$\mathscr{O}$-ADAPTED SLICE FILTRATIONS FOR INCOMPLETE TRANSFER SYSTEMS AND A GEOMETRIC FIXED POINT CRITERION FOR SLICE CONNECTIVITY

Abstract. Let G be a finite group and let $\mathscr{O}$ be the incomplete transfer system arising from an N_∞-operad (equivalently, from the transitive admissible H-sets H/K). We define an $\mathscr{O}$-adapted analogue of the Hill–Hopkins–Ravenel slice-connectivity filtration on the stable ∞-category Sp^G of genuine G-spectra by restricting the allowed slice cells to those built from $\mathscr{O}$-admissible orbits. For a genuine *connective* G-spectrum X and each integer $n \geq 0$, we prove a sharp criterion for $\mathscr{O}$-slice n-connectivity in terms of ordinary Postnikov-connectivity bounds on the geometric fixed points $\Phi^H(X)$ for all subgroups $H \leq G$. The scaling constant is the maximal admissible index

$$M_{\mathscr{O}}(H) \ := \ \max\{[H : K] \mid K \leq_{\mathscr{O}} H\}.$$

Precisely, X is $\mathscr{O}$-slice n-connective if and only if $\mathrm{conn}(\Phi^H(X)) \geq \lfloor n/M_{\mathscr{O}}(H) \rfloor$ for every $H \leq G$. A regular (even-cell) variant replaces $\lfloor\cdot\rfloor$ by $\lceil\cdot\rceil$.

Contents

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1. Introduction

The Hill–Hopkins–Ravenel slice filtration [2] refines the Postnikov filtration in genuine equivariant stable homotopy theory by measuring complexity using induced representation spheres. Separately, N_∞-operads of Blumberg–Hill [1] encode homotopy-commutative equivariant multiplicative structures with *incomplete* norms and transfers. Their orbit-level input is captured combinatorially by a *transfer system* on the subgroup lattice (see [5]).

In this paper we formalize a natural compatibility between these structures: given an incomplete transfer system $\mathscr{O}$, we restrict the usual slice cells to those built from $\mathscr{O}$-admissible transitive orbits. We then prove that, for connective G-spectra, the resulting notion of slice-connectivity is detected by geometric fixed points with an explicit scaling constant determined by $\mathscr{O}$.

Main result. For each $H \le G$, define the *maximal admissible index*

$$M_{\mathscr{O}}(H) \;:=\; \max\{[H:K] \mid K \le_{\mathscr{O}} H\}.$$

We define a decreasing family $\{\tau_{\ge n}^{G,\mathscr{O}}\}_{n\ge 0}$ of localizing subcategories of connective type (localizing preaisles) in Sp^G generated by certain induced spheres $G_+ \wedge_H S^{m\rho_{H/K}-\varepsilon}$, where $K \le_{\mathscr{O}} H$ and $\rho_{H/K} = \mathbb{R}[H/K]$ is the real permutation representation.

The notion of $\mathscr{O}$-slice n-connectivity is membership in $\tau_{\ge n}^{G,\mathscr{O}}$.

Theorem 1.1 (Geometric fixed point criterion)**.** *Let G be a finite group and $\mathscr{O}$ a transfer system on G arising from an N_∞-operad. Let $X \in \mathrm{Sp}^G$ be* connective *(in the genuine sense of Definition 7.1) and let $n \ge 0$. Then X is $\mathscr{O}$-slice n-connective if and only if for every subgroup $H \le G$,*

$$\mathrm{conn}\big(\Phi^H(X)\big) \;\ge\; \left\lfloor \frac{n}{M_{\mathscr{O}}(H)} \right\rfloor.$$

When $\mathscr{O}$ is complete (all transfers admissible), $M_{\mathscr{O}}(H) = |H|$ and the criterion recovers the usual geometric fixed point test for slice-connectivity (cf. [3, 7]). When $\mathscr{O}$ is trivial (only $H \le_{\mathscr{O}} H$), then $M_{\mathscr{O}}(H) = 1$ and the filtration reduces to simultaneous Postnikov connectivity of all geometric fixed points.

Organization. Sections 2–4 define transfer systems and the $\mathscr{O}$-adapted slice filtration. Sections 5–6 develop the necessary geometric fixed point calculations and orbit-counting bounds. Section 8 proves Theorem 8.1. A key point is a careful geometric reduction argument (Lemma 8.5), where we explicitly address and resolve a nontrivial closure issue involving smashing with $\widetilde{E}\mathcal{P}$.

Conventions. We work in a standard symmetric monoidal model for genuine G-spectra (e.g. orthogonal G-spectra in a complete universe), or equivalently in the underlying stable presentable ∞-category. All functors are derived, and all colimits are homotopy colimits.

For an ordinary spectrum E, we write $\mathrm{conn}(E) \geq q$ if $\pi_i(E) = 0$ for all $i < q$.

2. Transfer systems and maximal admissible index

2.1. Transfer systems.

Definition 2.1 (Transfer system). A *transfer system* on a finite group G is a relation $\leq_{\mathscr{O}}$ on the set of subgroups of G such that for all subgroups $K, H, L \leq G$:

(i) (Refines inclusion) $K \leq_{\mathscr{O}} H \Rightarrow K \leq H$.
(ii) (Reflexive) $H \leq_{\mathscr{O}} H$.
(iii) (Transitive) If $L \leq_{\mathscr{O}} K \leq_{\mathscr{O}} H$ then $L \leq_{\mathscr{O}} H$.
(iv) (Conjugation) If $K \leq_{\mathscr{O}} H$ then $gKg^{-1} \leq_{\mathscr{O}} gHg^{-1}$ for all $g \in G$.
(v) (Restriction/intersection) If $K \leq_{\mathscr{O}} H$ and $L \leq H$, then $K \cap L \leq_{\mathscr{O}} L$.

Remark 2.2. An N_∞-operad determines an indexing system of admissible finite H-sets [1], and by restricting to transitive H-sets one obtains a transfer system: the orbit H/K is admissible if and only if $K \leq_{\mathscr{O}} H$. See [5] for a detailed combinatorial account. In what follows we use only the axioms in Definition 2.1.

2.2. Maximal admissible index.

Definition 2.3 (Maximal admissible index). For each subgroup $H \leq G$, define

$$M_{\mathscr{O}}(H) \ := \ \max\{[H : K] \mid K \leq_{\mathscr{O}} H\} \ \in \ \mathbb{N}.$$

Lemma 2.4. *For all $H \leq G$, $M_{\mathscr{O}}(H) \geq 1$ and $M_{\mathscr{O}}(H)$ is conjugation invariant: $M_{\mathscr{O}}(gHg^{-1}) = M_{\mathscr{O}}(H)$. Moreover, if $L \leq H$, then $M_{\mathscr{O}}(L)$ is unchanged when computed using the restricted transfer system on H.*

Proof. The set $\{K \leq H \mid K \leq_{\mathscr{O}} H\}$ is finite and nonempty (it contains H by reflexivity), so the maximum exists and is at least $[H : H] = 1$. Conjugation invariance follows from Definition 2.1(iv) and the equality $[gHg^{-1} : gKg^{-1}] = [H : K]$. Restriction to H does not change which relations among subgroups of H hold, hence the maxima agree. □

Example 2.5. If $\mathscr{O}$ is complete (all subgroup inclusions are $\mathscr{O}$-admissible), then $M_{\mathscr{O}}(H) = |H|$. If $\mathscr{O}$ is trivial (only $H \leq_{\mathscr{O}} H$), then $M_{\mathscr{O}}(H) = 1$.

3. Localizing preaisles and connectivity

Slice-connective subcategories behave like connectivity conditions: they are closed under suspension and colimits but not generally under desuspension. We therefore use one-sided localizing subcategories.

Definition 3.1 (Localizing preaisle). Let $\mathcal{C}$ be a stable presentable ∞-category. A full subcategory $\tau \subseteq \mathcal{C}$ is a *localizing preaisle* if it is closed under:

(i) equivalences and retracts,
(ii) all small colimits,
(iii) suspension Σ,

(iv) extensions: if $A \to B \to C$ is a cofiber sequence with $A, C \in \tau$, then $B \in \tau$.

Given a set of objects $S \subseteq \mathcal{C}$, we write $\mathrm{Loc}(S)$ for the smallest localizing preaisle containing S.

Lemma 3.2 (Cofiber closure). *If τ is a localizing preaisle and $f : A \to B$ is a map with $A, B \in \tau$, then* $\mathrm{cofib}(f) \in \tau$.

Proof. Let $A \to B \to C$ be a cofiber sequence with $C = \mathrm{cofib}(f)$. Rotate to $B \to C \to \Sigma A$. Since $\Sigma A \in \tau$ and $B \in \tau$, extension closure gives $C \in \tau$. □

Definition 3.3 (Postnikov-connectivity). For $q \in \mathbb{Z}$, let $\mathrm{Sp}_{\geq q} \subseteq \mathrm{Sp}$ denote the full subcategory of q-connective spectra: $\pi_i(E) = 0$ for all $i < q$. Equivalently, $\mathrm{Sp}_{\geq q} = \mathrm{Loc}(\{S^k \mid k \geq q\})$. We write $\mathrm{conn}(E) \geq q$ to mean $E \in \mathrm{Sp}_{\geq q}$.

4. $\mathscr{O}$-slice cells and the $\mathscr{O}$-adapted slice filtration

4.1. **Permutation representations.** For $K \leq H$, let $\rho_{H/K} = \mathbb{R}[H/K]$ be the real permutation representation of H on the coset set H/K. Then $\dim(\rho_{H/K}) = [H : K]$. For a (virtual) real H-representation V, write S^V for the associated representation sphere.

4.2. **$\mathscr{O}$-slice cells.**

Definition 4.1 ($\mathscr{O}$-slice cells). Let $K \leq_{\mathscr{O}} H \leq G$, let $m \in \mathbb{N}$, and let $\varepsilon \in \{0, 1\}$, with the convention that $m \geq 1$ if $\varepsilon = 1$. Define the *$\mathscr{O}$-slice cell*

$$C(H, K; m, \varepsilon) \ := \ G_+ \wedge_H S^{m\rho_{H/K} - \varepsilon} \ \in \ \mathrm{Sp}^G,$$

and define its *$\mathscr{O}$-slice degree* by

$$\deg_{\mathscr{O}}\big(C(H, K; m, \varepsilon)\big) \ := \ m[H : K] - \varepsilon \ \in \ \mathbb{N}.$$

Remark 4.2 (Regular variant). Restricting to $\varepsilon = 0$ only yields the *regular* $\mathscr{O}$-slice filtration. In the regular case, the main criterion replaces $\lfloor \cdot \rfloor$ by $\lceil \cdot \rceil$; see Remark 9.1.

4.3. **The filtration.**

Definition 4.3 ($\mathscr{O}$-adapted slice filtration). For $n \in \mathbb{N}$, define $\tau^{G,\mathscr{O}}_{\geq n} \subseteq \mathrm{Sp}^G$ to be the localizing preaisle generated by all $\mathscr{O}$-slice cells of degree at least n:

$$\tau^{G,\mathscr{O}}_{\geq n} \ := \ \mathrm{Loc}\Big\{C(H, K; m, \varepsilon) \ \Big| \ K \leq_{\mathscr{O}} H \leq G,\ m \in \mathbb{N},\ \varepsilon \in \{0, 1\},\ \deg_{\mathscr{O}}(C(H, K; m, \varepsilon)) \geq n\Big\}.$$

A G-spectrum X is *$\mathscr{O}$-slice n-connective* if $X \in \tau^{G,\mathscr{O}}_{\geq n}$.

Remark 4.4. The filtration is decreasing: $\tau^{G,\mathscr{O}}_{\geq n+1} \subseteq \tau^{G,\mathscr{O}}_{\geq n}$. It is designed for connectivity: it is closed under suspension and colimits but not generally under desuspension.

5. Geometric fixed points

5.1. **Universal spaces for families.** For a finite group H, let $\mathcal{P}_H$ denote the family of proper subgroups of H. A universal $\mathcal{P}_H$-space $E\mathcal{P}_H$ is an H-CW complex characterized (up to H-equivalence) by

$$(E\mathcal{P}_H)^K \simeq * \quad \text{for } K \in \mathcal{P}_H, \qquad (E\mathcal{P}_H)^H = \varnothing.$$

Let $\widetilde{E}\mathcal{P}_H$ denote the cofiber of the based map $(E\mathcal{P}_H)_+ \to S^0$.

5.2. Definition and basic properties.

Definition 5.1 (Geometric fixed points)**.** For $H \le G$, the geometric fixed point functor $\Phi^H : \mathrm{Sp}^G \to \mathrm{Sp}$ is defined by

$$\Phi^H(X) \;:=\; \big(\widetilde{E}\mathcal{P}_H \wedge \mathrm{Res}^G_H X\big)^H.$$

Proposition 5.2 (Standard properties)**.** *For each $H \le G$, Φ^H is exact, preserves all colimits, and is strong symmetric monoidal. In particular, $\Phi^H(S^V) \simeq S^{V^H}$ for any finite-dimensional real G-representation V. Moreover, the family $\{\Phi^H\}_{H\le G}$ jointly detects equivalences.*

Proof. Exactness and colimit preservation are standard for geometric fixed points; see for example [6, Chapter 9] or [4, Chapter V]. Strong symmetric monoidality is also standard (geometric fixed points are built from smashing with $\widetilde{E}\mathcal{P}_H$ and then taking H-fixed points in a way compatible with the symmetric monoidal structure); see [6, Chapter 9]. Joint detection of equivalences by geometric fixed points is proved in many references; see [6, Theorem 9.1.11]. $\square$

5.3. Geometric fixed points kill proper induction.

Lemma 5.3. *Let H be a finite group and $K < H$ a proper subgroup. Then for any $Z \in \mathrm{Sp}^K$,*

$$\Phi^H\big(H_+ \wedge_K Z\big) \;\simeq\; 0.$$

Proof. Using Definition 5.1 and the projection formula for induction,

$$\Phi^H(H_+ \wedge_K Z) \simeq \big(\widetilde{E}\mathcal{P}_H \wedge H_+ \wedge_K Z\big)^H \simeq \Big(H_+ \wedge_K \big(\mathrm{Res}^H_K \widetilde{E}\mathcal{P}_H \wedge Z\big)\Big)^H.$$

Since $K < H$, every subgroup of K is proper in H, hence $\mathrm{Res}^H_K(E\mathcal{P}_H)$ is K-contractible and $\mathrm{Res}^H_K(\widetilde{E}\mathcal{P}_H) \simeq *$. Therefore the induced spectrum is contractible, hence its H-fixed points are contractible. $\square$

5.4. A double-coset formula.

Proposition 5.4 (Geometric fixed points of induction)**.** *Let $J \le G$, let $Y \in \mathrm{Sp}^J$, and let $H \le G$. Then there is a natural equivalence*

$$\Phi^H\big(G_+ \wedge_J Y\big) \;\simeq\; \bigvee_{\substack{[g]\in H\backslash G/J \\ g^{-1}Hg \le J}} \Phi^{g^{-1}Hg}(Y).$$

Proof. Restrict to H and apply the Mackey decomposition for $\mathrm{Res}^G_H \mathrm{Ind}^G_J$:

$$\mathrm{Res}^G_H(G_+ \wedge_J Y) \simeq \bigvee_{[g]\in H\backslash G/J} H_+ \wedge_{H\cap gJg^{-1}} \mathrm{Res}^{gJg^{-1}}_{H\cap gJg^{-1}}\, c_g(Y),$$

where c_g is conjugation by g. Applying Φ^H and using Lemma 5.3, all summands with $H \cap gJg^{-1} < H$ vanish. The surviving summands are exactly those with $H \le gJg^{-1}$, i.e. $g^{-1}Hg \le J$. For such g, the remaining term identifies (up to conjugation) with $\Phi^{g^{-1}Hg}(Y)$. $\square$

6. A combinatorial orbit bound and Φ^H of slice cells

We need a uniform lower bound on orbit counts of restricted admissible H-sets.

Lemma 6.1 (Orbit counting bound)**.** *Let $H \le J \le G$ and $K \le_{\mathscr{O}} J$. Then*

$$|H\backslash J/K| \;\ge\; \left\lceil \frac{[J:K]}{M_{\mathscr{O}}(H)} \right\rceil .$$

Proof. Decompose $\mathrm{Res}^J_H(J/K)$ into H-orbits:

$$\mathrm{Res}^J_H(J/K) \;\cong\; \coprod_{i\in I} H/L_i, \qquad L_i = H\cap g_iKg_i^{-1} \text{ for some } g_i\in J.$$

Since $K \le_{\mathscr{O}} J$, conjugation invariance gives $g_iKg_i^{-1} \le_{\mathscr{O}} J$. Then the restriction axiom implies $L_i = H\cap g_iKg_i^{-1} \le_{\mathscr{O}} H$. Hence $[H:L_i] \le M_{\mathscr{O}}(H)$ for all i. Counting cardinalities,

$$[J:K] = \sum_{i\in I}[H:L_i] \;\le\; |I|\cdot M_{\mathscr{O}}(H),$$

so $|I| \ge \lceil [J:K]/M_{\mathscr{O}}(H)\rceil$. Finally $|I| = |H\backslash J/K|$. □

Lemma 6.2 (Fixed vectors in permutation representations)**.** *Let L be a finite group and T a finite L-set. Then* $\dim(\mathbb{R}[T]^L) = |L\backslash T|$.

Proof. An element of $\mathbb{R}[T]$ is a real-valued function on T; it is L-fixed iff it is constant on each L-orbit. Thus $\mathbb{R}[T]^L \cong \mathbb{R}^{(L\backslash T)}$. □

Lemma 6.3 (An integer inequality)**.** *Let $m\in\mathbb{N}$, $a, M\in\mathbb{N}$ with $M\ge 1$, and $\varepsilon\in\{0,1\}$. Then*

$$m\left\lceil \frac{a}{M}\right\rceil - \varepsilon \;\ge\; \left\lfloor \frac{ma-\varepsilon}{M}\right\rfloor .$$

Proof. Let $t = \lceil a/M\rceil$, so $tM \ge a$. Then $mtM - \varepsilon \ge ma - \varepsilon$, hence $mt - \varepsilon/M \ge (ma-\varepsilon)/M$. Taking floors and using that $mt\in\mathbb{Z}$ and $0\le \varepsilon/M < 1$,

$$\left\lfloor \frac{ma-\varepsilon}{M}\right\rfloor \;\le\; \lfloor mt - \varepsilon/M\rfloor \;=\; mt-\varepsilon,$$

which is the desired inequality. □

Proposition 6.4 (Connectivity of Φ^H of $\mathscr{O}$-slice cells)**.** *Let $C = C(J,K;m,\varepsilon)$ be an $\mathscr{O}$-slice cell with* $\deg_{\mathscr{O}}(C) = m[J:K]-\varepsilon \ge n$. *Then for every subgroup $H\le G$,*

$$\mathrm{conn}\big(\Phi^H(C)\big) \;\ge\; \left\lfloor \frac{n}{M_{\mathscr{O}}(H)}\right\rfloor .$$

Proof. By Proposition 5.4, $\Phi^H(C)$ is a finite wedge of spectra of the form

$$\Phi^L\big(S^{m\rho_{J/K}-\varepsilon}\big) \;\simeq\; S^{m\dim((\rho_{J/K})^L)-\varepsilon},$$

where $L = g^{-1}Hg \le J$. By Lemma 6.2 with $T = J/K$, $\dim((\rho_{J/K})^L) = |L\backslash J/K|$. By Lemma 6.1,

$$|L\backslash J/K| \;\ge\; \left\lceil \frac{[J:K]}{M_{\mathscr{O}}(L)}\right\rceil = \left\lceil \frac{[J:K]}{M_{\mathscr{O}}(H)}\right\rceil,$$

using conjugation invariance of $M_{\mathscr{O}}(-)$ (Lemma 2.4). Therefore each sphere summand has dimension at least

$$m\left\lceil \frac{[J:K]}{M_{\mathscr{O}}(H)} \right\rceil - \varepsilon \;\geq\; \left\lfloor \frac{m[J:K]-\varepsilon}{M_{\mathscr{O}}(H)} \right\rfloor \;\geq\; \left\lfloor \frac{n}{M_{\mathscr{O}}(H)} \right\rfloor,$$

by Lemma 6.3 and the assumption $\deg_{\mathscr{O}}(C) \geq n$. A wedge of q-connective spectra is q-connective since $\mathrm{Sp}_{\geq q}$ is closed under colimits. ☐

7. Connective G-spectra

We impose a genuine connectivity hypothesis to avoid pathologies in isotropy separation arguments.

Definition 7.1 (Genuine connectivity)**.** A genuine G-spectrum $X \in \mathrm{Sp}^G$ is *connective* if for every subgroup $H \leq G$, the categorical fixed point spectrum X^H is 0-connective, i.e. $\pi_i(X^H) = 0$ for all $i < 0$.

Remark 7.2. Equivalently, the homotopy Mackey functors $\underline{\pi}_k(X)$ vanish for $k < 0$. This notion is strictly stronger than underlying (nonequivariant) connectivity.

8. Main theorem: $\mathscr{O}$-slice connectivity detected by geometric fixed points

For $H \leq G$ and $n \in \mathbb{N}$, define

$$q_{\mathscr{O}}(H,n) \;:=\; \left\lfloor \frac{n}{M_{\mathscr{O}}(H)} \right\rfloor \;\in\; \mathbb{N}.$$

Theorem 8.1 (Geometric fixed point characterization)**.** *Let G be a finite group and let $\mathscr{O}$ be a transfer system on G arising from an N_∞-operad. Let $X \in \mathrm{Sp}^G$ be connective and let $n \geq 0$. Then the following are equivalent:*

(1) $X \in \tau_{\geq n}^{G,\mathscr{O}}$.
(2) *For every subgroup $H \leq G$,*

$$\mathrm{conn}\big(\Phi^H(X)\big) \;\geq\; q_{\mathscr{O}}(H,n) = \left\lfloor \frac{n}{M_{\mathscr{O}}(H)} \right\rfloor.$$

8.1. A functoriality lemma.

Lemma 8.2 (Induction preserves $\mathscr{O}$-slice connectivity)**.** *Let $H \leq G$ and let $\mathscr{O}|_H$ denote the restricted transfer system on H. Then for every $n \geq 0$, induction $\mathrm{Ind}_H^G : \mathrm{Sp}^H \to \mathrm{Sp}^G$ carries $\tau_{\geq n}^{H,\mathscr{O}|_H}$ into $\tau_{\geq n}^{G,\mathscr{O}}$.*

Proof. Induction is exact and preserves all colimits, hence preserves the closure operations defining $\mathrm{Loc}(-)$. A generating $\mathscr{O}|_H$-cell has the form $H_+ \wedge_{H'} S^{m\rho_{H'/K}-\varepsilon}$ with $K \leq_{\mathscr{O}} H' \leq H$ and degree $\geq n$. Inducing to G yields

$$\mathrm{Ind}_H^G\big(H_+ \wedge_{H'} S^{m\rho_{H'/K}-\varepsilon}\big) \simeq G_+ \wedge_{H'} S^{m\rho_{H'/K}-\varepsilon},$$

which is a generating $\mathscr{O}$-cell of the same degree. ☐

8.2. **Isotropy separation and geometric spectra.** Let $\mathcal{P} = \mathcal{P}_G$ be the family of proper subgroups of G. The isotropy separation cofiber sequence is

$$(E\mathcal{P})_+ \wedge X \longrightarrow X \longrightarrow \widetilde{E}\mathcal{P} \wedge X. \tag{8.1}$$

Write $L_{\mathrm{geom}} = \widetilde{E}\mathcal{P} \wedge (-)$. A G-spectrum Y is *geometric* if $Y \simeq L_{\mathrm{geom}}(Y)$, equivalently if $\Phi^H(Y) \simeq 0$ for all proper $H < G$. Let $\mathrm{Sp}^G_{\mathrm{geom}} \subseteq \mathrm{Sp}^G$ denote the full subcategory of geometric G-spectra.

Lemma 8.3. *There is a natural equivalence $\Phi^G(\widetilde{E}\mathcal{P}) \simeq S^0$.*

Proof. Apply Φ^G to the cofiber sequence $(E\mathcal{P})_+ \to S^0 \to \widetilde{E}\mathcal{P}$. Every cell of $E\mathcal{P}$ has isotropy a proper subgroup, so $(E\mathcal{P})_+$ is built from induced spectra from proper subgroups. By Lemma 5.3 and exactness, $\Phi^G((E\mathcal{P})_+) \simeq 0$. Since $\Phi^G(S^0) \simeq S^0$, the cofiber identifies $\Phi^G(\widetilde{E}\mathcal{P}) \simeq S^0$. $\square$

Lemma 8.4 (Geometric fixed points give an equivalence on geometric spectra)**.** *The functor $\Phi^G : \mathrm{Sp}^G_{\mathrm{geom}} \to \mathrm{Sp}$ is an equivalence. A quasi-inverse $F : \mathrm{Sp} \to \mathrm{Sp}^G_{\mathrm{geom}}$ is given by*

$$F(E) \ := \ \widetilde{E}\mathcal{P} \wedge \mathrm{Inf}(E),$$

where $\mathrm{Inf} : \mathrm{Sp} \to \mathrm{Sp}^G$ *denotes inflation (trivial G-action).*

Proof. We first show $\Phi^G \circ F \simeq \mathrm{id}_{\mathrm{Sp}}$. Using Lemma 8.3 and strong monoidality of Φ^G,

$$\Phi^G(F(E)) \simeq \Phi^G(\widetilde{E}\mathcal{P}) \wedge \Phi^G(\mathrm{Inf}(E)) \simeq S^0 \wedge \Phi^G(\mathrm{Inf}(E)).$$

The composite $\Phi^G \circ \mathrm{Inf} : \mathrm{Sp} \to \mathrm{Sp}$ is exact and preserves all colimits (Proposition 5.2) and sends S^0 to S^0. In the stable presentable ∞-category Sp, any exact colimit-preserving endofunctor is determined (up to equivalence) by its value on S^0 (it is smashing with that value). Hence $\Phi^G \circ \mathrm{Inf} \simeq \mathrm{id}_{\mathrm{Sp}}$, so $\Phi^G(F(E)) \simeq E$.

Conversely, let $Y \in \mathrm{Sp}^G_{\mathrm{geom}}$. The unit map $F(\Phi^G(Y)) \to Y$ induces an equivalence on Φ^G by the previous paragraph. For every proper $H < G$, both $\Phi^H(F(\Phi^G(Y)))$ and $\Phi^H(Y)$ are zero because both spectra are geometric. Since geometric fixed points jointly detect equivalences (Proposition 5.2), the map $F(\Phi^G(Y)) \to Y$ is an equivalence. $\square$

8.3. **Geometric reduction for the slice-connective subcategory.**

Lemma 8.5 (Geometric reduction)**.** *Fix $n \geq 0$ and set $q = \lfloor n/M_{\mathscr{O}}(G) \rfloor$. Assume Theorem 8.1 holds for all proper subgroups of G. If $Y \in \mathrm{Sp}^G$ is geometric, then*

$$Y \in \tau^{G,\mathscr{O}}_{\geq n} \quad \Longleftrightarrow \quad \mathrm{conn}(\Phi^G(Y)) \geq q.$$

Proof. Let $\mathcal{U} \subseteq \mathrm{Sp}$ be the essential image $\mathcal{U} := \Phi^G(\tau^{G,\mathscr{O}}_{\geq n} \cap \mathrm{Sp}^G_{\mathrm{geom}})$. Since $\Phi^G : \mathrm{Sp}^G_{\mathrm{geom}} \to \mathrm{Sp}$ is an equivalence (Lemma 8.4) and $\tau^{G,\mathscr{O}}_{\geq n} \cap \mathrm{Sp}^G_{\mathrm{geom}}$ is a localizing preaisle in $\mathrm{Sp}^G_{\mathrm{geom}}$, the subcategory $\mathcal{U}$ is a localizing preaisle in Sp.

Step 1: $\mathcal{U} \subseteq \mathrm{Sp}_{\geq q}$. If $Y \in \tau^{G,\mathscr{O}}_{\geq n} \cap \mathrm{Sp}^G_{\mathrm{geom}}$, then $Y \in \tau^{G,\mathscr{O}}_{\geq n}$. By Proposition 6.4 and the argument (1) $\Rightarrow$ (2) from the proof of Theorem 8.1 below (which does not require induction), we obtain $\mathrm{conn}(\Phi^G(Y)) \geq \lfloor n/M_{\mathscr{O}}(G) \rfloor = q$. Thus $\Phi^G(Y) \in \mathrm{Sp}_{\geq q}$, proving $\mathcal{U} \subseteq \mathrm{Sp}_{\geq q}$.

Step 2: $\mathrm{Sp}_{\geq q} \subseteq \mathcal{U}$. Choose $K_{\max} \leq_{\mathscr{O}} G$ with $[G : K_{\max}] = M_{\mathscr{O}}(G)$; such a subgroup exists by Definition 2.3. Write $M := M_{\mathscr{O}}(G)$ and $n = qM + r$ with $0 \leq r < M$. For each integer $t \geq q$, define the odd $\mathscr{O}$-slice cell

$$C_t \;:=\; S^{(t+1)\rho_{G/K_{\max}}-1} \;\in\; \mathrm{Sp}^G.$$

Its $\mathscr{O}$-slice degree is $(t+1)M - 1$, and since $t \geq q$ we have

$$(t+1)M - 1 \;\geq\; (q+1)M - 1 \;=\; qM + (M-1) \;\geq\; qM + r \;=\; n,$$

hence $C_t \in \tau^{G,\mathscr{O}}_{\geq n}$ by Definition 4.3.

Set $Y_t := \widetilde{E}\mathcal{P} \wedge C_t$. Then Y_t is geometric, and

$$\Phi^G(Y_t) \;\simeq\; \Phi^G(\widetilde{E}\mathcal{P}) \wedge \Phi^G(C_t) \;\simeq\; S^0 \wedge S^t \;\simeq\; S^t,$$

using Lemma 8.3 and the fact that $(\rho_{G/K_{\max}})^G \cong \mathbb{R}$ is 1-dimensional.

It remains to prove $Y_t \in \tau^{G,\mathscr{O}}_{\geq n}$. This is the nontrivial point: *we do not assume that smashing with $\widetilde{E}\mathcal{P}$ preserves $\tau^{G,\mathscr{O}}_{\geq n}$*. Instead we prove membership via isotropy separation and the inductive hypothesis.

Consider the isotropy separation cofiber sequence (8.1) for $X = C_t$:

$$(E\mathcal{P})_+ \wedge C_t \longrightarrow C_t \longrightarrow \widetilde{E}\mathcal{P} \wedge C_t \;=\; Y_t.$$

We already know $C_t \in \tau^{G,\mathscr{O}}_{\geq n}$. By Lemma 3.2, it suffices to show $(E\mathcal{P})_+ \wedge C_t \in \tau^{G,\mathscr{O}}_{\geq n}$.

The based G-CW complex $(E\mathcal{P})_+$ is built from cells of the form $G/H_+ \wedge S^k$ with $H < G$ and $k \geq 0$. Smashing with C_t expresses $(E\mathcal{P})_+ \wedge C_t$ as a filtered colimit built from extensions of suspensions of spectra $G/H_+ \wedge C_t$ with $H < G$. Since $\tau^{G,\mathscr{O}}_{\geq n}$ is closed under suspensions, colimits, and extensions, it suffices to show $G/H_+ \wedge C_t \in \tau^{G,\mathscr{O}}_{\geq n}$ for each proper $H < G$.

For such H, we have $G/H_+ \wedge C_t \simeq \mathrm{Ind}^G_H(\mathrm{Res}^G_H C_t)$. We claim $\mathrm{Res}^G_H C_t \in \tau^{H,\mathscr{O}|_H}_{\geq n}$. Indeed:

- *Connectivity:* For any $L \leq H$, the categorical fixed points are $(\mathrm{Res}^G_H C_t)^L \simeq (C_t)^L \simeq S^{(t+1)\dim((\rho_{G/K_{\max}})^L)-1}$. Since $\dim((\rho_{G/K_{\max}})^L) = |L\backslash G/K_{\max}| \geq 1$, this sphere has dimension $\geq t \geq 0$, so $(\mathrm{Res}^G_H C_t)^L$ is 0-connective for all L, hence $\mathrm{Res}^G_H C_t$ is connective.
- *Geometric fixed point bounds:* For each $L \leq H$, we have $\Phi^L(\mathrm{Res}^G_H C_t) \simeq \Phi^L(C_t)$. Since C_t is itself a generating $\mathscr{O}$-slice cell of degree $\geq n$, Proposition 6.4 gives $\mathrm{conn}(\Phi^L(C_t)) \geq \lfloor n/M_{\mathscr{O}}(L) \rfloor$.

Thus $\mathrm{Res}^G_H C_t$ satisfies condition (2) of Theorem 8.1 for the proper subgroup H. By the inductive hypothesis (Theorem 8.1 for H), we conclude $\mathrm{Res}^G_H C_t \in \tau^{H,\mathscr{O}|_H}_{\geq n}$. Lemma 8.2 then yields $G/H_+ \wedge C_t \in \tau^{G,\mathscr{O}}_{\geq n}$, as required.

Therefore $(E\mathcal{P})_+ \wedge C_t \in \tau^{G,\mathscr{O}}_{\geq n}$, and hence $Y_t = \widetilde{E}\mathcal{P} \wedge C_t \in \tau^{G,\mathscr{O}}_{\geq n}$.

Consequently, $S^t \simeq \Phi^G(Y_t) \in \mathcal{U}$ for every $t \geq q$. Since $\mathrm{Sp}_{\geq q} = \mathrm{Loc}(\{S^t \mid t \geq q\})$ is the smallest localizing preaisle containing these spheres, we obtain $\mathrm{Sp}_{\geq q} \subseteq \mathcal{U}$.

Combining Steps 1 and 2 yields $\mathcal{U} = \mathrm{Sp}_{\geq q}$. Transporting back along the equivalence $\Phi^G : \mathrm{Sp}^G_{\mathrm{geom}} \simeq \mathrm{Sp}$ gives the claimed criterion for geometric Y. $\square$

8.4. **Proof of Theorem 8.1.**

Proof of Theorem 8.1. We prove (1) ⇒ (2) for all finite G and then (2) ⇒ (1) by induction on $|G|$.

(1) ⇒ (2). Assume $X \in \tau_{\geq n}^{G,\mathscr{O}}$. Fix $H \leq G$. Since Φ^H is exact and preserves colimits (Proposition 5.2), the image $\Phi^H(\tau_{\geq n}^{G,\mathscr{O}})$ is contained in the localizing preaisle generated by the spectra $\Phi^H(C)$ for $\mathscr{O}$-slice cells C of degree $\geq n$. By Proposition 6.4, each such $\Phi^H(C)$ is $q_{\mathscr{O}}(H,n) = \lfloor n/M_{\mathscr{O}}(H) \rfloor$-connective. Since $\mathrm{Sp}_{\geq q_{\mathscr{O}}(H,n)}$ is a localizing preaisle, we conclude $\mathrm{conn}(\Phi^H(X)) \geq \lfloor n/M_{\mathscr{O}}(H) \rfloor$.

(2) ⇒ (1). We argue by induction on $|G|$.

If $|G| = 1$, then $\mathrm{Sp}^G \simeq \mathrm{Sp}$, $M_{\mathscr{O}}(G) = 1$, and $\tau_{\geq n}^{G,\mathscr{O}} = \mathrm{Sp}_{\geq n}$ by Definition 4.3, so the statement is ordinary connectivity.

Assume $|G| > 1$ and that the theorem holds for all proper subgroups of G. Let $X \in \mathrm{Sp}^G$ be connective and satisfy condition (2). Consider isotropy separation (8.1). Since $\tau_{\geq n}^{G,\mathscr{O}}$ is extension-closed, it suffices to show that both $(E\mathcal{P})_+ \wedge X$ and $\widetilde{E}\mathcal{P} \wedge X$ lie in $\tau_{\geq n}^{G,\mathscr{O}}$.

Step 1: $(E\mathcal{P})_+ \wedge X \in \tau_{\geq n}^{G,\mathscr{O}}$. The based G-CW complex $(E\mathcal{P})_+$ is built from cells $G/H_+ \wedge S^k$ with $H < G$ and $k \geq 0$. Smashing with X expresses $(E\mathcal{P})_+ \wedge X$ as a filtered colimit built from extensions of suspensions of spectra $G/H_+ \wedge X$ with $H < G$. Since $\tau_{\geq n}^{G,\mathscr{O}}$ is closed under colimits, suspensions, and extensions, it suffices to show $G/H_+ \wedge X \in \tau_{\geq n}^{G,\mathscr{O}}$ for each proper $H < G$.

For such H, $G/H_+ \wedge X \simeq \mathrm{Ind}_H^G(\mathrm{Res}_H^G X)$. For every $L \leq H$, we have $\Phi^L(\mathrm{Res}_H^G X) \simeq \Phi^L(X)$, and condition (2) for X gives

$$\mathrm{conn}(\Phi^L(\mathrm{Res}_H^G X)) \;\geq\; \left\lfloor \frac{n}{M_{\mathscr{O}}(L)} \right\rfloor.$$

Moreover, $\mathrm{Res}_H^G X$ is connective as an H-spectrum. By the inductive hypothesis applied to H, we obtain $\mathrm{Res}_H^G X \in \tau_{\geq n}^{H,\mathscr{O}|_H}$. Lemma 8.2 then yields $G/H_+ \wedge X \in \tau_{\geq n}^{G,\mathscr{O}}$, completing Step 1.

Step 2: $\widetilde{E}\mathcal{P} \wedge X \in \tau_{\geq n}^{G,\mathscr{O}}$. Set $Y = \widetilde{E}\mathcal{P} \wedge X$. Then Y is geometric. Let $q = \lfloor n/M_{\mathscr{O}}(G) \rfloor$. By Lemma 8.3 and monoidality, $\Phi^G(Y) \simeq \Phi^G(X)$. Condition (2) with $H = G$ gives $\mathrm{conn}(\Phi^G(X)) \geq q$, hence $\mathrm{conn}(\Phi^G(Y)) \geq q$. Applying Lemma 8.5 shows $Y \in \tau_{\geq n}^{G,\mathscr{O}}$, completing Step 2.

Finally, since both ends of (8.1) lie in $\tau_{\geq n}^{G,\mathscr{O}}$, extension closure gives $X \in \tau_{\geq n}^{G,\mathscr{O}}$. □

9. Variants and special cases

Remark 9.1 (Regular $\mathscr{O}$-slice filtration and ceil-scaling)**.** Define the *regular* $\mathscr{O}$-slice filtration by restricting Definition 4.1 to $\varepsilon = 0$, i.e. generators $G_+ \wedge_H S^{m\rho_{H/K}}$ only. The same argument as above (with the obvious modifications) yields the analogous characterization: for connective X,

$$X \in \bar{\tau}_{\geq n}^{G,\mathscr{O}} \quad \Longleftrightarrow \quad \mathrm{conn}(\Phi^H(X)) \geq \left\lceil \frac{n}{M_{\mathscr{O}}(H)} \right\rceil \quad \text{for all } H \leq G.$$

In the complete-transfer case $M_{\mathscr{O}}(H) = |H|$, this recovers the Hill–Yarnall criterion for regular slice connectivity [3].

Remark 9.2 (Complete and trivial transfer systems)**.** If $\mathscr{O}$ is complete, then $M_{\mathscr{O}}(H) = |H|$ and Theorem 8.1 specializes to the familiar floor-scaling characterization of (full) slice-connectivity; compare [3, 7]. If $\mathscr{O}$ is trivial, then $M_{\mathscr{O}}(H) = 1$ and Theorem 8.1 becomes:

$$X \in \tau_{\geq n}^{G,\mathscr{O}} \iff \mathrm{conn}(\Phi^H(X)) \geq n \text{ for all } H \leq G,$$

i.e. the filtration reduces to simultaneous Postnikov connectivity of all geometric fixed points.

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