Chapter 3
Systems of differential equations

3.1 Existence and uniqueness

We consider the system of differential equations

$$\begin{aligned} x_1'(t) &= f_1(t, x_1(t), x_2(t), ...x_n(t)) \\ x_2'(t) &= f_2(t, x_1(t), x_2(t), ...x_n(t)) \\ &\vdots \\ x_n'(t) &= f_2(t, x_1(t), x_2(t), ...x_n(t)) \end{aligned}$$

A solution of this system is a vector valued function $\vec{x}(t) : [a,b] \to \mathbb{R}^n$ denoted by $\vec{x}(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$. We assume that $\vec{f} = (f_1, f_2, \cdots, f_n)$ is continuous function in its variables t and $\vec{x}$. We define norm (the distance of $\vec{x}$ from 0) of a vector $\vec{x}$ as

$$|\vec{x}| = \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{\frac{1}{2}}$$

Definition 3.1.1 *A vector valued function $\vec{f}(t, \vec{x})$ is said to be Lipschitz continuous in $\vec{x}$ if there exists constant $L > 0$ such that*

$$|\vec{f}(t, \vec{x}) - \vec{f}(t, \vec{y})| \le L|\vec{x} - \vec{y}|, \ \ \forall \vec{x}, \vec{y}, \ \&\forall t.$$

We have the following existence and uniqueness theorem is known as Picard's theorem:

Theorem 3.1.1 *Suppose $\vec{f}(t, x)$ is Lipschitz continuous in an open set around $(t_0, \vec{x}_0)$. Then the following IVP for the system*

$$\vec{x}'(t) = \vec{f}(t, \vec{x}), \ \vec{x}(t_0) = \vec{x}_0 \tag{1.1}$$

admits unique solution in a neighborhood of $(t_0, \vec{x}_0)$.

Solving this IVP is equivalent to solving the following system of integral equations:

$$\vec{x}(t) = \vec{x}_0 + \int_{t_0}^{t} \vec{f}(s, \vec{x}(s)) ds.$$

In this case we define the metric on the functions space $C(I)$ containing the all continuous vector valued functions $\vec{x}(t)$ as

$$d(\vec{x}, \vec{y}) = \max_{I} |\vec{x}(t) - \vec{y}(t)|$$

From this we can define the Picard iteration:

$$\vec{x}_n(t) = \vec{x}_0 + \int_{t_0}^{t} \vec{f}(s, \vec{x}_{n-1}(s))ds,\ \vec{x}_0(t) = \vec{x}_0,\ n = 1, 2, ...$$

The existence and uniqueness theorem in this case can be stated as follows:

Theorem 3.1.2 *Let $f(t,\vec{x})$ be Lipschitz continuous function in a "(n+1)-rectangle" $R = \{(t,\vec{x}) : |t-t_0| \leq a, |\vec{x}-\vec{x}_0| \leq b\}$ and let $M = \max_R |f(t,\vec{x})|$. Then the initial value problem in* (0.1) *admits unique solution in* $|t-t_0| \leq h = \min(a, \frac{b}{M})$.

Remark 3.1.1 *Here again we remark that local uniqueness implies global uniqueness and the interval of uniqueness does NOT depend on the Lipschitz constant.*

Corollary 3.1.1 *Suppose $f(t,\vec{x})$ is Lipschitz continuous on $[a,b]\times\mathbb{R}^n$, then the solution exists and is unique in $[a,b]$.*

We have the following Uniqueness theorem for Linear system of equations

Theorem 3.1.3 *Let A be an $n\times n$ matrix with real entries. Then the Initial Value Problem:*

$$\vec{x}'(t) = A\vec{x}(t),\ \vec{x}(t_0) = \vec{x}_0.$$

admits unique solution.

As in the scalar case, We have the following Peono's theorem on existence of solutions (uniqueness is not guaranteed)

Theorem 3.1.4 *Suppose $f(t,\vec{x})$ is continuous in an $(n+1)$-rectangle around $(t_0,\vec{x}_0)$. Then the initial Value Problem $\vec{x}' = f(t,\vec{x})$, $\vec{x}(t_0) = \vec{x}_0$ admits solution in a neighbourhood of t_0.*

3.1.1 Theory of Linear systems

In this section, we study the linearly independent solutions and the dimension of the solution space of linear systems of differential equations

$$\vec{x}'(t) = A\vec{x}(t)$$

where $\vec{x}(t) = (x_1(t), x_2(t), ..., x_n(t))^T$ and A is a $n\times n$ matrix with elements $a_{ij}(t), i, j = 1, 2...n$ are continuous functions. A solution of this system is a vector valued function $\vec{x}(t) : [a,b] \to \mathbb{R}^n$.

3.2 First order linear systems

Now let us convert the equation $x'' + a(t)x' + b(t)x = 0$ into first order system by defining

$$x_1 = x,\ x_2 = x_1'$$

Then the second order equation become the first order system

$$
\begin{aligned}
x_1' &= x_2 \\
x_2' &= x_1'' = x'' = -a_1 x_2 - a_2 x_1.
\end{aligned}
$$

So in the new variables the Wronskian becomes

$$
W(x,y) = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}
$$

Motivated from above, we define

Definition 3.2.1 *The Wronskian of n-vector valued functions, $\vec{x}^1(t), \vec{x}^2(t), ..., \vec{x}^n(t) : [a,b] \to \mathbb{R}^n$, is defined as*

$$
W(\vec{x}^1, \vec{x}^2, ..., \vec{x}^n)(t) = \begin{vmatrix} x_1^1(t) & x_1^2(t) & ... & x_1^n(t) \\ x_2^1(t) & x_2^2(t) & ... & x_2^n(t) \\ \vdots & \vdots & \vdots & \vdots \\ x_n^1(t) & x_n^2(t) & ... & x_n^n(t) \end{vmatrix}
$$

where $\vec{x}^i(t) = \left(x_1^i(t), x_2^i(t), ..., x_n^i(t)\right)^T$ for $i = 1, \cdots, n$.

Definition 3.2.2 *The vector valued functions, $\vec{x}^1(t), \vec{x}^2(t), \cdots, \vec{x}^n(t) : [a,b] \to \mathbb{R}^n$, are linearly dependent if there exists $c_1, c_2, \cdots, c_n$ (not all zero) such that*

$$
c_1 \vec{x}^1(t) + c_2 \vec{x}^2(t) + \cdots + c_n \vec{x}^n(t) = 0.
$$

That is, the following system of equations has non-trivial solution

$$
\begin{pmatrix} x_1^1(t) & x_1^2(t) & ... & x_1^n(t) \\ x_2^1(t) & x_2^2(t) & ... & x_2^n(t) \\ \vdots & \vdots & \vdots & \vdots \\ x_n^1(t) & x_n^2(t) & ... & x_n^n(t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{2.2}
$$

Then as an immediate consequence, we have the following

Theorem 3.2.1 *The vector valued functions, $\vec{x}^1(t), \vec{x}^2(t), \cdots, \vec{x}^n(t) : [a,b] \to \mathbb{R}^n$, are linearly dependent then $W(\vec{x}^1, \vec{x}^2, \cdots, \vec{x}^n)(t) = 0$ for all t.*

However the converse is not true.

Theorem 3.2.2 *Abel's formula: $\vec{x}^1(t), \vec{x}^2(t), ..., \vec{x}^n(t) : [a,b] \to \mathbb{R}^n$ be solutions of $\vec{x}' = A(t)\vec{x}$. Then their Wronskian is given by*

$$
W(t) = Cexp\left(\int_{t_0}^{t} (Tr(A(s)))ds\right)
$$

Proof. We give the proof for $n = 2$. In this case $\vec{x}^i, i = 1, 2$ satisfies the system

$$
\begin{pmatrix} (x_1^i)' \\ (x_2^i)' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1^i \\ x_2^i \end{pmatrix}.
$$

$$\begin{aligned}\frac{d}{dt}W(t) &= \begin{vmatrix} (x_1^1)' & (x_1^2)' \\ x_2^1 & x_2^2 \end{vmatrix} + \begin{vmatrix} x_1^1 & x_1^2 \\ (x_2^1)' & (x_2^2)' \end{vmatrix} \\ &= \begin{vmatrix} a_{11}x_1^1 + a_{12}x_2^1 & a_{11}x_1^2 + a_{12}x_2^2 \\ x_2^1 & x_2^2 \end{vmatrix} + \begin{vmatrix} x_1^1 & x_1^2 \\ a_{21}x_1^1 + a_{22}x_2^1 & a_{21}x_1^2 + a_{22}x_2^2 \end{vmatrix} \\ &= a_{11}W + a_{22}W = Tr(A)W. \end{aligned}$$

Integrating this, we get the required formula. □

Corollary 3.2.1 *Let $\vec{x}^1(t), \vec{x}^2(t), \cdots, \vec{x}^n(t) : [a,b] \to \mathbb{R}^n$ be solutions of $\vec{x}' = A(t)\vec{x}$. Then $W(\vec{x}^1, \vec{x}^2, \cdots, \vec{x}^n)(t_0) = 0$ for some t_0, implies $W(\vec{x}^1, \vec{x}^2, \cdots, \vec{x}^n)(t) = 0$ for all t.*

Now we can use the uniqueness theorem to show the following:

Theorem 3.2.3 *Let $\vec{x}^1(t), \vec{x}^2(t), ..., \vec{x}^n(t) : [a,b] \to \mathbb{R}^n$ be solutions of $\vec{x}' = A(t)\vec{x}$. Then $\vec{x}^1(t), \vec{x}^2(t), ..., \vec{x}^n(t)$ are linearly dependent $\iff$ $W(\vec{x}^1, \vec{x}^2, \cdots, \vec{x}^n)(t) = 0$ for all t.*

Proof. $\implies$ is easy. For the converse, if $W(\vec{x}^1, \vec{x}^2, \cdots, \vec{x}^n)(t_0) = 0$ implies the existence of non-trivial solution $(\alpha_1, \alpha_2, ...\alpha_n)$ to the system (2.2). Now we can define $\vec{x}(t) = \alpha_1\vec{x}^1 + \alpha_2\vec{x}^2 + ... + \alpha_n\vec{x}^n$. Then by the linearity, $\vec{x}(t)$ is a solution of $\vec{x}' = A\vec{x}$. We also have $\vec{x}(t_0) = 0$. Therefore, by uniqueness theorem, $x(t) \equiv 0$ implying $\vec{x}^1, \vec{x}^2, \cdots, \vec{x}^n$ are linearly dependent. □

Next theorem is about the "General solution"

Theorem 3.2.4 *Let $\vec{x}^1, \vec{x}^2, \cdots, \vec{x}^n : [a,b] \to \mathbb{R}^n$ be linearly independent solutions of $\vec{x}' = A(t)\vec{x}$. Then all solution of this system are in the linear span of $\vec{x}^1, \vec{x}^2, \cdots, \vec{x}^n$.*

Proof. Let $\vec{y}(t) = (y_1(t), y_2(t), \cdots, y_n(t))^T$ be any solution of $\vec{x}' = A\vec{x}$. Then using the fact that $\vec{x}^1, \vec{x}^2, \cdots, \vec{x}^n$ are linearly independent, we get that the system of equations

$$\begin{pmatrix} x_1^1(t_0) & x_1^2(t_0) & ... & x_1^n(t_0) \\ x_2^1(t_0) & x_2^2(t_0) & ... & x_2^n(t_0) \\ \vdots & \vdots & \vdots & \vdots \\ x_n^1(t_0) & x_n^2(t_0) & ... & x_n^n(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_1(t_0) \\ y_2(t_0) \\ \vdots \\ y_n(t_0) \end{pmatrix} \tag{2.3}$$

has unique solution $C = (\alpha_1, \alpha_2, ..., \alpha_n)$ (say). Now considering the function

$$\vec{z}(t) = \alpha_1\vec{x}^1(t) + \alpha_2\vec{x}^2(t) + ... + \alpha_n\vec{x}^n(t),$$

we see by (2.3) that $\vec{z}(t)$ satisfies $\vec{z}(t_0) = \vec{y}(t_0)$. Also by linearlity,

$$\vec{y}'(t) = A\vec{y},\ \vec{z}'(t) = A\vec{z}.$$

By the Uniqueness theorem for systems we get $\vec{y}(t) \equiv \vec{z}(t)$. □

Corollary 3.2.2 *The set of all solutions is a vector space of dimension n.*

3.2.1 Linear system with constant coefficients

We consider the homogeneous system

$$\vec{x}' = A\vec{x}$$

where (a_{ij}) are constants. In case of higher order equation, we found general solution by substituting $x(t) = e^{mt}$. This suggests that we try substituting $\vec{x}(t) = e^{\lambda t}\vec{v}$ in $\vec{x}' = A\vec{x}$. Then we get

$$\lambda e^{\lambda t}\vec{v} = e^{\lambda t}A\vec{v}.$$

This gives rise to the equation

$$(A - \lambda I)\vec{v} = 0,$$

where I is an $n \times n$ identity matrix. So it is clear now that λ is an eigenvalue and $\vec{v}$ is the corresponding eigenvector. We have the following cases

Case 1: A has distinct eigenvalues $\lambda_1, \lambda_2, ...\lambda_n$.

In this case, Let $\vec{v}^i$ be the eigenvector corresponding to λ_i. We consider $\vec{x}^1(t) = e^{\lambda_1 t}\vec{v}^1, \cdots, \vec{x}^n(t) = e^{\lambda_n t}\vec{v}^n$. Then $\vec{x}^1, \vec{x}^2, \cdots, \vec{x}^n$ are linearly independent as $\vec{x}^1, \vec{x}^2, \cdots, \vec{x}^n$ are linearly independent. i.e.,

$$W(\vec{x}^1, \vec{x}^2, \cdots, \vec{x}^n)(0) = \begin{vmatrix} v_1^1 & v_1^2 & ... & v_1^n \\ v_2^1 & v_2^2 & ... & v_2^n \\ \vdots & \vdots & \vdots & \vdots \\ v_n^1 & v_n^2 & ... & v_n^n \end{vmatrix} \neq 0.$$

Case 2: A has one (or more) eigenvalues repeated. But eigenvectors form a basis of $\mathbb{R}^n$

In this case, say $\lambda_1, \lambda_2, ..., \lambda_m$ are distinct eigenvalues $(m < n)$, and let $\vec{v}^1, \vec{v}^2, ..., \vec{v}^n$ are eigenvectors that from basis of $\mathbb{R}^n$. Then again, we can take $\vec{x}^1(t) = e^{\lambda_1 t}\vec{v}^1, \cdots, \vec{x}^m(t) = e^{\lambda_m t}\vec{v}^m, ..., \vec{x}^n(t) = e^{\lambda_n t}\vec{v}^n$. Then as in the previous case we see $W(\vec{x}^1, \vec{x}^2, \cdots, \vec{x}^n)(0) \neq 0$.

That is, if A is diagnolizable then $A = PDP^{-1}$ where D consists of eigenvalues of A. Then the system

$$\vec{x}' = A\vec{x} = PDP^{-1}\vec{x}$$

Multiplying by P^{-1}, we get

$$P^{-1}\vec{x}' = DP^{-1}\vec{x}$$

By writing $\vec{y} = P^{-1}\vec{x}$ we get

$$\vec{y}' = D\vec{y}$$

which is a noncoupled system whose general solution will be

$$\vec{y} = \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

where λ_i are egienvalues counted with multiplicity. Now the solution of the original system is

$$\vec{x} = P\vec{y}$$

where columns of P are eigen vectors of A.

Case 3: Geometric multiplicity of λ_i is not equal to algebraic multiplicity of λ_i.
Let λ be a repeated eigenvalue twice and $\vec{v}^1$ is the only eigenvector(L.I). Then we have $\vec{x}^1(t) = e^{\lambda t}\vec{v}^1$ is a solution and let $\vec{x}^2(t) = \vec{v}^1 te^{\lambda t} + \vec{u}e^{\lambda t}$ and determine $\vec{u}$ such that $\vec{x}^1, \vec{x}^2$ are linearly independent. Substituting $\vec{x}^2$ in the system, we get

$$\lambda te^{\lambda t}\vec{v}^1 + e^{\lambda t}\vec{v}^1 + \lambda e^{\lambda t}\vec{u} = A(te^{\lambda t}\vec{v}^1 + e^{\lambda t}\vec{u}) = A\vec{v}^1 te^{\lambda t} + A\vec{u}e^{\lambda t}$$

Since $A\vec{v}^1 = \lambda\vec{v}^1$, we have

$$\lambda te^{\lambda t}\vec{v}^1 + e^{\lambda t}\vec{v}^1 + \lambda e^{\lambda t}\vec{u} = \lambda\vec{v}^1 te^{\lambda t} + A\vec{u}e^{\lambda t}$$

Canceling $\lambda te^{\lambda t}\vec{v}^1$, we obtain $e^{\lambda t}\vec{v}^1 + \lambda e^{\lambda t}\vec{u} = A\vec{u}e^{\lambda t}$ and hence

$$\vec{v}^1 + \lambda I\vec{u} = A\vec{u}$$

That is, $\vec{u}$ is a solution of the system

$$(A - \lambda I)\vec{u} = \vec{v}^1$$

Then a natural question arises: **Does such $\vec{u}$ exist?**.

We will answer this question in the next section. First let us compute some examples.

Example 3.2.1 *Find all L.I.solutions of $\vec{x}' = A\vec{x}$ where $A = \begin{pmatrix} 4 & 3 & 1 \\ -4 & -4 & -2 \\ 8 & 12 & 6 \end{pmatrix}$.*

Eigenvalues are $\lambda_1 = \lambda_2 = \lambda_3 = 2$ and has only two L.I. eigenvectors that are

$$\vec{v}^1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \text{ and } \vec{v}^2 = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}.$$

So the two linearly independent solutions are $\vec{x}^1 = \vec{v}^1 e^{2t}, \vec{x}^2 = \vec{v}^2 e^{2t}$. The third linearly independent solution is of the form $(\vec{\alpha}t + \vec{\beta})e^{2t}$ where $\vec{\alpha}$ and $\vec{\beta}$ satisfies

$$(A - 2I)\vec{\alpha} = 0 \text{ and } (A - 2I)\vec{\beta} = \vec{\alpha}.$$

We take $\vec{\alpha} = k_1\vec{v}^1 + k_2\vec{v}^2$. Then $\vec{\alpha} = \begin{pmatrix} k_1 \\ k_2 \\ -2k_1 - 3k_2 \end{pmatrix}$ and $\vec{\beta}$ is a solution of

$$\begin{pmatrix} 2 & 3 & 1 \\ -4 & -6 & -2 \\ 8 & 12 & 4 \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \\ -2k_1 - 3k_2 \end{pmatrix}$$

First two equations imply $k_2 = -2k_1$. A simple non-trivial solution is $k_1 = 1, k_2 = -2$. With this choice, $\alpha = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$. With this choice of $\vec{\alpha}$ we compute $\vec{\beta}$ as solution of

$$\begin{pmatrix} 2 & 3 & 1 \\ -4 & -6 & -2 \\ 8 & 12 & 4 \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}.$$

A solution of this system is $\vec{\beta} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Therefore, third L.I. solution is $\vec{x}^3 = \begin{pmatrix} te^{2t} \\ -2te^{2t} \\ (4t+1)e^{2t} \end{pmatrix}$

□

Suppose A is a 3×3 matrix with only eigenvalue λ repeated thrice and has only one eigen vector $\vec{v}$. Then we have

$$\vec{x}^1 = \vec{v}e^{\lambda t}$$

and the second L.I. solution can be found as explained earlier as

$$\vec{x}^2 = \vec{v}te^{\lambda t} + \vec{u}e^{\lambda t}$$

where u satisfies

$$(A-\lambda I)\vec{u} = \vec{v}.$$

Now to find other linearly independent solution, we consider

$$\vec{x}^3 = \frac{t^2}{2}\vec{\eta}e^{\lambda t} + \vec{\rho}te^{\lambda t} + \vec{w}e^{\lambda t}$$

where we have to find the vectors $\vec{\eta}, \vec{\rho}, \vec{w}$. Substituting this in the equation $\vec{x}' = A\vec{x}$, we get

$$e^{\lambda t}\left(t\vec{\eta}I + \frac{\lambda}{2}t^2\vec{\eta} + \vec{\rho}I + t\lambda\vec{\rho} + \lambda\vec{w}\right) = e^{\lambda t}\left(\frac{t^2}{2}A\vec{\eta} + tA\vec{\rho} + A\vec{w}\right)$$

Equating the coefficients of t^2, t and constant vectors, we get

$$A\vec{\eta} = \lambda\vec{\eta},\ \ (A-\lambda I)\vec{\rho} = \vec{\eta},\ \ (A-\lambda I)\vec{w} = \vec{\rho}.$$

This implies $\vec{\eta} = c\vec{v}$ and $\vec{\rho} = c\vec{u}$, where $\vec{v}$ is an eigenvector corresponding to λ and $\vec{u}$ is found in $\vec{x}^2$ and $\vec{w}$ are the solutions of the above system $(A-\lambda I)\vec{w} = \vec{\rho}$.

Example 3.2.2 *Solve the system* $\vec{x}' = A\vec{x}$ *with* $A = \begin{pmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix}$.

Easy to see the eigenvalue is $\lambda = 2$ repeated thrice. There is only one eigen vector $\vec{v} = (1,0,0)^T$. So the first solution is

$$\vec{x}^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t}$$

The second solution is in the form $\vec{x}^2 = \vec{v}te^{2t} + \vec{u}e^{2t}$ where $\vec{u}$ satisfies $(A-2I)\vec{u} = \vec{v}$. So solving

$$\begin{pmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \delta \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

we get $\vec{u} = (0,1,0)^T$ as one solution. Hence the second L.I. solution is

$$\vec{x^2} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{2t}$$

To get the third L.I. solution we solve $(A-2I)\vec{w} = \vec{u}$. That is

$$\begin{pmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

This gives a solution $w = \frac{1}{5}(0,-6,1)^T$. Hence

$$\vec{x^3} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \frac{t^2}{2} e^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} te^{2t} + \frac{1}{5} \begin{pmatrix} 0 \\ -6 \\ 1 \end{pmatrix} e^{2t}$$

□

These vectors $\vec{u}, \vec{v}$ and $\vec{w}$ are called generalized eigen vectors. In these examples we see that such eigen vectors exist and form a basis of $\mathbb{R}^3$. It is indeed happens $(A-2I)z = \vec{w}$ has no solution in the above example. This motivates one to study the following:

3.2.2 Generalized eigen vectors and Jordan form

Definition 3.2.3 *Let A be a $n \times n$ matrix. Then a nontrivial vector u that solves the system*

$$(A-\lambda I)^k \vec{u} = 0,$$

for some positive integer k, is called generalized eigen vector associated to eigenvalue λ. The smallest such k is called index of the generalized eigen vector $\vec{u}$ or eigen vector of index k.

1. If $k = 1$ then we see that u, then $(A-\lambda I)\vec{u} = 0$ and therefore $\vec{u}$ is an eigen vector associated to the eigenvalue λ.
2. If $\vec{v}_k$ is a generalized eigen vector of index $k(>1)$ then $\vec{v}_{k-1}$ defined as

$$\vec{v}_{k-1} = (A-\lambda I)\vec{v}_k$$

is a generalized eigen vector of index $k-1$. Indeed,

$$(A-\lambda I)^{k-1}\vec{v}_{k-1} = (A-\lambda I)^k \vec{v}_k = 0.$$

But

$$(A-\lambda I)^{k-2}\vec{v}_{k-1} = (A-\lambda I)^{k-1}\vec{v}_k \neq 0.$$

Therefore we have

$$\vec{v}_1 = (A-\lambda I)\vec{v}_2 = (A-\lambda I)^2\vec{v}_3 = \cdots = (A-\lambda I)^{k-1}\vec{v}_k$$

Next we have the following:

Theorem 3.2.5 *Suppose $\vec{u}_i, i = 1,2,...n$ are generalized eigen vectors of an $n\times n$ matrix. Then they are linearly independent.*

Proof. We take $n = 2$ for simplicity and suppose $\vec{u}_1$ and $\vec{u}_2$ are generalized eigen vectors. Suppose $c_1\vec{u}_1 + c_2\vec{u}_2 = 0$. Then we will show that $c_2 = c_1 = 0$. We have that $(A-\lambda I)\vec{u}_2 = \vec{u}_1$. Therefore

$$c_1(A-\lambda I)\vec{u}_2 + c_2\vec{u}_2 = 0.$$

Multiplying left by $(A-\lambda I)$, we get

$$c_1(A-\lambda I)^2\vec{u}_2 + c_2(A-\lambda I)\vec{u}_2 = 0.$$

$$\implies c_2(A-\lambda I)\vec{u}_2 = 0 \implies c_2 = 0$$

Therefore $c_1\vec{u}_1 = 0$ implying $c_1 = 0$. □

Theorem 3.2.6 *Suppose $n = 2$ and λ is a repeated eigenvalue with geometric multiplicity strictly less than the algebraic multiplicity. Then there exist eigen vectors of index 1 and 2.*

Proof. We know that there exists an eigen vector u of index 1. By Caley-Hamilton theorem, we get $(A-\lambda I)^2 = 0$ implying dimension of $Ker(A-\lambda I)^2$ is 2. But dimension of $Ker(A-\lambda I)$ is 1. Therefore

$$Ker(A-\lambda I) \subsetneq Ker(A-\lambda I)^2 \tag{2.4}$$

Therefore there exists $\vec{z} \in Ker(A-\lambda I)^2$ such that $\vec{z} \notin Ker(A-\lambda I)$. That is

$$(A-\lambda I)\left((A-\lambda I)\vec{z}\right) = 0.$$

That is $(A-\lambda I)\vec{z}$ is also an eigen vector corresponding to λ and so $(A-\lambda I)\vec{z} \not\equiv 0$ and moreover it belongs to span of $\vec{u}$,

$$(A-\lambda I)\vec{z} = \alpha\vec{u}$$

for some $\alpha \in \mathbb{R}$. Therefore $\vec{v} = \dfrac{\vec{z}}{\alpha}$ is the eigen vector of index 2. □

Definition 3.2.4 *A 2×2 Jordan block associated to the eigenvalue λ is a 2×2 matrix of the form*

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Theorem 3.2.7 *Suppose $n = 2$ and λ is a repeated eigenvalue with geometric multiplicity strictly less than the algebraic multiplicity. Let u and v be the eigen vectors of index 1 and 2 respectively. Then*

$$A = PJP^{-1}$$

where $P = [\ \vec{u}\,|\ \vec{v}\ ]$ is a matrix whose columns are the generalized eigen vectors and J is the Jordan block of order 2.

Proof. We have $(A-\lambda I)\vec{u} = 0$ and $(A-\lambda I)\vec{v} = \vec{u}$. Taking the matrices

$$P = \begin{pmatrix} u_1 & v_2 \\ u_2 & v_2 \end{pmatrix} \text{ and } J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Using the fact that

$$A\vec{v} = (A - \lambda I + \lambda I)\vec{v} = (A - \lambda I)\vec{v} + \lambda\vec{v} = \vec{u} + \lambda\vec{v}$$

and

$$A\vec{u} = \lambda\vec{u}$$

we get

$$AP = [A\vec{u} \,|\, A\vec{v}] = [\lambda\vec{u} \,|\, \vec{u} + \lambda\vec{v}]$$

By direct computation we see that

$$PJ = [\lambda u \,|\, \vec{u} + \lambda\vec{v}]$$

Therefore $AP = PJ$ and hence the result. □

Indeed one can prove the following theorem:

Theorem 3.2.8 *Let A be a $n \times n$ matrix with real eigenvalues. Then A is similar to a real matrix in Jordan canonical form. More precisely, $A = PJP^{-1}$ with P and J real matrices, with J consists of eigenvalues of A and columns of P are generalized eigenvectors of A.*

Theorem 3.2.9 *If $A = PJP^{-1}$ where J is a matrix of Jordan canonical form. Suppose $\vec{y}$ is solution of*

$$\vec{y}' = J\vec{y}$$

Then $\vec{x} = P\vec{y}$ is a solution of

$$\vec{x}' = A\vec{x}.$$

Proof.

$$\vec{x}' = A\vec{x} \implies \vec{x}' = PJP^{-1}\vec{x} \implies P^{-1}\vec{x}' = JP^{-1}\vec{x} \implies \vec{y}' = J\vec{y}$$

□

Theorem 3.2.10 *Suppose J be a $k \times k$ matrix in Jordan canonical form*

$$J = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}$$

then the system $\vec{z}' = J\vec{z}$ has the solution

$$\vec{z} = e^{\lambda t} \begin{pmatrix} 1 & t & t^2/2! & \cdots & \frac{t^{k-1}}{(k-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{k-2}}{(k-2)!} \\ 0 & 0 & 1 & \cdots & \frac{t^{k-3}}{(k-3)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & t \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{k-1} \\ c_k \end{pmatrix} \tag{2.5}$$

where $c_1,\cdots c_n$ are arbitrary constants.

Proof. Case 1: Let us take $k = 2$. In this case the system looks like

$$\begin{aligned} z_1' &= \lambda z_1 + z_2 \\ z_2' &= \qquad \lambda z_2 \end{aligned}$$

It is easy to solve this system to see that

$$z_2 = c_2 e^{\lambda t},\ z_1 = e^{\lambda t}(c_1 + c_2 t)$$

That is

$$\vec{z} = e^{\lambda t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Case 2: $k = 3$. In this case the system of equations is

$$\begin{aligned} z_1' &= \lambda z_1 + z_2 \\ z_2' &= \qquad \lambda z_2 + z_3 \\ z_3' &= \qquad\qquad \lambda z_3 \end{aligned}$$

Using $z_3 = c_3 e^{\lambda t}$ and back substitution we will get

$$z_2 = e^{\lambda t}(c_2 + c_3 t),\ z_1 = e^{\lambda t}(c_1 + c_2 t + c_3 \frac{t^2}{2})$$

That is

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = e^{\lambda t}\begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Remark 3.2.1 *Suppose that $\vec{z}\,' = J\vec{z}$ where J is a matrix in Jordan canonical form but one consisting of several blocks. This systems decomposes into several systems, one corresponding to each block, and that these systems are uncoupled, so we may solve them each separately and then simply assemble these individual solutions together to obtain a solution of the general system.*

Definition 3.2.5 *Fundamental Matrix: The fundamental matrix of a system $\vec{x}\,' = A\vec{x}$ is a matrix valued function whose columns are linearly independent solutions of the system.*

Definition 3.2.6 *A fundamental matrix Φ is called principal fundamental matrix or evolution matrix if it satisfies $\Phi(0) = I$, where I is the identity matrix. That is Φ satisfies*

$$\Phi' = A\Phi,\ \Phi(0) = I.$$

Suppose $\Phi(t)$ is a principal fundamental matrix of the system $\vec{x}\,' = A\vec{x}$, then integrating from 0 to t we see that it satisfies

$$\Phi(t) = I + \int_0^t A(s)\Phi(s)ds,\ t > 0$$

Its unique solution is given by (iterating)

$$\Phi(t) = I + \int_0^t A(s)ds + \int_0^t \int_0^s A(\sigma_1)A(\sigma_2)d\sigma_2 d\sigma_1 + ...$$

This is called Peano-Baker series for principal fundamental matrix. The series converges for continuous $A(t)$:

Theorem 3.2.11 *Let I be an interval, with $0 \in I$. Suppose A is a continuous matrix function on I, and $\vec{b}$ is also continuous on I. Then, the initial value problem*

$$\vec{x}' = A\vec{x} + \vec{b} \;\; \vec{x}(0) = \vec{x}_0$$

has unique solution.

Proof. The proof of this theorem follows in the similar lines as that of Peano's theorem. We omit the proof here. □

In case of A is constant matrix. Then the series of principal fundamental matrix is

$$\begin{aligned}\Phi(t) &= I + \int_0^t A ds + \int_0^t \int_0^s A^2 d\tau ds + \cdots \\ &= I + At + A^2 \frac{t^2}{2!} + \cdots\end{aligned}$$

We can identify the series with that of exponential. This motivates us to define

Definition 3.2.7 *Exponential matrix: The exponential of a matrix A denoted by e^A is defined as $\Phi(1)$ where Φ satisfies*

$$\Phi' = A\Phi, \; \Phi(0) = I.$$

This function $\Phi(t)$ called exponential matrix function e^{At}. In case A is a diagonal matrix, then $A = PDP^{-1}$ where columns of P are eigen vectors. Then the series above reduces to

$$\begin{aligned}e^{At} = I + At + A^2\frac{t^2}{2!} + \cdots &= P\left(I + Dt + D^2\frac{t^2}{2!} + \cdots\right)P^{-1} \\ &= P\begin{pmatrix} e^{\lambda_1} t & 0 & \cdots & 0 \\ 0 & e^{\alpha_1 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & e^{\lambda_n t}\end{pmatrix}P^{-1} \\ &= Pe^{Dt}P^{-1}\end{aligned}$$

Similarly, if A is not a diagonal matrix, then

$$e^{At} = Pe^{Jt}P^{-1}$$

where e^{Jt} is the solutions defined in (2.5) and P is the matrix whose columns are generalized eigen vectors of the matrix A. The following are easy consequences of uniqueness theorem

Remark 3.2.2 1. *Suppose Φ_1 and Φ_2 are two fundamental matrices of the system $\vec{x}' = A\vec{x}$. Then there exists a constant matrix C such that $\Phi_2 = \Phi_1 C$.*

2. *If $\Phi(t)$ is any fundamental matrix then the principal fundamental matrix is given by $\Psi(t) = \Phi(t)\Phi^{-1}(0)$. This satisfies $\Psi' = A\Psi, \;\; \Psi(0) = I$. This motivates one to write $\Psi(t)$ as e^{At}.*

The above discussion summerizes to

Theorem 3.2.12 *Let P be the matrix with generalized eigen vectors of A as columns. Then* $A = PJP^{-1}$ *and*

$$e^{At} = Pe^{Jt}P^{-1}$$

Complex roots
In case of complex eigenvalues, say for $n = 2$, let $\alpha \pm i\beta$ are eigenvalues and $\vec{a} \pm i\vec{b}$ are eigen vectors. (for complex eigenvalues λ and $\overline{\lambda}$, the eigen vectors are z and $\overline{z}$). Then the general solution is

$$x = k_1(\vec{a} + i\vec{b})e^{(\alpha+i\beta)t} + k_2(\vec{a} - i\vec{b})e^{(\alpha-i\beta)t}$$

where k_1 and k_2 are arbitrary constants. Using the matrix methods, in this case we can use the Jordan form

$$P = [\ \vec{a} \mid \vec{b}\ ],\ \text{and}\ J = \begin{pmatrix} \alpha + i\beta & 0 \\ 0 & \alpha - i\beta \end{pmatrix}$$

to write the solution as

$$e^{At} = [\ \vec{a} \mid \vec{b}\ ] \begin{pmatrix} e^{(\alpha+i\beta)t} & 0 \\ 0 & e^{(\alpha-i\beta)t} \end{pmatrix} [\ \vec{a} \mid \vec{b}\ ]^{-1}$$

Example 3.2.3 *Find L.I. solutions of* $\vec{x}' = A\vec{x}$, *with* $A = \begin{pmatrix} 3 & 5 \\ -1 & -1 \end{pmatrix}$

Easy to see that $\lambda = 1 \pm i$ are eigenvalues. The eigen vector v associated with $\lambda = 1 + i$ is

$$v = \begin{pmatrix} -2 - i \\ 1 \end{pmatrix}$$

Therefore

$$P = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } e^{Jt} = \begin{pmatrix} e^{(1+i)t} & 0 \\ 0 & e^{(1-i)t} \end{pmatrix}$$

Hence

$$e^{At} = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{(1+i)t} & 0 \\ 0 & e^{(1-i)t} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$$

□

In case of complex repeated eigenvalues with geometric multiplicity less than algebraic multiplicity, the procedure of Jordan blocks will continue to hold.

3.3 Non homogeneous problems

The problem

$$\vec{x}' = A(t)\vec{x} + \vec{f}(t)$$

is called non-homogenous problem if $\vec{f}(t) \not\equiv 0$. Assuming that the homogeneous part of the problem $\vec{x}' = A(t)\vec{x}$ is solvable, we would like to study the general solution of the non-homogenous problem.

Scalar case: Consider the equation $x' + ax = f(t)$ with constant a and $f(t)$ continuous. Then we know that $x_h(t) = ce^{-at}$ is the general solution of $x' = ax$. Now for a solution of non-homogeneous equation, we consider $x_p(t) = c(t)e^{-at}$. Then substituting this in the equation (non-homogenous)

$$f(t) = x_p' + ax_p = c'(t)e^{-at} - ac(t)e^{-at} + ac(t)e^{-at} = c'e^{-at}$$

That is, $c'(t) = e^{at}f(t)$. Therefore, $x_p(t) = e^{-at}\int e^{-as}f(s)ds$ is a solution of non-homogenous equation.

We take the first order system $\vec{x}' = A\vec{x} + \vec{f}$. Let X be the fundamental matrix of the system $\vec{x}' = A\vec{x}$. Then we can expect the solution of non-homogeneous system is of the form

$$y = X\vec{u}$$

where $\vec{u}$ is a function of t. Substituting this in the system $\vec{x}' = A\vec{x} + \vec{f}$, we get

$$X\vec{u}' + X'\vec{u} = AX\vec{u} + \vec{f}$$

since, $X' = AX$, the above equation is reduced to

$$X\vec{u}' = \vec{f}$$

In other words, $\vec{u}' = X^{-1}\vec{f}$. Therefore, the variation of parameters formula for systems is

$$\vec{y} = X\vec{u} = X\int X^{-1}\vec{f}(t)dt \tag{3.6}$$

is a particular solution of the non-homogeneous system.

Example 3.3.1 *(Second order non-homogeneous system): Solve the problem $\vec{x}' = A\vec{x} + \vec{f}$ where*

$$A = \begin{pmatrix} 1 & 1 \\ -3 & 5 \end{pmatrix} \quad \text{and} \quad f(t) = \begin{pmatrix} 2t-2 \\ -4t \end{pmatrix}$$

The general solution of homogenous part is

$$c_1\begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} + c_2\begin{pmatrix} e^{4t} \\ 3e^{4t} \end{pmatrix}$$

So the fundamental solution X is

$$X = \begin{pmatrix} e^{2t} & e^{4t} \\ e^{2t} & 3e^{4t} \end{pmatrix}.$$

Then $X^{-1} = \frac{e^{-6t}}{2}\begin{pmatrix} 3e^{4t} & -e^{4t} \\ -e^{2t} & e^{2t} \end{pmatrix}$. Therefore,

$$X^{-1}\vec{f}(t) = \frac{e^{-6t}}{2}\begin{pmatrix} 3e^{4t} & -e^{4t} \\ -e^{2t} & e^{2t} \end{pmatrix}\begin{pmatrix} 2t-2 \\ -4t \end{pmatrix} = \begin{pmatrix} 5te^{-2t} - 3e^{-2t} \\ -3te^{-4t} + e^{-4t} \end{pmatrix}$$

Hence

$$\vec{u} = \int \begin{pmatrix} 5te^{-2t} - 3e^{-2t} \\ -3te^{-4t} + e^{-4t} \end{pmatrix} = \begin{pmatrix} \frac{-5}{2}te^{-2t} + \frac{1}{4}e^{-2t} \\ \frac{3}{4}te^{-4t} - \frac{1}{16}e^{-4t} \end{pmatrix}$$

Therefore, a particular solution is

$$\vec{y} = X\vec{u} = \begin{pmatrix} e^{2t} & e^{4t} \\ e^{2t} & 3e^{4t} \end{pmatrix} \begin{pmatrix} \frac{-5}{2}te^{-2t} + \frac{1}{4}e^{-2t} \\ \frac{3}{4}te^{-4t} - \frac{1}{16}e^{-4t} \end{pmatrix} = \begin{pmatrix} -\frac{7}{4}t + \frac{3}{16} \\ -\frac{1}{4}t + \frac{1}{16} \end{pmatrix}$$

We may also use the method of undetermined coefficients, to find a particular solution

$$x_p = \begin{pmatrix} at + b \\ ct + d \end{pmatrix}$$

and determine the constants a, b, c, d. Substituting this in the given system, we obtain

$$a = (at + b) + (ct + d) + 2t - 2$$
$$c = -3(at + b) + 5(ct + d) - 4t$$

This reduces to the solving the system of equations,

$$a + c + 2 = 0, 3a - 5c + 4 = 0, b + d - 2 - a = 0, c + 3b - 5d = 0$$

Solving the first two equations, we get $a = \frac{-7}{4}, c = \frac{-1}{4}$. Now from the other two equations, we find $b = \frac{3}{16}, d = \frac{1}{16}$. Therefore,

$$x_p = \begin{pmatrix} \frac{-7}{4}t + \frac{3}{16} \\ \frac{-1}{4}t + \frac{1}{16} \end{pmatrix}$$

3.4 Exercises

1. Find all linearly independent solutions of the system $\vec{x}' = A\vec{x}$ with A given below:

   (a) $\begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & -1 & -1 \\ 2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 4 & 6 & -1 \\ -1 & -2 & 1 \\ -2 & -8 & 4 \end{pmatrix}$

2. Suppose $u_1, u_2, ...u_n$ are generalized eigen vectors of A, then show that they are linearly independent.
3. Solve the non-homogeneous systems

   (a) $x' = x + 3y + 2t,\ y' = x - 2y - e^t$

   (b) $x' = x + 2y + e^t,\ y' = x - 2y - e^t$

4. Let A be Jordan block of order 3×3 and let $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Then show that $B^3 = 0$ and $e^{At} = e^{\lambda t}\left[I + tB + \frac{t^2}{2}B^2\right]$.
5. Let A be a nilpotent matrix of order k. Then show that the principal fundamental matrix of $\vec{x}' = A\vec{x}$ is equal to
$$e^{At} = I + tA + \cdots + \frac{t^k A^k}{k!}.$$
6. Compute the solution of $\vec{x}' = A(t)\vec{x}$ using Peano-Baker series where

$$A(t) = \begin{pmatrix} 0 & e^t \\ 0 & 0 \end{pmatrix}$$