Chapter 5
Stability Theory

We consider the system

$$\vec{x}' = \vec{f}(t, \vec{x}), \ \vec{x}(t_0) = \vec{x}_0 \tag{0.1}$$

The set of points $\{\vec{x}(t), \ t \in I\} \subset \mathbb{R}^n$ is called trajectory.

Definition 5.0.1 *A point $\vec{a}$ is called a critical point of the system* (0.1) *if* $\vec{f}(t, \vec{a}) = 0$.

If $\vec{a}$ is a critical point of the system (0.1) then $\vec{x}(t) = \vec{a}$ is a solution of (0.1). Also called as equilibrium point/solution.

Example 5.0.1 $x_1' = -x_1(x_2 - 1), \ \ x_2' = (x_2 - 1)(x_1 - 1),$
Easy to see that $x_2 = 1$ and $x_1 = c$, constant is the critical point.

Let $\vec{x}(t) = \vec{x}(t, t_0, \vec{x}_0)$ be the solution of the IVP (0.1) then we define

Definition 5.0.2 *An equilibrium point $\vec{a}$ of the system $\vec{x}' = \vec{f}(t, \vec{x})$ is stable if for all $\epsilon > 0$ there exists $\delta = \delta(\epsilon, t_0) > 0$ such that*

$$|\vec{x}_0 - \vec{a}| < \delta \implies |\vec{x}(t) - \vec{a}| < \epsilon, \ \forall \ t \geq t_0.$$

The point $\vec{a}$ is called uniformly stable if δ is independent of t_0.

Definition 5.0.3 *An equilibrium point $\vec{a}$ of the system $\vec{x}' = \vec{f}(t, \vec{x})$ is (locally) asymptotically stable if it is stable and there exists $\delta = \delta(t_0) > 0$ such that*

$$\vec{x}_0 \in B_\delta(\vec{a}) \implies \lim_{t \to \infty} \vec{x}(t) = \vec{a}.$$

if δ is independent of t_0 then it is called uniformly asymptotically stable.

Definition 5.0.4 *If $\vec{a}$ is not stable then we say $\vec{a}$ is unstable equilibrium point.*

5.1 Linear System

5.1.1 2×2 *Linear System*

We first study the case of linear system with constant coefficients

$$\vec{x}' = A\vec{x}$$

Then the equilibrium point is $\vec{a} = 0$. Then we have the following from the fundamental matrix

Theorem 5.1.1 *The critical point* 0 *is asymptotically stable if and only if all eigenvalues of A have negative real parts. If at least one eigenvalue has positive real part, then* 0 *is unstable.*

Let us understand the types of equilibrium/critical points in case of $n = 2$. There are 6 types of critical points. We already noted that the behaviour of the solutions of $\vec{x}' = A\vec{x}$ is same as the behaviour of solutions of its Jordan equivalent system $\vec{x}' = J\vec{x}$ where J is the Jordan block of A. Since we are in the case of two dimensions, we try to understand with the following examples

Example 5.1.1 *Improper Critical point: All trajectories, except two, have the same limiting direction of tangent. The two exceptional ones also have limiting direction of tangent but different from the first one. For example*

$$\vec{x}' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \vec{x}$$

Then the general solution is

$$\begin{aligned} \vec{y} &= c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t} \\ &= c_1 \vec{y}^1 + c_2 \vec{y}^2 \end{aligned}$$

$\vec{y}^2$ is exceptional directions and $\vec{y}^1$ is the limiting tangential direction.

Example 5.1.2 *Proper critical point: All directions are same. For example*

$$\vec{x}' = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \vec{x}, \qquad \lambda < 0$$

In this case the general solution is

$$\vec{y} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\lambda t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{\lambda t}$$

Example 5.1.3 *Saddle point: Two incoming trajectories and two outgoing trajectories. All other trajectories in a neighbourhood of critical point avoid the critical point. for example*

$$\vec{x}' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \vec{x}$$

the general solution is

$$\begin{aligned} \vec{y} &= c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} \\ &= c_1 \vec{y}^1 + c_2 \vec{y}^2 \end{aligned}$$

Example 5.1.4 *Degenerate critical point: All the trajectories have only one limiting tangential direction. This is the case when geometric multiplicity is NOT equal to the algebraic multiplicity of an eigenvalue. For example,*

$$\vec{x}' = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix} \vec{x}$$

In this case $\lambda = 3$ is the only eigenvalue with $(1,-1)^T$ as the eigen vector. The general solution here is

$$\begin{aligned} \vec{y} &= c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] e^{3t} \\ &= c_1 \vec{y}^1 + c_2 \vec{y}^2 \end{aligned}$$

Example 5.1.5 *Centre: The eigenvalues are of purely complex. In this case the critical point is enclosed by infinitely many closed paths. For example,*

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \vec{x}$$

The general solution in this case is

$$\vec{y} = c_1 \begin{pmatrix} 1 \\ 2i \end{pmatrix} e^{2it} + c_2 \begin{pmatrix} 1 \\ -2i \end{pmatrix} e^{-2it}$$

In this case we see that the system of equations is

$$y_1' = y_2,\ 4y_1 = -y_2'.$$

That is

$$4y_1 y_1' = -y_2 y_2'.$$

Integrating this, we get

$$2y_1^2 + \frac{1}{2} y_2^2 = constant$$

Example 5.1.6 *Spiral: A spiral point P about which trajectories spiral approaching P as $t \to \infty$. For example*

$$\vec{y}' = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \vec{y}$$

The eigenvalues are $-1 \pm i$. The general solution is

$$\vec{y} = c_1 \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-1+i)t} + c_2 \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(-1-i)t}$$

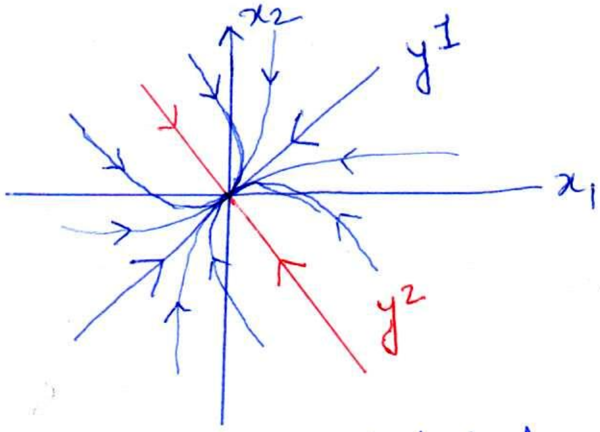
To see the spirality, we note from the system

$$y_1 y_1' + y_2 y_2' = -(y_1^2 + y_2^2)$$

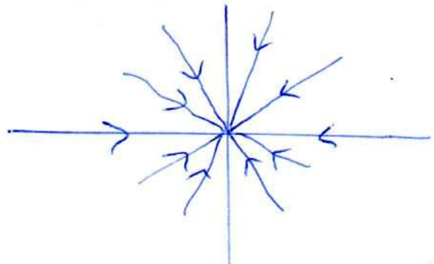
going to polar form (r,t): $r^2 = y_1^2 + y_2^2$, we see the above equation as

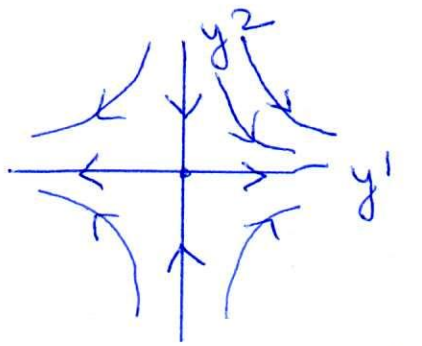
$$\frac{1}{2}(r^2)' = -r^2$$

This implies $r' = -r$. Therefore $r = ce^{-t}$. That is as $t \to \infty$, we get r, the distance from O, approaches 0.

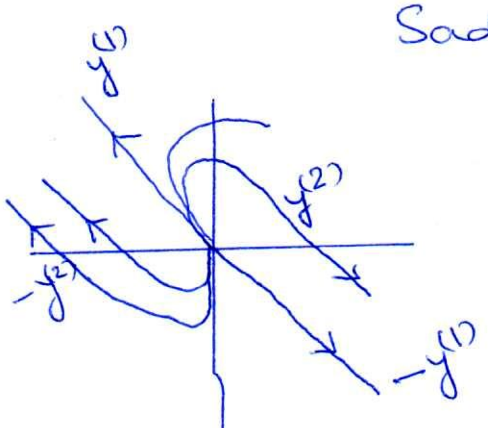


Improper critical point

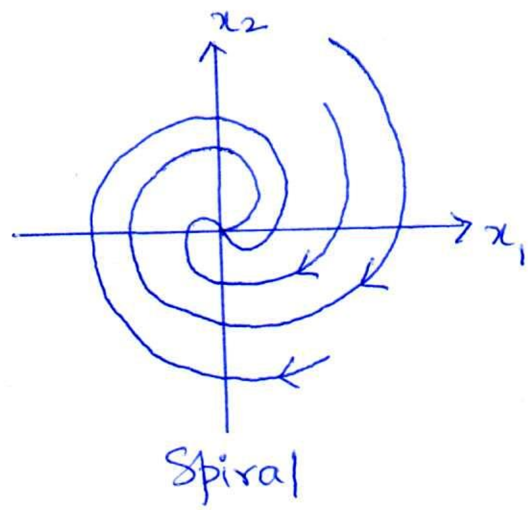

proper critical point



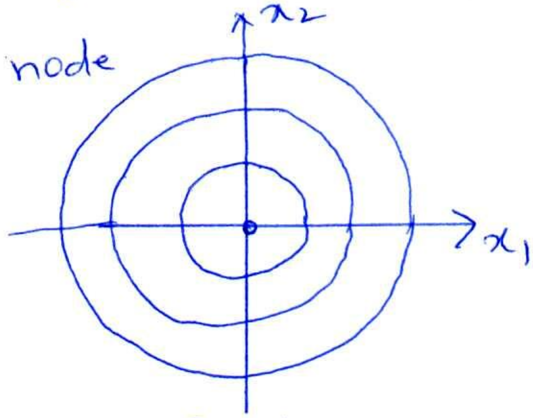
Saddle point



Degenerate node



Spiral



Centre

5.1.2 Asymptotic behaviour for $n \times n$ system

From the previous section we see that the solution of linear system

$$\vec{x}' = A\vec{x}, \ \vec{x}(t_0) = \vec{x}_0 \tag{1.2}$$

has solutions in the form

$$\vec{\phi}(t) = P \ diag[exp(B_j(t-t_0))] \ P^{-1}\vec{x}_0.$$

In the exponential notation, we write the solution as

$$\vec{\phi}(t) = e^{A(t-t_0)}\vec{x}_0, \tag{1.3}$$

where columns of P are the generalized eigen vectors and B_j are the Jordan blocks corresponding to eigen values λ_j. Each component of the solution of the system is a linear combination of functions of the form

$$(t-t_0)^\alpha exp[a(t-t_0)]\cos[b(t-t_0)] \quad \text{or } (t-t_0)^\alpha exp[a(t-t_0)]\sin[b(t-t_0)]$$

where $\lambda = a \pm ib$ are eigenvalues of the matrix A and α is less than the order of matrix. Let us assume that $\lambda_j = a_j \pm ib_j$ has multiplicity p_j. Then, there exists $t_1 > 0$ such that

$$||\vec{\phi}(t)|| \leq Ce^{at}t^p, \quad \forall t \geq t_1 \tag{1.4}$$

where $a = \max_j a_j$, $p = \max_j p_j$ and C is a suitable constant. Then we have the following theorem on the asymptotic behaviour:

Theorem 5.1.2 *The solutions of the system of equations in* (1.2) *satisfies the following:*

1. *Every solution of $\vec{x}(t)$ of the linear system tends to zero as $t \to \infty$ if and only if the real parts of the eigenvalues of the matrix A are negative.*
2. *Every solution of $\vec{x}(t)$ of the linear system is boundedif and only if the real parts of the repeated eigenvalues are negative and of simple eigenvalues are non-positive.*

Next we want to understand the asymptotic behaviour of the system like

$$\vec{x}' = A\vec{x} + B(t)\vec{x}, \ \vec{x}(0) = \vec{x}_0 \tag{1.5}$$

where A is a matrix with constant elements and $B(t)$ is matrix with variable elements. From (3.6), we can write the solution of (1.5) as follows: Let $\phi(t) = e^{At}$ is the solution of $\vec{x}' = A\vec{x}$, $x(0) = I$, then the solution of (1.5) is

$$\begin{aligned}\vec{x}(t) &= \phi(t)\vec{x}_0 + \int_0^t \phi(t-s)B(s)\vec{x}(s)ds \\ &= e^{At}\vec{x}_0 + \int_0^t e^{A(t-s)}B(s)\vec{x}(s)ds\end{aligned}$$

We have the folloiwng Grownwall's inequality

Theorem 5.1.3 *Let $u(t), p(t), q(t)$ be nonnegative continuous functions in th interval* $[t_0, t_1]$ *and let*

$$u(t) \le p(t) + \int_{t_0}^{t} q(s)u(s)ds,\ t \in [t_0, t_1].$$

Then we have

$$u(t) \le p(t) + \int_{t_0}^{t} p(\tau)q(\tau)exp\left(\int_{r}^{t} q(s)ds\right)d\tau,\ t_0 \le t \le t_1.$$

Moreover if $p(t)$ is nondecreasing, then

$$u(t) \le p(t)\ exp\left(\int_{0}^{t} q(s)ds\right)$$

Theorem 5.1.4 *Suppose the fundamental matrix e^{At} bounded and*

$$\int_{0}^{\infty} ||B(s)||ds < \infty, \tag{1.6}$$

then any solutions $x(t)$ of (1.5) *is bounded. Here $||B||$ is defined as*

$$||B|| = \sup_{j} ||B^j||,\ B^j \text{ is the column of } B.$$

Proof. Suppose there exists $c > 0$ such that

$$||e^{At}|| \le c,\ t \ge 0.$$

Therefore

$$\begin{aligned} ||\vec{x}(t)|| &\le ||\vec{x}_0||||e^{At}|| + \int_{0}^{t} ||e^{A(t-s)}||||B(s)||||\vec{x}(s)||ds \\ &\le c||\vec{x}_0|| + c\int_{0}^{t} ||B(s)||||\vec{x}(s)||ds \end{aligned}$$

Then by the Grownwall's inequality (taking $u(t) = ||\vec{x}(t)||, p(t) = ||\vec{x}_0||, q(s) = ||B(s)||$), we get

$$||\vec{x}(t)|| \le c||\vec{x}_0||exp\left(\int_{0}^{t} ||B(s)||ds\right)$$

Hence $||\vec{x}(t)||$ is bounded provided (1.6) holds. □

Theorem 5.1.5 *Suppose the real parts of eigenvalues of the matrix are negative and*

$$||B(t)|| \to 0 \ \text{ as } t \to \infty.$$

Then all solutions of (1.5) *converges to* 0 *as $t \to \infty$.*

Proof. There exists $\delta > 0$ such that

$$||e^{At}|| \le ce^{-\delta t},\ \ \forall\ t \ge 0.$$

For any $\epsilon > 0$ there exists $t_1 > 0$ such that

$$||B(t)|| < \epsilon,\ \ t \ge t_1.$$

From the variation of parameters formula, we get

$$||\vec{x}(t)|| \le ce^{-\delta t}||\vec{x}_0|| + c\int_0^t e^{-\delta(t-s)}||B(s)||||\vec{x}(s)||ds$$

Therefore,

$$\begin{aligned} ||\vec{x}(t)||e^{\delta t} &\le c||\vec{x}_0|| + c\int_0^{t_1} e^{\delta s}||B(s)||||\vec{x}(s)||ds + c\int_{t_1}^t e^{\delta s}||B(s)||||\vec{x}(s)||ds \\ &\le c||\vec{x}_0|| + c\int_0^{t_1} e^{\delta s}||B(s)||||\vec{x}(s)||ds + c\epsilon\int_{t_1}^t e^{\delta s}||\vec{x}(s)||ds \end{aligned}$$

Now taking $u(t) = e^{\delta t}||x(t)||$ and $C = c||x_0|| + c\int_0^{t_1} e^{\delta s}||B(s)||||x(s)||ds$ the above inequality is

$$u(t) \le C + c\epsilon\int_{t_1}^t u(s)ds, t \in I.$$

and applying Grownwall's inequality, we get

$$u(t) \le Ce^{\int_{t_1}^t c\epsilon ds} = Ce^{c\epsilon(t-t_1)}$$

Therefore,

$$||\vec{x}(t)|| \le e^{\delta t}Ce^{c\epsilon(t-t_1)} = Ce^{t(c\epsilon-\delta)-c\epsilon t_1}$$

So if we choose ϵ small we see that $||\vec{x}(t)|| \to 0$ as $t \to \infty$. □

5.2 Lyapunov Stability

In this section we describe the Lyapunov indirect method of stability through positive definite functions.

Definition 5.2.1 *We say an equilibrium point $\vec{x}_0$ is globally asymptotically asymptotically stable if it is stable for all initial conditions $\vec{x}_0 \in \mathbb{R}^n$.*

5.2.1 Stability theorems

Definition 5.2.2 *A real valued function $\lambda : [0,a) \to \mathbb{R}$ is said to be of class $K([0,a))$ if*

1. *λ is continuous*
2. *λ is strictly increasing*
3. *$\lambda(0) = 0$.*

Definition 5.2.3 *A function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be positive definite if*

1. *$V(0) = 0$*
2. *There exists $\lambda \in K([0,a))$ such that*

$$V(\vec{x}) \ge \lambda(|\vec{x}|), \ \forall \vec{x} \in B_a(0)$$

V is negative definite if $-V$ is positive definite.

Example 5.2.1 *$V:\mathbb{R}^2\to\mathbb{R}$ defined as*

$$V(\vec{x}) = |\vec{x}|^2 = x_1^2 + x_2^2$$

In this case we can take $\lambda(r) = r^2$, then $\lambda \in K((0,\infty))$ and $V(\vec{x}) = \lambda(|\vec{x}|)$.

Definition 5.2.4 Derivative along trajectories: *The derivative of Lyapunov function along trajectories is*

$$\dot{\boldsymbol{V}}(\vec{x}) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\frac{dx_i}{dt}$$

We have the following theorem of Lyapunov for stability:

Theorem 5.2.1 *Let 0 be an equilibrium point of the system $\vec{x}' = \vec{f}(\vec{x})$. If there exists a positive definite function V and $\dot{\boldsymbol{V}}(\vec{x}) \le 0$ for all $t \ge t_0$. Then 0 is a stable equilibrium point of the sytem.*

Proof. Positive definiteness of V implies there exists $\lambda \in K$ such that

$$V(\vec{x}) \ge \lambda(|\vec{x}|) \ge 0. \tag{2.7}$$

Let $\vec{x}$ be the solution of

$$\vec{x}' = \vec{f}(\vec{x}),\ \vec{x}(t_0) = \vec{x}_0.$$

Now

$$V(\vec{x}(t)) - V(\vec{x}_0) = \int_{t_0}^{t} \frac{dV(\vec{x}(s))}{ds}ds \le 0$$

That is,

$$V(\vec{x}(t)) \le V(\vec{x}_0)\ \ \forall t \ge t_0 \tag{2.8}$$

The continuity of V implies, for any given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|\vec{x}_0| < \delta \implies V(\vec{x}_0) < \lambda(\epsilon) \tag{2.9}$$

From (2.7), (2.8) and (2.9) we get

$$\lambda(|\vec{x}|) \le V(\vec{x}(t)) < \lambda(\epsilon)$$

Since λ is strictly increasing, we get $|\vec{x}(t)| < \epsilon$. Therefore we have

$$|\vec{x}_0| < \delta \implies |\vec{x}(t)| < \epsilon.$$

Hence 0 is locally asymptotically stable. □

Theorem 5.2.2 *Let 0 be an equilibrium point of the system $\vec{x}' = \vec{f}(\vec{x})$. If there exists a positive definite function V and $\dot{\boldsymbol{V}}(\vec{x})$ is negative definite. Then 0 is asymptotically stable equilibrium point of the sytem.*

Proof. Let $\vec{x}(t)$ be the solution of the IVP: $\vec{x}'(t) = \vec{f}(\vec{x}),\ \ \vec{x}(t_0) = x_0$. Suppose $\vec{x}(t) \nrightarrow 0$ as $t \to \infty$.
$V(\vec{x}(t))$ is decreasing, non-negative. So it converges to ϵ_0 (say) as $t \to \infty$.
(Suppose $V(x(t)) \to 0$ then $V(\vec{x}) \ge \lambda(|\vec{x}|)$ implies $\lambda(|\vec{x}|) \to 0$ and increasing nature of λ implies $|\vec{x}| \to 0$, a contradiction.)

Let $A = \{\vec{z} \in \mathbb{R}^n :\ \ \epsilon \le V(\vec{z}) \le V(x(0))\}$. Then A is compact subset of $\mathbb{R}^n$. Hence $\frac{dV}{dt}$ attains its supremum on A. That is

$$\sup_A \frac{dV}{dt} = -a < 0.$$

Therefore

$$\begin{aligned} V(\vec{x}(t)) &= V(\vec{x}_0) + \int_{t_0}^{t} \frac{dV(\vec{x}(s))}{ds} ds \\ &\leq V(\vec{x}_0) - a(t - t_0) < 0, \ \text{ for large } t \end{aligned}$$

Contradiction to the positive definiteness of V. □

Example 5.2.2 *Study the stability of the critical point* 0 *of the system*

$$x_1' = (x_1^2 + x_2^2)(x_2 - x_1);\ x_2' = -(x_1^2 + x_2^2)(x_1 + x_2)$$

Consider the positive definite function

$$V(x_1, x_2) = x_1^2 + x_2^2$$

Then

$$\begin{aligned} \frac{dV}{dt} &= 2x_1 x_1' + 2x_2 x_2' \\ &= 2(x_1^2 + x_2^2)(x_1 x_2 - x_1^2) + 2(x_1^2 + x_2^2)(-x_1 x_2 - x_2^2) \\ &= -2(x_1^2 + x_2^2)(x_1^2 + x_2^2) < 0 \end{aligned}$$

Hence 0 is globally asymptotically stable critical point. □

Example 5.2.3 *Study the stability of the critical point* 0 *of the system*

$$x_1' = -x_1 + x_2 + x_1(x_1^2 + x_2^2);\ x_2' = -x_1 - x_2 + x_2(x_1^2 + x_2^2)$$

Here again we take $V(x_1, x_2) = x_1^2 + x_2^2$. Then

$$\begin{aligned} \frac{dV}{dt} &= x_1 x_1' + x_2 x_2' \\ &= -(x_1^2 + x_2^2) + (x_1^2 + x_2^2)^2 \end{aligned}$$

Taking $r^2 = x_1^2 + x_2^2$ we see

$$\frac{dV}{dt} = -r^2 + r^4 = -r^2(1 - r^2) < 0 \ \ for\, r < 1$$

So if $|\vec{x}_0| < 1$ then $\frac{dV}{dt} < 0$. Once the trajectory enters the unit ball then it approaches zero as $t \to \infty$. In such cases we say 0 is locally asymptotically stable critical point. □

Example 5.2.4 *Damped Pendulum: Let x_1 be the angular displacement θ and $x_2 = x_1'$, the angular velocity. The motion of pendulum is governmened by*

$$x_1' = x_2;\ \ x_2' = -\sin x_1 - b x_2$$

The critical points are $(0,0), (\pm n\pi, 0)$. Let us consider the point $(0,0)$. We use the Lyapunov function

$$V(x_1, x_2) = \frac{b^2}{2} x_1^2 + b x_1 x_2 + x_2^2) + 2(1 - \cos x_1)$$

Then we can check that

$$\frac{dV}{dt} = -b(x_2^2 + x_1 \sin x_1) < 0,\ \forall\ (x_1, x_2) \in B_1(0)\backslash\{0\}.$$

Hence once the trajectory reaches insided the unit ball then it approaches zero as $t \to \infty$. Hence, $(0,0)$ is asymptotically stable.

There are Lyapunov functions that are of the form

$$V(x) = C_1 x_1^n + C_2 x_2^m$$

where $n, m \in \mathbb{N}$ and C_1, C_2 are positive constants. For example

Example 5.2.5 *Study the stability of the critical point* $(0,0)$

$$x_1' = -3x_1 - x_2;\ \ x_2' = -2x_2^3 + x_1^5$$

In this case we take the positive definite function

$$V(x) = \frac{1}{3}x_1^6 + x_2^2$$

Then

$$\frac{dV}{dt} = -2(3x_1^8 + 2x_2^4) < 0$$

Hence the system is asymptotically stable.

The following theorem is important for asymptotically linear systems. The relation below (2.10) is known as *Lyapunov linear matrix relation*

Theorem 5.2.3 *The following statements are equivalent*

1. *All eigenvalues of a matrix A have negative real parts*
2. *For any give symmetric positive definite matrix Q, there exists unique positive definite matrix P such that*
$$A^T P + PA = -Q \tag{2.10}$$
3. 0 *is asymptotically stable critical point of the system* $\vec{x}' = A\vec{x}$*.*

Proof. We know that 1 and 3 are equivalent. We will show $1 \implies 2$ and $2 \implies 3$.
$\underline{1 \implies 2}$: Suppose all the eigenvalues of A have negative real parts and let Q be a positive definite matrix. Then define

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt$$

Since the elements of the integrand matrix are all linear combinations of functions of the form $t^k e^{\alpha t}$ where α has negative real part. Therefore the integral converges. We will show that P is the required matrix.

$$A^T P + PA = \int_0^\infty A^T e^{A^T t} Q e^{At} + e^{A^T t} Q e^{At} A = \int_0^\infty \frac{d}{dt}\left(e^{A^T t} Q e^{At}\right) = \left[e^{A^T t} Q e^{At}\right]_0^\infty = -Q$$

Also it is easy to check for any $z \not\equiv 0$

$$z^T P z = \int_0^\infty z^T e^{A^T t} Q e^{At} z = \int_0^\infty \left(e^{At}\right)^T Q \left(e^{At}\right) > 0$$

To show the uniqueness, let P_1 and P_2 satisfy (2.10)

$$A^T P_1 + P_1 A = -Q$$
$$A^T P_2 + P_2 A = -Q$$

Then pre-multiply by $e^{A^T t}$ and post multiply by e^{At} and subtract to get

$$\begin{aligned} 0 &= e^{A^T t} A^T (P_1 - P_2) e^{At} + e^{A^T t} (P_1 - P_2) A e^{At} \\ &= \frac{d}{dt}\left(e^{A^T t}(P_1 - P_2)e^{At}\right) \end{aligned}$$

Therefore $e^{A^T t} A^T (P_1 - P_2) e^{At} = c$, constant. Taking $t = 0$ we get $P_1 - P_2 = c$. But taking limit $t \to \infty$ we get $c = 0$.

<u>$2 \implies 3$</u>: Take $Q = I$ and let P be the matrix satisfying (2.10). Now define the Lyapunov function V as

$$V(x) = x^T P x$$

Then V is positive definite function and

$$\frac{dV}{dt} = (x\,')^T P x + x^T P x\,' = (Ax)^T P x + x^T P A x = x^T (A^T P + PA) x = -x^T I x = -|x|^2 < 0$$

Therefore 0 is asymptotically stable critical point of the sytem. □

Now we will study the asymptotically linear systems. Consider the nonlinear system.

$$x' = f(x)$$

Let A be the matrix defined by

$$A = \left[\frac{\partial \vec{f}}{\partial x}\right]_{x=0} = \begin{pmatrix} \frac{\partial \vec{f_1}}{\partial x_1} & \frac{\partial \vec{f_1}}{\partial x_2} & \cdots & \frac{\partial \vec{f_1}}{\partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial \vec{f_n}}{\partial x_1} & \frac{\partial \vec{f_n}}{\partial x_2} & \cdots & \frac{\partial \vec{f_n}}{\partial x_n} \end{pmatrix}_{x=0}$$

Then by Taylor's theorem

$$f(x) = Ax + r(x)$$

where $|r(x)| \to 0$ as $|x| \to 0$. This motivates the following theorem

Theorem 5.2.4 *Suppose* 0 *is a critical point of the system*

$$x' = Ax + r(x) \tag{2.11}$$

where A and $r(x)$ *satisfy*

1. *All eigenvalues of A have negative real parts*
2. $\lim_{|x|\to 0} \frac{|r(x)|}{|x|} = 0.$

Then 0 *is asymptotically stable critical point of the system* (2.11)

Proof. By the above theorem taking $Q = I$ we get P, symmetric positive definite matrix such that (2.10) is satisfied. Consider the Lyapunov function $V(x) = x^T Px$. Then

$$\begin{aligned}\frac{dV}{dt} &= (x^T)' Px + x^T Px' \\ &= (Ax + r(x))^T Px + x^T P(Ax + r(x)) \\ &= x^T(A^T P + PA)x + r(x)^T Px + x^T Pr(x) \\ &= -|x|^2 + r(x)^T Px + x^T Pr(x)\end{aligned}$$

From 2, we estimate, for any $\epsilon > 0$ there exists δ such that

$$|x| < \delta \implies |r(x)| < \epsilon|x|$$

Therefore

$$\begin{aligned}|r(x)^T Px + x^T Pr(x)| &\leq |r(x)^T|||P|||x| + |x^T|||P|||r(x)| \\ &= 2|x||r(x)|||P|| \\ &< 2\epsilon|x|^2||P||\end{aligned}$$

Hence for small ϵ, we get

$$\frac{dV}{dt} \leq -|x|^2(1 - \epsilon||P||) < 0$$

That is, once the path enters the ball $B_\epsilon(0)$ then it approaches zero as $t \to \infty$. In this case we call) is locally asymptotically stable. □

Example 5.2.6 *We go back to our earlier example of Damped Pendulum. Let x_1 be the angular displacement θ and $x_2 = x_1'$, the angular velocity. The motion of pendulum is governmened by*

$$x_1' = x_2;\; x_2' = -\sin x_1 - bx_2$$

The critical points are $(0,0), (\pm n\pi, 0)$. First let us consider the point $(0,0)$:
The linearlized system is

$$\vec{x}' = A\vec{x} + r(\vec{x}) = \begin{pmatrix} 0 & 1 \\ -1 & -b \end{pmatrix}\vec{x} + r(\vec{x})$$

where $|r(x)| \sim |x|^2$ as $|x| \to 0$. The eigenvalues of the matrix is

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -1 & -b - \lambda \end{vmatrix} = \lambda^2 + \lambda b + 1$$

That is,

$$\lambda = -\frac{b}{2} \pm \frac{1}{2}\sqrt{b^2 - 4}.$$

Then we have the following

1. If $b^2 > 4$ then both eigenvalues are negative reals. $(0,0)$ is asymptotically stable.

2. If $b^2 = 4, b > 0$ then negative, repeated eigenvalue. $(0,0)$ is Asymptotically stable.
3. If $0 < b^2 < 4$ then eigen values are complex with negative real parts. The point $(0,0)$ is spiral asympotically stable point.

Now we turn to other critical points $(\pm\pi, 0)$: The linearization system is

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ 1 & -b \end{pmatrix} \vec{x} + r(\vec{x})$$

where $|r(x)| \sim |x|^2$. The eigenvalues of the system are

$$\lambda = -\frac{b}{2} \pm \frac{1}{2}\sqrt{b^2+4}$$

At least one of the eigenvalues is positive. So these points $(\pm\pi, 0)$ are unstable.

Remark 5.2.1 *The basic idea to take $Q = I$ or some other positive definite matrix and compute P satisfying* (2.10). *If P is uniquely dertermined and positive definite then the system is asympotically stable.*

5.2.2 Instability theorems

We have the following theorems on instability of equilibrium points

Theorem 5.2.5 *Suppose*

1. 0 *is an equilibrium point of the system* $x' = f(x)$
2. $V(x) : \mathbb{R}^n \to \mathbb{R}$ *be a continuously differentiable function such that* $V(0) = 0$.
3. $V(x_0) > 0$ *for some* x_0 *with small* $|x_0|$
4. $\dot{V}(x) > 0$ *on* $U = \{x \in B_r(0) : \ V(x) > 0\}$ *for some* $r > 0$.

Then 0 *is unstable.*

Proof. $V(\vec{x}_0) > 0$ and V is continuous implies $\vec{x}_0 \in int(U)$. Let $V(\vec{x}_0) = a > 0$ and let $\vec{x}(t)$ be the solution of $\vec{x}' = \vec{f}(\vec{x}),\ \vec{x}(t_0) = \vec{x}_0$.

Suppose $\vec{x}(t)$ lies inside U for t in some interval, then 4 implies

$$V(x(t)) \geq a.$$

Now let

$$\gamma = \min\left\{\frac{dV}{dt}(\vec{x}(t)),\ \vec{x}(t) \in U,\ V(\vec{x}(t)) \geq a\right\}$$

is achieved as $\frac{dV}{dt}$ is continuous and $\{x \in U : \ V(x(t)) \geq a\}$ is compact. Clearly $\gamma > 0$.

Then we have

$$V(\vec{x}(t)) = V(\vec{x}_0) + \int_{t_0}^{t} V(\vec{x}(s))ds \geq a + \gamma(t - t_0)$$

implying that $x(t)$ cannot stay in U as V is bounded in U (V being continuous function on $B_r(0)$). Moreover, since $V(\vec{x}(t)) \geq a$ implies $x(t)$ will leave U through the boundary $\|\vec{x}| = r$. This can happen for small $|\vec{x}_0|$. So 0 is unstable. □

Corollary 5.2.1 *Suppose V is positive definite and $\dot{\mathbf{V}}$ is also positive definite then* 0 *is unstable.*

Theorem 5.2.6 *Suppose*

1. 0 *is an equilibrium point of the system $\vec{x}' = \vec{f}(\vec{x})$*
2. *There exists $V, W : \mathbb{R}^n \to \mathbb{R}$, $\lambda, \epsilon > 0$ such that*

$$V(0) = 0, W(\vec{x}) \geq 0, \forall \vec{x} \in B_\epsilon(0)$$

$$\frac{d}{dt} V(x(t)) = \lambda V(x(t)) + W(x(t))$$

3. *For any small $\delta > 0$ there exists $\vec{x}_0$ such that $|\vec{x}_0| < \delta$ and $V(\vec{x}_0) > 0$.*

Then 0 *is unstable.*

Proof. Suppose 0 is stable. Then for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\vec{x}_0 \in B_\epsilon(0) \implies \vec{x}(t) \in B_\epsilon(0), \ \forall t \geq t_0$$

$V(x_0) > 0$, and $\frac{d}{dt} V(x(t)) = \lambda V(x(t)) + W(x(t)), W \geq 0$ implies $\frac{d}{dt} V(x(t)) \geq \lambda V(x(t))$. That is

$$\frac{d}{dt}(e^{-\lambda t}) V(x(t)) \geq 0$$

This implies

$$V(x(t)) \geq e^{\lambda t} V(x(0)), \ t \geq t_0$$

Therefore as $t \to \infty$

$$V(x_0) > 0 \implies x(t) \notin B_\epsilon(0)$$

This is a contradiction. □

For asymptotically linear systems, we have the following

Theorem 5.2.7 *Consider the system $\vec{x}' = Ax + r(x)$. Suppose*

1. *A has distinct eigenvalues and are non-zero.*
2. *At least one of the eigenvalues has positive real part.*
3. $\lim\limits_{|x| \to 0} \dfrac{|r(x)|}{|x|} = 0$.

Then 0 *is unstable.*

Proof. is left as an exercise.

Remark 5.2.2 *Note that we cannot say anything when we just know that the eigenvalues of the matrix have non-positive real parts. That is, nothing can be said if there are eigenvalues on the imaginary axis. For instance, a linear system with all its eigenvalues at* 0 *can be either stable or unstable. For instance* $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *versus* $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Even when linearized system is stable the nonlinear system can be unstable. For example $x' = -x^2$ and $x' = x^2$ with $x(t_0) = x_0$ has solutions $x(t) = \frac{1}{\frac{1}{x_0} + t}$ and $x(t) = \frac{1}{\frac{1}{x_0} + t}$. The first one is stable and second one is unstable (as it blows up for finite t).

Example 5.2.7 *Study the stability of the critical point* 0 *of the system*

$$x_1' = x_2 \quad x_2' = x_1 - x_2 \sin x_1$$

Let $V(x_1, x_2) = x_1 x_2$. Then V is continuously differentiable and $V(0) = 0$. In every neighbourhood of 0 there exists $\vec{x}$ such that $V(\vec{x}) > 0$. Also

$$\begin{aligned} \frac{dV}{dt} &= x_1' x_2 + x_1 x_2' \\ &= x_2^2 + x_1^2 - x_1 x_2 \sin x_1 \end{aligned}$$

Now going to the polar coordinates $x_1 = r\cos\theta, x_2 = r\sin\theta$, we see

$$\frac{dV}{dt} = r^2 - r^2 \sin(r\cos\theta)(\sin\theta\cos\theta) > 0$$

for small r. Therefore 0 is unstable. □

Example 5.2.8 *Study the stability of the critical point* 0 *of the system*

$$x_1' = 3x_1 + x_2^2; \quad x_2' = -2x_2 + x_1^3$$

Let $V = \frac{1}{2}(x_1^2 - x_2^2)$. Then $V > 0$ on x_1-axis. Also

$$\frac{dV}{dt} = (3x_1^2 + 2x_2^2) + (x_1 x_2^2 - x_1^3 x_2)$$

Taking $x_1 = r\cos\theta, x_2 = r\sin\theta$ we see that for small r

$$|x_1 x_2^2 - x_1^3 x_2| < 3x_1^2 + 2x_2^2$$

Therefore from the above relation we get

$$\frac{dV}{dt} > 0 \text{ on } B_r(0)$$

for small $r > 0$. Hence 0 is unstable. □

5.2.3 Non-autonomous systems

Stability for non-autonomous systems at times quite tricky and solutions can behave quite different from that of autonomous system. For example,

$$x_1' = x_2; \quad x_2' = -x_1 - (2 + e^t)x_2$$

The system has real eigenvalues that are negative for all $t > 0$:

$$\lambda = -\frac{-(2 + e^t) \pm \sqrt{(2 + e^t)^2 - 4}}{2}.$$

But the solution

$$x_1 = 1 + e^{-t},\ x_2 = -e^{-t}$$

does not converge to the origin as $t \to \infty$.

On the other hand we could find examples of systems which will have a positive eigenvalue but the system is asymptotically stable.

Example 5.2.9 (K. Josic-Rosenbaum, SIAM Review, 2008) *Consider the system $\vec{x}' = A(t)\vec{x}$ with the matrix*

$$A(t) = \begin{pmatrix} -1 - 9\cos^2(6t) + 12\sin(6t)\cos(6t) & 12\cos^2(6t) + 9\sin(6t)\cos(6t) \\ -12\sin^2(6t) + 9\sin(6t)\cos(6t) & -1 - 9\sin^2(6t) - 12\sin(6t)\cos(6t) \end{pmatrix}$$

It is not difficult to check that

$$Det(A + I) = 0 \ \ textand \ \ Det(A + 10I) = 0$$

implying -1 and -10 are eigenvalues. However one of solution is

$$x(t) = \begin{pmatrix} e^{2t}[\cos(6t) + 2\sin(6t)] + 2e^{-13t}[2\cos(6t) - \sin(6t)] \\ e^{2t}[2\cos(6t) - \sin(6t)] - 2e^{-13t}[\cos(6t) + 2\sin(6t)] \end{pmatrix}$$

which is unstable. □

So eigenvalues does not determine the asymptotic stability.

Consider the non-autonomous system

$$\vec{x}' = \vec{f}(t, \vec{x}),\ \ \vec{x}(t_0) = \vec{x}_0$$

Then we have the following Lyanpunov stability theorems

Theorem 5.2.8 *Suppose*

1. *0 is an equilibrium point of the system $\vec{x}' = \vec{f}(t, \vec{x})$*
2. *let $V(t, \vec{x}) : [t_0, \infty] \times B_r(0) \to \mathbb{R}$ be a continuous function such that*

$$W_1(\vec{x}) \le V(t, \vec{x}),\ \forall \vec{x} \in B_r(0)$$

   where W_1 is a positive definite function

3. *For all $t \ge t_0$ and $\vec{x} \in B_r(0)$,*

$$\frac{d}{dt}\left[V(t, \vec{x}(t))\right] \le 0$$

Then 0 is stable point. Moreover if there exists W_2 such that $V(x) \le W_2(\vec{x})$ then the 0 is uniformly stable.

Proof. We omit the proof. refer to the text book.

Theorem 5.2.9 *Suppose 1 and 2 of above theorem hold and there exists a positive definite function W_3 such that*

$$\frac{d}{dt}\left[V(t, \vec{x}(t))\right] \le -W_3(\vec{x})$$

$t \ge t_0$ and $\vec{x} \in B_r(0)$. Moreover if there exists W_2 such that $V(x) \le W_2(\vec{x})$ then the 0 is uniformly asymptotically stable.Then 0 is asymptotically stable.

Proof. We omit the proof. refer to the text book.

Example 5.2.10 *Show that if $p(t) \geq \frac{1}{2}$ then the system*

$$x_1' = x_2, \quad x_2' = -p(t)x_2 - e^{-t}x_1$$

is stable.

Let $V(t,x_1,x_2) = x_1^2 + e^t x_2^2$. Then $V \geq x_1^2 + x_2^2 := W(\vec{x})$ Also it is not difficult to check

$$\frac{d}{dt}V(x(t)) = e^t x_2^2(1-2p(t))$$

So if $p(t) \geq 1/2$ then $\frac{d}{dt}V(x(t)) \leq 0$. Hence system is stable. □

Example 5.2.11 *Study the stability of the equilibrium point of the system*

$$x_1' = -x_1 - e^{-2t}x_2, \quad x_2' = x_1 - x_2.$$

Let $V(t,x_1,x_2) = x_1^2 + (1+e^{-2t})x_2^2$. Then

$$x_1^2 + x_2^2 \leq V(t,x_1,x_2) \leq x_1^2 + 2x_2^2$$

Also,

$$\begin{aligned}\frac{d}{dt}V(x(t)) &= 2x_1(-x_1 - e^{-2t}x_2) + 2(1+e^{-2t})x_2(x_1 - x_2) - 2e^{-2t}x_2^2 \\ &= -2x_1^2 - 2x_2^2 + 2x_1x_2 - 4x_2^2e^{-2t} \\ &\leq -2(x_1^2 - x_1x_2 + x_2^2) = -2W_3\end{aligned}$$

where $W_3 = 2x_1^2 + 2x_2^2 - x_1x_2$ is a positive definite function. Hence 0 is uniformly asymptotically stable. □

5.3 Exercises

1. Find a such that the system $X' = AX$ with A given by

$$A = \begin{pmatrix} a & 0 & 0 \\ b_1 & b_2 & 0 \\ b_4 & b_5 & b_6 \end{pmatrix}$$

has a nontrivial solution $x(t)$ satisfying $|x(t)| \to 0$ as $t \to \infty$ for all $b_i \in \mathbb{R}$.

2. Find a such that all solutions of $X' = AX$ with

$$A = \begin{pmatrix} a-2 & 1 & 0 \\ -1 & a-2 & 0 \\ 0 & 0 & -a \end{pmatrix}$$

satisfy $\lim_{t\to\infty} |x(t)| = 0$.

3. Study the nature of critical points of the linear system $\vec{x}' = A\vec{x}$ by their phase-plane trajectory for the following A:

$(a)\begin{pmatrix} -2 & 1 \\ 7 & -4 \end{pmatrix} \quad (b)\begin{pmatrix} -1 & -1 \\ 4 & -1 \end{pmatrix} \quad (c)\begin{pmatrix} 1 & -1 \\ 3 & -1 \end{pmatrix} \quad (d)\begin{pmatrix} -1 & 4 \\ 1 & -5 \end{pmatrix}$

4. Find the equilibrium points and the nature of their stability

   a. $x_1' = -x_2 - x_1^3, \quad x_2' = x_1 - x_2^3$
   b. $x_1' = x_2 - x_1^3, \quad x_2' = x_1 - x_2^3$
   c. $x_1' = -2x_1 - 16x_2^4, \quad x_2' = -3x_2 + 12x_2^3$
   d. $x_1' = -x_1 + 2x_1^2 + x_2^2, \quad x_2' = -x_2 + x_2^2$
   e. $x_1' = -2x_1 - 3x_2 + x_1^2, \quad x_2' = x_1 + x_2$
   f. $x_1' = x_2 - x_1 - x_1x_2^2, \quad x_2' = -2x_1 - x_2 - x_2x_1^2$

5. Suppose the linearization around an equilibrium point is given by

$$\vec{x}' = A\vec{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\vec{x}$$

and suppose $det(A) > 0$ and $a, d < 0$ and $b, c \neq 0$. Then determine k such that $x_1^2 + kx_2^2$ is a Lyapunov function and 0 is asymptotically stable

6. Suppose the linearization around an equilibrium point is given by

$$\vec{x}' = A\vec{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\vec{x}$$

and suppose $det(A) > 0$ and $tr(A) < 0$. Then find B and C so that $V = x_1^2 + Bx_1x_2 + Cx_2^2$ is a Lyapunov function and 0 is asymptotically stable.

7. Let $V(x)$ be a function which is positive definite in $B_R(0)$ and $V\frac{dV}{dt}$ is continuous and ≤ 0 in $B_R(0)$, then show that 0 is stable.

Reference: M. Vidyasagar, Nonlinear systems analysis, Prenticehall, NJ,1993.