Several Variable Differential Calculus

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1 Motivations

The aim of studying the functions depending on several variables is to understand the functions which has several input variables and one or more output variables. For example, the following are Real valued functions of two variables x, y:

(1) $f(x, y) = x^2 + y^2$ is a real valued function defined over $\mathbb{R}^2$.

(2) $f(x, y) = \frac{xy}{x^2+y^2}$ is a real valued function defined over $\mathbb{R}^2 \backslash \{(0, 0)\}$

The real world problems like temperature distribution in a medium is a real valued function with more than 2 variables. The temperature function at time t and at point (x, y) has 3 variables. For example, the temperature distribution in a plate, (unit square) with zero temperature at the edges and initial temperature (at time $t = 0$)$T_0(x, y) = \sin \pi x \sin \pi y$, is $T(t, x, y) = e^{-\pi^2 kt} \sin \pi x \sin \pi y$.

Another important problem of physics is Sound waves and water waves. The function $u(x, t) = A \sin(kx - \omega t)$ represents the traveling wave of the initial wave front $\sin kx$. The Optimal cost functions, for example a manufacturing company wants to optimize the resources, for their produce, like man power, capital expenditure, raw materials etc. The cost function depends on these variables. Earning per share for Apple company (2005-2010) has been modeled by $z = 0.379x - 0.135y - 3.45$ where x is the sales and y is the share holders equity.

2 Limits

Let $\mathbb{R}^2$ denote the set of all points $(x, y) : x, y \in \mathbb{R}$. The open ball of radius r with center (x_0, y_0) is

$$B_r((x_0, y_0)) = \left\{(x, y) : \sqrt{(x - x_0)^2 + (y - y_0)^2} < r\right\}.$$

Definition 2.0.1. *A point (a, b) is said to be an interior point of a subset Ω of $\mathbb{R}^2$ if there exists r such that $B_r((a, b)) \subset A$. A subset Ω is called open if each point of Ω is an interior point of Ω. A subset Ω is said to be closed in its set compliment ($\mathbb{R}^2 \setminus \Omega$) is an open subset of $\mathbb{R}^2$.*

For example,

(1). The open ball of radius δ: $B_\delta((0, 0)) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ is an open set

(2). Union of open balls is also an open set.

(3). The closed ball of radius $r : \overline{B_r((0,0))} = \left\{(x,y) : \sqrt{|x|^2+|y|^2} \le r\right\}$ is a closed set.

A sequence $\{(x_n, y_n)\}$ is said to converge to a point (x,y) in $\mathbb{R}^2$ if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sqrt{|x_n - x|^2 + |y_n - y|^2} < \epsilon, \text{ for all } \ n \ge N.$$

Definition 2.0.2. *Let Ω be a open set in $\mathbb{R}^2$, $(a,b) \in \Omega$ and let f be a real valued function defined on Ω except possibly at (a,b). Then the limit $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |f(x,y) - L| < \epsilon.$$

Example 2.0.3. *Consider the function $f(x,y) = xy$. Then the value of $f(0,0) = 0$. It is easy to see by back calculation*

$$|xy - 0| = |x||y| \le 2\sqrt{x^2+y^2} < 2\delta.$$

Therefore, given any $\epsilon > 0$, we can choose δ scuh that $\delta < \frac{\epsilon}{2}$ to have

$$\sqrt{x^2+y^2} < \delta \implies |xy - 0| < \epsilon$$

Therefore $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

First guess for L: Simultaneous limits: As a first guess we use the simultaneous limit for the value of L. That is,

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = L_1(say)$$

Also, if the limit exists, then it is unique. Indeed, if L_1 and L_2 are two limits, then by choosing $\delta = \min\{\delta_1, \delta_2\}$, then by triangle inequality, we get

$$|L_1 - L_2| \le |L_1 - f(x,y)| + |f(x,y) - L_2| < \epsilon + \epsilon.$$

Since ϵ is arbitrary, we get $L_1 = L_2$.

Example 2.0.4. *Consider the function $f(x,y)$: $f(x,y) = \dfrac{4xy^2}{x^2+y^2}$.*

This function is defined in $\mathbb{R}^2\backslash\{(0,0)\}$. So we need a value of L to verify the definition. We see that the simultaneous limit is 0. So $L = 0$ is a candidate to be the limit. So, let $\epsilon > 0$ be given, then

$$\frac{4|xy^2|}{x^2+y^2} \le 4|x| \le 4\sqrt{x^2+y^2} < 4\delta.$$

Therefore we can choose $\delta = \frac{\epsilon}{4}$. For such δ, $|f(x,y) - 0| < \epsilon$. The continuity of the function can be seen from the plot given below in Figure 1 (generated using wolfram web-site)

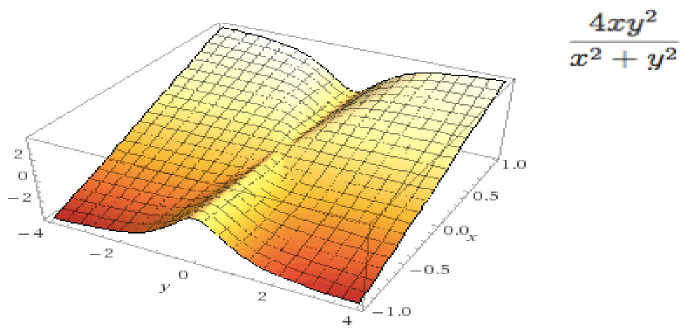


Figure 1:

Therefore, if limit exists then simultaneous limit (if exists) will be equal to the limit. However at times simultaneous limit may not exist but limit may exist. For example,

Example 2.0.5. *The function*

$$f(x,y) = \begin{cases} x\sin(\frac{1}{y}) + y\sin(\frac{1}{x}) & xy \neq 0 \\ 0 & xy = 0 \end{cases}$$

In this case we can easily see that simultaneous limits does not exist. But limit along $x = y$ will be zero. So this can be our choice for obtaining δ. By back calculation,

$$\left| x\sin(\frac{1}{y}) + y\sin(\frac{1}{x}) \right| \leq |x| + |y| \leq 2\sqrt{x^2+y^2} < 2\delta < e$$

So given any $\epsilon > 0$, we can choose $\delta = \epsilon/2$.

Example 2.0.6. *Finding limit through polar coordinates:*
Consider the function $f(x,y) = \dfrac{x^3}{x^2+y^2}$.

This function is defined in $\mathbb{R}^2 \backslash \{(0,0)\}$. Taking $x = r\cos\theta, y = r\sin\theta$, we get

$$|f(r,\theta)| = |r\cos^3\theta| \leq r \to 0 \;\; as \;\; r \to 0.$$

Using the triangle inequality (as in one variable limit), one can show that if limit exists, then it is unique. That is, the limit is independent of choice of path connecting (x,y) and (a,b). Geometrically, a piece of surface cannot lie in two different strips.

Example 2.0.7. *Example of function which has different limits along different straight lines.*
Consider the function $f(x,y) = \dfrac{xy}{x^2+y^2}$.

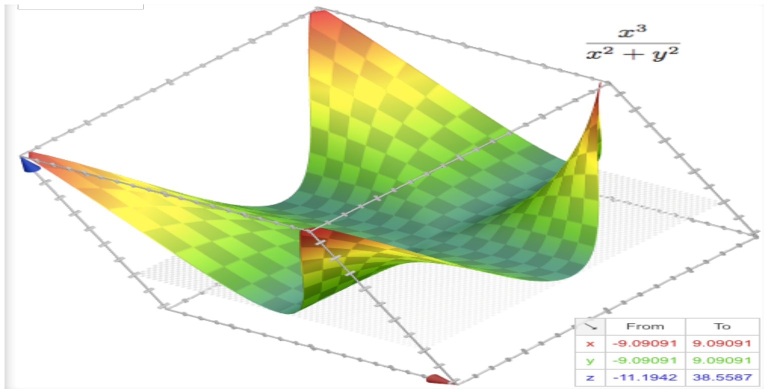


Figure 2:

Then along the straight lines $y = mx$, we get $f(x, mx) = \frac{m}{1+m^2}$. See this in the plot blow in figure 3, the height from through each line appraching the origin is different. Hence limit does not exist.

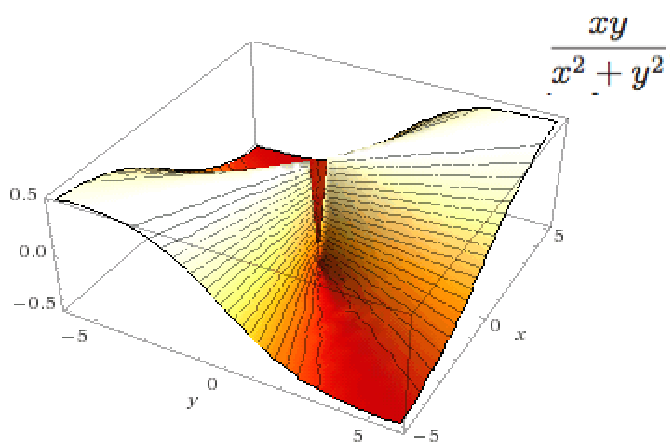


Figure 3:

Example 2.0.8. *Example of function which has different limits along different curves: Consider the function $f(x, y)$:*

$$f(x, y) = \frac{x^4 - y^2}{x^4 + y^2}$$

Then along the curves $y = mx^2$, we get $f(x, mx^2) = \frac{1-m^2}{1+m^2}$. Hence limit does not exist.

Example 2.0.9. *Example function where polar coordinates seem to give wrong conclusions Consider the function $f(x, y) = \dfrac{2x^2 y}{x^4 + y^2}$.*

Taking the path, $y = mx^2$, we see that the limit does not exist at $(0,0)$. Now taking $x = r\cos\theta, y = r\sin\theta$, we get

$$f(r,\theta) = \frac{2r\cos^2\theta\sin\theta}{r^2\cos^4\theta + \sin^2\theta}.$$

For any $r > 0$, the denominator is > 0. For each fixed r and taking $\theta \to 0$, we see it tends to

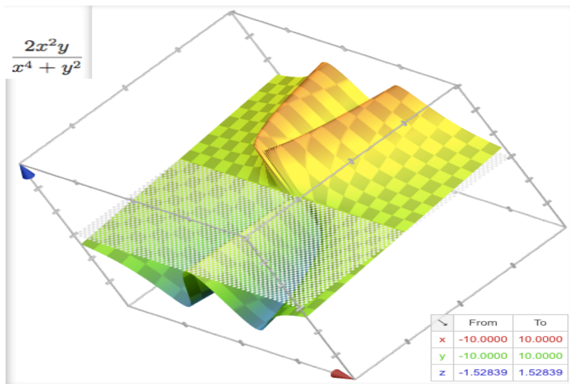


Figure 4:

0. Since $|\cos^2\theta\sin\theta| \leq 1$, we tend to think for a while that this limit goes to zero as $r \to 0$. However we know that the limit along the path $y = x^2$ is 1. So So taking this path in the polar form, we get the path $r\sin\theta = r^2\cos^2\theta$, (i.e., $r = \frac{\sin\theta}{\cos^2\theta}$). Along this path $r \to 0$ as $\theta \to 0$. Along this, we get

$$f(r,\theta) = \frac{2\sin^2\theta}{2\sin^2\theta} = 1.$$

Therefore the limit does not exist. Also see the above figure 4, where one could see the presence of sharp corners as one approaches origin from different sides.

3 Continuity and Partial derivatives

Let f be a real valued function defined in a ball around (a,b). Then

Definition 3.0.1. *f is said to be continuous at (a,b) if*

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

In other words,

$$f(a+h, b+k) \to f(a,b) \text{ as } h \to 0,\ k \to 0.$$

Example 3.0.2. *The function*

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & x^2+y^2 \neq 0 \\ 0 & x = y = 0 \end{cases}$$

Let $\epsilon > 0$. *Then* $|f(x,y) - 0| = |x|\frac{|y|}{\sqrt{x^2+y^2}} \leq |x|$. *So if we choose* $\delta = \epsilon$, *then* $|f(x,y)| \leq \epsilon$.

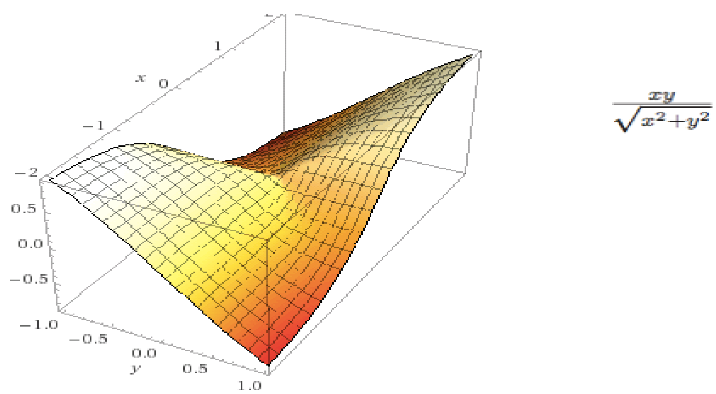


Figure 5:

Therefore, f is continuous at $(0,0)$.

Partial Derivatives: The partial derivative of f with respect to x at (a,b) is defined as

$$f_x(a,b) = \frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0} \frac{1}{h}\left(f(a+h,b) - f(a,b)\right).$$

similarly, the partial derivative with respect to y at (a,b) is defined as

$$f_y(a,b) = \frac{\partial f}{\partial y}(a,b) = \lim_{k\to 0} \frac{1}{k}\left(f(a,b+k) - f(a,b)\right).$$

In other words, if we treat the variable y as constant, and differentiate with respect to x, we get the partial derivative with respect to x. When computing a partial derivative with respect to x, we are looking at the instantaneous rate of change of f with respect to x, if we keep the variable y constant. Roughly speaking, we are asking: how does increasing x a tiny bit affect the value of the function f? The partial derivatives may be used as follows: If $f_x(a,b) > 0$ then we may say that at the point (a,b) if we move along positive x-direction then the function f increases.

Example 3.0.3. *Consider the function* $f(x,y) = x^2y$. *Then*

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$
$$= \lim_{h\to 0} \frac{(a+h)^2 b - a^2 b}{h} = 2ab + \lim_{h\to 0} hb = 2ab.$$

Example 3.0.4. *Consider the function*

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \not\equiv (0,0) \\ 0 & (x,y) \equiv (0,0) \end{cases}$$

As noted earlier, this is not a continuous function, but

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

Similarly, we can show that $f_y(0,0)$ *exists.*

Also for a continuous function, partial derivatives need not exist. For example $f(x,y) = |x| + |y|$. This is a continuous function at $(0,0)$. Indeed, for any $\epsilon > 0$, we can take $\delta < \epsilon/2$. But partial derivatives do not exist at $(0,0)$

Sufficient condition for continuity:

Theorem 3.0.5. *Suppose one of the partial derivatives exist at* (a,b) *and the other partial derivative is bounded in a neighborhood of* (a,b). *Then* $f(x,y)$ *is continuous at* (a,b).

Proof. Let f_y exists at (a,b). Then

$$f(a, b+k) - f(a,b) = kf_y(a,b) + \epsilon_1 k,$$

where $\epsilon_1 \to 0$ as $k \to 0$. Since f_x exists and bounded in a neighborhood of at (a,b),

$$\begin{aligned} f(a+h,b+k) - f(a,b) =& f(a+h,b+k) - f(a,b+k) + f(a,b+k) - f(a,b) \\ =& hf_x(a+\theta h, b+k) + kf_y(a,b) + \epsilon_1 k \\ \leq& hM + k|f_y(a,b)| + \epsilon_1 k \\ &\to 0 \text{ as } h,k \to 0. \end{aligned}$$

4 Directional derivatives

Let $\hat{p} = p_1\hat{i} + p_2\hat{j}$ be any **unit vector**. Then the directional derivative of $f(x, y)$ at (a, b) in the direction of $\hat{p}$ is

$$D_{\hat{p}}f(a, b) = \lim_{s\to 0} \frac{f(a + sp_1, b + sp_2) - f(a, b)}{s}.$$

Example 4.0.1. *Find the directional derivative of $f(x, y)$ at $P(1, 2)$ in the direction of unit vector $p = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$ for the function $f(x, y) = x^2 + xy$.*
From the definition,

$$\begin{aligned} D_{\hat{p}}f(1, 2) &= \lim_{s\to 0} \frac{f(1 + \frac{s}{\sqrt{2}}, 2 + \frac{s}{\sqrt{2}}) - f(1, 2)}{s} \\ &= \lim_{s\to 0} \frac{1}{s}\left(s^2 + s(2\sqrt{2} + \frac{1}{\sqrt{2}})\right) = 2\sqrt{2} + \frac{1}{\sqrt{2}} \end{aligned}$$

Next we would like to see if fhe existence of partial derivatives guarantee the existence of directional derivatives in all directions. The answer is negative. For example, take

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & x^2 + y^2 \neq 0 \\ 0 & x = y = 0 \end{cases}.$$

Let $\overrightarrow{p} = (p_1, p_2)$ such that $p_1^2 + p_2^2 = 1$. Then the directional derivative along p is

$$D_{\widehat{p}}f(0, 0) = \lim_{h\to 0} \frac{f(hp_1, hp_2) - f(0, 0)}{h} = \lim_{h\to 0} \frac{p_1 p_2}{h(p_1^2 + p_2^2)}$$

exist if and only if $p_1 = 0$ or $p_2 = 0$.

Also, the existence of all directional derivatives does not guarantee the continuity of the function. For example

Example 4.0.2. *The function*

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & (x, y) \not\equiv (0, 0) \\ 0 & x = y = 0 \end{cases}.$$

has directional derivatives in all directions at $(0, 0)$ but is not continuous at $(0, 0)$.

Let $\overrightarrow{p} = (p_1, p_2)$ such that $p_1^2 + p_2^2 = 1$. Then the directional derivative along p is

$$\begin{aligned} D_{\widehat{p}} f(0,0) &= \lim_{s\to 0} \frac{f(sp_1, sp_2) - f(0,0)}{s} \\ &= \lim_{s\to 0} \frac{s^3 p_1^2 p_2}{s(s^4 p_1^4 + s^2 p_2^2)} \\ &= \frac{p_1^2 p_2}{p_2^2} \text{ if } p_2 \neq 0 \end{aligned}$$

In case of $p_2 = 0$, we can compute the partial derivative w.r.t y to be 0. Therefore all the directional derivatives exist. But this function is not continuous ($y = mx^2$ and $x \to 0$). ///

So the concept of differentiability of a function is stronger than directional derivative.

5 Differentiability

First let us recall the concept of differentiability in one variable case. A function defined in an interval containing $x = a$ is said to be differentiable if the following limit exists

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = f'(a).$$

However we cannot write this for functions defined in $\mathbb{R}^2$ as we do not know how to divide by say (h,k). But the above definition can equivalently written as: f is differentiable at $x = a$ if there exists a constant $f'(a)$ (say) such that

$$f(a+h) - f(a) = hf'(a) + \epsilon h$$

where $\epsilon \to 0$ as $h \to 0$. So this motivates us to the following generalization:
Let D be an open subset of $\mathbb{R}^2$. Then we may think of defining a function $f(x,y) : D \to \mathbb{R}$ is differentiable at a point (a,b) of D if there exists C_1, C_2 and $\epsilon_1 = \epsilon(h,k), \epsilon_2 = \epsilon_2(h,k)$ such that

$$f(a+h, b+k) - f(a,b) = hC_1 + kC_2 + h\epsilon_1 + k\epsilon_2,$$

where $\epsilon_1, \epsilon_2 \to 0$ as $(h,k) \to (0,0)$.
Suppose if the above holds for a function f, then by taking $k = 0$ in the above definition, we have

$$f(a+hb) - f(a,b) = hC_1 + \epsilon_1 h$$

taking $h \to 0$, we obtain $C_1 = f_x(a,b)$. Similarly $C_2 = f_y(a,b)$. So we may define the differentiability as

Definition 5.0.1. *A function $f : D \to \mathbb{R}$ is differentiable at a point (a, b) of D if*

$$f(a+h, b+k) - f(a, b) - hf_x(a, b) - kf_y(a, b) = h\epsilon_1 + k\epsilon_2$$

where $\epsilon_1, \epsilon_2 \to 0$ as $(h, k) \to (0, 0)$.

Example 5.0.2. *Consider the function $f(x, y) = x^2 + y^2 + xy$. Then $f_x(0, 0) = f_y(0, 0) = 0$. Also*

$$f(h, k) - f(0, 0) - hf_x(0, 0) - kf_y(0, 0) = h^2 + k^2 + hk = 0h + 0k + \epsilon_1 h + \epsilon_2 k$$

where $\epsilon_1 = h + k, \epsilon_2 = k$. Therefore f is differentiable at $(0, 0)$.

Example 5.0.3. *Show that the following function $f(x, y)$ is is not differentiable at $(0, 0)$*

$$f(x, y) = \begin{cases} x \sin \frac{1}{x} + y \sin \frac{1}{y}, & xy \neq 0 \\ 0 & xy = 0 \end{cases}$$

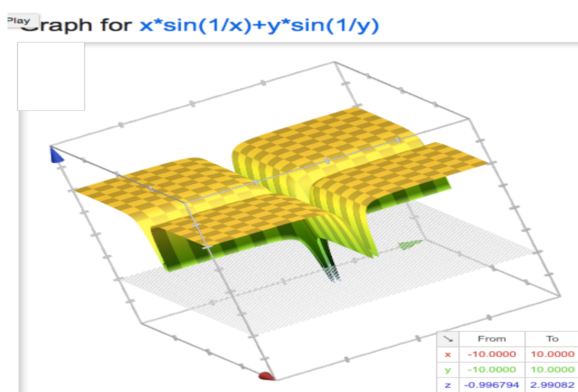


Figure 6:

Using the boundedness of sin and cos, we get $|f(x, y)| \leq |x| + |y| \leq 2\sqrt{x^2 + y^2}$ implies that f is continuous at $(0, 0)$. Also

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = 0.$$

$$f_y(0, k) = \lim_{k \to 0} \frac{f(0, k) - f(0, 0)}{k} = 0.$$

If f is differentiable, then there exists ϵ_1, ϵ_2 such that

$$f(h, k) - f(0, 0) = \epsilon_1 h + \epsilon_2 k$$

where $\epsilon_1, \epsilon_2 \to 0$ as $h, k \to 0$. Now taking $h = k$, we get

$$f(h,h) = (\epsilon_1 + \epsilon_2)h \implies 2h \sin\frac{1}{h} = h(\epsilon_1 + \epsilon_2).$$

So as $h \to 0$, we get $\sin\frac{1}{h} \to 0$, a contradiction.

Example 5.0.4. *Show that the function $f(x,y) = \sqrt{|xy|}$ is not differentiable at the origin.*

Easy to check the continuity (take $\delta = \epsilon$).

$$f_x(0,0) = \lim_{h\to 0} \frac{0-0}{h} = 0, \text{ and similar calculation shows } f_y(0.0) = 0$$

So if f is differentiable at $(0,0)$, then there exist, ϵ_1, ϵ_2 such that

$$f(h,k) = \epsilon_1 h + \epsilon_2 k.$$

Taking $h = k$, we get

$$|h| = (\epsilon_1 + \epsilon_2)h.$$

This implies that $(\epsilon_1 + \epsilon_2) \not\to 0$.

Notations:

1. $\Delta f = f(a+h, b+k) - f(a,b)$, the total variation of f
2. $df = hf_x(a,b) + kf_y(a,b)$, the total differential of f.
3. $\rho = \sqrt{h^2 + k^2}$

Then we have the following **Equivalent condition** for differentiability:

Theorem 5.0.5. *f is differentiable at (a,b)* $\iff \lim_{\rho\to 0} \frac{\Delta f - df}{\rho} = 0.$

Proof. Suppose $f(x,y)$ is differentiable. Then, there exists ϵ_1, ϵ_2 such that

$$f(a+h, b+k) - f(a,b) = hf_x(a,b) + kf_y(a,b) + h\epsilon_1 + k\epsilon_2,$$

where $\epsilon_1, \epsilon_2 \to 0$ as $(h,k) \to (0,0)$.
Therefore, we can write

$$\frac{\Delta f - df}{\rho} = \epsilon_1(\frac{h}{\rho}) + \epsilon_2(\frac{k}{\rho}),$$

where $df = hf_x(a,b) + kf_y(a,b)$ and $\rho = \sqrt{h^2 + k^2}$. Now since $|\frac{h}{\rho}| \leq 1$, $\frac{k}{\rho} \leq 1$ we get

$$\lim_{\rho\to 0} \frac{\Delta f - df}{\rho} = 0.$$

On the other hand, if $\lim_{\rho \to 0} \frac{\Delta f - df}{\rho} = 0$, then

$$\Delta f = df + \epsilon\rho, \quad \epsilon \to 0 \ \text{ as } h, k \to 0.$$

We may write $\epsilon\rho$ as

$$\begin{aligned} \epsilon\rho &= \frac{\epsilon\rho}{|h| + |k|}|h| + \frac{\epsilon\rho}{|h| + |k|}|k| \\ &= \frac{\epsilon\rho \ sgn(h)}{|h| + |k|}h + \frac{\epsilon\rho \ sgn(k)}{|h| + |k|}k \end{aligned}$$

Therefore, we can take $\epsilon_1 = \frac{\epsilon\rho \ sgn(h)}{|h|+|k|}$ and $\epsilon_2 = \frac{\epsilon\rho \ sgn(k)}{|h|+|k|}$.

Example 5.0.6. *Consider the function* $f(x, y) = \begin{cases} \frac{x^2y^2}{x^2+y^2} & (x, y) \not\equiv (0, 0) \\ 0 & x = y = 0 \end{cases}$.

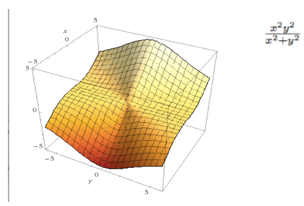


Figure 7:

Partial derivatives exist at $(0, 0)$ and $f_x(0, 0) = 0, f_y(0, 0) = 0$. By taking $h = \rho\cos\theta, k = \rho\sin\theta$, we get

$$\frac{\Delta f - df}{\rho} = \frac{h^2k^2}{\rho^3} = \frac{\rho^4\cos^2\theta\sin\theta}{\rho^3} = \rho\cos^2\theta\sin\theta.$$

Therefore, $\left|\frac{\Delta f - df}{\rho}\right| \leq \rho \to 0$ as $\rho \to 0$. Therefore f is differentiable at $(0, 0)$.

Example 5.0.7. *Consider* $f(x, y) = \begin{cases} \frac{x^2y}{x^2+y^2} & (x, y) \not\equiv 0 \\ 0 & x = y = 0 \end{cases}$.

Partial derivatives exist at $(0, 0)$ and $f_x(0, 0) = f_y(, 0) = 0$. By taking $h = \rho\cos\theta, k = \rho\sin\theta$, we get

$$\frac{\Delta f - df}{\rho} = \frac{h^2k}{\rho^3} = \frac{\rho^3\cos^2\theta\sin\theta}{\rho^3} = \cos^2\theta\sin\theta.$$

The limit does not exist. Therefore f is NOT differentiable at $(0, 0)$.

The following theorem is on **Sufficient condition** for differentiability:

Theorem 5.0.8. *Suppose $f_x(x, y)$ and $f_y(x, y)$ exist in an open neighborhood containing (a, b) and both functions are continuous at (a, b). Then f is differentiable at (a, b).*

Proof. Since $\frac{\partial f}{\partial y}$ is continuous at (a, b), there exists a neighborhood N(say) of (a, b) at every point of which f_y exists. We take $(a + h, b + k)$, a point of this neighborhood so that $(a + h, b), (a, b + k)$ also belongs to N.
We write

$$f(a+h, b+k) - f(a, b) = \{f(a+h, b+k) - f(a+h, b)\} + \{f(a+h, b) - f(a, b)\}.$$

Consider a function of one variable $\phi(y) = f(a + h, y)$.
Since f_y exists in $N, \phi(y)$ is differentiable with respect to y in the closed interval $[b, b+k]$ and as such we can apply Lagrange's Mean Value Theorem, for function of one variable y in this interval and thus obtain

$$\begin{aligned} \phi(b+k) - \phi(b) &= k\phi^{'}(b+\theta k), \;\; 0 < \theta < 1 \\ &= kf_y(a+h, b+\theta k) \end{aligned}$$

$$f(a+h, b+k) - f(a+h, b) = kf_y(a+h, b+\theta k), \;\; 0 < \theta < 1.$$

Now, if we write

$$f_y(a+h, b+\theta k) - f_y(a, b) = \epsilon_2 \text{ (a function of } h, k)$$

then from the fact that f_y is continuous at (a,b). we may obtain

$$\epsilon_2 \to 0 \; as \; (h, k) \to (0, 0).$$

Again because f_x exists at (a, b) implies

$$f(a+h, b) - f(a, b) = hf_x(a, b) + \epsilon_1 h,$$

where $\epsilon_1 \to 0$ as $h \to 0$. Combining all these we get

$$\begin{aligned} f(a+h, b+k) - f(a, b) &= k[f_y(a, b) + \epsilon_2] + hf_x(a, b) + \epsilon_1 h \\ &= hf_x(a, b) + kf_y(a, b) + \epsilon_1 h + \epsilon_2 k \end{aligned}$$

where ϵ_1, ϵ_2 are functions of (h, k) and they tend to zero as $(h, k) \to (0, 0)$.
This proves that $f(x, y)$ is differentiable at (a, b).

Remark 5.1. *The above proof still holds if f_y is continuous and f_x exists at (a, b).*

There are functions which are Differentiable but the partial derivatives need not be continuous. For example,

Example 5.0.9. *Consider the function*

$$f(x,y) = \begin{cases} x^3 \sin\frac{1}{x^2} + y^3 \sin\frac{1}{y^2} & xy \neq 0 \\ 0 & xy = 0. \end{cases}$$

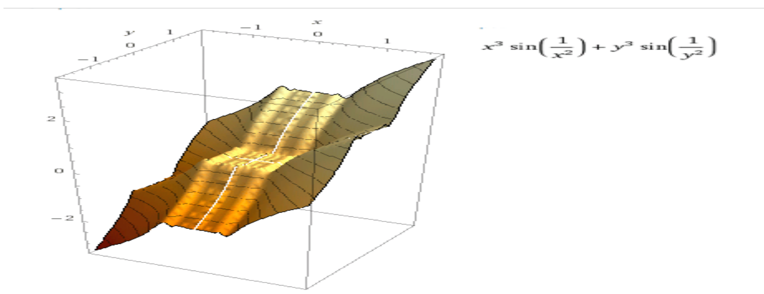


Figure 8:

Then

$$f_x(x,y) = \begin{cases} 3x^2 \sin\frac{1}{x^2} - 2\cos\frac{1}{x^2} & xy \neq 0 \\ 0 & xy = 0 \end{cases}$$

Also $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0$. So partial derivatives are not continuous at $(0,0)$.

$$\begin{aligned} f(\Delta x, \Delta y) &= (\Delta x)^3 \sin\frac{1}{(\Delta x)^2} + (\Delta y)^3 \sin\frac{1}{(\Delta y)^2} \\ &= 0 + 0 + \epsilon_1 \Delta x + \epsilon_2 \Delta y \end{aligned}$$

where $\epsilon_1 = (\Delta x)^2 \sin\frac{1}{(\Delta x)^2}$ and $\epsilon_2 = (\Delta y)^2 \sin\frac{1}{(\Delta y)^2}$. It is easy to check that $\epsilon_1, \epsilon_2 \to 0$. So f is differentiable at $(0,0)$.

There are continuous functions for which directional derivatives exist in any direction, but the function is not differentiable. For example,

Example 5.0.10. *Consider the function*

$$f(x,y) = \begin{cases} \frac{y}{|y|}\sqrt{x^2+y^2} & y \neq 0 \\ 0 & y = 0. \end{cases}$$

Chain rule:

Partial derivatives of composite functions: Let $z = F(u, v)$ and $u = \phi(x, y), v = \psi(x, y)$. Then $z = F(\phi(x, y), \psi(x, y))$ as a function of x, y. Suppose F, ϕ, ψ have continuous partial derivatives, then we can find the partial derivatives of z w.r.t x, y as follows: Let x be increased by Δx, keeping y constant. Then the increment in u is $\Delta_x u = u(x + \Delta x, y) - u(x, y)$ and similarly for v. Then the increment in z is (as z is differentiable as a function of u, v)

$$\Delta_x z := z(x + \Delta x, y + \Delta y) - z(x, y) = \frac{\partial F}{\partial u}\Delta_x u + \frac{\partial F}{\partial v}\Delta_x v + \epsilon_1 \Delta_x u + \epsilon_2 \Delta_x v$$

Now dividing by Δx

$$\frac{\Delta_x z}{\Delta x} = \frac{\partial F}{\partial u}\frac{\Delta_x u}{\Delta x} + \frac{\partial F}{\partial v}\frac{\Delta_x v}{\Delta x} + \epsilon_1 \frac{\Delta_x u}{\Delta x} + \epsilon_2 \frac{\Delta_x v}{\Delta x}$$

Taking $\Delta x \to 0$, we get

$$\begin{aligned}\frac{\partial z}{\partial x} =& \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} + (\lim_{\Delta x \to 0} \epsilon_1)\frac{\partial u}{\partial x} + (\lim_{\Delta x \to 0} \epsilon_2)\frac{\partial v}{\partial x} \\ =& \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x}\end{aligned}$$

similarly, one can show

$$\frac{\partial z}{\partial y} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y}.$$

Example 5.0.11. *Let $z = \ln(u^2 + v^2)$, where $u = e^{x+y^2}$ and $v = x^2 + y$. Then find $\frac{\partial z}{\partial x}$ at $x = 1, y = 1$.*
From the above Chain rule, taking $F(u, v) = \ln(u^2 + v^2)$ and differentiating F with u, $F_u = \frac{2u}{u^2+v}, F_v = \frac{1}{u^2+v}$. Also, $u_x = e^{x+y^2}$ and $v_x = 2x$. Therefore,

$$\frac{\partial z}{\partial x} = \frac{2u}{u^2 + v}e^{x+y^2} + \frac{2x}{u^2 + v}$$

At $x = 1, y = 1$, we have $u = e^2, v = 2$. Therefore,

$$\frac{\partial z}{\partial x} = \frac{2e^2}{2 + e^4}e^2 + \frac{2}{e^4 + 2} \quad at\ x = 1,\ y = 1.$$

Derivative of Implicitly defined function

Theorem 5.0.12. *Let $y = y(x)$ be defined as $F(x, y) = 0$, where F, F_x, F_y are continuous at (x_0, y_0) and $F_y(x_0, y_0) \neq 0$. Then $\frac{dy}{dx} = -\frac{F_x}{F_y}$ at (x_0, y_0).*

Proof. Increase x by Δx, then y receives Δy increment and $F(x + \Delta x, y + \Delta y) = 0$. Also

$$0 = \Delta F = F_x \Delta x + F_y \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where $\epsilon_1, \epsilon_2 \to 0$ as $\Delta x \to 0$. This is same as

$$\frac{\Delta y}{\Delta x} = -\frac{F_x + \epsilon_1}{F_y + \epsilon_2}$$

Now taking limit $\Delta x \to 0$, we get $\frac{dy}{dx} = -\frac{F_x}{F_y}$.

Example 5.0.13. *Find $\frac{dy}{dx}$ for the function $y = y(x)$ defined implicitly as $e^y - e^x + xy = 0$. Taking $F(x, y) = e^y - e^x + xy$, we obtain by differentiang partially with respect to x and y,*

$$F_x = -e^x + y, F_y = e^y + x.$$

Therefore, $\frac{dy}{dx} = \frac{e^x - y}{e^y + x}$.

Theorem 5.0.14. *If $f(x, y)$ is differentiable, then the directional derivative in the direction $\hat{p}$ at $(a, b)$is*

$$D_{\hat{p}} f(a, b) = \nabla f(a, b) \cdot \hat{p}.$$

Proof. Let $\hat{p} = (p_1, p_2)$. Then from the definition,

$$\lim_{s \to 0} \frac{f(a + sp_1, b + sp_2) - f(a, b)}{s} = \lim_{s \to 0} \frac{f(x(s), y(s)) - f(x(0), y(0))}{s}$$

where $x(s) = a + sp_1, y(s) = b + sp_2$.
From the chain rule,

$$\lim_{s \to 0} \frac{f(x(s), y(s)) - f(x(0), y(0))}{s} = \frac{\partial f}{\partial x}(a, b)\frac{dx}{ds} + \frac{\partial f}{\partial y}(a, b)\frac{dy}{ds} = \nabla f(a, b) \cdot (p_1, p_2).$$

///

The above proposition is again only sufficient condition. That is The formula $D_{\hat{p}} f = \nabla f \cdot \hat{p}$ can still hold even when function f is NOT differentiable. for example

$$f(x, y) = \begin{cases} \dfrac{x^2 y \sqrt{|y|}}{x^4 + y^2} & (x, y) \not\equiv (0, 0) \\ 0 & (x, y) \equiv (0, 0) \end{cases}$$

In this case it is easy to check from the definition that all directional derivatives at the origin are equal to zero. But the function is not differentiable at the origin. To show this take the polar coordinates $x = r\cos\theta, y = r\sin\theta$ to see that

$$\frac{\Delta f - df}{r} = \sqrt{r}\, \frac{\sin^2\theta \cos\theta \sqrt{|\sin\theta|}}{r^2\cos^2\theta + \sin^2\theta}.$$

Taking $r = \frac{\sin\theta}{\cos\theta}$ and taking $\theta \to 0$, we see that the above limit approaches infinity. ///

Definition 5.0.15. ***Gradient:*** *The gradient of a function $f(x, y)$ at the point (a, b) is the vector*

$$\nabla u = f_x \ \hat{\imath} + f_y \ \hat{\jmath} \ at\,(a, b).$$

Using the directional derivatives, we can also find the Direction of maximum rate of change. $D_{\hat{p}} f = \nabla f \cdot \hat{p} = |\nabla f| \cos\theta$ where θ is the angle between ∇u and $\hat{p}$. So the function f increases most rapidly when $\cos\theta = 1$ or when $\hat{p}$ is the direction of ∇f. The directional derivative in the direction $\frac{\nabla f}{|\nabla f|}$ is equal to $|\nabla f|$. So the maximal rate of change of f is $|\nabla f|$.
Similarly, f decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is $D_{\hat{p}} f = -|\nabla f|$. Finally, the direction of no change is when $\theta = \frac{\pi}{2}$. i.,e., $\hat{p} \perp \nabla f$.

Example 5.0.16. *Find the direction in which $f = x^2/2 + y^2/2$ increases and decreases most rapidly at the point $(1, 1)$. Also find the direction of zero change at $(1, 1)$.*

The function f is differentiable as $f_x = x, f_y = y$ are continuous function. Therefore

$$D_{\hat{p}} f = (x, y) \cdot \hat{p} = xp_1 + yp_2 = p_1 + p_2 \text{ at } (1, 1)$$

The direction of maximum change is ∇f. Therefore for direction of maximal change we take

$$\hat{p} = \frac{1}{x^2 + y^2}(x\hat{\imath} + y\hat{\jmath}) = \frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath} \text{ at } (1, 1).$$

Hence the maximum change at $(1, 1)$ is $\sqrt{2}$. Also the direction of zero change is

$$\hat{p} = \frac{1}{\sqrt{x^2 + y^2}}(-y, x) = \frac{-1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}, \text{ at } (1, 1).$$

6 Tangents and Normal to Level curves

Let $f(x, y)$ be differentiable and consider the level curve $f(x, y) = c$. Let $\vec{r}(t) = g(t)\hat{\imath} + h(t)\hat{\jmath}$ be its parametrization. For example $f(x, y) = x^2 + y^2$ has $x(t) = a\cos t, y(t) = a\sin t$ as level curve $x^2 + y^2 = a^2$, which is a circle of radius a. For any curve $\mathcal{C}$ with position vector $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$, the direction

$$\frac{1}{\Delta t}\left(\vec{r}(t + \Delta t) - \vec{r}(t)\right)$$

represents the direction of tangent. Therefore taking limit $\Delta t \to 0$ we get

$$\vec{r}\,'(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\vec{r}(t + \Delta t) - \vec{r}(t)\right) = x'(t)\hat{\imath} + y'(t)\hat{\jmath}.$$

That is, $\vec{r}\,'(t)$ represents the tangential direction. To get the normal direction, differentiating the equation $f(x(t), y(t)) = a^2$ with respect to t, we get

$$f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = 0.$$

Now since $\overrightarrow{r}'(t) = x'(t)\hat{i} + y'(t)\hat{j}$ is the tangent to the curve, we can infer from the above equation that ∇f is the direction of Normal. Hence we have

Equation of Normal at (a, b) is

$$x = a + f_x(a, b)t, y = b + f_y(a, b)t,\ t \in \mathbb{R}$$

Equation of Tangent is

$$(x - a)f_x(a, b) + (y - b)f_y(a, b) = 0.$$

Example 6.0.1. *Find the normal and tangent to* $\frac{x^2}{4} + y^2 = 2$ *at* $(-2, 1)$*.*

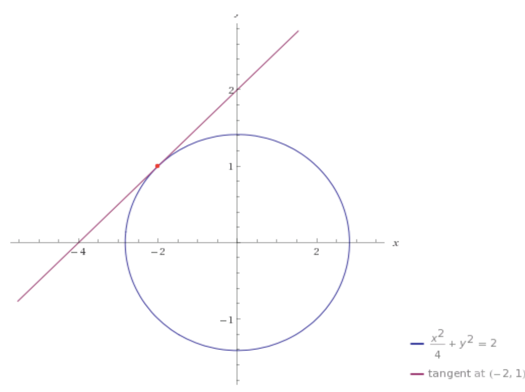


Figure 9:

We find $\nabla f = \frac{x}{2}\hat{i} + 2y\hat{j}\Big|_{(-2,1)} = -\hat{i} + 2\hat{j}$. Therefore, the Tangent line through $(-2, 1)$ is $-(x + 2) + 2(y - 1) = 0$.

Tangent Plane and Normal lines

Let $\overrightarrow{r}(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}$ is a smooth level curve(space curve) of the level surface $f(x, y, z) = c$. Then differentiating $f(x(t), y(t), z(t)) = c$ with respect to t and applying chain rule, we get

$$\nabla f(a, b, c) \cdot (x'(t), y'(t), z'(t)) = 0$$

This is same as

$$\nabla f \cdot \vec{r}\,'(t) = 0.$$

Now note that the vector ∇f in the above equation will be same for any curve $\vec{r}(t)$. In other words the only vector $\nabla(a, b, c)$ will work for any curve $\vec{r}(t)$. That is this vector is normal to tangents of all curves passing through the point (a, b, c). Therefore the equation of Normal line at (a, b, c) is

$$x = a + f_x(a, b, c)t, y = b + f_y(a, b, c)t, z = c + f_z(a, b, c)t$$

and the equation of Tangent plane

$$(x - a)f_x(a, b, c) + (y - b)f_y(a, b, c) + (z - c)f_z(a, b, c) = 0$$

Example 6.0.2. *Find the tangent plane and normal line of* $f(x, y, z) = x^2 + y^2 + z - 9 = 0$ *at* $(1, 2, 4)$.

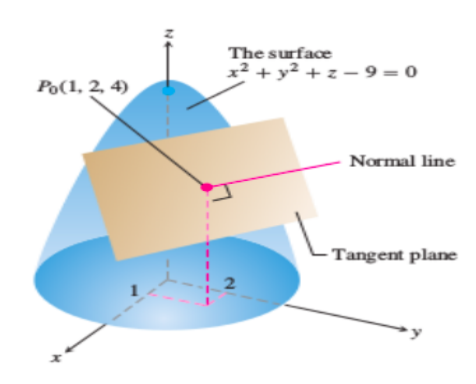


Figure 10:

From the given function, we obtain the gradient

$$\nabla f = 2x\hat{i} + 2y\hat{j} + \hat{k} \implies \nabla f(1, 2, 4) = 2\hat{i} + 4\hat{j} + \hat{k}$$

Hence the tangent plane is

$$2(x - 1) + 4(y - 2) + (z - 4) = 0$$

The normal line is

$$x = 1 + 2t, y = 2 + 4t, z = 4 + t$$

///

Example 6.0.3. *Find the tangent line to the curve of intersection of two surfaces* $f(x, y, z) = x^2 + y^2 - 2 = 0, g(x, y, z) = x + z - 4 = 0$.

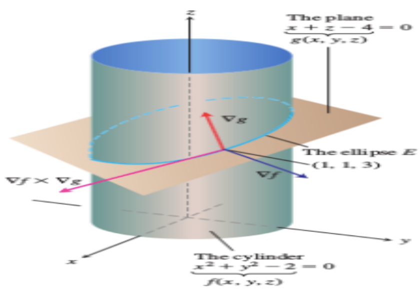


Figure 11:

The intersection of these two surfaces is an an ellipse on the plane $g = 0$. The direction of normal to $g(x, y, z) = 0$ at $(1, 1, 3)$ is $\hat{\imath} + \hat{k}$ and normal to $f(x, y, z) = 0$ is $2\hat{\imath} + 2\hat{\jmath}$. The required tangent line is orthogonal to both these normals. So the direction of tangent is

$$v = \nabla f \times \nabla g = 2\hat{\imath} - 2\hat{\jmath} - 2\hat{k}.$$

Tangent through $(1, 1, 3)$ is $x = 1 + 2t, y = 1 - 2t, z = 3 - 2t$. ///

Linearization: Let f be a differentiable function in a rectangle containing (a, b). The linearization of a function $f(x, y)$ at a point (a, b) where f is differentiable is the function

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

The approximation $f(x, y) \sim L(x, y)$ is the standard linear approximation of $f(x, y)$ at (a, b). Since the function is differentiable,

$$E(x, y) = f(x, y) - L(x, y) = \epsilon_1(x - a) + \epsilon_2(y - b)$$

where $\epsilon_1, \epsilon_2 \to 0$ as $x \to a, y \to b$. The error of this approximation is

$$|E(x, y)| \leq \frac{M}{2}(|x - a| + |y - b|)^2.$$

where $M = \max\{|f_{xx}|, |f_{xy}|, |f_{yy}|\}$.

Example 6.0.4. *Find the linearization and error in the approximation of* $f(x, y, z) = x^2 - xy + \frac{1}{2}y^2 + 3$ *at* $(3, 2)$.

We obtain from the given function,

$$f(3, 2) = 8, f_x(3, 2) = 4, f_y(3, 2) - 1.$$

So

$$L(x, y) = 8 + 4(x - 3) - (y - 2) = 4x - y - 2$$

Also $\max\{|f_{xx}|, |f_{xy}|, |f_{yy}|\} = 2$ and

$$|E(x, y)| \leq (|x-3| + |y-2|)^2.$$

7 Taylor's theorem

It is not always true that the second order mixed derivatives $f_{xy} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y})$ and $f_{yx} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})$ are equal. The following is the example

Example 7.0.1. $f(x, y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & x \neq 0, y \neq 0 \\ 0 & x = y = 0 \end{cases}.$

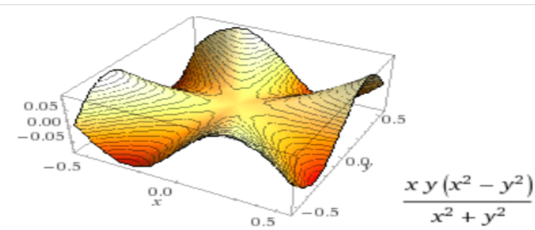


Figure 12:

Then

$$f_y(h, 0) = \lim_{k\to 0} \frac{f(h, k) - f(h, 0)}{k} = \lim_{k\to 0} \frac{1}{k} \frac{hk(h^2-k^2)}{h^2+k^2} = h$$

Also $f_y(0, 0) = 0$. Therefore,

$$\begin{aligned} f_{xy}(0, 0) &= \lim_{h\to 0} \frac{f_y(h, 0) - f_y(0, 0)}{h} \\ &= \lim_{h\to 0} \frac{h-0}{h} = 1 \end{aligned}$$

Now

$$f_x(0, k) = \lim_{h\to 0} \frac{f(h, k) - f(0, k)}{h} = \lim_{h\to 0} \frac{1}{h} \frac{hk(h^2-k^2)}{h^2+k^2} = -k$$

and

$$f_{yx}(0, 0) = \lim_{h\to 0} \frac{f_x(0, k) - f_x(0, 0)}{k} = -1.$$

///

The following theorem is on the sufficient condition for equality of mixed derivatives. We omit the proof.

Theorem 7.0.2. *If $f, f_x, f_y, f_{xy}, f_{yx}$ are continuous in a neighborhood of (a, b). Then $f_{xy}(a, b) = f_{yx}(a, b)$.*

But this is not a necessary condition as can be seen from the following example

Example 7.0.3. *Consider the function* $f(x, y) = \begin{cases} \frac{x^2y^2}{x^2+y^2} & x \neq 0, y \neq 0 \\ 0 & x = y = 0 \end{cases}$.
Here $f_{xy}(0, 0) = f_{yx}(0, 0)$ but they are not continuous at $(0, 0)$ (Try!).

Theorem 7.0.4. *Taylor's theorem: Suppose $f(x, y)$ and its partial derivatives through order $n+1$ are continuous throughout an open rectangular region R centered at a point (a, b). Then, throughout R,*

$$\begin{aligned} f(a+h, b+k) =& f(a, b) + (hf_x + kf_y)\Big|_{(a,b)} + \frac{1}{2!}(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy})\Big|_{(a,b)} \\ &+ \frac{1}{3!}(h^3 f_{xxx} + 3h^2 k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy})\Big|_{(a,b)} \\ &+ ... + \frac{1}{n!}(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})^n f\Big|_{(a,b)} + \frac{1}{(n+1)!}(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})^{n+1} f\Big|_{(a+ch,b+ck)} \end{aligned}$$

where $(a + ch, b + ck)$ is a point on the line segment joining (a, b) and $(a + h, b + k)$.

Proof. Proof follows by applying the Taylor's theorem, Chain rule on the one dimensional function $\phi(t) = f(x + ht, y + kt)$ at $t = 0$.
Error estimation: Similar to the one variable case, we can approximate a given function by its Taylor polynomial in two variables, for exaple,

Example 7.0.5. *The function $f(x, y) = x^2 - xy + y^2$ is approximated by a first degree Taylor's polynomial about the point $(2, 3)$. Find a square $|x - 2| < \delta$, $|y - 3| < \delta$ such that the error of approximation is less than or equal to* 0.1.

We have $f_x = 2x - y, f_y = 2y - x, f_{xx} = -1, f_{xy} = -1, f_{yy} = 2$. At the point (2, 3): $f_x = 1, f_y = 4$. So the linear polynomial $L(x, y)$ is $7 + (x - 2) + 4(y - 3)$. The maximum error in the first degree approximation is

$$|R| \leq \frac{B}{2}(|x - 2| + |y - 3|)^2$$

where $B = \max\{|f_{xx}|, |f_{xy}|, |f_{yy}|\} = \max\{2, 1, 2\} = 2$. Therefore, we want to determine δ such that

$$|R| \leq \frac{2}{2}(\delta + \delta)^2 < 0.1$$

Hence δ me chosen such that $0 < \delta < \sqrt{0.025}$.

8 Local maxima and minima in two variables

Let us consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ be a well defined function in an open set around the point (a, b).

Definition 8.0.1. *We say that the point (a, b) is a point of local maximum of f, if there exists $\delta > 0$ such that*

$$\sqrt{h^2 + k^2} < \delta \implies \Delta f = f(a + h, b + k) - f(a, b) \leq 0$$

The point (a, b) is a point of local minimum of f, if there exists $\delta > 0$ such that

$$\sqrt{h^2 + k^2} < \delta \implies \Delta f = f(a + h, b + k) - f(a, b) \geq 0$$

We say that the point (a, b) is a point of local extremum if either it is local maximum or local minimum.

Necessary condition: Suppose f is differntiable in an open set around the point (a, b) and the point (a, b) is a point of local extremum. Then by taking the one variable function $\psi(x) = f(x, b)$ we see that the point $x = a$ is a point of local extremum for ψ. Therefore, by the one variable calculus, $\psi'(x) = 0$. That is $\frac{\partial f}{\partial x}(a, b) = 0$. Similarly, we also get $\frac{\partial f}{\partial y}(a, b) = 0$.

Sufficient Condition: Suppose f is twice differentiable and the second order partial derivatives are continuous in an open set around the point (a, b). If the point is also a local extremum, then from the above neccessary condition all first order partial derivatives at the point (a, b) will be equal to zero. So to determine if the point is a point of local maximum or minimum, it is enough to determine the sign of $\Delta f = f(a + \Delta x, b + \Delta y) - f(a, b)$ for all small Δx and Δy. From the above Taylor's theorem for two variables,

$$\Delta f = \frac{1}{2}\left(f_{xx}(a, b)(\Delta x)^2 + 2f_{xy}(a, b)\Delta x\Delta y + f_{yy}(a, b)(\Delta y)^2\right) + \alpha(\Delta\rho)^3$$

where $\Delta\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$. We use the notation

$$A = f_{xx}, B = f_{xy}, C = f_{yy}$$

In polar form,

$$\Delta x = \Delta\rho\cos\phi, \Delta y = \Delta\rho\sin\phi$$

Then we have

$$\Delta f = \frac{1}{2}(\Delta\rho)^2\left(A\cos^2\phi + 2B\cos\phi\sin\phi + C\sin^2\phi + 2\alpha\Delta\rho\right).$$

Suppose $A \neq 0$, then

$$\Delta f = \frac{1}{2}(\Delta\rho)^2 \left(\frac{(A\cos\phi + B\sin\phi)^2 + (AC - B^2)\sin^2\phi}{A} + 2\alpha\Delta\rho \right)$$

Now we consider the 4 possible cases:
Case 1: Let $AC - B^2 > 0, A < 0$. Then $(A\cos\phi + B\sin\phi)^2 \geq 0, \sin^2\phi \geq 0$ implies

$$\Delta f = \frac{1}{2}(\Delta\rho)^2 \left(-m^2 + 2\alpha\Delta\rho \right)$$

where m is independent of $\Delta\rho, \alpha\Delta\rho \to 0$ as $\Delta\rho \to 0$. Hence for $\Delta\rho$ small, $\Delta f \leq 0$. Hence the point (a, b) is a point of local maximum
Case 2: Let $AC - B^2 > 0, A > 0$.
In this case

$$\Delta f = \frac{1}{2}(\Delta\rho)^2(m^2 + \alpha\Delta\rho)$$

So $\Delta f \geq 0$. Hence the point (a, b) is a point of local minimum.
Case 3:

1. Let $AC - B^2 < 0, A > 0$. When we move along $\phi = 0$, we have

   $$\Delta f = \frac{1}{2}(\Delta\rho)^2(A + 2\alpha\Delta\rho) > 0.$$

   for $\Delta\rho$ small. When we move along $\tan\phi_0 = -A/B$, then

   $$\Delta f = \frac{1}{2}(\Delta\rho)^2 \left(\frac{AC - B^2}{A} \sin^2\phi_0 + 2\alpha\Delta\rho \right) \leq 0$$

   for $\Delta\rho$ small. so we don't have constant sign along all directions. Hence (a, b) is neither a point of maximum nor a point of minimum. Such point (a, b) is called **Saddle point.**

2. Let $AC - B^2 < 0, A < 0$.
   Similar as above the sign along path $\phi = 0$ is non-positive and $\tan\phi_0 = -A/B$ is non-negative.

3. Let $AC - B^2 < 0, A = 0$.
   In this case $B \neq 0$ and

   $$\Delta f = \frac{1}{2}(\Delta\rho)^2 \left(\sin\phi(2B\cos\phi + C\sin\phi) + 2\alpha\Delta\rho \right)$$

   for small ϕ, $2B\cos\phi + C\sin\phi$ is close to $2B$, but $\sin\phi$ changes sign for $\phi > 0$ or $\phi < 0$. Again here (a, b) is a saddle point.

Case 4: Let $AC - B^2 = 0$.

Again in this case it is difficult to decide the sign of Δf. For instance, if $A \neq 0$,

$$\Delta f = \frac{1}{2}(\Delta\rho)^2 \left(\frac{(A\cos\phi + B\sin\phi)^2}{A} + 2\alpha\Delta\rho \right)$$

When $\phi = arctan(-A/B)$, the sign of Δf is determined by the sign of α. So Additional investigation is required. No conclusion can be made with $AC - B^2 = 0$.

We summarize the derivative test in two variables in the following:

step 1: Find the critical points by solving the equations

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0.$$

Step 2: At the critical points, the following holds:

S.No.	Condition	Nature
1	$AC - B^2 > 0, A > 0$	local minimum
2	$AC - B^2 > 0, A < 0$	local maximum
3	$AC - B^2 < 0$	Saddle point
4	$AC - B^2 = 0$	No conclusion

Example 8.0.2. *Find critical points and their nature of* $f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4$

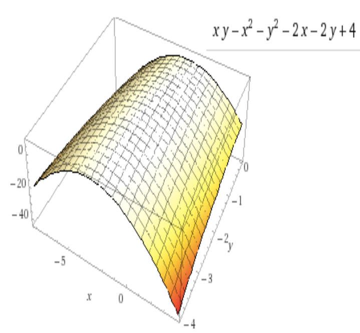


Figure 13:

The critical points are found using $f_x = y - 2x - 2 = 0, \quad f_y = x - 2y - 2 = 0$. Therefore, the point $(-2, -2)$ is the only critical point. Also $A = f_{xx} = -2, C = f_{yy} = -2$, and $B = f_{xy} = 1$. Therefore, $AC - B^2 = 3 > 0$ and $A = -2 < 0$. Therefore, $(-2, -2)$ is a point of local maximum.

The following is an example where the derivative test fails.

Example 8.0.3. *Consider the function* $f(x, y) = (x - y)^2$*, then* $f_x = 0, f_y = 0$ *implies* $x = y$.

Also, $AC - B^2 = 0$. Moreover, all third order partial derivatives are zero. so no further information can be expected from Taylor's theorem. However it is easy to see that $f(x,y) \geq 0$ and $f(x,x) = 0$ implying that all points on the line $y = x$ are the points of global minimum.

Example 8.0.4. *Find the critical points and their nature for the function $f(x,y) = x^3 + 3xy + y^3$.*

Critical points:

$$f_x = 3x^2 + 3y = 0, f_y = 3x + 3y^2 = 0 \implies x = 0, y = 0, \text{ and } x = -1, y = -1.$$

$$f_{xx} = 6x, f_{xy} = 6y, f_{yy} = 3$$

Therefore at $(0,0)$, $AC - B^2 < 0$. Therefore, $(0,0)$ is a saddle point.
At $(-1,-1)$, $AC - B^2 > 0, A < 0$. Therefore, $(-1,-1)$ is a point of local minimum.

9 Global maxima and Minima

Suppose we have a function defined on closed and bounded domain. Then we know that the maximum and minimum are achieved. This maximum and minimum may be in the interior or on the boundary. If it is in the interior then the derivatives of the function at these points must be zero. However on the boundary we cannot do the derivative test. For example $f(x) = x$ on $[0,1]$. We know that the minimum is at 0 and maximum is at 1. But the derivative is not zero at these points. We we do the following

1. Find all interior points critical points of $f(x,y)$. These are the points inside the open domain where partial derivatives are zero.

2. Restrict the function to the each piece of the boundary. This will be one variable function defined on closed interval I(say) and use the derivative test of one variable calculus to find the nature of critical points.

3. Find the end points of these intervals I and evaluate $f(x,y)$ at these points.

4. The global/Absolute maximum/minimum will be the maximum/minimum of f among all these points.

Example 9.0.1. *Find the absolute maxima and minima of $f(x,y) = 2 + 2x + 2y - x^2 - y^2$ on the triangular region in the first quadrant bounded by the lines $x = 0, y = 0, y = 9 - x$.*

Solution: $f_x = 2 - 2x = 0$, $f_y = 2 - 2y = 0$ implies that $x = 1, y = 1$ is the only critical point and $f(1,1) = 4$. $f_{xx} = -2, f_{yy} = -2, f_{xy} = 0$. Therefore, $AC - B^2 = 4 > 0$ and $A < 0$. So this is local maximum.

1. On the segment $y = 0$, $f(x, y) = f(x, 0) = 2 + 2x - x^2$ defined on $I = [0, 9]$. $f(0, 0) = 2, f(9, 0) = -61$ and the at the interior points where $f'(x, 0) = 2 - 2x = 0$. So $x = 1$ is the only critical point and $f(1, 0) = 3$.

2. On the segment $x = 0$, $f(0, y) = 2 + 2y - y^2$ and $f_y = 2 - 2y = 0$ implies $y = 1$ and $f(0, 1) = 3$.

3. On the segment $y = 9 - x$, we have $f(x, 9 - x) = -61 + 18x - 2x^2$ and the critical point is $x = 9/2$. At this point $f(9/2, 9/2) = -41/2$.

4. finally, $f(0, 0) = 2, f(9, 0) = f(0, 9) = -61$. so the global maximum is 4 at $(1, 1)$ and minimum is -61 at $(9, 0)$ and $(0, 9)$.

10 Constrained extremum

First, we describe the substitution method. Consider the problem of finding the shortest distance from origin to the plane $z = 2x + y - 5$.
Here we minimize the function $f(x, y, z) = x^2+y^2+z^2$ subject to the constraint $2x+y-z-5 = 0$. Substituting the constraint in the function, we get

$$h(x, y) = f(x, y, 2x + y - 5) = x^2 + y^2 + (2x + y - 5)^2.$$

The critical points of this function are

$$h_x = 2x + 2(2x + y - 5)(2) = 0,\ h_y = 2y + 2(2x + y - 5) = 0.$$

This leads to $x = 5/3, y = 5/6$. Then $z = 2x + y - 5$ implies $z = -5/6$. We can check that $AC - B^2 > 0$ and $A > 0$. So the point $(5/3, 5/6, -5/6)$ is a point of minimum.

Does this substitution method always work? The answer is **NO**. The following example explains

Example 10.0.1. *The shortest distance from origin to $x^2 - z^2 = 1$.*

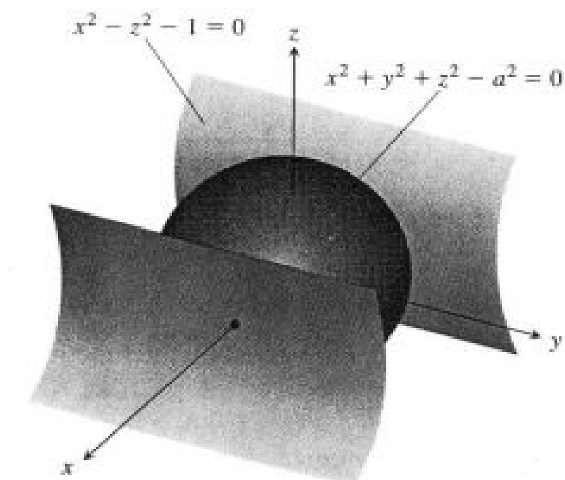
This is minimizing $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $x^2 - z^2 = 1$. Then substituting $z^2 = x^2 - 1$ in f, we get

$$h(x, y) = f(x, y, x^2 - 1) = 2x^2 + y^2 - 1$$

The critical points of the function are $h_x = 4x = 0, h_y = 2y = 0$. That is, $x = 0, y = 0, z^2 = -1$. But this point is not on the hyperbolic cylinder. To overcome this difficulty, we can substitute $x^2 = z^2 + 1$ in f and find that $z = y = 0$ and $x = \pm 1$. These points are on the hyperbolic cylinder and we can check that $AC - B^2 > 0, A > 0$. This implies the points are of local minimum nature.

In the substitution method, once we substitute the constraint in the minimizing function, then the the domain of the function will be the domain of the minimizing function. Then the critical points can belong to this domain which may not be the domain of constraints. This is overcome by the following:

Lagrange Multiplier Method:
Imagine a small sphere centered at the origin. Keep increasing the radius of the sphere untill the sphere touches the hyperbolic cylinder. The required smallest distance is the radius of that sphere which touches the cylinder. When the sphere touches the cylinder, both these surfaces has common tangent plane. So at the point of touching, both surfaces has normal proportional.



(Above picture is taken from Thomas Calculus book). That is $\nabla f = \lambda \nabla g$ for some λ. Now solving this equations along with $g = 0$ gives the points of extrema. In the above example, taking $f = x^2 + y^2 + z^2$ and $g = x^2 - z^2 - 1 = 0$, we get

$$2x\widehat{i} + 2y\widehat{j} + 2z\widehat{k} = \lambda(2x\widehat{i} - 2z\widehat{k})$$

This implies, $2x = 2\lambda x$, $\;2y = 0\;\; 2z = -2\lambda z$. $x = 0$ does not satisfy $g = 0$. So from first equation we get $\lambda = 1$. Then $2z = -2z$. That is $z = 0$, and $y = 0$. Therefore, the critical points are $(x, 0, 0)$. Substituting this in the constraint equation, we get $x = \pm 1$. Hence the points of extrema are $(\pm, 0, 0)$.

Caution: The Method of Lagrange multiplies gives only the points of extremum. To find the maxima or minima one has to compare the function values at these extremum points.

Method of Lagrange multipliers with many constraints in n variables

1. Number of constraints m should be less than the number of independent variables n say $g_1 = 0, g_2 = 0, ...g_m = 0$.
2. Write the Lagrange multiplier equation: $\nabla f = \sum_{i=1}^{m} \lambda_i \nabla g_i$.
3. Solve the set of $m + n$ equations to find the extremal points

$$\nabla f = \sum_{i=1}^{m} \lambda_i \nabla g_i, \quad g_i = 0, \ i = 1, 2, ...m$$

4. Once we have extremum points, compare the values of f at these points to determine the maxima and minima.

Example 10.0.2. *Let $\mathcal{C}$ be the intersection of the surfaces*

$$x^2 + 4y^2 + 4z^2 = 4, \ x + y + z = 0$$

Using the Lagrange multiplier method, determine the points that are nearest and farthest from the origin.

Solution: Problem translates into finding the extremum of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

subject to the conditions

$$x^2 + 4y^2 + 4z^2 = 4, \ x + y + z = 0.$$

This surface is closed and bounded. Hence f being a continuous function attains its global maximum and global minimum of this surface.
From the Lagrange multiplier method we get

$$2x = \lambda 2x + \mu, \ \ 2y = \lambda 8y + \mu, \ \ 2z = \lambda 8z + \mu. \tag{10.1}$$

From second and third equations of (10.1), we get

$$2(y - z) = 8\lambda(y - z) \implies y = z \ OR \ \lambda = 1/4$$

In case of $\lambda = 1/4$ using the first and second equations of (10.1), we get

$$2(y - x) = 2y - x/2 \implies x = 0$$

Then from the constraints we get

$$y = -z,\ 8y^2 = 4 \implies y = \pm 1/\sqrt{2}$$

Therefore we get the critical points

$$(0, 1\sqrt{2}, -1/\sqrt{2}),\ \ (0, -1/\sqrt{2}, -1/\sqrt{2})$$

In case of $y = z$, from constraints we get

$$x = -2y,\ y^2 = 1/3$$

Hence we get two more critical points

$$(2/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}),\ (-2/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$$

Therefore the nearest points from the origin is $(0, 1\sqrt{2}, -1/\sqrt{2}),\quad (0, -1/\sqrt{2}, -1/\sqrt{2})$ and farthest points are $(2/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}),\ (-2/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$.

Example 10.0.3. *Find the maximum and minimum of the function $f(x, y) = x^3 + 4y^2$ on $x^2 + y^2 = 1$.*

Here again the set of points on the circle is a closed and bounded set and the function f is continuous on this set. Therefore f attains its maximum and minimum on the circle. Now by Lagrange multiplier method, we get

$$3x^2 = 2\lambda x,\ \ 8y = 2\lambda y$$

Therefore either $x = 0$ or $x = \frac{2\lambda}{3}$. In case of $x = 0$, from the constraint we get $y = \pm 1$ and from the second equation we get $\lambda = 4$. So we get one set of points

$$(0, \pm 1)$$

Similarly when we solve the second equation, we get $y = 0$ and from the constraint we get $x = \pm 1$. Hence we get another set of points

$$(\pm 1, 0)$$

The case $x = \frac{2\lambda}{3}$ will not give any feasible points that satisfy the constraint.(check). Therefore we have the following extremum values

$$f(0, \pm 1) = 4, f(1, 0) = 1, f(-1, 0) = -1.$$

Therefore maximum of f is 4 and minimum of f is -1 on the circle $x^2 + y^2 = 1$.

Example 10.0.4. *Among all rectangular boxes with a fixed surface area of 30 sq. cms., find the dimension of the box that has largest volume.*

Let l, b and h denote the length, breadth and height, respectively, of a rectangular box. The problem is

$$\begin{aligned} &\max V(l,b,h) = lbh \\ \text{subject to}\quad &2(lb+bh+lh) = 30 \\ &l > 0, b > 0, h > 0. \end{aligned}$$

The Lagrangian is

$$L(l,b,h,\lambda) = (lbh) + \lambda(2(lb+bh+lh) - 30).$$

The Lagrange multiplier equation is

$$\nabla L(l,b,h,\lambda) = \nabla(lbh) + \lambda\nabla(lb+bh+lh-15) = 0.$$

Thus, we arrive at the following set of equations:

$$\begin{aligned} bh + \lambda(b+h) = 0; &\qquad lh + \lambda(l+h) = 0 \\ lb - \lambda(l+b) = 0; &\qquad lb + bh + lh = 15. \end{aligned}$$

Solving these equations, we get the following two solutions $(\sqrt{5}, \sqrt{5}, \sqrt{5})$ and $(-\sqrt{5}, -\sqrt{5}, -\sqrt{5})$. Since l, b and h are positive, the dimension of the box with largest volume is $\left(\sqrt{5}, \sqrt{5}, \sqrt{5}\right)$.

Example 10.0.5. *Find the maximum and minimum of $f(x,y) = 81x^2 + y^2$ subject to the constraint $g(x,y) = 4x^2 + y^2 = 9$.*

Here Lagrange multiplier method yields the critical points. However taking a parametrization of the ellipse g: $x(\theta) = \frac{3}{2}\cos\theta, y(\theta) = 3sin\theta$ and substituting in the function f, and applying the second derivative test would also get the required answer.

11 Applictions

Gasoline company

A petroleum company (Indiana) producing heating oil and gasoline from crude oil has the profit function

$$f(x,y) = -60 + 140x + 100y - 10x^2 - 8y^2 - 6xy$$

where x is the heating oil and y is the gasoline. Using the second derivative test we find the optimum at the the critical point $x = 5.77$ and $y = 4.08$ and the optimal profit (maximum of f) =548.45.

The same company if there is a constraint on the supply of crude oil. Say during some period they only have 200 units of crude oil and suppose crude oil (x) requires 20 liters and (y) requires 40 liters for making each unit. Then we have the following constraint

$$20x + 40y = 200$$

The solution of unconstrained problem $x = 5.77, y = 4.08$ in this case does not satisfy the constraint because it requires $20(5.77) + 40(4.08) = 278.6$. However we have only 200 units of crude oil. However solving the constrained problem we get

$$x = 5.56, \ y = 2.22$$

Sales of a company
A company (Bounds Inc) that manufactures a product has observed through a statistical data analysis that their sales are a function of amount of advertising in two different media: news papers (x) and magazines (y)

$$f(x, y) = 200x + 100y - 10x^2 - 20y^2 + 20xy$$

Using the second derivative test we can calculate the maximum sales at the optimum values of news papers magazine advertising. Suppose the advertisement budget is restricted to 20 then what would be the maximum sales at optimum levels. ($x + y = 20$).

By the second derivative test we find the optimal values as $x = 25$ and $y = 15$. The maximal sales will be 3500. Similarly, we can use Lagrange multiplier method to solve the constrained optimization problem:

$$\text{maximize} f(x, y) \ \text{ subject to } \ x + y = 20.$$

12 Mamima & Minima in 3 or more variables

We recall from the above Taylor's theorem:

$$f(x+h, y+k) - f(x, y) = \nabla f \cdot (h, k) + \frac{1}{2!} \begin{pmatrix} h \\ k \end{pmatrix} \begin{pmatrix} A & B \\ B & C \end{pmatrix} (h, k) + o(h^2 + k^2)$$

where $o(h^2+k^2)$ is such that

$$\frac{o(h^2+k^2)}{h^2+k^2} \to 0 \text{ as } h,k \to 0.$$

From here we see that if (x,y) is a critical point, then the sign of Left hand side depends on the sign of second term on the Right hand side. In terms of matrices, we can say that the second term is positive if $A>0, AC-B^2>0$. This is the above second derivative test. Such matrices are known as positive definite matrices. Equivalently, A matrix H is called positive definite matrix if all its eigenvalues are positive.

This process can be now generalized to functions of n variables:

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and let $f:\mathbb{R}^n \to \mathbb{R}$ be a sufficiently differentiable function in an open set around the point X_0. Then from the Taylor's theorem, we may write

$$f(X+X_0)-f(X_0) = \nabla f(X_0)\cdot X + X^T H X + o(\|X\|^2)$$

where H is the Hessian matrix given as

$$H = \begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_nx_1} & f_{x_nx_2} & \cdots & f_{x_nx_n} \end{pmatrix}$$

Here $f_{x_ix_j} = \frac{\partial^2 f}{\partial x_i \partial x_j}(X_0)$. Similar to the second derivative test as above we may get sufficient condition for points of local maxima and minima. For this we need

Definition 12.0.1. *A symmetric matrix H is called positive definite if $X^THX>0$ for all non-zero X and is called negative definite if $X^THX<0$ for all non-zero X.*
Equivalently, H is positive definite if $Det(H_i)>0$ for each $i=1,2,\cdots n$, where H_i is the submatrix $\begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1i} \\ h_{21} & h_{22} & \cdots & h_{2i} \\ \vdots & \vdots & \ddots & \vdots \\ h_{i1} & h_{i2} & \cdots & h_{ii} \end{pmatrix}$

We have the following characterization using eigenvalues

Theorem 12.0.2. *A symmetric Matrix H is positive definite if and only if all its eigenvalues are positive.*

Proof. The proof follows from the fact that symmetric matrices are diagonalizable. We leave this as an exercise.

Remark 12.1. *A symmetric matrix H is negative definite if all its eigenvalues are negative. In case H has both negative and positive eigenvalues, then the matrix H is called indefinite.*

As a consequence of these we have

Theorem 12.0.3. *A critical point X_0 is a point of local minimum if all eigenvalues of the Hessian matrix H at X_0 is positive. It is a point of local maximum if all its eigenvalues are negative. If H is indefinite then X_0 is a saddle point. If one of the eigenvalues is zero then we need more information.*

Example 12.0.4. *Consider the the problem of finding the maximum of the function $f(x, y, z) = xyz(16 - x - y - 2z)$.*

It is not difficult to check that the critical points are $(4, 2, 2), (x, 0, 0), (0, y, 0)$ and $(0, 0, z)$. The Hessian at the points $(x, 0, 0)$ is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 16 - x^2 \\ 0 & 16 - x^2 & 0 \end{pmatrix}$$

Without calculating all eigenvalues, we see that it has at least one zero eigenvalue. So we cannot decide the nature of the critical points (coordinate axis in this case) using the test. At the point $(4, 2, 2)$ the Hessian is equal to

$$\begin{pmatrix} -16 & -8 & -16 \\ -8 & -16 & -16 \\ -16 & -16 & -64 \end{pmatrix}$$

Now one can compute the eigenvalues of this matrix and see that all of them are negative. Therefore noting that $f(x, 0, 0) = 0, f(0, y, 0) f(0, 0, z) = 0$, we see that $f(4, 2, 2) = 16(16 - 4 - 2 - 4) = 160$ is the maximum value of the function.

References

1. G.B. Thomas, Thomas' Calculus,
2. https://www.wolframalpha.com
3. http://www.public-policy.org/ncpa/studies/s171/s171.html

13 Exercises

1. Examine if the limits as $(x,y)\to(0,0)$ exist?

$$(a)\ \begin{cases}\frac{x^3+y^3}{x^2-y^2} & x\neq y\\ 0 & x=y\end{cases}\quad (b)\ xy\left(\frac{x^2-y^2}{x^2+y^2}\right)\quad (c)\ \begin{cases}x\sin\frac{1}{y}+y\sin\frac{1}{x} & xy\neq 0\\ 0 & xy=0\end{cases}$$

$$(d)\ \frac{\sin(xy)}{x^2+y^2}\quad (e)\ \begin{cases}\frac{xy}{x^2+y} & x^2\neq -y\\ 0 & x^2=-y\end{cases}$$

2. Examine the continuity of the following functions.

$$(a)\ \begin{cases}\dfrac{xy^3}{x^2+y^6}\,, & (x,y)\neq(0,0)\\ 0\,, & \text{otherwise.}\end{cases}\qquad (b)\ \begin{cases}x^2+y^2\,, & x^2+y^2\leq 1\\ 0\,, & \text{otherwise.}\end{cases}$$

$$(c)\ \begin{cases}\dfrac{\sin^2(x-y)}{|x|+|y|}\,, & (x,y)\neq(0,0)\\ 0\,, & \text{otherwise.}\end{cases}\qquad (d)\ \begin{cases}\dfrac{x^2y^2}{x^2y^2+(x-y)^2}\,, & (x,y)\neq(0,0)\\ 0\,, & \text{otherwise.}\end{cases}$$

3. Discuss the differentiability of the following functions at $(0,0)$.

$$(a) f(x,y)=\begin{cases}x\sin\frac{1}{x}+y\sin\frac{1}{y} & xy\neq 0\\ 0xy=0\end{cases}\qquad (b)\ \begin{cases}\frac{xy}{\sqrt{x^2+y^2}} & x^2+y^2\neq 0\\ 0 & x=y=0\end{cases}$$

$$(c)\ \begin{cases}\frac{x^6-2y^4}{x^2+y^2} & x^2+y^2\neq 0\\ 0 & x=0, y=0\end{cases}$$

4. Let $f(x,y)=\dfrac{y}{|y|}\sqrt{x^2+y^2}$, $y\neq 0$ and $f(x,0)=0$. Show that f has all directional derivatives at $(0,0)$ but it is not differentiable at $(0,0)$.

5. Let $f(x,y)=\Big|\,|x|-|y|\,\Big|-|x|-|y|$. Is f continuous at $(0,0)$? Which directional derivatives of f exist at $(0,0)$? Is f differentiable at $(0,0)$? Give reasons.

6. Let $f(x,y)=\frac{1}{2}\ln(x^2+y^2)+tan^{-1}(\frac{y}{x})$, $P=(1,3)$. Find the direction in which $f(x,y)$ is increasing the fastest at P. Find the derivative of $f(x,y)$ in this direction.

7. A heat-seeking bug is a bug that always moves in the direction of the greatest increase in heat. Discuss the behavior of a heat seeking bug placed at a point $(2,1)$ on a metal plate heated so that the temperature at (x,y) is given by $T(x,y)=50y^2e^{\frac{-1}{5}(x^2+y^2)}$.

8. Suppose the gradient vector of the linear function $z=f(x,y)$ is $\nabla z=(5,-12)$. If $f(9,15)=17$, what is the value of $f(11,11)$?

9. Suppose $f(8,3)=24$ and $f_x(8,3)=-3.4$, $f_y(8,3)=4.2$. Estimate the values $f(9,3)$, $f(8,5)$, and $f(9,5)$. Explain how you got your estimates.

10. Find the quadratic Taylor's polynomial approximation of $e^{-x^2-2y^2}$ near $(0,0)$.

11. Using Taylor's formula, find quadratic and cubic approximations $e^x \sin y$ at origin. Estimate the error in approximations if $|x| \leq 0.1\,, |y| \leq 0.2$.

12. Find all the critical points of $f(x,y) = \sin x \sin y$ in the domain $-2 \leq x \leq 2,\ -2 \leq y \leq 2$.

13. Find local minima and local maxima points of the function $f(x,y) = xye^{-(x^2+y^2)}$.

14. Let $f(x,y) = (y-4x^2)(y-x^2)$. Verify that $(0,0)$ is a saddle point of f.

15. Let $f(x,y) = (x-y)^2$. Find all critical points of f and categorize them according as they are either saddle points or the location of local extreme values. Is the second derivative test useful in this case?

16. In each of the cases below, discriminant is zero. Find the critical points and their nature by imagining the surface $z = f(x,y)$ (a) $f(x,y) = x^2y^2$ $\quad (b) 1 - x^2y^2 \quad (c) f(x,y) = xy^2$.

17. Find the maximum and minimum values of $f(x,y) = 3x+4y$ subject to the constraint $x^2 + 4xy + 5y^2 = 10$.

18. Find the points on the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ which are nearest and farthest from the straight line $3x + y - 9 = 0$.

19. Find the maximum volume of a rectangular solid in the first octant $(x \geq 0, y \geq 0, z \geq 0)$ with one vertex at the origin and the other vertex on the plane $x + y + \dfrac{z}{2} = 1$.

20. Show that the closed cylinder (with lids) with the greatest surface area that can be inscribed in a sphere of radius a has the altitude $h = a\sqrt{2 - \frac{2}{\sqrt{5}}}$ and the radius of the base $r = \frac{a}{2}\sqrt{2 + \frac{2}{\sqrt{5}}}$.

21. A farmer wishes to build a rectangular storage bin, without a top, with a volume of 500 cubic meters. Find the dimensions of the bin that will minimize the amount of material needed in its construction.

22. A company produces steel boxes at three different plants in amounts $x,\ y$ and z, respectively, producing an annual revenue of $f(x,y,z) = 8xyz^2 - 200(x+y+z)$. The company is to produce 100 units annually. How should the production be distributed to maximize revenue?

14 Solutions

1. (a) Take two paths $y = \frac{x}{x+1}$ and $y = mx$. We will get different limiting values. So limit does not exist.

   (b) Take $x = r\cos\theta$ and $y = r\sin\theta$. Then

$$\left|xy\frac{(x^2-y^2)}{x^2+y^2}\right| = \left|r^2\sin\theta\cos\theta\cos 2\theta\right|$$
$$\leq \frac{x^2+y^2}{4} < \frac{\delta^2}{4} = \epsilon.$$

   If we take $\delta = 4\sqrt{\epsilon}$, we are done.

   (c)

$$|x\sin\frac{1}{y} + y\sin\frac{1}{x}| \leq (|x|+|y|) \leq 2(x^2+y^2)^{\frac{1}{2}} < \epsilon.$$

   If we take $\delta = \frac{\epsilon}{2}$ we are done.

   (d) Put $y = mx$, we have

$$\lim_{(x,y)\longrightarrow(0,0)} \frac{sin(xy)}{x^2+y^2} = \lim_{x\longrightarrow 0} \frac{\sin(mx^2)}{x^2(1+m^2)} = \frac{m}{1+m^2}.$$

   Hence it is not continuous at $(0,0)$.

2. (a) Take $x = my^3$ and show that the function is not continuous at $(0,0)$.

   (b) Clearly the function is continuous inside the circle and outside the circle. We need to check at the boundary of the circle, i.e. the points where $x^2+y^2 = 1$. In the polar form the function is $f(r,\theta) = \begin{cases} r^2 & r \leq 1 \\ 0 & r > 1 \end{cases}$. This is a function of one variable which is NOT continuous at $r = 1$.

   (c)

$$\left|\frac{\sin^2(x-y)}{|x|+|y|}\right| \leq \frac{|x-y|^2}{|x|+|y|} \leq \frac{(|x|+|y|)^2}{|x|+|y|} = (|x|+|y|) \leq 2(x^2+y^2)^{\frac{1}{2}} < \epsilon.$$

   If we take $\delta = \frac{\epsilon}{2}$ we are done.

   (d) Take $y = x$, we have $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2y^2+(x-y)^2} = \lim\limits_{x\to 0} \dfrac{x^4}{x^4} = 1 \neq 0$. Hence f is not continuous at $(0,0)$.

3. (a)Both the partial derivatives are 0 at point (0,0). But error term $E(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{f(x,y)}{\sqrt{x^2+y^2}} =$ $\lim\limits_{(x,y)\to(0,0)} \dfrac{x\sin\frac{1}{x} + y\sin\frac{1}{y}}{\sqrt{x^2+y^2}}$ fails to exist along $y = x$. Hence not differentiable.

(b) Both the partial derivatives are 0 at point (0,0). But error term $E(x,y) = \lim_{(x,y)\to(0,0)} \dfrac{f(x,y)}{\sqrt{x^2+y^2}} = \dfrac{xy}{x^2+y^2} = \dfrac{1}{2}$ by taking limit along the line y=x . Hence f is not differential at (0,0).

(c) Both the partial derivatives are 0 at point (0,0). The error term $E(x,y) = \lim_{(x,y)\to(0,0)} \dfrac{x^6 - 2k^4}{(h^2+k^2)^{3/2}} = 0$(use polar co-ordinates to show this). Hence f is differentiable at $(0,0)$.

4. In this case $f_x(0,0) = 0$ and $f_y(0,0) = 1$. For differentiability, the error term $E(x,y) = \lim_{(x,y)\to(0,0)} \dfrac{f(x,y)+y}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \dfrac{y}{\mid y \mid} + \lim_{(x,y)\to(0,0)} \dfrac{y}{\sqrt{x^2+y^2}}$ fails to exist. Hence f is not differentiable at (0,0).

   So, we can't apply the formula $D_u f = f_x(0,0)u_1 + f_y(0,0)u_2$. The directional derivative exists in each direction. This can be checked directly from the definition of directional derivative.

5. Given $\epsilon > 0$ we have to find a $\delta > 0$ such that for $0 < \sqrt{x^2+y^2} < \delta$ $\mid f(x,y) - 0 \mid < \epsilon$. Consider $\mid f(x,y) - 0 \mid \leq \mid\mid x \mid - \mid y \mid\mid + \mid x \mid + \mid y \mid \leq 2(\mid x \mid + \mid y \mid) \leq 4\sqrt{x^2+y^2}$. So take $\delta = \frac{\epsilon}{4}$ we have $\mid f(x,y) - 0 \mid < \epsilon$. Hence $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$ and the function is continuous at (0,0).

   Also both the the partial derivatives are 0 at point(0,0). For differentiability the error term $E(x,y) = \lim_{(x,y)\to(0,0)} \dfrac{f(x,y)}{\sqrt{x^2+y^2}} = -\sqrt{2}$ along y=x. Hence f is not differentiable at (0,0).

   So, we can't apply the formula $D_u f = f_x(0,0)u_1 + f_y(0,0)u_2$. Direction derivative of the function exists only in the direction of $(1,0)$ and $(0,1)$ and this can be checked from the definition.

6. $f_x(1,3) = -1/5,\ f_y(1,2) = 2/5$. Directional derivative is greatest when pointing in the direction of the gradient $(-1/5, 2/5)$. Hence, the direction is $-1/\sqrt{5}\ \widehat{i} + 2/\sqrt{5}\ \widehat{j}$. $f_{\widehat{u}}(1,3) = f_x(1,3)u_1 + f_y(1,3)u_2 = 1/\sqrt{5}$

7. $T_x(2,1) = -40/e$, $T_y(2,1) = 80/e$. Therefore, the bug will move in the direction $-1/\sqrt{5}$ $\widehat{i} + 2/\sqrt{5}\ \widehat{j}$. $T_{\widehat{u}}(2,1) = 40\sqrt{5}/e$

8. Suppose z $= ax + by + c$. Therefore, $5 = f_x(x,y) = a$ and $-12 = f_y(x,y) = b$. Since, $f(9,15) = 17$, $c = 152$. $f(11,11) = 5 \times 11 - 12 \times 11 + 152$.

9. Given $f(8,3) = 24,\ f_x(8,3) = -3.4,\ f_y(8,3) = 4.2$ then

$$f(9,3) = f(8,3) + 1.f_x(8,3) + 0.f_y(8,3) = 20.6$$

$$f(8,5) = f(8,3) + 0.f_x(8,3) + 2.f_y(8,3) = 32.4$$

$$f(9,5) = f(8,3) + 1.f_x(8,3) + 2.f_y(8,3) = 29$$

10.

$$\begin{aligned}
f(x,y) &= \exp^{-x^2-2y^2} \implies f(0,0) = 1 \\
f_x(x,y) &= \exp^{-x^2-2y^2}(-2x) \implies f_x(0,0) = 0 \\
f_y(x,y) &= \exp^{-x^2-2y^2}(-4y) \implies f_y(0,0) = 0 \\
f_{xx}(x,y) &= -2\exp^{-x^2-2y^2} + 4x^2 exp^{-x^2-2y^2} \implies f_{xx}(0,0) = -2 \\
f_{yy}(x,y) &= -4\exp^{-x^2-2y^2} + 16y^2 exp^{-x^2-2y^2} \implies f_{yy}(0,0) = -4 \\
f_{xy}(x,y) &= 8xy\exp^{-x^2-2y^2} \implies f_{xy}(0,0) = 0
\end{aligned}$$

Therefore,

$$\begin{aligned}
f(x,y) &\approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{x^2}{2}f_{xx}(0,0) + xyf_{xy}(0,0) + \frac{y^2}{2}f_{yy}(0,0) \\
&= 1 - x^2 - 2y^2
\end{aligned}$$

11. Given $f(x,y) = e^x \sin y$

Case (i) Quadratic Approximation:

$$\begin{aligned}
f(h,k) = f(0,0) + hf_x(0,0) + kf_y(0,0) + \frac{1}{2}[h^2 f_{xx}(0,0) + 2hkf_{xy}(0,0) \\
+ k^2 f_{yy}(0,0)] + R_2,
\end{aligned}$$

where R_2 is the error term given by

$$R_2 = \frac{1}{3!}[h^3 f_{xxx}(\theta h,\theta k) + 3h^2 k f_{xxy}(\theta h,\theta k) + 3hk^2 f_{xyy}(\theta h,\theta k) + k^3 f_{yyy}(\theta h,\theta k)]$$

for $0 \le \theta \le 1$. Thus $f(h,k) \simeq k + hk$. Also

$$\begin{aligned}
|R_2| &\le \frac{1}{3!}[|h|^3 e^{|h|} + 3h^2|k|e^{|h|} + 3|h|k^2 e^{|h|} + |k|^3 e^{|h|}] \\
&\le \frac{1}{3!}e^{0.1} \times 0.027
\end{aligned}$$

Case (ii) Cubic Approximation:

$$\begin{aligned}
f(h,k) \simeq f(0,0) + hf_x(0,0) + kf_y(0,0) + \frac{1}{2}[h^2 f_{xx}(0,0) + 2hkf_{xy}(0,0) + k^2 f_{yy}(0,0)] \\
+ \frac{1}{3!}[h^3 f_{xxx}(0,0) + 3h^2 k f_{xxy}(0,0) + 3hk^2 f_{xyy}(0,0) + k^3 f_{yyy}(0,0)] + R_3
\end{aligned}$$

where R_3 is the error term given by

$$R_3 = \frac{1}{4!}[h^4 f_{xxxx}(\theta h, \theta k) + 4h^3 k f_{xxxy}(\theta h, \theta k) + 6h^2k^2 f_{xxyy}(\theta h, \theta k) + 4hk^3 f_{xyyy}(\theta h, \theta k) + k^4 f_{yyyy}(\theta h, \theta k)],$$

for $0 \leq \theta \leq 1$. Thus $f(h,k) \simeq k + hk + \frac{1}{2}h^2k - \frac{1}{6}k^3$. Also R_3 can be estimated as in quadratic case.

12. Given $f(x,y) = \sin x \sin y$, $-2 \leq x \leq 2$ and $-2 \leq y \leq 2$. $f_x = 0$ implies $\cos x \sin y = 0$ and $f_y = 0$ implies $\sin x \cos y = 0$. Thus $x = \pm(2n+1)\frac{\pi}{2}$ or $y = \pm n\pi$ and $x = \pm n\pi$ or $y = \pm(2n+1)\frac{\pi}{2}$ i.e. $(x,y) = (\pm(2n+1)\frac{\pi}{2}, \pm(2n+1)\frac{\pi}{2})$ and $(\pm n\pi, \pm n\pi)$. Thus critical points in the domain are given by $(0,0)$, $(\frac{\pi}{2}, \frac{\pi}{2})$, $(-\frac{\pi}{2}, \frac{\pi}{2})$, $(\frac{\pi}{2}, -\frac{\pi}{2})$ and $(-\frac{\pi}{2}, -\frac{\pi}{2})$.

13. Given $f(x,y) = xye^{-(x^2+y^2)}$. Thus $f_x = ye^{-(x^2+y^2)}(1-2x^2) = 0$ implies $y = 0$ or $x = \pm\frac{1}{\sqrt{2}}$ and $f_y = xe^{-(x^2+y^2)}(1-2y^2) = 0$ implies $x = 0$ or $y = \pm\frac{1}{\sqrt{2}}$. Therefore critical points are $(0,0)$, $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ and $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.
Now $f_{xx} = ye^{-(x^2+y^2)}(4x^3 - 6x)$, $f_{yy} = xe^{-(x^2+y^2)}(4y^3 - 6y)$ and $f_{xy} = e^{-(x^2+y^2)}(1 - 2x^2)(1-2y^2)$. Thus

$$D = f_{xx}f_{yy} - f_{xy}^2 = e^{-2(x^2+y^2)}(-8x^2y^4 - 8x^4y^2 + 20x^2y^2 - 1 - 4x^4 + 4x^2 - 4y^4 + 4y^2).$$

Now one can check that $D < 0$ at $(0,0)$. Thus $(0,0)$ is saddle point. Also $D > 0$ at $(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}})$. Now as $A = f_{xx} < 0$ at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, these are points of local maxima. Also as $A = f_{xx} > 0$ at $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, these are points of local minima.

14. Given $f(x,y) = (y - 4x^2)(y - x^2)$. Thus $f_x = 16x^3 - 10xy = 0$ implies $x = 0$ or $8x^2 = 5y$ and $f_y = -5x^2 + 2y = 0$ implies $2y = 5x^2$. Thus $(0,0)$ is the only critical point. As $D = 0$, the second derivative test fails. Note that along the parabola $y = 5x^2$, $f(x,y) > 0$ while along $y = 2x^2$, $f(x,y) < 0$. Thus $(0,0)$ is a saddle point.

15. Given $f(x,y) = (x-y)^2$. Thus $f_x = 0$ and $f_y = 0$ implies $x = y$. Note that as $D = 0$ at (x,x), the second derivative test fails. Also note that $f(x,y) \geq 0$ and $f(x,y) = 0$ at $x = y$. Thus (x,x) are points of local minimum.

16. (a) Critical points are : $\{(x,0),(0,y) : x,y \in \mathbb{R}\}$. Since $f \geq 0$, thus they are points of minima.
(b) Critical points are : $\{(x,0),(0,y) : x,y \in \mathbb{R}\}$. Since $f \leq 1$, thus they are points of maxima.
(c) Critical points are : $\{(x,0) : x \in \mathbb{R}\}$. They are saddle points because consider a straight line passing through them, then along the points above x axis $f > 0$ whereas for points below, $f < 0$.

17. Given $f(x,y) = 3x + 4y$ and $g(x,y) = x^2 + 4xy + 5y^2 - 10$. Thus

$$L(x, y, \lambda) = 3x + 4y + \lambda(x^2 + 4xy + 5y^2 - 10).$$

So $L_x = 0$ implies $2\lambda x + 4\lambda y + 3 = 0$ and $L_y = 0$ implies $4\lambda x + 10\lambda y + 4 = 0$. Solving these we get $x = \frac{-7}{2\lambda}$ and $y = \frac{1}{\lambda}$. Now $g(x,y) = 0$ gives $\lambda = \mp\frac{1}{2}\sqrt{\frac{13}{10}}$. Hence critical points are $(-7\sqrt{\frac{10}{13}}, 2\sqrt{\frac{10}{13}})$ and $(7\sqrt{\frac{10}{13}}, -2\sqrt{\frac{10}{13}})$. Also $f(-7\sqrt{\frac{10}{13}}, 2\sqrt{\frac{10}{13}}) = -\sqrt{130}$ and $f(7\sqrt{\frac{10}{13}}, -2\sqrt{\frac{10}{13}}) = \sqrt{130}$. Thus $(-7\sqrt{\frac{10}{13}}, 2\sqrt{\frac{10}{13}})$ and is point of minima and $(7\sqrt{\frac{10}{13}}, -2\sqrt{\frac{10}{13}})$ is point of maxima under the constraint $g(x,y)$.

18. Note that here we find the point on ellipse corresponding to the shortest distance only. For this distance of a point (x,y) on the ellipse and a point (p,q) on the line is $\sqrt{(x-p)^2 + (y-q)^2}$.
$L(x, y, p, q, \lambda, \mu) = (x-p)^2 + (y-q)^2 + \lambda(x^2/4 + y^2/9 - 1) + \mu(3p + q - 9)$.

$$L_x = 2(x-p) + \lambda x/2 = 0 - (1)$$

$$L_y = 2(y-q) + 2\lambda y/9 = 0 - (2)$$

$$L_p = -2(x-p) + 3\mu = 0 - (3)$$

$$L_q = -2(y-q) + \mu = 0 - (4)$$

$$x^2/4 + y^2/9 = 1 - (5)$$

$$3p + q = 9 - (6)$$

$(1) + (3) \Rightarrow \lambda x/2 + 3\mu = 0 - (7)$
$(2) + (4) \Rightarrow 2\lambda y/9 + \mu = 0 - (8)$
$(7) - 3 * (8) \Rightarrow \lambda(3x - 4y) = 0 - (9)$
$\lambda \neq 0$ else contradiction, therefore $x = 4y/3 - (10)$
Put (10) in (5), we get $y = \pm 3/\sqrt{5}$, thus,$x = \pm 4/\sqrt{5}$.
Now using the equations $(1) - (6)$ find the points (p,q) corresponding to (x,y) and calculate the distance between them.

19. $L(x, y, z, \lambda) = xyz + \lambda(x + y + \frac{z}{2} - 1)$
$\nabla L(x, y, z, \lambda) = 0$

$$L_x = yz + \lambda = 0 - (1)$$

$$L_y = xz + \lambda = 0 - (2)$$

$$L_z = xy + \lambda/2 = 0 - (3)$$

$$x + y + \frac{z}{2} = 1 - (4)$$

Now, $(1) * x + (2) * y + (3) * z$ will give $\lambda = -3xyz - (5)$
$(1) - (2)$ will give either $z = 0$ or $x = y$, we discard $z = 0$ as otherwise volume would be 0. Therefore, $x = y$ and (3) will give $\lambda = -2x^2 - (6)$
Equating equation (5) and (6), we get $z = 2/3$ and equation (4) will give $x = y = 1/3$. Thus maximum volume is $2/27$.

20. The surface area of a closed cylinder with radius of the base r and altitude h is $A(r,h) = 2\pi rh + 2\pi r^2$. Since this is inscribed in a sphere of radius a, we have $r^2 + (\frac{h}{2})^2 = a^2$.

    We maximize $A(r,h)$ subject to the constraint $r^2 + (\frac{h}{2})^2 - a^2 = 0$. By the method of Lagrange multipliers, we get
    $2\pi h + 4\pi r - 2\lambda r = 0, \quad 2\pi r - \frac{1}{2}\lambda h = 0, \quad r^2 + (\frac{h}{2})^2 - a^2 = 0.$

    From the first two equations we have $2\pi h + 4\pi r - 2r\frac{4\pi r}{h} = 0$, or $h^2 + 2hr - 4r^2 = 0$. Thus $(h+r)^2 = 5r^2$, so that $h = (\sqrt{5} - 1)r$.
    Using this in the third equation we get $((\frac{\sqrt{5}-1}{2})^2 + 1)r^2 = a^2$, so that $r^2 = \frac{4a^2}{10-2\sqrt{5}} = \frac{5+\sqrt{5}}{10}a^2 = \frac{1}{4}(2 + \frac{2}{\sqrt{5}})a^2$.
    Thus $h^2 = (\sqrt{5}-1)^2 r^2 = \frac{2(\sqrt{5}-1)}{\sqrt{5}}a^2 = (2 - \frac{2}{\sqrt{5}})a^2$.

21. $L(x,y,z,\lambda) = 2(xz + yz) + xy + \lambda(xyz - 500)$
    $\nabla L(x,y,z,\lambda) = 0$

$$L_x = 2z + y + \lambda yz = 0 - (1)$$

$$L_y = 2z + x + \lambda xz = 0 - (2)$$

$$L_z = 2x + 2y + \lambda xy = 0 - (3)$$

$$xyz = 500 - (4)$$

Now, $(1) * x + (2) * y + (3) * z$ will give $2xz + xy + 2yz + 750\lambda = 0 - (5)$
$(1) - (2)$ will give either $x = y$ or $z = -1/\lambda$
$z \neq -1/\lambda$ by (2). Thus, $x = y$.
$(4) \Rightarrow x^2 z = 500 - (6)$
$(3) \Rightarrow x = 0$ (discard) or $x = -4/\lambda = y$
$(6) \Rightarrow z = 125\lambda^2/4$
$(2) \Rightarrow \lambda = -2/5$. Therefore, dimensions should be $(10, 10, 5)$.

22. $L(x,y,z,\lambda) = 8xyz^2 - 200(x + y + z) + \lambda(x + y + z - 100)$
    $\nabla L(x,y,z,\lambda) = 0$

$$L_x = 8yz^2 - 200 + \lambda = 0 - (1)$$

$$L_y = 8xz^2 - 200 + \lambda = 0 - (2)$$

$$L_z = 16xyz - 200 + \lambda = 0 - (3)$$

$$x + y + z = 100 - (4)$$

$(1) - (2) \Rightarrow z = 0 (discard)$ or $x = y$
$(3) - (2) \Rightarrow z = 2x$
$(4) \Rightarrow x = 25 = y$. Therefore, $z = 50$. Also, since continuous function on a closed and bounded set must attain its bounds ,thus it is the point of maxima.