UNIVERSAL DETERMINANTAL TESTS FOR SEPARABLE BLOCK SCALINGS OF STACKED 4-VIEW DETERMINANT TENSORS

ABSTRACT. Fix $n \geq 5$ and let $A^{(1)}, \ldots, A^{(n)} \in \mathbb{R}^{3\times 4}$ be Zariski-generic. For each quadruple $(\alpha, \beta, \gamma, \delta) \in [n]^4$ we form a tensor $Q^{(\alpha\beta\gamma\delta)} \in \mathbb{R}^{3\times 3\times 3\times 3}$ whose entries are 4×4 determinants of selected rows of $A^{(\alpha)}, A^{(\beta)}, A^{(\gamma)}, A^{(\delta)}$. Given a blockwise scaling tensor $\lambda \in \mathbb{R}^{n\times n\times n\times n}$ that vanishes precisely on the diagonal $\{(m, m, m, m)\}$, we construct a *universal* family of polynomial relations, with degree bounded independently of n, that vanish on the scaled family $\big(\lambda_{\alpha\beta\gamma\delta} Q^{(\alpha\beta\gamma\delta)}\big)$ if and only if λ factors as a pure tensor $u \otimes v \otimes w \otimes x$ on the off-diagonal. Concretely, we stack the blocks into a $(3n)^4$-tensor $\mathcal{Z}$ and take $\mathbf{F}_n$ to be the vector of all 5×5 minors of the four standard mode-flattenings of $\mathcal{Z}$. Each coordinate has degree 5 (independent of n), and for Zariski-generic cameras $\mathbf{F}_n(\mathcal{Z}) = 0$ holds if and only if $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$ for all $(\alpha, \beta, \gamma, \delta) \neq (m, m, m, m)$.

1. Introduction

Let $n \geq 5$. For each $\alpha \in [n] := \{1, \ldots, n\}$, let $A^{(\alpha)} \in \mathbb{R}^{3\times 4}$. For $\alpha, \beta, \gamma, \delta \in [n]$, define $Q^{(\alpha\beta\gamma\delta)} \in \mathbb{R}^{3\times 3\times 3\times 3}$ by

$$Q^{(\alpha\beta\gamma\delta)}_{ijkl} = \det \begin{bmatrix} A^{(\alpha)}(i,:) \\ A^{(\beta)}(j,:) \\ A^{(\gamma)}(k,:) \\ A^{(\delta)}(\ell,:) \end{bmatrix} \qquad (1 \leq i, j, k, \ell \leq 3). \tag{1}$$

Write the ith row of $A^{(\alpha)}$ as a row vector $a_{\alpha i} \in \mathbb{R}^4$.

We are interested in universal algebraic relations on the family $\{Q^{(\alpha\beta\gamma\delta)}\}$ under blockwise scaling. Let $\lambda \in \mathbb{R}^{n\times n\times n\times n}$ satisfy

$$\lambda_{\alpha\beta\gamma\delta} \neq 0 \iff (\alpha, \beta, \gamma, \delta) \neq (m, m, m, m) \ \text{ for all } m \in [n]. \tag{2}$$

Define $Z^{(\alpha\beta\gamma\delta)} := \lambda_{\alpha\beta\gamma\delta} Q^{(\alpha\beta\gamma\delta)}$. The diagonal blocks $Q^{(mmmm)}$ vanish identically:

$$Q^{(mmmm)} \equiv 0 \qquad \text{for all } m \in [n], \tag{3}$$

since any 4×4 matrix made from four rows of a 3×4 matrix repeats a row. Thus the values λ_{mmmm} do not affect the scaled data, and the factorization condition is meaningful only off the diagonal.

Problem. Does there exist a polynomial map $\mathbf{F} : \mathbb{R}^{81n^4} \to \mathbb{R}^N$ (for some N) such that:

1. $\mathbf{F}$ does not depend on $A^{(1)}, \ldots, A^{(n)}$;
2. the degrees of the coordinate functions of $\mathbf{F}$ do not depend on n;
3. for λ satisfying (2),

$$\mathbf{F}\big(\lambda_{\alpha\beta\gamma\delta} Q^{(\alpha\beta\gamma\delta)}\big) = 0 \iff \exists\, u, v, w, x \in (\mathbb{R}^*)^n : \ \lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta \ \text{ for all off-diagonal } (\alpha, \beta, \gamma, \delta).$$

We answer this in the affirmative, by a bounded-degree determinantal test on the mode-flattenings of an explicitly constructed stacked tensor.

2. Stacking and flattenings

2.1. **Stacked tensor.** Let $I := [n] \times \{1, 2, 3\}$, so $|I| = 3n$. Given tensors $T^{(\alpha\beta\gamma\delta)} \in \mathbb{R}^{3\times 3\times 3\times 3}$, define the stacked tensor $\mathcal{T} \in \mathbb{R}^{I\times I\times I\times I} \cong \mathbb{R}^{(3n)^4}$ by

$$\mathcal{T}_{(\alpha,i),(\beta,j),(\gamma,k),(\delta,\ell)} := T^{(\alpha\beta\gamma\delta)}_{ijkl}. \tag{4}$$

Stacking $\{Q^{(\alpha\beta\gamma\delta)}\}$ gives $\mathcal{Q}$, and stacking $\{Z^{(\alpha\beta\gamma\delta)}\}$ gives $\mathcal{Z}$.

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2.2. **Mode-flattenings and determinantal rank test.** For $r \in \{1,2,3,4\}$, let $M_r(\mathcal{T})$ denote the standard mode-r flattening of $\mathcal{T}$, i.e. the matrix obtained by using the rth index of $\mathcal{T}$ as the row index and concatenating the other three indices as the column index. Thus $M_r(\mathcal{T}) \in \mathbb{R}^{(3n)\times(3n)^3}$.

We use the basic determinantal criterion:

Lemma 2.1. *Let B be a matrix over a field. Then* $\operatorname{rank}(B) \le 4$ *if and only if every* 5×5 *minor of B vanishes.*

Proof. A matrix has rank at least 5 if and only if it contains a nonsingular 5×5 submatrix. □

3. The polynomial map $\mathbf{F}_n$

For fixed n, define $\mathbf{F}_n : \mathbb{R}^{(3n)^4} \cong \mathbb{R}^{81n^4} \to \mathbb{R}^{N_n}$ to be the polynomial map whose coordinates are *all* 5×5 minors of each of the four flattenings $M_1(\mathcal{T}), M_2(\mathcal{T}), M_3(\mathcal{T}), M_4(\mathcal{T})$ of the input tensor $\mathcal{T}$. (Here N_n is the total number of such minors; its value is irrelevant.)

Proposition 3.1. *For each n, the map $\mathbf{F}_n$ does not depend on $A^{(1)},\dots,A^{(n)}$, and each coordinate of $\mathbf{F}_n$ is homogeneous of degree 5. In particular, the degree bound is independent of n.*

Proof. Each coordinate is a 5×5 determinant in the entries of a flattening of the input tensor, hence is a homogeneous polynomial of degree 5 with coefficients in $\{0,\pm 1\}$. □

4. A Hodge-star factorization

4.1. **Definition of the $\star$-map.** Fix the standard dot product on $\mathbb{R}^4$. For any fixed $b,c,d \in \mathbb{R}^4$, the map $a \mapsto \det[a;b;c;d]$ is linear in a, hence there is a unique vector $\star(b\wedge c\wedge d) \in \mathbb{R}^4$ such that

$$\det\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = a\cdot \star(b\wedge c\wedge d) \qquad \text{for all } a \in \mathbb{R}^4. \tag{5}$$

This defines a linear map $\star : \Lambda^3(\mathbb{R}^4) \to \mathbb{R}^4$.

4.2. **Block matrices $W_{\beta\gamma\delta}$.** For each triple $(\beta,\gamma,\delta) \in [n]^3$, define the matrix

$$W_{\beta\gamma\delta} \in \mathbb{R}^{4\times 27}$$

whose columns are the vectors $\star(a_{\beta j}\wedge a_{\gamma k}\wedge a_{\delta\ell})$ for all $(j,k,\ell) \in \{1,2,3\}^3$ in some fixed order.

Lemma 4.1 (Block formula)**.** *Let $\mathcal{Q}$ be the stacked tensor of $\{Q^{(\alpha\beta\gamma\delta)}\}$. For each $(\alpha;\beta,\gamma,\delta)$ the (3×27) block of $M_1(\mathcal{Q})$ indexed by row-camera α and column-triple (β,γ,δ) equals*

$$\big(M_1(\mathcal{Q})\big)_{\alpha;\beta\gamma\delta} = A^{(\alpha)}\, W_{\beta\gamma\delta}. \tag{6}$$

Consequently, for $\mathcal{Z} = \lambda \odot \mathcal{Q}$,

$$\big(M_1(\mathcal{Z})\big)_{\alpha;\beta\gamma\delta} = \lambda_{\alpha\beta\gamma\delta}\, A^{(\alpha)}\, W_{\beta\gamma\delta}. \tag{7}$$

Proof. Fix $\alpha,\beta,\gamma,\delta$. The entry in row i and column (j,k,ℓ) of $A^{(\alpha)}W_{\beta\gamma\delta}$ equals $a_{\alpha i}\cdot\star(a_{\beta j}\wedge a_{\gamma k}\wedge a_{\delta\ell})$, which is $\det[a_{\alpha i}; a_{\beta j}; a_{\gamma k}; a_{\delta\ell}]$ by (5), i.e. $Q^{(\alpha\beta\gamma\delta)}_{ijkl}$. Stacking yields (6). Scaling yields (7). □

Corollary 4.2. *For any cameras $A^{(1)},\dots,A^{(n)}$,* $\operatorname{rank}(M_1(\mathcal{Q})) \le 4$. *By symmetry,* $\operatorname{rank}(M_r(\mathcal{Q})) \le 4$ *for $r = 1,2,3,4$.*

Proof. Equation (6) gives the factorization

$$M_1(\mathcal{Q}) = \underbrace{\begin{bmatrix} A^{(1)} \\ \vdots \\ A^{(n)} \end{bmatrix}}_{3n\times 4} \cdot \underbrace{\begin{bmatrix} W_{\beta\gamma\delta} \end{bmatrix}_{(\beta,\gamma,\delta)\in[n]^3}}_{4\times 27n^3},$$

hence $\operatorname{rank}(M_1(\mathcal{Q})) \leq 4$. The other modes follow by permuting the four indices in (1). □

5. Genericity: kernel vectors and ranks of $W_{\beta\gamma\delta}$

5.1. Zariski-genericity.

Definition 5.1. A property holds for *Zariski-generic* $(A^{(1)}, \dots, A^{(n)}) \in (\mathbb{R}^{3\times 4})^n$ if it holds on the complement of a proper Zariski-closed subset of the affine space $(\mathbb{R}^{3\times 4})^n \cong \mathbb{R}^{12n}$.

5.2. Kernel vectors.

Define

$$z_\alpha := \star(a_{\alpha 1} \wedge a_{\alpha 2} \wedge a_{\alpha 3}) \in \mathbb{R}^4. \tag{8}$$

Then $A^{(\alpha)} z_\alpha = 0$ by (5). If $\operatorname{rank}(A^{(\alpha)}) = 3$ then $\ker(A^{(\alpha)}) = \operatorname{span}\{z_\alpha\}$.

Lemma 5.2. *There exists a nonempty Zariski-open subset $U_n^{\ker} \subset (\mathbb{R}^{3\times 4})^n$ such that for all $(A^{(1)}, \dots, A^{(n)}) \in U_n^{\ker}$:*

(1) $\operatorname{rank}(A^{(\alpha)}) = 3$ *and* $\ker(A^{(\alpha)}) = \operatorname{span}\{z_\alpha\}$ *for all α;*
(2) *for $\alpha \neq \alpha'$, the kernel lines* $\operatorname{span}\{z_\alpha\}$ *and* $\operatorname{span}\{z_{\alpha'}\}$ *are distinct;*
(3) *for any three distinct indices $\alpha_1, \alpha_2, \alpha_3$, the vectors $z_{\alpha_1}, z_{\alpha_2}, z_{\alpha_3}$ are linearly independent.*

Proof. Each item defines a Zariski-open condition (nonvanishing of appropriate minors). Nonemptiness is witnessed by the moment-curve family: choose distinct $t_1, \dots, t_n \in \mathbb{R}$ and set

$$A(t) := \begin{bmatrix} -t & 1 & 0 & 0 \\ -t^2 & 0 & 1 & 0 \\ -t^3 & 0 & 0 & 1 \end{bmatrix}.$$

Then $\operatorname{rank}(A(t)) = 3$ and $\ker(A(t)) = \operatorname{span}\{(1, t, t^2, t^3)^T\}$, and any three such vectors are independent for distinct parameters (Vandermonde). □

5.3. Ranks of $W_{\beta\gamma\delta}$.

Lemma 5.3. *Fix $(\beta, \gamma, \delta) \in [n]^3$.*

(1) *If $\beta = \gamma = \delta = m$, then* $\operatorname{rank}(W_{mmm}) = 1$ *and* $\operatorname{colsp}(W_{mmm}) = \operatorname{span}\{z_m\}$.
(2) *If (β, γ, δ) are not all equal, then the condition* $\operatorname{rank}(W_{\beta\gamma\delta}) = 4$ *is Zariski-open and nonempty (hence holds for Zariski-generic cameras).*

Proof. (1) If $\beta = \gamma = \delta = m$, then each column of W_{mmm} is either 0 (when two of j, k, ℓ coincide) or $\pm\star(a_{m1} \wedge a_{m2} \wedge a_{m3}) = \pm z_m$ (when (j, k, ℓ) is a permutation of $(1, 2, 3)$). Thus $\operatorname{colsp}(W_{mmm}) = \operatorname{span}\{z_m\}$ and $\operatorname{rank}(W_{mmm}) = 1$ provided $\operatorname{rank}(A^{(m)}) = 3$.

(2) The condition $\operatorname{rank}(W_{\beta\gamma\delta}) = 4$ is the nonvanishing of some 4×4 minor of the 4×27 matrix $W_{\beta\gamma\delta}$, hence is Zariski-open. To show it is nonempty, it suffices to exhibit one assignment of the relevant matrices for which $\operatorname{rank}(W_{\beta\gamma\delta}) = 4$. There are two essential patterns up to permuting β, γ, δ.

Distinct cameras. Take three matrices

$$A^{(\beta)} = \begin{bmatrix} e_1^T \\ e_2^T \\ e_3^T \end{bmatrix}, \quad A^{(\gamma)} = \begin{bmatrix} e_1^T \\ e_2^T \\ e_4^T \end{bmatrix}, \quad A^{(\delta)} = \begin{bmatrix} e_1^T \\ e_3^T \\ e_4^T \end{bmatrix},$$

where $e_1, \dots, e_4$ is the standard basis of $\mathbb{R}^4$. Then among the columns of $W_{\beta\gamma\delta}$ appear the four vectors

$$\star(e_1 \wedge e_2 \wedge e_3) = -e_4, \qquad \star(e_1 \wedge e_2 \wedge e_4) = e_3, \qquad \star(e_1 \wedge e_4 \wedge e_3) = e_2, \qquad \star(e_3 \wedge e_2 \wedge e_4) = -e_1,$$

which form a basis of $\mathbb{R}^4$. (For example, using (5), $\det[e_4; e_1; e_2; e_3] = -1$, hence $e_4 \cdot \star(e_1 \wedge e_2 \wedge e_3) = -1$, so $\star(e_1 \wedge e_2 \wedge e_3) = -e_4$; the remaining identities are checked similarly.) Therefore $\operatorname{rank}(W_{\beta\gamma\delta}) = 4$ in this instance.

Two equal indices. Consider $(\beta, \gamma, \delta) = (m, m, m')$ with $m \neq m'$. Take

$$A^{(m)} = \begin{bmatrix} e_1^T \\ e_2^T \\ e_3^T \end{bmatrix}, \quad A^{(m')} = \begin{bmatrix} e_3^T \\ e_4^T \\ e_1^T \end{bmatrix}.$$

Then among the columns of $W_{mmm'}$ appear

$$\star(e_1 \wedge e_2 \wedge e_3) = -e_4, \qquad \star(e_1 \wedge e_2 \wedge e_4) = e_3, \qquad \star(e_1 \wedge e_3 \wedge e_4) = -e_2, \qquad \star(e_2 \wedge e_3 \wedge e_4) = e_1,$$

again a basis of $\mathbb{R}^4$, so $\operatorname{rank}(W_{mmm'}) = 4$ in this instance.

In either pattern we have produced a point at which $\operatorname{rank}(W_{\beta\gamma\delta}) = 4$, so the rank-4 locus is nonempty. Hence, by Zariski-openness, $\operatorname{rank}(W_{\beta\gamma\delta}) = 4$ holds for Zariski-generic cameras whenever (β, γ, δ) are not all equal. □

5.4. **A single generic open set.** For each triple (β, γ, δ) not all equal, let $U^{\mathrm{W}}_{\beta\gamma\delta}$ be the Zariski-open set on which $\operatorname{rank}(W_{\beta\gamma\delta}) = 4$. Define

$$U_n^{\mathrm{W}} := \bigcap_{\substack{(\beta,\gamma,\delta)\in[n]^3 \\ \text{not all equal}}} U^{\mathrm{W}}_{\beta\gamma\delta}, \qquad U_n := U_n^{\ker} \cap U_n^{\mathrm{W}}.$$

As $(\mathbb{R}^{3\times 4})^n$ is irreducible (its coordinate ring is a domain), any finite intersection of nonempty Zariski-open subsets is nonempty. Thus U_n is nonempty, Zariski-open, and dense.

From now on we assume $(A^{(1)}, \ldots, A^{(n)}) \in U_n$.

6. Main theorem and proof

Theorem 6.1. *Fix $n \geq 5$ and let $A^{(1)}, \ldots, A^{(n)} \in \mathbb{R}^{3\times 4}$ be Zariski-generic. Let λ satisfy* (2) *and let $\mathcal{Z} = \lambda \odot \mathcal{Q}$ be the stacked scaled tensor. Let $\mathbf{F}_n$ be as in §3. Then:*

(1) *$\mathbf{F}_n$ does not depend on the cameras.*
(2) *Every coordinate of $\mathbf{F}_n$ has degree 5, independent of n.*
(3) *One has $\mathbf{F}_n(\mathcal{Z}) = 0$ if and only if there exist $u, v, w, x \in (\mathbb{R}^*)^n$ such that*

$$\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta \qquad \text{for all } (\alpha, \beta, \gamma, \delta) \neq (m, m, m, m).$$

6.1. The "if" direction.

Proof of Theorem 6.1, "if" direction. Items (1) and (2) follow from Proposition 3.1.

Assume $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$ for all off-diagonal quadruples, with $u, v, w, x \in (\mathbb{R}^*)^n$. Let $\mathcal{Z} = \lambda \odot \mathcal{Q}$. In the mode-1 flattening, the $(\alpha; \beta, \gamma, \delta)$ block of $M_1(\mathcal{Z})$ equals $u_\alpha(v_\beta w_\gamma x_\delta)$ times the corresponding block of $M_1(\mathcal{Q})$, by (7). Thus $M_1(\mathcal{Z}) = D_1\, M_1(\mathcal{Q})\, E_1$ for suitable invertible diagonal matrices D_1, E_1, hence $\operatorname{rank}(M_1(\mathcal{Z})) = \operatorname{rank}(M_1(\mathcal{Q})) \leq 4$ by Corollary 4.2. Therefore all 5×5 minors of $M_1(\mathcal{Z})$ vanish.

The same argument applies to M_2, M_3, M_4 (each flattening block-scales by products of u, v, w, x along its row and column blocks), so all coordinates of $\mathbf{F}_n(\mathcal{Z})$ vanish. □

6.2. Mode-wise rank constraints force separability.

The remaining direction uses the special structure (7) and genericity.

Proposition 6.2. *Assume $(A^{(1)}, \ldots, A^{(n)}) \in U_n$ and λ satisfies* (2)*. If $\operatorname{rank}(M_1(\mathcal{Z})) \leq 4$, then there exist $u \in (\mathbb{R}^*)^n$ and $s \in (\mathbb{R}^*)^{n\times n\times n}$ such that*

$$\lambda_{\alpha\beta\gamma\delta} = u_\alpha\, s_{\beta\gamma\delta} \qquad \text{for all off-diagonal } (\alpha, \beta, \gamma, \delta). \tag{9}$$

Proof. Assume $\operatorname{rank}(M_1(\mathcal{Z})) \leq 4$. Then there exist matrices $C \in \mathbb{R}^{3n\times 4}$ and $V \in \mathbb{R}^{4\times 27n^3}$ such that $M_1(\mathcal{Z}) = CV$. Partition C into blocks $C_\alpha \in \mathbb{R}^{3\times 4}$ and V into blocks $V_{\beta\gamma\delta} \in \mathbb{R}^{4\times 27}$. Comparing with (7), for all $\alpha, \beta, \gamma, \delta$,

$$C_\alpha V_{\beta\gamma\delta} = \lambda_{\alpha\beta\gamma\delta}\, A^{(\alpha)} W_{\beta\gamma\delta}. \tag{10}$$

Choose a reference triple $(\beta_0, \gamma_0, \delta_0)$ with three distinct indices (possible since $n \geq 5$). Then $\operatorname{rank}(W_{\beta_0\gamma_0\delta_0}) = 4$ by Lemma 5.3, so there is a right inverse $W^+_{\beta_0\gamma_0\delta_0} \in \mathbb{R}^{27\times 4}$ with $W_{\beta_0\gamma_0\delta_0} W^+_{\beta_0\gamma_0\delta_0} = I_4$. Let

$$M := V_{\beta_0\gamma_0\delta_0} W^+_{\beta_0\gamma_0\delta_0} \in \mathbb{R}^{4\times 4}.$$

Multiplying (10) on the right by $W^+_{\beta_0\gamma_0\delta_0}$ yields

$$C_\alpha M = \lambda_{\alpha\beta_0\gamma_0\delta_0}\, A^{(\alpha)}. \tag{11}$$

Since $(\beta_0, \gamma_0, \delta_0)$ are distinct, $(\alpha, \beta_0, \gamma_0, \delta_0)$ is never diagonal, hence $\lambda_{\alpha\beta_0\gamma_0\delta_0} \neq 0$ for all α by (2).

Claim: M is invertible. If $0 \neq y \in \ker(M)$, then (11) gives $0 = C_\alpha M y = \lambda_{\alpha\beta_0\gamma_0\delta_0} A^{(\alpha)} y$, hence $A^{(\alpha)} y = 0$ for all α. Thus $y \in \bigcap_\alpha \ker(A^{(\alpha)})$. But for cameras in U_n, the kernel lines are $\ker(A^{(\alpha)}) = \operatorname{span}\{z_\alpha\}$ and are pairwise distinct (Lemma 5.2); the intersection of two distinct lines is $\{0\}$, so the intersection over all α is $\{0\}$. Contradiction. Hence $M \in \mathrm{GL}_4(\mathbb{R})$.

Solving (11) gives

$$C_\alpha = \lambda_{\alpha\beta_0\gamma_0\delta_0}\, A^{(\alpha)} M^{-1}. \tag{12}$$

Substitute (12) into (10), divide by $\lambda_{\alpha\beta_0\gamma_0\delta_0}$, and set

$$c_{\alpha;\beta\gamma\delta} := \frac{\lambda_{\alpha\beta\gamma\delta}}{\lambda_{\alpha\beta_0\gamma_0\delta_0}} \qquad \big((\alpha, \beta, \gamma, \delta) \text{ off-diagonal}\big),$$

to obtain

$$A^{(\alpha)}\Big(M^{-1} V_{\beta\gamma\delta} - c_{\alpha;\beta\gamma\delta}\, W_{\beta\gamma\delta}\Big) = 0. \tag{13}$$

Since $\ker(A^{(\alpha)}) = \operatorname{span}\{z_\alpha\}$ is one-dimensional, each column of the bracketed 4×27 matrix lies in $\operatorname{span}\{z_\alpha\}$. Hence there exists a row vector $r^\top_{\alpha;\beta\gamma\delta} \in \mathbb{R}^{1\times 27}$ such that

$$M^{-1} V_{\beta\gamma\delta} - c_{\alpha;\beta\gamma\delta}\, W_{\beta\gamma\delta} = z_\alpha r^\top_{\alpha;\beta\gamma\delta}. \tag{14}$$

Fix (β, γ, δ) and subtract (14) for two distinct cameras $\alpha_1 \neq \alpha_2$:

$$(c_{\alpha_1;\beta\gamma\delta} - c_{\alpha_2;\beta\gamma\delta}) W_{\beta\gamma\delta} = z_{\alpha_2} r^\top_{\alpha_2;\beta\gamma\delta} - z_{\alpha_1} r^\top_{\alpha_1;\beta\gamma\delta}. \tag{15}$$

The right-hand side has column space contained in $\operatorname{span}\{z_{\alpha_1}, z_{\alpha_2}\}$.

If (β, γ, δ) are not all equal, then $\operatorname{rank}(W_{\beta\gamma\delta}) = 4$ by Lemma 5.3. If the scalar on the left were nonzero, the left-hand side would have rank 4, contradicting the rank ≤ 2 of the right-hand side. Hence $c_{\alpha;\beta\gamma\delta}$ is independent of α.

If $(\beta, \gamma, \delta) = (m, m, m)$, then $\operatorname{colsp}(W_{mmm}) = \operatorname{span}\{z_m\}$ by Lemma 5.3. Choose $\alpha_1, \alpha_2 \in [n]\setminus\{m\}$ distinct (possible since $n \geq 5$). By Lemma 5.2(3), $z_m, z_{\alpha_1}, z_{\alpha_2}$ are linearly independent, so $\operatorname{span}\{z_m\} \cap \operatorname{span}\{z_{\alpha_1}, z_{\alpha_2}\} = \{0\}$. In (15), the left-hand side has column space in $\operatorname{span}\{z_m\}$ and the right-hand side has column space in $\operatorname{span}\{z_{\alpha_1}, z_{\alpha_2}\}$, so both sides must be zero, and thus $c_{\alpha_1;mmm} = c_{\alpha_2;mmm}$. Therefore $c_{\alpha;mmm}$ is independent of $\alpha \neq m$, which is exactly the range where (α, m, m, m) is off-diagonal.

We conclude: for each triple (β, γ, δ) there exists $s_{\beta\gamma\delta} \in \mathbb{R}^*$ such that $c_{\alpha;\beta\gamma\delta} = s_{\beta\gamma\delta}$ for all α with $(\alpha, \beta, \gamma, \delta)$ off-diagonal. Setting $u_\alpha := \lambda_{\alpha\beta_0\gamma_0\delta_0}$ gives (9). □

Proposition 6.3. *Assume $(A^{(1)}, \ldots, A^{(n)}) \in U_n$ and λ satisfies (2).*

1. *If $\operatorname{rank}(M_2(\mathcal{Z})) \leq 4$, then there exist $v \in (\mathbb{R}^*)^n$ and $t \in (\mathbb{R}^*)^{n\times n\times n}$ such that $\lambda_{\alpha\beta\gamma\delta} = v_\beta\, t_{\alpha\gamma\delta}$ for all off-diagonal $(\alpha, \beta, \gamma, \delta)$.*
2. *If $\operatorname{rank}(M_3(\mathcal{Z})) \leq 4$, then there exist $w \in (\mathbb{R}^*)^n$ and $p \in (\mathbb{R}^*)^{n\times n\times n}$ such that $\lambda_{\alpha\beta\gamma\delta} = w_\gamma\, p_{\alpha\beta\delta}$ for all off-diagonal $(\alpha, \beta, \gamma, \delta)$.*

(3) *If* $\operatorname{rank}(M_4(\mathcal{Z})) \leq 4$, *then there exist* $x \in (\mathbb{R}^*)^n$ *and* $q \in (\mathbb{R}^*)^{n\times n\times n}$ *such that* $\lambda_{\alpha\beta\gamma\delta} = x_\delta\, q_{\alpha\beta\gamma}$ *for all off-diagonal* $(\alpha, \beta, \gamma, \delta)$.

Proof. The argument is the same as Proposition 6.2 after permuting the roles of the four indices in (1). Concretely, for mode 2, one uses the identity

$$\det[a_{\alpha i}; a_{\beta j}; a_{\gamma k}; a_{\delta \ell}] = -\, a_{\beta j} \cdot \star(a_{\alpha i} \wedge a_{\gamma k} \wedge a_{\delta \ell}),$$

obtained by swapping the first two rows, and similarly for modes 3 and 4 (with the appropriate sign). These signs only multiply entire block-columns by ± 1 and therefore do not affect the rank arguments. The generic rank statements required for the corresponding block matrices reduce to Lemma 5.3. □

Lemma 6.4. *Assume* (2). *Suppose for all off-diagonal quadruples*

$$\lambda_{\alpha\beta\gamma\delta} = u_\alpha s_{\beta\gamma\delta} \quad \textit{and} \quad \lambda_{\alpha\beta\gamma\delta} = v_\beta t_{\alpha\gamma\delta},$$

with $u, v \in (\mathbb{R}^*)^n$. *Then there exists* $r \in (\mathbb{R}^*)^{n\times n}$ *such that*

$$\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta r_{\gamma\delta} \quad \textit{for all off-diagonal } (\alpha, \beta, \gamma, \delta).$$

Proof. Fix $(\gamma, \delta) \in [n]^2$. Choose $\alpha_0 \in [n] \setminus \{\gamma, \delta\}$ (possible since $n \geq 5$). Then for every β the quadruple $(\alpha_0, \beta, \gamma, \delta)$ is off-diagonal, so

$$u_{\alpha_0} s_{\beta\gamma\delta} = \lambda_{\alpha_0\beta\gamma\delta} = v_\beta t_{\alpha_0\gamma\delta}.$$

Define $r_{\gamma\delta} := t_{\alpha_0\gamma\delta}/u_{\alpha_0} \in \mathbb{R}^*$ to obtain $s_{\beta\gamma\delta} = v_\beta r_{\gamma\delta}$ for all β. Hence $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta r_{\gamma\delta}$ on the off-diagonal. □

Lemma 6.5. *Assume* (2). *Suppose for all off-diagonal quadruples*

$$\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta r_{\gamma\delta} \quad \textit{and} \quad \lambda_{\alpha\beta\gamma\delta} = w_\gamma p_{\alpha\beta\delta},$$

with $u, v, w \in (\mathbb{R}^*)^n$. *Then there exists* $x \in (\mathbb{R}^*)^n$ *such that*

$$\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta \quad \textit{for all off-diagonal } (\alpha, \beta, \gamma, \delta).$$

Proof. Choose $\alpha_0 \neq \beta_0$ (possible since $n \geq 5$). Then for all (γ, δ) the quadruple $(\alpha_0, \beta_0, \gamma, \delta)$ is off-diagonal, so

$$u_{\alpha_0} v_{\beta_0} r_{\gamma\delta} = \lambda_{\alpha_0\beta_0\gamma\delta} = w_\gamma p_{\alpha_0\beta_0\delta}.$$

Define

$$x_\delta := \frac{p_{\alpha_0\beta_0\delta}}{u_{\alpha_0} v_{\beta_0}} \in \mathbb{R}^*.$$

Then $r_{\gamma\delta} = w_\gamma x_\delta$ for all γ, δ, and substituting into $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta r_{\gamma\delta}$ gives the desired factorization. □

Proof of Theorem 6.1, "only if" direction. Assume $\mathbf{F}_n(\mathcal{Z}) = 0$. By Lemma 2.1, all 5×5 minors of each flattening vanish, hence $\operatorname{rank}(M_r(\mathcal{Z})) \leq 4$ for $r = 1, 2, 3, 4$.

Apply Proposition 6.2 to obtain $\lambda_{\alpha\beta\gamma\delta} = u_\alpha s_{\beta\gamma\delta}$ on the off-diagonal. Apply Proposition 6.3(1) to obtain $\lambda_{\alpha\beta\gamma\delta} = v_\beta t_{\alpha\gamma\delta}$ on the off-diagonal. Then Lemma 6.4 yields $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta r_{\gamma\delta}$ on the off-diagonal. Apply Proposition 6.3(2) to obtain $\lambda_{\alpha\beta\gamma\delta} = w_\gamma p_{\alpha\beta\delta}$ on the off-diagonal. Then Lemma 6.5 yields $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$ on the off-diagonal.

Finally, u, v, w, x lie in $(\mathbb{R}^*)^n$ because each is defined via off-diagonal entries of λ, which are nonzero by (2). □

7. Remarks

Remark 7.1 (Diagonal entries of λ). Since $Q^{(mmmm)} \equiv 0$, the diagonal values λ_{mmmm} do not affect $\mathcal{Z}$. Theorem 6.1 therefore asserts and proves factorization only on the off-diagonal, which is exactly the identifiable part.

Remark 7.2 (On the hypothesis $n \geq 5$). The argument in fact requires only: (i) existence of three distinct indices to form a reference triple; and (ii) for each m, existence of two indices distinct from m to handle the (m, m, m) triple in Proposition 6.2; together with the generic condition that any three kernel vectors z_α are independent. Thus, after minor bookkeeping, the proof works for $n \geq 3$. We retain $n \geq 5$ to match the problem statement.

Remark 7.3 (Degree). The degree 5 is dictated by the ambient dimension 4: the relevant flattenings have rank ≤ 4 precisely when all 5×5 minors vanish.