UNIFORM LATTICES WITH TORSION AND CLOSED MANIFOLDS WITH $\mathbb{Q}$-ACYCLIC UNIVERSAL COVERS: THE ODD-TORSION OBSTRUCTION AND THE INDEX AT INFINITY

Abstract. We clarify the status of the problem of realizing torsionful uniform lattices as fundamental groups of closed manifolds with $\mathbb{Q}$-acyclic universal cover. Compactly supported Lefschetz numbers do not vanish for fixed-point-free proper maps on noncompact manifolds: the translation $x \mapsto x+1$ on $\mathbb{R}$ has $L_c = -1$. This reflects an "index at infinity," made precise via Čech cohomology of the one-point compactification and a comparison theorem for manifolds. For uniform lattices, a deep theorem of Fowler rules out the case of odd prime torsion; the purely 2-primary case remains open. We also record Fowler's constructions showing that torsion (including 2-torsion) can occur in fundamental groups of closed manifolds with $\mathbb{Q}$-acyclic universal covers, so torsion-freeness is not a general topological consequence of $\mathbb{Q}$-acyclicity of the universal cover.

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1. Introduction

Let G be a real semisimple Lie group and $\Gamma < G$ a *uniform lattice*, i.e. Γ is discrete and G/Γ is compact. If Γ has torsion, then G/Γ is an orbifold rather than a manifold. This motivates the following question.

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Problem. *Suppose that* Γ *is a uniform lattice in a real semisimple Lie group, and that* Γ *contains an element of order* 2. *Is it possible for* Γ *to be the fundamental group of a closed manifold* M *whose universal cover* $\widetilde{M}$ *is* $\mathbb{Q}$*-acyclic?*

A common strategy is to compute $H_c^*(\widetilde{M};\mathbb{Q})$ by Poincaré duality with compact supports and then attempt to apply a Lefschetz-type fixed point theorem to finite-order deck transformations. Two facts govern the outcome:

- compactly supported Lefschetz numbers of proper maps detect a contribution from infinity, so fixed-point-free proper maps can have $L_c \neq 0$;
- nevertheless, for uniform lattices with *odd* torsion there is a genuine obstruction, proved by Fowler, using controlled surgery and ρ-invariants.

The goal of this note is to present a self-contained account of these points and to state the best current resolution of the lattice problem.

Notation. All manifolds are compact connected topological manifolds without boundary unless explicitly stated otherwise. Cohomology is singular cohomology with $\mathbb{Q}$-coefficients unless denoted by $\check{H}^*$ (Čech cohomology). For a locally compact Hausdorff space X, $H_c^*(X;\mathbb{Q})$ denotes compactly supported cohomology.

2. Compactly supported cohomology of a $\mathbb{Q}$-acyclic universal cover

2.1. Compactly supported cohomology and proper maps.

Definition 2.1 (Compactly supported cohomology)**.** Let X be locally compact Hausdorff. Define

$$H_c^i(X;\mathbb{Q}) \;:=\; \varinjlim_{K\subseteq X \text{ compact}} H^i(X, X\setminus K;\mathbb{Q}).$$

Lemma 2.2 (Proper maps act on H_c^*)**.** *If* $f : X \to Y$ *is proper between locally compact Hausdorff spaces, then pullback on relative cohomology induces natural maps*

$$f^* : H_c^i(Y;\mathbb{Q}) \to H_c^i(X;\mathbb{Q}) \qquad \text{for all } i \geq 0.$$

Proof. For each compact $K \subseteq Y$, properness implies $f^{-1}(K)$ is compact. Thus f is a map of pairs $(X, X\setminus f^{-1}(K)) \to (Y, Y\setminus K)$, inducing pullbacks $H^i(Y, Y\setminus K) \to H^i(X, X\setminus f^{-1}(K))$ compatible with the direct system in K. Passing to the direct limit gives f^* on H_c^i. □

2.2. Orientability and Poincaré duality.

Lemma 2.3. *If* X *is a connected simply connected topological manifold, then* X *is orientable.*

Proof. The orientation local system is classified by the sign homomorphism $\pi_1(X) \to \{\pm 1\}$ recording whether loops preserve or reverse local orientation. If $\pi_1(X) = 0$, this homomorphism is trivial, so the local system is constant and X is orientable. □

Theorem 2.4 (Poincaré duality with compact supports)**.** *Let* X *be an oriented* n*-manifold without boundary. Then cap product with the Borel–Moore fundamental class induces natural isomorphisms*

$$H_c^i(X;\mathbb{Q}) \cong H_{n-i}(X;\mathbb{Q}) \qquad \text{for all } i \geq 0.$$

Reference. See [5, Thm. 3.35]. □

2.3. The computation.

Definition 2.5 ($\mathbb{Q}$-acyclic)**.** A space X is $\mathbb{Q}$-*acyclic* if $\widetilde{H}_i(X;\mathbb{Q}) = 0$ for all $i \geq 0$.

Theorem 2.6 (Compactly supported cohomology of a $\mathbb{Q}$-acyclic universal cover)**.** *Let* M^n *be a closed connected* n*-manifold and let* $X = \widetilde{M}$ *be its universal cover. If* X *is* $\mathbb{Q}$*-acyclic, then*

$$H_c^i(X;\mathbb{Q}) \;\cong\; \begin{cases} \mathbb{Q}, & i = n, \\ 0, & i \neq n, \end{cases} \qquad \text{and hence} \qquad \chi_c(X;\mathbb{Q}) = (-1)^n.$$

Proof. The universal cover X is a connected simply connected n-manifold, hence orientable by Lemma 2.3. By Theorem 2.4,

$$H_c^i(X;\mathbb{Q}) \cong H_{n-i}(X;\mathbb{Q}).$$

Since X is $\mathbb{Q}$-acyclic, $H_j(X;\mathbb{Q}) = 0$ for $j > 0$ and $H_0(X;\mathbb{Q}) \cong \mathbb{Q}$. Therefore $H_c^i(X;\mathbb{Q}) = 0$ for $i < n$, while $H_c^n(X;\mathbb{Q}) \cong H_0(X;\mathbb{Q}) \cong \mathbb{Q}$. For $i > n$, $H_{n-i}(X;\mathbb{Q}) = 0$ in negative degree, so $H_c^i(X;\mathbb{Q}) = 0$. The Euler characteristic follows from the definition of χ_c. □

3. Proper maps and a fixed-point-free counterexample

3.1. **Compactly supported Lefschetz number.** Assume $H_c^i(X;\mathbb{Q})$ is finite-dimensional for all i and vanishes for $i \gg 0$. For a proper self-map $f : X \to X$, define the compactly supported Lefschetz number

$$L_c(f;X) := \sum_{i\geq 0}(-1)^i \operatorname{tr}\big(f^* : H_c^i(X;\mathbb{Q}) \to H_c^i(X;\mathbb{Q})\big) \in \mathbb{Q}.$$

3.2. **Proper homotopy invariance.**

Definition 3.1. A homotopy $H : X \times [0,1] \to Y$ between maps $f_0, f_1 : X \to Y$ is *proper* if H is proper.

Lemma 3.2 (Proper homotopy invariance)**.** *Let $f_0, f_1 : X \to Y$ be proper maps between locally compact Hausdorff spaces. If f_0 and f_1 are properly homotopic, then $f_0^* = f_1^* : H_c^i(Y;\mathbb{Q}) \to H_c^i(X;\mathbb{Q})$ for all i.*

Proof. Fix a compact $K \subseteq Y$ and a proper homotopy $H : X \times [0,1] \to Y$ between f_0 and f_1. Then $H^{-1}(K)$ is compact in $X \times [0,1]$, so its projection $L := \operatorname{pr}_X(H^{-1}(K)) \subseteq X$ is compact. If $x \notin L$, then $(x,t) \notin H^{-1}(K)$ for all t, hence $H(x,t) \notin K$ for all t. Thus H restricts to a homotopy of pairs

$$H : (X, X \setminus L) \times [0,1] \to (Y, Y \setminus K)$$

between f_0 and f_1 as maps of pairs $(X, X\setminus L) \to (Y, Y \setminus K)$. Homotopy invariance of relative cohomology implies equality of the induced maps $H^i(Y, Y\setminus K) \to H^i(X, X \setminus L)$. Passing to the direct limit over compact K yields $f_0^* = f_1^*$ on H_c^i. □

3.3. **The translation on $\mathbb{R}$.**

Theorem 3.3 (Fixed-point-free proper map with $L_c \neq 0$)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be the translation $f(x) = x + 1$. Then f is a fixed-point-free homeomorphism (hence proper) and*

$$L_c(f;\mathbb{R}) = -1 \neq 0.$$

Proof. The map f is a homeomorphism, hence proper, and $\operatorname{Fix}(f) = \varnothing$.

Compute $H_c^*(\mathbb{R};\mathbb{Q})$. Since $\mathbb{R}$ is an oriented 1-manifold, Theorem 2.4 gives $H_c^1(\mathbb{R};\mathbb{Q}) \cong H_0(\mathbb{R};\mathbb{Q}) \cong \mathbb{Q}$ and $H_c^i(\mathbb{R};\mathbb{Q}) = 0$ for $i \neq 1$. Hence $L_c(f;\mathbb{R}) = -\operatorname{tr}(f^*|H_c^1(\mathbb{R};\mathbb{Q}))$.

Define a homotopy $H : \mathbb{R} \times [0,1] \to \mathbb{R}$ by $H(x,t) = x + t$, from id to f. If $K \subseteq \mathbb{R}$ is compact, say $K \subseteq [-M, M]$, then $H^{-1}(K) \subseteq [-M-1, M] \times [0,1]$, which is compact. Thus H is a proper homotopy, so by Lemma 3.2 we have $f^* = \mathrm{id}^*$ on $H_c^1(\mathbb{R};\mathbb{Q})$. Therefore $\operatorname{tr}(f^*) = 1$ and $L_c(f;\mathbb{R}) = -1$. □

4. Čech cohomology at infinity and the "index at infinity"

Theorem 3.3 reflects a general phenomenon: for proper maps on noncompact spaces, compactly supported traces can record a contribution from infinity. To formalize this for universal covers with potentially wild end structure, it is convenient to use Čech cohomology on the one-point compactification.

4.1. **One-point compactification and proper extensions.** Let X be noncompact locally compact Hausdorff and let $X^+ = X \sqcup \{\infty\}$ be the one-point compactification. A neighborhood basis of ∞ is given by $U_K := (X \setminus K) \cup \{\infty\}$ as K ranges over compact subsets of X.

Lemma 4.1. *If $f : X \to X$ is proper, then f extends uniquely to a continuous map $f^+ : X^+ \to X^+$ with $f^+(\infty) = \infty$.*

Proof. Define $f^+|_X = f$ and $f^+(\infty) = \infty$. Continuity on X is clear. If U_K is a neighborhood of ∞ in X^+, then properness implies $f^{-1}(K)$ is compact, hence $U_{f^{-1}(K)}$ is a neighborhood of ∞ and $f^+(U_{f^{-1}(K)}) \subseteq U_K$. Thus f^+ is continuous at ∞. □

4.2. **Čech compact supports.** For a compact Hausdorff space Y, write $\check{H}^*(Y;\mathbb{Q})$ for Čech cohomology and $\widetilde{\check{H}}^*(Y;\mathbb{Q})$ for reduced Čech cohomology. Relative Čech cohomology satisfies tautness/continuity for closed subsets of compact Hausdorff spaces; see [6, §6.6–§6.8].

Definition 4.2 (Compactly supported Čech cohomology)**.** For locally compact Hausdorff X, define

$$\check{H}^i_c(X;\mathbb{Q}) := \widetilde{\check{H}}^i(X^+;\mathbb{Q}) \cong \check{H}^i(X^+,\{\infty\};\mathbb{Q}).$$

If $f : X \to X$ is proper, define the reduced Čech Lefschetz number of f^+ by

$$\widetilde{\check{L}}(f^+;X^+) := \sum_{i\ge 0}(-1)^i \operatorname{tr}\Big((f^+)^* : \widetilde{\check{H}}^i(X^+;\mathbb{Q}) \to \widetilde{\check{H}}^i(X^+;\mathbb{Q})\Big),$$

whenever these traces are defined.

4.3. **Comparison for manifolds.** On manifolds (more generally, locally contractible paracompact spaces), singular and Čech cohomology agree; see [6, §6.9] or [2, Ch. III]. We use this to compare singular compact supports with Čech compact supports.

Proposition 4.3 (Singular vs. Čech compact supports on manifolds)**.** *Let X be a topological manifold. Then the natural comparison maps yield canonical isomorphisms*

$$H^i_c(X;\mathbb{Q}) \cong \check{H}^i_c(X;\mathbb{Q}) \qquad \text{for all } i,$$

natural for proper maps. Consequently, for any proper $f : X \to X$,

$$L_c(f;X) = \widetilde{\check{L}}(f^+;X^+)$$

whenever the traces defining either side are finite.

Proof. For each compact $K \subseteq X$, both X and $X \setminus K$ are locally contractible and paracompact. The comparison theorem gives natural isomorphisms $H^i(X, X\setminus K;\mathbb{Q}) \cong \check{H}^i(X, X\setminus K;\mathbb{Q})$ for all i. Passing to direct limits over compact K yields a natural isomorphism

$$H^i_c(X;\mathbb{Q}) \cong \varinjlim_K \check{H}^i(X, X\setminus K;\mathbb{Q}).$$

On the other hand, by Definition 4.2 and tautness of Čech cohomology at the closed subset $\{\infty\} \subseteq X^+$, one has

$$\check{H}^i_c(X;\mathbb{Q}) \cong \check{H}^i(X^+,\{\infty\};\mathbb{Q}) \cong \varinjlim_K \check{H}^i(X^+, U_K;\mathbb{Q}).$$

Excision identifies $\check{H}^i(X^+, U_K;\mathbb{Q}) \cong \check{H}^i(X, X\setminus K;\mathbb{Q})$, hence $\check{H}^i_c(X;\mathbb{Q}) \cong \varinjlim_K \check{H}^i(X, X\setminus K;\mathbb{Q})$. Combining yields $H^i_c(X;\mathbb{Q}) \cong \check{H}^i_c(X;\mathbb{Q})$. Naturality for proper maps follows from naturality of the comparison map and Lemma 4.1. Equality of Lefschetz traces follows by taking traces through these canonical isomorphisms. □

Remark 4.4 (Why Čech matters at ∞)*.* The point ∞ in X^+ need not be locally contractible (for instance, X may have wildly embedded ends). Singular cohomology need not satisfy the continuity needed to identify $H^*_c(X)$ with $H^*(X^+,\{\infty\})$ in such generality. Čech (or Alexander–Spanier) cohomology is specifically designed to satisfy tautness for closed subsets of compact Hausdorff spaces; Proposition 4.3 then recovers the usual singular compactly supported theory for manifolds.

5. Finite free actions and the correct Euler characteristic formula

A second natural idea is to try to rule out torsion using "multiplicativity" of χ_c under finite covers. In the $\mathbb{Q}$-acyclic setting the correct statement is an averaging formula, not multiplicativity.

5.1. Transfer and invariants.

Lemma 5.1 (Transfer and invariants for H_c^*)**.** *Let $p : X \to Y$ be a finite regular covering of locally compact Hausdorff spaces with finite deck group G and $|G| < \infty$. Over $\mathbb{Q}$, pullback induces an isomorphism*

$$p^* : H_c^i(Y;\mathbb{Q}) \xrightarrow{\cong} H_c^i(X;\mathbb{Q})^G \qquad \text{for all } i,$$

onto the G-invariant subspace.

Proof. For each compact $K \subseteq Y$, the restriction $p : (X, X \setminus p^{-1}(K)) \to (Y, Y \setminus K)$ is a finite regular covering of pairs. The classical cochain-level transfer for finite coverings produces maps

$$\mathrm{tr}_K : H^i(X, X \setminus p^{-1}(K);\mathbb{Q}) \to H^i(Y, Y \setminus K;\mathbb{Q})$$

such that $\mathrm{tr}_K \circ p^* = |G| \cdot \mathrm{id}$ and $p^* \circ \mathrm{tr}_K = \sum_{g \in G} g^*$. These maps are compatible as K increases, hence pass to direct limits and define $\mathrm{tr} : H_c^i(X;\mathbb{Q}) \to H_c^i(Y;\mathbb{Q})$ satisfying the same identities.

Over $\mathbb{Q}$, $|G|$ is invertible. Thus p^* is injective since $\mathrm{tr} \circ p^* = |G|\mathrm{id}$. Moreover, if $v \in H_c^i(X;\mathbb{Q})^G$, then

$$p^*\left(\frac{1}{|G|}\,\mathrm{tr}(v)\right) = \frac{1}{|G|}\sum_{g \in G} g^* v = \frac{1}{|G|}\sum_{g \in G} v = v,$$

so v lies in the image of p^*. Hence p^* identifies $H_c^i(Y;\mathbb{Q})$ with the invariants. □

5.2. Averaging formula.

Proposition 5.2 (Averaging formula)**.** *Let X be a manifold such that $H_c^i(X;\mathbb{Q})$ is finite-dimensional for all i and vanishes for $i \gg 0$. Let a finite group G act freely on X by homeomorphisms, and set $Y := X/G$. Then*

$$\chi_c(Y;\mathbb{Q}) = \frac{1}{|G|}\sum_{g \in G} L_c(g; X).$$

Proof. Let $V^i := H_c^i(X;\mathbb{Q})$ and define the averaging operator $P^i := \frac{1}{|G|}\sum_{g \in G} g^* \in \mathrm{End}(V^i)$. Then P^i is the projection onto $(V^i)^G$, hence $\mathrm{tr}(P^i) = \dim_{\mathbb{Q}}(V^i)^G$.

Since the action is free and G is finite, $p : X \to Y$ is a finite regular covering with deck group G. By Lemma 5.1, $H_c^i(Y;\mathbb{Q}) \cong (V^i)^G$. Therefore

$$\begin{aligned}
\chi_c(Y;\mathbb{Q}) &= \sum_i (-1)^i \dim_{\mathbb{Q}} H_c^i(Y;\mathbb{Q}) = \sum_i (-1)^i \dim_{\mathbb{Q}}(V^i)^G \\
&= \sum_i (-1)^i \,\mathrm{tr}(P^i) = \frac{1}{|G|}\sum_{g \in G}\sum_i (-1)^i \,\mathrm{tr}\big(g^* \mid V^i\big) \\
&= \frac{1}{|G|}\sum_{g \in G} L_c(g; X).
\end{aligned}$$

□

Corollary 5.3 (No divisibility obstruction in the $\mathbb{Q}$-acyclic case)**.** *Let X be a $\mathbb{Q}$-acyclic n-manifold and let G be a finite group acting freely on X by homeomorphisms. Then there is a character $\lambda : G \to \{\pm 1\}$ such that*

$$\chi_c(X/G;\mathbb{Q}) = \frac{(-1)^n}{|G|}\sum_{g \in G} \lambda(g),$$

$$\chi_c(X/G;\mathbb{Q}) \in \{(-1)^n, 0\}.$$

Proof. By Theorem 2.6, $H^n_c(X;\mathbb{Q}) \cong \mathbb{Q}$ and $H^i_c(X;\mathbb{Q}) = 0$ for $i \neq n$. Hence each $g \in G$ acts on $H^n_c(X;\mathbb{Q})$ by multiplication by $\lambda(g) \in \mathbb{Q}^\times$. Since g has finite order, $\lambda(g)$ is a root of unity in $\mathbb{Q}$, so $\lambda(g) \in \{\pm 1\}$. Thus $L_c(g;X) = (-1)^n \lambda(g)$ and Proposition 5.2 gives the first displayed formula. If λ is trivial, the sum is $|G|$; if nontrivial, it is 0. □

6. Uniform lattices: the odd-torsion obstruction and the open 2-primary case

6.1. Fowler's odd-torsion obstruction.

Definition 6.1 ($\mathbb{Q}$-homology manifold)**.** A locally compact space X is a $\mathbb{Q}$-*homology* n-*manifold* if for every $x \in X$,

$$H_i(X, X \setminus \{x\};\mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & i = n, \\ 0, & i \neq n. \end{cases}$$

If, moreover, X is an absolute neighborhood retract (ANR), we call it an *ANR* $\mathbb{Q}$-*homology manifold*.

Every closed topological manifold is an ANR $\mathbb{Q}$-homology manifold.

Theorem 6.2 (Fowler)**.** *Let* Γ *be a uniform lattice in a real semisimple Lie group. If* Γ *contains an element of order* p *for some odd prime* p*, then there exists no closed ANR* $\mathbb{Q}$*-homology manifold* X *(in particular, no closed manifold) with* $\pi_1(X) \cong \Gamma$ *and* $\mathbb{Q}$*-acyclic universal cover.*

Reference. This is Fowler's Theorem 5.1.1 in [3, §5.1]. See also [7, §7] for context and [1] for a workshop report account. □

6.2. Torsion does occur in manifold groups with $\mathbb{Q}$-acyclic universal cover.

Theorem 6.3 (Fowler)**.** *For each integer* $n \geq 2$ *there exists a closed manifold* M *whose universal cover* $\widetilde{M}$ *is* $\mathbb{Q}$*-acyclic and such that* $\pi_1(M)$ *contains an element of order* n*. In particular, there exist examples with* 2*-torsion.*

Reference. This is Proposition 4.5.1 and Corollary 4.5.2 of [3, §4.5]. (A retraction onto a group containing $\mathbb{Z}/n$ forces n-torsion in $\pi_1(M)$.) □

6.3. Resolution of the Problem.

Corollary 6.4 (Best current answer)**.** *Let* Γ *be a uniform lattice in a real semisimple Lie group and assume* Γ *contains an element of order* 2*.*

(a) If Γ *contains an element of order* p *for some odd prime* p*, then* Γ *is* not *isomorphic to the fundamental group of any closed manifold whose universal cover is* $\mathbb{Q}$*-acyclic.*
(b) If all torsion in Γ *is* 2*-primary, the existence problem is open in general.*

Proof. (a) Apply Theorem 6.2. (b) This is the remaining case isolated in [3, §5.1] and discussed in [7, §7]. □

Remark 6.5. Theorems 3.3 and 4.3 explain why a naive Lefschetz/Euler characteristic approach cannot settle the open case: a finite-order deck transformation is a proper, fixed-point-free homeomorphism of a noncompact manifold, and L_c may be supported entirely at infinity.

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