HAMILTONIAN SMOOTHING OF FOUR-VALENT POLYHEDRAL LAGRANGIAN SURFACES IN $(\mathbb{R}^4, \omega_{\mathrm{st}})$

ABSTRACT. Let $K \subset (\mathbb{R}^4, \omega_{\mathrm{st}})$ be a finite 2–dimensional polyhedral complex which is a topological surface and whose 2–faces lie in affine Lagrangian planes. Assume that exactly four faces meet at every vertex. We prove that K admits a Lagrangian smoothing in the sense of the problem statement: there exists a Hamiltonian isotopy K_t of smooth embedded Lagrangian surfaces for $t \in (0, 1]$ extending to a topological isotopy on $[0, 1]$ with $K_0 = K$.

The proof has two ingredients. First, four-valence forces a rigid local symplectic splitting of every *generic* vertex cone into a product of planar corners; *fold* vertices (collinear opposite rays) are edge-type and are treated by the edge model. Second, edgewise gluing requires accommodating a nontrivial diagonal symplectic squeeze (holonomy) between transverse coordinate choices at edge endpoints; we construct an interpolating Lagrangian cylinder whose varying squeeze is compensated by a longitudinal conjugate-momentum shift. The resulting family is a Lagrangian isotopy, and we turn it into a Hamiltonian isotopy by an $O(t^2)$ Liouville/flux correction realized as a graph of a small closed 1-form inside a quantitative Weinstein neighborhood. The correcting form is built from restrictions of fixed ambient closed 1-forms, avoiding metric blow-up.

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1. INTRODUCTION

Write (q_1, q_2, p_1, p_2) for the standard linear coordinates on $\mathbb{R}^4$ and set

$$\omega_{\mathrm{st}} := dq_1 \wedge dp_1 + dq_2 \wedge dp_2, \qquad \lambda_{\mathrm{st}} := p_1\, dq_1 + p_2\, dq_2, \qquad d\lambda_{\mathrm{st}} = \omega_{\mathrm{st}}.$$

Definition 1.1 (Polyhedral Lagrangian surface)**.** A *polyhedral surface* $K \subset \mathbb{R}^4$ is a finite 2–dimensional polyhedral complex embedded in $\mathbb{R}^4$ such that every 2–cell (face) is a compact convex polygon contained in an affine 2–plane and such that K is a topological submanifold of $\mathbb{R}^4$ (every point has a neighborhood in K homeomorphic to an open disc).

We call K *polyhedral Lagrangian* if the affine span of each face is an affine Lagrangian plane in $(\mathbb{R}^4, \omega_{\mathrm{st}})$.

Definition 1.2 (Four-valent vertices)**.** A vertex v of K is *four-valent* if exactly four faces meet at v (equivalently, the link of v in K is a 4–cycle).

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Definition 1.3 (Lagrangian smoothing)**.** Let K be polyhedral Lagrangian. A *Lagrangian smoothing* of K is a family $\{K_t\}_{t\in(0,1]}$ of smooth embedded Lagrangian surfaces in $\mathbb{R}^4$ such that:

(i) $\{K_t\}_{t\in(0,1]}$ is a *Hamiltonian isotopy*: there exists a smooth compactly supported time-dependent Hamiltonian $H_t : \mathbb{R}^4 \to \mathbb{R}$ whose flow Φ_t satisfies $K_t = \Phi_t(K_{t_0})$ for all $t, t_0 \in (0, 1]$;

(ii) $\{K_t\}$ extends to a *topological isotopy* on $[0, 1]$ with $K_0 = K$.

Theorem 1.4. *Let $K \subset (\mathbb{R}^4, \omega_{\rm st})$ be a polyhedral Lagrangian surface such that every vertex is four-valent. Then K admits a Lagrangian smoothing in the sense of Definition 1.3.*

2. Four-valent vertex cones: planar, generic, and fold

Fix a vertex v of K. Since K is a topological surface and v is four-valent, there are four edge rays leaving v. Translate so $v = 0$ and let $r_1, r_2, r_3, r_4 \in \mathbb{R}^4$ be nonzero vectors along the four edges, ordered cyclically so that the four incident face cones lie in the planes

$$\Lambda_{12} := \operatorname{span}(r_1, r_2), \quad \Lambda_{23} := \operatorname{span}(r_2, r_3), \quad \Lambda_{34} := \operatorname{span}(r_3, r_4), \quad \Lambda_{41} := \operatorname{span}(r_4, r_1).$$

Each Λ_{ij} is Lagrangian, hence

$$\omega_{\rm st}(r_1, r_2) = \omega_{\rm st}(r_2, r_3) = \omega_{\rm st}(r_3, r_4) = \omega_{\rm st}(r_4, r_1) = 0. \tag{1}$$

Proposition 2.1 (Vertex trichotomy)**.** *Assume* (1)*. Exactly one of the following holds.*

(a) ***Planar case.*** *One has $\omega_{\rm st}(r_1, r_3) = \omega_{\rm st}(r_2, r_4) = 0$. Then r_1, r_2, r_3, r_4 lie in a single Lagrangian plane. In particular, the tangent cone is locally planar.*

(b) ***Generic case.*** *After cyclic relabeling, $\omega_{\rm st}(r_1, r_3) \neq 0$ and r_2, r_4 are linearly independent. Then*

$$V_1 := \operatorname{span}(r_1, r_3) \text{ is a symplectic } 2\text{–plane}, \qquad V_2 := V_1^{\omega_{\rm st}} = \operatorname{span}(r_2, r_4),$$

so $\mathbb{R}^4 = V_1 \oplus V_2$ is a symplectic orthogonal direct sum, and the vertex cone factors as a product of planar corners:

$$C_0K = C_1 \times C_2 \subset V_1 \times V_2,$$

where $C_1 = \mathbb{R}_{\geq 0}r_1 \cup \mathbb{R}_{\geq 0}r_3 \subset V_1$ and $C_2 = \mathbb{R}_{\geq 0}r_2 \cup \mathbb{R}_{\geq 0}r_4 \subset V_2$.

(c) ***Fold case.*** *After cyclic relabeling, $\omega_{\rm st}(r_1, r_3) \neq 0$ but r_2 and r_4 are collinear. Then $V_1 := \operatorname{span}(r_1, r_3)$ is symplectic and $V_2 := V_1^{\omega_{\rm st}}$ is symplectic, but the cone uses only a line $\ell \subset V_2$ spanned by r_2:*

$$C_0K = C_1 \times \ell,$$

with $C_1 = \mathbb{R}_{\geq 0}r_1 \cup \mathbb{R}_{\geq 0}r_3 \subset V_1$ and $\ell = \mathbb{R}r_2 = \mathbb{R}r_4 \subset V_2$. Geometrically, 0 is edge-type: a corner crossed with a line.

Proof. If $\omega_{\rm st}(r_1, r_3) = \omega_{\rm st}(r_2, r_4) = 0$, then together with (1) we have $\omega_{\rm st}(r_i, r_j) = 0$ for all pairs (i, j), so $W := \operatorname{span}(r_1, r_2, r_3, r_4)$ is isotropic. In a symplectic 4–space, $\dim W \leq 2$, hence the rays lie in a 2–plane. Since each Λ_{ij} is Lagrangian, this plane is Lagrangian. This is (a).

Otherwise at least one of $\omega_{\rm st}(r_1, r_3)$ or $\omega_{\rm st}(r_2, r_4)$ is nonzero. After cyclic relabeling assume $\omega_{\rm st}(r_1, r_3) \neq 0$. Then $V_1 := \operatorname{span}(r_1, r_3)$ is symplectic. From (1),

$$\omega_{\rm st}(r_2, r_1) = \omega_{\rm st}(r_2, r_3) = 0 \Rightarrow r_2 \in V_1^{\omega_{\rm st}}, \qquad \omega_{\rm st}(r_4, r_1) = \omega_{\rm st}(r_4, r_3) = 0 \Rightarrow r_4 \in V_1^{\omega_{\rm st}}.$$

Set $V_2 := V_1^{\omega_{\rm st}}$ (a symplectic plane). If r_2, r_4 are independent then $V_2 = \operatorname{span}(r_2, r_4)$ and the cone is $C_1 \times C_2$, giving (b). If r_2, r_4 are collinear then they span a line $\ell \subset V_2$ and the cone is $C_1 \times \ell$, giving (c). □

Remark 2.2. Fold vertices are not essential 4–dimensional vertex singularities; they are points on an edge-type singular locus. In the smoothing construction we treat fold vertices by the edge model (no vertex patch).

3. A planar corner smoothing curve

A key feature used later is a planar rounding curve whose coordinate product vanishes identically outside the rounding region.

Lemma 3.1 (A corner rounding with compactly supported product)**.** *Let $\sigma : \mathbb{R} \to [0, 1]$ be smooth with $\sigma(s) = 0$ for $s \leq 0$, $\sigma(s) = 1$ for $s \geq 1$, and σ flat at 0 (all derivatives vanish at 0). Fix $T = \log 2$. For $\rho > 0$ define $\gamma_\rho : \mathbb{R} \to \mathbb{R}^2$ by*

$$x_\rho(t) := \rho e^t\, \sigma(t + T), \qquad y_\rho(t) := \rho e^{-t}\, \sigma(T - t), \qquad \gamma_\rho(t) := (x_\rho(t), y_\rho(t)).$$

Then $\Gamma_\rho := \gamma_\rho(\mathbb{R})$ is a smooth properly embedded curve such that:

(a) Γ_ρ *agrees* exactly *with the union of coordinate rays*

$$\{(x,0) : x \geq 2\rho\} \ \cup \ \{(0,y) : y \geq 2\rho\}$$

outside the Euclidean ball $B_{2\rho}(0)$*;*

(b) *the coordinate product* $x_\rho(t)\, y_\rho(t)$ *is compactly supported in* $t \in [-T,T]$*; equivalently,*

$$x_\rho(t)y_\rho(t) = 0 \quad \textit{for } |t| \geq T;$$

(c) Γ_ρ *depends smoothly on* ρ *and converges (Hausdorff on compacts) to the union of coordinate rays as* $\rho \to 0$.

Moreover, Γ_ρ *is symmetric under swapping the coordinates* $(x,y) \mapsto (y,x)$.

Proof. For $t \geq T$ we have $T - t \leq 0$, hence $\sigma(T-t) = 0$ and $y_\rho(t) = 0$, while $t + T \geq 2T > 1$ so $\sigma(t+T) = 1$ and $x_\rho(t) = \rho e^t \geq \rho e^T = 2\rho$. Thus $\gamma_\rho([T,\infty)) = \{(x,0) : x \geq 2\rho\}$. Similarly, for $t \leq -T$ we have $t + T \leq 0$, hence $x_\rho(t) = 0$ and $y_\rho(t) = \rho e^{-t} \geq 2\rho$, so $\gamma_\rho((-\infty,-T]) = \{(0,y) : y \geq 2\rho\}$. This proves (a). Statement (b) follows because for $|t| \geq T$ one of $\sigma(t+T)$ or $\sigma(T-t)$ is identically 0.

Smoothness at $t = \pm T$ follows from flatness of σ at 0, which forces all derivatives of the cut-off coordinate to vanish at the transition and hence yields a smooth gluing to the axis rays. Proper embeddedness and smooth dependence on ρ are immediate. Symmetry follows from $\gamma_\rho(-t) = (y_\rho(t), x_\rho(t))$. □

4. Local Lagrangian smoothing patches and edge holonomy

4.1. **Generic vertex patches (product structure).** Let v be a generic vertex. By Proposition 2.1(b), $\mathbb{R}^4$ splits symplectically as $V_1 \oplus V_2$ and the vertex cone is $C_1 \times C_2$ with $C_i \subset V_i$ a planar corner.

Choose linear symplectic identifications $V_i \cong (\mathbb{R}^2, dq \wedge dp)$ sending C_i to the standard positive coordinate corner. Define the *vertex smoothing patch*

$$P_{v,\rho} \ := \ \Gamma_\rho \times \Gamma_\rho \ \subset \ V_1 \oplus V_2 \cong \mathbb{R}^4.$$

Lemma 4.1. *$P_{v,\rho}$ is a smooth embedded Lagrangian surface. Moreover, $P_{v,\rho}$ agrees exactly with the cone C_0K outside the product neighborhood $(B_{2\rho} \subset V_1) \times (B_{2\rho} \subset V_2)$. In particular, along any incident edge-ray direction it equals a constant-squeeze cylinder (a ray times Γ_ρ) beyond distance 2ρ from the vertex.*

Proof. The tangent space of $\Gamma_\rho \times \Gamma_\rho$ splits as a direct sum of two 1–dimensional spaces, so $\omega_{\mathrm{st}} = \omega_{V_1} \oplus \omega_{V_2}$ vanishes on it. Agreement with the cone follows from Lemma 3.1(a) in each factor. The final sentence is the same observation applied to one factor at a time. □

4.2. **Edge normal form and diagonal squeezes.** Let e be an edge whose interior points lie in the 1–skeleton of K. Exactly two faces meet along e; let their affine spans be Lagrangian planes Λ^+, Λ^- intersecting along the line ℓ containing e. We call e *singular* if $\Lambda^+ \neq \Lambda^-$ (equivalently, K is not smooth along the interior of e).

Lemma 4.2 (Symplectic normal form along a singular edge)**.** *Let e be a singular edge with line ℓ and adjacent Lagrangian planes $\Lambda^\pm$. There exists an affine symplectic coordinate chart*

$$\Phi_e : (\mathbb{R}^4, \omega_{\mathrm{st}}) \to (\mathbb{R}^4, \omega_{\mathrm{st}}), \qquad (q_1,q_2,p_1,p_2) \textit{ coordinates},$$

sending ℓ to the q_1–axis $\{q_2 = p_1 = p_2 = 0\}$ and sending the two face planes to

$$\Phi_e(\Lambda^+) = \{p_1 = p_2 = 0\}, \qquad \Phi_e(\Lambda^-) = \{p_1 = q_2 = 0\}.$$

In these coordinates, the transverse symplectic plane is $W_e := \{q_1 = p_1 = 0\} \cong (\mathbb{R}^2, dq_2 \wedge dp_2)$, and the two transverse face rays are the positive q_2-axis and the positive p_2-axis.

Proof. Choose a nonzero $e_1 \in \ell$. Choose $e_2 \in \Lambda^+$ with $\Lambda^+ = \mathrm{span}(e_1,e_2)$. Choose $f_2 \in \Lambda^-$ with $\Lambda^- = \mathrm{span}(e_1,f_2)$ and scale so that $\omega_{\mathrm{st}}(e_2,f_2) = 1$. Complete (e_1,e_2,f_2) to a symplectic basis (e_1,f_1,e_2,f_2) and send it to the standard basis; add a translation. □

At each endpoint of a singular edge e, any local coordinate identification of W_e that sends the two transverse face rays to the *positive* coordinate rays differs from the edge chart identification by a diagonal symplectic squeeze.

Lemma 4.3 (Quadrant-preserving symplectic maps are diagonal squeezes)**.** *Let $A \in \mathrm{Sp}(2,\mathbb{R}) \cong \mathrm{SL}(2,\mathbb{R})$ preserve the set*

$$(\mathbb{R}_{\geq 0} e_1) \ \cup \ (\mathbb{R}_{\geq 0} e_2) \subset \mathbb{R}^2,$$

where $e_1 = (1, 0)$ *and* $e_2 = (0, 1)$. *Then*

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \textit{for some } \lambda > 0.$$

Proof. Since A is invertible, it permutes the two rays $\mathbb{R}_{\geq 0}e_1$ and $\mathbb{R}_{\geq 0}e_2$. If A swapped them, then $A(e_1) = ae_2$ and $A(e_2) = be_1$ with $a, b > 0$, hence $\det A = -ab < 0$, contradicting $\det A = 1$. Therefore A preserves each ray: $A(e_1) = \lambda e_1$ and $A(e_2) = \mu e_2$ with $\lambda, \mu > 0$. Since $\det A = \lambda\mu = 1$, we have $\mu = \lambda^{-1}$. □

4.3. **The interpolating Lagrangian edge patch (holonomy compensation).** Let e be a singular edge and fix an edge chart Φ_e as in Lemma 4.2. Let the q_1–coordinate range of the edge segment be $[0, L_e]$.

Fix $\rho > 0$ and parametrize $\Gamma_\rho \subset \mathbb{R}^2_{(q_2,p_2)}$ by $u \mapsto (x(u), y(u))$ so that $(x(u), y(u)) \in \Gamma_\rho$. By Lemma 3.1(b), the product $x(u)y(u)$ has compact support in u.

At each endpoint of e, the transverse identification induced by the local model sends Γ_ρ to a diagonally squeezed curve $(q_2, p_2) = (\lambda x, \lambda^{-1}y)$ for some $\lambda > 0$ (Lemma 4.3). Let $\lambda_- = \lambda_e(0)$ and $\lambda_+ = \lambda_e(L_e)$ denote the endpoint squeeze factors.

Choose a smooth function $\lambda_e : [0, L_e] \to (0, \infty)$ such that:

(a) $\lambda_e(s) = \lambda_-$ for s near 0 and $\lambda_e(s) = \lambda_+$ for s near L_e (hence $\lambda_e'(s) = 0$ near endpoints);
(b) λ_e is bounded above and below by positive constants depending only on the finite polyhedron K.

Definition 4.4 (Interpolating edge patch)**.** Define a map $F_{e,\rho} : [0, L_e] \times \mathbb{R} \to \mathbb{R}^4_{(q_1,q_2,p_1,p_2)}$ by

$$\begin{aligned} q_1(s,u) &:= s, \\ q_2(s,u) &:= \lambda_e(s)\, x(u), \\ p_2(s,u) &:= \lambda_e(s)^{-1}\, y(u), \\ p_1(s,u) &:= -\frac{\lambda_e'(s)}{\lambda_e(s)}\, x(u)\, y(u). \end{aligned}$$

Let $P_{e,\rho} := \Phi_e^{-1}\big(F_{e,\rho}([0, L_e] \times \mathbb{R})\big) \subset \mathbb{R}^4$.

Lemma 4.5 (Lagrangian property and exact matching)**.** $P_{e,\rho}$ *is a smooth embedded Lagrangian surface. Moreover:*

(a) $p_1(s,u) \equiv 0$ *for* $|u|$ *sufficiently large, hence* $P_{e,\rho}$ *agrees* exactly *with the original polyhedral faces away from a transverse neighborhood of radius* $O(\rho)$*;*
(b) since $\lambda_e' \equiv 0$ *near* $s = 0$ *and* $s = L_e$*, we have* $p_1 \equiv 0$ *near the endpoints and the transverse profiles there are exactly the endpoint-squeezed curves* $(q_2, p_2) = (\lambda_\pm x(u), \lambda_\pm^{-1} y(u))$.

Proof. Smoothness and embeddedness are immediate since $q_1 = s$ separates s–levels and all functions are smooth; properness in u follows from properness of Γ_ρ.

To check Lagrangian, compute in the edge chart:

$$\omega_{\mathrm{st}} = dq_1 \wedge dp_1 + dq_2 \wedge dp_2.$$

We have $dq_1 = ds$, and

$$dp_1 = -\left(\frac{\lambda_e'}{\lambda_e}\right)' xy\, ds - \frac{\lambda_e'}{\lambda_e}\,(x'y + y'x)\, du,$$

where primes on x, y denote derivatives in u. Hence

$$dq_1 \wedge dp_1 = -\frac{\lambda_e'}{\lambda_e}\,(x'y + y'x)\, ds \wedge du = -\frac{\lambda_e'}{\lambda_e}\,\frac{d}{du}(xy)\, ds \wedge du.$$

Next,

$$dq_2 = \lambda_e' x\, ds + \lambda_e x'\, du, \qquad dp_2 = -\lambda_e^{-2}\lambda_e' y\, ds + \lambda_e^{-1} y'\, du,$$

so

$$\begin{aligned} dq_2 \wedge dp_2 &= (\lambda_e' x\, ds + \lambda_e x'\, du) \wedge (-\lambda_e^{-2}\lambda_e' y\, ds + \lambda_e^{-1} y'\, du) \\ &= \big(\lambda_e'\lambda_e^{-1} xy' + \lambda_e'\lambda_e^{-1} x'y\big)\, ds \wedge du \\ &= \frac{\lambda_e'}{\lambda_e}\,\frac{d}{du}(xy)\, ds \wedge du. \end{aligned}$$

Therefore $dq_1 \wedge dp_1 + dq_2 \wedge dp_2 = 0$, so $F_{e,\rho}^* \omega_{\mathrm{st}} = 0$ and $P_{e,\rho}$ is Lagrangian.

For (a), Lemma 3.1(b) gives $xy = 0$ for $|u|$ large, hence $p_1 = 0$ there and (q_2, p_2) lies on one of the coordinate rays; this is precisely the unmodified face model. Part (b) is immediate from $\lambda_e' \equiv 0$ near endpoints. □

Remark 4.6. The diagonal squeeze factors $\lambda_\pm$ need not be 1 and can have nontrivial holonomy along cycles in the 1–skeleton. The correction term $p_1 = -(\lambda_e'/\lambda_e)\, xy$ compensates the transverse symplectic error created by varying the squeeze; compact support of xy ensures exact matching to faces and endpoint models.

5. Global Lagrangian smoothing

Let $\mathcal{V}_{\mathrm{gen}}$ be the set of generic vertices (Proposition 2.1(b)) and let $\mathcal{E}_{\mathrm{sing}}$ be the set of singular edges (those whose adjacent face planes are distinct).

5.1. **Choice of control scale.** Because K is a finite embedded polyhedral complex, there is a positive separation between disjoint closed cells. Fix $\delta > 0$ sufficiently small so that:

(a) closed balls $\overline{B}_{10\delta}(v)$ around distinct vertices are disjoint;
(b) tubular neighborhoods of radius 10δ around disjoint singular edges are disjoint, and any overlaps occur only in balls around shared endpoints;
(c) δ is smaller than half the minimum distance between disjoint closed cells of the complex (so local modifications at scale $\leq 10\delta$ cannot create new intersections);
(d) δ is sufficiently small relative to the fixed geometric constants of the local models so that the Hamiltonian correction of Section 6 fits inside the Weinstein neighborhoods for all $t \in (0,1]$ (see Definition 6.7).

Define the smoothing scale
$$\rho(t) := \delta t, \qquad t \in (0,1].$$

5.2. **Definition of the Lagrangian smoothed surfaces L_t.** We define L_t by modifying K inside disjoint control neighborhoods around $\mathcal{V}_{\mathrm{gen}} \cup \mathcal{E}_{\mathrm{sing}}$.

(i) Generic vertices. For each $v \in \mathcal{V}_{\mathrm{gen}}$, choose a symplectic affine chart Φ_v identifying a neighborhood of v with a neighborhood of 0 in the product splitting $V_1 \oplus V_2$ of Proposition 2.1(b), and set
$$L_t \cap B_{5\rho(t)}(v) \;:=\; \Phi_v^{-1}\big(P_{v,\rho(t)}\big) \cap B_{5\rho(t)}(v),$$
where $P_{v,\rho}$ is the vertex patch from Lemma 4.1.

(ii) Singular edges. Let $e \in \mathcal{E}_{\mathrm{sing}}$ have endpoints v_-, v_+. Define the truncated edge segment
$$e_t := e \setminus \bigcup_{v \in \{v_-, v_+\} \cap \mathcal{V}_{\mathrm{gen}}} B_{3\rho(t)}(v). \tag{2}$$
(Thus we remove neighborhoods only around generic endpoints.) Choose an edge chart Φ_e as in Lemma 4.2 on a neighborhood of e and define L_t inside the tube $N_{5\rho(t)}(e_t)$ by the interpolating edge patch $P_{e,\rho(t)}$ from Definition 4.4, where λ_e is chosen so that:

(a) if an endpoint $v \in \mathcal{V}_{\mathrm{gen}}$ is truncated, then on the overlap with $B_{5\rho(t)}(v)$ the patch matches the constant-squeeze cylinder determined by the vertex patch (Lemma 4.1);
(b) if an endpoint is fold, the local geometry is edge-type (Proposition 2.1(c)), and the adjacent singular edges are collinear; we choose the edge charts consistently near such a fold point so that the corresponding edge patches glue smoothly there (this is possible because the two incident face planes on either side of the singular line agree).

(iii) Away from the singular locus. Set $L_t = K$ outside the union of the above vertex balls and edge tubes.

Proposition 5.1. *For each $t \in (0,1]$, L_t is a smooth embedded Lagrangian surface. The family $t \mapsto L_t$ is smooth for $t \in (0,1]$ and extends to a topological isotopy on $[0,1]$ with $L_0 = K$.*

Proof. Each local patch is smooth and Lagrangian (Lemmas 4.1 and 4.5). On overlaps between vertex neighborhoods and edge tubes, the vertex patch has stabilized to a constant-squeeze cylinder with $p_1 = 0$ beyond radius $2\rho(t)$, while the edge patch has $\lambda_e' \equiv 0$ near endpoints and uses the same squeezed transverse profile; hence the definitions agree by subset equality and glue smoothly. At fold points, the local model is edge-type, and consistent choice of edge charts ensures the edge patches meet smoothly.

Embeddedness follows from the choice of δ: all modifications remain in disjoint control neighborhoods and cannot create new intersections. Smooth dependence on t follows from smooth dependence of $\Gamma_{\rho(t)}$

and λ_e on $\rho(t)$ and from the finiteness of the construction. As $t \to 0$, the modified regions shrink to the singular locus and L_t converges to K in Hausdorff distance; collapsing the rounded pieces gives a continuous isotopy of embeddings, hence the topological extension with $L_0 = K$. □

6. Hamiltonian normalization

6.1. **Liouville class and Hamiltonian criterion.** For a smooth embedded Lagrangian surface $L \subset (\mathbb{R}^4, \omega_{\mathrm{st}})$ define its *Liouville class*

$$\mathfrak{a}(L) := [\lambda_{\mathrm{st}}|_L] \in H^1(L;\mathbb{R}).$$

Lemma 6.1 (Flux criterion in an exact symplectic manifold)**.** *Let $\iota_t : \Sigma \to (\mathbb{R}^4, \omega_{\mathrm{st}})$ be a smooth family of embeddings for $t \in (0,1]$ such that $L_t := \iota_t(\Sigma)$ is Lagrangian for all t. Set*

$$a_t := [\iota_t^* \lambda_{\mathrm{st}}] \in H^1(\Sigma;\mathbb{R}).$$

Then the isotopy L_t is induced by a compactly supported Hamiltonian isotopy of $\mathbb{R}^4$ if and only if a_t is constant in t.

Proof. Differentiate:

$$\frac{d}{dt}\,\iota_t^*\lambda_{\mathrm{st}} = \iota_t^*(\mathcal{L}_{\partial_t \iota_t}\lambda_{\mathrm{st}}) = d(\cdots) + \iota_t^*(\iota_{\partial_t\iota_t}\omega_{\mathrm{st}}).$$

Thus $\frac{d}{dt}a_t = [\,\iota_t^*(\iota_{\partial_t\iota_t}\omega_{\mathrm{st}})\,]$. If a_t is constant, the class vanishes, hence the closed form $\iota_t^*(\iota_{\partial_t\iota_t}\omega_{\mathrm{st}})$ is exact and can be extended to a compactly supported Hamiltonian producing the isotopy (using a Weinstein neighborhood and a cutoff). Conversely, Hamiltonian isotopy preserves $[\lambda_{\mathrm{st}}|_{L_t}]$. □

6.2. **Quadratic variation of periods.** Let L_t be as in Proposition 5.1. Fix smooth embeddings $\iota_t : \Sigma \to \mathbb{R}^4$ with image L_t for $t > 0$, varying smoothly in t (possible since L_t is a smooth isotopy for $t > 0$).

Lemma 6.2 (Cauchy estimate for periods)**.** *There exists $C > 0$ such that for all $0 < s < t \le 1$ and every smooth loop $\gamma \subset \Sigma$ transverse to the preimage of the singular locus of K, one has*

$$\left| \int_\gamma \iota_t^*\lambda_{\mathrm{st}} - \int_\gamma \iota_s^*\lambda_{\mathrm{st}} \right| \;\le\; C\,(\rho(t)^2 - \rho(s)^2).$$

In particular, the classes $a_t := [\iota_t^*\lambda_{\mathrm{st}}] \in H^1(\Sigma;\mathbb{R})$ converge as $t \to 0$ to a limit a_0, with $a_0 - a_t = O(\rho(t)^2)$.

Proof. Consider the annulus $A = [s,t] \times S^1$ mapped by $F(\tau, u) = \iota_\tau(\gamma(u))$. By Stokes,

$$\int_\gamma \iota_t^*\lambda_{\mathrm{st}} - \int_\gamma \iota_s^*\lambda_{\mathrm{st}} = \int_A F^*\omega_{\mathrm{st}}.$$

Thus the difference is bounded by $\|\omega_{\mathrm{st}}\|$ times the Euclidean area swept by F.

By construction, ι_τ is stationary outside the union of control neighborhoods around the singular locus, of transverse radius $O(\rho(\tau))$. Transversality of γ implies it meets these neighborhoods in finitely many arcs, each of length $O(\rho(\tau))$, uniformly in τ. The deformation speed $|\partial_\tau \iota_\tau|$ is $O(\rho'(\tau)) = O(\delta)$ since the local models scale linearly with ρ. Hence the swept area is $O(\rho(\tau)\rho'(\tau))\,d\tau$, and summing over finitely many neighborhoods yields

$$\mathrm{Area}(F(A)) \le C_1 \int_s^t \rho(\tau)\rho'(\tau)\,d\tau = C_2(\rho(t)^2 - \rho(s)^2),$$

proving the estimate. Convergence of a_t follows by evaluating on a homology basis represented by such transverse loops (general position). □

6.3. **Ambient cohomology basis and a uniformly small correcting form.** Let $U \subset \mathbb{R}^4$ be a sufficiently small open neighborhood of K which deformation retracts onto K (a regular neighborhood in the PL sense). Then $H^1(U;\mathbb{R}) \cong H^1(K;\mathbb{R})$.

Lemma 6.3 (Ambient closed forms dual to loops)**.** *There exist smooth closed 1–forms $\beta_1, \dots, \beta_b \in \Omega^1(U)$ whose classes form a basis of $H^1(U;\mathbb{R})$ and loops $\gamma_1, \dots, \gamma_b \subset K$ representing a basis of $H_1(K;\mathbb{Z})/\mathrm{tors}$ such that*

$$\int_{\gamma_j} \beta_i = \delta_{ij}.$$

Proof. Choose any basis of $H^1(U;\mathbb{R})$ and represent it by smooth closed forms by de Rham. Choose loops γ_j representing a basis of $H_1(K;\mathbb{Z})/\mathrm{tors}$. The de Rham pairing gives a perfect duality between $H^1(U;\mathbb{R})$ and $H_1(U;\mathbb{R})$; replace the chosen forms by the dual basis to achieve the Kronecker normalization. □

For $t > 0$ small we have $L_t \subset U$ and the topological isotopy transports each loop $\gamma_j \subset K$ to a loop $\gamma_j(t) \subset L_t$ which is isotopic to γ_j *inside* U. Define real numbers

$$A_j(t) := \int_{\gamma_j(t)} \lambda_{\mathrm{st}}, \qquad A_j(0) := \lim_{t\to 0} A_j(t),$$

where the limit exists by Lemma 6.2. Set $c_j(t) := A_j(0) - A_j(t)$.

Lemma 6.4 (Closed-form correction with ambient C^0 control)**.** *Define*

$$\alpha_t := \sum_{j=1}^{b} c_j(t)\,\beta_j\Big|_{L_t} \in \Omega^1(L_t).$$

Then α_t is closed, $[\lambda_{\mathrm{st}}|_{L_t}] + [\alpha_t]$ is independent of t, and there is a constant C_α (independent of δ) such that

$$\|\alpha_t\|_{C^0(L_t)} \le C_\alpha\,\rho(t)^2,$$

where the norm is computed using the ambient Euclidean metric on $\mathbb{R}^4$.

Proof. Closedness follows from $d\beta_j = 0$.

Since $\gamma_j(t)$ is isotopic to γ_j inside U and β_i is closed on U, Stokes' theorem implies the period is *constant*:

$$\int_{\gamma_j(t)} \beta_i = \int_{\gamma_j} \beta_i = \delta_{ij} \qquad \text{for all } t \in (0,1].$$

Therefore, for each j,

$$\int_{\gamma_j(t)} \alpha_t = \sum_{i=1}^{b} c_i(t) \int_{\gamma_j(t)} \beta_i = c_j(t) = A_j(0) - A_j(t).$$

Equivalently, the class $[\alpha_t] \in H^1(L_t;\mathbb{R})$ is exactly the class difference between the limiting Liouville periods and the current ones, so $[\lambda_{\mathrm{st}}|_{L_t}] + [\alpha_t]$ has constant periods on the transported basis loops and hence is independent of t.

By Lemma 6.2, $|c_j(t)| = |A_j(0) - A_j(t)| = O(\rho(t)^2)$ with a constant independent of δ (it depends only on the scale-1 local models and finitely many charts). Since β_j are fixed smooth forms on the relatively compact set U, $\|\beta_j\|_{C^0(U)} < \infty$. Thus

$$\|\alpha_t\|_{C^0(L_t)} \le \sum_{j=1}^{b} |c_j(t)|\,\|\beta_j\|_{C^0(U)} \le C_\alpha\,\rho(t)^2.$$

□

6.4. **Quantitative Weinstein neighborhood and global definition of K_t.** We require a Weinstein neighborhood large enough to contain the graph of α_t for all $t \in (0,1]$.

Lemma 6.5 (Weinstein radius $\gtrsim \rho(t)$, uniformly in t)**.** *There exists a constant $c_{\mathrm{W}} > 0$ (independent of δ) such that for every $t \in (0,1]$ the Lagrangian surface L_t admits:*

(a) an embedded Euclidean tubular neighborhood of radius at least $c_{\mathrm{W}}\rho(t)$;

(b) a Weinstein neighborhood symplectomorphism

$$\Psi_t : \big(D^*_{c_{\mathrm{W}}\rho(t)} L_t,\, d\theta\big) \;\longrightarrow\; \big(\mathbb{R}^4, \omega_{\mathrm{st}}\big)$$

defined on the cotangent disk bundle of radius $c_{\mathrm{W}}\rho(t)$ (with the Euclidean dual norm), mapping the zero section to L_t. Moreover, Ψ_t can be chosen smoothly in t on $(0,1]$ in the sense that the induced map on the total space

$$\big\{(t,x,\xi) : t \in (0,1],\, x \in L_t,\, |\xi| < c_{\mathrm{W}}\rho(t)\big\} \to \mathbb{R}^4$$

is smooth.

Proof. On each face region L_t is planar, hence has zero second fundamental form. Inside each smoothing neighborhood, L_t is obtained by inserting one of finitely many smooth template patches whose geometry is fixed up to scaling by $\rho(t)$ in transverse directions; therefore the norm of the second fundamental form is bounded by $C/\rho(t)$ for a constant C depending only on the scale-1 templates. Consequently, the principal curvatures are bounded by $C/\rho(t)$.

For a smooth embedded submanifold of Euclidean space with principal curvatures bounded by κ, the normal exponential map is non-singular on normal disks of radius $< 1/\kappa$; hence a tubular neighborhood exists of radius $\geq c_1/\kappa$. Applying this gives a tubular radius $\geq c_2\rho(t)$. The choice of δ in Section 5 ensures that $c_2\rho(t) \leq c_2\delta$ is smaller than the separation scale between disjoint cell neighborhoods, so the tube is globally embedded. This proves (a) with $c_W \leq c_2$.

Given such a tubular neighborhood, the standard proof of the Weinstein neighborhood theorem identifies T^*L_t with the symplectic normal bundle using ω_{st}, composes with the tubular embedding into $\mathbb{R}^4$, and then applies Moser's method on a sufficiently small disk bundle to correct the pulled-back symplectic form to $d\theta$. Since all bounds and domains are uniform after scaling by $\rho(t)$ and only finitely many local templates are used, this yields a symplectomorphism defined at least on a disk bundle of radius $c_W\rho(t)$, with c_W independent of δ. The parameter dependence is smooth on $(0, 1]$ by the parameter-dependent Moser argument. □

Lemma 6.6 (Liouville class shift on graphs)**.** *Let $L \subset (\mathbb{R}^4, \omega_{\mathrm{st}} = d\lambda_{\mathrm{st}})$ be a smooth embedded Lagrangian and let $\Psi : (V, d\theta) \to (\mathbb{R}^4, \omega_{\mathrm{st}})$ be a Weinstein neighborhood symplectomorphism, with Ψ restricting to the inclusion of the zero section. Then for any sufficiently small closed* 1*–form α on L with* $\mathrm{graph}(\alpha) \subset V$*, the Lagrangian $L_\alpha := \Psi(\mathrm{graph}(\alpha))$ satisfies*

$$[\lambda_{\mathrm{st}}|_{L_\alpha}] = [\lambda_{\mathrm{st}}|_L] + [\alpha] \in H^1(L; \mathbb{R}).$$

Proof. On T^*L the canonical 1–form θ restricts to α on $\mathrm{graph}(\alpha)$. The closed form $\Psi^*\lambda_{\mathrm{st}} - (\theta + \pi^*(\lambda_{\mathrm{st}}|_L))$ vanishes on the zero section and is exact on V (since V deformation retracts to the zero section). Restricting to $\mathrm{graph}(\alpha)$ gives the stated cohomology identity. □

Definition 6.7 (Hamiltonian-normalized smoothing family)**.** Let C_α be as in Lemma 6.4 and c_W as in Lemma 6.5. These constants depend only on the fixed scale-1 local models and the fixed ambient forms β_j, and are independent of δ. By the choice of δ in Section 5(d), we may assume

$$C_\alpha\, \delta < \tfrac{1}{2} c_W. \tag{3}$$

Then for all $t \in (0, 1]$,

$$\|\alpha_t\|_{C^0(L_t)} \leq C_\alpha \rho(t)^2 = C_\alpha \delta^2 t^2 \leq C_\alpha \delta\, \rho(t) < \tfrac{1}{2} c_W \rho(t),$$

so $\mathrm{graph}(\alpha_t) \subset D^*_{c_W\rho(t)} L_t$.

Define

$$K_t := \Psi_t(\mathrm{graph}(\alpha_t)) \subset \mathbb{R}^4, \qquad t \in (0, 1],$$

with Ψ_t from Lemma 6.5.

Proposition 6.8 (Constant Liouville class)**.** *The family K_t has constant Liouville class:*

$$[\lambda_{\mathrm{st}}|_{K_t}] \textit{ is independent of } t \in (0, 1].$$

Proof. By Lemma 6.6,

$$[\lambda_{\mathrm{st}}|_{K_t}] = [\lambda_{\mathrm{st}}|_{L_t}] + [\alpha_t].$$

By Lemma 6.4, the right-hand side is independent of t. □

Theorem 6.9 (Completion of Theorem 1.4)**.** *The family $\{K_t\}_{t \in (0,1]}$ of Definition 6.7 is a Hamiltonian isotopy of smooth embedded Lagrangian surfaces. Moreover, it extends to a topological isotopy on* $[0, 1]$ *with $K_0 = K$.*

Proof. By Lemma 6.5 and smooth dependence of α_t, the family K_t is a smooth Lagrangian isotopy on $(0, 1]$. Proposition 6.8 shows its Liouville class is constant, hence Lemma 6.1 implies the isotopy is Hamiltonian.

As $t \to 0$, $L_t \to K$ by Proposition 5.1, and $\|\alpha_t\|_{C^0} = O(\rho(t)^2) \to 0$, so K_t remains within $o(\rho(t))$ of L_t and also converges to K; thus the topological isotopy extends with $K_0 = K$. □

Proof of Theorem 1.4. Combine Proposition 5.1 and Theorem 6.9. □

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