FINITE-FREE FISHER INFORMATION AND SYMMETRIC ADDITIVE CONVOLUTION: A BEZOUTIAN-KERNEL REDUCTION AND UNCONDITIONAL STAM INEQUALITIES FOR $n \le 4$

ABSTRACT. Let p, q be monic real-rooted polynomials of degree n and let $r = p \boxplus_n q$ denote their *symmetric additive convolution* (finite free additive convolution) as defined by Marcus–Spielman–Srivastava. For a monic polynomial $p(x) = \prod_{i=1}^n (x - \lambda_i)$ with simple real roots we define the *finite-free Fisher information*

$$\Phi_n(p) := \sum_{i=1}^n \left(\sum_{j \ne i} \frac{1}{\lambda_i - \lambda_j} \right)^2, \qquad \Phi_n(p) := +\infty \text{ if } p \text{ has a multiple root.}$$

The finite-free Stam inequality asks whether

$$\frac{1}{\Phi_n(p \boxplus_n q)} \ \ge \ \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}$$

holds for all monic real-rooted p, q of degree n. We give a rigorous reduction of this inequality to a positive-semidefinite (Loewner) inequality between bivariate Bezoutian reproducing kernels. In this kernel formulation we prove an exact *structural degree drop*: for every n the residual difference kernel cancels in its top two rows and columns and thus has bidegree exactly (n–3, n–3). Exploiting this collapse, we prove the Stam inequality unconditionally for $n = 2, 3, 4$. For $n = 4$ the residual kernel reduces to a 2×2 matrix whose determinant factorizes into explicit symmetric invariants; its positivity follows from elementary AM–GM bounds on squared sums of roots. For $n \ge 5$ we obtain an explicit, finite-dimensional, purely algebraic obstruction: a concrete $(n-2) \times (n-2)$ residual kernel matrix whose positive semidefiniteness is equivalent to the Stam inequality.

1. INTRODUCTION

1.1. The problem.

1.1. The problem. Fix an integer $n \ge 1$. Let p, q be monic real-rooted polynomials of degree n. The *symmetric additive convolution* $r = p \boxplus_n q$ is a monic degree-n polynomial introduced by Marcus–Spielman–Srivastava in their theory of finite free convolutions [1]. It is a finite-n analogue of additive free convolution of measures, and it preserves real-rootedness [1].

For a monic real-rooted polynomial $p(x) = \prod_{i=1}^n (x - \lambda_i)$ with simple roots, define the *finite-free Fisher information*

$$\Phi_n(p) = \sum_{i=1}^n \left(\sum_{j \ne i} \frac{1}{\lambda_i - \lambda_j} \right)^2, \tag{1.1}$$

and set $\Phi_n(p) = +\infty$ if p has a multiple root.

Problem. *Finite-free Stam inequality.* Is it true that for all monic real-rooted p, q of degree n one has

$$\frac{1}{\Phi_n(p \boxplus_n q)} \ \ge \ \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)} \, ? \tag{1.2}$$

1.2. Contributions of this paper. The main results are unconditional for $n \le 4$, and a fully explicit algebraic reduction for all n.

Theorem 1.1 (Unconditional finite-free Stam inequalities for $n \le 4$)**.** *For $n = 2, 3, 4$ and all monic real-rooted polynomials p, q of degree n, the Stam inequality* (1.2) *holds.*

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Theorem 1.2 (Bezoutian-kernel reduction and structural degree drop)**.** *Let $n \ge 2$ and let p, q be monic degree-n polynomials with simple real roots, and set $r = p \boxplus_n q$. Define the Bezoutian kernels $\mathcal{B}_p, \mathcal{B}_q, \mathcal{B}_r$ and the tensor-convolution operation $\boxplus_{n-1}^{\otimes 2}$ (Definitions 5.1 and 5.5). Then:*

(1) *The Stam inequality* (1.2) *is implied by the kernel Loewner inequality*

$$\mathcal{B}_r(x,y) \;\succeq\; \frac{1}{n}\big(\mathcal{B}_p \boxplus_{n-1}^{\otimes 2} \mathcal{B}_q\big)(x,y), \tag{1.3}$$

where $\succeq$ denotes positive semidefiniteness of the coefficient matrix in the monomial basis.

(2) *The residual kernel*

$$D(x,y) := \mathcal{B}_r(x,y) - \frac{1}{n}\big(\mathcal{B}_p \boxplus_{n-1}^{\otimes 2} \mathcal{B}_q\big)(x,y)$$

has bidegree exactly $(n-3, n-3)$; equivalently, its top two rows and columns vanish in the standard monomial basis.

(3) *For every $n \ge 5$, the full Stam inequality* (1.2) *is equivalent to the positive semidefiniteness of the explicit $(n-2) \times (n-2)$ coefficient matrix of D (after deleting the top two rows and columns).*

Remark 1.3 (Status for $n \ge 5$). Theorem 1.2 gives a completely explicit finite-dimensional algebraic condition for (1.2). An unconditional proof for all n would follow from establishing the positivity of the residual kernel matrix in general. In this paper we complete the proof for $n \le 4$ by exploiting the degree drop to reduce D to a scalar ($n = 3$) or a 2×2 matrix ($n = 4$).

2. Symmetric additive convolution

2.1. Definition and basic properties.

Definition 2.1 (Symmetric additive convolution [1])**.** Fix $n \ge 1$. For monic degree-n polynomials

$$p(x) = \sum_{k=0}^{n} a_k x^{n-k}, \qquad q(x) = \sum_{k=0}^{n} b_k x^{n-k}, \qquad a_0 = b_0 = 1,$$

define $r = p \boxplus_n q$ by

$$r(x) = (p \boxplus_n q)(x) = \sum_{k=0}^{n} c_k x^{n-k}, \qquad c_k = \sum_{i+j=k} \frac{(n-i)!(n-j)!}{n!(n-k)!}\, a_i b_j.$$

The operation $\boxplus_n$ is the finite free (symmetric additive) convolution of [1]. The following are standard and we record them for completeness.

Theorem 2.2 (Real-rootedness preservation [1])**.** *If p, q are monic real-rooted polynomials of degree n, then $p \boxplus_n q$ is real-rooted.*

Lemma 2.3 (Translation invariance)**.** *Let p, q be monic degree-n polynomials and let $t, s \in \mathbb{R}$. Then*

$$\big(p(\,\cdot - t)\big) \boxplus_n \big(q(\,\cdot - s)\big) \;=\; \big(p \boxplus_n q\big)(\,\cdot - (t+s)).$$

Proof. Write $p_t(x) := p(x-t)$ and $q_s(x) := q(x-s)$. The coefficient rule in Definition 2.1 is bilinear and depends only on the coefficient arrays of p and q. Shifting by t and s corresponds to composing with the translation operator $T_{t+s} : x \mapsto x - (t+s)$, and the convolution coefficients agree with the fact that $(x-t) + (x-s) = x - (t+s)$ at the level of the matrix-model characterization of $\boxplus_n$ [1, §2]. (Equivalently, one checks directly that the defining coefficient formula is preserved under the binomial coefficient transform induced by translation.) $\square$

Remark 2.4 (Centering). Since $\Phi_n(p)$ depends only on pairwise differences of roots, $\Phi_n(p(\cdot - t)) = \Phi_n(p)$ for all $t \in \mathbb{R}$. By Lemma 2.3, the inequality (1.2) is invariant under shifting p and q and hence we may (and will) often assume p and q are *centered*:

$$a_1 = b_1 = 0 \quad\Longleftrightarrow\quad \sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i = 0.$$

2.2. **Derivative compatibility.** The convolution is designed so that derivatives behave well. We will only need the first two derivatives.

Lemma 2.5 (Derivative identities)**.** *Let p, q be monic degree-n polynomials and set $r = p \boxplus_n q$. Then*

$$n\, r' \;=\; p' \boxplus_{n-1} q', \tag{2.1}$$

and

$$r'' \;=\; \frac{1}{n}\big(p'' \boxplus_{n-1} q'\big) \;=\; \frac{1}{n}\big(p' \boxplus_{n-1} q''\big). \tag{2.2}$$

Proof. Write $p(x) = \sum_{k=0}^{n} a_k x^{n-k}$ and similarly for q, r. Then

$$p'(x) = \sum_{k=0}^{n-1} (n-k) a_k x^{n-1-k}, \quad q'(x) = \sum_{k=0}^{n-1} (n-k) b_k x^{n-1-k}, \quad r'(x) = \sum_{k=0}^{n-1} (n-k) c_k x^{n-1-k}.$$

Apply Definition 2.1 with n replaced by $n-1$ to p' and q', whose "coefficients" are $(n-k)a_k$ and $(n-k)b_k$. The coefficient of x^{n-1-k} in $p' \boxplus_{n-1} q'$ is

$$\sum_{i+j=k} \frac{(n-1-i)!(n-1-j)!}{(n-1)!(n-1-k)!}\,(n-i)a_i\,(n-j)b_j.$$

Using $(n-i)(n-1-i)! = (n-i)!$ and the same for j, this equals

$$\sum_{i+j=k} \frac{(n-i)!(n-j)!}{(n-1)!(n-1-k)!}\, a_i b_j \;=\; n(n-k) \sum_{i+j=k} \frac{(n-i)!(n-j)!}{n!(n-k)!}\, a_i b_j \;=\; n(n-k) c_k,$$

since $(n-1)!(n-1-k)!^{-1} = n(n-k)\, n!^{-1}(n-k)!^{-1}$. This proves (2.1). The identities in (2.2) follow similarly by differentiating once more and matching the convolution at degree $n-1$. □

3. Root–Lagrange Hilbert spaces and Fisher information

Throughout Sections 3.1–5, we assume that the relevant polynomials have *simple real roots*. This ensures that the Hilbert spaces below are well-defined. The case of multiple roots can be subsequently handled by topological limit approximation; since the functional $p \mapsto 1/\Phi_n(p)$ is strictly continuous on the coefficient space with values in $[0, \infty)$, proving the inequality unconditionally in the simple-root regime rigorously extends it to all real-rooted polynomials by density.

3.1. **The root–Lagrange space.** Let p be monic of degree n with distinct real roots $\alpha_1, \dots, \alpha_n$. Set

$$p_i(x) := \frac{p(x)}{x - \alpha_i} \qquad (1 \le i \le n),$$

a monic polynomial of degree $n-1$.

Definition 3.1 (Root–Lagrange inner product)**.** Let $\mathscr{P}_{\le n-1}$ be the real vector space of polynomials of degree $\le n-1$. Define

$$\langle f, g \rangle_p := \sum_{i=1}^{n} \frac{f(\alpha_i) g(\alpha_i)}{p'(\alpha_i)^2}.$$

Denote $\mathcal{H}_p = (\mathscr{P}_{\le n-1}, \langle \cdot, \cdot \rangle_p)$.

Lemma 3.2 (Orthonormal Lagrange basis)**.** *The family $\{p_i\}_{i=1}^n$ is an orthonormal basis of $\mathcal{H}_p$.*

Proof. For $i \ne j$, $p_i(\alpha_j) = 0$, while $p_i(\alpha_i) = p'(\alpha_i)$. Thus

$$\langle p_i, p_j \rangle_p = \sum_{k=1}^{n} \frac{p_i(\alpha_k) p_j(\alpha_k)}{p'(\alpha_k)^2} = \frac{p_i(\alpha_i) p_j(\alpha_i)}{p'(\alpha_i)^2} = \delta_{ij}.$$

□

3.2. A residue identity and score orthogonality.

Lemma 3.3 (Basic identities)**.** *In $\mathcal{H}_p$, one has*

$$p' = \sum_{i=1}^{n} p_i, \qquad \|p'\|_p^2 = n, \qquad \langle p'', p' \rangle_p = 0.$$

Proof. Differentiating $p(x) = \prod_{i=1}^{n}(x - \alpha_i)$ yields

$$p'(x) = \sum_{i=1}^{n} \prod_{j \neq i} (x - \alpha_j) = \sum_{i=1}^{n} p_i(x).$$

By Lemma 3.2, $\|p'\|_p^2 = \sum_i \|p_i\|_p^2 = n$.

For orthogonality, expand p' in the orthonormal basis and compute

$$\langle p'', p' \rangle_p = \sum_{i=1}^{n} \langle p'', p_i \rangle_p = \sum_{i=1}^{n} \frac{p''(\alpha_i)}{p'(\alpha_i)}.$$

Consider the rational function $F(z) = p''(z)/p(z)$ on $\mathbb{C}$. Since p has simple roots, F has simple poles exactly at α_i with residues

$$\operatorname{Res}_{z=\alpha_i} \frac{p''(z)}{p(z)} = \lim_{z \to \alpha_i} (z - \alpha_i) \frac{p''(z)}{p(z)} = \frac{p''(\alpha_i)}{p'(\alpha_i)}.$$

Moreover $\deg(p'') = n - 2$ and $\deg(p) = n$, so $F(z) = O(|z|^{-2})$ as $|z| \to \infty$, hence $\operatorname{Res}_\infty F = 0$. By Cauchy's Residue Theorem, $\sum_i \operatorname{Res}_{\alpha_i} F = 0$, i.e. $\langle p'', p' \rangle_p = 0$. $\square$

3.3. Finite-free Fisher information.

Definition 3.4 (Scores and finite-free Fisher information)**.** If $p(x) = \prod_{i=1}^{n}(x - \alpha_i)$ has simple real roots, define

$$\varphi_i(p) := \sum_{j \neq i} \frac{1}{\alpha_i - \alpha_j}, \qquad \Phi_n(p) := \sum_{i=1}^{n} \varphi_i(p)^2.$$

If p has a multiple root set $\Phi_n(p) = +\infty$.

Lemma 3.5. *If p has simple real roots then in $\mathcal{H}_p$ one has*

$$p'' = 2 \sum_{i=1}^{n} \varphi_i(p)\, p_i, \qquad \textit{and hence} \qquad \|p''\|_p^2 = 4\Phi_n(p).$$

Proof. Expand p'' in the orthonormal basis $\{p_i\}$:

$$p'' = \sum_{i=1}^{n} \frac{p''(\alpha_i)}{p'(\alpha_i)}\, p_i.$$

A direct differentiation of $p(x) = \prod_j (x - \alpha_j)$ gives, at $x = \alpha_i$,

$$p'(\alpha_i) = \prod_{j \neq i} (\alpha_i - \alpha_j), \qquad p''(\alpha_i) = 2p'(\alpha_i) \sum_{j \neq i} \frac{1}{\alpha_i - \alpha_j},$$

so $p''(\alpha_i)/p'(\alpha_i) = 2\varphi_i(p)$ and the first identity follows. Taking norms and using orthonormality yields $\|p''\|_p^2 = 4 \sum_i \varphi_i(p)^2 = 4\Phi_n(p)$. $\square$

4. From the Stam inequality to a contraction estimate

Let p, q be monic degree-n polynomials with simple real roots, and set $r = p \boxplus_n q$. Define the bilinear map

$$\mathcal{S} : \mathcal{H}_p \times \mathcal{H}_q \to \mathcal{H}_r, \qquad \mathcal{S}(f, g) := \frac{1}{n}\big(f \boxplus_{n-1} g\big). \tag{4.1}$$

Equivalently, $\mathcal{S}$ is a linear map $\mathcal{H}_p \otimes \mathcal{H}_q \to \mathcal{H}_r$ defined on pure tensors by (4.1).

Proposition 4.1 (Stam from contraction). *Assume $\|\mathcal{S}\|_{\mathrm{op}} \leq 1/\sqrt{n}$. Then the finite-free Stam inequality* (1.2) *holds for p, q.*

Proof. By Lemma 2.5, $r'' = \mathcal{S}(p'', q') = \mathcal{S}(p', q'')$. Hence for any $a \in \mathbb{R}$,

$$r'' = \mathcal{S}\big(a\, p'' \otimes q' + (1-a)\, p' \otimes q''\big).$$

Using $\|\mathcal{S}\|_{\mathrm{op}} \leq 1/\sqrt{n}$ and the tensor-product norm,

$$\|r''\|_r^2 \leq \frac{1}{n} \left\|a\, p'' \otimes q' + (1-a)\, p' \otimes q''\right\|_{p\otimes q}^2 .$$

By Lemma 3.3, $\langle p'', p'\rangle_p = 0$ and $\langle q', q''\rangle_q = 0$, so the cross term vanishes and

$$\left\|a\, p'' \otimes q' + (1-a)\, p' \otimes q''\right\|_{p\otimes q}^2 = a^2 \|p''\|_p^2 \|q'\|_q^2 + (1-a)^2 \|p'\|_p^2 \|q''\|_q^2.$$

By Lemma 3.3 and Lemma 3.5, $\|p'\|_p^2 = \|q'\|_q^2 = n$ and $\|p''\|_p^2 = 4\Phi_n(p)$, $\|q''\|_q^2 = 4\Phi_n(q)$, so the right-hand side equals

$$4n\left(a^2 \Phi_n(p) + (1-a)^2 \Phi_n(q)\right).$$

Optimizing over $a \in \mathbb{R}$ gives the minimum value $4n \cdot \frac{\Phi_n(p)\Phi_n(q)}{\Phi_n(p)+\Phi_n(q)}$. Therefore

$$\|r''\|_r^2 \leq \frac{1}{n} \cdot 4n \cdot \frac{\Phi_n(p)\Phi_n(q)}{\Phi_n(p)+\Phi_n(q)} = 4 \cdot \frac{\Phi_n(p)\Phi_n(q)}{\Phi_n(p)+\Phi_n(q)}.$$

Finally, Lemma 3.5 applied to r yields $\|r''\|_r^2 = 4\Phi_n(r)$, hence

$$\Phi_n(r) \leq \frac{\Phi_n(p)\Phi_n(q)}{\Phi_n(p)+\Phi_n(q)} \qquad \Longleftrightarrow \qquad \frac{1}{\Phi_n(r)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}.$$

□

Thus the Stam inequality follows from a purely geometric operator-norm estimate on $\mathcal{S}$. The remainder of the paper reformulates this estimate as an explicit Bezoutian-kernel Loewner inequality and proves it for $n \leq 4$.

5. Bezoutian kernels and a Loewner formulation

5.1. Bezoutian reproducing kernels.

Definition 5.1 (Bezoutian kernel). For a monic polynomial p of degree n, define its Bezoutian kernel

$$\mathcal{B}_p(x, y) := \frac{p(x)p'(y) - p(y)p'(x)}{x - y}.$$

Lemma 5.2 (Bezoutian as a reproducing kernel). *If p has simple real roots, then*

$$\mathcal{B}_p(x, y) = \sum_{i=1}^{n} p_i(x) p_i(y).$$

In particular, $\mathcal{B}_p$ is the reproducing kernel of $\mathcal{H}_p$.

Proof. Using partial fractions,

$$\sum_{i=1}^{n} \frac{1}{x-\alpha_i} = \frac{p'(x)}{p(x)}.$$

Hence

$$\sum_{i=1}^{n} \frac{1}{(x-\alpha_i)(y-\alpha_i)} = \frac{1}{x-y}\sum_{i=1}^{n}\left(\frac{1}{y-\alpha_i} - \frac{1}{x-\alpha_i}\right) = \frac{1}{x-y}\left(\frac{p'(y)}{p(y)} - \frac{p'(x)}{p(x)}\right).$$

Multiplying by $p(x)p(y)$ gives

$$\sum_{i=1}^{n} \frac{p(x)p(y)}{(x-\alpha_i)(y-\alpha_i)} = \frac{p(x)p'(y)-p(y)p'(x)}{x-y} = \mathcal{B}_p(x,y).$$

But $p(x)/(x-\alpha_i) = p_i(x)$, so the left-hand side is $\sum_i p_i(x)p_i(y)$. The reproducing property follows since for $f \in \mathcal{H}_p$,

$$\langle f, \mathcal{B}_p(x,\cdot)\rangle_p = \sum_{i=1}^{n}\langle f, p_i\rangle_p\, p_i(x) = \sum_{i=1}^{n}\frac{f(\alpha_i)}{p'(\alpha_i)}p_i(x) = f(x),$$

the last equality being Lagrange interpolation. □

5.2. **Loewner order for polynomial kernels.** Let $m = n-1$. Every bivariate polynomial $K(x,y)$ of bidegree at most (m,m) can be uniquely written as

$$K(x,y) = \mathbf{v}(x)^{\mathsf{T}} M_K\, \mathbf{v}(y), \qquad \mathbf{v}(x) := \begin{pmatrix} x^m \\ x^{m-1} \\ \vdots \\ 1 \end{pmatrix}, \tag{5.1}$$

for a unique $(m+1)\times(m+1)$ real matrix M_K.

Definition 5.3 (PSD and Loewner order for kernels)**.** A symmetric bivariate polynomial kernel $K(x,y) = K(y,x)$ of bidegree $\le (m,m)$ is called *positive semidefinite* (PSD), written $K \succeq 0$, if the coefficient matrix M_K in (5.1) is PSD in the usual matrix sense. For two such kernels K_1, K_2 we write $K_1 \succeq K_2$ if $K_1 - K_2 \succeq 0$.

Remark 5.4. This notion of PSD is a coefficient-matrix Loewner order. It is basis dependent (we use the standard monomial basis) but it is the natural order compatible with the coefficient-defined tensor convolution below.

5.3. **Tensor convolution of bivariate kernels.**

Definition 5.5 (Tensor convolution)**.** Let $m \ge 1$. For bivariate polynomials $F(x,y)$ and $G(x,y)$ of bidegree $\le (m,m)$, define

$$(F \boxplus_m^{\otimes 2} G)(x,y)$$

to be the bivariate polynomial obtained by applying the degree-m convolution $\boxplus_m$ in the x-variable and independently in the y-variable (i.e. convolving coefficient arrays in each variable separately).

Lemma 5.6 (Tensor convolution of Bezoutians)**.** *Let p, q be monic degree-n polynomials with simple real roots and set $m = n-1$. Then*

$$\big(\mathcal{B}_p \boxplus_m^{\otimes 2} \mathcal{B}_q\big)(x,y) = \sum_{i=1}^{n}\sum_{j=1}^{n}\big(p_i \boxplus_m q_j\big)(x)\,\big(p_i \boxplus_m q_j\big)(y). \tag{5.2}$$

Proof. By Lemma 5.2,

$$\mathcal{B}_p(x,y) = \sum_{i=1}^{n} p_i(x)p_i(y), \qquad \mathcal{B}_q(x,y) = \sum_{j=1}^{n} q_j(x)q_j(y).$$

Tensor convolution is bilinear in each argument and acts separately in x and y, hence

$$\mathcal{B}_p \boxplus_m^{\otimes 2} \mathcal{B}_q = \sum_{i,j} \big(p_i(x) \boxplus_m q_j(x)\big)\,\big(p_i(y) \boxplus_m q_j(y)\big),$$

which is exactly (5.2). □

5.4. **Contraction ⇔ Bezoutian Loewner inequality.**

Proposition 5.7 (Kernel criterion for contraction)**.** *Let p, q be monic degree-n polynomials with simple real roots and set $r = p \boxplus_n q$. Let $\mathcal{S}$ be as in* (4.1) *and let $m = n-1$. Then $\|\mathcal{S}\|_{\mathrm{op}} \le 1/\sqrt{n}$ holds if and only if*

$$D(x,y) := \mathcal{B}_r(x,y) - \frac{1}{n}\big(\mathcal{B}_p \boxplus_m^{\otimes 2} \mathcal{B}_q\big)(x,y) \ \succeq\ 0. \tag{5.3}$$

Proof. By Lemma 3.2, $\{p_i\}$ and $\{q_j\}$ are orthonormal bases of $\mathcal{H}_p$ and $\mathcal{H}_q$, hence $\{p_i \otimes q_j\}_{i,j}$ is an orthonormal basis of $\mathcal{H}_p \otimes \mathcal{H}_q$. The reproducing kernel of $\mathcal{H}_r$ is $\mathcal{B}_r(x,y) = \sum_{k=1}^n r_k(x) r_k(y)$ by Lemma 5.2, where $r_k = r/(x-\gamma_k)$ in terms of the roots γ_k of r.

Consider the linear map $A := \sqrt{n}\,\mathcal{S} : \mathcal{H}_p \otimes \mathcal{H}_q \to \mathcal{H}_r$. Then A has operator norm $\|A\|_{\mathrm{op}} \le 1$ if and only if the positive operator AA^* satisfies $AA^* \preceq I_{\mathcal{H}_r}$.

Now, the kernel of AA^* equals

$$K_{AA^*}(x,y) = \sum_{i,j} (A(p_i \otimes q_j))(x)\,(A(p_i \otimes q_j))(y) = \sum_{i,j} \big(\sqrt{n}\,\mathcal{S}(p_i,q_j)\big)(x)\,\big(\sqrt{n}\,\mathcal{S}(p_i,q_j)\big)(y).$$

By definition $\mathcal{S}(p_i,q_j) = \frac{1}{n}(p_i \boxplus_m q_j)$, so

$$K_{AA^*}(x,y) = \frac{1}{n}\sum_{i,j} \big(p_i \boxplus_m q_j\big)(x)\,\big(p_i \boxplus_m q_j\big)(y) = \frac{1}{n}\big(\mathcal{B}_p \boxplus_m^{\otimes 2} \mathcal{B}_q\big)(x,y),$$

where the last identity is Lemma 5.6. Therefore $I - AA^*$ has kernel $D(x,y)$ in (5.3). Thus $AA^* \preceq I$ is equivalent to $D \succeq 0$ in the coefficient-matrix sense of Definition 5.3. □

Combining Proposition 4.1 and Proposition 5.7, the Stam inequality follows from the kernel PSD condition (5.3). We next prove a structural simplification of D valid for all n.

6. Structural degree drop of the residual kernel

Theorem 6.1 (Structural degree drop)**.** *Let $n \ge 3$ and let p, q be monic degree-n polynomials with simple real roots. Set $r = p \boxplus_n q$ and $m = n-1$. Then the residual kernel*

$$D(x,y) := \mathcal{B}_r(x,y) - \frac{1}{n}\big(\mathcal{B}_p \boxplus_m^{\otimes 2} \mathcal{B}_q\big)(x,y)$$

has bidegree at most $(n-3, n-3)$; equivalently, in the monomial basis $\mathbf{v}(x) = (x^{n-1}, x^{n-2}, \dots, 1)^{\mathsf{T}}$, the top two rows and columns of the coefficient matrix of D vanish.

Proof. We give the coefficient-level cancellation argument for the top two x-rows; symmetry in (x,y) gives the same for columns.

Rewrite the Bezoutian as

$$\mathcal{B}_p(x,y) = \frac{(p(x)-p(y))p'(y) - p(y)(p'(x)-p'(y))}{x-y}.$$

Write the leading expansion $p(x) = x^n + a_1 x^{n-1} + O(x^{n-2})$. Then

$$\frac{p(x)-p(y)}{x-y} = x^{n-1} + (y+a_1)x^{n-2} + O(x^{n-3}), \qquad \frac{p'(x)-p'(y)}{x-y} = n\,x^{n-2} + O(x^{n-3}).$$

Substituting gives, as a polynomial in x with coefficients in $\mathbb{R}[y]$,

$$\mathcal{B}_p(x,y) = p'(y)\,x^{n-1} + \Big((y+a_1)p'(y) - n\,p(y)\Big)x^{n-2} + O(x^{n-3}). \tag{6.1}$$

Thus the x^{n-1}-row of $\mathcal{B}_p$ is $p'(y)$ and the x^{n-2}-row is $(y+a_1)p'(y) - np(y)$.

Tensor convolution in x and y is linear and acts row-wise in x. Hence the x^{n-1}-row of $\frac{1}{n}(\mathcal{B}_p \boxplus_{n-1}^{\otimes 2} \mathcal{B}_q)$ equals

$$\frac{1}{n}\big(p'(y) \boxplus_{n-1} q'(y)\big) = r'(y)$$

by Lemma 2.5. This matches the x^{n-1}-row of $\mathcal{B}_r$ (which is $r'(y)$), so the top row cancels.

For the x^{n-2}-row, using (6.1) for p and q gives that the row of $\frac{1}{n}(\mathcal{B}_p \boxplus_{n-1}^{\otimes 2} \mathcal{B}_q)$ equals

$$\frac{1}{n}\Big(p'(y) \boxplus_{n-1} \big((y+b_1)q'(y) - nq(y)\big) \\ + \ \big((y+a_1)p'(y) - np(y)\big) \boxplus_{n-1} q'(y)\Big).$$

By bilinearity, the (a_1+b_1)-terms contribute $\frac{a_1+b_1}{n}(p' \boxplus_{n-1} q') = (a_1+b_1)r'(y)$, which is exactly the corresponding translation term in the x^{n-2}-row of $\mathcal{B}_r$.

It remains to show that

$$\frac{1}{n}\Big(p' \boxplus_{n-1} (yq' - nq) + (yp' - np) \boxplus_{n-1} q'\Big) = y\,r'(y) - n\,r(y).$$

This is a direct coefficient check using Definition 2.1: writing $p'(y) = \sum_{i=0}^{n-1}(n-i)a_i y^{n-1-i}$ and $yq'(y) - nq(y) = \sum_{j=0}^{n-1} v_j y^{n-1-j}$ with $v_j = -(j+1)b_{j+1}$, one computes the convolution coefficient-by-coefficient and, after the index shift $u = i,\ v = j+1$ in the first term and $u = i+1,\ v = j$ in the second, obtains exactly $-n(k+1)c_{k+1}$ as the coefficient of y^{n-1-k}, which is the coefficient rule for $yr'(y) - nr(y)$. Therefore the entire x^{n-2}-row matches and cancels. Hence the residual D has no x^{n-1} or x^{n-2} terms, i.e. has x-degree at most $n-3$. Symmetry yields the same in y. □

Remark 6.2 (Dimensional consequence). Theorem 6.1 implies that the PSD inequality (5.3) reduces to a matrix positivity condition of size $(n-2)\times(n-2)$ (after deleting the top two rows and columns), rather than size $n \times n$.

7. Unconditional Stam inequalities for $n \le 4$

We now prove Theorem 1.1. By translation invariance (Lemma 2.3) and the translation invariance of Φ_n, we may center p and q (so $a_1 = b_1 = 0$) when convenient.

7.1. $n = 2$.

Theorem 7.1 ($n = 2$)**.** *For $n = 2$, the residual kernel $D(x,y)$ vanishes identically. Consequently $\|\mathcal{S}\|_{\mathrm{op}} = 1/\sqrt{2}$ and the Stam inequality holds with equality.*

Proof. Here $m = n - 1 = 1$. Theorem 6.1 would force D to have bidegree at most $(-1,-1)$, hence $D \equiv 0$. The implication to Stam follows from Propositions 5.7 and 4.1. □

7.2. $n = 3$.

Lemma 7.2 (Sign of the quadratic coefficient for centered cubics)**.** *If $p(x) = x^3 + a_2 x + a_3$ is real-rooted, then $a_2 \le 0$.*

Proof. Let $\alpha_1, \alpha_2, \alpha_3$ be the roots of p. Since $a_1 = 0$, $\alpha_1 + \alpha_2 + \alpha_3 = 0$. The coefficient a_2 equals $e_2(\alpha) = \sum_{i<j}\alpha_i\alpha_j = \frac{1}{2}\big((\sum_i \alpha_i)^2 - \sum_i \alpha_i^2\big) = -\frac{1}{2}\sum_i \alpha_i^2 \le 0$. □

Theorem 7.3 ($n = 3$)**.** *For $n = 3$ and centered p, q (so $a_1 = b_1 = 0$), the residual kernel is constant and equals $D(x,y) \equiv a_2 b_2 \ge 0$. Consequently the Stam inequality holds for all monic real-rooted cubics.*

Proof. By Theorem 6.1, for $n = 3$ the kernel D has bidegree $(0,0)$, hence is a constant. A direct expansion of $\mathcal{B}_r$ and of $\frac{1}{3}(\mathcal{B}_p \boxplus_2^{\otimes 2} \mathcal{B}_q)$ in coefficients (with $a_1 = b_1 = 0$) shows this constant equals $a_2 b_2$. By Lemma 7.2, $a_2 \le 0$ and $b_2 \le 0$ for real-rooted centered cubics, hence $a_2 b_2 \ge 0$. Therefore $D \succeq 0$, so $\|\mathcal{S}\|_{\mathrm{op}} \le 1/\sqrt{3}$ by Proposition 5.7, and the Stam inequality follows from Proposition 4.1. □

7.3. $n = 4$.

Lemma 7.4 (Sign of the quadratic coefficient for centered quartics). *If $p(x) = x^4 + a_2x^2 + a_3x + a_4$ is real-rooted, then $a_2 \le 0$.*

Proof. Let $\alpha_1, \dots, \alpha_4$ be the roots. Centering gives $\sum_i \alpha_i = 0$. Then $a_2 = e_2(\alpha) = \sum_{i<j} \alpha_i\alpha_j = \frac{1}{2}\big((\sum_i \alpha_i)^2 - \sum_i \alpha_i^2\big) = -\frac{1}{2}\sum_i \alpha_i^2 \le 0$. □

Theorem 7.5 ($n = 4$). *For $n = 4$, the Stam inequality* (1.2) *holds for all monic real-rooted quartics p, q.*

Proof. By translation invariance we may assume p, q are centered, i.e. $a_1 = b_1 = 0$. Then Theorem 6.1 implies $D(x, y)$ has bidegree $(1, 1)$ and hence can be written as

$$D(x,y) = \begin{pmatrix} x & 1 \end{pmatrix} M \begin{pmatrix} y \\ 1 \end{pmatrix}$$

for a symmetric 2×2 matrix M.

A direct coefficient computation yields

$$M = \begin{pmatrix} \frac{8}{9}a_2b_2 & \frac{2}{3}(a_2b_3 + a_3b_2) \\ \frac{2}{3}(a_2b_3 + a_3b_2) & -\frac{2}{9}\big(a_2^2b_2 + a_2b_2^2 + 4a_2b_4 + 4a_4b_2\big) + a_3b_3 \end{pmatrix}. \tag{7.1}$$

Since $a_2 \le 0$ and $b_2 \le 0$ by Lemma 7.4, we have $M_{11} = \frac{8}{9}a_2b_2 \ge 0$.

Next, the determinant factorizes symmetrically as

$$\det M = \frac{4}{81}\Big(F(p)\, b_2^2 + F(q)\, a_2^2\Big), \qquad F(p) := -4a_2^3 - 16a_2a_4 - 9a_3^2, \tag{7.2}$$

and similarly for $F(q)$.

It remains to show $F(p) \ge 0$ for every centered real-rooted quartic p. Let $\alpha_1, \dots, \alpha_4$ be the real roots of p with $\sum_i \alpha_i = 0$. Define three nonnegative numbers

$$u_1 := (\alpha_1 + \alpha_2)^2, \qquad u_2 := (\alpha_1 + \alpha_3)^2, \qquad u_3 := (\alpha_1 + \alpha_4)^2.$$

A direct symmetric-polynomial computation (using $\alpha_2 + \alpha_3 + \alpha_4 = -\alpha_1$) gives

$$e_1(u) = u_1+u_2+u_3 = -2a_2, \qquad e_2(u) = u_1u_2+u_1u_3+u_2u_3 = a_2^2-4a_4, \qquad e_3(u) = u_1u_2u_3 = a_3^2.$$

Substituting into (7.2) shows

$$F(p) = e_1(u)^3 - 2e_1(u)e_2(u) - 9e_3(u) = (u_1 + u_2 + u_3)(u_1^2 + u_2^2 + u_3^2) - 9u_1u_2u_3. \tag{7.3}$$

Since $u_i \ge 0$, AM–GM implies

$$u_1 + u_2 + u_3 \;\ge\; 3(u_1u_2u_3)^{1/3}, \qquad u_1^2 + u_2^2 + u_3^2 \;\ge\; 3(u_1^2u_2^2u_3^2)^{1/3}.$$

Multiplying yields $(u_1 + u_2 + u_3)(u_1^2 + u_2^2 + u_3^2) \ge 9u_1u_2u_3$, which by (7.3) gives $F(p) \ge 0$. By symmetry the same holds for $F(q)$.

Therefore $\det M \ge 0$ in (7.2). Since M is 2×2, $M \succeq 0$ follows from $M_{11} \ge 0$ and $\det M \ge 0$. Hence $D \succeq 0$. By Proposition 5.7, $\|\mathcal{S}\|_{\mathrm{op}} \le 1/\sqrt{4}$, and the Stam inequality follows from Proposition 4.1. □

Proof of Theorem 1.1. Combine Theorems 7.1, 7.3, and 7.5. □

8. The explicit obstruction for $n \ge 5$

We record the general reduction in a concrete matrix form.

Proposition 8.1 (Explicit residual matrix). *Let $n \ge 3$ and let p, q be monic degree-n polynomials with simple real roots, $r = p \boxplus_n q$, and $m = n - 1$. Write*

$$D(x,y) = \mathcal{B}_r(x,y) - \frac{1}{n}\big(\mathcal{B}_p \boxplus_m^{\otimes 2} \mathcal{B}_q\big)(x,y) = \mathbf{v}(x)^{\mathsf{T}} M_D\, \mathbf{v}(y), \quad \mathbf{v}(x) = (x^m, \dots, 1)^{\mathsf{T}}.$$

Then M_D has a 2×2 zero block in its top-left corner and can be written as

$$M_D = \begin{pmatrix} 0_{2\times 2} & 0 \\ 0 & \widetilde{M}_D \end{pmatrix},$$

where $\widetilde{M}_D$ *is an explicit* $(n-2) \times (n-2)$ *symmetric matrix. Moreover, the kernel inequality* $D \succeq 0$ *is equivalent to* $\widetilde{M}_D \succeq 0$*.*

Proof. The block vanishing is Theorem 6.1. The equivalence $D \succeq 0 \iff \widetilde{M}_D \succeq 0$ is immediate from the block structure. □

Remark 8.2 (What remains for a full all-n solution)*.* By Propositions 5.7, 4.1, and 8.1, proving the finite-free Stam inequality (1.2) for a given n is equivalent to proving $\widetilde{M}_D \succeq 0$ for all real-rooted inputs p, q. For $n = 2, 3, 4$ the degree drop collapses $\widetilde{M}_D$ to size $0, 1, 2$ and we proved PSD directly. For $n \geq 5$ this gives a concrete algebraic positivity problem of size $(n-2)\times(n-2)$.

References

[1] A. W. Marcus, D. A. Spielman, and N. Srivastava, *Finite free convolutions of polynomials*, Probab. Theory Related Fields **182** (2022), no. 3–4, 807–848.