Chapter 1
First order equations

1.1 Semilinear equations

Semilinear equations are the equations in the form $x'(t) = f(t,x)$, where $f : \Omega \to \mathbb{R}$ is a real valued function and Ω is a domain in $\mathbb{R}^2$.

The problem of finding $x(t) \in C^1$ for a given (t_0, x_0) satisfying

$$x' = f(t,x), \ x(t_0) = x_0 \tag{1.1}$$

is called Cauchy problem or initial value problem. Here the domain Ω of $f(t,x)$ is a subset of $\mathbb{R}^2$ and the solution is a function $x(t) \in C^1((a,b))$ that satisfies the given equation. The domain of definition of solution could be very small even if the function $f(t,x)$ is defined in the whole of $\mathbb{R}^2$.
The basic motivation to understand the IVP (1.1) comes from system of linear algebraic equations

$$Ax = b, \ b \in \mathbb{R}^n, A = A_{n\times n} \text{ matrix}.$$

The following 3 possibilities occur:

1. Solution exists and is unique
2. There is No solution
3. There are infinitely many solutions.

That is, if a solution exists then it is either unique OR there are infinitely many solutions. We should expect this to happen in case of (1.1). We shall establish this for (1.1) in this section.
Another motivation comes from geometry. This we will discuss while studying the singular solutions in section 2.

Geometrically, a solution of this equation is a curve that is contained in Ω such that the tangent at each point $(t^*, x(t^*))$ on the curve has slope equal to $f(t^*, x(t^*))$. Integrating the equation from t_0 to t, we get

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s))ds.$$

So it enough to find $x(t)$ satisfying above integral equation. The iteration scheme: for $k = 0, 1, 2, 3, \cdots$

$$x_0(t) = x_0,\ x_{k+1}(t) = x_0 + \int_{t_0}^{t} f(s, x_k(s))ds$$

is known as Picard iterations. Suppose if $x_k(t)$ "converges" to $x(t)$, then by taking limit in the iteration scheme we see that $x(t)$ is the required solution. This is possible only when limit and integration is interchangeable. Mathematically,
Suppose we have a sequence of continuous functions $\{h_k\}$ that converges point wise to $h(x)$. Then one asks the question: Is $h(x)$ continuous and

$$\lim_{k\to\infty} \int h_k(x)dx = \int h(x)dx?$$

The answer is NO. In fact, we can take the well known "hat functions" defined as

$$h_k(x) = \begin{cases} 0 & x \notin (-\frac{1}{k}, \frac{1}{k}) \\ k^2x + k & [-\frac{1}{k}, 0] \\ -k^2x + k & [0, \frac{1}{k}] \end{cases}$$

Then $\int h_k(x)dx = 1$ but $h_k(x) \to h(x) = 0$ for $x \neq 0$ and h is not a well defined function at 0. Such functions h are called Dirac delta distribution or "generalized function".

1.1.1 Uniform Convergence

Definition 1.1.1 *A sequence of functions $\{f_n\}$ on the interval I is said to converge uniformly to the function f on I, if given $\epsilon > 0$, there exists $N \in \mathbb{N}$ depending only on ϵ such that*

$$|f_n(x) - f(x)| < \epsilon \textit{ for all } n \geq N, \textit{ for all } x \in I.$$

An immediate consequence of this definition is the following

Theorem 1.1.1 *Let $f_n(x) \to f(x)$ point wise on I and let $M_n = \sup_I |f_n(x) - f(x)|$. Then*

$$f_n \to f \textit{ uniformly } \iff M_n \to 0, \textit{ as } n \to \infty.$$

Proof. Follows from the definition. □
We also have the following Cauchy criterion for uniform convergence

Definition 1.1.2 *A sequence $\{f_n\}$ defined on an interval I is called uniformly Cauchy if for each $\epsilon > 0$, there exists N such that*

$$|f_n(x) - f_m(x)| < \epsilon, \textit{ for all } m, n \geq N, \textit{ for all } x \in I$$

Theorem 1.1.2 *A sequence $\{f_n\}$ defined on an interval I converges uniformly on I if and only if $\{f_n\}\}$ is uniformly Cauchy.*

Proof. Suppose $f_n \to f$ uniformly on I. Then for all $n, m \geq N$, we have

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \epsilon + \epsilon, \text{ for all } x \in I.$$

For the converse, let $\{f_n\}$ be uniformly Cauchy sequence. Then for each $\epsilon > 0$, there exists N such that

$$|f_n(x) - f_m(x)| < \epsilon, \text{ for all } m, n \geq N, \forall x \in I.$$

Then we see that $f_n(x)$ is a Cauchy sequence and therefore there exists $f(x)$ such that $f_n(x) \to f(x)$ point wise for each $x \in I$. Since $\{f_n\}$ is uniformly Cauchy, for each $x \in I$, $n_x \geq N$ such that

$$|f_{n_x}(x) - f(x)| < \epsilon/2.$$

Hence for all $n \geq N$

$$\begin{aligned}|f_n(x) - f(x)| &= |f_n(x) - f_{n_x}(x) + f_{n_x}(x) - f(x)| \\ &\leq |f_n(x) - f_{n_x}(x)| + |f_{n_x}(x) - f(x)| < \epsilon\end{aligned}$$

for all $x \in I$. □

Example 1.1.1 *Let $\{f_n\}$ be a sequence of continuous functions that converges uniformly on the closed bounded interval $[a,b]$. For each $n \in \mathbb{N}$, let*

$$F_n(x) = \int_a^x f_n(t)dt, \ a < x < b.$$

Then $\{F_n\}$ converges uniformly on $[a,b]$.

Proof. Given that the sequence $\{f_n\}$ is a Cauchy sequence. Hence

$$\begin{aligned}|F_n(x) - F_m(x)| &\leq \int_a^b |f_n(t) - f_m(t)|dt \\ &< \sup_{[a,b]} |f_n(x) - f_m(x)|(b-a) < \epsilon(b-a).\end{aligned}$$

Therefore, $\{F_n\}$ is uniformly Cauchy sequence and hence converges uniformly. □

The following theorem is an important consequence of uniform convergence.

Theorem 1.1.3 *Suppose $\{f_n\}$ be a sequence in $C(I)$, I is an interval, such that $f_n \to f$ uniformly on I, then $f(x)$ is continuous on I.*

Proof. Let x_0 be an arbitrary point in I. We will show f is continuous at x_0. Let $\epsilon > 0$ be given. Since $f_n(x) \to f(x)$ uniformly, then there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \epsilon/3, \text{ for every } x \in I, \text{and for all } n \geq N.$$

Since f_n is continuous at x_0, for this ϵ, there exists $\delta > 0$ such that

$$|x - x_0| < \delta \implies |f_n(x) - f_n(x_0)| < \epsilon/3.$$

Then for $x \in \{x : |x - x_0| < \delta\}$, and $n \geq N$, we have

$$\begin{aligned}|f(x)-f(x_0)| &= |f(x)-f_n(x)+f_n(x)-f_n(x_0)+f_n(x_0)-f(x_0)| \\ &< |f(x)-f_n(x)|+|f_n(x)-f_n(x_0)|+|f_n(x_0)-f(x_0)| \\ &< \epsilon/3+\epsilon/3+\epsilon/3=\epsilon.\end{aligned}$$

Therefore f is continuous at x_0.

Corollary 1.1.1 *Suppose $\{f_n(x)\}$ be a sequence in $C(I)$, I is a closed, bounded interval, such that $f_n(x) \to f(x)$ uniformly on I then*

$$\int_I f_n(x)dx \to \int_I f(x)dx$$

Proof. Follows from

$$\left|\int_I f_n(x)-f(x)dx\right| \le \left(\sup_I |f_n(x)-f(x)|\right) \mathit{length}(I).$$

□

1.1.2 Picard-Lindelof theorem

The most important and fundamental theorem on existence and uniqueness for Initial value problems (1.1) is Picard-Lindelof's theorem. Let Ω be an open subset of $\mathbb{R}^2$.

Definition 1.1.3 Lipschitz condition: *A continuous function $f:\Omega \to \mathbb{R}$ is said to satisfy Lipschitz condition if there exists $L>0$ such that*

$$|f(t,x_1)-f(t,x_2)| \le L|x_1-x_2|, \text{ for all } (t,x_1),(t,x_2) \in \Omega. \tag{1.2}$$

f is said to be locally Lipschitz with respect to the point (t_0,x_0) if (1.2) holds in all small open balls around $(t_0,x_0) \in \Omega$ which are not necessarily the entire domain Ω.

Remark 1.1.1 *Suppose $f(t,x)$ is continuous and $\frac{\partial f}{\partial x}$, the partial derivative with respect to x, is bounded in $\Omega \subset \mathbb{R}^2$. Then f is Lipschitz continuous. Indeed, by Mean value theorem, there exists $x^* \in (x_1,x_2)$ such that,*

$$|f(t,x_1)-f(t,x_2)| = \left|\frac{\partial f}{\partial x}(t,x^*)(x_2-x_1)\right| \le M|x_2-x_1|.$$

Examples 1.1.1 1. *Constant functions, linear (in x) functions*

2. $f(t,x)=|x|$
   follows from the inequality
   $$||a|-|b|| \le |a-b|, \ \forall\, a,b \in \mathbb{R}$$
3. $f(t,x)=x^2,\ t\in\mathbb{R},\ |x|\le 1$
   follows from
   $$|f(t,x)-f(t,y)| \le |x-y||x+y| \le 2|x-y|.$$
4. *Suppose $f(t,y)$ and $g(t,z)$ are Lipschitz with Lipschitz constants L_f and L_g respectively, then the composition $h(t,x)=f(t,g(t,x))$ is also Lipschitz:*
   $$\begin{aligned}|h(t,x)-h(t,y)| &\le L_f|g(t,x)-g(t,y)| \\ &\le L_fL_g|x-y|.\end{aligned}$$

for example $h(t,y) = |\sin y|$. *Here h is composition of* $|.|$ *and* $\sin(.)$.

We have the following local existence and uniqueness theorem:

Theorem 1.1.4 *Let* (t_0, x_0) *be a given interior point of* Ω. *If f is locally Lipschitz with respect to x in neighbourhood of* (t_0, x_0). *Then the Cauchy problem* (1.1) *has a unique solution defined in a suitable neighbourhood of* t_0.
More precisely, if f is Lipschitz in the rectangle $R = \{(t,x) : |t - t_0| \le a,\ |x - x_0| \le b\}$ *and* $M = \sup_R |f(t,x)|$, *solution exists and is unique in the interval* $(t_0 - h, t_0 + h)$ *where* $h = \min\left\{a, \frac{b}{M}\right\}$.

Example 1.1.2 *Consider the IVP:* $x' = x,\ x(0) = 1$.
Taking $x_0(t) = 1$. *The Picard's iterations are*

$$\begin{aligned}
x_1(t) &= 1 + \int_0^t ds = 1 + t \\
x_2(t) &= 1 + \int_0^t (1+s) ds = 1 + t + \frac{t^2}{2} \\
&\vdots \\
x_n(t) &= 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!}
\end{aligned}$$

Hence the limit of $x_n(t) = e^t$ *which is the solution of the IVP.*

Proof. of theorem: Let us denote the interval $I_h := [t_0 - h, t_0 + h]$ and let L be the Lipschitz constant, then we first
Claim 1: $x(t)$ is a solution of IVP (1.1) if and only if $x(t)$ satisfies

$$x(t) = x_0 + \int_{t_0}^t f(s, x(s))\, ds. \tag{1.3}$$

Let $x(t)$ be a solution. Then integrating the equation $x' = f(t,x)$ between t_0 and t,

$$\int_{t_0}^t x'(s) ds = \int_{t_0}^t f(s, x(s)) ds.$$

This immediately implies (1.3). Conversely, let $x(t)$ satisfies the equation (1.3). We define

$$\phi(t) = \int_{t_0}^t f(s, \phi(s)) ds.$$

Then by Fundamental theorem of calculus, ϕ is differentiable and $\phi'(t) = f(t, \phi(t))$. Therefore $x_0 + \phi(t)$ satisfies IVP.
Now define the successive approximations:

$$\begin{aligned}
x_0(t) &= x_0 \\
x_{k+1}(t) &= x_0 + \int_{t_0}^t f(s, x_k(s)) ds, \quad k = 0, 1, 2, \cdots
\end{aligned}$$

Claim 2: $\{x_k(t)\}$ is well-defined and $\{(t, x_k(t)) : t \in I_h\} \subset R$ for all $k = 0, 1, 2....$ To show this, by induction, it is clear that $(t, x_0(t)) \subset R$. Assume for $x_{k-1}(t)$. Therefore, from the definition of M and R we have

$\max_{s\in I_h}|f(s,x_{k-1}(s)| \le M$. Then for k,

$$\begin{aligned}|x_k(t)-x_0| &\le \int_{t_0}^{t}|f(s,x_{k-1}(s))|ds\\ &\le M\int_{t_0}^{t}ds \le Mh \le b.\end{aligned}$$

Hence the claim follows.

Claim 3: For all $t\in I_h$
$$|x_k(t)-x_{k-1}(t)| \le \frac{M}{L}\frac{|t-t_0|^k}{k!}L^k,$$
where L is the Lipschitz constant.

For $k=1$,
$$|x_1(s)-x_0| = |\int_{t_0}^{s} f(r,x_0)dr| \le M\,|s-t_0|.$$
and for $k=2$,
$$\begin{aligned}|x_2(t)-x_1(t)| &= \Big|\int_{t_0}^{t} f(s,x_1)-f(s,x_0)ds\Big|\\ &\le L\int_{t_0}^{t}|x_1(s)-x_0(s)|ds\\ &\le ML\int_{t_0}^{t}|s-t_0|ds = \frac{ML}{2}|t-t_0|^2.\end{aligned}$$

By induction, assume for $k-1$. Replacing arguments, we find
$$|x_{k+1}-x_k| \le \int_{t_0}^{t}|f(s,x_k)-f(s,x_{k-1})|\,ds \le L\int_{t_0}^{t}|x_k(s)-x_{k-1}(s)|\,ds.$$

Using induction hypothesis, forall $t\in I_h$,
$$\begin{aligned}|x_{k+1}-x_k| &\le L\,\frac{M}{L}\,\frac{L^k}{k!}\int_{t_0}^{t}|s-t_0|^k\\ &\le \frac{M}{L}\,\frac{L^{k+1}}{k!}\,\frac{|t-t_0|^{k+1}}{k+1} = \frac{M}{L}\,\frac{|t-t_0|^{k+1}L^{k+1}}{(k+1)!}.\end{aligned}$$

Therefore
$$\max_{t\in I_h}|x_{k+1}(t)-x_k(t)| \le \frac{M}{L}\,\frac{h^{k+1}L^{k+1}}{(k+1)!}.$$

Now we may choose n_0 large so that $\frac{hL}{n_0+2}<\frac{1}{2}$. Then for $m>n>n_0$,

$$\begin{aligned}|x_m(t)-x_n(t)| &\le |x_m-x_{m-1}|+\cdots+|x_{n+1}-x_n|\\ &\le \frac{(hL)^m}{m!}+\cdots+\frac{(hL)^{n+2}}{(n+2)!}+\frac{(hL)^{n+1}}{(n+1)!}\\ &= \frac{(hL)^{n+1}}{(n+1)!}\left[1+\frac{hL}{n+2}+\frac{(hL)^2}{(n+2)(n+3)}+\cdots+\frac{(hL)^{m-n-1}}{(n+2)...m}\right]\end{aligned}$$

Now note that the expression in the bracket on RHS is dominated by a geometric series with $r = 1/2$ and $a = 1$ (for $n \geq n_0$) and hence is bounded by 2 and $\frac{(hL)^{n+1}}{(n+1)!}$ converges to 0 as $n \to \infty$. Hence, $\{x_k\}$ is a uniformly Cauchy sequence. Hence $\{x_k\}$ converges uniformly. That is, there exists a continuous function $x(t)$ such that $x_{k+1}(t) \to x(t)$ uniformly in I_h. Since $f(s,x)$ is Lipschitz, $f(t,x_{k+1}) \to f(t,x)$ uniformly. Therefore,

$$x(t) = \lim_{k\to\infty} x_{k+1}(t) = x_0 + \lim_{k\to\infty} \int_{t_0}^{t} f(s,x_k)ds = x_0 + \int_{t_0}^{t} f(s,x)ds.$$

Uniqueness

Consider the interval $I_\delta = \{t : |t - t_0| < \delta\}$ such that $L\delta < 1$. Let $x(t), y(t)$ be two solutions. Then

$$x(t) - y(t) = \int_{t_0}^{t} (f(s,x(s)) - f(s,y(s)))ds.$$

Therefore,

$$|x(t) - y(t)| \leq L \int_{t_0}^{t} |x(s) - y(s)|ds \leq L\left(\max_{t\in I_\delta} |x(t) - y(t)|\right)|t - t_0|$$

Therefore taking max on both sides, we get

$$\max_{t\in I_\delta} |x(t) - y(t)| \leq (L\,\delta) \max_{t\in I_\delta} |x(t) - y(t)|.$$

This implies $1 \leq L\,\delta$, which is a contradiction. Hence $x(t) \equiv y(t)$ in $|t - t_0| \leq \delta$. In particular $x(t_0 \pm \delta) = y(t_0 \pm \delta)$. Now we can repeat the procedure in $[t_0 + \delta, t_0 + 2\delta]$ and $[t_0 - 2\delta, t_0 - \delta]$. Continuing like this, after a finite steps we get $x(t) \equiv y(t)$ in I_h. $\square$

Remark 1.1.2 *What we notice here is local uniqueness implies uniqueness in the entire domain of existence of solution. This phenomena of initial value problems is known as local uniqueness implies global uniqueness. This local uniqueness arguments can be adopted to the maximal interval of existence of solution which need not be I_h.*

The conditions of Picard-Lindelof theorem can be relaxed for uniqueness, for instance Osgood condition. For existence of solutions, continuity of f is enough (Peano's theorem).

Definition 1.1.4 *A continuous nondecreasing function $\eta : [0,\infty) \to [0,\infty)$ with $\eta(0) = 0$ and $\eta(t) > 0$ for $t > 0$ is said to satisfy Osgood's condition if*

$$\int_0^1 \frac{dt}{\eta(t)} = \infty.$$

Theorem 1.1.5 *(W.F.Osgood, Monatsch Math. Phys. 9, 1898)*
Suppose $f(t,x)$ satisfies

$$|f(t,x_1) - f(t,x_2)| \leq \eta(|x_1 - x_2|), \textit{for all } (t,x_1),(t,x_2) \in \Omega \tag{1.4}$$

where η satisfies the Osgood condition. Then there is a $\delta > 0$ for which there exists a unique solution of the IVP(1.1) (with $t_0 = 0$) in $(0,\delta)$.

Example 1.1.3 *for example, $f(t,x) = x\log x$ does not satisfy Lipschitz condition but satisfies the condition* (1.4).

Remark 1.1.3 *limiting cases lead to logx: Easy to see that x^α when $0<\alpha<1$ does not satisfy the Osgood condition and when $\alpha \geq 1$ it satisfies Osgood condition. So $x\log x$ may be seen as limiting phenomena. This happens in Hardy's inequality also. In 2 dimensions the Hardy potential is $|x|^2\log|x|$ But in higher dimensions it is $|x|^2$.*

We have the following **Global existence theorem:**

Theorem 1.1.6 *Let $\Omega = [a,c]\times\mathbb{R}$ and let $f(t,x)$ be Lipschitz in Ω. Then the IVP (1.1) has a unique solution in $[a,c]$.*

Proof. In this case b can be chosen any number and therefore h in the above theorem is equal to the length of interval $[a,c]$. □

Corollary 1.1.2 *If $\Omega = I\times\mathbb{R}$ where $I=\mathbb{R}$ or $[a,\infty)$ or $(-\infty,c]$ and f is Lipschitz on Ω, then the solution exists globally. That is in the whole of I.*

Example 1.1.4 *The IVP $y' = |\sin x|,\ x(0)=0$ has solution in the whole $\mathbb{R}$.*

So one would ask the question: Does Picard iterations converge if Lipschitz continuity is not satisfied by $f(t,x)$. More generally is Lipschitz continuity necessary for existence and uniqueness? The answer lies in the following remarks:

Remark 1.1.4 Lipschitz condition is only Sufficient but not necessary
(Muller, 1927, Math. Z)
Consider the IVP

$$x' = f(t,x),\ x(0)=0,$$

where the function

$$f(t,x) = \begin{cases} 0 & t\leq 0, x\in\mathbb{R} \\ 2t & t>0, x<0 \\ 2t-\frac{4x}{t} & t>0, 0\leq x\leq t^2 \\ -2t & t>0, x>t^2 \end{cases}$$

Then the $f(t,x)$ is continuous near $(0,0)$ but NOT Lipschitz. We claim that no sub-sequence of Picard iterations converge to a solution. In the next theorem we will show tht the solution is unique.

Proof. To see the that f is not Lipschitz

$$f(t,0)-f(t,t^2) = 4t \implies \frac{f(t,0)-f(t,t^2)}{t^2} \to \infty$$

as $t\to 0$. The Picard iterations:

$$\begin{aligned} x_1(t) &= \int_0^t f(s,0)ds = \int_0^t 2s ds = t^2 \\ x_2(t) &= \int_0^t f(s,s^2)ds = -\int_0^t 2s ds = -t^2 \\ x_3(t) &= \int_0^t f(s,-s^2) = \int_0^t 2s ds = t^2 \end{aligned}$$

Therefore, we get $x_{2n}=t^2, x_{2n+1}=-t^2$. However the limits of sub-sequences t^2 and $-t^2$ does not satisfy the IVP.

Uniqueness follows from the following

Theorem 1.1.7 *Suppose $f(t,x)$ is decreasing in x. Then the IVP $x' = f(t,x), x(0) = 0$ has at most one solution.*

Proof. *Suppose $\phi(t)$ and $\psi(t)$ are two solutions such that for some $\theta > 0$*

$$\phi(\theta) < \psi(\theta)$$

Now the set $S = \{t \in [0,\theta) : \phi(t) = \psi(t)\}$ is non-empty and closed. If σ is the supremum of S then $\phi(\sigma) = \psi(\sigma)$ and

$$\phi(t) < \psi(t),\ t \in (\sigma,\theta]$$

Then by MVT there exists t_1 such that

$$\phi'(t_1) - \psi'(t_1) < 0.$$

This contradicts the following:

$$\phi' - \psi' = f(t,\phi) - f(t,\psi) \geq 0,\ t \in (\sigma,\theta].$$

thanks to the fact that $f(t,x)$ is decreasing in x. Hence the solution is unique. □

Example 1.1.5 *Show that $y = 0$ is the only solution of IVP $y' = -y^{2/3},\ y(0) = 0$.*
The function $f(t,y) = -y^{2/3}$ is decreasing and hence it must have only one solution. We know that $y = 0$ is already a solution.

Example 1.1.6 *Initial value problem may still have unique solution even if $f(t,y)$ is not Lipschitz. For example*

$$y' = y\sin(1/y),\ y(0) = 0.$$

Here $f(t,y) = y\sin(1/y)$ is not Lipschitz around $y = 0$ but it is Lipschitz at any other point other than zero. In fact, by taking $x_n = \frac{1}{\frac{\pi}{2}+2n\pi}, y_n = \frac{1}{\frac{3\pi}{2}+2n\pi}$, we see that as $n \to \infty$,

$$\frac{|f(x_n) - f(y_n)|}{|x_n - y_n|} \to \infty.$$

We claim that $y = 0$ is the only solution. Indeed, if there is any other solution y that is positive (say) at a point. Then it has to intersect the lines $\frac{1}{n\pi}, n \in \mathbb{N}$ and these lines are solutions of $y' = y\sin(1/y)$. By considering IVP with initial value at the intersection points, we see by Picard's theorem all these problems with initial value $y_0 > 0$ shall have unique solution. But we already have two solutions y and $y = \frac{1}{n\pi}$. That is a contradiction. □
Next we recall the inverse function theorem:

Theorem 1.1.8 *if f is a continuously differentiable function with nonzero derivative at the point a then f is bijective onto the image of f in a neighborhood of a. Moreover, the inverse is continuously differentiable near $f(a)$, and the derivative of the inverse function at $f(a)$ is the reciprocal of the derivative of f at a,*

$$(f^{-1})'(f(a)) = \frac{1}{f'(a)}$$

Theorem 1.1.9 *Suppose $f(x)$ is just continuous but not Lipschitz continuous and $f(x) > \beta > 0$ in the whole of $\mathbb{R}$. Then the initial value problem*

$$x' = f(x),\ x(\tau) = \xi$$

has unique solution.

Proof. Define,

$$F(x) = \tau + \int_{\xi}^{x} \frac{ds}{f(s)}$$

Then $F(\xi) = \tau$, F is differentiable, strictly increasing and so one-one and onto from $\mathbb{R}$ to (A_-, A_+), where $A_- = \lim_{x\to-\infty} F(x)$, and $A_+ = \lim_{x\to\infty} F(x)$. Therefore by inverse function theorem, inverse of F exists and is differentiable. Let $\phi : (A_-, A_+) \to \mathbb{R}$ be its inverse. Then ϕ is a solution of IVP. Indeed, $F(\xi) = \tau \implies \phi(\tau) = F^{-1}(\tau) = \xi$ and

$$\phi'(t) = \frac{1}{F'(F^{-1}(t))} = f(\phi(t)).$$

Claim: ϕ is the unique solution
Suppose ψ is another solution in an interval containing τ. Then

$$[F(\psi(t))]' = F'(\psi(t))\psi'(t) = \frac{\psi'(t)}{f(\psi(t))} = 1$$

implying $t + c = F(\psi(t))$ and taking $t = \tau$

$$\tau + c = F(\psi(\tau)) = F(\xi) = \tau \implies c = 0.$$

Therefore,

$$t = F(\psi(t)) = F(\phi(t))$$

Since F is 1-1 and onto, we get $\phi(t) \equiv \psi(t)$. $\square$
Later in the next section, we will prove the following Peano's existence theorem where continuity of $f(t,x)$ is sufficient to ensure existence of solution of IVP.

Theorem 1.1.10 *Suppose $f : D \to \mathbb{R}$ be a continuous function on the domain $D \subset \mathbb{R}^2$. Then the IVP* (1.1) *has a solution in a neighbourhood of t_0 for any given initial point $(t_0, x_0) \in D$.*

Example 1.1.7 *As a consequence of the above two theorems, the IVP $x' = 1 + x^{\frac{2}{3}}$, $x(0) = 0$. admits one and only one solution.*

Theorem 1.1.10 implies that the IVP considered in the Muller's example admits solution.

1.1.3 Maximal interval of existence

From the above theorem, the solution exists and is unique in the interval I_h. This motivates us to to explore if this is maximal interval of existence of solution.
Let us consider the equation $x' = x^2$, $x(t_0) = x_0 \neq 0$. By the above theorem, this problem has unique solution. A solution of this equation is $x(t) = \frac{-1}{t - t_0 - \frac{1}{x_0}}$. So this function has discontinuity at $t_0 + \frac{1}{x_0}$. If $x_0 > 0$, we take the positive branch if $\frac{1}{t - t_0 - \frac{1}{x_0}}$ for $t < t_0 + \frac{1}{x_0}$. For $x_0 < 0$, we take the other branch that is negative. Now we see from the above theorem with $t_0 = 0$, by taking $R = \{(t,x) : |t| \leq a, |x - x_0| \leq b\}$. Then $M = \max_R x^2 \leq (b + x_0)^2$, and $h = \min\left\{a, \frac{b}{(b+x_0)^2}\right\}$. Now maximizing the function $g(x) = \frac{x}{(x+x_0)^2}$, we see $\max g(x) = \frac{1}{4x_0}$. This interval is smaller than the interval where solution is defined. Indeed, we have the following theorem on maximal interval of existence of solution.

Theorem 1.1.11 *Let $x(t)$ be a solution of $x' = f(x,t)$. Suppose, for simplicity, that the set Ω where f is defined is all of $\mathbb{R}^2$. If J, the maximal interval existence of solution, is not all of $\mathbb{R}$, then it cannot be closed.*

Proof. Assume by contradiction, let $x_0(t)$ be a solution and the maximal interval of existence is $[\alpha,\beta]$. Then $x_0(\beta)$ is well defined and let $x_1(t)$ be the solution of $x' = f(t,x), x(\beta) = x_0(\beta)$. Such solution exists by the local existence theorem in $[\beta,\beta+\delta]$. Now we can define

$$x(t) = \begin{cases} x_0(t) & \beta-\delta < t \le \beta \\ x_1(t) & \beta \le t < \beta+\delta. \end{cases}$$

Now we claim that $x(t)$ is a solution of $x' = f(t,x)$. Easy to see the continuity of x at β. So if is enough to show that $x(t)$ is differentiable at β. The left derivative $(x')^-(\beta)$ is

$$(x')^-(\beta) = \lim_{h\to 0} \frac{x(\beta)-x(\beta-h)}{h} = \lim_{h\to 0^-} \frac{x_0(\beta)-x_0(\beta-h)}{h} = x_0'(\beta) = f(\beta, x_0(\beta))$$

$$(x')^+(\beta) = \lim_{h\to 0} \frac{x(\beta+h)-x(\beta)}{h} = \lim_{h\to 0^+} \frac{x_1(\beta+h)-x_1(\beta)}{h} = x_1'(\beta) = f(\beta, x_1(\beta))$$

Now since x is continuous, we have $f(\beta, x_1(\beta) = f(\beta, x_0(\beta))$ and the conclusion follows. □

An important consequence of this theorem is

Theorem 1.1.12 *If $x(t)$ is bounded, monotone function and $f(t,x)$ is continuous on $\mathbb{R}^2$, then solution of the IVP* (1.1) *exists globally everywhere on $\mathbb{R}$.*

Proof. Follows from the fact that bounded monotone function has limit. □

Example 1.1.8 *Consider the function $f(t,x) \in C(\mathbb{R}^2)$ which is either positive or negative for all (t,x). Then consider the IVP:*

$$x' = \frac{f(t,x)}{1+|f(t,x)|},\ x(0) = x_0.$$

Then it is easy to see that $|x'(t)| < 1$ for all t. Therefore the function $x(t)$ cannot approach infinity in finite time. Also $x' > 0$ or < 0 implies x is monotone. Therefore the maximal interval of existence is the whole space.

For example, $x' = \frac{x^2}{1+x^2},\ x(0) = 0$ admits unique solution in the whole of $\mathbb{R}$.

1.1.4 Continuous dependence

Here we discuss the third point of well-posedness namely the continuous dependence of solutions on the initial data. Let us start our discussion with the following example.
Consider the problem $x' = \alpha x^2,\ x(0) = x_0 + \epsilon$. From the above existence uniqueness theorems, we see that problem admits unique solution. We can compute this solution as

$$x(t,\epsilon) = \frac{x_0+\epsilon}{1-\alpha(x_0+\epsilon)t}.$$

This is a continuous function of both parameters α and ϵ. Thus a small change in the initial data or parameters of the equation produces a small change in the solution at least for times near the initial times. In fact, here we see that the blowup point $t^* = \frac{1}{\alpha(x_0+\epsilon)}$ depends on the initial data and parameter α of the equation. Thus, as we approach the singularity, solutions that started out very close to each other will get arbitrarily far apart. This shows that the continuous dependence does not prevent the solutions from exhibiting chaotic behavior.

Theorem 1.1.13 Grownwall's inequality: *Let $x(t)$ be a differentiable function and suppose*

$$|x(t)| \le \delta + K\int_{t_0}^{t} |x(s)|ds + \epsilon(t-t_0).$$

Then

$$|x(t)| \le \frac{\delta}{K}(e^{K|t-t_0|}-1) + \frac{\epsilon}{K^2}(e^{K|t-t_0|}-1) - \frac{\epsilon}{K}|t-t_0|$$

for all t

Proof. Let $E(t) = \int_{t_0}^{t} |x(s)|ds$. Then

$$E'(t) = |x(t)| \le \delta + KE(t) + \epsilon(t-t_0)$$

$$i.e.,\ E'(t) - KE(t) \le \delta + \epsilon(t-t_0).$$

Now multiplying by $e^{-K(t-t_0)}$ and integrating from t_0 to t, one gets

$$E(t) \le \frac{\delta}{K}(e^{K(t-t_0)}-1) - \frac{\epsilon}{K^2}(K(t-t_0)) + \frac{\epsilon}{K^2}e^{K(t-t_0)} - \frac{\epsilon}{K}(t-t_0)$$

Hence the proof. □

Theorem 1.1.14 *Solutions of IVP* (1.1) *depends continuously on the initial data. That is if x and y are solutions of*

$$x' = f(t,x), x(t_0) = x_0,\ \ y' = f(t,y) + \epsilon,\ y(t_0) = x_0 + \delta$$

then

$$(|\epsilon| + |\delta|) \to 0 \implies \max_{t\in I_h} |x(t) - y(t)| \to 0.$$

Proof. By the integral representation, $x(t)$ and $y(t)$ satisfy the integral equations

$$x(t) = x_0 + \int_{t_0}^{t} f(s,x(s))ds,\ \ y(t) = x_0 + \delta + \int_{t_0}^{t} (f(s,y(s)) + \epsilon)ds.$$

Subtracting and taking absolute value, we get

$$|x(t) - y(t)| \le \delta + \int_{t_0}^{t} |f(s,x(s)) - f(s,y(s)) - \epsilon|ds.$$

Now by Lipschitz continuity of $f(t,s)$ we get

$$|x(t) - y(t)| \le \delta + \int_{t_0}^{t} |x(s) - y(s)|ds + \epsilon(t-t_0).$$

Now by above inequality, we get

$$|x(t) - y(t)| \le \delta e^{|t-t_0|} + \epsilon(e^{|t-t_0|} - 1).$$

So if $t \in I_h$, we have

$$\max_{t \in I_h} |x(t) - y(t)| \leq (\delta + \epsilon)\, e^h$$

which implies that the maximum change in the solution goes to zero as $\delta, \epsilon \to 0$. ◻

1.1.5 Topological methods

We can also use the topological arguments such as contraction maps and their fixed points to show the existence and uniqueness of solutions. We recall the following Banach fixed point theorem
To start, we recall the following fixed point theorem from calculus.

Theorem 1.1.15 *Suppose $f : [0,1] \to [0,1]$ be a continuous function. Then f has fixed point in* $[0,1]$.

Proof. Proof follows by taking $g(x) = f(x) - x$ and applying Intermediate value theorem. ◻
The above theorem in case of Metric spaces requires some strong assumptions:

Definition 1.1.5 *Let (X,d) be a metric space. We call a map $T : X \to X$ a contraction if there exists a constant $L < 1$ such that*

$$d(Tx, Ty) \leq Ld(x,y)\ \forall x, y \in X.$$

Then we have the following Banach fixed point theorem:

Theorem 1.1.16 *Let (X,d) be a complete metric space and $T : X \to X$ be a contraction mapping. Then T has unique fixed point. i.e., There exists unique x such that $T(x) = x$.*

Proof. First let us show that there is at most one solution. Suppose x and y are two fixed points then

$$d(x,y) = d(Tx,Ty) \leq Ld(x,y) \implies (1-L)d(x,y) \leq 0$$

Hence $L < 1$ implies $d(x,y) = 0$ and $x = y$.
To show the existence, define the sequence $x_n = T(x_{n-1})$ starting with any element of $x_0 \in X$. Then for $m > n > 2$

$$\begin{aligned} d(x_n, x_{n-1}) &= d(Tx_{n-1}, Tx_{n-2}) \leq Ld(x_{n-1}, x_{n-2}) = Ld(Tx_{n-2}, Tx_{n-3}) \\ &\leq L^2 d(x_{n-2}, x_{n-3}) \\ &\vdots \\ &\leq L^{n-1} d(x_1, x_0). \end{aligned}$$

Then

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \ldots + d(x_{n+1}, x_n) \\ &= L^n (1 + L + L^2 + \ldots) d(x_1, x_0) \\ &< L^n \frac{1}{1-L} d(x_1, x_0) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence in the complete metric space X. Hence $\{x_n\}$ converges to x. Now taking limit $n \to \infty$ in $x_n = T(x_{n-1})$ we see that x is a fixed point. ◻

Remark 1.1.5 *Suppose if T^k has unique fixed point for some $k \in \mathbb{N}$, then T has unique fixed point. Indeed, if x is the fixed point of T^k, then $x = \lim_{n\to\infty} (T^k)^n x_0, x_0 \in X$. Also, $\lim_{n\to\infty} (T^k)^n T x_0 = x$. Hence*

$$x = \lim_{n\to\infty} (T^k)^n T x_0 = \lim_{n\to\infty} T (T^k)^n x_0 = T(\lim_{n\to\infty} (T^k)^n x_0) = T x.$$

Proof of local existence and uniqueness theorem(Picard-Lindelof):
Now we take the metric space $X = \{x \in C(I_h) : |x(t) - x_0| \le b\}$, with uniform metric $d(f,g) = \max_{I_h} |f(t) - g(t)|$ and the operator T as

$$T(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) ds$$

Since f is continuous, it is easy to show that $T(x)(t)$ is a continuous function and by Lipschitz continuity of $f(t,x)$,

$$|T(x)(t) - x_0| = \left| \int_{t_0}^{t} f(s,x) ds \right| \le M|t - t_0| \le Mh \le M \min(a, \frac{b}{M}) \le b.$$

implying $T(X) \subset X$. To show the contraction of T^k,

$$\begin{aligned} |T(x_1)(t) - T(x_2)(t)| &\le \int_{t_0}^{t} |f(s,x_1) - f(s,x_2)| ds \\ &\le L \int_{t_0}^{t} |x_1(s) - x_2(s)| ds \\ &\le L \max_{|s-t_0|\le h} |x_1(s) - x_2(s)||t - t_0| \qquad (1.5) \\ &\le (Lh)\ d(x_1, x_2). \end{aligned}$$

Therefore, taking $\max_{I_h}$ on L.H.S, we get

$$d(T(x_1), T(x_2)) \le (Lh)\ d(x_1, x_2)$$

Now consider T^2, using (1.5)

$$\begin{aligned} |T^2(x_1)(t) - T^2(x_2)(t)| &\le \int_{t_0}^{t} |f(s, Tx_1) - f(s, Tx_2)| ds \\ &\le L \int_{t_0}^{t} |Tx_1(s) - Tx_2(s)| ds \\ &\le L^2 \int_{t_0}^{t} \left(\max_{|t-t_0|\le h} |x_1(s) - x_2(s)| \right) |s - t_0| ds \\ &\le L^2 d(x_1, x_2) \frac{|t - t_0|^2}{2}. \end{aligned}$$

Therefore,

$$d(T^2 x_1, T^2 x_2) \le \frac{(Lh)^2}{2}\ d(x_1, x_2).$$

Following this for k, we have

$$d(T^k x_1, T^k x_2) \le \frac{(L|t - t_0|)^k}{k!}\ d(x_1, x_2).$$

Now as $\frac{(L|t-t_0|)^k}{k!} \to 0$ as $k \to \infty$, we get T^k is contraction for some k. Therefore, by the above theorem, T^k has a unique fixed point. Hence T has a unique fixed point. That is,

$$x(t) = T(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds. \square$$

We can also prove the following global existence and uniqueness theorem using the above method:

Theorem 1.1.17 *Let $\Omega = [a,b]\times\mathbb{R}$ and let $f(t,x)$ be Lipschitz in Ω. Then the IVP* (1.1) *has unique solution in* $[a,b]$. $\square$

Proof. Proof follows by taking $X = C([a,b])$ and noting that T defined above maps X into itself and T^k is a contraction for some k. $\square$

Another important result on Cauchy problem is the existence theorem (with no guarantee of uniqueness) due to Peano. To prove this we recall the following results on compact subsets of $C(I)$, I is closed and bounded interval. Then

Definition 1.1.6 *A subset $\mathcal{A} \subset C(I)$ is equi-continuous if, for a given $\epsilon > 0$ there exists $\delta > 0$ such that*

$$|x-y| < \delta \implies |f(x) - f(y)| < \epsilon, \ \forall f \in \mathcal{A}.$$

Definition 1.1.7 *A subset $\mathcal{A}$ of a metric space (X,d) is compact if every sequence $\{x_k\}$ has a convergent subsequence.*

The following theorem is the famous Ascoli-Arzela theorem on the characterization of compact subsets of $C(I)$:

Theorem 1.1.18 *A subset $\mathcal{A} \subset C(I)$ is compact if and only if $\mathcal{A}$ is bounded and equi-continuous.*

Example 1.1.9 *Suppose $\{u_n\}$ is a sequence such that $\max_I |u_n(x)| < M$ and $\max_I |u_n'(x)| < M$ for some $M > 0$, then $\mathcal{A} = \{u_n(x) : n = 1,2,3....\}$ is compact.*

Example 1.1.10 *Let $K = \{v(x) : v(x) = \int_a^x u(t)dt, \ \|u\|_\infty \le 1\}$. Then K is compact in* $C([a,b])$

Definition 1.1.8 *A subset $\mathcal{A}$ of $C(I)$ is a convex subset if for every $f, g \in C(I)$ there holds $tf(t) + (1-t)g(t) \in C(I)$ for all $t \in (0,1)$.*

We will use the following Schauder fixed point theorem for $C(I_h)$ to show the existence of solution.

Theorem 1.1.19 *Let $\mathcal{A}$ be a compact, convex subset of $C(I)$ and $T : C(I) \to C(I)$ be a continuous operator such that $T(\mathcal{A}) \subset \mathcal{A}$. Then T has a fixed point.*

Theorem 1.1.20 Peano's theorem
Suppose $f(t,x)$ is a continuous function in a neighbourhood of (t_0, x_0). Then the IVP (1.1) *has a solution in a neighbourhood of t_0.*

Proof. Without loss of generality, assume that $M \ge 1$ and f is continuous on $R = \{(t,x) : |t-t_0| \le a, |x-x_0| \le a\}$. Then $h = \min\left(a, \frac{a}{M}\right) = \frac{a}{M}$. Define the set $\mathcal{A} = \{u \in C(I_h), |u(t) - x_0| \le a, |u(t_1) - u(t_2)| \le M|t_1 - t_2|\}$. Then it is easy to see by triangle inequality, that $\mathcal{A}$ is convex, bounded subset of $C(I_h)$. Also, $\mathcal{A}$ is compact subset of $C(I_h)$. So we will have fixed point provided $T(\mathcal{A}) \subset \mathcal{A}$. It is again easy, indeed

$$|T(u)(t) - x_0| \le \int_{t_0}^{t} |f(s, u(s)|ds \le Mh \le a$$

and

$$|Tu(t_1) - Tu(t_2)| \leq \int_{t_1}^{t_2} |f(s,u(s))|ds \leq M|t_1 - t_2|.$$

Hence the proof. □

Example 1.1.11 Continuity of $f(t,x)$ is only sufficient but not necessary:
Consider the problem $y' = f(t,y),\ y(0) = 0$ with

$$f(t,y) = \begin{cases} \frac{2ty}{t^2+y^2} & (t,y) = (0,0) \\ 0 & (t,y) = (0,0) \end{cases}$$

Here it is not difficult to check that the function $f(t,x)$ is not continuous at $(0,0)$. But $y \equiv 0$ is a solution of the problem.

Remark 1.1.6 *Peano's theorem only ensures existence and uniqueness of local solution even if the function $f(t,x)$ is continuous in the whole space. For example the IVP $y' = y^2,\ y(t_0) = x_0$ does not have solution in the whole space.*

Proof of Peano's theorem through approximations:
There are other ways to prove the Peano's theorem. Let R be the rectangle around (t_0,x_0) and $\rho \in C^\infty(\mathbb{R})$ with support in $[-1,1]$ and $\rho_\epsilon(x) = \frac{1}{\epsilon}\rho(\frac{x-x_0}{\epsilon})$. Then ρ_ϵ is family of C_c^∞ functions with support in $B_\epsilon(x_0)$, $\int_{\mathbb{R}} \rho_\epsilon = 1$. We define

$$f_\epsilon(t,x) = (\rho_\epsilon * f)(x) = \int_{\mathbb{R}} \rho_\epsilon(x-y)f(t,y)dy.$$

Then it is not difficult to show that $f_\epsilon \to f$ uniformly on compact subsets of I_h. Also, $\sup_{I_h}|f_\epsilon(t,x(t))| \leq \sup_{I_h}|f(t,x(t))| \leq M$. Then the initial value problems

$$x' = f_\epsilon(t,x),\ x(t_0) = x_0 \tag{1.6}$$

has unique solution $x_\epsilon(t)$ for each ϵ, thanks to Picard's theorem. This $x_\epsilon(t)$ also satisfies the integral equation

$$x_\epsilon(t) = x_0 + \int_{t_0}^{t} f_\epsilon(s,x_\epsilon)ds \tag{1.7}$$

From (1.7) one can show that $\{x_\epsilon\}$ is uniformly bounded and from the differential equation in (1.6), we see that $\{x_\epsilon'\}$ are bounded. Hence by Ascoli-Arzela theorem $\{x_\epsilon\}$ is compact in $C(I_h)$. Hence this sequence converges uniformly to $x(t)$(say). Then taking the limit in the integral equation satisfied by x_ϵ, we see that $x(t)$ is a solution. The smooth sequence of functions $\{f_\epsilon\}$ can be defined using the so called mollifiers. This is known as regularization technique. This method can be easily adopted to integral equations, Partial differential equations and operator equations.

Example 1.1.12 *Consider the problem $y' = y^{\frac{1}{2}},\ y(0) = 0$. Then one could see that $y_1 \equiv 0$ and $y_2 = \frac{t^2}{4}$ are two solutions. Also, for any $\alpha > 0$ we can define*

$$y_\alpha(t) = \begin{cases} \frac{(t-\alpha)^2}{4} & t > \alpha \\ 0 & t \leq \alpha \end{cases}$$

is a solution of the initial value problem.

From the above discussion we tend to think that if IVP has more than one solution then it has infinitely many solutions. Indeed this is the case. The following theorem is known as Helmut Kneser's Theorem:

Theorem 1.1.21 *Suppose $f(t,x)$ is continuous function in a neighbourhood of (t_0,x_0). Consider the IVP* (1.1). *Then the set S, defined as*

$$S = \{x(s) : \text{ IVP has solution in } [t_0, s]\}$$

is connected.

Proof. Suppose $x_1(t)$ and $x_2(t)$ are two solutions in $[t_0,\sigma]$ for some σ such that $x_1(\sigma) < x_2(\sigma)$. Let η be such that

$$x_1(\sigma) < \eta < x_2(\sigma).$$

Then the IVP

$$x' = f(t,x), \; x(\sigma) = \eta$$

has a solution ξ for $t \le \sigma$. As soon as the graph of ξ hits the graph of $x_1(t)$ or $x_2(t)$, say at t_1, $\xi(t_1) = x_1(t)$ then define

$$\tilde{\xi}(t) = \begin{cases} x_1(t) & [t_0,t_1] \\ \xi(t) & [t_1,\sigma] \end{cases}$$

then $\tilde{\xi}$ is also solution of the IVP in $[t_0,\sigma]$. This is how we obtain continuum of solutions □

Summary of Existence and uniqueness (local Existence uniqueness):
There exists a unique local solutions of the IVP (1.1) if one of the following holds

1. $f(t,x)$ is locally Lipschitz at (t_0,x_0) in the variable x
2. $f(t,x)$ satisfies Osgood growth condition (1.4)
3. $f(t,x)$ is continuous and decreasing in x
4. $f(t,x) = f(x)$ is continuous and strictly positive in $\mathbb{R}$

(global existence and uniqueness): We have

1. If $f(t,x)$ is globally Lipschitz then solution exists and is unique in $\mathbb{R}$
2. If the solution is bounded and monotone then solution exists and unique in $\mathbb{R}$ even if f is not globally Lipschitz.
3. If $f(t,x)$ is continuous every where then solution need not exist every where.
4. local uniqueness implies uniqueness in the maximal interval of existence.

1.2 Fully Nonlinear equations

Let $F : [a,b]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a continuous function. Then the first order equations of the form

$$F(t,y,y') = 0$$

are called fully nonlinear equations. There are many methods to find solutions of such equations such as separation of variables, exact differential, linear homogeneous and special type of equations that can be transformed into linear equation viz. Bernoulli equation etc. Here we are interested in the new class of solutions called *Singular solutions.*

1.2.1 Singular solutions

In this section we will the role of geometry in understanding certain important calss of solutions known as singular solutions.
An important feature of nonlinear equations is the existence of singular solutions that are not expressed in the one parameter family of solutions (general solution). A function $\phi(t)$ is called singular solution of ODE $F(t,y,y')=0$, if uniqueness of solution is violated at each point of the domain of the equation. That is, a solution $x=\gamma(t)$ is called singular solution if for each (t_0,x_0) with $\gamma(t_0)=x_0$, there exists a solution $\psi(t)\neq\gamma(t)$ passing through (t_0,x_0). In particular, $\gamma(t)$ and $\psi(t)$ have the same derivative at $t=t_0$. Since this holds for every pint (t_0,x_0) this means that $x=\gamma(t)$ is the envelope of a family of solutions of $F(t,x,x')=0$. For example, $y'=2\sqrt{y}$ has general solution $y=(t+c)^2$. Besides this $y\equiv 0$ is also a solution. However this is not contained in the general solution. Also it is easy to check that more than one solution curve passes through each point of $y\equiv 0$. Hence, $y\equiv 0$ is the singular solution. There are few strategies to find singular solution.

Let $\phi(t,y,c)=0$ be a one parameter family of solution of $F(t,y,y')=0$.

Example 1.2.1 *Find singular solution of* $y'=\sqrt{1-y^2}$.

By separation of variables, one can find the solution to be $\phi(t,c)=\sin(t+c)$ with $t+c\in[\frac{\pi}{2},\frac{-\pi}{2}]$. Now the envelop of $\phi(t,c)$ can be calculated to be

$$\cos(t+c)=0, \implies t+c=\pm\frac{\pi}{2}$$

Therefore $y\equiv\pm 1$ is a solution. □
If ϕ and its partial derivatives are continuous then the envelope of family of curves $\phi(t,y,c)=0$ is defined by the system of equations

$$\phi(t,y,c)=0,\ \frac{\partial\phi}{\partial c}=0.$$

Let $\phi(t,c)$ is a one parameter family of solutions of $F(t,y,y')=0$, then differentiating $F(t,\phi(t,c),\partial_t\phi(t,c))=0$ with respect to c, by chain rule, one gets

$$F_y\ \frac{\partial\phi}{\partial c}+F_{y'}\ \frac{\partial^2\phi}{\partial t\partial c}=0$$

If $F_y\not\equiv 0$ and $F_{y'}=0$, then $\phi_c(t,c)=0$. So the envelop of $\phi(t,c)$ at times can be obtained by solving $F=0, F_{y'}=0$ if $F_y\neq 0$. However there are situations where this may not be enough(see the example 1.2.4).

Example 1.2.2 *find singular solution of* $y=(y')^2-3ty'+3t^2$.

The general solution is $y=ct+c^2+t^2$

$$\phi=y-(ct+c^2+t^2)$$

$$\frac{\partial \phi}{\partial c} = -t - 2c = 0 \quad \Rightarrow \quad c = \frac{-t}{2}$$

Therefore, $y = t(-t/2) + t^2/4 + t^2 = 3t^2/4$ and $y_2 = 3t^2/4$ is the singular solution.

As noted above, if $F(t,y,y')$ and its partial derivatives $F_y, F_{y'}$ are continuous in the domain of differential equation. Then the singular solution can be found by solving the equations

$$F(t,y,y') = 0, \;\; F_{y'} = 0$$

So taking $F = (y')^2 - 3ty' + 3t^2 - y$, we have

$$F_{y'} = 2y' - 3t, \; F_y = -1 \neq 0$$

Therefore $2y' - 3t = 0$ implies $y = \frac{3}{4}t^2$ □

Example 1.2.3 *Find the singular solution of* $1 + (y')^2 = \dfrac{1}{y^2}$.

Writing the system

$$F = y^2\left[1 + (y')^2\right]$$

$$F_{y'} = 2y'y^2 = 0 \text{ } or \text{ } y' = 0$$

putting this in $F(t,y,y') = 0$ gives $y^2 = 1$ *or* $y = \pm 1$. It is easy to check that $y = \pm 1$ satisfies the equation.

Example 1.2.4 *Investigate the singular solution of* $(y')^2(1-y)^2 = 2 - y$.

$$F = (y')^2(1-y)^2 - (2-y) = 0$$

Differentiating the equation w.r.t. y',

$$F_{y'} = 2(y')(1-y)^2 = 0, \;\; F_y = 2(y')^2(1-y) + 1.$$

Eliminating y' from the system, we get the equations

$$(y')^2(1-y)^2 = 2 - y, \;\; y'(1-y)^2 = 0.$$

Then we obtain

$$(y')^2 = \frac{2-y}{(1-y)^2}, \;\; \Longrightarrow \;\; \frac{2-y}{(1-y)^2}(1-y)^4 = 0 \;\; or (1-y)^2(2-y) = 0.$$

Also note that $\frac{\partial F}{\partial y} = (y')^2(1-y) + 1 \neq 0$ implies $y = 1$ or $y' = 0$. Since $y = 1$ is not a solution, so $y = 2$ is the singular solution. □

Clairaut Equation

A typical equation where singular solution can arise is the clairaut equation:

$$x = tx' + g(x')$$

where say $g \in C(R)$

$$F(tx, x') = tx' + g(x') - x = 0$$

$$F_{x'} = t + g'(x') = 0$$

If we assume that g' is invertible and let $h = (g')^{-1}$, then $t + g'(x') = 0$ implies $x' = h(-t)$. substituting this in the equation we find

$$x(t) = th(-t) + g(h(-t))$$

Example 1.2.5 *Solve the equation $x = tx' + (x')^2$.*

Here $g(x') = (x')^2$ and $g'(x') = 2x'$ is invertible. Solving $2x' = -t$ we find $x' = \dfrac{-t}{2}$. Hence,

$$\vartheta(t) = t\frac{-t}{2} + g\Big(\frac{-t}{2}\Big) = \frac{-t^2}{2} + (\frac{-t}{2})^2 = \frac{-t^2}{4}$$

and this turns out to be the envelop of family of solution.

$$x(t) = ct + c^2 \quad c \in \mathbb{R}$$

1.3 Exercises

1. Determine which of the following functions are Lipschitz around the given point 0.
   (a) $f(t,x) = |x|$ (b) $f(t,x) = |x|^p,\ 0 < p < 1,$ (c) $f(t,x) = |\sin x|$.
2. If $y(t)$ be a solution of $y'(t) = y^2(1-\cos y) + t^2 \sin y$ that vanishes at a point t_0 then show that $y(t) \equiv 0$.
   Consider the following Initial value problems
3. which of the following problems has unique global (in whole of $\mathbb{R}$) solution?
   (a) $y' = y^2 \sin\left(\dfrac{1}{y}\right),\ y(0) = 0$ (b) $y' = \tan^{-1}(e^{-y}),\ y(0) = 0$
4. Suppose that $\eta(0) = 0, \eta(u) > 0$ for $u > 0$ and $\eta'(0)$ exists exists. Show that η satisfies the Osgood's condition.
5. Find the solution of IVP: $x' = x|x|,\ x(0) = 0$.
6. Determine the existence and uniqueness for the IVP

$$x' = 2t - 2\sqrt{x^+},\ x(0) = 0$$

   where $x^+(t) = \max\{x(t), 0\}$. Show that no sub sequence of Picard iterations converge to solution.
7. Show that the IVP: $x'' = f(t,x),\ x(t_0) = x_0,\ x'(t_0) = x_1$ is equivalent to

$$x(t) = x_0 + (t-t_0)x_1 + \int_{t_0}^{t} (t-s) f(s, x(s)) ds,\ t \in J$$

   where J is the interval of existence of solution.
8. Find all $x_0 \geq 0$ such that the problem $x' = x^{2/3}, x(0) = a$ has a unique solution.
9. Which of the following problems have unique solutions
   a. $y' = e^{-y},\ y(0) = 1,$ (b) $y' = 1 + e^{y^2},\ y(0) = 0,$ (c) $y' = e^y, y(0) = 0$
10. Consider the problem $x' = x - x^2,\ x(0) = x_0$. Show that if $0 < x_0 < 1$ then solution exists for all t and $0 \leq x(t) \leq 1$ for all t.

11. Show that the following IVP has solution in the whole of $\mathbb{R}$ and is unique

$$y' = \frac{1-y^4}{1+y^2},\ y(0) = 0.$$

Also prove that $-1 < y(t) < 1$ for all t and there is no t_0 where $|y(t_0)| = 1$.

12. Let $f(x,t)$ be Lipschitz continuous from $\mathbb{R}$ to $\mathbb{R}$ and let $x(t)$ be solution of IVP: $x' = f(t,x),\ x(t_0) = x_0$. Suppose $x_1(t)$ and $x_2(t)$ satisfies

$$x_1'(t) \leq f(t,x_1(t)),\ x_1(t_0) \leq x_0$$
$$x_2'(t) \geq f(t,x_2(t)),\ x_2(t_0) \geq x_0.$$

Then show that $x_1(t) \leq x(t) \leq x_2(t)$ for all $t \geq t_0$

13. Suppose $f(s)$ is continuous, increasing function and let $z(t), w(t)$ satisfy the relations

$$z(t) \leq w(t)$$
$$z'(t) \leq f(z(t)),\ z(0) = a$$
$$w'(t) \geq f(w(t)),\ w(0) = a.$$

Then show that the sequence defined by

$$x_1(t) = z(t)$$
$$x_{n+1}'(t) = f(x_n(t)),\ x_{n+1}(0) = a,\ n = 1,2,\cdots$$

converges to a solution of $x'(t) = f(x(t)),\ x(0) = a$.

14. What can we say about existence and uniqueness of the following problems?

   a. $e^{x'} = x,\ x(t_0) = x_0$ $\quad$ (b) $\ln x' = x^2,\ x(t_0) = x_0$

15. Which of the following equations has solution in the whole of $\mathbb{R}$:
(a) $x' = \sin x$ (b) $x' = \ln(1+x^2)$ (c) $x' = \max\{1,x\}, x(0) = 1$
16. Show that the solutions of $x' = -\cos(tx)$ are even.
17. Find singular solution of $(x')^2 - tx' + x = 0$