Chapter 6
Boundary Value Problems

In this chapter we deal with the boundary value problems as

$$\mathcal{L}u = a_0(t)u'' + a_1(t)u' + a_2(t)u(t) = f(t) \tag{0.1}$$

satisfying the boundary conditions

$$B_1(u(a), u'(a), u(b)u'(b)) = 0, \;\; B_2(u(a), u'(a), u(b)u'(b)) = 0. \tag{0.2}$$

where B_1 and B_2 are linear functions of its variables and $a_0(t) > 0$ in $[a,b]$ and $a_1(t), a_2(t)$ are continuous functions on $[a,b]$.
Multiplying the equation (0.1) (taking $f = 0$) with v and try to make it exact, we get

$$a_0(t)u'' + a_1(t)u' + a_2(t)u(t) = 0$$

This may be written in the exact form

$$(va_0u')' + (S\,u)' = 0 \iff S = -v'a_0 - va_0' + va_1, \; S' = va_2$$

That is,

$$va_2 = S' = -v''a_0 - 2v'a_0' - va_0'' + v'a_1 + va_1'$$

That is v satisfies

$$\mathcal{L}^*(v) := a_0v'' + v'(2a_0' - a_1) + v(a_2 + a_0'' - a_1') = 0 \tag{0.3}$$

The equation in (0.4) is called the adjoint equation and $\mathcal{L}^*$ is called adjoint of $\mathcal{L}$. It is easy to see that if we can solve the adjoint equation then we can solve the given equation, for example,

Example 6.0.1 *consider the equation* $u'' - (2t + \frac{3}{t})u' - 4u = 0$

Here $a_0 = 1, a_1 = -(2t + \frac{3}{t})$ and $a_2 = -4$. Then the adjoint equation is

$$v'' + (2t + \frac{3}{t})v' - (2 + \frac{3}{t^2})v = 0$$

It is easy to see that $v = t$ is a solution. Then we see that

$$S = -4 - 2t^2$$

The given equation in exact form becomes

$$(tu')' - ((4+2t^2)u)' = 0$$

Integrating this equation we get

$$u' - (\frac{4}{t} + 2t)u = C$$

a first order equation that is easy to solve. □

Solvability features of Boundary value problems are similar to the solvability features of the system of linear equations $Ax = b$. We have the following:
Let A be an $n \times n$ matrix whose eigenvectors form a basis of $\mathbb{R}^n$. Let $\mathcal{B} = \{e_1, e_2, e_3, ...e_n\}$ be an orthonormal basis of eigenvectors of A. That is $\langle e_i, e_j \rangle = 0$ if $i \neq j$ and $\langle e_i, e_i \rangle = 1$.
Now since $\mathcal{B}$ is a basis of $\mathbb{R}^n$ and $b \in \mathbb{R}^n$, we have $b = \sum_{i=1}^{n} b_i e_i$, where $b_i = \langle b, e_i \rangle$. We assume the solution to be

$$x = x_1 e_1 + x_2 e_2 + ... + x_n e_n$$

So it is enough to solve for $x_i's$. Now substituting this in the equation $Ax = b$ we get

$$A \sum_{i=1}^{n} x_i e_i = \sum_{i=1}^{n} b_i ei \implies \sum_{i=1}^{n} x_i A e_i = \sum_{i=1}^{n} b_i e_i$$

Therefore,

$$\sum_{i=1}^{n} \lambda_i x_i e_i = \sum_{i=1}^{n} b_i e_i$$

Since $e_1, e_2, ...e_n$ are linearly independent, we get

$$\lambda_i x_i = b_i, \;\; i = 1, 2, ...n. \tag{0.4}$$

From the equation (0.4), we get the following assertions:

1. If zero is not an eigenvalue of A then the problem $Ax = b$ has unique solution $x = \sum_{i=1}^{n} \frac{b_i}{\lambda_i} e_i$. That is, if A is non-singular, then we have unique solution.
2. If $\lambda_m = 0$ for some m. Then solutions exist if $b_m = 0$, in which case x_m is arbitrary. Since $\mathcal{B}$ is orthonormal, $b_m = \langle b, e_m \rangle = 0$. Also e_m satisfies $Ae_m = 0$. That is solutions exists if b is orthogonal to every solution of $Ax = 0$.
3. On the other hand, if b is othogonal to every non-trivial solution of $Ax = 0$. Then 0 is an eigenvalue λ_m(say) of A and $b_m = \langle b, e_m \rangle = 0$. Hence x_m is arbitrary in the equation (0.4) and infinitely many solutions exist.
4. It is also clear that the problem $Ax = b$ has NO SOLUTION if $\lambda_m = 0$ and $\langle b, e_m \rangle \neq 0$.

The above method tells us that once we know the eigenvalues and eigenvectors forms a basis of $\mathbb{R}^n$, then it is easy to determine the solvability of the system. We will show that we can generalize the above approach to differential equations to solve a non-homogeneous boundary value problem. These are known as **Fredholm Alternative** theorems.

6.1 Oscillation theory

In this section we will study the qualitative properties of solutions like oscillatory behaviour of solutions. We consider the equation

$$x'' + p(t)x = 0, \tag{1.5}$$

where $p(t)$ is continuous everywhere. We note that if $x(t)$ is a solution of

$$x'' + p(t)x' + q(t)x = 0. \tag{1.6}$$

Then $y(t) = x(t)e^{\frac{1}{2}\int p(t)dt}$ satisfies the equation

$$y'' + (q - \frac{p^2}{4} + \frac{1}{2}p')y = 0. \tag{1.7}$$

So the zeros of $x(t)$ and $y(t)$ are same. We have the following defitions.

Definition 6.1.1 *We say that a nontrivial solution $x(t)$ of (1.5) is oscillatory if for any number T, $x(t)$ has infinitely many zeros in the interval (T,∞). We also call the equation (1.5) is oscillatory if it has an oscillatory solution.*

We have the following theorem which is a consequence of uniqueness theorem.

Theorem 6.1.1 *Let $x_1(t)$ and $x_2(t)$ are two linearly independent solutions of (1.5) and let a and b are two consecutive zeros of $x_1(t)$ with $a < b$. Then $x_2(t)$ has exactly one zero in the interval (a,b).*

Proof. Since $x_1(a) = 0 = x_1(b)$, we note that $x_2(a) \neq 0$ and $x_2(b) \neq 0$. Otherwise Wronskian of x_1, x_2 is zero. Without loss of generality, we assume that $x_1(t) > 0$ in (a,b). Suppose by contradiction, $x_2(t) > 0$ in $[a,b]$. Let

$$h(t) = \frac{x_1(t)}{x_2(t)}.$$

Then h is differentiable on (a,b) and $h(a) = h(b) = 0$. Therefore, by Rolle's theorem, there exists $c \in (a,b)$ such that $h'(c) = 0$. But $h'(c) = 0$ implies $x_2(c)x_1'(c) - x_1(c)x_2'(c) = 0$, which implies Wronskian of x_1, x_2 is zero at c. This is a contradiction. So $x_2(t)$ has a zero in (a,b).
Suppose $x_2(t)$ has two zeros t_1, t_2 in (a,b). Then change the roles of x_1 and x_2 in the above discussion to obtain a zero d between (t_1,t_2) of $x_1(t)$. This contradicts that a and b are two consecutive zeros of $x_1(t)$. □

Corollary 6.1.1 *If (1.5) has one oscillatory solution, then all solutions are oscillatory.*

Example 6.1.1 *Show that between any two zeros of $\cos t$ there is a zero of $2\sin t - 3\cos t$.*

Follows from Theorem 6.1.1 by noting that both functions are solutions of $x'' + x = 0$. □

Next we study the number of zeros of solutions and their dependance on the coefficients. First let us consider the equation

$$x'' + k^2 x = 0$$

The linearly independent solutions are $x_1(t) = \sin kt$ and $x_2(t) = \cos kt$, which are oscillatory. To start with let us take $k = 1$. Then the resultant solution $x_1(t) = \sin t$ has exactly one zero in $(0,2\pi)$. If we take $k = 2$, then $x_1(t) = \sin 2t$ has three zeros in $(0,2\pi)$. So we notice that as k increases the zeros are moving towards left and the number of zeros in the interval $(0,2\pi)$ increases. In otherwords, the larger the coefficient k is the solutions oscillates more rapidly. In the general case this is the consequence of the following theorem.

Theorem 6.1.2 *Consider the two equations*

$$x'' + p(t)x = 0 \tag{1.8}$$
$$y'' + q(t)y = 0. \tag{1.9}$$

Suppose that $x(t)$ is a nontrivial solution of (1.8) with consecutive zeros at a and b. Assume further that $p(t), q(t) \in C([a,b])$ such that $p(t) \le q(t)$, $p(t) \not\equiv q(t)$. If $y(t)$ is any nontrivial solution of (1.9) such that $y(a) = 0$, then there exists another zero c of $y(t)$ in (a,b).

Proof. Assume that the assertion of the theorem is false. Without loss of generality, we can assume that $x(t) > 0$ on (a,b), otherwise replace x by $-x$. Similarly, we can also assume that $y(t) > 0$ on (a,b). Multiplying the first equation by $y(t)$ and second equation by $x(t)$ and subtracting, integrating from a to b, we get

$$\int_a^b y(t)x''(t) - x(t)y''(t) + (p(t) - q(t))x(t)y(t) = 0 \tag{1.10}$$

Integrating by parts,

$$[xy' - yx']_a^b = \int_a^b (q-p)(t)x(t)y(t)dt.$$

Since $x(a) = x(b) = y(a) = 0$, the above equation can be written as

$$y(b)x'(b) = \int_a^b (q(t) - p(t))x(t)y(t)dt.$$

The right side of above equation is strictly positive and left side is non-positive as $y(b) \ge 0$, and $x'(b) \le 0$, a contradiction. □

Corollary 6.1.2 *If $l = \lim_{t\to\infty} Q(t) > 0$, then $x'' + Q(t)x = 0$ is an oscillatory equation.*

Proof. Since $\lim_{t\to\infty} Q(t) = l > 0$, we can choose a number T such that for $t \ge T$, $Q(t) > \frac{l}{2}$. Now using the Theorem 6.1.2 with $p(t) = \frac{l}{2}$ and $q(t) = Q(t)$, we get that for $t \ge T$ every solution of $x'' + Q(t)x = 0$ has a zero between any two zeros of solution of $x'' + \frac{l}{2}x = 0$. The assertion follows from the fact that $x'' + \frac{l}{2}x = 0$ is oscillatory. □

Example 6.1.2 *Show that*

$$x'' + \frac{2t^6 - 2t^3 - 1}{t^6 + 1}x = 0$$

is an oscillatory equation.

Proof follows from the above corollary as $\lim_{t\to\infty} \frac{2t^6 - 2t^3 - 1}{t^6 + 1} = 2$. □

Corollary 6.1.3 *If $P(t) \le 0, P(t) \not\equiv 0$, then no solution of $x'' + P(t)x = 0$ can have more than one zero.*

Proof. We will apply theorem 6.1.2 with $p(t) = P(t)$ and $q(t) = 0$. Let t_1, t_2 are two zeros of solution of $x'' + P(t)x = 0$. By theorem 6.1.2, every solution of $y'' = 0$ has a zero between t_1 and t_2, which is impossible as the general solution of $y'' = 0$ is $at + b$ which cannot have two zeros. This contradiction proves the theorem. □

6.2 Eigenvalue problems

In this section, we will study the Sturm-Liouville eigenvalue problems and their applications to the boundary value problems of type (0.1)-(0.2). where $B_1 = c_1x(a) + c_2x(b) + c_3x'(a) + c_4x'(b) = 0$ and $B_2 = d_1x(a) + d_2x(b) + d_3x'(a) + d_4x'(b) = 0$. The constants c_i are such that not all of them are zero. Similarly, not all d_i are zero.

For simplicity, we will study the existence and properties of eigenvalues for the following eigenvalue problem:

$$\begin{aligned}(p(t)x'(t))' + \lambda q(t)x(t) = 0,\ t \in (a,b)\\ x(a) = 0,\ x(b) = 0.\end{aligned} \tag{2.1}$$

where $p \in C^1(a,b),\ p(t) > 0$ in $[a,b]$ and $q \in C([a,b])$. It is clear that $x(t) = 0$ is a solution of (2.1).

Definition 6.2.1 *A scalar λ is called an eigenvalue if the problem* (2.1) *has non-trivial solution $x(t)$. The function $x(t)$ is called the eigen function corresponding to λ.*

Example 6.2.1 *Find eigenvalues of the SLP:*

$$x'' + \lambda x = 0,\ x(a) = x(b) = 0.$$

Solution: case 1: First we look for negative eigenvalues. So if $\lambda < 0$. Then the characteristic polynomial is $m^2 + \lambda = 0$. So the general solution is $x(t) = c_1e^{-|\lambda|t} + c_2e^{|\lambda|t}$. Substituting the boundary conditions $x(a) = x(b) = 0$, we get

$$\begin{aligned}c_1e^{-\sqrt{|\lambda|}a} + c_2e^{\sqrt{|\lambda|}a} = 0\\ c_2e^{-\sqrt{|\lambda|}b} + c_2e^{\sqrt{|\lambda|}b} = 0\end{aligned}$$

The above equations has unique solution $c_1 = c_2 = 0$. Therefore, there is no negative eigenvalue for the problem.

Case 2: If $\lambda = 0$. Then the general solution is $x(t) = A + Bt$. This is a linear function and cannot vanish at two points on $x-$ axis unless it is zero. So $\lambda = 0$ is not an eigenvalue.

Case 3: If $\lambda > 0$. Then the general solution is $x_\lambda(t) = c_1 \sin\sqrt{\lambda}t + c_2 \cos\sqrt{\lambda}t$. Imposing the boundary conditions $x(a) = x(b) = 0$.

$$\begin{aligned}c_1 \sin\sqrt{\lambda}a + c_2 \cos\sqrt{\lambda}a = 0\\ c_1 \sin\sqrt{\lambda}b + c_2 \cos\sqrt{\lambda}b = 0\end{aligned}$$

This system has nontrival solution if and only if

$$\begin{vmatrix}\sin\sqrt{\lambda}a & \cos\sqrt{\lambda}a\\ \sin\sqrt{\lambda}b & \cos\sqrt{\lambda}b\end{vmatrix} = \sin\sqrt{\lambda}(a-b) = 0.$$

Therefore, $\sqrt{\lambda}(a-b) = k\pi, k = 1,2,...$ Then for any $\lambda_k = \frac{k^2\pi^2}{(b-a)^2}, k = 1,2,3...$ In case of $a = 0, b = \pi$, eigenvalues are $\lambda_k = k^2, k = 1,2,...$ The eigenfunctions are $x_k(t) = C\sin kt,\ C \neq 0$ a constant. □

Next we will study some qualitative properties of eigen functions. First, we will show the following:

Lemma 6.2.1 *If $q(t) > 0$ in $[a,b]$. Then the eigenvalue λ is positive.*

Proof. Multiplying the equation with $x(t)$ and integrate by parts, we find

$$-\lambda \int_a^b q(t)x^2(t)dt = \int_a^b (px(t))' x(t)dt$$
$$= (p(b)x'(b))x(b) - (p(a)x'(a))x(a) - \int_a^b p(t)x'(t)x'(t)dt$$

Since $x(a) = x(b) = 0$, we infer

$$\lambda \int_a^b q(t)x^2(t)dt = \int_a^b p(t)(x'(t))^2 dt$$

Since $p(t) > 0$ and $x(t) \not\equiv 0$, the right side is strictly positive. Hence $\lambda > 0$. □

Lemma 6.2.2 *Let$\lambda_1 \neq \lambda_2$ are two different eigenvalues and let ϕ_1, ϕ_2 be the the corresponding eigenfunctions. Then $\int_a^b \phi_\lambda \phi_\mu q(t)dt = 0$.*

Proof. Multiplying $(p\phi_1')' + \lambda_1\phi_1 = 0$ by ϕ_2 and integrating by parts, we obtain

$$\int_a^b p\phi_1'(t)\phi_2'(t)dt = \lambda_1 \int_a^b \phi_1(t)\phi_2(t)q(t)dt \tag{2.2}$$

Similarly, multiplying $(p\phi_2')' + \lambda_2\phi_2 = 0$ by ϕ_1 and integrate by parts, we get

$$\int_a^b p\phi_1'(t)\phi_2'(t)dt = \lambda_2 \int_a^b \phi_1(t)\phi_2(t)q(t)dt \tag{2.3}$$

Subtracting (2.2) from (2.3) we get

$$(\lambda_1 - \lambda_2) \int_a^b \phi_1(t)\phi_2(t)q(t)dt = 0. \tag{2.4}$$

Proof follows as $\lambda_1 \neq \lambda_2$. □

Corollary 6.2.1 *Eigenfunctions corresponding to different eigenvalues are linearly independent.*

Proof. If $\phi_2 = \alpha\phi_1$ for some real number $\alpha \neq 0$ we would have from (2.4)

$$\int_a^b \phi_1(t)\phi_2(t)q(t)dt = \alpha \int_a^b q(t)\phi_1^2(t)dt = 0,$$

a contradiction. □

Lemma 6.2.3 *Let $q(t)$ be a positive continuous function that satisfies*

$$0 < m^2 < q(t) < M^2$$

on a closed and bounded interval $[a,b]$. If $y(t)$ is a nontrivial solution $y'' + \lambda^2 q(t)y = 0$ on this interval, and if t_1 and t_2 are successive zeros of $y(t)$, then

$$\frac{\pi}{\lambda M} < t_2 - t_1 < \frac{\pi}{\lambda m} \tag{2.5}$$

Furthermore, if $y(t)$ vanishes at a and b, and $n-1$ points in (a,b), then

$$\frac{\lambda m(b-a)}{\pi} < n < \frac{\lambda M(b-a)}{\pi}. \tag{2.6}$$

Proof. Let $z(t)$ be the solution of $z'' + m^2\lambda^2 z = 0$. A nontrivial solution of this that vanishes at t_1 is $z(t) = \sin m\lambda(t-t_1)$. Since the next zero of $z(t)$ is $t_1 + \frac{\pi}{\lambda m}$ and Theorem6.1.2 tells us that t_2 must occur before this. So we have $t_2 < t_1 + \frac{\pi}{\lambda m}$ or $t_2 - t_1 < \frac{\pi}{\lambda m}$.
To prove the other side of (2.5), take $w(t)$, solution of $w'' + M^2\lambda^2 w = 0$. A non-trivial solution of this that vanishes at t_2 is $w(t) = \sin M\lambda(t-t_2)$. The zero before this is $t_2 - \frac{\pi}{\lambda M}$ and Theorem6.1.2 tells us that $t_2 - \frac{\pi}{\lambda M}$ must lie between t_1 and t_2. So we have $t_2 - \frac{\pi}{\lambda M} > t_1$ or $t_2 - t_1 > \frac{\pi}{\lambda M}$.
To prove (2.6), we first observe that there are n subintervals between $n+1$ zeros, so by (2.5) we have $b-a =$ the sum of the lengths of the n subintervals $< \frac{n\pi}{\lambda m}$, and therefore, $\frac{m\lambda(b-a)}{\pi} < n$. In the same way, we see that $b-a > \frac{n\pi}{M\lambda}$, so $n < \frac{M\lambda(b-a)}{\pi}$. □
Finally, we prove the following theorem on existence of sequence of eigenvalues.

Theorem 6.2.1 *Suppose $q(t) > 0$. Then there exist infinitely many positive eigenvalues λ_k of (2.1) such that $0 < \lambda_1 < \lambda_2 < \lambda_3 < ... < \lambda_k < \lambda_{k+1} < ....$ Moreover $\lambda_k \to \infty$.*

Proof. For each λ, let $y_\lambda(t)$ be the unique solution of $y'' + \lambda q(t)y = 0, y(a) = 0, y'(a) = 1$. It is clear from Theorem 6.1.2 that y_λ has no zeros to the right of a when $\lambda \le 0$. Our plan is to watch the oscillation behaviour of $y_\lambda(t)$ as λ increases from 0. By continuity of $q(t)$ on $[a,b]$ there exists m, M such that $0 < m^2 < q(t) < M^2$. Thus, by Theorem 6.1.2, $y_\lambda(t)$ oscillates more rapidly on $[a,b]$ than solutions of $z'' + \lambda^2 m^2 z = 0$, and less rapidly than solutions of $w'' + \lambda^2 M^2 w = 0$.

When λ is small say $\frac{\pi}{\lambda M} \ge b-a$: If t' is the immediate zero right of a, then by the previour theorem

$$t' - a > \frac{\pi}{\lambda M}$$

Since $\frac{\pi}{\lambda M} \ge b-a$, we get $t' > b$. Hence the function $y_\lambda(t)$ has no zeros in $[a,b]$ to the right of a;
When λ increases such that $\frac{\pi}{\lambda m} \le b-a$: then by above lemma, if t'' is the next zero of $y_\lambda(t)$,

$$t'' - a < \frac{\pi}{\lambda m}$$

Since $\frac{\pi}{\lambda m} \le b-a$, we get $t' < b$. Hence y_λ has at least one zero in $[a,b]$.
Also from (2.5), we see that as $\lambda \to \infty$, we get $t_2 - t_1 \to 0$ and the number of zeros of y_λ increases. Also, (2.5) also ensures that $\lambda \mapsto \alpha_k(\lambda)$, where $\alpha_k(\lambda)$ is the k^{th} zero of $y_\lambda(t)$, is continuous and increasing. Therefore, as λ starts from zero and increases to ∞, there are many values $\lambda_1, \lambda_2, ...\lambda_n$ ... for which a zero of $y_\lambda(t)$ reaches b and subsequently enters the interval (a,b) so that $y_\lambda(t)$ vanishes at a and b and has $n-1$ zeros in (a,b). Also, notice that λ_1 is such that y_{λ_1} has no zero in (a,b). i.e., The eigenfunction corresponding to smallest eigenvalue is positive. □

6.3 Method of eigenfunction expansions

As outlined in the introduction, in this section we shall study the method of solving boundary value problems using the eigenvalues and eigenfunctions of the operator.

Theorem 6.3.1 *The set of all eigen functions span the space C. That is, any $\psi \in C$ may be written*

$$\psi(t) = \sum_{i=1}^{\infty} \frac{\langle \psi, \phi_i \rangle}{\langle \phi_i, \phi_i \rangle} \phi_i(t)$$

and the series converges absolutely and uniformly.
Moreover if $f \in C([a,b])$ then the series converges with respect to the metric $d(f,g) = \left(\int_a^b |f-g|^2\right)^{1/2}$.

The series above is called **Fourier Series** of ψ. Proof of this theorem is again beyond the scope of this course.
We have the following on the differentiation of Fourier series can be proved using integration by parts.

Theorem 6.3.2 *suppose u is a solution of the equation*

$$(pu')' = f, \; u(0) = 0, \; u(1) = 0$$

and let λ_n and $\{\phi_n\}$ be the sequence of eigen values and eigen functions respectively of the corresponding eigenvalue problem:

$$(pu')' = \lambda u, \; u(0) = 0, \; u(1) = 0$$

If $u(t) = \sum c_n \phi_n$ then

$$(pu')'(t) = \sum \lambda_n c_n \phi_n$$

Proof. W.l.g assume that $\{\phi_n\}$ is ortho normal. Since u' is also continuous function, by the above theorem we can write $(pu')'(t) = \sum b_n \phi_n$ where

$$b_n = \int_0^1 (pu')'(t) \phi_n(t) dt$$

Now by integration by parts and using $u(0) = u(1) = 0$

$$\begin{aligned} b_n &= \int_0^1 (pu')'(t) \phi_n(t) dt \\ &= pu' \phi_n \Big|_0^1 - \int_0^1 pu' \phi_n' \\ &= -\int_0^1 pu' \phi_n' \end{aligned}$$

Once again using integration by parts and using the fact that $\phi_n(0) = \phi_n(1) = 0$ we get

$$b_n = \int_0^1 u (p\phi_n')' = \lambda_n \int_0^1 u \phi_n = \lambda_n c_n$$

□

Let $\mathcal{L}$ be second order linear differential operator and consider the boundary conditions $B_1 = 0, B_2 = 0$. Let $\{\lambda_n\}_{n=1}^{\infty}, \{\phi_n\}_{n=1}^{\infty}$ be sequences of eigenvalues and eigen functions. That is

$$\mathcal{L}\phi_n = \lambda_n \phi_n, \text{ in } (a,b) \; B_1(\phi_n) = 0, B_2(\phi_n) = 0$$

Now consider the boundary value problem: $f(t) \in C$,

$$\mathcal{L}u = f(t),\ t \in (a,b),\ \ B_1(u) = 0,\ B_2(u) = 0$$

By above theorem

$$f(t) = \sum_{n=1}^{\infty} f_n \phi_n(t),\ \text{where } f_n = \frac{\int_a^b f(t)\phi_n(t)dt}{\int_a^b (\phi_n)^2(t)dt}$$

Let $u(t) = \sum_{n=1}^{\infty} c_n \phi_n$ be the solution. Then assuming that the series converges absolutely and uniformly, we get

$$\mathcal{L}u = \sum c_n \mathcal{L}(\phi_n) = \sum c_n \lambda_n \phi_n$$

Then $u(t)$ solves the problem if

$$\sum c_n \lambda_n \phi_n = \sum f_n \phi_n$$

From this, we obtain the relation

$$c_n \lambda_n = f_n$$

So from this, we see that

1. If $\lambda_n \neq 0$ for all n, then $c_n = \frac{f_n}{\lambda_n}$ gives the unique solution.
2. If $\lambda_m = 0$ for some m, then solution exists if and only if $\int_a^b f(t)\phi_m(t)dt = 0$.

Example 6.3.1 *Solve the boundary value problem:* $u'' = t,\ u(0) = 0, u(1) = 0.$

Solution: The eigenvalue problem: $u'' = \lambda u,\ u(0) = u(1) = 0$ and the eigenvalues are $\lambda_n = -n^2\pi^2$ and eigen functions are $\phi_n = \sin(n\pi t)$. Then

$$f(t) = \sum_{n=1}^{\infty} \frac{\int_0^1 t \sin n\pi t dt}{\int_0^1 \sin^2 n\pi t dt} = \sum_{n=1}^{\infty} \frac{\frac{(-1)^{n+1}}{n\pi}}{\frac{1}{2}} \sin n\pi t = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin n\pi t$$

Let $u(t) = \sum_{n=1}^{\infty} c_n \sin n\pi t$. Then $u'' = -\sum c_n(n^2\pi^2)\sin n\pi t$. substituting this in the equation and comparing the coefficients, we get

$$c_n = -\frac{2(-1)^{n+1}}{n^3\pi^3}$$

Therefore,

$$u(t) = \sum_{n=0}^{\infty} \frac{2(-1)^n}{n^3\pi^3} \sin n\pi t$$

Now it is easy to check that this series converges uniformly, as $\sum |c_n| = \frac{2}{\pi^3} \sum \frac{1}{n^3} < \infty$. □

Example 6.3.2 *Solve the boundary value problem:* $u'' = 1,\ t \in (0,1)\ u(0) = 0, u'(1) = 0$

Solution: Consider the eigenvalue problem: $u'' = \lambda u,\ u(0) = 0, u'(1) = 0$. It is not difficult to show that

$$\lambda_n = -\frac{(2n-1)^2\pi}{4},\ \ \phi_n(t) = \sin \frac{2n-1}{2}\pi t$$

The Fourier coefficients of $f(t) = 1$ are

$$f_n = \int_0^1 \sin \frac{(2n-1)\pi t}{2} dt = \frac{4}{(2n-1)\pi}$$

So by assuming that the solution $u(t) = \sum c_n \sin \frac{(2n-1)\pi t}{2}$ we get

$$u'' = -\sum c_n \frac{(2n-1)^2\pi^2}{4} \sin \frac{(2n-1)\pi t}{2}$$

substituting this in the equation, we get

$$c_n = -\frac{16}{(2n-1)^3\pi^3}.$$

As in the previous problem, we can show that the corresponding series converges absolutely and uniformly. □

6.4 Singular boundary value Problems

In this section we aim to study the eigenvalue problems in the special case when $p(t)$ is zero for some t and/or the interval is unbounded. Consider the Sturm-Liouville eigenvalue problem (SLP)

$$\begin{aligned} -(p(t)y')' + q(t)y &= \lambda w(t)y, \quad t \in [a,b] \\ B_1(a) := A_1 u(a) + A_2 u'(a) &= 0 \\ B_2(a) := C_1 u(b) + C_2 u'(b) &= 0. \end{aligned}$$

Definition 6.4.1 *The (SLP) is called regular if* $(pw)(a) \neq 0$ *and* $(pw)(b) \neq 0$.

The following cases are called singular eigenvalue problem

Case 1 : $(pw)(a) = 0$. In this case $B_1(a)$ can be dropped.
Case 2 : $(pw)(b) = 0$. In this case $B_2(b)$ can dropped.
Case 3 : $(pw)(a) = 0, (pw)(b) = 0$. Then both the boundary conditions can be dropped.
In all the above cases, we call $y(t)$ a solution if y and y' are defined at the boundary points.
Case 4 : The interval $[a,b]$ may be infinite. In this case we assume that $y(t) \in H_0^1$.

Variational method

Let us look at the matrix A that has symmetric and positive definite. If λ is an eigen value and x is an eigen vector, then we may write

$$\lambda = \frac{\langle Ax, x\rangle}{\langle x, x\rangle}$$

Then the smallest eigenvalue λ_1 can be found as

$$\lambda_1 = \inf_{x\in\mathbb{R}^n} \frac{\langle Ax, x\rangle}{\langle x, x\rangle}$$

where $\langle x, x\rangle = x^T x,\ x \in \mathbb{R}^n$. So $\langle Ax, x\rangle$ is polynomial of order n. To see that this minimizer is achieved, we consider an equivalent problem

$$\lambda_1 = \min_{x\in\mathbb{R}^n}\{\langle Ax, x\rangle : \langle x, x\rangle = 1\}.$$

Suppose, $\{x_k\}$ be a sequence such that $\langle x_k, x_k\rangle = 1$ and $\langle Ax_k, x_k\rangle \to \lambda_1$. Then boundedness of the sequence implies that there is a subsequence which converges to x and $\langle Ax_k, x_k\rangle \to \langle Ax, x\rangle$. Therefore, $\langle x, x\rangle = 1$ and $\langle Ax, x\rangle = \lambda_1$. Once this minimizer is achieved, then one can use Lagrange multiplier theorem to conclude

$\langle Ax, y\rangle = \lambda_1\langle x, y\rangle$.
To obtain the second smallest eigenvalue of A, we minimize over the ortohogonal compliment of the first eigenspace. That is, let E_1 be the eigen space corresponding to λ_1. Then $\mathbb{R}^n = E_1 \bigoplus H_2$ where H_2 is the orthogonal compliment of E_1. We consider the minimization problem

$$\lambda_2 = \min_{x\in H_2} \frac{\langle Ax, x\rangle}{\langle x, x\rangle}.$$

Following the above argument, one can show that the minimizer exists. We can follow this procedure to find all eigenvalues.

Remark 6.4.1 *In this proof we used the fact that every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence. This is not true in case of infinite dimensional spaces like $C^1(a,b)$.*

The eigenvalue problems in ODE's also has variational structure. Multiplying the equation with ϕ such that $\phi(a) = \phi(b) = 0$ and integrating by parts, we get

$$\int_a^b p(t)y'(t)\phi'(t) = \lambda \int_a^b q(t)y(t)\phi(t)dt$$

If (λ_1, ϕ_1) is an eigen pair then taking $y = \phi_1$ and $\phi = \phi_1$ we get

$$\int_a^b p(t)(\phi'(t))^2 = \lambda_1 \int_a^b q(t)\phi_1^2 dt$$

It follows that

$$\lambda_1(q) = \frac{\int_a^b p(t)(\phi_1')^2(t)dt}{\int_a^b \phi_1^2(t)q(t)dt}.$$

Moreover it is clear that for all $\phi \in C^1(a,b)$ with $\phi(a) = \phi(b) = 0$, we have

$$\lambda_1(q) \leq \frac{\int_a^b p(t)(\phi')^2(t)dt}{\int_a^b q\phi^2(t)dt}. \tag{4.7}$$

We may take the metric space $C^1((a,b))$ with respect to the metric

$$d_1(f,g) = \left(\int_a^b |f(t) - g(t)|^2 q(t)\, dt + \int_a^b |f'(t) - g'(t)|^2 p(t)\, dt\right)^{1/2}$$

we remark here that the space C^1 is not complete with respect to the metric d_1. Therefore Ascoli-Arzela theorem is not useful. But we can consider the the completion of this space, say $H_0^1(a,b)$. This space H_0^1 is a complete inner product space. It is interesting to study the compact subsets of this space H_0^1. Once we understand this compactness we may define the smallest eigenvalue as in the matrix case

$$\lambda_1 = \min_{H_0^1(a,b)} \frac{\int_a^b p(t)(\phi_1')^2(t)dt}{\int_a^b \phi_1^2(t)q(t)dt}$$

and the rest of the eivenvalues.
We start our discussion from (4.7). From this equation, one can show that

$$\lambda(q) = \inf_{\phi \in H_0^1(a,b)} \left\{ \int_a^b p(t)(\phi')^2(t)dt : \int_a^b \phi^2(t)q(t)dt = 1 \right\}.$$

Proof of this requires good amount of functional analysis. We give an idea of the proof this characterization in case of Symmetric positive definite matrix. Let A be an $n \times n$ positive definite symmetric matrix.

Similarly, in case of Boundary value problems, we have the following characterization for $k \geq 2$

$$\lambda_k = \min_{\phi \in H_k} \left\{ \int_a^b p(t)(\phi')^2(t)dt : \int_a^b q(t)(\phi)^2(t)dt \right\},$$

where H_k is the orthogonal compliment of $span\{\phi_1, \phi_2, ...\phi_{k-1}\}$

Proof of this involves Hilbert space theory and is beyond the scope of this course.

6.5 Eigenvalue problems of Mathematical physics

Consider the Legendre equation

$$(1-t^2)y'' - 2ty' + \nu(\nu+1)y = 0.$$

We know that there are two linearly independent power series solutions $P_\nu(t)$ and $Q_\nu(t)$ which converge in $(-1, 1)$. This equation may be written as

$$-\Big((1-t^2)y'\Big)' = \lambda\nu(\nu+1)y.$$

So comparing with (SLP), we have $p(t) = 1-t^2, q = 0$ and $w(t) = 1$. $p(\pm 1) = 0$. This is a singular eigenvalue problems and as we mentioned, both boundary conditions can be dropped. We will see that if the solution is an infinite series, then it does not converge at ± 1. Indeed, the series solution is determined by the recurrence relation

$$c_{k+2} = \frac{(k-\nu)(\nu+k+1)}{(k+2)(k+1)} c_k, k = 0, 1, 2...$$

So, the ratio of two consecutive terms in the power series at $t = 1$ will be,

$$\begin{aligned} \frac{c_{k+2}}{c_k} &= \frac{(k-\nu)(k+\nu+1)}{(k+2)(k+1)} \\ &\geq \frac{1}{k+1}\left(\frac{(k-\nu)(\nu+k+1)}{k+2}\right) \\ &\geq \frac{k-\nu}{k+1} \sim \frac{k}{k+1}. \end{aligned}$$

The series $\sum \frac{1}{n}$ also will have the same ratio $\frac{a_{k+1}}{a_k} = \frac{k}{k+1}$. Hence the series solution does not converge at $t = 1$. A similar argument shows that the series solution does not converge at $t = -1$.

Another way of defining the series solution at $t = 1$ is to find the solution around $t = 1$ with series in the form $\sum c_k(t-1)^k$. In this case, $t = 1$ is a regular singular point and so taking the solution in the form $\sum c_k(1-t)^{k+r}$, we see that the equation equivalently can be written as

$$(t-1)^2y'' + (t-1)\frac{2t}{t+1}y' + \frac{1-t}{1+t}\nu(\nu+1)y = 0$$

So the indicial polynomial is $r(r-1)+\alpha_0 r+\beta_0 = 0$ where α_0 is the constant term in the series expansion of $\frac{2t}{t+1}$ at $t = 1$ and β_0 is the constant term in the series expansion of $\frac{1-t}{1+t}$. So $\alpha_0 = 1, \beta_0 = 0$. Therefore indicial polynomial is $r^2 = 0$. Therefore one of the solutions has $log(t-1)$ as a factor which is not defined at $t = 1$. The other solution is given by $\sum a_k(t-1)^k$, where

$$a_{k+1} = \frac{(\nu-k)(\nu+k+1)}{2(k+1)^2}a_k,\ k \geq 0.$$

It is easy to see by Ratio test that the series converges in $|t-1| < 2$. But this solution is not defined at $t = -1$. To see this,

$$\frac{a_{k+1}(-2)^{k+1}}{a_k(-2)^k} = \frac{(\nu-k)(\nu+k+1)}{2(k+1)^2}(-2) = \frac{(k-\nu)(k+\nu+1)}{(k+1)^2}$$

If ν is not a positive integer, then as earlier one can show that this ratio is similar to the ratio of $\sum \frac{1}{n}$.

Therefore, the infinite series solution that is defined at $t = 1$ does not converge at $t = -1$. Similarly, we can show that the infinite series solution that is defined at $t = -1$ is not defined at $t = 1$. Therefore, the only solution that is defined at $t = \pm 1$ is the polynomial solution which exists only when ν is a positive integer. Hence we proved the following:

Theorem 6.5.1 *The sequence of eigenvalues and eigenfunctions of the singular SLP:*

$$((1-t^2)y')' + n(n+1)y = 0,\ \ -1 < t < 1$$

are $\lambda_n = n(n+1), \phi_n = P_n(t)$.

Hermite and Laguerre polynomials

The equation $y'' - 2ty' + 2\alpha y = 0,\ t \in \mathbb{R}$ is the Hermite equation. This may be written in the self-adjoint form as

$$-(e^{t^2}y')' = 2\alpha e^{-t^2}y,\ t \in \mathbb{R}.$$

Here $p(t) = e^{-t^2}, q = 0, w = e^{-t^2}, \lambda = 2\alpha$. This is singular SLP because the domain in unbounded. We can find two linearly independent power series solutions for each real number α. If we write $x(t) = \sum c_k t^k$, then the coefficients are determined by the recurrence relation is

$$a_{n+2} = -\frac{2(\alpha-n)}{(n+1)(n+2)}a_n$$

So, if $\alpha = m$ a positive integer, then one of the solution is a polynomial. So for each m, we get a polynomial solution $H_m(t)$ which can be defined as

$$H_m(t) = (-1)^m e^{t^2}\frac{d^m}{dt^m}e^{-t^2}$$

In this case, it can be shown that if α is not a positive integer, then the infinite series solution $H_\alpha(t)$ is not eigenfunction in the sense that it does not belong to $L^2(\mathbb{R}, e^{-t^2})$. i.e.,

$$\int_{\mathbb{R}} H_\alpha^2(t)e^{-t^2}dt < \infty \text{ if and only if } \alpha = 1, 2, 3, 4, ...$$

Laguerre equation: The equation $ty'' + (1-t)y' + au,\ t \in [0,\infty)$ is the Laguerre equation. In the self adjoint form, this equation is

$$-(te^{-t}y')' = ae^{-t}y,\ t \in [0,\infty).$$

Here $p(t) = t$ and the problem is singular as $p(0) = 0$ and interval is infinite. The polynomial solutions can be obtained in similar fashion as in the above cases if $\lambda = n$ a positive integer. One can show that the only solutions defined at $t = 0$ and $L^2([0,\infty) : e^{-t})$ are polynomial solutions.

Bessel functions and eigenvalue problems

Consider the Bessel equation $t^2y'' + xy' + (t^2 - n^2)y = 0$. Taking the transformation, $t = az$, this equation transforms to

$$-\frac{d}{dz}(z\frac{dy}{dz}) - \frac{n^2}{z}y = a^2zy.$$

This is in the form of $-(py')' + qy = \lambda w(z)y$ with $p(z) = z, q(z) = \frac{n^2}{z}, w(z) = z$ and $\lambda = a^2$. The solution is $J_n(az)$. So if we consider the eigenvalue problem

$$-\frac{d}{dz}(z\frac{dy}{dz}) - \frac{n^2}{z}y = \lambda zy, \text{ in } [0,c]$$
$$y(c) = 0.$$

Then the eigenvalues are $\lambda_j = \alpha_j^2$ and eigenfunctions are $\phi_j = J_n(\alpha_j t)$ where α_j are roots of $J_n(\alpha c) = 0$.

6.6 Green's functions

The following is known as Langrange identity

$$\int_a^b (\mathcal{L}u)v - u(\mathcal{L}^*v)dt = [J(u,v)]_a^b$$

where

$$J(u,v) = a_0(vu' - uv') + (a_1 - a_0')uv$$

Definition 6.6.1 *An operator $\mathcal{L}$ is called self adjoint if $\mathcal{L} = \mathcal{L}^*$.*

In case of (0.1), the operator is self-adjoint if $a_0' = a_1$. In this case we may write the operator

$$\mathcal{L}u = (pu')' + qu,\ \ J = p(vu' - uv')$$

where p is a differentiable function and q is continuous function such that $p(t) > 0$ in the given interval $[a,b]$.

We consider the self-adjoint operator $\mathcal{L}$. The boundary conditions $B_1(u)(0) = 0$ and $B_2(u)(1) = 0$ are called separated boundary conditions like

$$B_1(u)(0) = c_1u(0) + c_2u'(0)$$

and

$$B_2(u)(1) = c_3u(1) + c_4u(1) = 0$$

for some constants c_1, c_2, c_3 and c_4.

Notation: We denote the BVP $\mathcal{L}u = f,\ B_1(u) = 0, B_2(u) = 0$ as $(\mathcal{L}, B_1, B_2)$.

Definition 6.6.2 *The problem $(\mathcal{L}^*, B_1^*, B_2^*)$ is called adjoint of $(\mathcal{L}, B_1, B_2)$ if $[J(u,v)]_a^b = 0$. That is,*

$$\int_a^b [\mathcal{L}u](t)v(t)dt = \int_0^1 u(t)[\mathcal{L}^* v](t)dt$$

for all continuous functions satisfying $B_1(u) = 0, B_2(u) = 0$ and $B_1^*(v) = 0,\ B_2^*(v) = 0$.

Example 6.6.1 *Consider the boundary value problem*

$$L(y) = y'' + y' - 2y = 0,\ B_1(y) = y(0) - y'(0) = 0,\ \ B_2(y) = y(1) - y'(1) = 0.$$

For any z, using integration by parts, we get

$$\begin{aligned}\int_0^1 (y'' + y' - 2y)\,z dt &= (zy' - yz' + yz)\Big|_0^1 + \int_0^1 (z'' - z' - 2z)\,y dt \\ &= y(1)\left(2z(1) - z'(1)\right) + y(0)\left(z'(0) - 2z(0)\right) + \int_0^1 (z'' - z' - 2z)\,y dt\end{aligned}$$

Therefore $L^*(z) = z'' - z' - 2z$, $B_1^* = 2z(0) - z'(0) = 0$, $B_2^* = 2z(1) - z'(1)$. □

Definition 6.6.3 *The problem $(\mathcal{L}, B_1, B_2)$ is called self-adjoint if $\mathcal{L} = \mathcal{L}^*, B_1^* = B_1, B_2^* = B_2$.*

6.6.1 Green's functions

We consider the operator $\mathcal{L}$ of the form $\mathcal{L}(y) = (py')' + qy$ for some positive and differentiable function p and continuous function q. Our aim is to solve the boundary value problem for a given $f \in C(a,b)$

$$\mathcal{L}u = f(x),\ B_1(u) = C_1u(0) + c_2u'(0) = 0,\ \ B_2(u) = C_3u(1) + C_4u'(1) = 0.$$

where $(C_1, C_2) \not\equiv 0$ and $(C_3, C_4) \not\equiv 0$.

Definition 6.6.4 *A function $G(t,\xi)$ is called Green's function if it satisfies*

(1) $G(t,\xi) = G(\xi,t)$ for all $t,\xi \in [0,1]$
(2) $G(t,\xi)$ is twice differentiable for $t < \xi$ and $t > \xi$ but continuous at all $t = \xi$
(3) $G(t,\xi)$ satisfies $B_1(G) = 0$ and $B_2(G)(1) = 0$ in the variable t
(4) The jump in the derivative of G, $[\frac{\partial G}{\partial t}]\Big|_{t=\xi} = \frac{\partial G}{\partial t}\Big|_{\xi^+} - \frac{\partial G}{\partial t}\Big|_{\xi^-} = \frac{1}{p(t)}$.
(5) $\mathcal{L}(G) = 0$ for $t \neq \xi$.

We show the existence of solutions with respect to the following homogeneous problem:

$$\mathcal{L}(u) = 0 \textit{ in } (0,1),\ B_1(u)(0) = 0,\ B_2(u)(1) = 0 \tag{6.1}$$

Remark 6.6.1 *If u and v satisfy the equation $(pw')' + qw = 0$ then $p(t)W(u,v)(t)$ is constant.*

Proof.

$$\begin{aligned}(pW)' &= (puv' - pu'v)' \\ &= u(pv')' + u'pv' - v(pu')' - v'pu' \\ &= -quv + quv = 0.\end{aligned}$$

Hence pW is constant. □

We have the following existence theorem

Theorem 6.6.1 *If* (6.1) *has only trivial solution, then there exists a unique Green's function.*

Proof. Suppose there are two such functions $G^1(t,\xi)$ and $G^2(t,\xi)$. Then by taking $G(t,\xi) = (G^1 - G^2)(t,\xi)$, we see that G satisfies the equation for $t \neq \xi$ and $B_1(G)(0) = B_2(G)(1) = 0$. Moreover the jump in $\frac{\partial G}{\partial t}$ is 0. Now we see that

$$pu'' = -p'u' - qu$$

implies that the jump in $p\frac{\partial^2 G}{\partial^2 t}(t,\xi)$ is 0. Therefore, G satisfies $\mathcal{L}u = 0$ along with both boundary conditions. Hence $G(t,\xi) \equiv 0$. To prove the existence, let $u_1(t)$ be a solution of $\mathcal{L}u_1 = 0, B_1(u_1)(0) = 0$ and let u_2 be linearly independent from u_1 and satisfy $\mathcal{L}u_2 = 0, B_2(u)(1) = 0$. Let

$$G(t,\xi) = \begin{cases} \frac{u_1(t)u_2(\xi)}{pW} & t < \xi \\ \frac{u_1(\xi)u_2(t)}{pW} & t \geq \xi \end{cases} \tag{6.2}$$

where W is the Wronskian of u_1 and u_2. Then it is not difficult to verify that $G(t,\xi)$ satisfies $(1)-(5)$ and hence is the Green's function. □

Theorem 6.6.2 *If* (6.1) *has only trivial solution, Then the non-homogeneous problem*

$$\mathcal{L}(u) = f(t),\ B_1(u)(0) = 0,\ B_2(u)(1) = 0 \tag{6.3}$$

has unique solution $u(x)$ given by

$$u(t) = \int_0^1 G(t,\xi)f(\xi)d\xi.$$

Proof. Proof follows by writing

$$G(t,\xi) = \begin{cases} G_1(t,\xi) & t < \xi \\ G_2(t,\xi) & t > \xi \end{cases}$$

where $G_1(.,\xi)$ satisfied the first boundary condition and $G_2(.,\xi)$ satisfies the second boundary condition along with $\mathcal{L}G_1(.,\xi) = 0$, for $t < \xi$ and $\mathcal{L}G_2(.,\xi) = 0$ for $t > \xi$. Writing

$$u(t) = \int_0^t G_1(t,\xi)f(\xi)d\xi + \int_t^1 G_2(t,\xi)f(\xi)d\xi.$$

Now using Newton-Leibniz formula, it is easy to see that

$$u'(t) = \int_0^t \frac{\partial}{\partial t}G_1(t,\xi)f(\xi)d\xi + \int_t^1 \frac{\partial}{\partial t}G_2(t,\xi)f(\xi)d\xi + f(t)\text{Jump in } G \text{ } at\ t = \xi$$

Since G is continuous at $t = \xi$, we get Jump in G at $t = \xi$ is zero and hence differentiating again

$$u''(t) = \int_0^t \frac{\partial^2}{\partial t^2} G_1(t,\xi) f(\xi) d\xi + \int_t^1 \frac{\partial^2}{\partial t^2} G_2(t,\xi) f(\xi) d\xi + f(t)\text{Jump in } \frac{\partial G}{\partial t} \; at \; t = \xi$$

Substituting this in the equation we gt

$$\begin{aligned}\mathcal{L}u &= \int_0^t \mathcal{L}[G_1(.,\xi)] f(\xi) d\xi + \int_t^1 \mathcal{L}[G_2(.,\xi)] f(\xi) d\xi + [p(t)] \frac{1}{p(t)} f(t) \\ &= f(t)\end{aligned}$$

Hence the proof. □

Example 6.6.2 *Find the Green's function of* $u'' = 0,\ 0 < t < 1, u(0) = u(1) = 0$ *if it exists*

The general solution is $u(t) = at + b$. Then $0 = u(0) = b$ and $0 = u(1) = a$. Thus $u(t) \equiv 0$. Therefore Green's function exists and is unique. Then using (6.2), with $u_1 = t, u_2 = t - 1$ we have $W = 1$ and

$$G(t,\xi) = \begin{cases} t(1-\xi) & t \le \xi \\ \xi(1-t), & t \ge \xi. \end{cases}$$

□

It is not difficult to prove the above existence result for problem with mixed boundary conditions. We explain the construction through the following example

Example 6.6.3

$$\mathcal{L}(y) = y'',\ B_1(y) = y(0) + y(1) = 0,\ B_2(y) = y'(0) + y'(1) = 0$$

It is easy to see that $y = 0$ is the only solution of the homogeneous problem. Now let us assume the Green's function in the form

$$G(t,\xi) = \begin{cases} A + Bt & t \le \xi \\ C + Dt & t \ge \xi \end{cases}$$

We will find A, B, C, D such that G satisfies (1)-(4) in the definition of Green's function.

$$\begin{aligned} B_1(G) = 0 &\implies G(0,\xi) + G(1,\xi) = 0 \\ &\implies A + C + D = 0 \end{aligned} \tag{6.4}$$

$$\begin{aligned} B_2(G) = 0 &\implies \frac{\partial G}{\partial t}(0,\xi) + \frac{\partial G}{\partial t}(1,\xi) = 0 \\ &\implies B + D = 0 \end{aligned} \tag{6.5}$$

G is continuous at $t = \xi$ implies

$$(C_A) + (D - B)\xi = 0 \tag{6.6}$$

$$\left[\frac{\partial G}{\partial x}\right]_{x=\xi} = \frac{1}{p} \implies D - B = 1 \tag{6.7}$$

$$(6.5) \text{ and } (6.7) \implies D = \frac{1}{2} \text{ and } B = -\frac{1}{2}$$

Using other two equations we get

$$C = -\frac{\xi}{2} - \frac{1}{4}, A = \frac{\xi}{2} - \frac{1}{4}$$

Hence

$$G(t, xi) = \begin{cases} \frac{\xi}{2} - \frac{1}{4} - \frac{t}{2} & t \le \xi \\ -\frac{\xi}{2} - \frac{1}{4} + \frac{t}{2} & t \ge \xi. \end{cases}$$

□

In general, we can consider any two linearly independent solutions u_1 and u_2 of $\mathcal{L}u = 0$ and write the Green's function as

$$G(x,t) = \begin{cases} Au_1(t) + Bu_2(t) & t \le \xi \\ Cu_1(t) + Du_2(t) & t \ge \xi. \end{cases}$$

and find the constants A, B, C and D so that G satisfies the conditions (1) to (4) of the definition of Green's function.

We have the following important property of Green's function

Theorem 6.6.3 *If $(\mathcal{L}, B_1, B_2)$ is self adjoint then the Greens function is symmetric.*

Proof. Suppose $g(t,s)$ is the Green function for the problem $\mathcal{L}u = f(t), B_1(u) = 0, B_2(u) = 0$ and $h(t,s)$ is the Greens function for $\mathcal{L}^* u = f(t), B_1^*(u) = 0, B_2^*(u) = 0$. Then

$$\mathcal{L}[g](t,s) = \delta(t-s), B_1(g) = 0, B_2(g) = 0$$
$$\mathcal{L}^*[h](t,s) = \delta(t-s), B_1^*(h) = 0, B_2^*(h) = 0$$

Let $v(t)$ be the solution of adjoint problem. That is

$$v(t) = \int_a^b h(t,s) f(s) ds \tag{6.8}$$

Now multiply the equation $L[g] = d(t-s)$ with v and integrate

$$\begin{aligned} v(s) = \int_a^b \delta(t-s) v(t) dt &= \int_a^b \mathcal{L}[g](t,s) v(t) dt \\ &= \int_a^b g(t,s) \mathcal{L}^* v dt \\ &= \int_a^b g(t,s) f(t) dt \end{aligned}$$

That is

$$v(t) = \int_a^b g(s,t) f(s) ds$$

Therefore from (6.8) we get $h(t,s) = g(s,t)$. For self-adjoint problem $g = h$ implies Green's function is symmetric. □

6.6.2 Modified Green's functions

We turn to study the case when (6.1) has non-trivial solution. In this case we know that the non-homogeneous problem will have infinitely many solutions. Indeed we have

Theorem 6.6.4 *Let u_0 be a nontrivial solution of* (6.1). *Then the non-homogeneous problem* (6.3) *has a solution if and only if*

$$\int_a^b f(t)u_0(t)dt = 0.$$

Proof. $\Longrightarrow$: Let $u(t)$ be solution of (6.3). Then self-adjoint nature of the problem,

$$\int_a^b f(t)u_0(t) = \int_a^b [\mathcal{L}u]u_0 dt = \int_a^b u[\mathcal{L}u_0]dt = 0$$

Conversely, let u_0 be the solution of (6.1). We can construct a function v of $\mathcal{L}u = 0$ such that $B_1(u)(0) \neq 0$, $B_2(u)(1) \neq 0$. Then u_0 and v are linearly independent. Then by the variation of parameters formula, a particular solution u_p of $\mathcal{L}u = f$ is given by

$$u_p(t) = y_1(t)v(t) + y_2(t)u_0(t)$$

such that

$$y_1(t) = \frac{-1}{pW}\int_a^t u_0(\xi)f(\xi)d\xi, \text{ and } y_2(t) = \frac{1}{pW}\int_a^t v(\xi)f(\xi)d\xi$$

Then the general solution of $\mathcal{L}u = f$ is given by

$$u(t) = u_p(t) + c_1 v(t) + c_2 u_0(t)$$

Then substituting the boundary conditions, we have

$$0 = B_1(u) = B_1(u_p) + c_1 B_1(v) \tag{6.9}$$

$$0 = B_2(u) = B_2(u_p) + c_1 B_2(v) \tag{6.10}$$

We note that $u_p'(x) = y_1 v' + y_2 u_0'$. Thus, $u_p(a) = u_p'(a) = 0$, and $u_p(b) = y_2(b)u_0(b)$, $u_p'(b) = y_2(b)u_0'(b)$ since $y_1(b) = \frac{1}{pW}\int_a^b u_0(\xi)f(\xi)d\xi = 0$ (from $\int f u_0 = 0$). Then

$$B_1(u_p) = c_1 u_p(a) + c_2 u_p'(a) = 0$$

and

$$B_2(u_p) = c_3 u_p(b) + c_4 u_p'(b) = y_2(b)B_2(u_0) = 0.$$

From (6.9), $c_1 = 0$ since $B_1(v) \neq 0$ and $B_2(v) \neq 0$ and c_2 is arbitrary. Hence,

$$u(t) = \frac{1}{pW}\int_a^t [u_0(\xi)v(t) - v(\xi)u_0(t)]f(\xi)d\xi + c_2 u_0(t)$$

is a solution of (6.3) for any c_2. □

This motivates us to the case when $(\mathcal{L}, B_1, B_2)$ is self-adjoint and homogeneous problem has non-trivial solution, we want to write the solution

$$u(t) = \int_a^b G(t,\xi)f(\xi)d\xi$$

We will see below that it is not possible to have a $G(t,xi)$ as in the previous case. So We will develop a more general theory of Green's functions. The Green's function satisfies

$$\mathcal{L}[G](t,\xi) = 0,\ t \neq \xi$$

When $t = \xi$ we saw that the derivative has jump discontinuity. Therefore, the second derivative will a more general function known as Dirac delta function. This Dirac delta distribution is understood as the derivative of well known Heaviside function

$$H(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

we see that the derivative is zero everywhere except $t = 0$. At $t = 0$ there is a infinite slope, though we know that H is not differentiable at $t = 0$. This derivative is known as Dirac delta distribution

$$\delta(t) = H'(t)$$

We understand this function by some properties of approximations

$$f_n(t) = \begin{cases} 0, & |t| > \frac{1}{n} \\ \frac{n}{2} & |t| < \frac{1}{n} \end{cases}$$

Then $\int_{\mathbb{R}} f_n(t)dt = 1$ and $f_n(t) \to 0$ for $t \neq 0$ and the limit is undefined at $t = 0$. The limit function is called Dirac delta function $\delta(t)$. This limit function satisfies the following

1. $\delta(t) = 0$ for $t \neq 0$
2. $\int_{\mathbb{R}} \delta(t)dt = 1$
3. $\int_{\mathbb{R}} \delta(t-a)f(t)dt = f(a)$ for any continuous function f

If a function f is differentiable then integration by parts, we have

$$\int_{\mathbb{R}} f'(t)\phi(t)dt = -\int_{\mathbb{R}} f(t)\phi'(t)$$

for all differentiable functions ϕ that vanish outside a compact interval. Motivated from this we may define the generalized derivative of a function as follows

Definition 6.6.5 *A function g is called generalized derivative of f if the following holds for all* $\phi \in C_c^1(\mathbb{R})$.

$$\int_{\mathbb{R}} g(t)\phi(t)dt = -\int_{\mathbb{R}} f(t)\phi'(t)$$

In this case we formally write $g(t) = f'(t)$

By this definition we can write the derivative of $H(t)$ as

$$\int_{\mathbb{R}} H'(t)\phi(t)dt = -\int_{\mathbb{R}} H(t)\phi'(t) = -\int_0^{\infty} \phi'(t)dt = \int_{\mathbb{R}} \delta(t)\phi(t)dt$$

Therefore $H'(t) = \delta(t)$.

In this sense we say Green's function satisfies the differential equation

$$\mathcal{L}[G](t,\xi) = \delta(t-\xi)$$

If the homogeneous problem has a non-trivial solution $u_0(t)$. Then y theorem 6.6.4, such G will exists if

$$\int_a^b \delta(t-\xi)u_0(\xi)d\xi = 0$$

But by the above properties of Dirac delta, this integral is equal to $u_0(t)$. Therefore Green's function does not exist in this case. So we take the following corrected problem

$$\mathcal{L}[G_m](t,\xi) = \delta(t-\xi) - u_0(t)u_0(\xi) \tag{6.11}$$

where u_0 is the normalized nontrivial solution such that $\int_a^b u_0^2(\xi)d\xi = 1$. Again by theorem 6.6.4 the equation (6.11) will admit solution if

$$\int_a^b \left[\delta(t-\xi) - u_0(t)u_0(\xi)\right] u_0(\xi)d\xi = 0$$

It is easy to see that the above holds thanks to the following facts

$$\int_a^b \delta(t-\xi)u_0(\xi)d\xi = u_0(t), \ \int_a^b u_0^2(\xi)d\xi = 1.$$

The solution $G_m(t,\xi)$ to the problem (6.11) is called modified Green's function.

If $u(t)$ is a solution of

$$\mathcal{L}u = f(t), \text{ in } (a,b), \ B_1(u) = 0, B_2(u) = 0$$

and G_m is a solution of (6.11). Then by Lagrange's identity we have

$$\begin{aligned}
\int_a^b (u\mathcal{L}G_m - G_m\mathcal{L}u)\,dt &= \int_a^b u(t)\left[\delta(t-\xi) - u_0(t)u_0(\xi)\right]dt - \int_a^b G_m(t,\xi)f(t)dt \\
&= u(\xi) - \left(\int_a^b u(t)u_1(t)dt\right)u_1(\xi) - \int_a^b G_m(t,\xi)f(t)dt \\
&= u(\xi) - Cu_1(\xi) - \int_a^b G_m(t,\xi)f(t)dt
\end{aligned}$$

where $C = \left(\int_a^b u(t)u_1(t)dt\right)$. The left side is zero as u and G_m both satisfy the boundary conditions. Therefore by the symmetry of G_m we get

$$u(t) = Cu_1(t) + \int_a^b G_m(t,\xi)f(\xi)d\xi.$$

where C is an arbitrary constant. By taking $C = 0$ we get

$$u(t) = \int_a^b G_m(t,\xi)f(\xi)d\xi.$$

Construction of modified Green's function

1. Let u_0 be a nontrivial function that satisfies

$$\mathcal{L}u = 0,\ B_1(u) = 0,\ B_2(u) = 0 \quad \text{and} \quad \int_a^b u_0^2 dt = 1,$$

and let v_0 be a solution of

$$\mathcal{L}v_0 = u_0$$

2. Let u_1 and u_2 be two linearly independent solutions of $\mathcal{L}u = 0$.
3. Consider the function

$$G_m(t,\xi) = \begin{cases} C_1u_1 + C_2u_2 - u_0(\xi)v_0(t) & t < \xi \\ C_3u_1 + C_4u_2 - u_0(\xi)v_0(t) & t > \xi \end{cases}$$

4. Find C_i's satisfying

   a. $B_1[G_m](.,\xi) = 0$ and $B_2[G_m](.,\xi) = 0$
   b. $G_m(t,\xi)$ is continuous at $t = \xi$
   c. $\frac{\partial G_m}{\partial t}$ has jump at $t = \xi$ equal to $\frac{1}{p}$

   Then it is easy to see that

$$\mathcal{L}[G_m](t,\xi) = \delta(t-\xi) - u_0(t)u_0(\xi) \tag{6.12}$$

Theorem 6.6.5 *Suppose f satisfies $\int_a^b f(t)u_0(t) = 0$ and G is as in (1) - (4) above. Then*

$$u(t) = \int_a^b G_m(t,\xi)f(\xi)d\xi$$

satisfies $\mathcal{L}u = f,\ B_1(u) = 0,\ B_2(u) = 0.$

Proof. From (6.12),

$$\begin{aligned}\mathcal{L}u &= \int_a^b \delta(t-\xi)f(\xi) - \int_a^t u_0(t)u_0(\xi)f(\xi)d\xi - \int_t^b u_0(t)u_0(\xi)f(\xi)d\xi \\ &= f(t) - u_0(t)\left[\int_a^t u_0(\xi)f(\xi)d\xi + \int_t^b u_0(\xi)f(\xi)d\xi\right] \\ &= f(t) - u_0(t)\int_a^b f(\xi)u_0(\xi)d\xi = f(t)\end{aligned}$$

thanks to the assumption that $\int fu_0 = 0$. Hence the proof. □

Theorem 6.6.6 *The modified Greens function G_m is symmetric if*

$$\int_a^b G_m(t,s)u_0(t)dt = 0,\ \forall\ s.$$

Proof. For s_1, s_2, consider $G_m(t,s_1), G_m(t,s_2)$ that satisfy

$$\mathcal{L}G_m(t,s_1) = \delta(t-s_1) - u_0(t)u_0(s_1) \tag{6.13}$$

$$\mathcal{L}G_m(t,s_2) = \delta(t-s_2) - u_0(t)u_0(s_2) \tag{6.14}$$

Cross multiply and integrate from a to b, we get

$$\begin{aligned}
&\int_a^b G_m(t,s_2)\mathcal{L}[G_m(t,s_1)] - G_m(t,s_1)\mathcal{L}[G_m(t,s_2)]dt \\
&\qquad\qquad = \int_a^b [\delta(t-s_1)G_m(t,s_2) - \delta(t-s_2)G_m(t,s_1)]dt \\
&\qquad\qquad\quad - \int_a^b [G_m(t,s_2)u_0(t)u_0(s_1) - G_m(t,s_1)u_0(t)u_0(s_2)]\,dt \\
&\qquad\qquad = G_m(s_1,s_2) - G_m(s_2,s_1) \\
&\qquad\qquad\quad - u_0(s_1)\int_a^b G_m(t,s_2)u_0(t)dt + u_0(s_2)\int_a^b G_m(t,s_1)u_0(t)dt \\
&\qquad\qquad = G_m(s_1,s_2) - G_m(s_2,s_1) + 0
\end{aligned}$$

thanks to the hypothesis $\int_a^b G_m(t,s)u_0(t)dt = 0$. From the Lagrange's identity the left side is zero. Therefore $G_m(s_1,s_2) = G_m(s_2,s_1)$ for all s_1, s_2. □

Example 6.6.4 *Consider the problem* $u'' = \sin 2\pi t,\ 0 < t < 1,\ u'(0) = u'(1) = 0.$

We can directly compute the normalized solution of homogeneous problem $u_0(t) = 1$ and a particular solution of $y'' = 1$ is $\frac{t^2}{2}$. Then the modified Green's function

$$G_m(t,\xi) = \begin{cases} A + Bt - \frac{t^2}{2} & t < \xi \\ C + Dt - \frac{t^2}{2} & t > \xi \end{cases}$$

$$\begin{aligned}
&\frac{\partial G}{\partial t}(0,\xi) = 0 \implies B = 0 \\
&\frac{\partial G}{\partial t}(1,\xi) = 0 \implies D = 1 \\
&G(t,\xi) \text{ is continuous at } t = \xi \implies C - A + \xi = 0 \\
&\frac{\partial G}{\partial t} \text{ has jump discontinuity at } \frac{1}{p} \implies D = 1
\end{aligned}$$

Therefore

$$G_m(t,\xi) = \begin{cases} A - \frac{t^2}{2} & t < \xi \\ A - \xi + t - \frac{t^2}{2} & t > \xi \end{cases}$$

Now by the symmetry condition

$$\int_0^1 G(t,\xi)dt = 0$$

implies

$$\int_0^1 (A - \frac{t^2}{2})dt + \int_\xi^1 (A - \xi + t - \frac{t^2}{2})dt = 0 \implies A = \xi - \frac{\xi^2}{2} - \frac{1}{3}$$

Hence

$$G_m(t,\xi) = \begin{cases} \xi - \frac{t^2+\xi^2}{2} - \frac{1}{3} & t < \xi \\ t - \frac{t^2+\xi^2}{2} - \frac{1}{3} & t > \xi \end{cases}.$$

6.6.3 Non self-adjoint problems

From the previous cases we observe the following

Theorem 6.6.7 *Suppose if the homogeneous problem admits a non-trivial solution then its adjoint problem also admits a non-trivial solution.*

Theorem 6.6.8 *Suppose if the homogeneous problem admits only trivial solution then there exists a unique Green's function.*

Theorem 6.6.9 *The non-homogeneous problem admits a solution if and only if*

$$\int_a^b f u_0 = 0$$

for all u_0 *such that* $\mathcal{L}^* u_0 = 0, B_1^*(u_0) = 0, B_2^*(u_0) = 0.$

In this case again we would like to construct a function $G(t,\xi)$ satisfying

$$\mathcal{L}[G] = \delta(t-\xi) - \sum_{i=1}^{m} v_i(x) v_i(\xi)$$

such a function exists. Indeed, let w be a solution of homogeneous adjoint probem, then $w = \sum c_i v_i$ and

$$\int_a^b \delta(t-\xi)w(\xi)d\xi - \sum_{i=1}^{m} \int_a^b v_i(t)v_i(\xi)w(\xi)d\xi = w(t) - \sum_{i=1}^{m} v_i(t) \int_a^b \sum_{l=1}^{m} v_i(\xi)c_l v_l(\xi)d\xi$$
$$= w(t) - \sum_{i=1}^{m} v_i(t)c_i = w(t) - w(t) = 0.$$

Construction of modified Green's function:

1. Find orthonormal basis $\{v_1, v_2, ...v_m\}$ of the solution set of adjoint problem
$$\mathcal{L}^* u = 0,\ B_1(u) = 0,\ B_2(u) = 0,\ \int_a^b v_i^2 = 1.$$
2. Let u_i be solution of $\mathcal{L} u_i = v_i$. Then consider
$$G(t,\xi) = \begin{cases} Ay_1 + By_2 - \sum_{i=1}^m u_i(t)v_i(t) & t < \xi \\ Cy_1 + Dy_2 - \sum_{i=1}^m u_i(t)v_i(t) \end{cases}$$
where y_1 and y_2 are two linearly independent solutions of $L(y) = 0$.

Example 6.6.5 *Consider the problem* $\mathcal{L}y = y''$, $B_1(y) = y(0) + y(1) = 0$, $B_2(y) = y'(0) - y'(1) = 0$

It is easy to check that the adjoint problem is

$$\mathcal{L}^* z = z'',\ B_1^*(z) = z(1) - z(0)\ \ B_2^*(z) = z'(1) + z'(1)$$

Then $v_0(t) = 1$ is the solution of adjoint problem. To construct Green's function we need to solve

$$\mathcal{L}u_0 = v_0$$

Therefore $u_0(t) = \frac{t^2}{2}$. So we take

$$G(t,\xi) = \begin{cases} A + Bt - \frac{t^2}{2} & t < \xi \\ C + Dt - \frac{t^2}{2} & t > \xi \end{cases}$$

Now the 4 conditions of Green's function imply

$$G(0,\xi) + G(1,\xi) = 0 \implies A + C + D = \frac{1}{2} \tag{6.15}$$

$$\frac{\partial G}{\partial x}(0,\xi) + \frac{paG}{\partial x}(1,\xi) = 0 \implies B - D + 1 = 0 \tag{6.16}$$

$$G\text{is continuous at } t = \xi \implies C - A + (D - B)\xi = 0 \tag{6.17}$$

$$\frac{\partial G}{\partial x}\text{has jump discontinuity at} t = \xi \implies D - B = 1 \tag{6.18}$$

Therefore the Green's function is

$$G(t,\xi) = \begin{cases} -\frac{B}{2} - \frac{1}{4} + \frac{\xi}{2} + Bt - \frac{t^2}{2} & t < \xi \\ -\frac{B}{2} - \frac{1}{4} - \frac{\xi}{2} + Bt + t - \frac{t^2}{2} & t > \xi \end{cases}$$

□

If we take the function $f(t) = \begin{cases} 1 & 0 \le t \le \frac{1}{2} \\ -1 & \frac{1}{2} < t \le 1 \end{cases}$. Then a solution of the problem

$$y'' = f(t), B_1(y) = y(0) + y(1) = 0,\ B_2(y) = y'(0) - y'(1) = 0$$

is given by

$$\begin{aligned} u(t) = \int_0^1 G(t,\xi)d\xi &= \int_0^t \left(-\frac{B}{2} - \frac{1}{4} - \frac{\xi}{2} + Bt + t - \frac{t^2}{2}\right)d\xi + \int_t^1 \left(-\frac{B}{2} - \frac{1}{4} + \frac{\xi}{2} + Bt - \frac{t^2}{2}\right)d\xi \\ &= t\int_0^t \cos\pi\xi d\xi + \int_x^1 \frac{\xi}{2}\cos\pi\xi d\xi - \int_0^t \frac{\xi}{2}\cos\pi\xi d\xi \\ &= \frac{1}{\pi^2}\cos\pi\xi. \end{aligned}$$

6.7 Exercises

1. Determine which of the followng equations are oscillatory:

   $(a)\ y'' + t^2 y = 0 \quad (b)\ y'' + \sqrt{t^6 + 3t^5 + 1}y = 0 \quad (c)\ y'' + \frac{t^2+1}{t^2+5}y = 0$

2. Determine the oscillation of $y'' + y' + y = 0$.
3. Determine the oscillation of $y'' - \frac{1}{4}ty' + y = 0$
4. Solve the following BVPs using eigenfunction expansions, shifting the data if necessary.

   $(a)\ -u'' = 1\ \ u(0) = u(1) = 0 \quad (b)\ -u'' = e^x\ u(0) = u(1) = 1$

$(c)\ -u'' = t,\ u(0) = u(1) = 0 \quad (d)\ -u'' + 2u = 1,\ u(0) = u(1) = 0$

$(e)\ -u'' = t^2,\ u(0) = u(1) = 0 \quad (f)\ -u'' + 2u = \frac{1}{2} - t,\ u(0) = u(1) = 0.$

5. suppose $\mathcal{L}u$ if $(\mathcal{L}, B_1, B_2)$ is self adjoint then show that integration by parts can be used to compute the Fourier coefficients of $\mathcal{L}u$.
6. Solve the following BVPs using eigenfunction expansion method:
   a. $-u'' = 1,\ 0 < t < 1,\ u'(0) = 1,\ u'(1) = 1$
   b. $u'' + u = t(1-t),\ u'(0) = 0, u'(1) = 0.$
7. Find the Green's function for the following problems, if it exists
   a. $u'' + u = 0,\ 0 < t < \pi,\ u(0) = u'(\pi) = 0$
   b. $u'' - u = 0,\ 0 < t < \pi,\ u(0) = u'(\pi) = 0$
   c. $u'' = 0,\ 0 < t < 1,\ 2u(0) + u'(0) = 0, u(1) = 0$
8. Solve the following problems using Green's function method and eigen function expansion methods
   a. $y'' - y = \cos t,\ y(0) = y(\pi) = 0$
   b. $y'' = \sin 2t,\ y'(0) = y'(\pi) = 0$
   c. $y'' - 4y = 1,\ y(0) = y(\pi) = 0$

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