CONDUCTOR-BOUNDED WHITTAKER TEST VECTORS FOR SHIFTED LOCAL RANKIN–SELBERG INTEGRALS ON $\mathrm{GL}_{n+1} \times \mathrm{GL}_n$

Abstract. Let F be a non-archimedean local field, fix an additive character ψ of conductor $\mathfrak{o}$, and let Π be a generic irreducible admissible representation of $\mathrm{GL}_{n+1}(F)$. For each integer $m \geq 0$ we construct a Whittaker function $W_m \in \mathcal{W}(\Pi, \psi^{-1})$ depending only on (Π, ψ, m) with the following uniform property: for every generic irreducible admissible representation π of $\mathrm{GL}_n(F)$ with conductor exponent $a(\pi) \leq m$, for the normalized Whittaker newform $V \in \mathcal{W}(\pi, \psi)$, and for any generator Q of the inverse conductor ideal $\mathfrak{q}(\pi)^{-1}$, the shifted local Rankin–Selberg integral

$$\int_{N_n\backslash \mathrm{GL}_n(F)} W_m(\mathrm{diag}(g,1)\, u_Q)\, V(g)\, |\det g|^{s-\frac{1}{2}}\, dg, \qquad u_Q := I_{n+1} + QE_{n,n+1},$$

is absolutely convergent for all $s \in \mathbb{C}$ and evaluates to the explicit nonzero constant $\psi^{-1}(Q) \cdot \mathrm{vol}(N_n(\mathfrak{o})\backslash K_1(\mathfrak{p}^m))$. In particular, for each fixed π one obtains a nonvanishing pair (W, V) by taking $m = a(\pi)$. We also record a topological obstruction showing that the "exact mirabolic" subgroup in $\mathrm{GL}_n(\mathfrak{o})$ has empty interior (for all $n \geq 1$), so it cannot support nonzero locally constant functions.

1. Introduction: the shifted integral and the quantifiers

Let F be a non-archimedean local field with ring of integers $\mathfrak{o}$, maximal ideal $\mathfrak{p}$, and residue cardinality q. Fix a nontrivial additive character $\psi : F \to \mathbb{C}^\times$ of conductor $\mathfrak{o}$ (i.e. ψ is trivial on $\mathfrak{o}$ and nontrivial on $\mathfrak{p}^{-1}$). For $r \geq 1$, let $N_r \subset \mathrm{GL}_r(F)$ be the standard upper-triangular unipotent subgroup and let $\psi_r : N_r \to \mathbb{C}^\times$ be the standard generic character $\psi_r(u) = \psi(\sum_{i=1}^{r-1} u_{i,i+1})$.

Let Π be a generic irreducible admissible representation of $\mathrm{GL}_{n+1}(F)$, realized in its ψ_{n+1}^{-1}-Whittaker model $\mathcal{W}(\Pi, \psi_{n+1}^{-1})$. Let π be a generic irreducible admissible representation of $\mathrm{GL}_n(F)$, realized in its ψ_n-Whittaker model $\mathcal{W}(\pi, \psi_n)$. Let $\mathfrak{q}(\pi) = \mathfrak{p}^{a(\pi)}$ be its conductor ideal (so $a(\pi) \geq 0$ is the conductor exponent), and choose $Q \in F^\times$ with $Q\mathfrak{o} = \mathfrak{q}(\pi)^{-1} = \mathfrak{p}^{-a(\pi)}$.

Define

$$u_Q := I_{n+1} + QE_{n,n+1} \in \mathrm{GL}_{n+1}(F), \qquad \iota : \mathrm{GL}_n(F) \hookrightarrow \mathrm{GL}_{n+1}(F),\ \iota(g) = \mathrm{diag}(g, 1).$$

For $W \in \mathcal{W}(\Pi, \psi_{n+1}^{-1})$ and $V \in \mathcal{W}(\pi, \psi_n)$, consider the shifted local Rankin–Selberg integral

$$Z(W, V, s; Q) := \int_{N_n\backslash \mathrm{GL}_n(F)} W(\iota(g)u_Q)\, V(g)\, |\det g|^{s-\frac{1}{2}}\, dg. \tag{1}$$

The natural existence question can be read in (at least) two ways:

(Pairwise problem). For fixed (Π, π), must there exist $W \in \mathcal{W}(\Pi, \psi_{n+1}^{-1})$ and $V \in \mathcal{W}(\pi, \psi_n)$ such that $Z(W, V, s; Q)$ is finite and nonzero for all $s \in \mathbb{C}$?

(Universal-W problem). For fixed Π, must there exist a single $W \in \mathcal{W}(\Pi, \psi_{n+1}^{-1})$ such that *for every* generic π there exists $V \in \mathcal{W}(\pi, \psi_n)$ with $Z(W, V, s; Q)$ finite and nonzero for all $s \in \mathbb{C}$?

The *universal-W problem* is substantially stronger (it places a uniformity constraint on W across all conductor exponents). The present note establishes a precise, strong *conductor-bounded* uniformity statement. This resolves the pairwise problem immediately (by choosing the bound $m = a(\pi)$), and it clarifies exactly where conductor dependence enters.

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Theorem 1.1 (Main theorem: conductor-bounded uniform test vectors)**.** *Fix $n \geq 1$ and a generic irreducible admissible representation Π of $\mathrm{GL}_{n+1}(F)$. For each integer $m \geq 0$ there exists a Whittaker function $W_m \in \mathcal{W}(\Pi, \psi_{n+1}^{-1})$ with the following property:*

For every generic irreducible admissible representation π of $\mathrm{GL}_n(F)$ with conductor exponent $a(\pi) \leq m$, let $V \in \mathcal{W}(\pi, \psi_n)$ be the normalized Whittaker newform and let $Q \in F^\times$ satisfy $Q\mathfrak{o} = \mathfrak{p}^{-a(\pi)}$. Then the shifted integral (1) is absolutely convergent for every $s \in \mathbb{C}$ and equals

$$Z(W_m, V, s; Q) = \psi^{-1}(Q) \cdot \mathrm{vol}\big(N_n(\mathfrak{o})\backslash K_1(\mathfrak{p}^m)\big),$$

a nonzero constant independent of s.

Corollary 1.2 (Resolution of the pairwise existence problem)**.** *For every pair (Π, π) as above, taking $m = a(\pi)$ and $W = W_m$ from Theorem 1.1, and taking V to be the normalized Whittaker newform of π, the integral $Z(W, V, s; Q)$ is finite and nonzero for all $s \in \mathbb{C}$.*

The construction of W_m uses only standard features of the Kirillov model (restriction of Whittaker functions to the mirabolic subgroup) and an elementary "phase collapse" computation forced by the inequality $m \geq a(\pi)$.

2. Notation

Fix Haar measures once and for all; the numerical value of the volume terms depends on these choices, but positivity and nonvanishing do not.

Let $K_n = \mathrm{GL}_n(\mathfrak{o})$. For $m \geq 0$ define the usual congruence subgroup

$$K_1(\mathfrak{p}^m) := \{k \in K_n : \ e_n k \equiv e_n \pmod{\mathfrak{p}^m}\}, \qquad e_n = (0, \dots, 0, 1) \in F^n. \tag{2}$$

Equivalently, $k \in K_1(\mathfrak{p}^m)$ if and only if $k_{n,j} \in \mathfrak{p}^m$ for $1 \leq j < n$ and $k_{n,n} \in 1 + \mathfrak{p}^m$. Each $K_1(\mathfrak{p}^m)$ is open and compact in $\mathrm{GL}_n(F)$.

3. A smoothness obstruction: the exact mirabolic has empty interior

This section is not used in the proof of Theorem 1.1, but it explains why one cannot hope to build a compactly supported test function by insisting the last row be *exactly* e_n.

Definition 3.1. Let $n \geq 1$. The *exact mirabolic* subgroup of $K_n = \mathrm{GL}_n(\mathfrak{o})$ is

$$K_n^{\mathrm{mir}} := \{k \in K_n : \ e_n k = e_n\}.$$

Lemma 3.2. *For every $n \geq 1$, the subgroup K_n^{mir} is closed and has empty interior in K_n. For $n \geq 2$ one moreover has*

$$K_n^{\mathrm{mir}} = \bigcap_{m \geq 1} K_1(\mathfrak{p}^m).$$

Proof. For $n \geq 2$, the condition $e_n k = e_n$ is equivalent to the congruences $e_n k \equiv e_n \pmod{\mathfrak{p}^m}$ for all $m \geq 1$, because $\bigcap_{m \geq 1} \mathfrak{p}^m = \{0\}$ in $\mathfrak{o}$. This gives $K_n^{\mathrm{mir}} = \bigcap_{m \geq 1} K_1(\mathfrak{p}^m)$, hence K_n^{mir} is closed.

Each $K_1(\mathfrak{p}^m)$ is open. For $n \geq 2$ the inclusions are strict, so the intersection cannot contain any open neighborhood (of the identity, hence of any point). Thus it has empty interior.

For $n = 1$, one has $K_1 = \mathfrak{o}^\times$ and $K_1^{\mathrm{mir}} = \{1\}$, which is closed but not open in the non-discrete compact group $\mathfrak{o}^\times$. Hence it also has empty interior. □

Proposition 3.3. *Let G be a totally disconnected locally compact group and let $H \subseteq G$ be a closed subset with empty interior. Then every locally constant function $f : G \to \mathbb{C}$ supported on H is identically zero.*

Proof. If $f(x) \neq 0$ at some x, then local constancy gives an open neighborhood U of x on which f is constant and nonzero, hence $U \subseteq \mathrm{supp}(f) \subseteq H$, contradicting that H has empty interior. □

4. The shift u_Q and the Whittaker character computation

Lemma 4.1 (Shift factor)**.** *Let $W \in \mathcal{W}(\Pi, \psi_{n+1}^{-1})$. For all $g \in \mathrm{GL}_n(F)$ and all $Q \in F$,*

$$W(\mathrm{diag}(g,1)\,u_Q) = \psi^{-1}(Q\,g_{n,n})\,W(\mathrm{diag}(g,1)).$$

Proof. Set $\iota(g) = \mathrm{diag}(g,1)$. Write

$$\iota(g)\,u_Q = \big(\iota(g)\,u_Q\,\iota(g)^{-1}\big)\,\iota(g).$$

Since $u_Q = I_{n+1} + QE_{n,n+1}$, a direct computation gives

$$\iota(g)\,E_{n,n+1}\,\iota(g)^{-1} = \sum_{i=1}^{n} g_{i,n}\,E_{i,n+1},$$

hence

$$x := \iota(g)\,u_Q\,\iota(g)^{-1} = I_{n+1} + Q\sum_{i=1}^{n} g_{i,n}E_{i,n+1} \in N_{n+1}.$$

The generic character ψ_{n+1} depends only on the superdiagonal entries $(i, i+1)$. Among the matrix units $E_{i,n+1}$, only $E_{n,n+1}$ lies on the superdiagonal; thus

$$\psi_{n+1}(x) = \psi(Q\,g_{n,n}).$$

Using Whittaker equivariance $W(ux) = \psi_{n+1}^{-1}(u)\,W(x)$ for $u \in N_{n+1}$ yields

$$W(\iota(g)u_Q) = W(x\,\iota(g)) = \psi_{n+1}^{-1}(x)\,W(\iota(g)) = \psi^{-1}(Q\,g_{n,n})\,W(\iota(g)),$$

as claimed. □

5. Kirillov models and extension of compactly supported data

Let $P_{n+1} \subset \mathrm{GL}_{n+1}(F)$ be the *mirabolic* subgroup (matrices whose last row equals e_{n+1}). Every element of P_{n+1} has a unique block decomposition

$$p = \begin{pmatrix} g & x \\ 0 & 1 \end{pmatrix}, \qquad g \in \mathrm{GL}_n(F),\ x \in F^n.$$

Definition 5.1. Let $H \subseteq G$ be closed and χ a character of H. We write $\text{c-Ind}_H^G(\chi)$ for the space of locally constant functions $f : G \to \mathbb{C}$ such that $f(hg) = \chi(h)f(g)$ for $h \in H$, and whose support is compact modulo H.

The following identification is a standard and elementary observation.

Lemma 5.2. *Restriction to the Levi embedding $\iota : \mathrm{GL}_n(F) \hookrightarrow P_{n+1}$ induces an isomorphism*

$$\text{c-Ind}_{N_{n+1}}^{P_{n+1}}(\psi_{n+1}^{-1}) \xrightarrow{\ \sim\ } \text{c-Ind}_{N_n}^{\mathrm{GL}_n(F)}(\psi_n^{-1}), \qquad F \longmapsto \big(g \mapsto F(\mathrm{diag}(g,1))\big).$$

Proof. Given $f \in \text{c-Ind}_{N_n}^{\mathrm{GL}_n(F)}(\psi_n^{-1})$, define $F : P_{n+1} \to \mathbb{C}$ by

$$F\left(\begin{pmatrix} g & x \\ 0 & 1 \end{pmatrix}\right) := \psi^{-1}(x_n)\,f(g),$$

where x_n is the last coordinate of the column vector x. One checks directly that F is locally constant, compactly supported modulo N_{n+1} if and only if f is compactly supported modulo N_n, and that F satisfies $F(up) = \psi_{n+1}^{-1}(u)F(p)$ for all $u \in N_{n+1}$. Moreover, $F(\mathrm{diag}(g,1)) = f(g)$. This constructs an inverse to restriction, and the two maps are inverse isomorphisms. □

We now invoke the Kirillov model theorem for p-adic GL_{n+1} in the precise form needed here.

Theorem 5.3 (Gelfand–Kazhdan; Kirillov model contains compact induction)**.** *Let Π be a generic irreducible admissible representation of $\mathrm{GL}_{n+1}(F)$ and realize it in its Whittaker model $\mathcal{W}(\Pi, \psi_{n+1}^{-1}) \subset \mathrm{Ind}_{N_{n+1}}^{\mathrm{GL}_{n+1}(F)}(\psi_{n+1}^{-1})$. Then the restriction map to the mirabolic subgroup,*

$$\mathrm{Res}_{P_{n+1}} : \mathcal{W}(\Pi, \psi_{n+1}^{-1}) \longrightarrow \mathrm{Ind}_{N_{n+1}}^{P_{n+1}}(\psi_{n+1}^{-1}), \qquad W \mapsto W|_{P_{n+1}},$$

has image containing the compact induction $\text{c-Ind}_{N_{n+1}}^{P_{n+1}}(\psi_{n+1}^{-1})$.

Proof (reference). This is a classical Kirillov model statement for p-adic GL_{n+1} due to Gelfand–Kazhdan [2]. For a modern discussion of Kirillov models via Bernstein–Zelevinsky derivatives and restriction to mirabolic subgroups, see also [4, §3]. (We do not reprove the Kirillov model theorem here.) □

Corollary 5.4 (Extension to the embedded Levi)**.** *For every $f \in \text{c-Ind}_{N_n}^{\mathrm{GL}_n(F)}(\psi_n^{-1})$ there exists $W \in \mathcal{W}(\Pi, \psi_{n+1}^{-1})$ such that*

$$W(\mathrm{diag}(g,1)) = f(g) \qquad \textit{for all } g \in \mathrm{GL}_n(F).$$

Proof. By Lemma 5.2, f corresponds to a function $F \in \text{c-Ind}_{N_{n+1}}^{P_{n+1}}(\psi_{n+1}^{-1})$ whose restriction to $\mathrm{diag}(\mathrm{GL}_n(F),1)$ is f. By Theorem 5.3 there exists $W \in \mathcal{W}(\Pi, \psi_{n+1}^{-1})$ with $W|_{P_{n+1}} = F$, hence $W(\mathrm{diag}(g,1)) = f(g)$ for all g. □

6. A conductor-bounded test function on GL_n and its lift to GL_{n+1}

Fix $m \geq 0$. Define $f_m : \mathrm{GL}_n(F) \to \mathbb{C}$ by

$$f_m(uk) = \psi_n^{-1}(u) \quad (u \in N_n,\ k \in K_1(\mathfrak{p}^m)), \qquad f_m(g) = 0 \quad \text{if } g \notin N_n K_1(\mathfrak{p}^m). \tag{3}$$

Lemma 6.1. *The function f_m is well-defined and lies in $\text{c-Ind}_{N_n}^{\mathrm{GL}_n(F)}(\psi_n^{-1})$. Moreover, $N_n \cap K_1(\mathfrak{p}^m) = N_n(\mathfrak{o})$.*

Proof. If $uk = u'k'$ with $u, u' \in N_n$ and $k, k' \in K_1(\mathfrak{p}^m)$, then $k^{-1}k' = u^{-1}u' \in N_n \cap K_1(\mathfrak{p}^m)$. Since $K_1(\mathfrak{p}^m) \subseteq \mathrm{GL}_n(\mathfrak{o})$, we have $N_n \cap K_1(\mathfrak{p}^m) \subseteq N_n(\mathfrak{o})$. Conversely, any $u \in N_n(\mathfrak{o})$ has last row e_n, hence lies in $K_1(\mathfrak{p}^m)$, so $N_n(\mathfrak{o}) \subseteq N_n \cap K_1(\mathfrak{p}^m)$. Thus $N_n \cap K_1(\mathfrak{p}^m) = N_n(\mathfrak{o})$.

Because ψ has conductor $\mathfrak{o}$, it is trivial on $\mathfrak{o}$, hence ψ_n is trivial on $N_n(\mathfrak{o})$. Therefore $\psi_n^{-1}(u) = \psi_n^{-1}(u')$, and (3) is well-defined.

The support of f_m modulo N_n is contained in $K_1(\mathfrak{p}^m)$, which is compact, and f_m is locally constant (right $K_1(\mathfrak{p}^m)$-invariant). Hence $f_m \in \text{c-Ind}_{N_n}^{\mathrm{GL}_n(F)}(\psi_n^{-1})$. □

Proposition 6.2. *For each $m \geq 0$ there exists $W_m \in \mathcal{W}(\Pi, \psi_{n+1}^{-1})$ such that*

$$W_m(\mathrm{diag}(g,1)) = f_m(g) \qquad (g \in \mathrm{GL}_n(F)).$$

Proof. Apply Corollary 5.4 to $f = f_m$. □

7. Whittaker newforms on GL_n

Let π be a generic irreducible admissible representation of $\mathrm{GL}_n(F)$ with conductor ideal $\mathfrak{q}(\pi) = \mathfrak{p}^{a(\pi)}$.

Proposition 7.1 (Whittaker newforms)**.** *There exists $V \in \mathcal{W}(\pi, \psi_n)$ (unique up to scalars) that is right $K_1(\mathfrak{p}^{a(\pi)})$-invariant. Moreover, one may normalize it so that $V(1) = 1$, and then $V(k) = 1$ for all $k \in K_1(\mathfrak{p}^{a(\pi)})$.*

Proof (reference). The existence and uniqueness (up to scalars) of the $K_1(\mathfrak{p}^{a(\pi)})$-fixed newvector is standard local newform theory for GL_n [3]. Translating the newvector to the Whittaker model and normalizing $V(1) = 1$ requires that the corresponding Whittaker newform does not vanish at the identity, a nontrivial fact proved in explicit newform computations; see, e.g., [6] and [5]. For such a normalized V, right invariance implies $V(k) = V(1) = 1$ on $K_1(\mathfrak{p}^{a(\pi)})$. □

8. Phase collapse and evaluation of the shifted integral

Lemma 8.1. *Let $m \geq 0$ and let $a \geq 0$. Suppose $Q \in \mathfrak{p}^{-a}$. If $m \geq a$ and $k \in K_1(\mathfrak{p}^m)$, then*

$$\psi(Q\, k_{n,n}) = \psi(Q).$$

Proof. For $k \in K_1(\mathfrak{p}^m)$ we have $k_{n,n} \equiv 1 \pmod{\mathfrak{p}^m}$, hence $k_{n,n} = 1 + t$ with $t \in \mathfrak{p}^m$. Then $Qk_{n,n} = Q + Qt$ with $Qt \in \mathfrak{p}^{m-a} \subseteq \mathfrak{o}$ because $m \geq a$. Since ψ is trivial on $\mathfrak{o}$, we get $\psi(Q + Qt) = \psi(Q)$. □

Proof of Theorem 1.1. Let W_m be as in Proposition 6.2 and V be the normalized Whittaker newform of π. By Lemma 4.1 and the defining property of W_m,

$$W_m(\mathrm{diag}(g,1)u_Q) = \psi^{-1}(Qg_{n,n})\, W_m(\mathrm{diag}(g,1)) = \psi^{-1}(Qg_{n,n})\, f_m(g).$$

Hence the integrand in (1) is supported on $N_nK_1(\mathfrak{p}^m)$. Since $K_1(\mathfrak{p}^m)$ is compact, the quotient $N_n\backslash N_nK_1(\mathfrak{p}^m)$ is compact; thus the integral is absolutely convergent for all $s \in \mathbb{C}$.

On the support, write $g = uk$ with $u \in N_n$ and $k \in K_1(\mathfrak{p}^m)$. Because the last row of u is e_n, one has $(uk)_{n,n} = k_{n,n}$. Moreover $\det(u) = 1$ and $k \in \mathrm{GL}_n(\mathfrak{o})$, hence $|\det g| = |\det k| = 1$, so $|\det g|^{s-\frac{1}{2}} \equiv 1$.

Using the definition of f_m and the Whittaker transformation of V,

$$f_m(uk) = \psi_n^{-1}(u), \qquad V(uk) = \psi_n(u)\, V(k),$$

so $f_m(uk)\, V(uk) = V(k)$. Also Lemma 8.1 (with $a = a(\pi) \le m$) gives $\psi(Qk_{n,n}) = \psi(Q)$, hence $\psi^{-1}(Q(uk)_{n,n}) = \psi^{-1}(Q)$. Therefore, on $N_nK_1(\mathfrak{p}^m)$ the integrand equals the constant $\psi^{-1}(Q)\, V(k)$.

Since $N_n \cap K_1(\mathfrak{p}^m) = N_n(\mathfrak{o})$ by Lemma 6.1, we have the standard identification

$$N_n\backslash N_nK_1(\mathfrak{p}^m) \;\simeq\; (N_n \cap K_1(\mathfrak{p}^m))\backslash K_1(\mathfrak{p}^m) \;=\; N_n(\mathfrak{o})\backslash K_1(\mathfrak{p}^m),$$

and hence

$$Z(W_m, V, s; Q) = \psi^{-1}(Q) \int_{N_n(\mathfrak{o})\backslash K_1(\mathfrak{p}^m)} V(k)\, dk.$$

Finally, since $a(\pi) \le m$, we have $K_1(\mathfrak{p}^m) \subseteq K_1(\mathfrak{p}^{a(\pi)})$, so V is right $K_1(\mathfrak{p}^m)$-invariant and normalized by $V(1) = 1$; thus $V(k) = 1$ for all $k \in K_1(\mathfrak{p}^m)$. Therefore

$$Z(W_m, V, s; Q) = \psi^{-1}(Q) \cdot \mathrm{vol}\big(N_n(\mathfrak{o})\backslash K_1(\mathfrak{p}^m)\big),$$

which is a nonzero constant (a nonzero complex unit times a strictly positive real volume). □

Proof of Corollary 1.2. Given π, take $m = a(\pi)$ and apply Theorem 1.1. □

Remark 8.2 (On Haar measures). The constant $\mathrm{vol}(N_n(\mathfrak{o})\backslash K_1(\mathfrak{p}^m))$ depends on Haar normalizations. However, it is always finite and strictly positive; hence the nonvanishing is independent of the choice of measures.

Remark 8.3 (On the universal-W problem). Theorem 1.1 produces a family $\{W_m\}_{m\ge 0}$ uniform for representations π with $a(\pi) \le m$. It does *not* assert the existence of a single W working simultaneously for all conductors. The topological obstruction in §3 shows why naive attempts to force compact support by restricting to an "exact mirabolic" subgroup cannot work in the smooth setting.

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