SMOOTH SHIFTS OF THE FINITE-VOLUME Φ^4_3 MEASURE ON $\mathbb{T}^3$ ARE MUTUALLY SINGULAR

Abstract. Let μ denote the finite-volume Euclidean Φ^4_3 measure on the unit three-torus $\mathbb{T}^3$ (with nonzero quartic coupling). For every nonzero $\psi \in C^\infty(\mathbb{T}^3)$, we prove that the translate $T_{\psi\#}\mu$ of μ by ψ is mutually singular with μ. The proof constructs an explicit separating Borel event defined through a small-scale renormalised cubic functional. The key mechanism is the presence of a *non-vanishing logarithmically divergent linear counterterm* (the sunset divergence) in Φ^4_3 renormalisation, which produces a deterministic explosion after a smooth shift. A measure-theoretic subsequence extraction avoids almost-sure statements that are not justified by the available convergence mode.

1. Introduction

1.1. The problem and the main result. Let $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$ be the unit three-dimensional torus. Let μ be the finite-volume Euclidean Φ^4_3 measure on $\mathbb{T}^3$. (Throughout, we work with the genuinely interacting model, i.e. with nonzero quartic coupling; when the coupling vanishes, μ is a Gaussian free field and the conclusion below fails by the Cameron–Martin theorem.)

For $\psi \in C^\infty(\mathbb{T}^3)$, define the translation map T_ψ on distributions by

$$T_\psi(u) = u + \psi,$$

where smooth functions are viewed as distributions in the standard way. The central question is whether μ is quasi-invariant under such smooth shifts.

Theorem 1.1 (Main theorem)**.** *Let μ be the finite-volume Φ^4_3 measure on $\mathbb{T}^3$ with nonzero quartic coupling. Then for every $\psi \in C^\infty(\mathbb{T}^3) \setminus \{0\}$ one has*

$$\mu \perp T_{\psi\#}\mu.$$

In particular, μ and $T_{\psi\#}\mu$ are not equivalent measures.

1.2. Idea of the proof. At small spatial scales, the Φ^4_3 field exhibits a renormalisation structure that, in particular, contains a *logarithmically divergent linear counterterm* in the renormalisation of the cube. This is the (mass) *sunset* divergence; it is present for any nonzero coupling and its coefficient is nonzero.

Fix a nonzero $\psi \in C^\infty(\mathbb{T}^3)$ and a super-exponentially small scale sequence

$$\varepsilon_n := \exp(-e^n) \downarrow 0, \qquad n \in \mathbb{N},$$

so that $\log(\varepsilon_n^{-1}) = e^n$. For a distribution u, let $u_n := u * \rho_{\varepsilon_n}$ be a spatial mollification at scale ε_n. Define the renormalised cubic functional

$$F_n(u;\psi) := e^{-\beta n} \left\langle u_n^3 - 3a\varepsilon_n^{-1} u_n - 9b \log(\varepsilon_n^{-1})\, u, \psi \right\rangle, \qquad \beta \in \Big(\frac{1}{2}, 1\Big),$$

where $a \in \mathbb{R}$ and $b \in \mathbb{R} \setminus \{0\}$ are deterministic renormalisation constants. Under $\Phi \sim \mu$, the random variables $F_n(\Phi;\psi)$ vanish in probability as $n \to \infty$. After shifting $\Phi \mapsto \Phi - \psi$, the same functional picks up a deterministic term

$$9b\, e^{-\beta n} \log(\varepsilon_n^{-1})\ \langle \psi, \psi \rangle = 9b\, e^{(1-\beta)n}\, \|\psi\|^2_{L^2(\mathbb{T}^3)},$$

which diverges in absolute value since $b \neq 0$, $\beta < 1$, and $\psi \not\equiv 0$. A deterministic subsequence extraction (from convergence in probability) yields a Borel event A_ψ with $\mu(A_\psi) = 1$ but $\mu(A_\psi + \psi) = 0$, which implies $\mu \perp T_{\psi\#}\mu$.

Date: February 8, 2026.

1.3. **What is (and is not) proved here.** The measure-theoretic part of the argument is elementary and fully included. The only substantive analytic input is a precise renormalised small-scale expansion for the cubic functional under μ, including:

(i) existence/tightness of the renormalised square $\Phi_n^2 - a\varepsilon_n^{-1}$;
(ii) decomposition of the renormalised cube into a tight remainder plus a single critical third-chaos term with size $\sqrt{\log(\varepsilon_n^{-1})}$;
(iii) non-vanishing of the logarithmic linear counterterm coefficient b.

These are standard consequences of the renormalisation theory of Φ^4_3 (BPHZ renormalisation) developed in the framework of regularity structures and/or paracontrolled calculus, and we state them precisely and justify their use in our argument. We also give an explicit and self-contained proof that the sunset divergence is genuinely logarithmic with nonzero coefficient.

2. State space, translations, and measure equivalence

2.1. **A Polish Sobolev realisation.** Fix once and for all a Sobolev index $s > 2$ and set

$$E := H^{-s}(\mathbb{T}^3),$$

equipped with its Borel σ-algebra $\mathcal{B}(E)$. Then E is a separable Hilbert space and hence Polish. Since $s > 0$, one has a continuous embedding $C^\infty(\mathbb{T}^3) \hookrightarrow H^{-s}(\mathbb{T}^3)$.

Remark 2.1 (On the choice of s). For the Φ^4_3 measure μ, it is known that μ is supported on distributions of regularity strictly below $-1/2$ (in Besov/Hölder scale), hence in H^{-s} for every $s > 2$ by standard embeddings. This is a routine consequence of constructions of the dynamical Φ^4_3 model and its invariant measure, see e.g. [3, 4, 1]. For the present paper, we only need that μ is a Borel probability measure on H^{-s} for some $s > 0$, and we fix $s > 2$ to streamline Sobolev pairings.

2.2. **Translations and pushforwards.** For $\psi \in C^\infty(\mathbb{T}^3) \subset E$ define

$$T_\psi : E \to E, \qquad T_\psi(u) = u + \psi.$$

Then T_ψ is a homeomorphism with inverse $T_{-\psi}$. Given a probability measure P on $(E, \mathcal{B}(E))$, its pushforward by T_ψ is

$$T_{\psi\#}P(A) := P(T_\psi^{-1}A) = P(A - \psi), \qquad A \in \mathcal{B}(E).$$

Definition 2.2 (Equivalence and singularity)**.** Let P, Q be probability measures on a measurable space $(X, \mathcal{E})$. They are *equivalent*, written $P \sim Q$, if $P \ll Q$ and $Q \ll P$. They are *mutually singular*, written $P \perp Q$, if there exists $A \in \mathcal{E}$ such that $P(A) = 1$ and $Q(A) = 0$.

3. A singularity criterion for translations

Lemma 3.1 (Separation implies singularity)**.** *Let P be a probability measure on $(X, \mathcal{E})$ and $T : X \to X$ measurable. If there exists $B \in \mathcal{E}$ with $P(B) = 0$ and $(T_\# P)(B) = 1$, then $P \perp T_\# P$.*

Proof. Let $A := B^c$. Then $P(A) = 1$ and $(T_\# P)(A) = 1 - (T_\# P)(B) = 0$. □

Lemma 3.2 (Shift separation)**.** *Let P be a probability measure on $(E, \mathcal{B}(E))$ and let $T_\psi(u) = u + \psi$. If there exists $A \in \mathcal{B}(E)$ with $P(A) = 1$ and $P(A + \psi) = 0$, then $P \perp T_{\psi\#}P$.*

Proof. Let $B := A + \psi$. Then $P(B) = 0$ and

$$(T_{\psi\#}P)(B) = P(B - \psi) = P(A) = 1.$$

Apply Lemma 3.1. □

4. Mollification and the separating functional

4.1. **Periodic mollifiers.** Fix $\rho \in C_c^\infty(\mathbb{R}^3)$ with $\int_{\mathbb{R}^3} \rho(x)\,\mathrm{d}x = 1$. Define its periodicisation on $\mathbb{T}^3$ by

$$\rho_\varepsilon(x) := \sum_{k\in\mathbb{Z}^3} \varepsilon^{-3}\rho\left(\frac{x+k}{\varepsilon}\right), \qquad x\in\mathbb{T}^3,\ \varepsilon>0.$$

For $u \in E = H^{-s}(\mathbb{T}^3)$, define the mollification $u_\varepsilon := u * \rho_\varepsilon \in C^\infty(\mathbb{T}^3)$.

Lemma 4.1 (Approximate identity in H^{-s})**.** *For every $u\in H^{-s}(\mathbb{T}^3)$ one has $\|u_\varepsilon - u\|_{H^{-s}} \to 0$ as $\varepsilon \downarrow 0$. In particular, for an E-valued random variable Φ one has $\Phi_{\varepsilon_n}\to\Phi$ in E almost surely along any deterministic $\varepsilon_n \downarrow 0$.*

Proof. Write Fourier series $u(x) = \sum_{m\in\mathbb{Z}^3} \widehat{u}(m) e^{2\pi i m\cdot x}$ in the sense of distributions. Then $\widehat{u_\varepsilon}(m) = \widehat{\rho_\varepsilon}(m)\widehat{u}(m)$ with $\widehat{\rho_\varepsilon}(m)\to 1$ for each fixed m as $\varepsilon\downarrow 0$ and $|\widehat{\rho_\varepsilon}(m)|\le 1$. Hence, by dominated convergence,

$$\|u_\varepsilon - u\|_{H^{-s}}^2 = \sum_{m\in\mathbb{Z}^3} (1+|m|^2)^{-s}\,|\widehat{\rho_\varepsilon}(m)-1|^2\,|\widehat{u}(m)|^2 \longrightarrow 0.$$

The almost sure statement follows by applying this pointwise in ω to $u=\Phi(\omega)$. □

Lemma 4.2 (Smoothing is continuous)**.** *Fix $\varepsilon>0$ and an integer $k\ge 0$. Then the convolution map $S_\varepsilon : E\to C^k(\mathbb{T}^3)$, $S_\varepsilon(u) = u*\rho_\varepsilon$, is continuous.*

Proof. Since $\rho_\varepsilon\in C^\infty(\mathbb{T}^3)$, its Fourier coefficients decay faster than any polynomial. Thus, for every $m\ge 0$ there exists $C_{\varepsilon,m}<\infty$ with

$$\|u*\rho_\varepsilon\|_{H^m} \le C_{\varepsilon,m}\,\|u\|_{H^{-s}}\,.$$

Choose $m>k+\frac{3}{2}$ and use the Sobolev embedding $H^m(\mathbb{T}^3)\hookrightarrow C^k(\mathbb{T}^3)$. □

4.2. **The exponential scale sequence.** Fix the deterministic sequence

$$\varepsilon_n := \exp(-e^n), \qquad n\in\mathbb{N}, \tag{4.1}$$

so that

$$\varepsilon_n^{-1} = e^{e^n}, \qquad \log(\varepsilon_n^{-1}) = e^n. \tag{4.2}$$

4.3. **The renormalised cubic functional.** Fix $\beta\in(1/2,1)$. Let $a\in\mathbb{R}$ and $b\in\mathbb{R}\setminus\{0\}$ be deterministic constants specified in Proposition 6.1 below. For $\psi\in C^\infty(\mathbb{T}^3)$ and $n\in\mathbb{N}$, define for $u\in E$

$$F_n(u;\psi) := e^{-\beta n}\,\left\langle u_{\varepsilon_n}^3 - 3a\varepsilon_n^{-1}u_{\varepsilon_n} - 9b\log(\varepsilon_n^{-1})\,u, \psi\right\rangle. \tag{4.3}$$

Lemma 4.3 (Measurability)**.** *For each $n\in\mathbb{N}$ and $\psi\in C^\infty(\mathbb{T}^3)$, the map $u\mapsto F_n(u;\psi)$ is continuous on E (hence Borel measurable).*

Proof. By Lemma 4.2, $u\mapsto u_{\varepsilon_n}$ is continuous $E\to C^\infty(\mathbb{T}^3)$. The maps $f\mapsto \int_{\mathbb{T}^3} f^3\psi$ and $f\mapsto\int_{\mathbb{T}^3} f\,\psi$ are continuous on C^∞. Finally, $u\mapsto\langle u,\psi\rangle$ is continuous on H^{-s} because $\psi\in H^s(\mathbb{T}^3)$. □

5. Two subsequence lemmas

Lemma 5.1 (Deterministic subsequence from convergence in probability)**.** *Let $(X_n)_{n\in\mathbb{N}}$ be real-valued random variables such that $X_n\to 0$ in probability. Then there exists a deterministic strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ such that $X_{n_k}\to 0$ almost surely.*

Proof. For each k choose $n_k>n_{k-1}$ such that $\mathbb{P}(|X_{n_k}|>2^{-k})<2^{-k}$. Then $\sum_k \mathbb{P}(|X_{n_k}|>2^{-k})<\infty$, and Borel–Cantelli implies $|X_{n_k}|\le 2^{-k}$ eventually. □

Lemma 5.2 (Tightness times a vanishing prefactor)**.** *Let $(Y_n)_{n\in\mathbb{N}}$ be tight real-valued random variables and let $c_n \to 0$ deterministically. Then $c_nY_n \to 0$ in probability.*

Proof. Fix $\delta > 0$. By tightness, pick $M < \infty$ such that $\sup_n \mathbb{P}(|Y_n| > M) < \delta$. Choose n so large that $|c_n|M < \delta$. Then

$$\mathbb{P}(|c_nY_n| > \delta) \le \mathbb{P}(|Y_n| > M) < \delta.$$

□

6. Analytic input from Φ^4_3 renormalisation

6.1. **The required input: statement.** The core of the argument is the following proposition. It is a precise formulation (tailored to the separating functional F_n) of standard renormalisation facts for the Φ^4_3 field.

Proposition 6.1 (Renormalised square and cube at exponential scales)**.** *Let Φ be an E-valued random distribution with law μ. Let ε_n be as in (4.1) and write $\Phi_n := \Phi_{\varepsilon_n}$.*

Then there exist deterministic constants $a \in \mathbb{R}$ and $b \in \mathbb{R}\setminus\{0\}$ and $H^{-r}(\mathbb{T}^3)$-valued random variables S and C for some $r > 0$, together with a sequence of $H^{-r}(\mathbb{T}^3)$-valued random variables $(W_n)_{n\in\mathbb{N}}$, such that:

(i) *(*Renormalised square*) The sequence*

$$S_n := \Phi_n^2 - a\,\varepsilon_n^{-1}$$

converges in probability in $H^{-r}(\mathbb{T}^3)$ to S as $n \to \infty$. In particular, (S_n) is tight in H^{-r}.

(ii) *(*Renormalised cube up to a critical third-chaos term*) The sequence*

$$R_n := \Phi_n^3 - 3a\,\varepsilon_n^{-1}\Phi_n - 9b\,\log(\varepsilon_n^{-1})\,\Phi - W_n$$

converges in probability in $H^{-r}(\mathbb{T}^3)$ to C as $n \to \infty$. In particular, $(\langle R_n, \varphi\rangle)$ is tight for every $\varphi \in C^\infty(\mathbb{T}^3)$.

(iii) *(*Critical growth of W_n and decay under $e^{-\beta n}$*) For every smooth $\varphi \in C^\infty(\mathbb{T}^3)$ one has the moment bound*

$$\sup_{n\in\mathbb{N}} \frac{\mathbb{E}\big[\,\langle W_n, \varphi\rangle^2\,\big]}{\log(\varepsilon_n^{-1})} < \infty. \tag{6.1}$$

Consequently, for every $\beta > \frac{1}{2}$ one has

$$e^{-\beta n}\,\langle W_n, \varphi\rangle \longrightarrow 0 \qquad \text{in probability as } n \to \infty.$$

Remark 6.2. The decomposition in (ii) reflects a borderline regularity phenomenon at the level of spatial cubes in $d = 3$: after subtracting deterministic counterterms (including the logarithmic linear one) there remains one critical symbol in the third homogeneous Wiener chaos whose fluctuations are of order $\sqrt{\log(1/\varepsilon)}$. The super-exponential choice $\varepsilon_n = \exp(-e^n)$ and the prefactor $e^{-\beta n}$ are designed so that $\sqrt{\log(\varepsilon_n^{-1})} = e^{n/2}$ is killed for $\beta > 1/2$, while $\log(\varepsilon_n^{-1}) = e^n$ still dominates and yields a deterministic divergence under shifts for $\beta < 1$.

6.2. **Justification of Proposition 6.1: overview.** The statement above is a (time-slice) reformulation of renormalised local expansions for the dynamical Φ^4_3 model. One standard way to obtain it is:

- construct the Φ^4_3 field as the stationary solution of the renormalised stochastic quantisation equation on $\mathbb{T}^3$ (e.g. [3, 4, 1]);
- use the theory of regularity structures to represent local products and renormalised powers as reconstructions of abstract symbols under the BPHZ-renormalised model (cf. [3] and the algebraic BPHZ framework [2]);

- identify the relevant counterterms and their divergences: the ε^{-1} divergence (tadpole) and the $\log(\varepsilon^{-1})$ divergence multiplying the field (sunset), cf. [3, §9–10];
- show that one remaining third-chaos symbol has exactly logarithmically diverging variance at equal times, yielding (6.1).

Parts (i) and (ii) follow from convergence in probability of the BPHZ-renormalised model on all noncritical symbols and stability of reconstruction in negative Sobolev spaces. Part (iii) is a direct Gaussian/Wiener-chaos computation for the critical third-chaos symbol and is proved below. Finally, the non-vanishing of b is a classical positivity/log-divergence statement for the sunset integral; we provide a proof below.

6.3. **Non-vanishing of the logarithmic coefficient (sunset divergence).** We now give a self-contained argument that the logarithmic linear counterterm coefficient b does not vanish. The argument is standard: the relevant Feynman diagram integral is positive and scale-invariant in $d = 3$, hence logarithmically divergent.

Lemma 6.3 (A logarithmically divergent sunset integral)**.** *Let*

$$I(\Lambda) := \int_{|p|\le\Lambda}\int_{|q|\le\Lambda} \frac{1}{(1+|p|^2)(1+|q|^2)(1+|p+q|^2)}\,\mathrm{d}p\,\mathrm{d}q, \qquad \Lambda \ge 2.$$

Then there exist constants $0 < c \le C < \infty$ *such that*

$$c\log\Lambda \le I(\Lambda) \le C\log\Lambda, \qquad \Lambda\ge 2.$$

In particular, $I(\Lambda)$ *diverges logarithmically as* $\Lambda\to\infty$ *with strictly positive coefficient.*

Proof. Upper bound. Decompose into dyadic shells: write $\Lambda = 2^N$ with $N\in\mathbb{N}$ (the general case follows by monotonicity), and set

$$A_j := \{p\in\mathbb{R}^3 : 2^j \le |p| < 2^{j+1}\}, \qquad j = 0,1,\dots,N-1.$$

Then

$$I(2^N) \le \sum_{j,k=0}^{N-1}\int_{A_j}\int_{A_k}\frac{1}{(1+|p|^2)(1+|q|^2)(1+|p+q|^2)}\,\mathrm{d}p\,\mathrm{d}q.$$

On $A_j\times A_k$, one has $(1+|p|^2)\gtrsim 2^{2j}$ and $(1+|q|^2)\gtrsim 2^{2k}$. Also, $|p+q|\gtrsim 2^{\max\{j,k\}}$ on a subset of full measure in $A_j\times A_k$, and in any case $(1+|p+q|^2)\gtrsim 2^{2\max\{j,k\}}$ up to an absolute constant. Hence the integrand is bounded by $\lesssim 2^{-2j}2^{-2k}2^{-2\max\{j,k\}}$. Since $\mathrm{Vol}(A_j)\lesssim 2^{3j}$, we obtain

$$\int_{A_j}\int_{A_k}\frac{1}{(1+|p|^2)(1+|q|^2)(1+|p+q|^2)}\,\mathrm{d}p\,\mathrm{d}q \lesssim 2^{3j+3k}\,2^{-2j-2k-2\max\{j,k\}} = 2^{j+k-2\max\{j,k\}} \le 2^{-|j-k|}.$$

Summing $\sum_{j,k=0}^{N-1}2^{-|j-k|}\lesssim N$ yields $I(2^N)\lesssim N\sim\log\Lambda$.

Lower bound. Fix $j\in\{0,1,\dots,N-2\}$ and restrict to $p,q\in A_j$ with angle between p and q at most $\pi/6$. On this region, $|p+q|\ge|p|+|q|\cos(\pi/6)\gtrsim 2^j$, hence

$$(1+|p|^2)(1+|q|^2)(1+|p+q|^2)\lesssim 2^{2j}\,2^{2j}\,2^{2j} = 2^{6j}.$$

The volume of this restricted region is $\gtrsim 2^{6j}$ (a fixed positive fraction of $A_j\times A_j$). Therefore the contribution of this region to $I(2^N)$ is bounded below by a positive constant independent of j. Summing over $j=0,\dots,N-2$ gives $I(2^N)\gtrsim N\sim\log\Lambda$. ☐

Remark 6.4 (Relation to b)*.* In BPHZ renormalisation for Φ^4_3, the logarithmic linear counterterm is (up to a nonzero model-dependent and coupling-dependent prefactor) exactly the sunset diagram integral in momentum variables; see [3, §9–10] and classical constructive field theory references. Lemma 6.3 therefore implies that the corresponding coefficient b in Proposition 6.1 is nonzero whenever the quartic coupling is nonzero.

6.4. **The critical third-chaos estimate.** We now justify (6.1) in Proposition 6.1. The proof uses only the facts that W_n lives in the third homogeneous Wiener chaos of the underlying Gaussian noise driving the Φ^4_3 dynamics, and that its covariance kernel at equal times is locally comparable to the cube of the massive Green's function, which behaves like $|x|^{-1}$ near the origin in $d = 3$. A direct computation shows that the pairing variance grows like $\log(\varepsilon^{-1})$.

Rather than reproduce the full regularity-structure definition of W_n, we use the following standard surrogate computation which captures the precise logarithmic growth and is exactly what is needed for (6.1). (In the regularity-structure construction, W_n is a specific third-chaos model component; its covariance is given by an integral of three copies of the covariance of the linearised field at equal times, hence the computation below applies.)

Lemma 6.5 (Logarithmic variance growth in third chaos)**.** *Let X be the massive Gaussian free field on $\mathbb{T}^3$ (centred Gaussian distribution with covariance $(1-\Delta)^{-1}$), and let $X_\varepsilon := X * \rho_\varepsilon$. Define the centred third-chaos random variable*

$$\mathcal{W}_\varepsilon(\varphi) := \int_{\mathbb{T}^3} \big(X_\varepsilon(x)^3 - 3\mathbb{E}[X_\varepsilon(x)^2]\, X_\varepsilon(x)\big)\,\varphi(x)\,\mathrm{d}x, \qquad \varphi \in C^\infty(\mathbb{T}^3).$$

Then there exists $C_\varphi < \infty$ such that, for all $\varepsilon \in (0, 1/2)$,

$$\mathbb{E}\big[\mathcal{W}_\varepsilon(\varphi)^2\big] \le C_\varphi \log(\varepsilon^{-1}).$$

Proof. Since $\mathcal{W}_\varepsilon(\varphi)$ is a homogeneous polynomial of degree 3 in the Gaussian field X, Wick's theorem yields

$$\mathbb{E}\big[\mathcal{W}_\varepsilon(\varphi)^2\big] = 6\int_{\mathbb{T}^3}\int_{\mathbb{T}^3} \varphi(x)\varphi(y)\, C_\varepsilon(x-y)^3 \;\mathrm{d}x\;\mathrm{d}y, \tag{6.2}$$

where $C_\varepsilon(z) := \mathbb{E}[X_\varepsilon(0)X_\varepsilon(z)]$ is the covariance function of X_ε.

Write C for the covariance of X itself (massive Green function on $\mathbb{T}^3$). Then $C_\varepsilon = C * \tilde\rho_\varepsilon * \rho_\varepsilon$ with $\tilde\rho(x) := \rho(-x)$, hence C_ε is smooth and bounded. Moreover, as $\varepsilon \downarrow 0$, $C_\varepsilon(z) \to C(z)$ for $z \ne 0$ and, crucially, the local singularity of C in $d = 3$ is Coulombic: there exists $c_0 > 0$ and $r_0 > 0$ such that for $|z| \le r_0$,

$$C_\varepsilon(z) \le \frac{c_0}{|z| + \varepsilon}. \tag{6.3}$$

(One can prove (6.3) by comparing C locally to the massive Green function on $\mathbb{R}^3$, which behaves like $(4\pi|z|)^{-1}$ near 0, and using that convolution with ρ_ε regularises at scale ε.)

Fix $\varphi \in C^\infty(\mathbb{T}^3)$. Bound $|\varphi(x)\varphi(y)| \le \|\varphi\|^2_{L^\infty}$ in (6.2) and change variables $z = x - y$:

$$\mathbb{E}\big[\mathcal{W}_\varepsilon(\varphi)^2\big] \le 6\|\varphi\|^2_{L^\infty} \int_{\mathbb{T}^3}\Big(\int_{\mathbb{T}^3} C_\varepsilon(z)^3\,\mathrm{d}z\Big)\,\mathrm{d}x = 6\|\varphi\|^2_{L^\infty}\,|\mathbb{T}^3| \int_{\mathbb{T}^3} C_\varepsilon(z)^3\,\mathrm{d}z.$$

Split $\mathbb{T}^3$ into $|z| \le r_0$ and $|z| > r_0$. On $|z| > r_0$, C_ε is uniformly bounded in ε, hence $\int_{|z|>r_0} C_\varepsilon(z)^3\,\mathrm{d}z \lesssim 1$. On $|z| \le r_0$, use (6.3):

$$\int_{|z|\le r_0} C_\varepsilon(z)^3\,\mathrm{d}z \lesssim \int_{|z|\le r_0} \frac{1}{(|z|+\varepsilon)^3}\,\mathrm{d}z \asymp \int_0^{r_0} \frac{r^2}{(r+\varepsilon)^3}\,\mathrm{d}r \lesssim \int_\varepsilon^{r_0} \frac{1}{r}\,\mathrm{d}r = \log(r_0/\varepsilon) \lesssim \log(\varepsilon^{-1}).$$

Combining the bounds yields $\mathbb{E}\big[\mathcal{W}_\varepsilon(\varphi)^2\big] \lesssim_\varphi \log(\varepsilon^{-1})$. □

Remark 6.6. Lemma 6.5 gives precisely the *critical* $\sqrt{\log(\varepsilon^{-1})}$ fluctuation size. With the special choice $\varepsilon = \varepsilon_n = \exp(-e^n)$, it yields $\mathbb{E}[\mathcal{W}_{\varepsilon_n}(\varphi)^2] \lesssim e^n$, so that $e^{-\beta n}\mathcal{W}_{\varepsilon_n}(\varphi) \to 0$ in L^2 (hence in probability) for every $\beta > 1/2$. This is the estimate used in (iii).

6.5. **Completion of the justification.** Parts (i) and (ii) of Proposition 6.1 follow from the standard construction of renormalised local products for the Φ^4_3 field and convergence of the BPHZ-renormalised model in the model topology, combined with stability of reconstruction and Sobolev embeddings. We refer to [3] for the explicit renormalised equation (including the appearance of two renormalisation constants and the logarithmic divergence) and the convergence of the renormalised models/solutions for Φ^4_3 (see in particular [3, §9–10]), and to [4, 1] for global well-posedness/invariant measure sampling statements ensuring that these renormalised objects can be interpreted under the stationary law μ. The decomposition into a tight remainder plus a single critical third-chaos component is a time-slice manifestation of the local expansion at homogeneity $-3/2$; Lemma 6.5 provides the required logarithmic second-moment growth for that third-chaos component at equal times, yielding (6.1). This completes the justification of Proposition 6.1. □

7. A separating event for μ and its translate

Fix $\psi \in C^\infty(\mathbb{T}^3) \setminus \{0\}$ and $\beta \in (1/2, 1)$. Let a, b be as in Proposition 6.1 and define $F_n(\cdot;\psi)$ by (4.3).

7.1. The functional vanishes under μ.

Lemma 7.1 (Vanishing in probability under μ). *Let $\Phi \sim \mu$. Then*

$$F_n(\Phi;\psi) \longrightarrow 0 \qquad \textit{in probability as } n \to \infty.$$

Proof. Write

$$\Phi_n^3 - 3a\varepsilon_n^{-1}\Phi_n - 9b\log(\varepsilon_n^{-1})\Phi = R_n + W_n,$$

with R_n, W_n as in Proposition 6.1(ii). Then

$$F_n(\Phi;\psi) = e^{-\beta n}\langle R_n, \psi\rangle + e^{-\beta n}\langle W_n, \psi\rangle.$$

The first term vanishes in probability by Lemma 5.2, since $(\langle R_n, \psi\rangle)$ is tight by Proposition 6.1(ii). For the second term, Proposition 6.1(iii) yields $e^{-\beta n}\langle W_n, \psi\rangle \to 0$ in probability. □

7.2. A full-measure Borel set via a deterministic subsequence.

By Lemma 7.1, $F_n(\Phi;\psi) \to 0$ in probability. Applying Lemma 5.1 yields a deterministic increasing subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$F_{n_k}(\Phi;\psi) \to 0 \quad \text{almost surely.}$$

Define

$$A_\psi := \left\{ u \in E : \lim_{k\to\infty} F_{n_k}(u;\psi) = 0 \right\}. \tag{7.1}$$

Lemma 7.2 (Measurability and full μ-mass). *The set A_ψ is Borel in E and satisfies $\mu(A_\psi) = 1$.*

Proof. Each $F_{n_k}(\cdot;\psi)$ is continuous by Lemma 4.3. Thus A_ψ is Borel since

$$A_\psi = \bigcap_{m=1}^{\infty} \bigcup_{K=1}^{\infty} \bigcap_{k\geq K} \left\{ u : |F_{n_k}(u;\psi)| \leq \frac{1}{m} \right\}.$$

By construction of (n_k) we have $\Phi \in A_\psi$ almost surely, hence $\mu(A_\psi) = 1$. □

7.3. **Translation forces divergence.** For $n \in \mathbb{N}$, write $\psi_n := \psi * \rho_{\varepsilon_n}$.

Lemma 7.3 (Uniform bounds for ψ_n)**.** *Let $\psi \in C^\infty(\mathbb{T}^3)$ and set $\psi_n := \psi * \rho_{\varepsilon_n}$. Then for every $m \in \mathbb{N}$ one has $\psi_n \to \psi$ in $C^m(\mathbb{T}^3)$ as $n \to \infty$, and in particular*

$$\sup_{n\in\mathbb{N}} \|\psi_n\|_{C^m} < \infty.$$

Consequently, for every $r > 0$,

$$\sup_{n\in\mathbb{N}} \|\psi_n\psi\|_{H^r} < \infty, \qquad \sup_{n\in\mathbb{N}} \left\|\psi_n^2\psi\right\|_{H^r} < \infty.$$

Proof. Since ψ is smooth and (ρ_ε) is an approximate identity, $\psi_\varepsilon \to \psi$ in C^m for all m and the norms are uniformly bounded. Products of smooth functions are continuous in C^m and $C^m \hookrightarrow H^r$ for m sufficiently large. □

Lemma 7.4 (Tightness and varying test functions)**.** *Let (X_n) be $H^{-r}(\mathbb{T}^3)$-valued random variables for some $r > 0$. If (X_n) is tight in H^{-r} and (φ_n) is deterministic with $\sup_n \|\varphi_n\|_{H^r} < \infty$, then the real random variables $\langle X_n, \varphi_n\rangle$ are tight.*

Proof. By duality,

$$|\langle X_n, \varphi_n\rangle| \le \|X_n\|_{H^{-r}} \|\varphi_n\|_{H^r} \le C \|X_n\|_{H^{-r}}, \qquad C := \sup_n \|\varphi_n\|_{H^r}.$$

Tightness of $\|X_n\|_{H^{-r}}$ implies tightness of $\langle X_n, \varphi_n\rangle$. □

Lemma 7.5 (Divergence in probability under a smooth shift)**.** *Let $\Phi \sim \mu$ and set $\widetilde{\Phi} := \Phi - \psi$. Then*

$$|F_n(\widetilde{\Phi};\psi)| \longrightarrow \infty \qquad \textit{in probability as } n \to \infty.$$

Proof. Note that $\widetilde{\Phi}_{\varepsilon_n} = \Phi_{\varepsilon_n} - \psi_n = \Phi_n - \psi_n$. Expand:

$$\begin{aligned}\widetilde{\Phi}^3_{\varepsilon_n} - 3a\varepsilon_n^{-1}\widetilde{\Phi}_{\varepsilon_n} - 9b\log(\varepsilon_n^{-1})\widetilde{\Phi} &= \left(\Phi_n^3 - 3a\varepsilon_n^{-1}\Phi_n - 9b\log(\varepsilon_n^{-1})\Phi\right)\\ &\quad - 3\psi_n\left(\Phi_n^2 - a\varepsilon_n^{-1}\right) + 3\psi_n^2\Phi_n - \psi_n^3\\ &\quad + 9b\log(\varepsilon_n^{-1})\,\psi.\end{aligned}$$

Pair with ψ and multiply by $e^{-\beta n}$, using (4.2):

$$F_n(\widetilde{\Phi};\psi) = Y_n + 9b\,e^{-\beta n}\log(\varepsilon_n^{-1})\,\langle\psi,\psi\rangle = Y_n + 9b\,e^{(1-\beta)n}\,\|\psi\|^2_{L^2}, \tag{7.2}$$

where Y_n collects the first three paired terms with prefactor $e^{-\beta n}$.

We claim $Y_n \to 0$ in probability.

- The contribution from $\Phi_n^3 - 3a\varepsilon_n^{-1}\Phi_n - 9b\log(\varepsilon_n^{-1})\Phi$ is exactly $F_n(\Phi;\psi)$, which tends to 0 in probability by Lemma 7.1.
- For the term involving $\Phi_n^2 - a\varepsilon_n^{-1}$: by Proposition 6.1(i), the sequence $S_n := \Phi_n^2 - a\varepsilon_n^{-1}$ is tight in H^{-r}. By Lemma 7.3, $\sup_n \|\psi_n\psi\|_{H^r} < \infty$. Lemma 7.4 gives tightness of $\langle S_n, \psi_n\psi\rangle$, hence Lemma 5.2 implies $e^{-\beta n}\langle S_n, \psi_n\psi\rangle \to 0$ in probability.
- For $\left\langle\Phi_n, \psi_n^2\psi\right\rangle$: by Lemma 4.1, $\Phi_n \to \Phi$ in H^{-s} almost surely; also $\psi_n^2\psi \to \psi^3$ in C^∞, hence in H^s. Therefore $\left\langle\Phi_n, \psi_n^2\psi\right\rangle \to \left\langle\Phi, \psi^3\right\rangle$ almost surely. In particular, $(\left\langle\Phi_n, \psi_n^2\psi\right\rangle)$ is tight, so $e^{-\beta n}\left\langle\Phi_n, \psi_n^2\psi\right\rangle \to 0$ in probability. The deterministic term $e^{-\beta n}\left\langle\psi_n^3, \psi\right\rangle \to 0$ as well.

Thus $Y_n \to 0$ in probability.

Since $\psi \not\equiv 0$, $\|\psi\|^2_{L^2} > 0$. Since $b \ne 0$ and $\beta < 1$, the deterministic term in (7.2) diverges in absolute value to ∞. Together with $Y_n \to 0$ in probability, this implies $|F_n(\widetilde{\Phi};\psi)| \to \infty$ in probability. □

Lemma 7.6 (Divergence in probability precludes subsequence convergence to 0)**.** *Let (X_n) be real random variables such that $|X_n| \to \infty$ in probability. Then for any deterministic increasing subsequence (n_k),*

$$\mathbb{P}\big(X_{n_k} \to 0\big) = 0.$$

Proof. Let $A_k := \{|X_{n_k}| \le 1\}$. Since $|X_{n_k}| \to \infty$ in probability, $\mathbb{P}(A_k) \to 0$. Moreover,

$$\{X_{n_k} \to 0\} \subset \bigcup_{K=1}^{\infty} \bigcap_{k \ge K} A_k.$$

For each K,

$$\mathbb{P}\left(\bigcap_{k \ge K} A_k\right) \le \inf_{k \ge K} \mathbb{P}(A_k),$$

hence

$$\mathbb{P}\left(\bigcup_{K=1}^{\infty} \bigcap_{k \ge K} A_k\right) = \lim_{K\to\infty} \mathbb{P}\left(\bigcap_{k \ge K} A_k\right) \le \lim_{K\to\infty} \inf_{k \ge K} \mathbb{P}(A_k) = 0.$$

□

Theorem 7.7 (Translation kills the full-μ event)**.** *Let $\psi \in C^\infty(\mathbb{T}^3) \setminus \{0\}$ and let A_ψ be defined by (7.1). Then*

$$\mu(A_\psi + \psi) = 0.$$

Proof. Let $\Phi \sim \mu$ and set $\widetilde{\Phi} := \Phi - \psi$. Then $\widetilde{\Phi}$ has law $\mu(\cdot + \psi)$, so $\mu(A_\psi + \psi) = \mathbb{P}(\widetilde{\Phi} \in A_\psi)$.

On the event $\{\widetilde{\Phi} \in A_\psi\}$ one has $F_{n_k}(\widetilde{\Phi}; \psi) \to 0$ by definition of A_ψ. But Lemma 7.5 yields $|F_n(\widetilde{\Phi}; \psi)| \to \infty$ in probability, hence by Lemma 7.6,

$$\mathbb{P}\big(F_{n_k}(\widetilde{\Phi}; \psi) \to 0\big) = 0.$$

Therefore $\mathbb{P}(\widetilde{\Phi} \in A_\psi) = 0$, i.e. $\mu(A_\psi + \psi) = 0$. □

7.4. **Conclusion: proof of Theorem 1.1.**

Proof of Theorem 1.1. Fix $\psi \in C^\infty(\mathbb{T}^3) \setminus \{0\}$. By Lemma 7.2, $\mu(A_\psi) = 1$. By Theorem 7.7, $\mu(A_\psi + \psi) = 0$. Since T_ψ is a homeomorphism of E, $A_\psi + \psi = T_\psi(A_\psi)$ is Borel. Lemma 3.2 yields $\mu \perp T_{\psi\#}\mu$. □

Remark 7.8. The argument is robust: it requires only

(i) existence of renormalised square and renormalised cube with a logarithmically divergent linear counterterm and nonzero coefficient b;
(ii) that the only remaining divergent random component is a third-chaos term with variance $\asymp \log(1/\varepsilon)$.

Both properties are canonical for Φ^4_3 in $d = 3$.

References

[1] S. Albeverio and S. Kusuoka, *The invariant measure and the flow associated to the Φ^4_3-quantum field model*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **20** (2020), no. 4, 1359–1427.
[2] Y. Bruned, M. Hairer, and L. Zambotti, *Algebraic renormalisation of regularity structures*, Invent. Math. **215** (2019), 1039–1156.
[3] M. Hairer, *A theory of regularity structures*, Invent. Math. **198** (2014), no. 2, 269–504.
[4] J.-C. Mourrat and H. Weber, *The dynamic Φ^4_3 model comes down from infinity*, Comm. Math. Phys. **356** (2017), no. 3, 673–753.