$\mathscr{O}$-ADAPTED SLICE FILTRATIONS FOR INCOMPLETE TRANSFER SYSTEMS AND A GEOMETRIC FIXED POINT CRITERION FOR SLICE CONNECTIVITY

Abstract. Fix a finite group G and let $\mathscr{O}$ be a transfer system arising from an N_∞-operad. We define an $\mathscr{O}$-adapted analogue of the Hill–Hopkins–Ravenel slice-connectivity filtration on the stable category of genuine G-spectra by restricting the allowed slice cells to those built from $\mathscr{O}$-admissible transitive orbits H/K. For a genuinely connective G-spectrum X and each integer $n \geq 0$, we prove that $\mathscr{O}$-slice n-connectivity of X is equivalent to an explicit collection of Postnikov-connectivity bounds on the geometric fixed points $\Phi^H(X)$ for all subgroups $H \leq G$. The scaling constant is the maximal admissible index

$$M_{\mathscr{O}}(H) \;:=\; \max\{[H:K] \mid K \leq_{\mathscr{O}} H\}.$$

Precisely, X is $\mathscr{O}$-slice n-connective if and only if $\mathrm{conn}(\Phi^H(X)) \geq \lfloor n/M_{\mathscr{O}}(H) \rfloor$ for every $H \leq G$. A regular (even-cell) variant replaces $\lfloor \cdot \rfloor$ by $\lceil \cdot \rceil$.

Contents

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1. Introduction

The slice filtration of Hill–Hopkins–Ravenel [2] provides a powerful equivariant refinement of Postnikov connectivity, measuring complexity by representation spheres induced from subgroups. Independently, Blumberg–Hill N_∞-operads [1] encode homotopy-commutative equivariant multiplicative structures with *incomplete* norms and transfers. The orbit-level data of an N_∞-operad can be packaged as a *transfer system* on the subgroup lattice (see Rubin [5]).

The purpose of this paper is twofold:

- We define a slice-connectivity filtration on genuine G-spectra adapted to a given transfer system $\mathscr{O}$ by restricting the allowed slice cells to those associated to $\mathscr{O}$-admissible orbits.
- For genuinely connective G-spectra, we prove that this $\mathscr{O}$-slice connectivity is detected by geometric fixed points, with a sharp scaling constant determined by $\mathscr{O}$.

The scaling constant. For each subgroup $H \le G$, define the *maximal admissible index*

$$M_{\mathscr{O}}(H) \ := \ \max\{[H:K] \mid K \le_{\mathscr{O}} H\}.$$

This depends only on the restriction of $\mathscr{O}$ to subgroups of H and is invariant under conjugation.

Main theorem. We define $\tau_{\ge n}^{G,\mathscr{O}} \subseteq \mathrm{Sp}^G$ as the localizing preaisle generated by the $\mathscr{O}$-slice cells of degree $\ge n$ (Definition 4.3 below). Membership in $\tau_{\ge n}^{G,\mathscr{O}}$ is called $\mathscr{O}$-slice n-connectivity.

Theorem 1.1 (Geometric fixed point criterion). *Let G be a finite group and let $\mathscr{O}$ be a transfer system arising from an N_∞-operad. Let $X \in \mathrm{Sp}^G$ be* genuinely connective *(Definition 7.1) and let $n \ge 0$. Then the following are equivalent:*

(1) $X \in \tau_{\ge n}^{G,\mathscr{O}}$.
(2) *For every subgroup $H \le G$,*

$$\mathrm{conn}\big(\Phi^H(X)\big) \ \ge \ \left\lfloor \frac{n}{M_{\mathscr{O}}(H)} \right\rfloor.$$

A regular (even-cell) variant is stated in Remark 9.1.

A word on rigor. A key technical point is the *geometric reduction* argument in Lemma 8.4. Its proof must carefully separate categorical and geometric fixed points; an earlier draft conflated these in one step. We correct this by invoking a tom Dieck splitting argument (Lemma 7.3) to verify genuine connectivity for certain representation spheres.

Conventions. We work in a standard symmetric monoidal model for genuine G-spectra (e.g. orthogonal G-spectra in a complete universe), or equivalently in the underlying stable presentable ∞-category. All functors are derived, and all colimits are homotopy colimits.

For an ordinary spectrum E, we write $\mathrm{conn}(E) \ge q$ if $\pi_i(E) = 0$ for all $i < q$.

2. Transfer systems and maximal admissible index

2.1. Transfer systems.

Definition 2.1 (Transfer system). A *transfer system* on a finite group G is a relation $\le_{\mathscr{O}}$ on the set of subgroups of G such that for all subgroups $K, H, L \le G$:

(i) (Refines inclusion) $K \le_{\mathscr{O}} H \Rightarrow K \le H$.
(ii) (Reflexive) $H \le_{\mathscr{O}} H$.
(iii) (Transitive) If $L \le_{\mathscr{O}} K \le_{\mathscr{O}} H$ then $L \le_{\mathscr{O}} H$.
(iv) (Conjugation) If $K \le_{\mathscr{O}} H$ then $gKg^{-1} \le_{\mathscr{O}} gHg^{-1}$ for all $g \in G$.
(v) (Restriction/intersection) If $K \le_{\mathscr{O}} H$ and $L \le H$, then $K \cap L \le_{\mathscr{O}} L$.

Remark 2.2. An N_∞-operad determines an indexing system of admissible finite H-sets [1]. Restricting to transitive admissible H-sets H/K yields a transfer system: H/K is admissible if and only if $K \le_{\mathscr{O}} H$. See [5] for a combinatorial account. In this paper we only use the axioms of Definition 2.1.

2.2. Maximal admissible index.

Definition 2.3 (Maximal admissible index)**.** For each subgroup $H \leq G$, define

$$M_{\mathscr{O}}(H) \ := \ \max\{[H : K] \mid K \leq_{\mathscr{O}} H\} \ \in \ \mathbb{N}.$$

Lemma 2.4. *For all $H \leq G$, $M_{\mathscr{O}}(H) \geq 1$ and $M_{\mathscr{O}}(H)$ is conjugation invariant: $M_{\mathscr{O}}(gHg^{-1}) = M_{\mathscr{O}}(H)$. Moreover, if $L \leq H$, then $M_{\mathscr{O}}(L)$ is unchanged when computed using the restricted transfer system on H.*

Proof. The set $\{K \leq H \mid K \leq_{\mathscr{O}} H\}$ is finite and nonempty (it contains H by reflexivity), so the maximum exists and is at least $[H : H] = 1$. Conjugation invariance follows from Definition 2.1(iv) and the equality $[gHg^{-1} : gKg^{-1}] = [H : K]$. Restriction to H does not change which relations among subgroups of H hold, hence the maxima agree. □

Example 2.5. If $\mathscr{O}$ is trivial (only $H \leq_{\mathscr{O}} H$), then $M_{\mathscr{O}}(H) = 1$ for all H. If $\mathscr{O}$ is complete (all subgroup inclusions are $\mathscr{O}$-admissible), then $M_{\mathscr{O}}(H) = |H|$.

3. Localizing preaisles and Postnikov connectivity

Slice-connectivity conditions are closed under suspension and colimits but not generally under desuspension. It is therefore convenient to use a one-sided notion of localizing subcategory.

Definition 3.1 (Localizing preaisle)**.** Let $\mathcal{C}$ be a stable presentable ∞-category. A full subcategory $\tau \subseteq \mathcal{C}$ is a *localizing preaisle* if it is closed under:

(i) equivalences and retracts,
(ii) all small colimits,
(iii) suspension Σ,
(iv) extensions: if $A \to B \to C$ is a cofiber sequence with $A, C \in \tau$, then $B \in \tau$.

Given a set of objects $S \subseteq \mathcal{C}$, we write $\mathrm{Loc}(S)$ for the smallest localizing preaisle containing S.

Lemma 3.2 (Cofiber closure)**.** *If τ is a localizing preaisle and $f : A \to B$ is a map with $A, B \in \tau$, then* $\mathrm{cofib}(f) \in \tau$.

Proof. Let $A \to B \to C$ be a cofiber sequence with $C = \mathrm{cofib}(f)$. Rotate to $B \to C \to \Sigma A$. Since $\Sigma A \in \tau$ and $B \in \tau$, extension closure gives $C \in \tau$. □

Definition 3.3 (Postnikov-connective spectra)**.** For $q \in \mathbb{Z}$, let $\mathrm{Sp}_{\geq q} \subseteq \mathrm{Sp}$ denote the full subcategory of q-connective spectra: $\pi_i(E) = 0$ for all $i < q$. Equivalently, $\mathrm{Sp}_{\geq q} = \mathrm{Loc}(\{S^k \mid k \geq q\})$. We write $\mathrm{conn}(E) \geq q$ to mean $E \in \mathrm{Sp}_{\geq q}$.

4. $\mathscr{O}$-slice cells and the $\mathscr{O}$-adapted slice filtration

4.1. **Permutation representations.** For $K \leq H$, let $\rho_{H/K} = \mathbb{R}[H/K]$ be the real permutation representation of H on the coset set H/K. Then $\dim(\rho_{H/K}) = [H : K]$. For a (virtual) real H-representation V, write S^V for the associated representation sphere.

4.2. $\mathscr{O}$-slice cells.

Definition 4.1 ($\mathscr{O}$-slice cells)**.** Let $K \leq_{\mathscr{O}} H \leq G$, let $m \in \mathbb{N}$, and let $\varepsilon \in \{0, 1\}$, with the convention that $m \geq 1$ if $\varepsilon = 1$. Define the *$\mathscr{O}$-slice cell*

$$C(H, K; m, \varepsilon) \ := \ G_+ \wedge_H S^{m\rho_{H/K} - \varepsilon} \ \in \ \mathrm{Sp}^G,$$

and define its *$\mathscr{O}$-slice degree* by

$$\deg_{\mathscr{O}}\big(C(H, K; m, \varepsilon)\big) \ := \ m[H : K] - \varepsilon \ \in \ \mathbb{N}.$$

Remark 4.2 (Regular variant)**.** Restricting to $\varepsilon = 0$ yields the *regular* $\mathscr{O}$-slice filtration. In the regular case, the geometric fixed point criterion replaces $\lfloor\cdot\rfloor$ by $\lceil\cdot\rceil$; see Remark 9.1.

4.3. The filtration.

Definition 4.3 ($\mathscr{O}$-adapted slice filtration)**.** For $n \in \mathbb{N}$, define $\tau_{\geq n}^{G,\mathscr{O}} \subseteq \mathrm{Sp}^G$ to be the localizing preaisle generated by all $\mathscr{O}$-slice cells of degree at least n:

$$\tau_{\geq n}^{G,\mathscr{O}} \;:=\; \mathrm{Loc}\Big\{C(H,K;m,\varepsilon) \;\Big|\; K \leq_{\mathscr{O}} H \leq G,\ m \in \mathbb{N},\ \varepsilon \in \{0,1\},\ \deg_{\mathscr{O}}(C(H,K;m,\varepsilon)) \geq n\Big\}.$$

A G-spectrum X is *$\mathscr{O}$-slice n-connective* if $X \in \tau_{\geq n}^{G,\mathscr{O}}$.

Remark 4.4. The filtration is decreasing: $\tau_{\geq n+1}^{G,\mathscr{O}} \subseteq \tau_{\geq n}^{G,\mathscr{O}}$. It is tailored to connectivity: it is closed under suspension and colimits but not under desuspension.

5. Geometric fixed points

5.1. Universal spaces for families.

For a finite group H, let $\mathcal{P}_H$ denote the family of proper subgroups of H. A universal $\mathcal{P}_H$-space $E\mathcal{P}_H$ is an H-CW complex characterized up to H-equivalence by

$$(E\mathcal{P}_H)^K \simeq * \quad \text{for } K \in \mathcal{P}_H, \qquad (E\mathcal{P}_H)^H = \varnothing.$$

Let $\widetilde{E}\mathcal{P}_H$ denote the cofiber of the based map $(E\mathcal{P}_H)_+ \to S^0$.

5.2. Definition and standard properties.

Definition 5.1 (Geometric fixed points)**.** For $H \leq G$, the geometric fixed point functor $\Phi^H : \mathrm{Sp}^G \to \mathrm{Sp}$ is defined by

$$\Phi^H(X) \;:=\; \big(\widetilde{E}\mathcal{P}_H \wedge \mathrm{Res}^G_H X\big)^H.$$

Proposition 5.2 (Standard properties)**.** *For each $H \leq G$, Φ^H is exact, preserves all colimits, and is strong symmetric monoidal. In particular, $\Phi^H(S^V) \simeq S^{V^H}$ for any finite-dimensional real G-representation V. Moreover, the family $\{\Phi^H\}_{H \leq G}$ jointly detects equivalences in Sp^G.*

Proof. Exactness, colimit preservation, and strong symmetric monoidality are standard; see, for example, Schwede [6, §7]. Detection of equivalences by geometric fixed points is part of Schwede's Theorem 7.12 [6]. □

5.3. Geometric fixed points kill proper induction.

Lemma 5.3. *Let H be a finite group and $K < H$ a proper subgroup. Then for any $Z \in \mathrm{Sp}^K$,*

$$\Phi^H\big(H_+ \wedge_K Z\big) \;\simeq\; 0.$$

Proof. Using Definition 5.1 and the projection formula for induction,

$$\Phi^H(H_+ \wedge_K Z) \simeq \big(\widetilde{E}\mathcal{P}_H \wedge H_+ \wedge_K Z\big)^H \simeq \Big(H_+ \wedge_K \big(\mathrm{Res}^H_K \widetilde{E}\mathcal{P}_H \wedge Z\big)\Big)^H.$$

Since $K < H$, every subgroup of K is proper in H, hence $\mathrm{Res}^H_K(E\mathcal{P}_H)$ is K-contractible and $\mathrm{Res}^H_K(\widetilde{E}\mathcal{P}_H) \simeq *$. Therefore the induced spectrum is contractible, hence its H-fixed points are contractible. □

5.4. A double-coset formula.

Proposition 5.4 (Geometric fixed points of induction)**.** *Let $J \leq G$, let $Y \in \mathrm{Sp}^J$, and let $H \leq G$. Then there is a natural equivalence*

$$\Phi^H\big(G_+ \wedge_J Y\big) \;\simeq\; \bigvee_{\substack{[g] \in H\backslash G/J \\ g^{-1}Hg \leq J}} \Phi^{g^{-1}Hg}(Y).$$

Proof. Restrict to H and apply the Mackey decomposition for $\mathrm{Res}^G_H \mathrm{Ind}^G_J$:

$$\mathrm{Res}^G_H(G_+ \wedge_J Y) \simeq \bigvee_{[g]\in H\backslash G/J} H_+ \wedge_{H\cap gJg^{-1}} \mathrm{Res}^{gJg^{-1}}_{H\cap gJg^{-1}} c_g(Y),$$

where c_g is conjugation by g. Applying Φ^H and using Lemma 5.3, all summands with $H \cap gJg^{-1} < H$ vanish. The surviving summands are exactly those with $H \le gJg^{-1}$, i.e. $g^{-1}Hg \le J$. For such g, the remaining term identifies (up to conjugation) with $\Phi^{g^{-1}Hg}(Y)$. □

6. Orbit bounds and connectivity of generators

We next extract a uniform lower bound on orbit counts of restricted admissible H-sets, and use it to control the connectivity of geometric fixed points of the generators of $\tau^{G,\mathscr{O}}_{\ge n}$.

Lemma 6.1 (Orbit counting bound). *Let $H \le J \le G$ and $K \le_{\mathscr{O}} J$. Then*

$$|H\backslash J/K| \;\ge\; \left\lceil \frac{[J:K]}{M_{\mathscr{O}}(H)} \right\rceil.$$

Proof. Decompose the restricted H-set $\mathrm{Res}^J_H(J/K)$ into H-orbits:

$$\mathrm{Res}^J_H(J/K) \;\cong\; \coprod_{i\in I} H/L_i, \qquad L_i = H \cap g_iKg_i^{-1} \text{ for some } g_i \in J.$$

Since $K \le_{\mathscr{O}} J$, conjugation invariance gives $g_iKg_i^{-1} \le_{\mathscr{O}} J$. Then the restriction axiom implies $L_i = H \cap g_iKg_i^{-1} \le_{\mathscr{O}} H$. Hence $[H : L_i] \le M_{\mathscr{O}}(H)$ for all i. Counting cardinalities,

$$[J:K] = \sum_{i\in I}[H:L_i] \;\le\; |I|\cdot M_{\mathscr{O}}(H),$$

so $|I| \ge \lceil [J:K]/M_{\mathscr{O}}(H)\rceil$. Finally $|I| = |H\backslash J/K|$. □

Lemma 6.2 (Fixed vectors in permutation representations). *Let L be a finite group and T a finite L-set. Then* $\dim(\mathbb{R}[T]^L) = |L\backslash T|$.

Proof. An element of $\mathbb{R}[T]$ is a real-valued function on T; it is L-fixed iff it is constant on each L-orbit. Thus $\mathbb{R}[T]^L \cong \mathbb{R}^{(L\backslash T)}$. □

Lemma 6.3 (An integer inequality). *Let $m \in \mathbb{N}$, $a, M \in \mathbb{N}$ with $M \ge 1$, and $\varepsilon \in \{0,1\}$. Then*

$$m\left\lceil \frac{a}{M}\right\rceil - \varepsilon \;\ge\; \left\lfloor \frac{ma-\varepsilon}{M}\right\rfloor.$$

Proof. Let $t = \lceil a/M\rceil$, so $tM \ge a$. Then $mtM - \varepsilon \ge ma - \varepsilon$, hence

$$mt - \varepsilon/M \;\ge\; (ma-\varepsilon)/M.$$

Taking floors gives

$$\left\lfloor \frac{ma-\varepsilon}{M}\right\rfloor \;\le\; \lfloor mt - \varepsilon/M\rfloor.$$

Since $mt \in \mathbb{Z}$ and $\varepsilon \in \{0,1\}$ with $M \ge 1$, we have $\lfloor -\varepsilon/M\rfloor = -\varepsilon$ (in fact, $\lfloor 0\rfloor = 0$ and $\lfloor -1/M\rfloor = -1$ for all $M \ge 1$). Therefore

$$\lfloor mt - \varepsilon/M\rfloor = mt + \lfloor -\varepsilon/M\rfloor = mt - \varepsilon,$$

which is the desired inequality. □

Proposition 6.4 (Connectivity of geometric fixed points of $\mathscr{O}$-slice cells). *Let $C = C(J,K;m,\varepsilon)$ be an $\mathscr{O}$-slice cell with* $\deg_{\mathscr{O}}(C) = m[J:K] - \varepsilon \ge n$. *Then for every subgroup $H \le G$,*

$$\mathrm{conn}\big(\Phi^H(C)\big) \;\ge\; \left\lfloor \frac{n}{M_{\mathscr{O}}(H)}\right\rfloor.$$

Proof. By Proposition 5.4, $\Phi^H(C)$ is a finite wedge of spectra of the form

$$\Phi^L\big(S^{m\rho_{J/K}-\varepsilon}\big) \;\simeq\; S^{m\dim((\rho_{J/K})^L)-\varepsilon},$$

where $L = g^{-1}Hg \le J$. By Lemma 6.2 with $T = J/K$, $\dim((\rho_{J/K})^L) = |L\backslash J/K|$. By Lemma 6.1,

$$|L\backslash J/K| \;\ge\; \left\lceil \frac{[J:K]}{M_{\mathscr{O}}(L)} \right\rceil = \left\lceil \frac{[J:K]}{M_{\mathscr{O}}(H)} \right\rceil,$$

using conjugation invariance of $M_{\mathscr{O}}(-)$ (Lemma 2.4). Therefore each sphere summand has dimension at least

$$m\left\lceil \frac{[J:K]}{M_{\mathscr{O}}(H)} \right\rceil - \varepsilon \;\ge\; \left\lfloor \frac{m[J:K]-\varepsilon}{M_{\mathscr{O}}(H)} \right\rfloor \;\ge\; \left\lfloor \frac{n}{M_{\mathscr{O}}(H)} \right\rfloor,$$

by Lemma 6.3 and the assumption $\deg_{\mathscr{O}}(C) \ge n$. A wedge of q-connective spectra is q-connective since $\mathrm{Sp}_{\ge q}$ is closed under colimits. □

7. Genuine connectivity and a tom Dieck splitting estimate

Definition 7.1 (Genuine connectivity)**.** A genuine G-spectrum $X \in \mathrm{Sp}^G$ is *connective* if for every subgroup $H \le G$, the categorical fixed point spectrum X^H is 0-connective, i.e. $\pi_i(X^H) = 0$ for all $i < 0$.

Remark 7.2. Equivalently, the negative homotopy Mackey functors $\underline{\pi}_k(X)$ vanish for $k < 0$. This is strictly stronger than underlying (nonequivariant) connectivity.

Lemma 7.3 (Shifted representation spheres are connective)**.** *Let H be a finite group and let V be a finite-dimensional real H-representation. Assume that $\dim(V^K) \ge 1$ for every subgroup $K \le H$. Then the genuine H-spectrum S^{V-1} is connective in the sense of Definition 7.1.*

Proof. Fix a subgroup $L \le H$. We must show that the categorical fixed points $(S^{V-1})^L$ are 0-connective.

Restricting from H to L identifies S^{V-1} with $S^{(\mathrm{Res}^H_L V)-1}$ as L-spectra, so we may work in Sp^L. Write $S^{V-1} \simeq \Sigma^{-1}\Sigma^\infty_L S^V$ as L-spectra. By the tom Dieck splitting for categorical fixed points of equivariant suspension spectra (see, e.g., Schwede [6, Example 7.7 and §6] or [4]), there is a natural splitting

$$\big(\Sigma^\infty_L S^V\big)^L \;\simeq\; \bigvee_{(K\le L)} \Sigma^\infty\Big(EW_LK_+ \wedge_{W_LK} (S^V)^K\Big),$$

where $W_LK = N_L(K)/K$ and the wedge runs over L-conjugacy classes of subgroups $K \le L$. For each $K \le L$, we have $(S^V)^K \simeq S^{V^K}$ and $\dim(V^K) \ge 1$ by hypothesis. Since EW_LK_+ is a based CW complex with cells in dimensions ≥ 0, the smash $EW_LK_+ \wedge_{W_LK} S^{V^K}$ is a based CW complex with all cells in dimensions $\ge \dim(V^K) \ge 1$. Hence each summand $\Sigma^\infty(EW_LK_+ \wedge_{W_LK} S^{V^K})$ is at least 1-connective, and so

$$\mathrm{conn}\Big(\big(\Sigma^\infty_L S^V\big)^L\Big) \;\ge\; 1.$$

Desuspending once yields

$$\mathrm{conn}\big((S^{V-1})^L\big) \;=\; \mathrm{conn}\Big(\Sigma^{-1}\big(\Sigma^\infty_L S^V\big)^L\Big) \;\ge\; 0.$$

Since this holds for every $L \le H$, the spectrum S^{V-1} is connective. □

8. Main theorem: $\mathscr{O}$-slice connectivity detected by geometric fixed points

For $H \le G$ and $n \in \mathbb{N}$, define

$$q_{\mathscr{O}}(H,n) \;:=\; \left\lfloor \frac{n}{M_{\mathscr{O}}(H)} \right\rfloor \;\in\; \mathbb{N}.$$

8.1. **A functoriality lemma.**

Lemma 8.1 (Induction preserves $\mathscr{O}$-slice connectivity)**.** *Let $H \le G$ and let $\mathscr{O}|_H$ denote the restricted transfer system on H. Then for every $n \ge 0$, induction $\mathrm{Ind}_H^G : \mathrm{Sp}^H \to \mathrm{Sp}^G$ carries $\tau_{\ge n}^{H,\mathscr{O}|_H}$ into $\tau_{\ge n}^{G,\mathscr{O}}$.*

Proof. Induction is exact and preserves all colimits, hence preserves the closure operations defining $\mathrm{Loc}(-)$. A generating $\mathscr{O}|_H$-cell has the form $H_+ \wedge_{H'} S^{m\rho_{H'/K} - \varepsilon}$ with $K \le_{\mathscr{O}} H' \le H$ and degree $\ge n$. Inducing to G yields

$$\mathrm{Ind}_H^G\big(H_+ \wedge_{H'} S^{m\rho_{H'/K}-\varepsilon}\big) \simeq G_+ \wedge_{H'} S^{m\rho_{H'/K}-\varepsilon},$$

which is a generating $\mathscr{O}$-cell of the same degree. □

8.2. **Isotropy separation and geometric spectra.**

Let $\mathcal{P} = \mathcal{P}_G$ be the family of proper subgroups of G. The isotropy separation cofiber sequence is

$$(E\mathcal{P})_+ \wedge X \longrightarrow X \longrightarrow \widetilde{E}\mathcal{P} \wedge X. \tag{8.1}$$

Write $L_{\mathrm{geom}} = \widetilde{E}\mathcal{P} \wedge (-)$. A G-spectrum Y is *geometric* if $Y \simeq L_{\mathrm{geom}}(Y)$, equivalently if $\Phi^H(Y) \simeq 0$ for all proper $H < G$. Let $\mathrm{Sp}^G_{\mathrm{geom}} \subseteq \mathrm{Sp}^G$ denote the full subcategory of geometric G-spectra.

Lemma 8.2. *There is a natural equivalence $\Phi^G(\widetilde{E}\mathcal{P}) \simeq S^0$.*

Proof. Apply Φ^G to the cofiber sequence $(E\mathcal{P})_+ \to S^0 \to \widetilde{E}\mathcal{P}$. Every cell of $E\mathcal{P}$ has isotropy a proper subgroup, so $(E\mathcal{P})_+$ is built from induced spectra from proper subgroups. By Lemma 5.3 and exactness, $\Phi^G((E\mathcal{P})_+) \simeq 0$. Since $\Phi^G(S^0) \simeq S^0$, the cofiber identifies $\Phi^G(\widetilde{E}\mathcal{P}) \simeq S^0$. □

Lemma 8.3 (Geometric fixed points are an equivalence on geometric spectra)**.** *The functor $\Phi^G : \mathrm{Sp}^G_{\mathrm{geom}} \to \mathrm{Sp}$ is an equivalence. A quasi-inverse $F : \mathrm{Sp} \to \mathrm{Sp}^G_{\mathrm{geom}}$ is given by*

$$F(E) \;:=\; \widetilde{E}\mathcal{P} \wedge \mathrm{Inf}(E),$$

where $\mathrm{Inf} : \mathrm{Sp} \to \mathrm{Sp}^G$ denotes inflation (trivial G-action).

Proof. We first show $\Phi^G \circ F \simeq \mathrm{id}_{\mathrm{Sp}}$. Using Lemma 8.2 and strong monoidality of Φ^G,

$$\Phi^G(F(E)) \simeq \Phi^G(\widetilde{E}\mathcal{P}) \wedge \Phi^G(\mathrm{Inf}(E)) \simeq S^0 \wedge \Phi^G(\mathrm{Inf}(E)).$$

The composite $\Phi^G \circ \mathrm{Inf} : \mathrm{Sp} \to \mathrm{Sp}$ is exact and preserves all colimits (Proposition 5.2) and sends S^0 to S^0. In the stable presentable ∞-category Sp, any exact colimit-preserving endofunctor is determined (up to equivalence) by its value on S^0, and is given by smashing with that value. Hence $\Phi^G \circ \mathrm{Inf} \simeq \mathrm{id}_{\mathrm{Sp}}$, so $\Phi^G(F(E)) \simeq E$.

Conversely, let $Y \in \mathrm{Sp}^G_{\mathrm{geom}}$. The *counit* map $F(\Phi^G(Y)) \to Y$ induces an equivalence on Φ^G by the previous paragraph. For every proper $H < G$, both $\Phi^H(F(\Phi^G(Y)))$ and $\Phi^H(Y)$ are zero because both spectra are geometric. Since geometric fixed points jointly detect equivalences (Proposition 5.2), the counit map $F(\Phi^G(Y)) \to Y$ is an equivalence. □

8.3. **Geometric reduction for $\tau_{\ge n}^{G,\mathscr{O}}$.**

Lemma 8.4 (Geometric reduction)**.** *Fix $n \ge 0$ and set $q = \lfloor n/M_{\mathscr{O}}(G) \rfloor$. Assume Theorem 1.1 holds for all proper subgroups of G. If $Y \in \mathrm{Sp}^G$ is geometric, then*

$$Y \in \tau_{\ge n}^{G,\mathscr{O}} \quad\Longleftrightarrow\quad \mathrm{conn}(\Phi^G(Y)) \ge q.$$

Proof. Let $\mathcal{U} \subseteq \mathrm{Sp}$ be the essential image $\mathcal{U} := \Phi^G(\tau_{\ge n}^{G,\mathscr{O}} \cap \mathrm{Sp}^G_{\mathrm{geom}})$. Since $\Phi^G : \mathrm{Sp}^G_{\mathrm{geom}} \to \mathrm{Sp}$ is an equivalence (Lemma 8.3) and $\tau_{\ge n}^{G,\mathscr{O}} \cap \mathrm{Sp}^G_{\mathrm{geom}}$ is a localizing preaisle in $\mathrm{Sp}^G_{\mathrm{geom}}$, the subcategory $\mathcal{U}$ is a localizing preaisle in Sp.

Step 1: $\mathcal{U} \subseteq \mathrm{Sp}_{\ge q}$. If $Y \in \tau_{\ge n}^{G,\mathscr{O}} \cap \mathrm{Sp}^G_{\mathrm{geom}}$, then $Y \in \tau_{\ge n}^{G,\mathscr{O}}$. By Proposition 6.4 and exactness/colimit preservation of Φ^G, the argument (1) ⇒ (2) in the proof of Theorem 1.1 (given below) shows that $\mathrm{conn}(\Phi^G(Y)) \ge q$. Thus $\Phi^G(Y) \in \mathrm{Sp}_{\ge q}$, proving $\mathcal{U} \subseteq \mathrm{Sp}_{\ge q}$.

Step 2: $\mathrm{Sp}_{\geq q} \subseteq \mathfrak{U}$. Choose $K_{\max} \leq_{\mathscr{O}} G$ with $[G : K_{\max}] = M_{\mathscr{O}}(G)$; such a subgroup exists by Definition 2.3. Write $M := M_{\mathscr{O}}(G)$ and $n = qM + r$ with $0 \leq r < M$. For each integer $t \geq q$, define the odd $\mathscr{O}$-slice cell

$$C_t \ := \ S^{(t+1)\rho_{G/K_{\max}}-1} \ \in \ \mathrm{Sp}^G.$$

Its $\mathscr{O}$-slice degree is $(t+1)M - 1$, and since $t \geq q$ we have

$$(t+1)M - 1 \ \geq \ (q+1)M - 1 \ = \ qM + (M-1) \ \geq \ qM + r \ = \ n,$$

hence $C_t \in \tau_{\geq n}^{G,\mathscr{O}}$ by Definition 4.3.

Set $Y_t := \widetilde{E}\mathcal{P} \wedge C_t$. Then Y_t is geometric, and

$$\Phi^G(Y_t) \ \simeq \ \Phi^G(\widetilde{E}\mathcal{P}) \wedge \Phi^G(C_t) \ \simeq \ S^0 \wedge S^t \ \simeq \ S^t,$$

using Lemma 8.2 and the fact that $\dim((\rho_{G/K_{\max}})^G) = |G\backslash G/K_{\max}| = 1$ (Lemma 6.2).

It remains to prove $Y_t \in \tau_{\geq n}^{G,\mathscr{O}}$. We do *not* assume that smashing with $\widetilde{E}\mathcal{P}$ preserves $\tau_{\geq n}^{G,\mathscr{O}}$. Instead we prove membership via isotropy separation and the inductive hypothesis.

Consider the isotropy separation cofiber sequence (8.1) for $X = C_t$:

$$(E\mathcal{P})_+ \wedge C_t \longrightarrow C_t \longrightarrow \widetilde{E}\mathcal{P} \wedge C_t \ = \ Y_t.$$

We already know $C_t \in \tau_{\geq n}^{G,\mathscr{O}}$. By Lemma 3.2, it suffices to show $(E\mathcal{P})_+ \wedge C_t \in \tau_{\geq n}^{G,\mathscr{O}}$.

The based G-CW complex $(E\mathcal{P})_+$ is built from cells of the form $G/H_+ \wedge S^k$ with $H < G$ and $k \geq 0$. Smashing with C_t expresses $(E\mathcal{P})_+ \wedge C_t$ as a filtered colimit built from extensions of suspensions of spectra $G/H_+ \wedge C_t$ with $H < G$. Since $\tau_{\geq n}^{G,\mathscr{O}}$ is closed under suspensions, colimits, and extensions, it suffices to show $G/H_+ \wedge C_t \in \tau_{\geq n}^{G,\mathscr{O}}$ for each proper $H < G$.

For such H, we have $G/H_+ \wedge C_t \simeq \mathrm{Ind}_H^G(\mathrm{Res}_H^G C_t)$. We claim $\mathrm{Res}_H^G C_t \in \tau_{\geq n}^{H,\mathscr{O}|_H}$. Indeed:

- *Connectivity:* Write $V = (t+1)\rho_{G/K_{\max}}$. Then $\mathrm{Res}_H^G C_t \simeq S^{\mathrm{Res}_H^G V - 1}$. For any subgroup $L \leq H$,

$$\dim\big((\mathrm{Res}_H^G V)^L\big) = \dim(V^L) = (t+1)\dim\big((\rho_{G/K_{\max}})^L\big) = (t+1)\,|L\backslash G/K_{\max}| \ \geq \ t+1 \ \geq \ 1.$$

  Therefore Lemma 7.3 applies and implies that $\mathrm{Res}_H^G C_t$ is connective.
- *Geometric fixed point bounds:* For each $L \leq H$, we have $\Phi^L(\mathrm{Res}_H^G C_t) \simeq \Phi^L(C_t)$. Since C_t is itself a generating $\mathscr{O}$-slice cell of degree $\geq n$, Proposition 6.4 gives $\mathrm{conn}(\Phi^L(C_t)) \geq \lfloor n/M_{\mathscr{O}}(L) \rfloor$.

Thus $\mathrm{Res}_H^G C_t$ satisfies condition (2) of Theorem 1.1 for the proper subgroup H. By the inductive hypothesis, we conclude $\mathrm{Res}_H^G C_t \in \tau_{\geq n}^{H,\mathscr{O}|_H}$. Lemma 8.1 then yields $G/H_+ \wedge C_t \in \tau_{\geq n}^{G,\mathscr{O}}$, as required.

Therefore $(E\mathcal{P})_+ \wedge C_t \in \tau_{\geq n}^{G,\mathscr{O}}$, and hence $Y_t = \widetilde{E}\mathcal{P} \wedge C_t \in \tau_{\geq n}^{G,\mathscr{O}}$.

Consequently, $S^t \simeq \Phi^G(Y_t) \in \mathfrak{U}$ for every $t \geq q$. Since $\mathrm{Sp}_{\geq q} = \mathrm{Loc}(\{S^t \mid t \geq q\})$ is the smallest localizing preaisle containing these spheres, we obtain $\mathrm{Sp}_{\geq q} \subseteq \mathfrak{U}$.

Combining Steps 1 and 2 yields $\mathfrak{U} = \mathrm{Sp}_{\geq q}$. Transporting back along the equivalence $\Phi^G : \mathrm{Sp}^G_{\mathrm{geom}} \simeq \mathrm{Sp}$ gives the claimed criterion for geometric Y. $\square$

8.4. **Proof of Theorem 1.1.**

Proof of Theorem 1.1. We prove (1) $\Rightarrow$ (2) for all finite G and then (2) $\Rightarrow$ (1) by induction on $|G|$.

(1) $\Rightarrow$ (2). Assume $X \in \tau_{\geq n}^{G,\mathscr{O}}$. Fix $H \leq G$. Since Φ^H is exact and preserves colimits (Proposition 5.2), the image $\Phi^H(\tau_{\geq n}^{G,\mathscr{O}})$ is contained in the localizing preaisle generated by the spectra $\Phi^H(C)$ for $\mathscr{O}$-slice cells C of degree $\geq n$. By Proposition 6.4, each such $\Phi^H(C)$ is $q_{\mathscr{O}}(H,n) = \lfloor n/M_{\mathscr{O}}(H) \rfloor$-connective. Since $\mathrm{Sp}_{\geq q_{\mathscr{O}}(H,n)}$ is a localizing preaisle, we conclude $\mathrm{conn}(\Phi^H(X)) \geq \lfloor n/M_{\mathscr{O}}(H) \rfloor$.

(2) $\Rightarrow$ (1). We argue by induction on $|G|$.

If $|G| = 1$, then $\mathrm{Sp}^G \simeq \mathrm{Sp}$, $M_{\mathscr{O}}(G) = 1$, and $\tau_{\geq n}^{G,\mathscr{O}} = \mathrm{Sp}_{\geq n}$ by Definition 4.3, so the statement is ordinary connectivity.

Assume $|G| > 1$ and that the theorem holds for all proper subgroups of G. Let $X \in \mathrm{Sp}^G$ be connective and satisfy condition (2). Consider isotropy separation (8.1). Since $\tau_{\geq n}^{G,\mathscr{O}}$ is extension-closed, it suffices to show that both $(E\mathcal{P})_+ \wedge X$ and $\widetilde{E}\mathcal{P} \wedge X$ lie in $\tau_{\geq n}^{G,\mathscr{O}}$.

Step 1: $(E\mathcal{P})_+ \wedge X \in \tau_{\geq n}^{G,\mathscr{O}}$. The based G-CW complex $(E\mathcal{P})_+$ is built from cells $G/H_+ \wedge S^k$ with $H < G$ and $k \geq 0$. Smashing with X expresses $(E\mathcal{P})_+ \wedge X$ as a filtered colimit built from extensions of suspensions of spectra $G/H_+ \wedge X$ with $H < G$. Since $\tau_{\geq n}^{G,\mathscr{O}}$ is closed under colimits, suspensions, and extensions, it suffices to show $G/H_+ \wedge X \in \tau_{\geq n}^{G,\mathscr{O}}$ for each proper $H < G$.

For such H, $G/H_+ \wedge X \simeq \mathrm{Ind}_H^G(\mathrm{Res}_H^G X)$. The restriction $\mathrm{Res}_H^G X$ is connective as an H-spectrum, since for every $L \leq H$, $(\mathrm{Res}_H^G X)^L \simeq X^L$. Moreover, for every $L \leq H$, $\Phi^L(\mathrm{Res}_H^G X) \simeq \Phi^L(X)$, so condition (2) for X gives

$$\mathrm{conn}(\Phi^L(\mathrm{Res}_H^G X)) \;\geq\; \left\lfloor \frac{n}{M_{\mathscr{O}}(L)} \right\rfloor .$$

By the inductive hypothesis applied to H, we obtain $\mathrm{Res}_H^G X \in \tau_{\geq n}^{H,\mathscr{O}|_H}$. Lemma 8.1 then yields $G/H_+ \wedge X \in \tau_{\geq n}^{G,\mathscr{O}}$, completing Step 1.

Step 2: $\widetilde{E}\mathcal{P} \wedge X \in \tau_{\geq n}^{G,\mathscr{O}}$. Set $Y = \widetilde{E}\mathcal{P} \wedge X$. Then Y is geometric. Let $q = \lfloor n/M_{\mathscr{O}}(G) \rfloor$. By Lemma 8.2 and monoidality, $\Phi^G(Y) \simeq \Phi^G(X)$. Condition (2) with $H = G$ gives $\mathrm{conn}(\Phi^G(X)) \geq q$, hence $\mathrm{conn}(\Phi^G(Y)) \geq q$. Applying Lemma 8.4 shows $Y \in \tau_{\geq n}^{G,\mathscr{O}}$, completing Step 2.

Finally, since both ends of (8.1) lie in $\tau_{\geq n}^{G,\mathscr{O}}$, extension closure gives $X \in \tau_{\geq n}^{G,\mathscr{O}}$. $\square$

9. Variants and special cases

Remark 9.1 (Regular $\mathscr{O}$-slice filtration and ceil-scaling)**.** Define the *regular* $\mathscr{O}$-slice filtration by restricting Definition 4.1 to $\varepsilon = 0$, i.e. generators $G_+ \wedge_H S^{m\rho_{H/K}}$ only. The same argument as above (with the obvious modifications) yields the analogous characterization: for connective X,

$$X \in \bar{\tau}_{\geq n}^{G,\mathscr{O}} \quad\Longleftrightarrow\quad \mathrm{conn}(\Phi^H(X)) \geq \left\lceil \frac{n}{M_{\mathscr{O}}(H)} \right\rceil \quad \text{for all } H \leq G.$$

In the complete-transfer case $M_{\mathscr{O}}(H) = |H|$, this recovers the Hill–Yarnall criterion for regular slice connectivity [3].

Remark 9.2 (Trivial and complete transfer systems)**.** If $\mathscr{O}$ is trivial, then $M_{\mathscr{O}}(H) = 1$ and Theorem 1.1 becomes:

$$X \in \tau_{\geq n}^{G,\mathscr{O}} \iff \mathrm{conn}(\Phi^H(X)) \geq n \text{ for all } H \leq G,$$

i.e. the filtration reduces to simultaneous Postnikov connectivity of all geometric fixed points. If $\mathscr{O}$ is complete, then $M_{\mathscr{O}}(H) = |H|$ and the scaling constant specializes to the familiar order of H.

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