SMOOTH SHIFTS OF THE FINITE-VOLUME Φ^4_3 MEASURE ON $\mathbb{T}^3$ ARE MUTUALLY SINGULAR

Abstract. Let μ be the finite-volume Euclidean Φ^4_3 measure on the three-dimensional unit torus $\mathbb{T}^3$ at nonzero quartic coupling. For any nonzero smooth function $\psi \in C^\infty(\mathbb{T}^3)$, we prove that μ and its translate $T_{\psi\#}\mu$ (pushforward under $u \mapsto u + \psi$) are mutually singular. The proof constructs an explicit separating Borel event using a renormalised cubic functional at super-exponential mollification scales. The key mechanism is the nontrivial logarithmically divergent linear counterterm ("sunset" divergence) in three dimensions; after a smooth shift, this term produces a deterministic blow-up. All auxiliary analytic estimates used in the separation argument (dyadic sunset bound, mollified Coulomb estimate, third-chaos logarithmic variance) are proved here in a self-contained manner. The only non-elementary input is a standard renormalised small-scale expansion for Φ^2_n and Φ^3_n under μ, which is stated precisely and attributed to published results in the theory of regularity structures and stochastic quantisation.

1. Introduction

1.1. **Problem.** Let $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$ be the unit three-dimensional torus. Let μ denote the (finite-volume) Euclidean Φ^4_3 measure on $\mathcal{D}'(\mathbb{T}^3)$ at nonzero quartic coupling. The existence and basic properties of this measure (as an invariant measure for the stochastic quantisation dynamics, supported on negative-regularity distributions) are established in several published works; see, for example, [8, 6, 2, 3]. Fix $\psi \in C^\infty(\mathbb{T}^3)$, $\psi \not\equiv 0$, and define the shift map

$$T_\psi : \mathcal{D}'(\mathbb{T}^3) \to \mathcal{D}'(\mathbb{T}^3), \qquad T_\psi(u) = u + \psi,$$

viewing smooth functions as distributions. The problem is to determine whether μ and $T_{\psi\#}\mu$ are equivalent measures (same null sets).

1.2. **Main result.**

Theorem 1.1. *Let μ be the finite-volume Φ^4_3 measure on $\mathbb{T}^3$ at nonzero quartic coupling. Then for every $\psi \in C^\infty(\mathbb{T}^3) \setminus \{0\}$,*

$$\mu \perp T_{\psi\#}\mu.$$

In particular, μ is not *quasi-invariant under any nontrivial smooth translation.*

Remark 1.2 (Gaussian case). When the quartic coupling is 0, μ is the massive Gaussian free field measure; then the Cameron–Martin theorem implies quasi-invariance under smooth shifts. Theorem 1.1 concerns the genuinely interacting model.

1.3. **Mechanism: logarithmic linear counterterm.** The proof exploits the renormalisation structure of the Φ^4_3 model in $d = 3$. Beyond the ε^{-1} tadpole divergence, renormalisation produces a *logarithmically divergent linear counterterm* (sunset divergence). For a carefully chosen renormalised cubic functional at scale ε, this logarithmic term becomes a deterministic quantity proportional to $\log(\varepsilon^{-1})\|\psi\|^2_{L^2}$ after shifting by ψ. Evaluating the functional along a super-exponentially decaying sequence of scales separates the original and shifted measures by a Borel set defined via almost-sure subsequence convergence.

2. State space and translations

2.1. **A Polish realisation.** Fix $s > 2$ and set

$$E := H^{-s}(\mathbb{T}^3),$$

with Borel σ-algebra $\mathcal{B}(E)$. Then E is a separable Hilbert space and hence Polish, and $C^\infty(\mathbb{T}^3) \hookrightarrow H^{-s}(\mathbb{T}^3)$ continuously.

Date: February 13, 2026.

2.2. **Translations and pushforwards.** For $\psi \in C^\infty(\mathbb{T}^3) \subset E$, define $T_\psi : E \to E$ by $T_\psi(u) = u + \psi$. Then T_ψ is a homeomorphism with inverse $T_{-\psi}$. For a probability measure P on $(E, \mathcal{B}(E))$, define its pushforward by

$$T_{\psi\#}P(A) := P(T_\psi^{-1}A) = P(A - \psi), \qquad A \in \mathcal{B}(E).$$

Definition 2.1 (Equivalence and singularity)**.** Let P, Q be probability measures on a measurable space $(X, \mathcal{E})$. They are *equivalent*, written $P \sim Q$, if $P \ll Q$ and $Q \ll P$. They are *mutually singular*, written $P \perp Q$, if there exists $A \in \mathcal{E}$ such that $P(A) = 1$ and $Q(A) = 0$.

3. A separation criterion

Lemma 3.1 (Shift separation)**.** *Let P be a probability measure on $(E, \mathcal{B}(E))$ and let $T_\psi(u) = u + \psi$. If there exists $A \in \mathcal{B}(E)$ such that $P(A) = 1$ and $P(A + \psi) = 0$, then $P \perp T_{\psi\#}P$.*

Proof. Set $B := A + \psi$. Then $P(B) = 0$ and

$$(T_{\psi\#}P)(B) = P(B - \psi) = P(A) = 1.$$

Hence $P \perp T_{\psi\#}P$. □

4. Mollifiers and approximation

4.1. **Periodic mollifiers with controlled support.** Fix once and for all $\rho \in C_c^\infty(\mathbb{R}^3)$ such that

$$\rho \geq 0, \qquad \int_{\mathbb{R}^3} \rho(x)\,\mathrm{d}x = 1, \qquad \operatorname{supp}(\rho) \subset B(0, 1/2). \tag{4.1}$$

For $\varepsilon > 0$, define its periodicisation on $\mathbb{T}^3$ by

$$\rho_\varepsilon(x) := \sum_{k \in \mathbb{Z}^3} \varepsilon^{-3} \rho\left(\frac{x + k}{\varepsilon}\right), \qquad x \in \mathbb{T}^3. \tag{4.2}$$

For $u \in E$, define the mollification $u_\varepsilon := u * \rho_\varepsilon \in C^\infty(\mathbb{T}^3)$.

Remark 4.1. Because $\operatorname{supp}(\rho) \subset B(0, 1/2)$, for every $\varepsilon \in (0, 1)$ and every representative $x \in [-\frac{1}{2}, \frac{1}{2})^3$, the sum in (4.2) contains only the term $k = 0$. In particular, for such ε, one has $\operatorname{supp}(\rho_\varepsilon) \subset B(0, \varepsilon/2)$ (viewed inside $\mathbb{T}^3$), and therefore if $\eta := \tilde{\rho} * \rho$ with $\tilde{\rho}(x) = \rho(-x)$ then $\operatorname{supp}(\eta) \subset B(0, 1)$ and $\operatorname{supp}(\eta_\varepsilon) \subset B(0, \varepsilon)$.

4.2. **Approximate identity in Sobolev spaces.**

Lemma 4.2 (Approximate identity in H^{-s})**.** *For every $u \in H^{-s}(\mathbb{T}^3)$ one has $\|u_\varepsilon - u\|_{H^{-s}} \to 0$ as $\varepsilon \downarrow 0$. In particular, if Φ is an E-valued random variable, then $\Phi_{\varepsilon_n} \to \Phi$ in E almost surely along any deterministic $\varepsilon_n \downarrow 0$.*

Proof. Write $u(x) = \sum_{m \in \mathbb{Z}^3} \widehat{u}(m) e^{2\pi i m \cdot x}$ in $\mathcal{D}'(\mathbb{T}^3)$. Then $\widehat{u_\varepsilon}(m) = \widehat{\rho_\varepsilon}(m)\widehat{u}(m)$ and $\widehat{\rho_\varepsilon}(m) \to 1$ as $\varepsilon \downarrow 0$ for each fixed m. Moreover, since $\rho_\varepsilon \in L^1(\mathbb{T}^3)$,

$$|\widehat{\rho_\varepsilon}(m)| \leq \|\rho_\varepsilon\|_{L^1(\mathbb{T}^3)} = \|\rho\|_{L^1(\mathbb{R}^3)} =: M < \infty.$$

Hence $|\widehat{\rho_\varepsilon}(m) - 1| \leq M + 1$. By dominated convergence,

$$\|u_\varepsilon - u\|_{H^{-s}}^2 = \sum_{m \in \mathbb{Z}^3} (1 + |m|^2)^{-s} |\widehat{\rho_\varepsilon}(m) - 1|^2 |\widehat{u}(m)|^2 \longrightarrow 0,$$

since $(1 + |m|^2)^{-s}|\widehat{u}(m)|^2 \in \ell^1$ and the summand is dominated by $(M + 1)^2(1 + |m|^2)^{-s}|\widehat{u}(m)|^2$. The almost sure statement follows by applying the deterministic convergence pointwise to $u = \Phi(\omega)$. □

4.3. A quantitative mollification error.

Lemma 4.3 (One derivative gain yields an ε-rate)**.** *Let $s \in \mathbb{R}$ and $u \in H^{-s}(\mathbb{T}^3)$. Then there exists $C = C(\rho, s) < \infty$ such that for all $\varepsilon \in (0, 1)$,*

$$\|u_\varepsilon - u\|_{H^{-s-1}} \leq C\,\varepsilon\,\|u\|_{H^{-s}}\,.$$

Proof. In Fourier variables,

$$\|u_\varepsilon - u\|^2_{H^{-s-1}} = \sum_{m\in\mathbb{Z}^3} (1 + |m|^2)^{-s-1}|\widehat{\rho_\varepsilon}(m) - 1|^2|\widehat{u}(m)|^2.$$

Since $\rho \in C_c^\infty(\mathbb{R}^3)$, its Fourier transform $\widehat{\rho}$ is smooth and satisfies $|\widehat{\rho}(\xi) - 1| \leq C_1|\xi|$ for all ξ. As $\widehat{\rho_\varepsilon}(m) = \widehat{\rho}(\varepsilon m)$, we obtain $|\widehat{\rho_\varepsilon}(m) - 1| \leq C_1\varepsilon|m|$. Thus

$$(1 + |m|^2)^{-s-1}|\widehat{\rho_\varepsilon}(m) - 1|^2 \leq C_1^2\varepsilon^2(1 + |m|^2)^{-s-1}|m|^2 \leq C_1^2\varepsilon^2(1 + |m|^2)^{-s}.$$

Summing yields $\|u_\varepsilon - u\|_{H^{-s-1}} \leq C_1\varepsilon\,\|u\|_{H^{-s}}$. □

5. The separating functional

5.1. Super-exponential scales.

Fix

$$\varepsilon_n := \exp(-e^n), \qquad n \in \mathbb{N}, \tag{5.1}$$

so that

$$\varepsilon_n^{-1} = e^{e^n}, \qquad \log(\varepsilon_n^{-1}) = e^n. \tag{5.2}$$

5.2. Definition.

Fix $\beta \in (1/2, 1)$ and $\psi \in C^\infty(\mathbb{T}^3)$. For $u \in E$ set $u_n := u_{\varepsilon_n}$ and define

$$F_n(u; \psi) := e^{-\beta n}\,\langle u_n^3 - 3a\,\varepsilon_n^{-1}u_n - 9b\,\log(\varepsilon_n^{-1})\,u,\,\psi\rangle\,, \tag{5.3}$$

where $a \in \mathbb{R}$ and $b \in \mathbb{R} \setminus \{0\}$ are deterministic renormalisation coefficients specified in Proposition 7.1.

Remark 5.1 (Functional design)*.* The form of (5.3) is tailored to isolate the sunset divergence under translation:

- the ε_n^{-1} counterterm is paired with the mollified field u_n so that, after replacing u by $u - \psi$, the ε_n^{-1} contribution cancels exactly against the algebraic expansion of $(u_n - \psi_n)^3$;
- the logarithmic counterterm is paired with the *unmollified* field u, so that $u \mapsto u - \psi$ produces a clean deterministic addendum $+9b\log(\varepsilon_n^{-1})\psi$ with no regularisation remainder.

A quantitative remark explaining why one may replace the standard u_n by u in the logarithmic term at these scales is given in Remark 7.3.

Lemma 5.2 (Continuity of F_n)**.** *For each $n \in \mathbb{N}$ and each $\psi \in C^\infty(\mathbb{T}^3)$, the map $u \mapsto F_n(u; \psi)$ is continuous on E.*

Proof. The map $u \mapsto u_n$ is continuous $H^{-s} \to C^\infty$ (convolution with a fixed smooth kernel is smoothing). The maps $f \mapsto \int_{\mathbb{T}^3} f^3\psi$ and $f \mapsto \int_{\mathbb{T}^3} f\psi$ are continuous on $C^\infty(\mathbb{T}^3)$. Finally $u \mapsto \langle u,\,\psi\rangle$ is continuous on H^{-s} since $\psi \in H^s$. □

6. Two probabilistic lemmas

Lemma 6.1 (Deterministic subsequence from convergence in probability)**.** *Let $(X_n)_{n\in\mathbb{N}}$ be real-valued random variables such that $X_n \to 0$ in probability. Then there exists a deterministic strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ such that $X_{n_k} \to 0$ almost surely.*

Proof. Choose $n_k > n_{k-1}$ such that $\mathbb{P}(|X_{n_k}| > 2^{-k}) < 2^{-k}$. Then $\sum_k \mathbb{P}(|X_{n_k}| > 2^{-k}) < \infty$ and Borel–Cantelli yields $|X_{n_k}| \leq 2^{-k}$ eventually. □

Lemma 6.2 (Tightness times a vanishing prefactor)**.** *Let $(Y_n)_{n\in\mathbb{N}}$ be tight real-valued random variables and let $c_n \to 0$ deterministically. Then $c_nY_n \to 0$ in probability.*

Proof. Fix $\delta > 0$. By tightness, choose $M < \infty$ such that $\sup_n \mathbb{P}(|Y_n| > M) < \delta$. Choose n such that $|c_n|M < \delta$. Then $\mathbb{P}(|c_nY_n| > \delta) \leq \mathbb{P}(|Y_n| > M) < \delta$. □

7. Analytic input from Φ^4_3 renormalisation

7.1. Statement.

Let Φ be an E-valued random distribution with law μ. Write $\Phi_n := \Phi_{\varepsilon_n}$.

Proposition 7.1 (Renormalised square/cube at super-exponential scales)**.** *There exist deterministic constants $a \in \mathbb{R}$ and $b \in \mathbb{R} \setminus \{0\}$, a Sobolev index $r > 0$, and $H^{-r}(\mathbb{T}^3)$-valued random variables S and C, together with $H^{-r}(\mathbb{T}^3)$-valued random variables $(W_n)_{n\in\mathbb{N}}$, such that:*

(i) *(*Renormalised square*)*

$$S_n := \Phi_n^2 - a\,\varepsilon_n^{-1}$$

converges in probability in $H^{-r}(\mathbb{T}^3)$ to S. In particular, (S_n) is tight in H^{-r}.

(ii) *(*Renormalised cube up to a critical third-chaos term*)*

$$R_n := \Phi_n^3 - 3a\,\varepsilon_n^{-1}\Phi_n - 9b\,\log(\varepsilon_n^{-1})\,\Phi - W_n$$

converges in probability in $H^{-r}(\mathbb{T}^3)$ to C. In particular, for every $\varphi \in C^\infty(\mathbb{T}^3)$, the real sequence $(\langle R_n, \varphi\rangle)$ is tight.

(iii) *(*Critical third-chaos growth*) For every $\varphi \in C^\infty(\mathbb{T}^3)$ there exists $C_\varphi < \infty$ such that*

$$\mathbb{E}\big[\,\langle W_n, \varphi\rangle^2\,\big] \le C_\varphi\,\log(\varepsilon_n^{-1}) \qquad \text{for all } n \in \mathbb{N}. \tag{7.1}$$

Consequently, for every $\beta > \frac{1}{2}$,

$$e^{-\beta n}\,\langle W_n, \varphi\rangle \longrightarrow 0 \quad \text{in } L^2 \text{ (hence in probability).}$$

Remark 7.2 (Published sources for Proposition 7.1)*.* Proposition 7.1 is a time-slice reformulation of the renormalised local expansions for the stochastic quantisation equation for Φ^4_3 obtained via regularity structures and BPHZ renormalisation. The underlying existence of the renormalised model and identification of counterterms, including the logarithmic linear counterterm, is established in [5] (see in particular the renormalisation discussion in the Φ^4_3 setting in [5, §10]) and in the algebraic BPHZ framework [1]. The global stochastic quantisation dynamics and its invariant measure are treated in [8] and in the discretisation/invariance approach of [6]; see also the measure construction via Girsanov in [2] and the paracontrolled approach [3]. The only quantitative estimate beyond these structural results that is used explicitly below is the logarithmic second-moment growth (7.1) for the critical third-chaos component, for which we provide a self-contained model computation in Lemma 9.1. The non-vanishing $b \neq 0$ is addressed in Lemma 8.1 and Lemma 8.2.

7.2. Replacing Φ_n by Φ in the logarithmic term.

Remark 7.3 (A harmless substitution at super-exponential scales)*.* Let $\psi \in C^\infty(\mathbb{T}^3)$ and $\Phi \in H^{-s}(\mathbb{T}^3)$ with $s > 2$. By Lemma 4.3 and Sobolev duality,

$$\log(\varepsilon_n^{-1})\,|\,\langle \Phi_n - \Phi,\, \psi\rangle\,| \le \log(\varepsilon_n^{-1})\;\|\Phi_n - \Phi\|_{H^{-s-1}}\,\|\psi\|_{H^{s+1}} \lesssim \log(\varepsilon_n^{-1})\,\varepsilon_n\;\|\Phi\|_{H^{-s}}\,.$$

Using (5.2), $\log(\varepsilon_n^{-1})\varepsilon_n = e^n e^{-e^n} \to 0$ deterministically. Hence $\log(\varepsilon_n^{-1})\,\langle \Phi_n - \Phi,\, \psi\rangle \to 0$ almost surely. In particular, evaluating the logarithmic counterterm on Φ instead of Φ_n modifies $F_n(\Phi;\psi)$ by a term vanishing almost surely (hence in probability).

8. Sunset divergence and covariance bounds

8.1. Sunset integral.

Lemma 8.1 (A logarithmically divergent sunset integral)**.** *Let*

$$I(\Lambda) := \int_{|p|\le\Lambda}\int_{|q|\le\Lambda} \frac{1}{(1+|p|^2)(1+|q|^2)(1+|p+q|^2)}\,\mathrm{d}p\,\mathrm{d}q, \qquad \Lambda \ge 2.$$

Then there exist constants $0 < c \le C < \infty$ such that

$$c\,\log\Lambda \le I(\Lambda) \le C\,\log\Lambda, \qquad \Lambda \ge 2.$$

In particular, $I(\Lambda)$ diverges logarithmically as $\Lambda \to \infty$ with strictly positive coefficient.

Proof. We treat $\Lambda = 2^N$ with $N \in \mathbb{N}$; monotonicity then yields the general case.

Dyadic decomposition (including the unit ball). Define

$$A_0 := \{p \in \mathbb{R}^3 : |p| < 2\}, \qquad A_j := \{p \in \mathbb{R}^3 : 2^j \le |p| < 2^{j+1}\}, \quad j = 1, \dots, N-1.$$

Set

$$I_{j,k} := \int_{A_j} \int_{A_k} \frac{1}{(1+|p|^2)(1+|q|^2)(1+|p+q|^2)} \, \mathrm{d}p \, \mathrm{d}q.$$

Then $I(2^N) \le \sum_{j,k=0}^{N-1} I_{j,k}$.

Upper bound. By symmetry it suffices to bound $I_{j,k}$ with $j \ge k$. We distinguish two regimes.

Case 1: off-diagonal, $j \ge k+2$. For $p \in A_j$ and $q \in A_k$ we have $|p| \ge 2^j$ and $|q| < 2^{k+1} \le 2^{j-1}$, hence $|p+q| \ge |p| - |q| > 2^{j-1}$. Therefore the integrand is $\lesssim 2^{-4j-2k}$ and $\mathrm{Vol}(A_\ell) \lesssim 2^{3\ell}$ for $\ell \ge 1$, $\mathrm{Vol}(A_0) \lesssim 1$, giving

$$I_{j,k} \lesssim 2^{3j+3k} \, 2^{-4j-2k} = 2^{k-j} = 2^{-|j-k|}.$$

Case 2: near-diagonal, $j \in \{k, k+1\}$. We factor the first two denominators and bound the remaining one by a convolution estimate:

$$I_{j,k} \lesssim 2^{-2j} \, 2^{-2k} \int_{A_k} \left(\int_{A_j} \frac{1}{1+|p+q|^2} \, \mathrm{d}p \right) \mathrm{d}q.$$

For fixed $q \in A_k$, the change of variables $u = p+q$ shows that $|u| \le |p|+|q| \le 2^{j+1}+2^{k+1} \le 2^{j+2}$, hence

$$\int_{A_j} \frac{1}{1+|p+q|^2} \, \mathrm{d}p \le \int_{|u| \le 2^{j+2}} \frac{1}{1+|u|^2} \, \mathrm{d}u \lesssim 2^j.$$

It follows that $I_{j,k} \lesssim 2^{-2j} 2^{-2k} \, 2^j \, \mathrm{Vol}(A_k) \lesssim 2^{k-j} \lesssim 1$, and since $|j-k| \le 1$ this implies $I_{j,k} \lesssim 2^{-|j-k|}$.

Summation. Thus $I_{j,k} \lesssim 2^{-|j-k|}$ for $0 \le k \le j \le N-1$, hence

$$I(2^N) \lesssim \sum_{j,k=0}^{N-1} 2^{-|j-k|} \lesssim N \sim \log \Lambda.$$

Lower bound. Restrict to $j \in \{1, \dots, N-2\}$ and to $p, q \in A_j$ with angle $\le \pi/6$. Then $|p+q| \gtrsim 2^j$ and the integrand is $\gtrsim 2^{-6j}$ on a region of volume $\gtrsim 2^{6j}$, so $I_{j,j} \gtrsim 1$ uniformly. Summing over j yields $I(2^N) \gtrsim N \sim \log \Lambda$. □

8.2. **Non-vanishing of the logarithmic coefficient.**

Lemma 8.2 (Non-vanishing of the sunset coefficient)**.** *In the BPHZ renormalisation of the dynamical Φ^4_3 model, the linear logarithmic counterterm coefficient is proportional to the sunset integral of Lemma 8.1 and is therefore nonzero when the quartic coupling is nonzero. In particular, the coefficient b in Proposition 7.1 satisfies $b \ne 0$.*

Proof. The BPHZ renormalisation procedure associates to each superficially divergent diagram a counterterm. In the Φ^4_3 model, the unique divergent two-loop one-point diagram contributing to the linear counterterm is the sunset diagram; its value is an integral of a product of three massive propagators. This identification (and the presence of a logarithmically divergent linear counterterm) is standard in the Φ^4_3 regularity-structures treatment; see [5, §10] and the algebraic BPHZ formalism [1]. Lemma 8.1 shows that the corresponding integral diverges like $c \log \Lambda$ with $c > 0$. The prefactor is quadratic in the quartic coupling and hence nonzero for nonzero coupling. Therefore $b \ne 0$. □

8.3. **Metric notation on $\mathbb{T}^3$.** Let $d_{\mathbb{T}^3}$ denote the standard flat distance on $\mathbb{T}^3$:

$$d_{\mathbb{T}^3}(x,y) := \min_{k\in\mathbb{Z}^3} |(x-y)+k|_{\mathbb{R}^3}.$$

We write

$$|x| := d_{\mathbb{T}^3}(x,0), \qquad |x-y| := d_{\mathbb{T}^3}(x,y).$$

Note that $|x|$ is the Euclidean norm of the unique representative of x in $[-\frac{1}{2},\frac{1}{2})^3$.

8.4. **Coulombic bound and mollified version.** Let $\mathcal{C}:\mathbb{T}^3\to\mathbb{R}$ be the covariance kernel of the massive Gaussian free field on $\mathbb{T}^3$, i.e. the Green function of $(1-\Delta)$:

$$(1-\Delta)\mathcal{C} = \delta_0 \quad \text{in } \mathcal{D}'(\mathbb{T}^3).$$

Lemma 8.3 (Coulomb singularity)**.** *There exist constants $c_0<\infty$ and $r_0\in(0,\sqrt{3}/2)$ such that for all $x\in\mathbb{T}^3$ with $0<|x|\le r_0$,*

$$0\le \mathcal{C}(x) \le \frac{c_0}{|x|}.$$

Proof. Let g denote the fundamental solution of $(1-\Delta)$ on $\mathbb{R}^3$ (Yukawa potential); it is classical that

$$g(z) = \frac{1}{4\pi}\frac{e^{-|z|}}{|z|}, \qquad z\in\mathbb{R}^3\setminus\{0\},$$

and $(1-\Delta)g=\delta_0$ in $\mathcal{D}'(\mathbb{R}^3)$ (see e.g. [4, Ch. 2]). Define its periodisation

$$G(x) := \sum_{k\in\mathbb{Z}^3} g(x+k), \qquad x\in\mathbb{R}^3.$$

The sum converges absolutely and uniformly on compact sets away from $\mathbb{Z}^3$ since g decays exponentially, so G defines a smooth periodic function on $\mathbb{R}^3\setminus\mathbb{Z}^3$, hence a smooth function on $\mathbb{T}^3\setminus\{0\}$. One checks by testing against smooth periodic test functions that $(1-\Delta)G=\delta_0$ on $\mathbb{T}^3$. Since $(1-\Delta)$ has trivial kernel on $\mathbb{T}^3$, the Green function is unique; thus $\mathcal{C}=G$.

For $|x|\le r_0$ with $r_0<1/4$, write $\mathcal{C}(x)=g(x)+\sum_{k\ne 0} g(x+k)$. The second sum is uniformly bounded in x for $|x|\le r_0$, while $g(x)\le (4\pi)^{-1}|x|^{-1}$. Absorb the bounded remainder into $c_0/|x|$ (since $|x|\le r_0$) to obtain the stated bound. □

Let $\eta := \tilde\rho * \rho$ with $\tilde\rho(x)=\rho(-x)$ and let $\eta_\varepsilon = \tilde\rho_\varepsilon * \rho_\varepsilon$ on $\mathbb{T}^3$. Define $\mathcal{C}_\varepsilon := \mathcal{C} * \eta_\varepsilon$.

Lemma 8.4 (Mollified Coulomb bound)**.** *There exists $C<\infty$ such that for all $\varepsilon\in(0,1/6)$ and all $x\in\mathbb{T}^3$,*

$$0\le \mathcal{C}_\varepsilon(x) \le \frac{C}{|x|+\varepsilon}.$$

Proof. Nonnegativity follows from $\mathcal{C}\ge 0$ and $\eta_\varepsilon\ge 0$. Fix $\varepsilon\in(0,1/6)$. By Remark 4.1, $\mathrm{supp}(\eta_\varepsilon)\subset\{y\in\mathbb{T}^3 : |y|\le\varepsilon\}$.

Since $\mathcal{C}$ is smooth on $\mathbb{T}^3\setminus\{0\}$ and $\mathbb{T}^3\setminus B(0,r_0)$ is compact, the function $\mathcal{C}$ is bounded there. Let

$$C_1 := \sup\{\mathcal{C}(z) : |z|\ge r_0\} < \infty.$$

Then for all $z\ne 0$,

$$\mathcal{C}(z) \le \frac{c_0}{|z|} + C_1, \tag{8.1}$$

using Lemma 8.3 when $|z|\le r_0$ and the definition of C_1 when $|z|\ge r_0$.

We consider two cases.

Case 1: $|x|\ge 2\varepsilon$. For any y with $\eta_\varepsilon(y)\ne 0$, we have $|y|\le\varepsilon$. By the reverse triangle inequality for the metric $d_{\mathbb{T}^3}$,

$$|x-y| = d_{\mathbb{T}^3}(x,y) \ge d_{\mathbb{T}^3}(x,0) - d_{\mathbb{T}^3}(y,0) = |x|-|y| \ge |x|-\varepsilon \ge |x|/2.$$

Applying (8.1) with $z=x-y$ yields

$$\mathcal{C}(x-y) \le \frac{c_0}{|x-y|} + C_1 \le \frac{2c_0}{|x|} + C_1.$$

Therefore, since $\int_{\mathbb{T}^3} \eta_\varepsilon = 1$,

$$\mathcal{C}_\varepsilon(x) = \int_{\mathbb{T}^3} \mathcal{C}(x-y)\eta_\varepsilon(y)\,\mathrm{d}y \le \frac{2c_0}{|x|} + C_1 \le \frac{C}{|x|+\varepsilon}.$$

Case 2: $|x| < 2\varepsilon$. Then $|x|+\varepsilon \le 3\varepsilon$. Using (8.1),

$$\mathcal{C}_\varepsilon(x) \le \int_{\mathbb{T}^3} \frac{c_0}{|x-y|}\eta_\varepsilon(y)\,\mathrm{d}y + C_1.$$

Because $|x| < 2\varepsilon$ and η_ε is supported in $\{|y| \le \varepsilon\}$, every such y admits a representative in $[-1/2,1/2)^3$ with $|y|_{\mathbb{R}^3} \le \varepsilon$. Similarly, choose the representative of x in $[-1/2,1/2)^3$, and note that $|x|_{\mathbb{R}^3} = |x| < 2\varepsilon$ implies $|x_i| < 2\varepsilon$ for each coordinate. Thus for each coordinate i,

$$|x_i - y_i| \le |x_i| + |y_i| < 2\varepsilon + \varepsilon = 3\varepsilon < \frac{1}{2},$$

since $\varepsilon < 1/6$. Hence in this case the torus distance equals the Euclidean distance between these representatives: $|x-y| = |x-y|_{\mathbb{R}^3}$. We may therefore lift the integral to $\mathbb{R}^3$ and substitute $y = \varepsilon u$:

$$\int_{|y|\le\varepsilon} \frac{1}{|x-y|}\eta_\varepsilon(y)\,\mathrm{d}y = \frac{1}{\varepsilon}\int_{\mathbb{R}^3} \frac{1}{|x/\varepsilon - u|}\eta(u)\,\mathrm{d}u.$$

The function $g(w) := \int_{\mathbb{R}^3} \frac{1}{|w-u|}\eta(u)\,\mathrm{d}u$ is the Newtonian potential of a smooth compactly supported function, hence finite and bounded on $\mathbb{R}^3$. Let $M := \sup_{w\in\mathbb{R}^3} g(w) < \infty$. Then

$$\mathcal{C}_\varepsilon(x) \le \frac{c_0 M}{\varepsilon} + C_1 \le \frac{C}{\varepsilon} \le \frac{3C}{|x|+\varepsilon}.$$

Combining both cases yields $\mathcal{C}_\varepsilon(x) \le C/(|x|+\varepsilon)$. □

9. A critical third-chaos estimate

Lemma 9.1 (Logarithmic variance growth in third chaos)**.** *Let X be the massive Gaussian free field on $\mathbb{T}^3$ with covariance $\mathcal{C}$. Let $X_\varepsilon := X * \rho_\varepsilon$ and define, for $\varphi \in C^\infty(\mathbb{T}^3)$,*

$$\mathcal{W}_\varepsilon(\varphi) := \int_{\mathbb{T}^3} \Big(X_\varepsilon(x)^3 - 3\mathbb{E}[X_\varepsilon(x)^2]\,X_\varepsilon(x)\Big)\varphi(x)\,\mathrm{d}x.$$

Then for every $\varphi \in C^\infty(\mathbb{T}^3)$ there exists $C_\varphi < \infty$ such that for all $\varepsilon \in (0,1/6)$,

$$\mathbb{E}\big[\mathcal{W}_\varepsilon(\varphi)^2\big] \le C_\varphi \log(\varepsilon^{-1}).$$

Proof. Since $\mathcal{W}_\varepsilon(\varphi)$ is a third-homogeneous polynomial in a centred Gaussian field, Wick's theorem yields

$$\mathbb{E}\big[\mathcal{W}_\varepsilon(\varphi)^2\big] = 6\int_{\mathbb{T}^3}\int_{\mathbb{T}^3} \varphi(x)\varphi(y)\,\mathcal{C}_\varepsilon(x-y)^3\,\mathrm{d}x\,\mathrm{d}y, \tag{9.1}$$

see e.g. [7, Ch. 2]. Bounding $|\varphi(x)\varphi(y)| \le \|\varphi\|^2_{L^\infty}$ and changing variables $z = x-y$ gives

$$\mathbb{E}\big[\mathcal{W}_\varepsilon(\varphi)^2\big] \le 6\|\varphi\|^2_{L^\infty}\,|\mathbb{T}^3| \int_{\mathbb{T}^3} \mathcal{C}_\varepsilon(z)^3\,\mathrm{d}z.$$

By Lemma 8.4, $\mathcal{C}_\varepsilon(z) \lesssim (|z|+\varepsilon)^{-1}$. Split $\mathbb{T}^3$ into $|z| \le 1/4$ and $|z| > 1/4$. On $|z| > 1/4$, $\mathcal{C}_\varepsilon$ is uniformly bounded in ε, hence the contribution is $O(1)$. On $|z| \le 1/4$,

$$\int_{|z|\le 1/4} \mathcal{C}_\varepsilon(z)^3\,\mathrm{d}z \lesssim \int_{|z|\le 1/4} \frac{1}{(|z|+\varepsilon)^3}\,\mathrm{d}z \asymp \int_0^{1/4} \frac{r^2}{(r+\varepsilon)^3}\,\mathrm{d}r \lesssim \int_\varepsilon^{1/4} \frac{1}{r}\,\mathrm{d}r \lesssim \log(\varepsilon^{-1}).$$

This proves the claim. □

10. Separating event and proof of Theorem 1.1

Fix $\psi \in C^\infty(\mathbb{T}^3)\setminus\{0\}$ and $\beta \in (1/2,1)$. Let $F_n(\cdot;\psi)$ be defined by (5.3).

10.1. Vanishing under μ.

Lemma 10.1 (Vanishing in probability under μ)**.** *Let $\Phi \sim \mu$. Then $F_n(\Phi;\psi) \to 0$ in probability as $n \to \infty$.*

Proof. By Proposition 7.1(ii),

$$\Phi_n^3 - 3a\varepsilon_n^{-1}\Phi_n - 9b\log(\varepsilon_n^{-1})\Phi = R_n + W_n,$$

where $\langle R_n, \psi\rangle$ is tight and $\mathbb{E}[\langle W_n, \psi\rangle^2] \lesssim \log(\varepsilon_n^{-1})$. Thus

$$F_n(\Phi;\psi) = e^{-\beta n}\langle R_n, \psi\rangle + e^{-\beta n}\langle W_n, \psi\rangle .$$

The first term tends to 0 in probability by Lemma 6.2. For the second term,

$$\mathbb{E}\big[|e^{-\beta n}\langle W_n, \psi\rangle|^2\big] \lesssim e^{-2\beta n}\log(\varepsilon_n^{-1}) = e^{-2\beta n}e^n = e^{-(2\beta-1)n} \to 0,$$

so it tends to 0 in L^2, hence in probability. □

10.2. A full-μ Borel set via deterministic subsequence.

By Lemma 10.1 and Lemma 6.1, there exists a deterministic subsequence (n_k) such that $F_{n_k}(\Phi;\psi) \to 0$ almost surely. Define

$$A_\psi := \Big\{u \in E : \lim_{k\to\infty} F_{n_k}(u;\psi) = 0\Big\}.$$

Lemma 10.2. *The set A_ψ is Borel in E and satisfies $\mu(A_\psi) = 1$.*

Proof. Each $F_{n_k}(\cdot;\psi)$ is continuous, hence Borel. Therefore A_ψ is Borel since it is a countable Boolean combination of inverse images of closed intervals. By construction, $\Phi \in A_\psi$ almost surely. □

10.3. Translation forces divergence.

For $n \in \mathbb{N}$ write $\psi_n := \psi_{\varepsilon_n}$.

Lemma 10.3 (Uniform bounds for ψ_n)**.** *For every $m \in \mathbb{N}$, $\psi_n \to \psi$ in $C^m(\mathbb{T}^3)$ and $\sup_n \|\psi_n\|_{C^m} < \infty$. Consequently, for every $r > 0$,*

$$\sup_n \|\psi_n\psi\|_{H^r} < \infty, \qquad \sup_n \big\|\psi_n^2\psi\big\|_{H^r} < \infty.$$

Proof. Standard properties of approximate identities yield convergence in C^m and uniform boundedness. The Sobolev bounds follow by $C^m \hookrightarrow H^r$ for large m and continuity of multiplication in C^m. □

Lemma 10.4 (Tightness with varying tests)**.** *Let (X_n) be tight in $H^{-r}(\mathbb{T}^3)$ for some $r > 0$, and let (φ_n) be deterministic with $\sup_n \|\varphi_n\|_{H^r} < \infty$. Then $(\langle X_n, \varphi_n\rangle)$ is tight in $\mathbb{R}$.*

Proof. By duality $|\langle X_n, \varphi_n\rangle| \le \|X_n\|_{H^{-r}}\|\varphi_n\|_{H^r} \le C\|X_n\|_{H^{-r}}$ with $C = \sup_n \|\varphi_n\|_{H^r}$. □

Lemma 10.5 (Divergence under a smooth shift)**.** *Let $\Phi \sim \mu$ and define $\widetilde{\Phi} := \Phi - \psi$. Then $|F_n(\widetilde{\Phi};\psi)| \to \infty$ in probability as $n \to \infty$.*

Proof. Since $(\Phi - \psi)_n = \Phi_n - \psi_n$,

$$\begin{aligned}
&(\Phi_n - \psi_n)^3 - 3a\varepsilon_n^{-1}(\Phi_n - \psi_n) - 9b\log(\varepsilon_n^{-1})(\Phi - \psi) \\
&\quad = \Big(\Phi_n^3 - 3a\varepsilon_n^{-1}\Phi_n - 9b\log(\varepsilon_n^{-1})\Phi\Big) - 3\psi_n\Big(\Phi_n^2 - a\varepsilon_n^{-1}\Big) + 3\psi_n^2\Phi_n - \psi_n^3 + 9b\log(\varepsilon_n^{-1})\psi.
\end{aligned}$$

Pair with ψ and multiply by $e^{-\beta n}$:

$$F_n(\widetilde{\Phi};\psi) = Y_n + 9b\,e^{-\beta n}\log(\varepsilon_n^{-1})\,\|\psi\|_{L^2}^2,$$

where Y_n is the sum of the other four terms with prefactor $e^{-\beta n}$. Exactly as in the standard decomposition:

- $e^{-\beta n}\big\langle \Phi_n^3 - 3a\varepsilon_n^{-1}\Phi_n - 9b\log(\varepsilon_n^{-1})\Phi, \psi\big\rangle = F_n(\Phi;\psi) \to 0$ in probability (Lemma 10.1);
- $\Phi_n^2 - a\varepsilon_n^{-1}$ is tight in H^{-r} by Proposition 7.1(i), and $\psi_n\psi$ is uniformly bounded in H^r (Lemma 10.3), hence $e^{-\beta n}\big\langle \Phi_n^2 - a\varepsilon_n^{-1}, \psi_n\psi\big\rangle \to 0$ in probability by Lemmas 10.4 and 6.2;
- $\big\langle \Phi_n, \psi_n^2\psi\big\rangle$ is tight (since $\Phi_n \to \Phi$ in H^{-s} a.s. and $\psi_n^2\psi \to \psi^3$ in H^s with uniform bounds), hence $e^{-\beta n}\big\langle \Phi_n, \psi_n^2\psi\big\rangle \to 0$ in probability by Lemma 6.2;

- $\langle \psi_n^3, \psi \rangle$ is bounded deterministically, hence $e^{-\beta n} \langle \psi_n^3, \psi \rangle \to 0$.

Thus $Y_n \to 0$ in probability. Using $\log(\varepsilon_n^{-1}) = e^n$ yields the deterministic divergence

$$9b\, e^{-\beta n} \log(\varepsilon_n^{-1}) \, \|\psi\|_{L^2}^2 = 9b\, e^{(1-\beta)n} \, \|\psi\|_{L^2}^2 \to \pm\infty,$$

since $b \neq 0$ (Lemma 8.2) and $\beta < 1$. Hence $|F_n(\widetilde{\Phi}; \psi)| \to \infty$ in probability. □

Lemma 10.6. *Let (X_n) be real random variables such that $|X_n| \to \infty$ in probability. Then for any deterministic increasing subsequence (n_k),*

$$\mathbb{P}(X_{n_k} \to 0) = 0.$$

Proof. Let $A_k := \{|X_{n_k}| \leq 1\}$. Then $\mathbb{P}(A_k) \to 0$. Moreover $\{X_{n_k} \to 0\} \subset \bigcup_{K=1}^{\infty} \bigcap_{k \geq K} A_k$. For each K, $\mathbb{P}(\bigcap_{k \geq K} A_k) \leq \inf_{k \geq K} \mathbb{P}(A_k)$, which tends to 0 as $K \to \infty$. □

Theorem 10.7. *With A_ψ as above, one has $\mu(A_\psi + \psi) = 0$.*

Proof. Let $\Phi \sim \mu$ and set $\widetilde{\Phi} = \Phi - \psi$. Then $\widetilde{\Phi}$ has law $\mu(\cdot + \psi)$, so $\mu(A_\psi + \psi) = \mathbb{P}(\widetilde{\Phi} \in A_\psi)$. On $\{\widetilde{\Phi} \in A_\psi\}$ one has $F_{n_k}(\widetilde{\Phi}; \psi) \to 0$ by definition of A_ψ. But by Lemma 10.5, $|F_n(\widetilde{\Phi}; \psi)| \to \infty$ in probability, hence Lemma 10.6 gives $\mathbb{P}(F_{n_k}(\widetilde{\Phi}; \psi) \to 0) = 0$. Thus $\mathbb{P}(\widetilde{\Phi} \in A_\psi) = 0$. □

Proof of Theorem 1.1. By Lemma 10.2, $\mu(A_\psi) = 1$. By Theorem 10.7, $\mu(A_\psi + \psi) = 0$. Lemma 3.1 yields $\mu \perp T_{\psi\#}\mu$. □

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