Chapter 4
Series solutions

4.1 Taylor Series solutions

In this section we will study the power series solutions of second order linear differential equations with variable coefficients, viz.,

$$y'' + a(t)y' + b(t)y = f(t)$$

where $a(t), b(t)$ and $f(t)$ are given functions. To start with let us recall the Taylor series of a function $y(t)$ around the point t_0:

$$y(t) = \sum_{k=0}^{\infty} \frac{y^{(k)}(t_0)}{k!}(t-t_0)^k.$$

where $y^{(k)}(t_0) = \frac{d^k y}{dx^k}(t_0)$. So if we know the derivatives of the function at a point then we can formally write its Taylor series. Let us take an example of IVP:

$$y' = y, \; y(0) = 1.$$

Then we can see from the equation $y'(0) = y(0) = 1$. We can differentiate the equation to get second derivative

$$y'' = y' = y = 1 \text{ at } 0.$$

Similarly we can find all derivatives:

$$y^{(m)}(0) = 1, \text{ for all } m \geq 3$$

Therefore its Taylor series is

$$y(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} = e^t.$$

This is the solution of the IVP.

Question: Can we do this always?

Consider another example

$$y' + y = h(t) = \begin{cases} e^{-\frac{1}{t^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}, \; y(0) = 0.$$

In this case, we have

$$y^{(m)}(0) = 0, \ \forall m \in \mathbb{N}.$$

Therefore Taylor series of $y(t)$ is identically equal to zero. However by integrating the equation we can write the solution as

$$y(t) = e^{-t} \int_0^t e^s h(s) ds$$

this is not equal to zero for any $t > 0$. Therefore the formal Taylor series expansion need not be a solution of IVP.
From our experience in Calculus, we know that $h(t)$ is not *real analytic*. So that could be the problem. To make our ideas clear let us recall some basics on power series.

Definition 4.1.1 *A function $g(t)$ defined in an interval I containing the point t_0 is called* **real analytic** *at t_0 if it can be represented as power series around t_0. That is there exist constants $c_k, k = 0, 1, 2...$ such that the following series converges for $|t - t_0| < r$, for some $r > 0$.*

$$g(t) = \sum_{k=0}^{\infty} c_k (t - t_0)^k.$$

Remark 4.1.1 *If g is real analytic function at a point t_0, then $c_k = \frac{g^{(k)}(t_0)}{k!}$. That is g is equal to its Taylor series. In other words, if $g(x)$ is equal to its Taylor series in an interval around a point t_0, we say $g(t)$ is real analytic at t_0.*

We state the following characterization of real analytic functions.

Theorem 4.1.1 *The following are equivalent*

1. *$f(t)$ is real analytic at $t = a$*
2. *For every small interval I containing a, there exist constants $r > 0$ and $C > 0$ such that for all $k \in \mathbb{N} \cup \{0\}$:*

$$\left| f^{(k)}(t) \right| \leq C \frac{k!}{r^k} \ \forall t \in I.$$

(2) $\implies$ (1): The remainder of Taylor's theorem R_n can be estimated as

$$\begin{aligned} |R_n(t)| &\leq \frac{C}{r^{n+1}} |t - a|^{n+1} \\ &= C \left(\frac{|t - a|}{r} \right)^{n+1} \to 0 \text{ if } |t - a| < r. \end{aligned}$$

We could use Root test/Ratio test to get the convergence of power/Taylor series. □

4.1.1 Power Series

We recall the following results from calculus about the power series. Given a sequence of real numbers $\{a_k\}_{k=0}^{\infty}$, the series $\sum_{k=0}^{\infty} a_k (t - t_0)^k$ is called *power series* with center t_0. It is easy to see that a power series converges for $t = t_0$. Power series is a function of x provided it converges for t. If a power series converges, then the domain of convergence is either a bounded interval or the whole of $\mathbb{R}$. So it is natural to study the largest interval where the power series converges.

Theorem 4.1.2 *If $\sum a_k t^k$ converges at $t = r$, then $\sum a_k t^k$ converges for $|t| < |r|$.*

Proof. We can find $C > 0$ such that $|a_k r^k| \le C$ for all k. Then

$$|a_k t^k| = |a_k r^k| \left|\frac{t}{r}\right|^k \le C \left|\frac{t}{r}\right|^k.$$

Conclusion follows from comparison theorem.

Theorem 4.1.3 *Consider the power series $\sum_{k=0}^{\infty} a_k t^k$. Suppose*

$$\beta = \limsup_k \sqrt[k]{|a_k|}$$

and $R = \frac{1}{\beta}$ (We define $R = 0$ if $\beta = \infty$ and $R = \infty$ if $\beta = 0$). Then

1. *$\sum_{k=0}^{\infty} a_k t^k$ converges absolutely for $|t| < R$*
2. *$\sum_{k=0}^{\infty} a_k t^k$ diverges for $|t| > R$.*
3. *No conclusion if $|t| = R$.*

In case the limit exists in the definition of β, then

$$\beta = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|.$$

Within the interval of convergence we can differentiate the series term-by-term and integrate term-by-term. Indeed we have

Theorem 4.1.4 *Suppose that the power series $f(t) = \sum_{k=0}^{\infty} a_k t^k$ for $|t| < R$ has radius of convergence $R > 0$. Then $f(t)$ is differentiable in $|t| < R$ and $f'(t) = \sum_{k=1}^{\infty} k a_k t^{k-1}$ for $|t| < R$.*

Proof. It is not difficult to check that the series of $f'(t)$ converges in $|t| < R$. Indeed, using the fact that $|k|^{1/k} \to 1$ as $k \to \infty$,

$$\lim_{k\to\infty} |k a_k|^{1/k} = \lim_{k\to\infty} |a_k|^{1/k} = \frac{1}{R}.$$

Next we will show that the power series of f is differentiable and its derivative is the series given in $f'(t)$:

$$\frac{f(t+h) - f(t)}{h} - f'(t) = \sum_{k=2}^{\infty} a_k \left[\frac{(t+h)^k - t^k}{h} - k t^{k-1}\right]$$

Now we recall the identity: For $a, b \in \mathbb{R}$ and $k \in \mathbb{N}$,

$$\frac{a^k - b^k}{a - b} = a^{k-1} + a^{k-2} b + \cdots + a b^{k-2} + b^{k-1} \tag{1.1}$$

Differentiating the identity (1.1) with respect to b, we get

$$\frac{a^k - b^k - k b^{k-1}(a-b)}{(a-b)^2} = a^{k-2} + 2a^{k-3} b + 3a^{k-4} b^2 + \cdots + (k-1) b^{k-2}$$

Now taking $a = t+h$ and $b = t$, and taking $|t+h| < r < R, |t| < r < R$, we get

$$\begin{aligned}\left|(t+h)^k - t^k - kt^{k-1}h\right| &= \left|h^2\left[(t+h)^{k-2} + (t+h)^{k-1}x + \cdots + (k-1)t^{k-2}\right]\right| \\ &\leq |h|^2 r^{k-2}(1+2+\cdots+(k-1)) \\ &\leq |h|^2 r^{k-2}k(k-1).\end{aligned}$$

Therefore

$$\begin{aligned}\left|\frac{f(t+h)-f(t)}{h} - f'(t)\right| &\leq \frac{1}{|h|}\sum_{k=2}^{\infty}|a_n|\left|(t+h)^k - t^k - kt^{k-1}h\right| \\ &\leq |h|\sum_{k=2}^{\infty}|a_n|k(k-1)r^{k-2} \\ &\leq |h|C \to 0 \; as \; h \to 0,\end{aligned}$$

where C is the limit of the series $\sum_{k=2}^{\infty} a_k k(k-1)r^{k-2}$. This series converges as $r < R$. □

Proof. of (1.1): We can prove this by induction. For $k = 2$ it is easy to see this. Assume for k, then for $k+1$, we may write

$$\frac{a^{k+1}-b^{k+1}}{a-b} = \frac{a^{k+1}-ab^k+ab^k-b^{k+1}}{a-b} = \frac{a(a^k-b^k)}{a-b} + b^k$$

Now using the relation satisfied by $a^k - b^k$ we see that (1.1) holds for $k+1$. □
An important consequence of this is

Theorem 4.1.5 *If $\sum a_k t^k \equiv 0$ then $a_k = 0$ for all k*

Proof. Taking $x = 0$ we get $a_0 = 0$. Then by above theorem we can differentiate the series to get

$$\sum ka_k t^{k-1} \equiv 0$$

Again taking $t = 0$ we get $a_1 = 0$. Proceeding this way we get $a_k = 0$ for all k. □

Remark 4.1.2 *If $f(t) = \sum a_k t^k$ then f is real analytic at $t = 0$. Indeed, it is not easy to show that $a_k = \frac{f^{(k)}}{k!}$.*

Moreover the following theorem holds

Theorem 4.1.6 *If $f(t) = \sum a_k t^k$ converges in $|t| < R$ then $f(t)$ is real analytic in $|t| < R$.*

For a proof of this we refer to the book of W. Rudin.

All the above theorems basically show that if a power series converges with positive radius of convergence then it defines a real analytic function within its radius of convergence. This motivates us to look for solutions of differential equations that are real analytic functions.

4.1.2 Real analytic solutions

Let us consider the example: $y'' = y$. Let us assume that the function $y(t)$ is equal to its power series $\sum c_k t^k$. Then by the above theorem 4.1.4, we get

$$y'(t) = \sum_{k=1}^{\infty} k c_k t^{k-1}, \quad y''(t) = \sum_{k=2}^{\infty} k(k-1) c_k t^{k-2}.$$

Substituting these into the equation $y'' - y = 0$ we get

$$\begin{aligned} 0 &= \sum_{k=2}^{\infty} k(k-1) t^{k-2} c_k - \sum_{k=0}^{\infty} c_k t^k \\ &= \sum_{k=0}^{\infty} \left[(k+2)(k+1) c_{k+2} - c_k \right] t^k \end{aligned}$$

From this, we get the recurrence relation

$$c_{k+2} = \frac{c_k}{(k+2)(k+1)},\ k = 0, 1, 2, 3...$$

Therefore

$$\begin{aligned} k = 0 : &\implies c_2 = \frac{c_0}{2} \\ k = 1 : &\implies c_3 = \frac{c_1}{3 \cdot 2} \\ k = 2 : &\implies c_4 = \frac{c_2}{4 \cdot 3} = \frac{c_0}{4!} \\ k = 3 : &\implies c_5 = \frac{c_1}{5!} \end{aligned}$$

Iterating this we get

$$c_{2k} = \frac{c_0}{(2k)!},\ c_{2k+1} = \frac{c_1}{(2k+1)!}$$

Therefore, if $c_0 = c_1$ we get

$$y(x) = c_0 \sum \frac{t^k}{k!} = c_0 e^t$$

and if $c_1 = -c_0$ then $y(t) = c_0 e^{-t}$. Since linear second order equation can have only two linearly independent solutions, e^t and e^{-t} are the only linearly independent solutions. □

Let us consider another example with variable coefficients.

Example 4.1.1 $y'' - 2ty' + 2y = 0$.

Writing $y(t) = \sum a_k t^k$ and differentiating term by term we get

$$\begin{aligned} ty' &= \sum_{k=1}^{\infty} k a_k t^k \\ y'' &= \sum_{k=2}^{\infty} k(k-1) a_k t^{k-2} = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} t^k \end{aligned}$$

Substituting in the equation we get

$$y'' - 2ty' + 2y = \sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2} - 2ka_k + 2a_k]t^k = 0$$

Then by Theorem 4.1.5 above we get

$$(k+2)(k+1)a_{k+2} - 2ka_k + 2a_k = 0, \ \forall k$$

That is

$$2a_2 + 2a_0 = 0 \implies a_2 = -a_0$$

and

$$a_{k+2} = \frac{2(k-1)}{(k+2)(k+1)}a_k$$

Now $k = 1$ implies $a_3 = 0 \cdot a_1 = 0$ and hence

$$a_{2k+1} = 0 \text{ for all } k = 0,1,2..$$

Also $k = 2$ implies $a_4 = \frac{2a_2}{12} = -\frac{a_0}{6}$. All other even coefficients can be calculated as multiple of a_0. Therefore we get

$$y(t) = a_1 t + a_0(1 - \frac{t^4}{6} + ...)$$

The two L.I. solutions are $y_1(t) = t$ and $y_2(t) = 1 - \frac{t^4}{6} + ...$ To see the convergence of y_2 we use the recurrence relation. The coefficients of y_2 satisfy

$$\left|\frac{a_{k+2}}{a_k}\right| = \frac{2(k-1)}{(k+2)(k+1)} \to 0 \text{ as } k \to \infty.$$

Hence by ratio test, power series converges with radius of convergence equals to ∞. □

Then it is natural to ask the following:

Q1: Can we apply this method for equations like $y'' + e^t y = 0$?
Q2: What is the most general class of coefficients for which the power series solution exists?
To answer the first question, we need the following Cauchy product

Definition 4.1.2 *Let $\sum_{k=0}^{\infty} a_k t^k$ and $\sum_{k=0}^{\infty} b_k t^k$ be two power series. Then the Cauchy product of these two series is defined as*

$$\sum_{k=0}^{\infty} c_k t^k = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} a_j b_{k-j}\right)t^k$$

Theorem 4.1.7 *If $\sum a_k t^k$ and $\sum b_k t^k$ converges for $|t| < R$ then their Cauchy product $\sum c_k t^k$ converges for $|t| < R$.*

Proof. Suppose Γ_n be the sequence of n^{th} partial sums of the series $\sum |c_k|$. Then

$$
\begin{aligned}
\Gamma_N &= \sum_{k=0}^{N} |c_k| = \sum_{k=0}^{N} \left| \sum_{j=0}^{k} a_j b_{k-j} \right| \le \sum_{k=0}^{N} \sum_{j=0}^{k} |a_j||b_{k-j}| \\
&= \sum_{k=0}^{N} \sum_{j+l=k} |a_j||b_l| = \sum_{j+l=N} |a_j||b_l| \\
&= \left(\sum_{j\le N} |a_j| \right) \left(\sum_{l\le N} |b_l| \right)
\end{aligned}
$$

Trivial modification for power series will complete the proof. The RHS of the last inequality is basically consists of all the products

$$|a_0||b_0|, |a_0||b_1|, \cdots |a_0||b_N|;$$

$$|a_1||b_0|, |a_1||b_1|, \cdots |a_1||b_N|;$$

$$|a_N||b_0|, |a_N||b_1|, \cdots |a_N||b_N|$$

where as the each element ($|c_k|$) in the L.H.S sum contains only few of them. □

Remark 4.1.3 *It is possible to have series with finite radius of convergence and their product has infinite radius of convergence. For example take* $f(t) = \frac{1+t}{1-t}$ *and* $g(t) = \frac{1-t}{1+t}$. f *has radius of convergence* 1 *but the product has* ∞.

Theorem 4.1.8 *Suppose* $a(t)$ *and* $b(t)$ *are real analytic in* $|t| < R$. *Then all solutions of the equation*

$$y'' + a(t)y' + b(t)y = 0$$

are real analytic in $|t| < R$.

Proof. Let $a(t) = \sum \alpha_k t^k$ and $b(t) = \sum \beta_k t^k$. There exists constant $M > 0$ such that for $r < R$

$$|\alpha_k| r^k \le M, \quad |\beta_k| r^k \le M, \quad k \ge 0. \tag{1.2}$$

Assuming the solution in the form $\phi(x) = \sum c_k t^k$, we have

$$a(t)\phi'(t) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \alpha_{k-j}(j+1)c_{j+1} \right) t^k, \quad b(t)\phi(t) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \beta_{k-j} c_j \right) t^k,$$

Substituting these in the equation we get

$$L(\phi) = \sum_{k=0}^{\infty} \left[(k+2)(k+1)c_{k+2} + \sum_{j=0}^{k} \alpha_{k-j}(j+1)c_{j+1} + \sum_{j=0}^{k} \beta_{k-j} c_j \right] t^k = 0$$

Therefore we obtain the recurrence relation

$$(k+2)(k+1)c_{k+2} = -\sum_{j=0}^{k} \alpha_{k-j}(j+1)c_{j+1} - \sum_{j=0}^{k} \beta_{k-j} c_j, \; k = 0, 1, 2, \cdots$$

Then using (1.2), we have

$$
\begin{aligned}
(k+2)(k+1)|c_{k+2}| &\le \frac{M}{r^k}\sum_{j=0}^{k}\Big[(j+1)|c_{j+1}|+|c_j|\Big]r^j \\
&\le \frac{M}{r^k}\sum_{j=0}^{k}\Big[(j+1)|c_{j+1}|+|c_j|\Big]r^j + M|c_{k+1}|r. \qquad (1.3)
\end{aligned}
$$

Now let us define

$$A_0 = |c_0|,\ A_1 = |c_1|,$$

and for $A_k, k \ge 2$, by

$$(k+2)(k+1)A_{k+2} = \frac{M}{r^k}\sum_{j=0}^{k}\Big[(j+1)A_{j+1}+A_j\Big]r^j + MA_{k+1}r,\ k=0,1,2,\cdots \qquad (1.4)$$

Now comparing (1.3) and (1.4), by induction

$$|c_k| \le A_k, \quad A_k \ge 0$$

Now from (1.4) we have

$$
\begin{aligned}
k(k+1)A_{k+1} &= \frac{M}{r^{k-1}}\sum_{j=0}^{k-1}\Big[(j+1)A_{j+1}+A_j\Big]r^j + MA_k r \\
k(k-1)A_k &= \frac{M}{r^{k-2}}\sum_{j=0}^{k-2}\Big[(j+1)A_{j+1}+A_j\Big]r^j + MA_{k-1} r
\end{aligned}
$$

From these, we obtain

$$
\begin{aligned}
rk(k+1)A_{k+1} &= \frac{M}{r^{k-2}}\sum_{j=0}^{k-2}\Big[(j+1)A_{j+1}+A_j\Big]r^j + M[kA_k + A_{k-1}]r + MA_k r^2 \\
&= k(k-1)A_k - MA_{k-1}r + MkA_k r + MA_{k-1}r + MA_k r^2 \\
&= \Big[k(k-1) + Mkr + Mr^2\Big]A_k.
\end{aligned}
$$

Therefore

$$\left|\frac{A_{k+1}t^{k+1}}{A_k t^k}\right| = \frac{k(k-1)+Mkr+Mr^2}{r(k+1)k}|t| \to \frac{|t|}{r}$$

as $k \to \infty$. Thus by Ratio test the series $\sum c_k t^k$ converges for $|t| < r$. □

4.1.3 Legendre Equation

One of the important differential equation that appears in Mathematical physics is the Legendre equation. In Quantum Mechanics the dynamics of one electron atom is governed by Scrodinger equation. In the symmetric case this equation reduces to Legendre equation. Another important application is in the study of flow around the outside of a puff of hot gas rising through the air. The Legendre equations is:

$$(1-t^2)y'' - 2ty' + p(p+1)y = 0,\ p \in \mathbb{R}.$$

If we write this equation in the normal form as

$$y'' - \frac{2t}{(1-t^2)}y' + \frac{p(p+1)}{(1-t^2)}y = 0$$

then we see that the functions a_1 and a_2 given by

$$a_1(t) = \frac{-2t}{(1-t^2)}, \text{ and } a_2(t) = \frac{p(p+1)}{(1-t^2)},$$

are real analytic at $t = 0$ with the series converges in $(-1,1)$. Therefore by the above theorem the problem will have two linearly independent real analytic solutions. We can write the serieses of a_1 and a_2 as

$$a_1(t) = (-2t)\sum_{k=0}^{\infty} t^{2k}, \;\; a_2(t) = \sum_{k=0}^{\infty} p(p+1)t^{2k}, \;\; \text{for } |t| < 1.$$

By assuming $y(t) = \sum c_k t^k$ we see that

$$\begin{aligned}
(-2t)y'(t) &= \sum_{k=0}^{\infty} -2kc_k t^k \\
y''(t) &= \sum_{k=2}^{\infty} k(k-1)c_k t^{k-2} = \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}t^k \\
-t^2 y'' &= \sum_{k=0}^{\infty} -k(k-1)c_k t^k
\end{aligned}$$

Therefore

$$\begin{aligned}
(1-t^2)y'' - 2ty' + p(p+1)y &= \sum_{k=0}^{\infty}\left[(k+2)(k+1)c_{k+2} - k(k-1)c_k - 2kc_k + p(p+1)c_k\right]t^k \\
&= \sum_{k=0}^{\infty}\left[(k+2)(k+1)c_{k+2} + (p+k+1)(p-k)c_k\right]t^k
\end{aligned}$$

Therefore by Theorem 4.1.5 we get the recurrence relation

$$(k+2)(k+1)c_{k+2} + (p+k+1)(p-k)c_k = 0, \; k = 0,1,2,... \tag{1.5}$$

Hence we get

$$\begin{aligned}
c_2 &= -\frac{p(p+1)}{2}c_0, & c_3 &= -\frac{(p+2)(p-1)}{3\cdot 2}c_1 \\
c_4 &= \frac{(p+3)(p+1)p(p-2)}{4\cdot 3\cdot 2}c_0, & c_5 &= \frac{(p+4)(p+2)(p-1)(p-3)}{5\cdot 4\cdot 3\cdot 2}c_1.
\end{aligned}$$

Therefore we can write the solution as

$$\begin{aligned}
y(t) &= c_0\left(1 - \frac{p(p+1)}{2!}t^2 + ...\right) + c_1\left(t - \frac{(p+2)(p-1)}{3!}t^3 + ...\right) \\
&= c_0\phi_1(t) + c_1\phi_2(t).
\end{aligned}$$

The adjacent coefficients of the series in ϕ_1 and ϕ_2 are related by (1.5) and hence

$$\left|\frac{c_{k+2}}{c_k}\right| = \left|\frac{(p+k+1)(p-k)}{(k+2)(k+1)}\right| \to 1, \text{ as } k \to \infty.$$

By ratio test the series converges for $|t| < 1$. Also,

$$W(\phi_1, \phi_2)(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

Therefore ϕ and ϕ_2 are two linearly independent solutions.

Legendre Polynomials: We note that when p is a non-negative even integer $p = 2m, m = 0, 1, 2..$ then ϕ_1 has only a finite number of non-zero terms viz a polynomial of order $2m$. for example

$$p = 0 : \implies \phi_1(t) = 1$$
$$p = 2 : \implies \phi_1(t) = 1 - 3t^2$$

Similarly when p is a positive odd integer $p = 2m+1$ then ϕ_2 is a polynomial of order $2m+1, m = 0, 1, 2...$ For example,

$$p = 1 : \implies \phi_2(t) = t$$
$$p = 3 : \implies \phi_2(t) = t - \frac{5}{3}t^3$$

Definition 4.1.3 *The polynomial solution P_n of degree n of Legendre equation satisfying $P_n(1) = 1$ is called n^{th} Legendre polynomial. These are explicitly given by*

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(t^2-1)^n\right].$$

4.2 Frobenius solutions

Let $a(t) : \mathbb{R}\backslash\{t_0\} \to \mathbb{R}$ be a continuous function. The point t_0 is called a singular point if $|a(t)| \to \infty$ as $t \to t_0$. Now we define

Definition 4.2.1 Regular singular point: *A point $t = t_0$ is a a regular singular point for the second order equation*

$$y'' + a(t)y' + b(t)y = 0$$

if $(t-t_0)a(t)$ and $(t-t_0)^2 b(t)$ are real analytic at $t = t_0$.

For simplicity if we take $t_0 = 0$ then a second order equation with a regular singular point at $t = 0$ has the form

$$t^2 y'' + ta(t)y' + b(t)y = 0$$

where a, b are real analytic at 0.

Important Idea: Assume $y(t) = t^r \sum_{k=0}^{\infty} c_k t^k$ and find the coefficients c_k by substituting y in the equation

Let us illustrate the method for

Example 4.2.1 *Consider the equation $L(y) = t^2 y'' + \frac{3}{2}ty' + ty = 0$.*

Solution: Let us assume $y(t) = t^r \sum c_k t^k$. Then

$$\begin{aligned} y'(t) &= \sum c_k(k+r)t^{k+r-1} \\ y''(t) &= \sum c_k(k+r)(k+r-1)t^{k+r-2}. \end{aligned}$$

Therefore substituting in the equation

$$\begin{aligned} 0 = L(y) &= \sum_{k=0}^{\infty}\left((k+r)(k+r-1)c_k + \frac{3}{2}(k+r)c_k\right)t^{k+r} + \sum_{k=1}^{\infty} c_{k-1}t^{k+r} \\ &= \left(r(r-1)+\frac{3}{2}r\right)c_0 t^r + \sum_{k=1}^{\infty}\left[\left((k+r)(k+r-1)+\frac{3}{2}(k+r)\right)c_k + c_{k-1}\right]t^{k+r} \\ &= p(r)c_0 x^r + \sum_{k=1}^{\infty}\left[p(r+k)c_k + c_{k-1}\right]t^{k+r} = 0. \end{aligned}$$

here $p(r) = r(r-1) + \frac{3}{2}r$ is called **indicial polynomial**. Now by Theorem 4.1.5, we get

$$p(r) = 0 \ (c_0 \neq 0, \ why?)$$

and the recurrence relation

$$p(r+k)c_k = -c_{k-1}. \tag{2.6}$$

This implies $r_1 = 0, r_2 = -\frac{1}{2}$.

$$p(r+k)c_k = -c_{k-1} \implies c_k = \frac{(-1)^k c_0}{p(r+k)p(r+k-1)...p(r+1)}, \ k = 1,2,...$$

1^{st} **Solution:** Take $r = r_1 = 0$. Then $p(r+k) = p(k) \neq 0$ for all $k = 1,2,..$ (since the only other root is $-\frac{1}{2}$). Hence all other coefficients c_k can be computed using

$$c_k = \frac{(-1)^k c_0}{p(k)p(k-1)...p(1)}, \ k = 1,2,...$$

Hence the first solution is

$$\phi_1(t) = c_0 + c_0 \sum_{k=1}^{\infty} \frac{(-1)^k t^k}{p(k)p(k-1)...p(1)}$$

2^{nd} **Solution:** Taking $r = r_2 = -\frac{1}{2}$, we see that $p(r+k) = p(k-\frac{1}{2}) \neq 0$ for any $k = 1,2,...$ since the other root is 0. Hence all other coefficients c_k can be computed using

$$c_k = \frac{(-1)^k c_0}{p(k-\frac{1}{2})p(k-\frac{3}{2})...p(\frac{1}{2})}, k = 1,2,3,$$

and the solution will be

$$\phi_2(t) = c_0 t^{-\frac{1}{2}} + c_0 t^{-\frac{1}{2}} \sum_{k=1}^{\infty} \frac{(-1)^k t^k}{p(k-\frac{1}{2})p(k-\frac{3}{2})...p(\frac{1}{2})}$$

Convergence: The recurrence relation (2.6) implies

$$\left|\frac{c_{k+1}}{c_k}\right| \leq \left|\frac{1}{p(k+r+1)}\right| \to 0 \text{ as } k \to \infty.$$

Hence by ratio test the series converges for all t. However the first solution is real analytic and the second solution has singularity at $t = 0$. $\square$

Now let us look at the most general case where $a(t)$ and $b(t)$ are real analytic. So let us assume $a(t) = \sum \alpha_k t^k$, $b(t) = \sum \beta_k t^k$ and $y(t) = \sum c_k t^{k+r}$ then

$$
\begin{aligned}
y'(t) &= \sum (k+r) c_k t^{k+r-1} \\
y''(t) &= \sum (k+r-1)(k+r) c_k t^{k+r-2} \\
b(t)y(t) &= t^r \left(\sum c_k t^k\right)\left(\sum \beta_k t^k\right) = t^r \sum \tilde{\beta}_k t^k, \quad \tilde{\beta}_k = \sum_{j=0}^{k} c_j \beta_{k-j} \\
ta(t)y'(t) &= t^r \left(\sum (k+r) c_k t^k\right)\left(\sum \alpha_k t^k\right) = t^r \sum \tilde{\alpha}_k t^k, \quad \tilde{\alpha}_k = \sum_{j=0}^{k} (j+r) c_j \alpha_{k-j}.
\end{aligned}
$$

Substituting this in $L(y) = 0$ we get

$$
\begin{aligned}
0 = L(y) = t^2 y'' + ta(t)y' + b(t)y &= t^r \sum \left[(k+r)(k+r-1)c_k + \tilde{\alpha}_k + \tilde{\beta}_k\right] t^k \\
&= \sum_{k=0}^{\infty} \left[(k+r)(k+r-1)c_k + \sum_{j=0}^{k} (j+r) c_j \alpha_{k-j} + \sum_{j=0}^{k} c_j \beta_{k-j}\right] t^{k+r}. \\
&= [r(r-1) + r\alpha_0 + \beta_0] c_0 t^r \\
&+ \sum_{k=1}^{\infty} \left\{ [(k+r)(k+r-1) + (k+r)\alpha_0 + \beta_0] c_k + \sum_{j=1}^{k-1} (j+r) c_j \alpha_{k-j} + \sum_{j=1}^{k-1} c_j \beta_{k-j} \right\} t^{k+r} \\
&= p(r) c_0 x^r + \sum_{k=1}^{\infty} \left\{ p(r+k) c_k + \sum_{j=1}^{k-1} (j+r) c_j \alpha_{k-j} + \sum_{j=1}^{k-1} c_j \beta_{k-j} \right\} t^{k+r}
\end{aligned}
$$

Therefore by Theorem 4.1.5, we get

$$
p(r) = 0 (\text{since } c \neq 0, \text{ why?})
$$

and

$$
c_k p(r+k) + \sum_{j=1}^{k-1} (j+r) c_j \alpha_{k-j} + \sum_{j=1}^{k-1} c_j \beta_{k-j} = 0, \ k = 0, 1, 2 \ldots
$$

where $p(r) = r(r-1) + r\alpha_0 + \beta_0$ is called the **indicial polynomial.**
By assuming $c_0 \neq 0$, we get

$$
p(r) = 0 \implies r = r_1 \text{ and } r = r_2
$$

For each of the values of r_1 and r_2 we see that $c_k, k = 1, 2, 3..$ satisfies

$$
p(r+k) c_k = -\sum_{j=1}^{k-1} \left[(j+r)\alpha_{k-j} + \beta_{k-j}\right] c_j
$$

So if $p(r+k)$ is not zero for any k, then the above equation determines the c_k uniquely and hence the solutions. But this may not be the case as r_2 can be equal to $r_1 + k$ for some k. Therefore we have the following several cases:

Case I: $r_1 \neq r_2,\ r_1 < r_2,\ r_2 \neq r_1 + k$ **for any** k**.**

In this case all coefficients $c_k, k = 1, 2, ...$ can be computed using

$$c_k(r) = \frac{-1}{p(r+k)} \sum_{j=0}^{k-1} \Big[(j+r)\alpha_{k-j} + \beta_{k-j} \Big] c_j \tag{2.7}$$

The linearly independent solutions are

$$\phi_1(t) = c_0 t^{r_1} + t^{r_1} \sum_{k=1}^{\infty} c_k(r_1) t^k,\ c_0 \neq 0$$
$$\phi_2(x) = c_0 t^{r_2} + t^{r_2} \sum_{k=1}^{\infty} c_k(r_2) t^k,\ c_0 \neq 0$$

An example of this type is computed above.
Case II: $r_1 = r_2$
In this case $p(r_1) = 0$ and $p'(r_1) = 0$. Writing the solution $y(t)$ as

$$y(t) = c_0 t^r + \sum_{k=1}^{\infty} c_k(r) t^{k+r}$$

we may see $L(y)$ as a function of t and r. By following (2.7), we define $c_k(r) = A_k(r)$ where $A_k(r)$ are chosen such that

$$A_0 = c_0,$$
$$A_k(r) = \frac{-1}{p(r+k)} \sum_{j=0}^{k-1} \Big[(j+r)\alpha_{k-j} + \beta_{k-j} \Big] A_j,\ k = 1, 2, 3...$$

With these coefficients in y, we get

$$L(y) = A_0 p(r) t^r.$$

Therefore

$$L\left(\frac{\partial}{\partial r} y\right) = \frac{\partial}{\partial r} L(y)(t, r) = c_0 (p'(r) + (\log r) p(r)) t^r = 0, \text{ at } r = r_1.$$

Therefore if ϕ_1 is the solution corresponding to $r = r_1$ then $\frac{\partial \phi_1}{\partial r}$ at $r = r_1$ is the second linearly independent solution. Therefore,

$$\phi_1(t) = \sum_{k=0}^{\infty} c_k(r_1) t^{k+r_1} = c_0 t^{r_1} + \sum_{k=1}^{\infty} c_k(r_1) t^{k+r_1}$$

$$\begin{aligned} \phi_2(t) = \frac{\partial \phi_1}{\partial r}\bigg|_{r=r_1} &= t^{r_1} \sum_{k=1}^{\infty} c_k'(r_1) t^k + (\log t) t^{r_1} \sum_{k=0}^{\infty} c_k(r_1) t^k \\ &= t^{r_1} \sum_{k=1}^{\infty} c_k'(r_1) t^k + (\log t)\phi_1(t) \end{aligned} \tag{2.8}$$

Practically, one assumes the following for calculating the second solution

$$\phi_2(t) = t^{r_1} \sum_{k=1}^{\infty} b_k t^k + (\log t)\phi_1(t)$$

and obtain the coefficients b_k by substituting this in the given equation.

Case III: Roots differ by integer., i.e., $r_2 = r_1 + m$ for some $m \in \mathbb{N}$
Recall that the relation satisfied by c_k:

$$p(r+k)c_k = -\sum_{j=0}^{k-1} \left[(j+r)\alpha_{k-j} + \beta_{k-j} \right] c_j := D_k \tag{2.9}$$

Since $r_2 > r_1$ and p is second order polynomial $p(r_2 + k) \neq 0$ for all $k \in \mathbb{N}$. Therefore all coefficients can be computed using (2.9) to write the series solution of first solution ϕ_1. Now to find second solution we note that

$$p(r_2) = p(r_1 + m) = 0$$

Therefore it is not clear how to compute c_m from (2.9). However in **some special cases** D_m on the Right hand side of (2.9) also may become zero. In that case we can take c_m to be arbitrary and other coefficients can be calculated using (2.9). For example

Example 4.2.2 $t^2 y'' + 2t^2 y' - 2y' = 0$

In this case by taking $y(t) = \sum c_k t^{k+r}$ and substituting in the equation, we get

$$\begin{aligned} 0 =& t^2 y'' + 2t^2 y' - 2y' = \sum_{k=0}^{\infty} (k+r)(k+r-1) c_k t^{k+r} + \sum_{k=0}^{\infty} 2(k+r) c_k t^{k+r+1} - 2 \sum_{k=0}^{\infty} c_k t^{k+r} \\ =& r(r-1) c_0 t^r + (-2) c_0 t^r + \sum_{k=1}^{\infty} \left[(k+r)(k+r-1) c_k + 2(k+r-1) c_{k-1} - 2c_k \right] t^{k+r} \end{aligned}$$

Therefore indicial polynomial is $p(r) = r^2 - r - 2$. The roots are $r_1 = -1$, and $r_2 = 2$. That is $r_2 = r_1 + 3$ implying $p(r_1 + 3) = 0$. So there could be difficulty in calculating c_3.
From the recurrence relation

$$((k+r)(k+r-1) - 2)\, c_k = -2(k+r-1) c_{k-1}$$

Taking $r = 2$ we can find all coefficients to get the first solution in the form

$$\phi_1(t) = t^2 \sum_{k=0}^{\infty} c_k t^k = t^2 (1 + c_1 x + \cdots)$$

Now taking $r = -1$ we get the relation

$$[(k-1)(k-2)-2]\,c_k = -2(k-2)c_{k-1},\ k = 1,2,3...$$

Therefore

$$\begin{aligned}
k = 1 :&\implies -2c_1 = (-2)(-1)c_0 = 2c_0 = 2 \implies c_1 = -1\\
k = 2 :&\implies (-2)c_2 = (-2)(0)c_1 \implies c_2 = 0\\
k = 3 :&\implies (2-2)c_3 = (-2)(1)c_2 = 0
\end{aligned}$$

implying c_3 is arbitrary. So we may choose $c_3 = 0$. From the recurrence relation we will get $c_k = 0$ for all $k \geq 4$. hence the second solution is

$$\phi_2(t) = t^{-1}(1-t)$$

On the other hand one may choose $c_3 \neq 0$. If $c_3 = 1$ then $c_4, c_5, ...$ can be found from the recurrence relation and the resultant solution will be

$$\begin{aligned}
y(t) &= t^{-1}(1-t) + t^{-1}(t^3 + c_4 t^4 + \cdots)\\
&= t^{-1}(1-t) + t^2(1 + c_4 t + \cdots) = \phi_2(t) + C\phi_1(t)
\end{aligned}$$

Hence for any choice of c_3 we will get only ϕ_1 and ϕ_2 above are the only linearly independent solutions. □

In general to find second solution we use $\phi_2 = v\phi_1$ and reducing the order we get

$$\begin{aligned}
v' &= \frac{1}{\phi_1^2} e^{-\int a_1(t)dt} = \frac{1}{t^{2r_1}(c_0 + c_1 t + \cdots)^2} e^{-\int\left(\alpha_0 t^{-1} + \alpha_1 + \alpha_2 t + \cdots\right)},\ \ a_1(x) = \frac{a(x)}{x}\\
&= \frac{1}{t^{2r_1}(c_0 + c_1 x + \cdots)^2} e^{-\alpha_0 \log t - \alpha_1 t + \cdots}\\
&= \frac{t^{-\alpha_0}}{t^{2r_1}(c_0 + c_1 t + \cdots)^2} e^{-\alpha_1 t + \cdots} = \frac{1}{t^{2r_1 + \alpha_0}} g(t)
\end{aligned}$$

where $g(t) = \dfrac{e^{-\alpha_1 t + \cdots}}{(c_0 + c_1 t + \cdots)^2}$. Since $g(0) = \frac{1}{c_0^2} \neq 0$, $g(t)$ is real analytic at 0. This implies

$$g(t) = \sum b_k t^k$$

Therefore by defining $n := 2r_1 + \alpha_0$

$$v'(t) = b_0 t^{-n} + b_1 t^{-n+1} + \cdots + b_{n-1} t^{-1} + b_n + \cdots$$

This implies

$$v(t) = b_0 \frac{t^{-n+1}}{-n+1} + \cdots + b_{n-1} \log t + b_n t + \cdots$$

Hence the second solution ϕ_2 is

$$\begin{aligned}\phi_2(t) = v\phi_1 &= b_{n-1}(\log t)\phi_1 + t^{r_1}(c_0 + c_1 t + \cdots)(b_0 \frac{t^{-n+1}}{-n+1} + \cdots)\\ &= b_{n-1}(\log t)\phi_1 + t^{r_1 - n+1}(c_0 + c_1 x + \cdots)(\frac{b_0}{-n+1} + \cdots)\\ &= b_{n-1}(\log t)\phi_1 + t^{r_2}\sum d_k t^k.\end{aligned}$$

To show $r_2 = 1 - n + r_1$ we notice that

$$p(r) = r(r-1) + r\alpha_0 + \beta_0, \text{ also } p(r) = (r - r_1)(r - r_2)$$

Sum of the roots $= \alpha_0 - 1 = -(r_1 + r_2) \implies r_2 = 1 - \alpha_0 - r_1 - 1$. Now substituting $n = 2r_1 + \alpha_0$ we get $r_2 = 1 - n + r_1$. There in this case we substitute

$$\phi_2(t) = b_{n-1}(\log t)\phi_1 + t^{r_2}\sum_{k=0}^{\infty} d_k t^k$$

in the equation and find the unknowns b_{n-1} & $d_k, k = 1,2,3...$ Note that in this case b_{n-1} can be equal to zero which is the special case discussed above.

In case of coefficients having regular singular points, we have the following theorem:

Theorem 4.2.1 *Suppose $a(t)$ and $b(t)$ are real analytic at $t = 0$ having radius of convergence R. Then the equation*

$$t^2 y'' + ta(t)y' + b(t)y = 0$$

has a solution in the form $t^r \sum c_k t^k$ (for some $r \in \mathbb{R}$) and the radius of convergence R. Other Linearly independent solution can be obtained by reduction of order.

Proof. From the previous discussion we get the recurrence relation

$$p(r+k)c_k(r) = -\sum_{j=1}^{k-1}\Big[(j+r)\alpha_{k-j} + \beta_{k-j}\Big]c_j(r)$$

where $p(r) = (r - r_1)(r - r_2)$ and r_1, r_2 are the roots of $p(r)$. Then

$$p(r_1 + k) = k(k + r_1 - r_2),\ p(r_2 + k) = k(k + r_2 - r_1)$$

and therefore,

$$|p(r_1 + k)| \geq k(k - |r_1 - r_2|),\ |p(r_1 + k)| \geq k(k - |r_1 - r_2|)$$

Then using the estimate $|\alpha_j| \leq M\rho^{-j}, |\beta_j| \leq M\rho^{-j}$ we get

$$k(k - |r_1 - r_2|)|c_k(r_1)| \leq M\sum_{j=1}^{k-1}(j + 1 + |r_1|)\rho^{j-k}|c_j(r_1)|$$

Let $N - 1 \leq |r_1 - r_2| \leq N$ then define

$$\gamma_0 = c_0(r_1) = 1,\ \gamma_k = |c_k(r_1)|,\ k = 1,2,\cdots N-1.$$

and γ_k, for $k = N, N+1...$

$$k(k-|r_1-r_2|)\gamma_k = M\sum_{j=1}^{k-1}(j+1+|r_1|)\rho^{j-k}\gamma_j \tag{2.10}$$

Now it is easy to see that

$$|c_k(r_1)) \leq \gamma_k,\ k = 1,2,...$$

Now to prove the convergence of $\sum \gamma_k t^k$, replacing k by $k+1$ in (2.10) we get

$$\begin{aligned}
\rho(k+1)(k+1-|r_1-r_2|)\gamma_{k+1} &= M\sum_{j=1}^{k}(j+1+|r_1|)\rho^{j-k}\gamma_j \\
&= M\sum_{j=1}^{k-1}(j+1+|r_1|)\rho^{j-k}\gamma_j + M(k+1+|r_1|)\gamma_k \\
&= k(k-|r_1-r_2|)\gamma_k + M(k+1+|r_1|)\gamma_k
\end{aligned}$$

Therefore, as $k \to \infty$,

$$\left|\frac{\gamma_{k+1}t^{k+1}}{\gamma_k t^k}\right| = \frac{k(k-|r_1-r_2|)+M(k+1+|r_1|)}{\rho(k+1)(k+1-|r_1-r_2|)} \to \frac{|t|}{\rho}$$

Hence the series converges for $|t| < \rho$. □

4.2.1 Bessel's equation

Bessel's equation arises in many physical models such as heat conduction in cylindrical shape objects, dynamics of floating bodies, patterns of acoustical radiation etc.. The following equation with regular singular point at $t = 0$

$$t^2y'' + ty' + (t^2 - p^2)y = 0,\ p \in \mathbb{R}$$

is called Bessel's equation of order p. Here $a(t) = 1$ and $b(t) = t^2 - p^2$ implying $\alpha_0 = 1$ and $\beta_0 = -p^2$. Therefore indicial polynomial is

$$r(r-1) + \alpha_0 r + \beta_0 = r^2 - r + r - p^2 = r^2 - p^2.$$

Therefore the roots of indicial polynomial are $r_1 = p$ and $r_2 = -p$.This leads to 3 cases:

1. $p = 0$, repeated roots
2. $r_1 - r_2 = 2p \neq n$, for any $n \in \mathbb{N}$
3. $r_1 - r_2 = 2p = n$, for some $n \in \mathbb{N}$.

Case 1: $p = 0$**:** Substituting $\phi(t) = \sum a_k t^{k+r}$ in the equation

$$\begin{aligned}
L(\phi) =& \sum_{k=0}^{\infty} a_k\left[(r+k)(r+k-1)+(r+k)\right]t^{r+k} + \sum_{k=0}^{\infty} a_k t^{k+r+2} \\
=& a_0\left[r(r-1)+r\right]t^r + a_1[(r+1)r+(r+1)]t^{r+1} \\
&+ \sum_{k=2}^{\infty}\{a_k[(r+k)(r+k-1)+(r+k)]+a_{k-2}\}t^{k+r} = 0.
\end{aligned}$$

Therefore indicial polynomial $p(r) = r^2$ and roots are 0. Moreover,

$$a_1(r(r+1)+(r+1)) = 0$$
$$a_k(r) = -\frac{a_{k-2}(r)}{(r+k)^2}, \quad k \geq 2.$$

Taking $r = r_1 = 0$ we get

$$a_1 = 0 \implies a_3 = a_5 = a_7 = 0, \cdots$$
$$a_k = -\frac{a_{k-2}}{k^2}, \quad k = 2,4,6,8,\cdots$$

Taking $k = 2m, m \in \mathbb{N}$ we get

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m}(m!)^2}, \ m = 1,2,3,\cdots$$

Hence

$$\phi_1(t) = a_0\left[1+\sum_{m=1}^{\infty}\frac{(-1)^m t^{2m}}{2^{2m}(m!)^2}\right], \ t > 0.$$

This is known as the Bessel function of the first kind of order zero and is denoted by $J_0(t)$.
The second solution is given by (recall (2.8))

$$\phi_2(t) = J_0(t)\log t + \sum_{m=1}^{\infty} a'_{2m}(0)t^{2m}$$

From the recurrence relation we get

$$a_{2m}(r) = \frac{(-1)^m a_0}{(r+2)^2(r+4)^2\cdots(r+2m-2)^2(r+2m)^2}$$

The computation of $a'_{2m}(r)$ can be done as follows: If

$$f(t) = (t-\alpha_1)^{\beta_1}(t-\alpha_2)^{\beta_2}(x-\alpha_3)^{\beta_3}\cdots(t-\alpha_n)^{\beta_n}$$

Then

$$\frac{f'(t)}{f(t)} = \frac{\beta_1}{t-\alpha_1}+\frac{\beta_2}{t-\alpha_2}+\cdots+\frac{\beta_n}{t-\alpha_n}$$

Applying this to $a_{2m}(r)$ we get

$$\frac{a'_{2m}(r)}{a_{2m}(r)} = -2\left(\frac{1}{r+2}+\frac{1}{r+4}+\cdots+\frac{1}{r+2m}\right)$$

Taking $r = 0$ we get

$$a'_{2m}(0) = -2\left[\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2m}\right]a_{2m}(0).$$

Therefore, we get

$$\phi_2(t) = J_0(t)\log t + \sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{2^{2m}(m!)^2}\left[1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\right]t^{2m}$$

□

Bessel equation of order $\frac{1}{2}$: When $p = \frac{1}{2}$ the Bessel equation becomes

$$L(y) = t^2y''+ty'+(t^2-\frac{1}{4})y = 0.$$

Then substituting $\phi = \sum a_k t^{k+r}$ we obtain

$$L(\phi) = \sum_{k=0}^{\infty}\left[(r+k)(r+k-1)+(r+k)-\frac{1}{4}\right]a_k t^{k+r} + \sum_{k=0}^{\infty} a_k t^{r+k+2}$$
$$= (r^2-\frac{1}{4})a_0 t^r + [(r+1)^2-\frac{1}{4}]a_1 t^{r+1} + \sum_{k=2}^{\infty}\left\{[(r+k)^2-\frac{1}{4}]a_k + a_{k-2}\right\}t^{k+r} = 0. \tag{2.11}$$

Then the roots of indicial equation are $r_1 = \frac{1}{2}$ and $r_2 = -\frac{1}{2}$. Hence the roots differ by 1. The recurrence relation is

$$[(r+k)^2-\frac{1}{4}]a_k = -a_{k-2},\ k \geq 2.$$

When $r = \frac{1}{2}$ we find that the coefficient of t^{r+1} is zero only when $a_1 = 0$. Therefore from the recurrence relation we get $a_{2k+1} = 0$ for all n. Further in this case

$$a_k = -\frac{a_{k-2}}{k(k+1)},\ k = 2,4,6,\cdots$$

Then for $m = 1,2,3\cdots$

$$a_{2m} = \frac{(-1)^m a_0}{(2m+1)!}$$

Hence taking $a_0 = 1$ we obtain

$$\phi_1(t) = t^{1/2}\sum_{m=1}^{\infty}\frac{(-1)^m t^{2m}}{(2m+1)!} = t^{-1/2}\sum_{m=0}^{\infty}\frac{(-1)^m t^{2m+1}}{(2m+1)!} = t^{-1/2}\sin t.$$

The Bessel function of order half is defined as

$$J_{1/2}(t) = \sqrt{\frac{2}{\pi}}\,\phi_1(t) = \sqrt{\frac{2}{\pi t}}\,\sin t,\ t>0$$

Corresponding to the root $r_2 = -\frac{1}{2}$ it is possible that we may have difficulty in computing a_1 as $r_1 - r_2 = 1$. However from (2.11), we see that the coefficients of t^r and t^{r+1} are both zero regardless of choice of a_0 and a_1. Therefore we can choose them arbitrarily. Thus no logarithmic term is needed to obtain a second solution in this case. Therefore in this case we get

$$a_{2m} = \frac{(-1)^m a_0}{(2m)!},\ \ a_{2m+1} = \frac{(-1)^m a_1}{(2m+1)!},\ m = 1,2,\cdots$$

Hence the solution is

$$\phi_2(t) = t^{-\frac{1}{2}}\left[a_0\sum_{m=0}^{\infty}\frac{(-1)^m t^{2m}}{(2m)!} + a_1\sum_{m=0}^{\infty}\frac{(-1)^m t^{2m+1}}{(2m+1)!}\right]$$
$$= t^{-\frac{1}{2}}(a_0\cos t + a_1\sin t).$$

The second linearly independent solution of Bessel equation of order half is usually taken to be the solution ϕ_2 when $a_1 = 0$. That is denoted by $J_{-1/2}$:

$$J_{-1/2}(t) = \left(\frac{2}{\pi t}\right)^{1/2}\cos t,\ t>0.$$

The case when $2p$ is not a natural number, it is straight forward to calculate the coefficients as in example 1.2. In this case the Bessel functions are denoted by J_p and J_{-p}.

Case: $2p \neq n$ **for any** $n \in \mathbb{N}$
In this case substituting $\phi(t) = \sum c_k t^{k+r}, c_0 \neq 0$ in the equation, we find for $r = p$

$$L(\phi) = 0.c_0 t^p + \left[(p+1)^2 - p^2\right] c_1 t^{p+1} + t^p \sum_{k=2}^{\infty} \left\{[(p+k)^2 - p^2]c_k + c_{k-2}\right\} t^k = 0.$$

Thus we have $c_1 = 0$, and

$$k(2p+k)c_k + c_{k-2} = 0, \; k = 2,3,4\cdots$$

$$c_1 = 0 \implies c_3 = c_5 = \cdots = 0.$$

$$c_{2m} = \frac{(-1)^m c_0}{2^{2m} m!(p+1)(p+2)\cdots(p+m)}$$

Hence the first solution is

$$\phi_1(x) = c_0 t^p + c_0 t^p \sum_{m=1}^{\infty} \frac{(-1)^m t^{2m}}{2^{2m} m!(p+1)(p+2)\cdots(p+m)}$$

By choosing $c_0 = \frac{1}{2^p \Gamma(p+1)}$, we get the Bessel function of first kind

$$J_p(t) = \left(\frac{t}{2}\right)^p \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+p+1)} \left(\frac{t}{2}\right)^{2m}, \; Re(p) \geq 0.$$

To find the second solution linearly independent solution first we see that from the recurrence relation we have

$$k(k-2p)c_k = -c_{k-2}$$

and $c_1 = 0$ implies $c_{2m+1} = 0$ for all $m \in \mathbb{N}$. Hence for non-zero coefficients $k = 2m$ and $2m - p \neq 0$ for all m. That is the recurrence relation implies all coefficients can be calculated if $-2p + 2m \neq 0$ for all m. That is provided p is not a positive integer. For all other p, all coefficients can be determined and is obtained by simply replacing p by $-p$.

$$J_{-p}(t) = \left(\frac{t}{2}\right)^{-p} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m-p+1)} \left(\frac{t}{2}\right)^{2m}$$

Since $\Gamma(m-p+1)$ exists for $m = 0,1,2,\cdots$ provided p is not a positive integer, we see that J_{-p} is well defined. That means the only case left is the case when p is a positive integer say $p = n$. In this case the second solution is in the form

$$\phi_2(t) = t^{-n} \sum_{k=0}^{\infty} c_k t^k + c(\log t) J_n(t).$$

We can substitute in the equation and compute the values of c and c_n.

4.3 Cauchy-Kovalevski theory

In this section we will study the existence of real analytic solutions for more general equations that are not linear. We call such equations as quasilinear equations. The existence of solutions for initial value problems of such equations is addressed by the Picard's theorem. Here we are interested in real analyticity of solutions.

Theorem 4.3.1 *Cauchy Kovalevski theorem Let $f:(-1,1)\to\mathbb{R}$ be a real analytic in some neighbourhood of* 0 *and $u(t)$ is the unique solution of the IVP:*

$$\frac{du}{dt}=f(u(t)),\ u(0)=0.$$

Then u is also real analytic in a neighbourhood of 0.

Proof. By repeated differentiation we see that

$$\begin{aligned}\frac{d^2u}{dt^2}&=f'(u)\frac{du}{dx}=f'(u)f(u)\\ \frac{d^3u}{dt^3}&=f''(u)f^2(u)+(f'(u))^2f(u)\\ &\vdots\\ \frac{d^nu}{dt^n}&=p_n(f(u),f'(u),f''(u),...,f^{(n-1)}(u))\end{aligned}$$

where p_n is a polynomial in n variables with all non-negative integer coefficients. It is easy to see that

$$p_2(x,y)=xy,\ p_3(x,y,z)=x^2z+xy^2$$

and so on. Note that these polynomials do not depend on $f(t)$. Also,

$$\frac{d^nu}{dt^n}(0)=p_n(f(0),f'(0),f''(0),...,f^{(n-1)}(0)).$$

Since the coefficients in the polynomial are non-negative, we get

$$\begin{aligned}\left|\frac{d^nu}{dt^n}(0)\right|&=\left|p_n(f(0),f'(0),f''(0),...,f^{(n-1)}(0))\right|\\ &\le p_n\Big(|f(0)|,|f'(0)|,|f''(0)|,...,|f^{(n-1)}(0)|\Big).\end{aligned}\tag{3.12}$$

Now suppose there exists a real analytic function $g(t)$ such that

$$|f^{(n)}(0)|\le g^{(n)}(0),\ \forall n \text{ and}$$
$$\frac{dv}{dt}=g(v(t)),\ v(0)=0$$
has real analytic solution with radius of convergence $R(say)$

Then by above observation as in (3.12),

$$\begin{aligned}\frac{d^nv}{dx^n}(0)&=p_n\Big(g(0),g'(0),g''(0),...,g^{(n-1)}(0)\Big)\\ &\ge p_n\Big(|f(0)|,|f'(0)|,|f''(0)|,...,|f^{(n-1)}(0)|\Big)\\ &\ge\left|\frac{d^nu}{dx^n}(0)\right|\end{aligned}$$

because

$$p_n\Big(|f(0)|,|f'(0)|,|f''(0)|,...,|f^{(n-1)}(0)|\Big) \le p_n\Big(g(0),g'(0),g''(0),...,g^{(n-1)}(0)\Big) \tag{3.13}$$

Since v is real analytic with radius of convergence R)(say), then for any $\rho < R$,

$$\sum_{n=0}^{\infty}\frac{1}{n!}|u^{(n)}(0)|\rho^n \le \sum_{n=0}^{\infty}\frac{1}{n!}v^{(n)}(0)\rho^n < \infty$$

and hence the following function is well defined for $|t| < \rho$,

$$w(t) := \sum_{n=0}^{\infty}\frac{1}{n!}p_n\Big(f(0),f'(0),f''(0),...,f^{(n-1)}(0)\Big)t^n = \sum_{n=0}^{\infty}\frac{1}{n!}u^{(n)}(0)t^n. \tag{3.14}$$

Now since expression in (3.14) is real analytic, it is easy to check that $w(x)$ is a solution by comparing the derivatives at 0.

To complete the proof, we show the existence of v and g. For this, we know that $\sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(0)\rho^n$ converges. Then there exists a constant $C > 0$ such that

$$\left|\frac{1}{n!}f^{(n)}(0)\right| \le C\rho^{-n},\ \forall\ n.$$

Now we define

$$g(t) = \sum_{n=0}^{\infty}C\left(\frac{t}{\rho}\right)^n = C\frac{1}{1-\dfrac{t}{\rho}} = \frac{C\rho}{\rho - t}$$

Clearly, $|f^{(n)}(0)| \le g^{(n)}(0)$. Consider the problem

$$\frac{dv}{dt} = \frac{C\rho}{\rho - v},\ v(0) = 0$$

i.e., $vv' - \rho v' + C\rho = 0$. In other words, $\frac{1}{2}d(v^2) - d(\rho v) + C\rho = 0$. Hence $v(t) = \rho - \sqrt{\rho^2 - 2C\rho t}$ that satisfies also $v(0) = 0$. Now it is easy to see that v is real analytic for $|t| < \frac{\rho}{2C}$. □

Example 4.3.1 *Consider the initial value problem* $y' = y^2,\ y(0) = 1$

Here $f(t) = t^2$ is a polynomial and hence analytic at 0. Therefore we can compute the solution by Taylor series

$$y'(0) = 1,\ y''(0) = 2[yy'](0) = 2,\ y^{(3)}(0) = 2[yy'' + (y')^2](0) = 6, \cdots$$

Hence

$$y(t) = 1 + t + t^2 + t^3 + \cdots = \frac{1}{(1-t)}.$$

Example 4.3.2 *Consider the problem* $y' = 1 + \sqrt{|y|},\ y(0) = 0.$

In this case we cannot apply Tarylor's series as $f(t) = 1 + \sqrt{|t|}$ is not real analytic at 0. □

Higher order equations

The method of majorants can be used to show the real analyticity of solutions of higher order equations of the type:

$$y^{(n)} = f(y, y', \cdots, y^{(n-1)}),\ y(x_0) = y_0,\ y'(x_0) = y_1, \cdots y^{(n-1)}(x_0) = y_{n-1}.$$

Here f is assumed to be real analytic in its variables. That is it can be expressed as Taylor series in $n-1$ variables. From the equation, and the given data we can write the formal Taylor series expansion of y around x_0. The proof of C-K theorem can proved by converting this initial value problem into system by taking the variables:

$$y^1 = y, y^2 = y' = (y^1)', \cdots, y^n = (y^{(n-1)})'$$

Then the sytem can be written for the vector $Y = (y^1, y^2, \cdots, y^n)$ as

$$(y^1)' = y_2,\ (y^2)' = y_3, ...(y^n)' = y^{(n)} = f(y^1, y^2, ...y^n)$$

That is,

$$Y' = AY,\ Y(x_0) = Y_0$$

where A is $n \times n$ matrix with real analytic elements

$$\begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & f(y^1, y^2, ...y^n) \end{pmatrix}$$

In case of second order equation, $n = 2$, we have the system

$$\begin{pmatrix} (y^1)' \\ (y^2)' \end{pmatrix} = \begin{pmatrix} 0 & y^2 \\ 0 & f(y^1, y^2) \end{pmatrix}$$

We can majorize each element of the matrix and construct the majorized problem to show the convergence of solutions of the initial value problem. Using this, we can write series solution of

Example 4.3.3 *Find solution of* $y'' = y^2,\ y(0) = 1,\ y'(0) = 1$

Since $f(y) = y^2$ is real analytic, we can find Taylor series solution:

$$y(t) = 1 + t + t^2 + \cdots$$

and this series converges in an interval around $t = 0$. □

4.4 Exercises

1. Find two linearly independent series solutions of

   (a) $y'' - ty' + 2y = 0$ (b) $y'' + 3t^2y' - 2ty = 0$
   (c) $y'' + t^2y' + t^2y = 0$ (d) $(1+t^2)y'' + y = 0$

2. Consider the Chebyshev equation

$$(1-t^2)y'' - ty' + \alpha^2 y = 0,\ \alpha \in \mathbb{R}$$

   a. Compute two linearly independent series solutions for $|t| < 1$.
   b. Show that for each non-negative $\alpha = n$ there is a polynomial solution of degree n

3. Consider the Hermite equation
$$y'' - 2ty' + 2\alpha y = 0,\ \alpha \in \mathbb{R}$$
   a. Compute two linearly independent series solutions.
   b. Show that for each non-negative $\alpha = n$ there is a polynomial solution of degree n.
4. Show (directly using product formula of derivatives) that P_n defined in (4.1.3) satisfies Lengendre equation with $p = n$ and $P_n(1) = 1$.
5. Show that
$$\int_{-1}^{1} P_n(t)P_m(t)dt = \begin{cases} 0, & (n \neq m) \\ \frac{2}{2n+1} & n = m. \end{cases}$$
6. Show that there are constants $c_0, c_1, c_2, ...c_n$ such that
$$t^n = c_0 P_0(t) + c_1 P_1(t) + ... + c_n P_n(t)$$
7. Find the weight function $w(x)$ such that the Hermite polynomials $H_n(t)$ are orthogonal with respect to $w(t)$. That is
$$\int_{\mathbb{R}} H_m(t)H_n(t)w(t)dt = 0,\ n \neq m.$$
8. Find the weight function $w(t)$ such that the Chebyshev polynomials $C_n(t)$ are orthogonal with respect to $w(t)$.
9. Find solutions Laguerre equation $ty'' + (1-t)y' + py = 0,\ p \in \mathbb{R}$. Compute $w(t)$ so that the polynomial solutions (for $p = n$) are orthogonal with respect to $w(t)$.
10. Find all solutions of the following equations for $t > 0$:

    (a) $t^2y'' + 2ty' - 6y = 0$ (b) $2t^2y'' + ty' - y = 0$ (c) $t^2y'' - 5ty' + 9y = 0$
11. Find all solutions of the following equations for $x > 0$:

    (a) $3t^2y'' + 5ty' + 3ty = 0$ (b) $t^2y'' + 3ty' + (1+t)y = 0$
    (c) $t^2y'' - 2t(t+1)y' + 2(t+1)y = 0$
12. For $\lambda > 0$ let $\phi_\lambda(t) = \sqrt{t}J_0(\lambda t)$, then show that
$$\phi_\lambda'' + \frac{1}{4t^2}\phi_\lambda = -\lambda^2\phi_\lambda$$
    and
$$\int_0^1 \phi_\lambda(t)\phi_\mu(t)dx = \begin{cases} 0 & \lambda \neq \mu. \\ \frac{1}{2}[J_0']^2 & \lambda = \mu, J_0(\lambda) = 0 \end{cases}$$
13. If $\lambda > 0$ is such that $J_0(\lambda) = 0$ then show that $J_0'(\lambda) \neq 0$.
14. Show that J_0 has an infinitely many positive zeros. Moreover the set of positive zeros of J_0 are denumerable.
    (Hint: If r_n are zeros of J_0 that converge to r. Then
$$J_0'(r) = \lim_{n\to\infty} \frac{J_0(r_n) - J_0(r)}{r_n - r_0} = 0$$
    Hence from the equation we get $J_0''(r) = 0$. Iterating the equation we get $J_0^{(m)}(r) = 0$ for all m. Therefore $J_0 \equiv 0$ in an interval around r.)

15. show the following identities

    (a) $J_{-n}(t) = (-1)^n J_n(t)$ (b) $(t^p J_p)'(t) = t^p J_{p-1}(t)$
    (c) $(t^{-p} J_p)'(t) = -t^{-p} J_{p+1}(t)$ (d)$J_{p-1}(t) - J_{p+1}(t) = 2J_p'(t)$
    (e) $J_{p-1}(t) + J_{p+1}(t) = 2pt^{-1} J_p(t)$

16. Show that between any two positive zeros of J_p there is a zero of J_{p+1} and vice versa.
17. Find the first few terms of series solution and justify why the series is a solution for the following initial value problems?

    a. $y' = y^2 \sin y,\ y(0) = 1$
    b. $y' = 1 - e^{y^2},\ y(0) = 0$
    c. $y'' = e^{y^2}, y(0) = 1, y'(0) = 1.$

Reference text book:

1. E.A. Coddington, An introduction to ODE, PHI, 2003.
2. Fritz John, Partial Differential equations, Spinger, 1970.