SCHUR COMPLEMENTS AND OBSTRUCTIONS FOR LARGE ε-LIGHT VERTEX SETS IN GRAPHS

Abstract. Let $G = (V, E)$ be a finite undirected graph (possibly disconnected) with Laplacian L. For $S \subseteq V$, let $G_S = (V, E(S, S))$ be the graph on the same vertex set but with only edges whose endpoints both lie in S, and let L_S be its Laplacian (equivalently, the Laplacian of the induced subgraph $G[S]$, padded with isolated vertices in $V \setminus S$). Fix $\varepsilon \in (0, 1)$. We call S *ε-light* if $\varepsilon L - L_S \succeq 0$.

We study the question of whether there exists a universal constant $c > 0$ such that every graph contains an ε-light set of size at least $c\,\varepsilon\,|V|$ for every $\varepsilon \in (0, 1)$. We provide (i) careful kernel/projection bookkeeping for disconnected graphs and equivalent normalized formulations; (ii) sharp extremal examples showing any such c must satisfy $c \le \frac{1}{2}$ and a complete characterization for complete graphs; (iii) explicit counterexamples showing that natural "edgewise effective resistance" and "vertex-star linearization" strategies do not certify ε-lightness; and (iv) an exact variational and Schur complement (Kron reduction) characterization that isolates the true obstruction: internal energy in $G[S]$ must be dominated by the effective coupling of S to $V \setminus S$. Finally, we document why current interlacing-polynomial methods [2] do not directly apply to vertex-induced edge selection, by exhibiting a K_3 example where the expected characteristic polynomial has nonreal roots.

Contents

1. Introduction

Let $G = (V, E)$ be a finite, undirected, unweighted graph with $n := |V|$. Its (combinatorial) Laplacian is the symmetric matrix $L \in \mathbb{R}^{V \times V}$ defined by

$$L_{uu} = \deg(u), \qquad L_{uv} = \begin{cases} -1, & \{u, v\} \in E, \\ 0, & \text{otherwise.} \end{cases}$$

Equivalently,

$$x^\top L x = \sum_{\{u,v\} \in E} (x_u - x_v)^2 \qquad (x \in \mathbb{R}^V).$$

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For a subset $S \subseteq V$, define $G_S = (V, E(S,S))$, the graph obtained by keeping only those edges with both endpoints in S. Let L_S be its Laplacian. Note that L_S is the Laplacian of the induced subgraph $G[S]$, padded with isolated vertices outside S.

Definition 1.1 (ε-light set)**.** Fix $\varepsilon \in (0,1)$. A set $S \subseteq V$ is *ε-light* if

$$\varepsilon L - L_S \succeq 0.$$

The motivating question is as follows.

Conjecture 1.2 (Linear-size ε-light subsets)**.** *There exists a universal constant $c > 0$ such that for every finite graph $G = (V, E)$ and every $\varepsilon \in (0,1)$, there exists an ε-light subset $S \subseteq V$ satisfying*

$$|S| \;\geq\; c\,\varepsilon\,|V|.$$

At the time of writing we do not know whether Conjecture 1.2 is true. The purpose of this note is to (a) formalize the correct operator-theoretic viewpoint (via Schur complements/Kron reduction) and (b) rigorously record why several seemingly natural approaches fail.

1.1. A sharp universal upper bound.

Proposition 1.3 (Any universal constant must satisfy $c \leq \frac{1}{2}$)**.** *If Conjecture 1.2 holds for a universal constant c, then necessarily $c \leq \frac{1}{2}$.*

Proof. Let G be a perfect matching on n vertices, where n is even. Then L is block diagonal with $n/2$ identical 2×2 blocks $\left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)$. Fix $\varepsilon \in (0,1)$ and let $S \subseteq V$. If S contains both endpoints of some matched edge $e = \{u, v\}$, then, with $x := \mathbf{e}_u - \mathbf{e}_v$, we have $x^\top L_S x = x^\top L x > 0$ (indeed both equal 4), so $L_S \preceq \varepsilon L$ fails unless $\varepsilon \geq 1$. Therefore, for $\varepsilon \in (0,1)$, every ε-light set must contain at most one endpoint from each matched edge, hence $|S| \leq n/2$. The requirement $c\,\varepsilon\, n \leq n/2$ for all $\varepsilon \in (0,1)$ forces $c \leq 1/2$. ☐

1.2. Complete graphs.

Complete graphs show that ε-lightness can force $|S|$ to be *at most* on the order of εn.

Proposition 1.4 (Complete graphs)**.** *Let $G = K_n$ and $\varepsilon \in (0,1)$.*

(i) If $|S| \geq 2$, then S is ε-light if and only if $|S| \leq \varepsilon n$.
(ii) If $|S| \in \{0,1\}$, then $L_S = 0$ and hence S is ε-light for every $\varepsilon > 0$.

Proof. If $|S| \in \{0,1\}$ then there are no edges with both endpoints in S, so $L_S = 0$.

Assume $|S| \geq 2$. The Laplacian of K_n is $L = nI - \mathbf{1}\mathbf{1}^\top$. The padded induced Laplacian L_S equals the Laplacian of $K_{|S|}$ on coordinates in S and is 0 on $V \setminus S$. Consider the subspace

$$W := \{x \in \mathbb{R}^V : \ \mathrm{supp}(x) \subseteq S,\ \mathbf{1}^\top x = 0\}.$$

On W, one has $Lx = nx$ and $L_S x = |S|x$. Therefore the maximal generalized eigenvalue of the pencil (L_S, L) equals $|S|/n$, and $L_S \preceq \varepsilon L$ holds if and only if $|S|/n \leq \varepsilon$. ☐

2. Preliminaries

2.1. Edge Laplacians and padding.

Fix an arbitrary orientation of edges. For $e = (u, v)$ write $b_e := \mathbf{e}_u - \mathbf{e}_v \in \mathbb{R}^V$. Then the Laplacian decomposes as

$$L = \sum_{e \in E} b_e b_e^\top.$$

If $F \subseteq E$ is any edge set, we write $L_F := \sum_{e \in F} b_e b_e^\top$; this is the Laplacian of the graph (V, F). In particular, for $S \subseteq V$,

$$L_S = L_{E(S,S)} = \sum_{e \in E(S,S)} b_e b_e^\top,$$

and thus $L_S \preceq L$ for every S.

2.2. Kernel, harmonic subspace, and pseudoinverses.

Let $\ker(L)$ denote the kernel of L. It is standard that $\ker(L)$ is the subspace of vectors constant on each connected component of G. Let $\mathcal{H} := \ker(L)^\perp$ and let $\Pi_{\mathcal{H}}$ be the orthogonal projection onto $\mathcal{H}$.

We use L^+ for the Moore–Penrose pseudoinverse, and $L^{1/2}$, $L^{+1/2}$ for the unique PSD square roots of L and L^+. We will use the identities

$$LL^+ = L^+L = \Pi_{\mathcal{H}}, \qquad L^{1/2}L^{+1/2} = L^{+1/2}L^{1/2} = \Pi_{\mathcal{H}},$$

and the fact that if $A \succeq 0$ satisfies $\ker(L) \subseteq \ker(A)$, then $\Pi_{\mathcal{H}} A \Pi_{\mathcal{H}} = A$.

3. Equivalent formulations and kernel bookkeeping

A technical pitfall in disconnected graphs is that substituting $x = L^{1/2}y$ yields $L^{+1/2}x = \Pi_{\mathcal{H}}y$, not y. This section records the correct equivalences.

Lemma 3.1 (Equivalent normalizations)**.** *Let $G = (V, E)$ have Laplacian L, let $S \subseteq V$, and let L_S be the Laplacian of $G_S = (V, E(S, S))$. Let $\mathcal{H} = \ker(L)^{\perp}$ and $\Pi_{\mathcal{H}}$ be the orthogonal projection onto $\mathcal{H}$. The following are equivalent:*

(a) $L_S \preceq \varepsilon L$.
(b) $L^{+1/2}L_S L^{+1/2} \preceq \varepsilon \Pi_{\mathcal{H}}$.
(c) For all $x \in \mathcal{H}$, $x^\top L_S x \le \varepsilon\, x^\top Lx$.

Proof. (a) $\Rightarrow$ (b) follows by conjugating with $L^{+1/2}$ and using $L^{+1/2}LL^{+1/2} = \Pi_{\mathcal{H}}$. (b) $\Rightarrow$ (c) is immediate since $\Pi_{\mathcal{H}}x = x$ for $x \in \mathcal{H}$.

(c) $\Rightarrow$ (a): let $y \in \mathbb{R}^V$ be arbitrary and decompose $y = y_{\mathcal{H}} + y_0$ where $y_{\mathcal{H}} := \Pi_{\mathcal{H}}y \in \mathcal{H}$ and $y_0 := y - \Pi_{\mathcal{H}}y \in \ker(L)$. Then $y^\top Ly = y_{\mathcal{H}}^\top L y_{\mathcal{H}}$ and, crucially, $L_S y_0 = 0$. Indeed, $y_0 \in \ker(L)$ means y_0 is constant on each connected component of G; every component of G_S is contained in a component of G (since G_S is a subgraph with the same vertex set), hence y_0 is constant on each component of G_S and thus $y_0 \in \ker(L_S)$. Therefore

$$y^\top(\varepsilon L - L_S)y = y_{\mathcal{H}}^\top(\varepsilon L - L_S)y_{\mathcal{H}} \ge 0$$

by hypothesis (c), proving (a). □

4. Two classical “naive” strategies and explicit counterexamples

This section records two pitfalls that arise in attempts to prove Conjecture 1.2.

4.1. Edgewise effective resistance does not control ε-lightness.

A tempting heuristic is that if every internal edge $e \in E(S, S)$ has small effective resistance in G, then L_S should be dominated by εL. The following explicit computation shows this is false.

Example 4.1 (Bounded effective resistances but not ε-light)**.** Let G have vertex set $\{1, 2, 3, 4, 5, 6\}$. The induced subgraph on $\{1, 2, 3, 4\}$ is a clique K_4. Vertices 5 and 6 are adjacent to each of $1, 2, 3, 4$, and there is no edge between 5 and 6. Let $S := \{1, 2, 3, 4\}$.

Then every internal edge $e \in E(S, S)$ satisfies $R^G_{\mathrm{eff}}(e) = \frac{1}{3}$, yet S is not ε-light for $\varepsilon = \frac{1}{2}$:

$$\lambda_{\max}\big(L^{-1/2}L_S L^{-1/2}\big) \;\ge\; \frac{2}{3} \;>\; \frac{1}{2}.$$

Proof. Consider the internal edge $e = \{1, 2\}$ and vector $b := \mathbf{e}_1 - \mathbf{e}_2$. Vertices 1 and 2 have degree 5 in G, and a direct calculation gives $Lb = 6b$. Hence $R^G_{\mathrm{eff}}(e) = b^\top L^+ b = \frac{1}{6}\|b\|^2 = \frac{1}{3}$.

In the induced subgraph $G[S] = K_4$, vertices 1 and 2 have degree 3, and similarly $L_S b = 4b$. Therefore

$$\frac{b^\top L_S b}{b^\top Lb} = \frac{4\|b\|^2}{6\|b\|^2} = \frac{2}{3},$$

so $L_S \preceq \varepsilon L$ fails for any $\varepsilon < 2/3$, in particular for $\varepsilon = 1/2$. □

4.2. Vertex-star “linearization” overcharges boundary edges.

Another common strategy is to upper bound L_S by a sum of star Laplacians $\sum_{u\in S} L_u$. This necessarily introduces boundary leakage: edges crossing $(S, V \setminus S)$ appear linearly even though they do not appear in L_S.

For each $u \in V$ let L_u denote the Laplacian of the star graph consisting of all edges incident to u. Let ∂S denote the edge boundary, and let $L_{\partial S}$ denote the Laplacian of the cut graph $(V, \partial S)$. The exact identity

$$\sum_{u\in S} L_u \;=\; 2L_S \;+\; L_{\partial S} \tag{1}$$

implies

$$L_S = \frac{1}{2}\sum_{u\in S} L_u - \frac{1}{2}L_{\partial S} \;\preceq\; \frac{1}{2}\sum_{u\in S} L_u,$$

but the inequality $L_S \preceq \frac{1}{2}\sum_{u\in S} L_u$ can be far from tight.

Example 4.2 (Boundary leakage in $K_{m,m}$)**.** Let $G = K_{m,m}$ with bipartition $V = A \sqcup B$. Take $S = A$. Then $L_S = 0$ (hence S is ε-light for every $\varepsilon > 0$), but

$$\frac{1}{2}\sum_{u\in S} L_u = \frac{1}{2}L,$$

so any certification of the form $\frac{1}{2}\sum_{u\in S} L_u \preceq \varepsilon L$ would force $\varepsilon \geq 1/2$ despite the fact that $L_S = 0$.

Proof. Since A is an independent set, $E(S,S) = \emptyset$ and $L_S = 0$.

Every edge of $K_{m,m}$ has exactly one endpoint in A, so $\sum_{u\in A} L_u$ counts each edge Laplacian $b_e b_e^\top$ exactly once. Hence $\sum_{u\in A} L_u = \sum_{e\in E} b_e b_e^\top = L$, giving the claim. ☐

5. A normalized obstruction and a clean conjugation

Define the normalized operator

$$M(S) := L^{+1/2} L_S L^{+1/2}.$$

By Lemma 3.1, S is ε-light if and only if

$$M(S) \preceq \varepsilon \Pi_{\mathcal{H}}.$$

The mapping $S \mapsto M(S)$ is quadratic in vertex indicators and is the source of the major difficulty.

5.1. A projection subtlety for disconnected graphs.

Proposition 5.1 (Projection bookkeeping for boundary operators)**.** *Let $S \subseteq V$ and let $L_{\partial S}$ denote the Laplacian of the cut graph $(V, \partial S)$. Let $y \in \mathbb{R}^V$ be arbitrary and set $x := L^{1/2} y \in \mathcal{H}$. Then $L^{+1/2} x = \Pi_{\mathcal{H}} y$ and*

$$x^\top L^{+1/2} L_{\partial S} L^{+1/2} x = (\Pi_{\mathcal{H}} y)^\top L_{\partial S} (\Pi_{\mathcal{H}} y).$$

Moreover, if $y_0 := y - \Pi_{\mathcal{H}} y \in \ker(L)$ then $y_0 \in \ker(L_S) \cap \ker(L_{\partial S})$ and hence

$$L_S(\Pi_{\mathcal{H}} y) = L_S y, \qquad L_{\partial S}(\Pi_{\mathcal{H}} y) = L_{\partial S} y.$$

Proof. The identity $L^{+1/2} L^{1/2} = \Pi_{\mathcal{H}}$ gives $L^{+1/2} x = L^{+1/2} L^{1/2} y = \Pi_{\mathcal{H}} y$. Then

$$x^\top L^{+1/2} L_{\partial S} L^{+1/2} x = (L^{+1/2} x)^\top L_{\partial S} (L^{+1/2} x) = (\Pi_{\mathcal{H}} y)^\top L_{\partial S} (\Pi_{\mathcal{H}} y).$$

If $y_0 \in \ker(L)$ then y_0 is constant on each connected component of G and hence is constant on each connected component of G_S and of the cut graph $(V, \partial S)$ (both are subgraphs on V). Therefore $y_0 \in \ker(L_S) \cap \ker(L_{\partial S})$, so $L_S y_0 = L_{\partial S} y_0 = 0$ and the final identities follow. ☐

5.2. Star-linearization equivalence.

Define $A_u := L^{+1/2} L_u L^{+1/2}$.

Proposition 5.2 (Star-linearization conjugation)**.** *If $\sum_{u\in S} A_u \preceq 2\varepsilon\, \Pi_{\mathcal{H}}$, then*

$$\frac{1}{2}\sum_{u\in S} L_u \preceq \varepsilon\, L.$$

Proof. Conjugate $\sum_{u\in S} A_u \preceq 2\varepsilon \Pi_{\mathcal{H}}$ by $L^{1/2}$:

$$L^{1/2}\Big(\sum_{u\in S} L^{+1/2} L_u L^{+1/2}\Big) L^{1/2} \preceq 2\varepsilon\, L^{1/2} \Pi_{\mathcal{H}} L^{1/2}.$$

Using $L^{1/2} L^{+1/2} = \Pi_{\mathcal{H}}$ and $L^{1/2} \Pi_{\mathcal{H}} L^{1/2} = L$, the left-hand side becomes $\sum_{u\in S} \Pi_{\mathcal{H}} L_u \Pi_{\mathcal{H}}$. Since $\ker(L) \subseteq \ker(L_u)$ for every u (vectors constant on components of G are constant on the star at u), we have $\Pi_{\mathcal{H}} L_u \Pi_{\mathcal{H}} = L_u$. Thus $\sum_{u\in S} L_u \preceq 2\varepsilon L$, i.e., $\frac{1}{2}\sum_{u\in S} L_u \preceq \varepsilon L$. ☐

6. Schur complements and the correct variational dual

The identity (1) shows that any attempt to replace L_S by a vertex-linear sum $\sum_{u\in S} L_u$ must necessarily interact with boundary edges. The correct operator that eliminates the complement is a Schur complement (Kron reduction) [1].

6.1. Block notation and a key decomposition.

Fix $S \subseteq V$ and let $T := V \setminus S$. Write the Laplacian in block form

$$L = \begin{pmatrix} L_{SS} & L_{ST} \\ L_{TS} & L_{TT} \end{pmatrix}.$$

Let $L_{G[S]}$ denote the *unpadded* Laplacian of the induced subgraph $G[S]$ (an $|S| \times |S|$ matrix). Then

$$L_{SS} = L_{G[S]} + D_{\partial S}, \tag{2}$$

where $D_{\partial S}$ is the diagonal matrix of boundary degrees on S (i.e. $(D_{\partial S})_{uu} = |\{v \in T : \{u,v\} \in E\}|$).

6.2. **Variational characterization of the Schur complement.**

Lemma 6.1 (Schur complement via energy minimization)**.** *Let $L \succeq 0$ be a graph Laplacian, partitioned into blocks as above. Fix $x_S \in \mathbb{R}^S$ and consider*

$$\min_{x_T \in \mathbb{R}^T} \begin{pmatrix} x_S \\ x_T \end{pmatrix}^\top \begin{pmatrix} L_{SS} & L_{ST} \\ L_{TS} & L_{TT} \end{pmatrix} \begin{pmatrix} x_S \\ x_T \end{pmatrix}. \tag{3}$$

Then the minimum is attained, equals

$$x_S^\top \Big(L_{SS} - L_{ST} L_{TT}^+ L_{TS} \Big) x_S,$$

and a canonical minimizer is $x_T^\star = -L_{TT}^+ L_{TS} x_S$.

Proof. Expand the objective:

$$Q(x_T) = x_S^\top L_{SS} x_S + 2 x_T^\top L_{TS} x_S + x_T^\top L_{TT} x_T.$$

We first show $\mathrm{im}(L_{TS}) \subseteq \mathrm{im}(L_{TT})$, which guarantees solvability of $L_{TT} x_T = -L_{TS} x_S$. Let $z \in \ker(L_{TT})$. For any $t \in \mathbb{R}$ consider $v = (x_S^\top, t z^\top)^\top$. Since $L \succeq 0$,

$$0 \le v^\top L v = x_S^\top L_{SS} x_S + 2t\, z^\top L_{TS} x_S + t^2 z^\top L_{TT} z = x_S^\top L_{SS} x_S + 2t\, z^\top L_{TS} x_S$$

for all $t \in \mathbb{R}$. Hence the linear term vanishes: $z^\top L_{TS} x_S = 0$ for all x_S, so $z^\top L_{TS} = 0$. Thus $\ker(L_{TT}) \subseteq \ker(L_{ST})$, and taking orthogonal complements yields $\mathrm{im}(L_{TS}) \subseteq \mathrm{im}(L_{TT})$.

Therefore $L_{TT} x_T = -L_{TS} x_S$ is solvable. Any such solution is a stationary point of Q, and since $L_{TT} \succeq 0$ the function Q is convex in x_T, so any stationary point is a minimizer. The pseudoinverse choice $x_T^\star = -L_{TT}^+ L_{TS} x_S$ is a solution because $L_{TT} L_{TT}^+$ is the orthogonal projection onto $\mathrm{im}(L_{TT}) \supseteq \mathrm{im}(L_{TS})$. Substituting $x_T^\star$ into Q yields the asserted minimum value. □

6.3. **Kron reduction and ε-lightness.**

Definition 6.2 (Kron reduction)**.** The *Kron-reduced Laplacian* (Schur complement) of L onto S is

$$L_{\mathrm{Kron}}(S) \;:=\; L_{SS} - L_{ST} L_{TT}^+ L_{TS}.$$

Theorem 6.3 (ε-lightness via Kron reduction)**.** *Let $G = (V, E)$ have Laplacian L. Let $S \subseteq V$ with complement T and induced Laplacian $L_{G[S]}$. Then S is ε-light if and only if*

$$L_{G[S]} \preceq \varepsilon\, L_{\mathrm{Kron}}(S). \tag{4}$$

Equivalently, using (2),

$$(1-\varepsilon)\, L_{G[S]} \preceq \varepsilon \Big(D_{\partial S} - L_{ST} L_{TT}^+ L_{TS} \Big). \tag{5}$$

Proof. By Lemma 3.1(c), S is ε-light if and only if for all $x \in \mathbb{R}^V$,

$$x^\top L_S x \le \varepsilon\, x^\top L x.$$

Since L_S only contains edges with both endpoints in S, we have $x^\top L_S x = x_S^\top L_{G[S]} x_S$. Fix x_S and minimize the right-hand side over x_T using Lemma 6.1. We obtain

$$x_S^\top L_{G[S]} x_S \le \varepsilon \cdot \min_{x_T} \begin{pmatrix} x_S \\ x_T \end{pmatrix}^\top L \begin{pmatrix} x_S \\ x_T \end{pmatrix} = \varepsilon\, x_S^\top L_{\mathrm{Kron}}(S) x_S$$

for all $x_S \in \mathbb{R}^S$, which is exactly (4). Rearranging using $L_{\mathrm{Kron}}(S) = L_{SS} - L_{ST} L_{TT}^+ L_{TS}$ and (2) yields (5). □

Remark 6.4 (Interpretation). Equation (5) isolates the true analytic content of ε-lightness: internal Laplacian energy in $G[S]$ must be dominated by the *effective coupling of S to its complement* captured by $D_{\partial S} - L_{ST} L_{TT}^+ L_{TS}$, rather than by naive linear surrogates that overcount boundary edges.

7. Obstructions to current polynomial/interlacing approaches

A natural hope is to partition V into r parts and show that some part S_i satisfies $L_{G[S_i]} \preceq O(1/r)\, L$, then take $r \approx 1/\varepsilon$ and pick the largest part. This resembles matrix paving and suggests using interlacing-polynomial methods [2]. However, two independent obstacles arise: (i) arithmetic leakage in rounding r from ε; and (ii) the induced-subgraph Laplacian depends quadratically on vertex indicators, violating the independence required for the mixed characteristic polynomial method.

7.1. A rounding leak in dyadic reductions.

Lemma 7.1 (A ceiling-function leak). *Assume hypothetically that for every integer $r \geq 2$ and every graph H on n vertices, there exists $S \subseteq V(H)$ with*

$$|S| \geq \frac{n}{r} \qquad \text{and} \qquad L_{H[S]} \preceq \frac{2}{r} L_H.$$

Then Conjecture 1.2 would hold with the universal constant $c = \frac{1}{3}$. Moreover, this dyadic rounding mechanism alone cannot yield $c = \frac{1}{2}$.

Proof. Fix $\varepsilon \in (0,1)$ and set $r := \lceil 2/\varepsilon \rceil$. Then $2/r \leq \varepsilon$, so the assumed S is ε-light. Its size satisfies $|S| \geq n/r$. Since $r < \frac{2}{\varepsilon} + 1 = \frac{2+\varepsilon}{\varepsilon}$, we have

$$\frac{1}{r} > \frac{\varepsilon}{2+\varepsilon} \geq \frac{\varepsilon}{3}.$$

Thus $|S| \geq (\varepsilon/3)n$, i.e. $c = 1/3$. As $\varepsilon \to 1$, the bound $\varepsilon/(2+\varepsilon) \to 1/3$, showing that this mechanism cannot recover $c = 1/2$. □

7.2. Quadratic dependence breaks interlacing families.

In the MSS framework [2], one controls the spectrum of a random sum of independent rank-one PSD matrices by analyzing a mixed characteristic polynomial and exploiting real-rootedness and interlacing. For vertex-induced subgraphs, edge indicators are *quadratic* in vertex indicators and highly dependent.

We record an explicit K_3 computation showing that even the *expected* characteristic polynomial can have complex roots.

Example 7.2 (Complex roots in an expected induced-subgraph characteristic polynomial). Let $G = K_3$ and assign each vertex independently to one of $r = 2$ parts. Let S be the first part and define $M(S) := L^{+1/2} L_S L^{+1/2}$ acting on the harmonic subspace $\mathcal{H} = \ker(L)^\perp$ (which has dimension 2 here). Then

$$\mathbb{E}\big[\det(xI_{\mathcal{H}} - M(S))\big] = x^2 - \frac{1}{2}x + \frac{1}{8},$$

whose roots are $\frac{1}{4} \pm i\frac{1}{4}$ and hence are nonreal.

Proof. For K_3, the nonzero Laplacian eigenvalues equal 3, so on $\mathcal{H}$ one has $L^{+1/2} = \frac{1}{\sqrt{3}} I_{\mathcal{H}}$.

There are three cases for the random set S:

- If $|S| \leq 1$, then $L_S = 0$ and hence $M(S) = 0$ on $\mathcal{H}$, so $\det(xI_{\mathcal{H}} - M(S)) = x^2$.
- If $|S| = 2$, then $G[S]$ is a single edge. The unpadded edge Laplacian has nonzero eigenvalue 2 on its (one-dimensional) orthogonal complement of constants; after padding and restricting to $\mathcal{H}$, the resulting operator has eigenvalues $\{2/3, 0\}$ on $\mathcal{H}$. Thus $\det(xI_{\mathcal{H}} - M(S)) = x(x-2/3) = x^2 - \frac{2}{3}x$.
- If $|S| = 3$, then $L_S = L$ and $M(S) = L^{+1/2} L L^{+1/2} = \Pi_{\mathcal{H}}$, which is $I_{\mathcal{H}}$ on $\mathcal{H}$. Thus $\det(xI_{\mathcal{H}} - M(S)) = (x-1)^2 = x^2 - 2x + 1$.

Under independent 2-coloring, $\mathbb{P}(|S| = 3) = 1/8$, $\mathbb{P}(|S| = 2) = 3/8$, and $\mathbb{P}(|S| \leq 1) = 4/8$. Therefore

$$\mathbb{E}[\det(xI_{\mathcal{H}} - M(S))] = \frac{1}{8}(x^2 - 2x + 1) + \frac{3}{8}\Big(x^2 - \frac{2}{3}x\Big) + \frac{4}{8}x^2 = x^2 - \frac{1}{2}x + \frac{1}{8}.$$

Its discriminant is $(-1/2)^2 - 4 \cdot (1/8) = 1/4 - 1/2 = -1/4 < 0$, so the roots are nonreal. □

Remark 7.3 (Consequences for interlacing). If a family of real-rooted polynomials has a common interlacing, then every convex combination is real-rooted. Therefore Example 7.2 certifies that the characteristic polynomials arising from vertex-induced parts in this simplest instance do not admit the interlacing structure required by the MSS method. This does not preclude other polynomial techniques, but it rules out a direct import of [2].

8. Discussion and open directions

The Schur complement characterization in Theorem 6.3 suggests that Conjecture 1.2 is fundamentally a statement about finding a large terminal set S for which the internal Laplacian is dominated by the Kron-reduced Laplacian of the full graph onto S. Any successful proof must simultaneously:

- control the quadratic dependence of internal edges on vertex selection, and
- exploit the effective boundary operator $D_{\partial S} - L_{ST} L_{TT}^{+} L_{TS}$ rather than a lossy linear surrogate.

We hope the framework and counterexamples here serve as a clean starting point.

References

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