}z^{\Delta+2} + (\Delta+1)\Delta A_{(1)\mu}z^{\Delta+3} + (\Delta+2)(\Delta+1)[A_{(2)\mu} + \partial_{\vec{x}}^2 A_{(0)\mu}]z^{\Delta+4} + \mathcal{O}(z^{\Delta+5}) = 0, \mu \neq z \\ z^{\Delta+n+2}\partial_t A_{(n)t} = z^{\Delta+n+2}(\partial_x A_{(n)x} + \partial_y A_{(n)y}) \end{cases}. \tag{61}$$

Following the same procedure of scalar case, we will solve for Δ and the recurrence relation,

$$\Delta = 0, 1 \qquad\qquad A_{(n+2)\mu} = -\partial_{\vec{x}}^2 A_{(n)\mu}, \mu \neq z. \tag{62}$$

Subsequently, there are 2 series acting as 2 independent basis to the solution, where we still denote the smaller Δ as Δ_-, or explicitly, $\Delta_- = 0$, and its counterpart as $\Delta_+ = 1$

$$A_\mu = z^{\Delta_-} A^-_{(n)\mu} z^n + z^{\Delta_+} A^+_{(n)\mu} z^n, \quad \mu \neq z. \tag{63}$$

But straightforwardly one will find the 2 series degenerate severely, since coefficients $A^-_{(n+1)\mu}$ and $A^+_{(n)\mu}$ govern same power. Taking advantage of the recurrence relation, we redefine the 2 series as

$$A_\mu = A^-_{(2n)\mu} z^{2n} + A^+_{(2n)\mu} z^{2n+1}, \quad \mu \neq z. \tag{64}$$

Therefore, by same argument as before, an imposing boundary condition at $z \to \infty$ builds the relation between A_0^+, A_0^- and endow them with the characters of *source* and VEV in holography discussion. More significantly, either A_0^+ or A_0^- being the VEV will give a well-defined inner product in AdS spacetime, though is not shown here, such that we can choose source freely.

After going through all these discussions, now we can choose the source we need at will, therefore turn on the $A^-_{(0)\mu}$ as source, which gives the boundary one-point correlation function by

$$\langle O^i \rangle_{A^-_{(0)i}} = -\lim_{\epsilon\to 0} \frac{1}{\sqrt{\gamma}} \frac{\delta S_E}{\delta A^-_{(0)i}}\Big|_{\epsilon=z} = -\frac{1}{4\pi G}\lim_{\epsilon\to 0}\frac{1}{\sqrt{\gamma}}\sqrt{g^0} n_\mu F^{\mu i}\Big|_{\epsilon=z} = -\frac{1}{4\pi G}\lim_{\epsilon\to 0}\frac{1}{z^3} n_\mu F^{\mu i}\Big|_{\epsilon=z}, \tag{65}$$

as given in 2306.16187 but with Euclidean signature, and $S_E = \frac{1}{16\pi G}\int F_{ab}F^{ab}$. A further calculation using the fact that $n_\mu = n_z = -\frac{1}{z}$, $\gamma_{ij} = z^2 g_{ij}$ yields

$$-\frac{1}{4\pi G}\frac{1}{z^3}(-\frac{1}{z})z^4 \tilde{F}_{zi} = \frac{1}{4\pi G}\partial_z \tilde{A}_i, \quad i = t, \theta, \varphi, \tag{66}$$

notice that the metric is Euclidean.

It is seemingly promising that we can recover the renormalized one point correlator in pure AdS with A_a solution given in the accelerating-AdS case. For one thing, after removing the acceleration in $\{t, z, \theta, \phi\}$ coordinate system vector field A_a reads

$$A_a = -\frac{e}{\kappa} z dt_a - g\cos\theta K d\varphi_a, \tag{67}$$

which can be rewrite as

$$A_\mu dx^\mu = A^-_{(0)\varphi} d\varphi + A^+_{(0)t} z dt. \tag{68}$$

and only $A^+_{(0)t}$ is possible to survive in 65, while other terms all vanish because they have z as a factor. This tells us that,

$$\langle O^t \rangle_{A^-_{(0)i}} = -\frac{1}{4\pi G}\lim_{\epsilon\to 0}\frac{\delta S_E}{\delta A^-_{(0)t}}\Big|_{\epsilon=z} = \frac{1}{4\pi G}A^+_{(0)t} = -\frac{e}{4\pi G\kappa} \tag{69}$$

where ℓ is set as 1 at first. Besides, A_a satisfies Lorenz gauge $\nabla_a A^a$ and $A_z = 0$, which implies it is also a solution to the Pure AdS equation of motion for Maxwell field. On the other hand, looking back at the j^i in $AlAdS$, which is calculated in a covariant way,

$$j^t = \frac{e\kappa}{4\pi G}, \qquad j^\varphi = \frac{g\alpha}{4\pi GK} \tag{70}$$

after ℓ is also fixed as 1, and the removal of acceleration gives

$$j^t = \frac{e\kappa}{4\pi G}. \tag{71}$$

where in all κ is a time scaling, and can be set as 1 in this simpler case, and if we do this, then the result is different by a minus from the renormalized current - which comes from g_{tt}, as the 70 is calculated with Lorenzian signature while the renormalized current is calculated with Euclidean signature.

References

[1] Vijay Balasubramanian and Per Kraus. A stress tensor for anti-de sitter gravity. *Commun.Math.Phys. 208 (1999) 413-428*, February 1999.

[2] Davide Cassani, Jerome P. Gauntlett, Dario Martelli, and James Sparks. Thermodynamics of accelerating and supersymmetric ads_4 black holes. *Phys. Rev. D 104, 086005 (2021)*, June 2021.

[3] Ruth Gregory. Accelerating black holes. December 2017.

[4] Ruth Gregory and Andrew Scoins. Accelerating black hole chemistry. *Physics Letters B*, 796:191–195, sep 2019.

[5] Thomas Hartman, Jorrit Kruthoff, Edgar Shaghoulian, and Amirhossein Tajdini. Holography at finite cutoff with a t^2 deformation. July 2018.

[6] I.. H. Ford Mukunda Aryal and Alexander Vilenkin. Cosmic strings and black holes. *Physics Review D*, 1986.

[7] Kostas Skenderis. Lecture notes on holographic renormalization. *Class.Quant.Grav.19:5849-5876,2002*, September 2002.