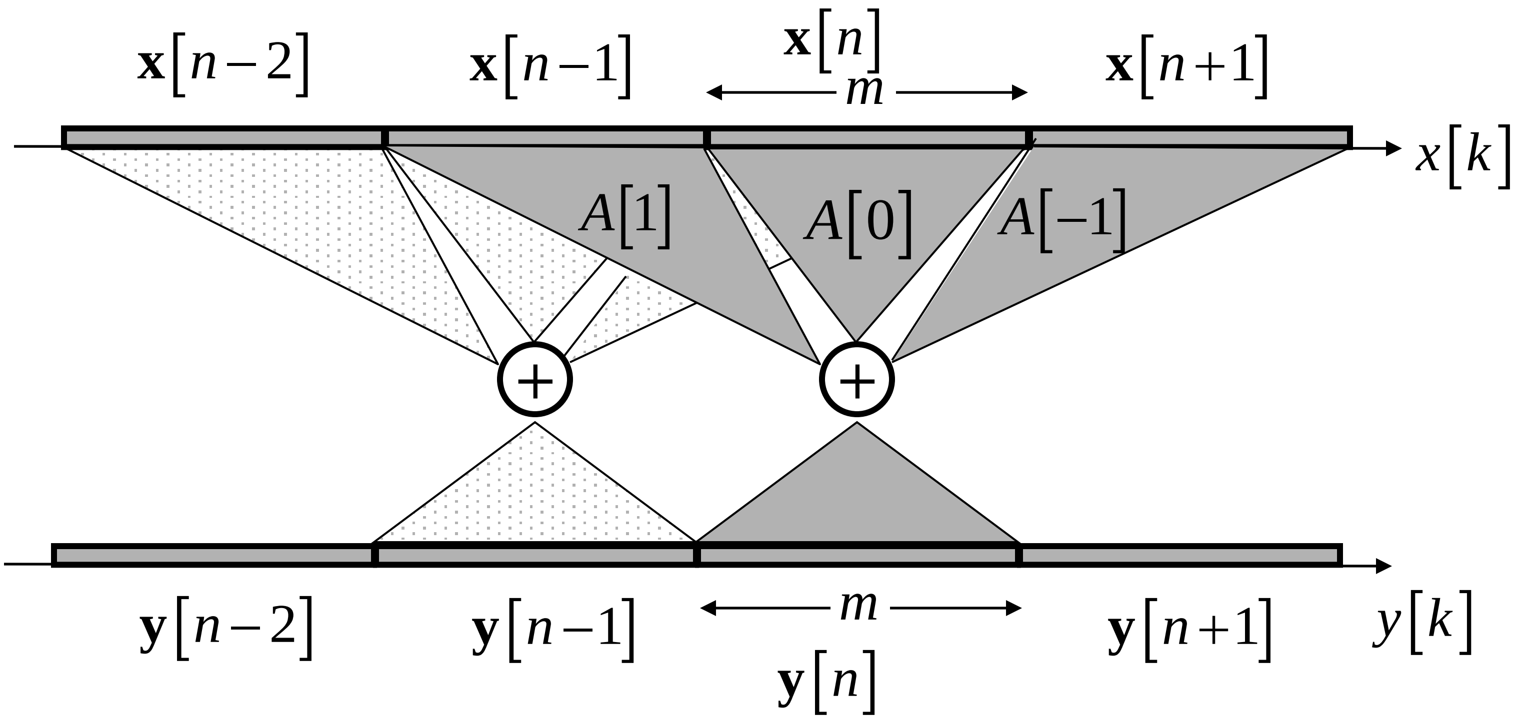
Subband and wavelet coding

- Vector convolution, convolutional transforms
- Filter banks vs. vector space interpretation
- Orthogonal and biorthogonal subband transforms
- DCT as a filter bank
- Lapped Orthogonal Transform (LOT)
- Discrete Wavelet Transform (DWT)
- Quadrature mirror filters and conjugate quadrature filters
- Lifting implementation/design of the DWT
- Embedded zero-tree coding of wavelet coefficients

Subband coding: motivation

- Coding with block-wise transform introduces visible blocking artifacts, as bit-rate decreases.
- Can we, somehow, overlap adjacent blocks,
  - thereby smoothing block boundaries,
  - but without increasing the number of transform coefficients?
- Solution: subband transform.

Vector convolution



Forward transform

$$\mathbf{y}[n] = \sum_{i \in \mathbb{Z}} A[i] \cdot \mathbf{x}[n-i]$$

Inverse transform

$$\mathbf{x}[n] = \sum_{i \in \mathbb{Z}} S[i] \cdot \mathbf{y}[n-i]$$

Perfect reconstruction condition

- Original domain

$$\sum_{i\in\mathbb{Z}} S[n-i]\cdot A[i] = \sum_{i\in\mathbb{Z}} A[n-i]\cdot S[i] = I\cdot\delta[n]$$

- z-transform: "polyphase matrices"

$$\mathbf{H}(z) = \sum_{i\in\mathbb{Z}} A[i]\cdot z^{-i} \qquad\qquad \mathbf{G}(z) = \sum_{i\in\mathbb{Z}} S[i]\cdot z^{-i}$$

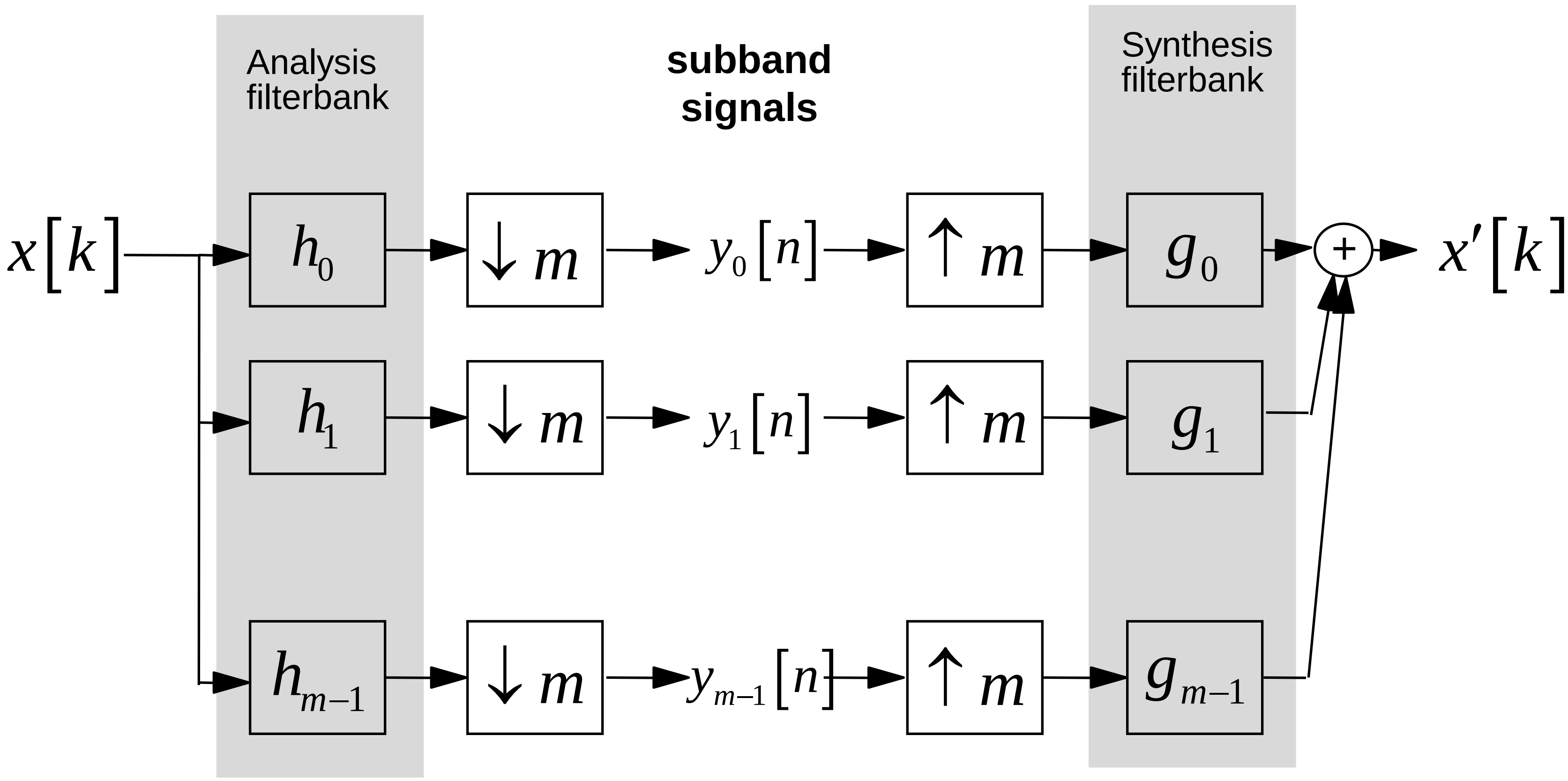
- Perfect reconstruction condition in the z-domain

$$\boxed{\mathbf{G}(z)\mathbf{H}(z) = \mathbf{H}(z)\mathbf{G}(z) = I \quad \Leftrightarrow \quad \mathbf{G}(z) = \left(\mathbf{H}(z)\right)^{-1}}$$

- Example, m=2

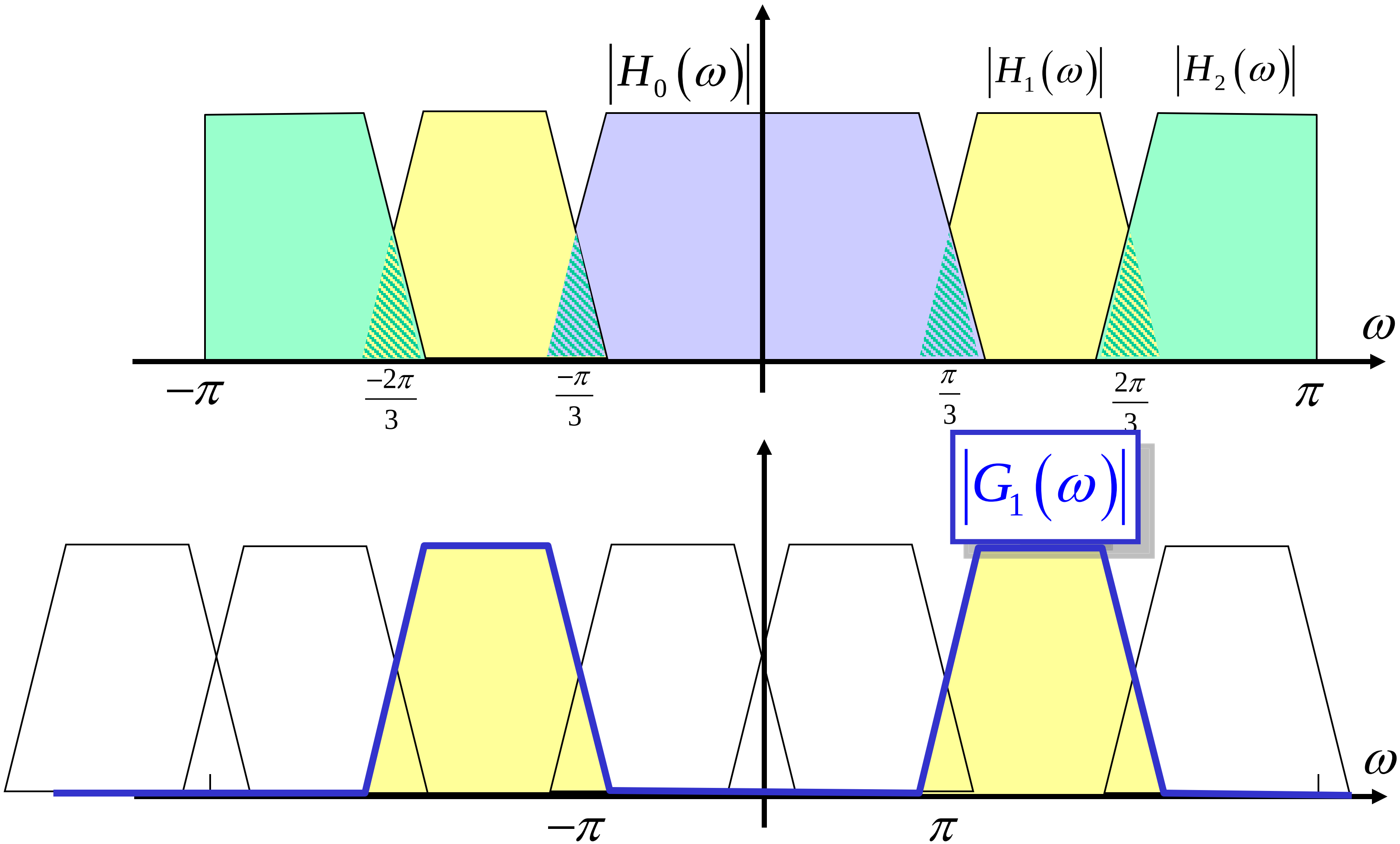
$$\begin{pmatrix} G_{00}(z) & G_{01}(z) \\ G_{10}(z) & G_{11}(z) \end{pmatrix} = \frac{\begin{pmatrix} H_{11}(z) & -H_{01}(z) \\ -H_{10}(z) & H_{00}(z) \end{pmatrix}}{H_{00}(z)H_{11}(z) - H_{01}(z)H_{10}(z)}$$

Filter bank interpretation of convolutional transform



$$h_q[mi-j] = \left(A[i]\right)_{q,j}\ ;\ 0 \le j,q < m \qquad\qquad g_q[mi+j] = \left(S[i]\right)_{j,q}\ ;\ 0 \le j,q < m$$

Frequency domain perspective



Vector space interpretation

- Subband decomposition is the projection of the input onto a set of "analysis vectors" in the Hilbert space of square summable sequences
- Consider signal in channel q

"Analysis vector"
n denotes shift

$$y_q[n] = \sum_k h_q[k] \cdot x[mn-k] = \sum_k h_q[mn-k] \cdot x[k] = \left\langle \mathbf{x}, \mathbf{a}_q^{(n)} \right\rangle$$

- Synthesis filterbank is linear combination of synthesis "basis vectors"

$$x[k] = \sum_{q=0}^{m-1} \sum_n y_q[n] \cdot g_q[k-mn]$$

$$\mathbf{x} = \sum_{q=0}^{m-1} \sum_n y_q[n] \cdot \mathbf{s}_q^{(n)}$$

"Synthesis vector"
n denotes shift

Orthonormal subband transforms

- Orthonormal expansion

$$\boxed{\mathbf{x} = \sum_{q=0}^{m} \sum_{n\in\mathbb{Z}} \left\langle \mathbf{x}, \mathbf{s}_q^{(n)} \right\rangle \mathbf{s}_q^{(n)}}$$

- Analysis and synthesis vectors are identical!

$$\mathbf{a}_q^{(n)} = \mathbf{s}_q^{(n)} \qquad A[i] = S^T[-i] \qquad h_q[k] = g_q[-k]$$

<u>Vector space</u> <u>Convolutional transform</u> <u>Filter bank</u>

Biorthogonal transforms

- Analysis vectors and synthesis vectors are not necessarily each orthogonal, but each analysis vector must be orthogonal to all but one synthesis vectors (and vice versa)

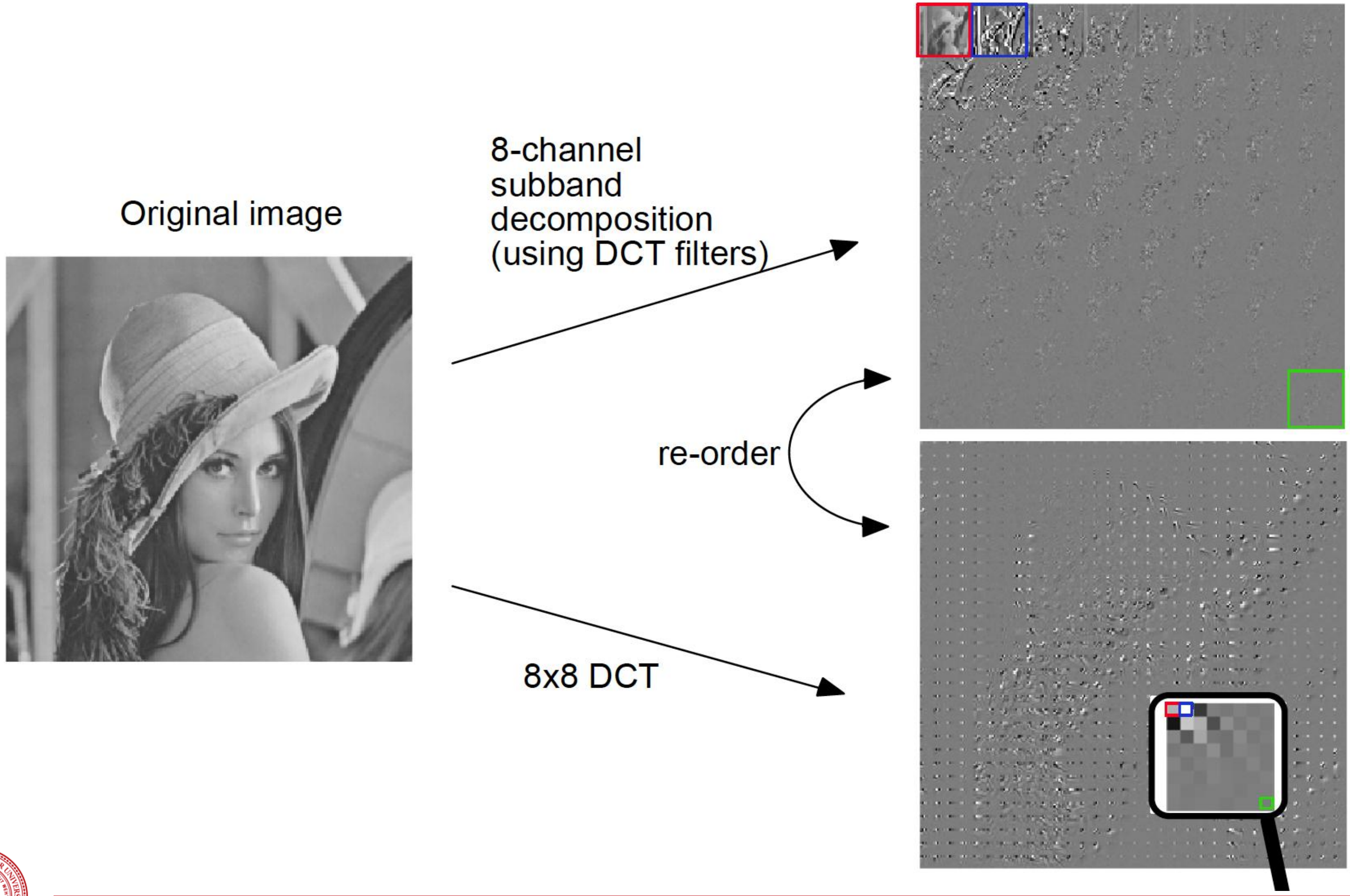
$$\left\langle \mathbf{s}_{q_1}^{(n_1)}, \mathbf{a}_{q_2}^{(n_2)} \right\rangle = \delta\left[q_1 - q_2\right]\delta\left[n_1 - n_2\right] \quad , \quad 0 \le q_1, q_2 < m, \quad n_1, n_2 \in \mathbb{Z}$$

- Equivalent to perfect reconstruction
- Important for linear-phase FIR filters, since lapped orthogonal transforms with linear phase do not exist.

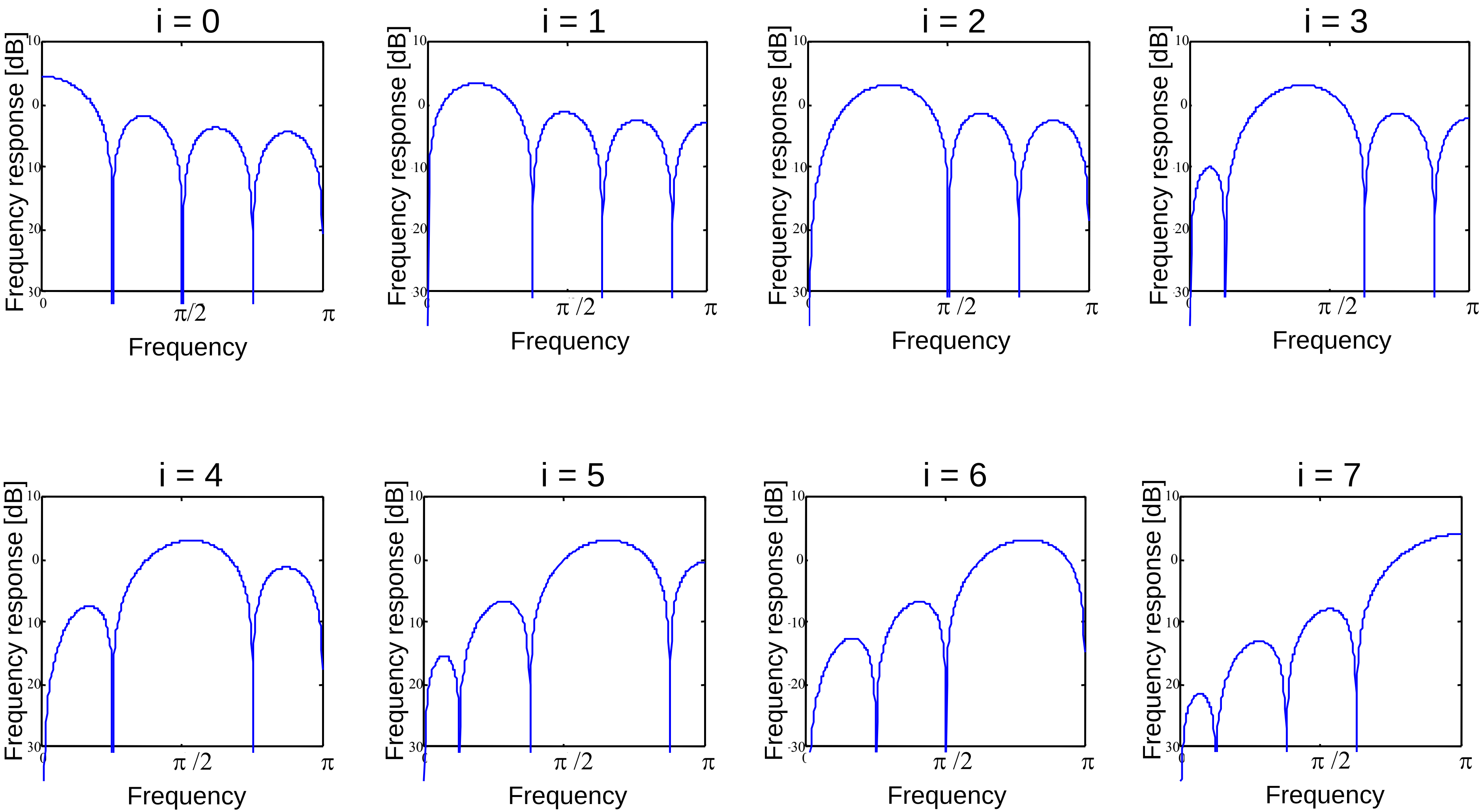
Subbands vs. block-wise transform

- Blockwise transforms are a special case of subband decompositions with:
  - Number of bands m = order of transform N
  - Length of impulse responses of analysis/synthesis filters $\leq m$
- Filters used in subband coders are **not** in general orthogonal.
- Linear phase is desirable for images.

Subbands vs. block-wise transform (cont.)



Frequency response of a DCT of order *N*=8



Lapped Orthogonal Transform

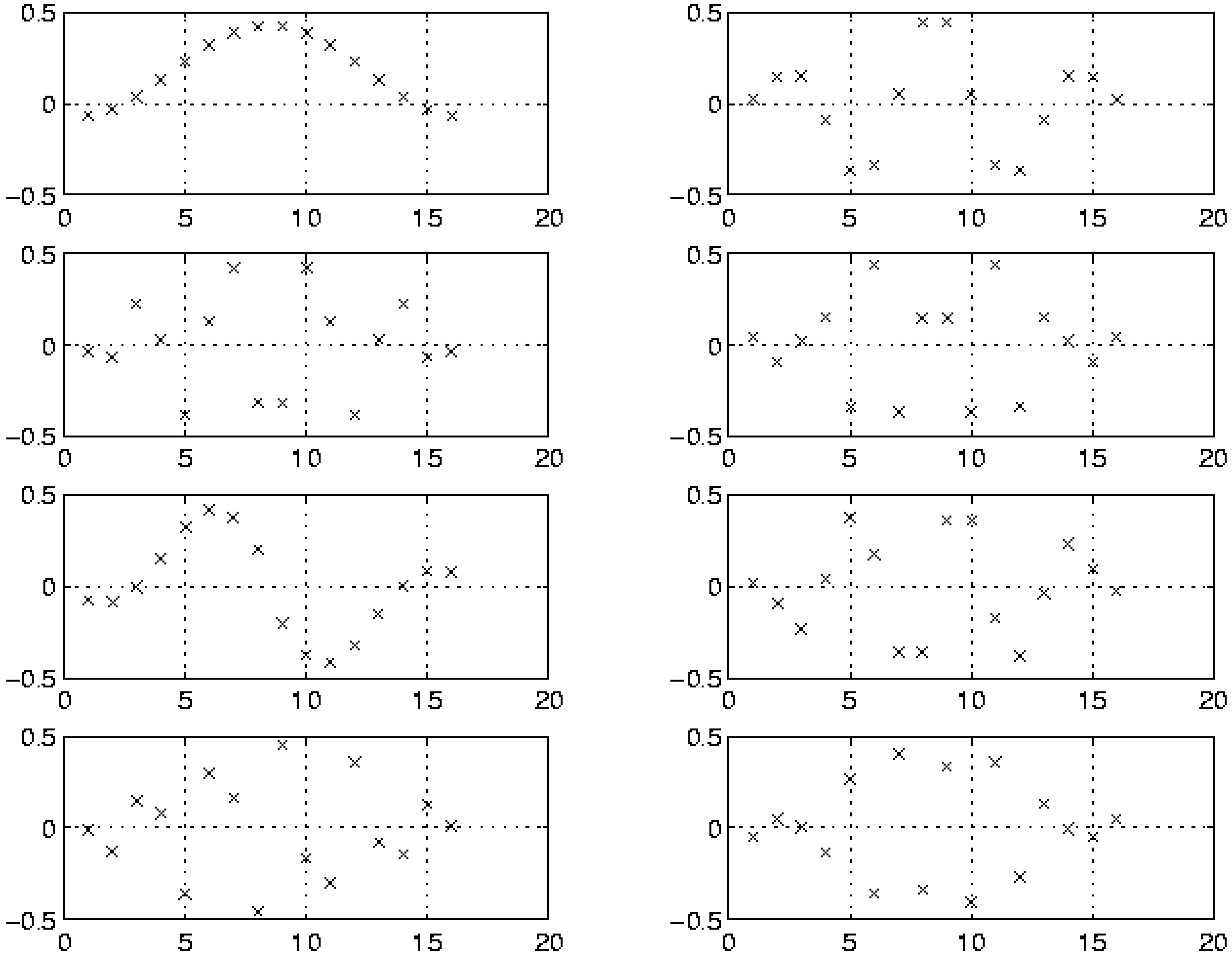
- Orthonormal convolutional transform with perfect reconstruction

$$\boxed{\sum_i A^T[i]A[n+i] = I \cdot \delta[n]} \qquad \boxed{A^T[i] = S[-i]}$$

- Lapped orthogonal transform (LOT): only $A[0]$ and $A[1]$ non-zero, hence

$$A^T[0]A[0] + A^T[1]A[1] = \begin{pmatrix} A[0] \\ A[1] \end{pmatrix}^T \begin{pmatrix} A[0] \\ A[1] \end{pmatrix} = I \qquad A^T[0]A[1] = 0$$

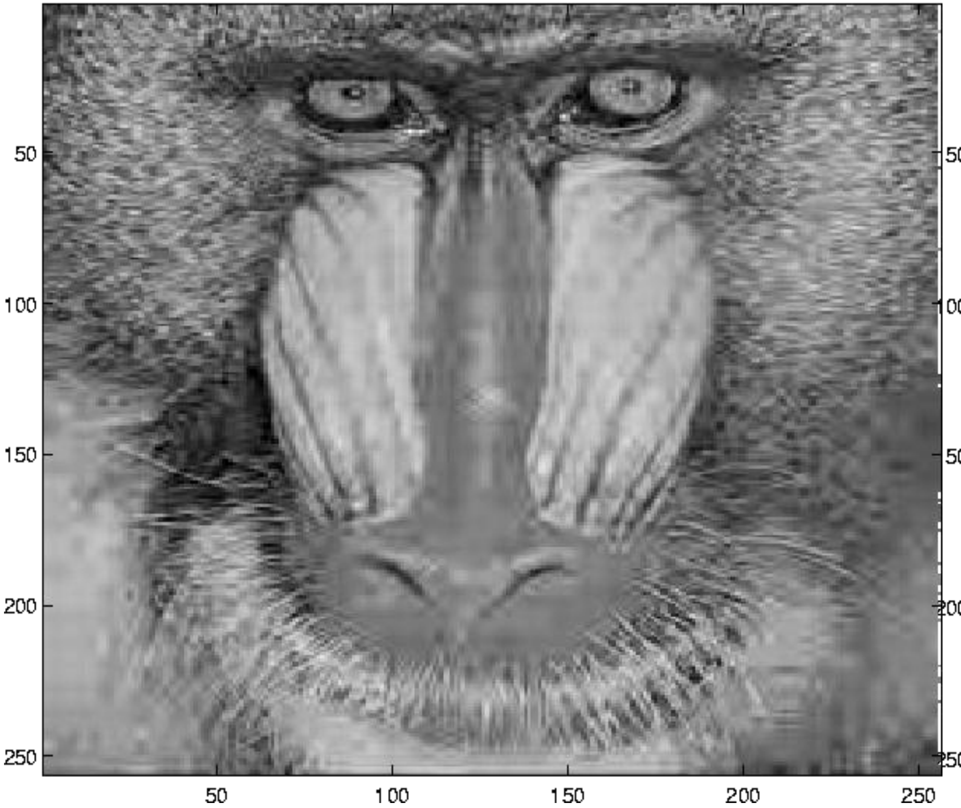
Example LOT basis functions, m=8



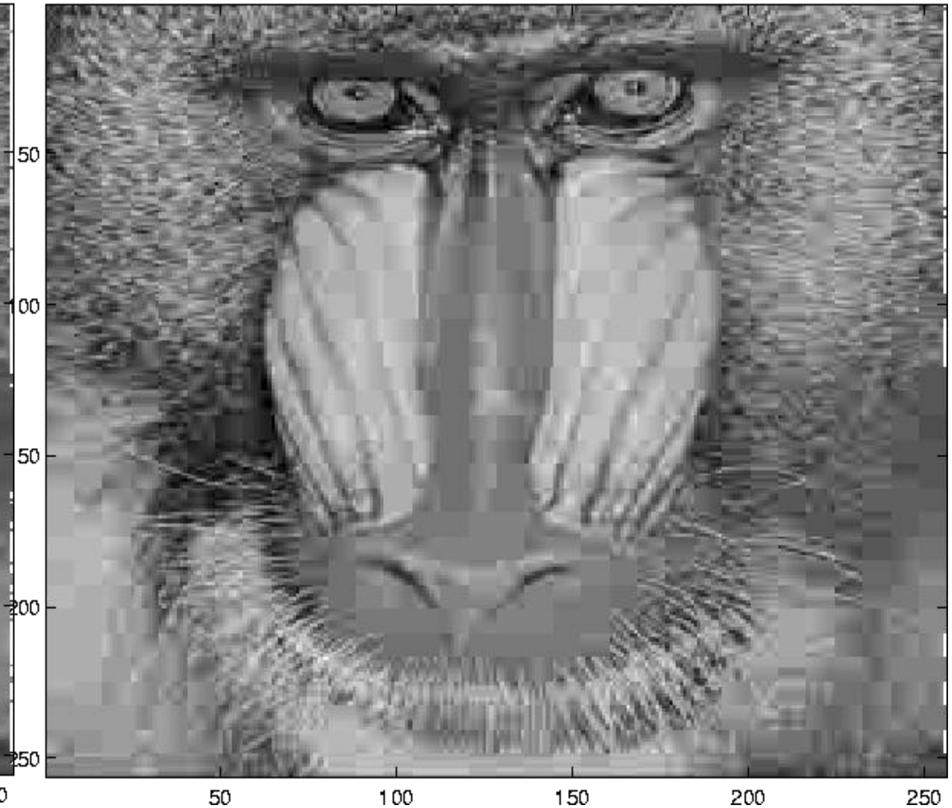
[G. Levinsky, EE392C class project, 1997]

LOT vs. DCT coding

LOT
quantizer step size 70
entropy 0.426 bpp



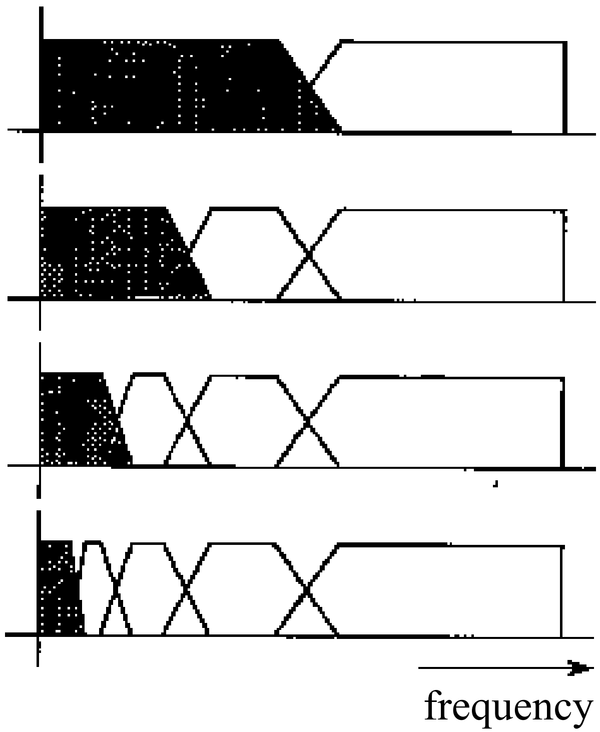
DCT
quantizer step size 70
entropy 0.453 bpp



[G. Levinsky, EE392C class project, 1997]

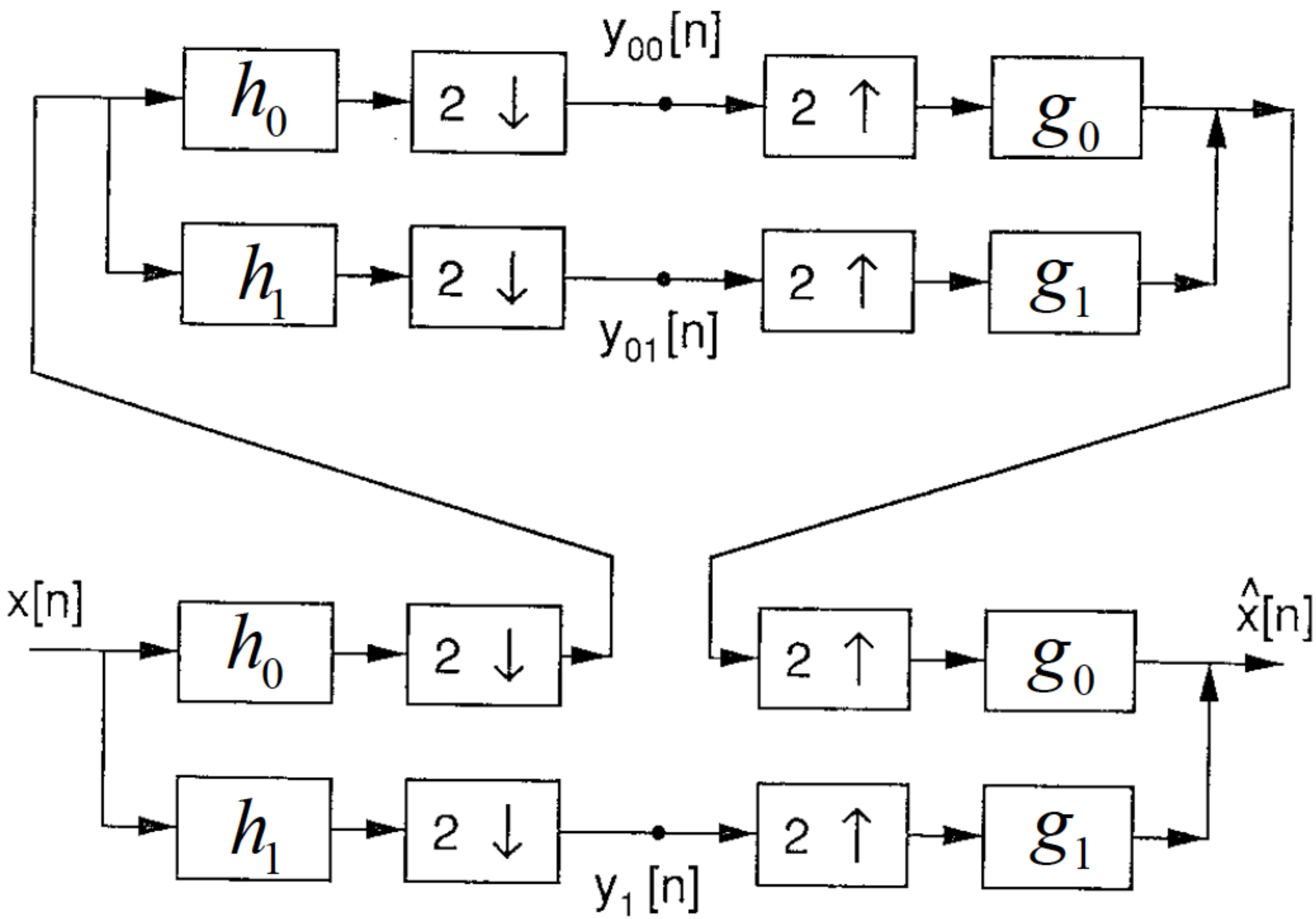
Discrete Wavelet Transform

- Recursive application of a two-band filter bank to the lowpass band of the previous stage yields octave band splitting:

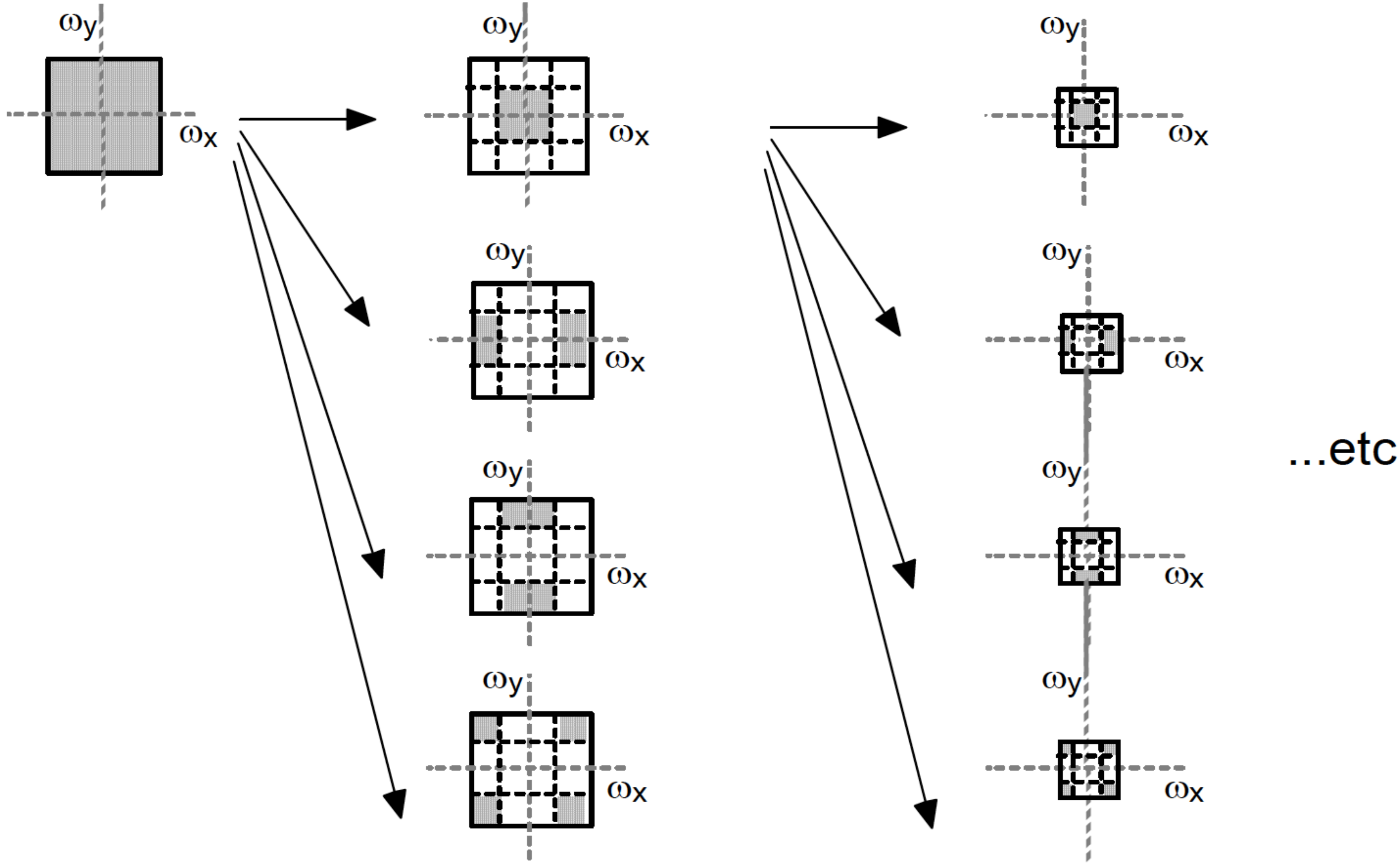


- Same concept can be derived from wavelet theory: Discrete Wavelet Transform (DWT)

Cascaded analysis / synthesis filterbanks



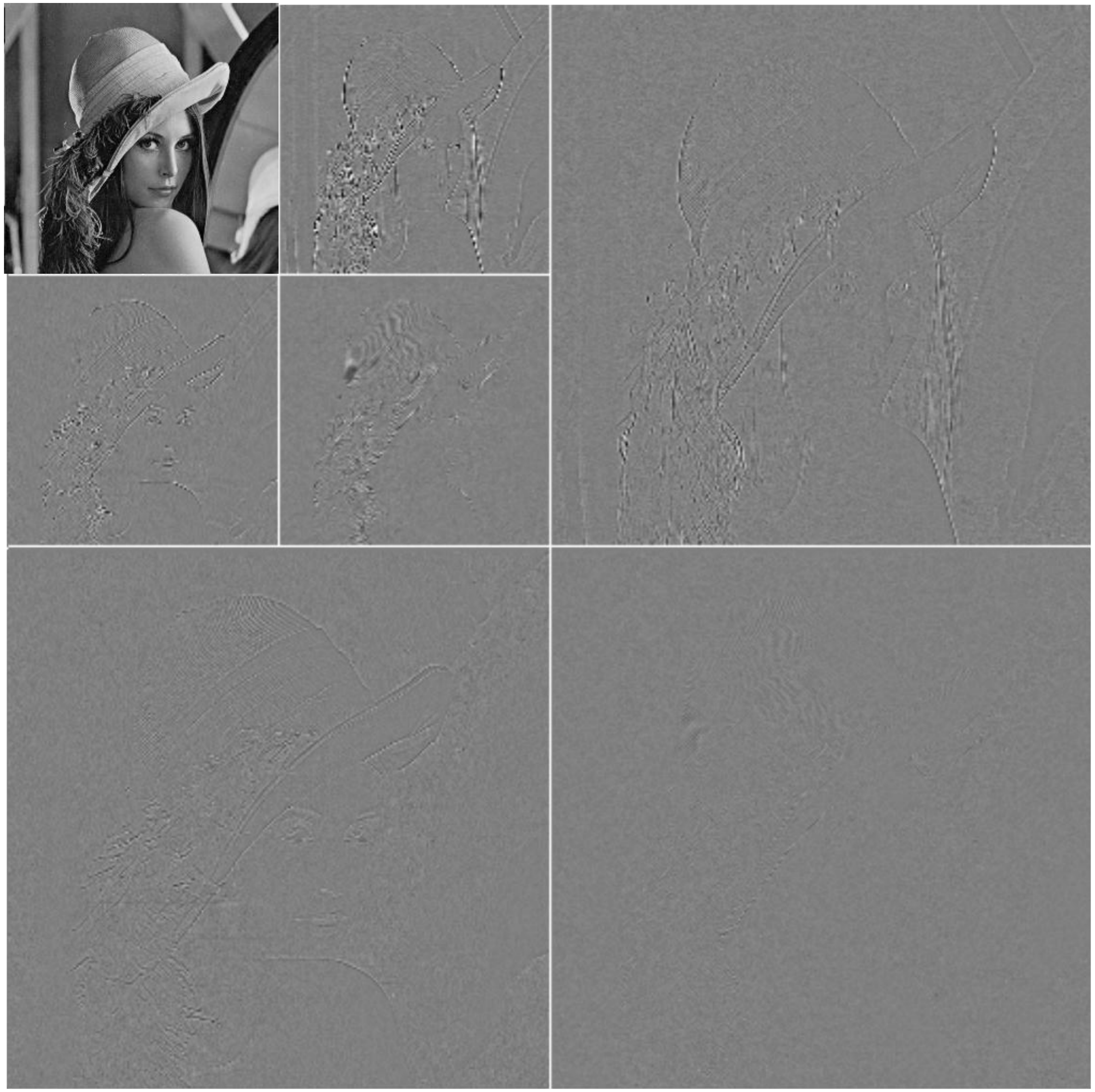
2-d Discrete Wavelet Transform



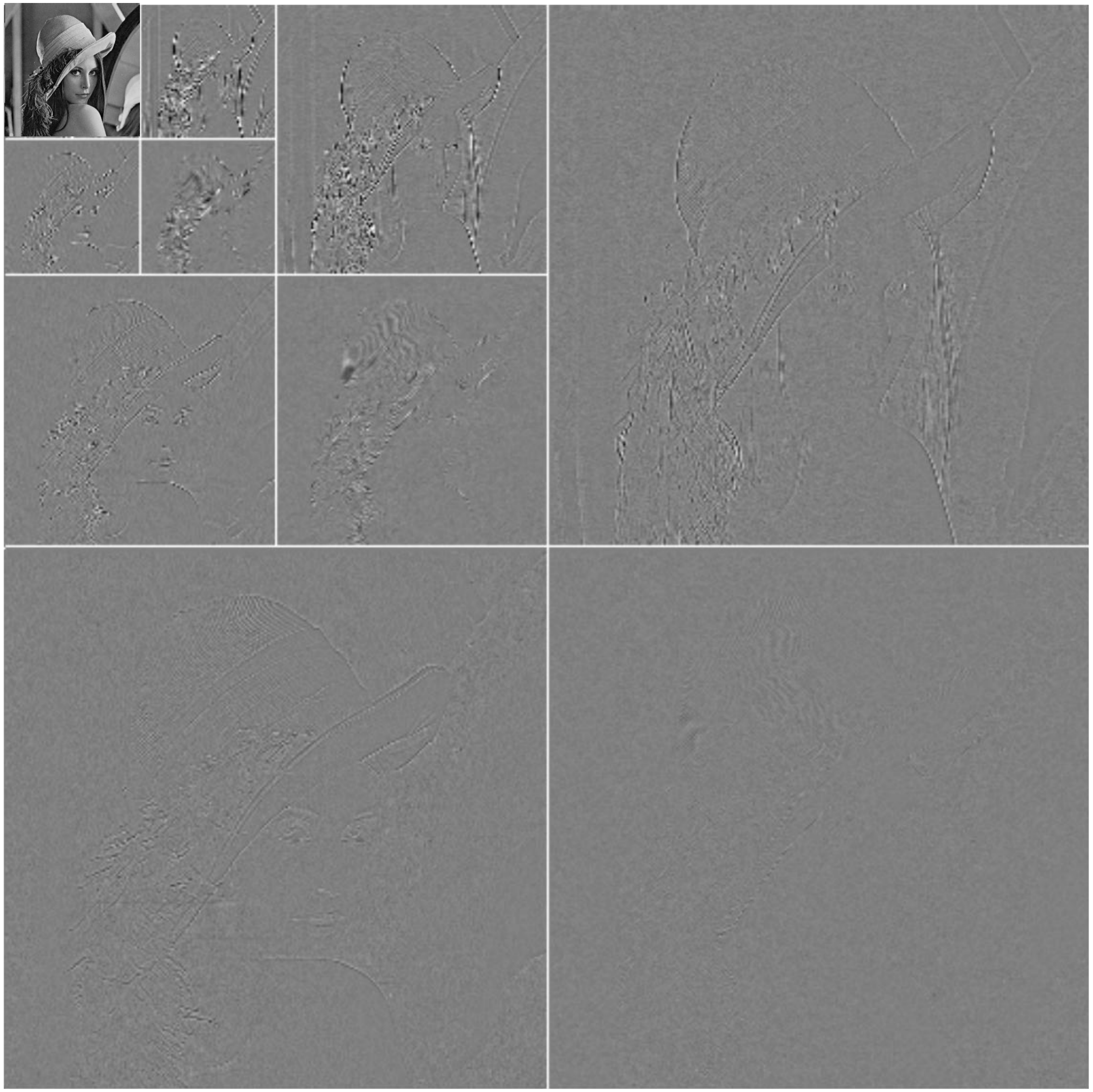
2-d Discrete Wavelet Transform example

2-d Discrete Wavelet Transform example

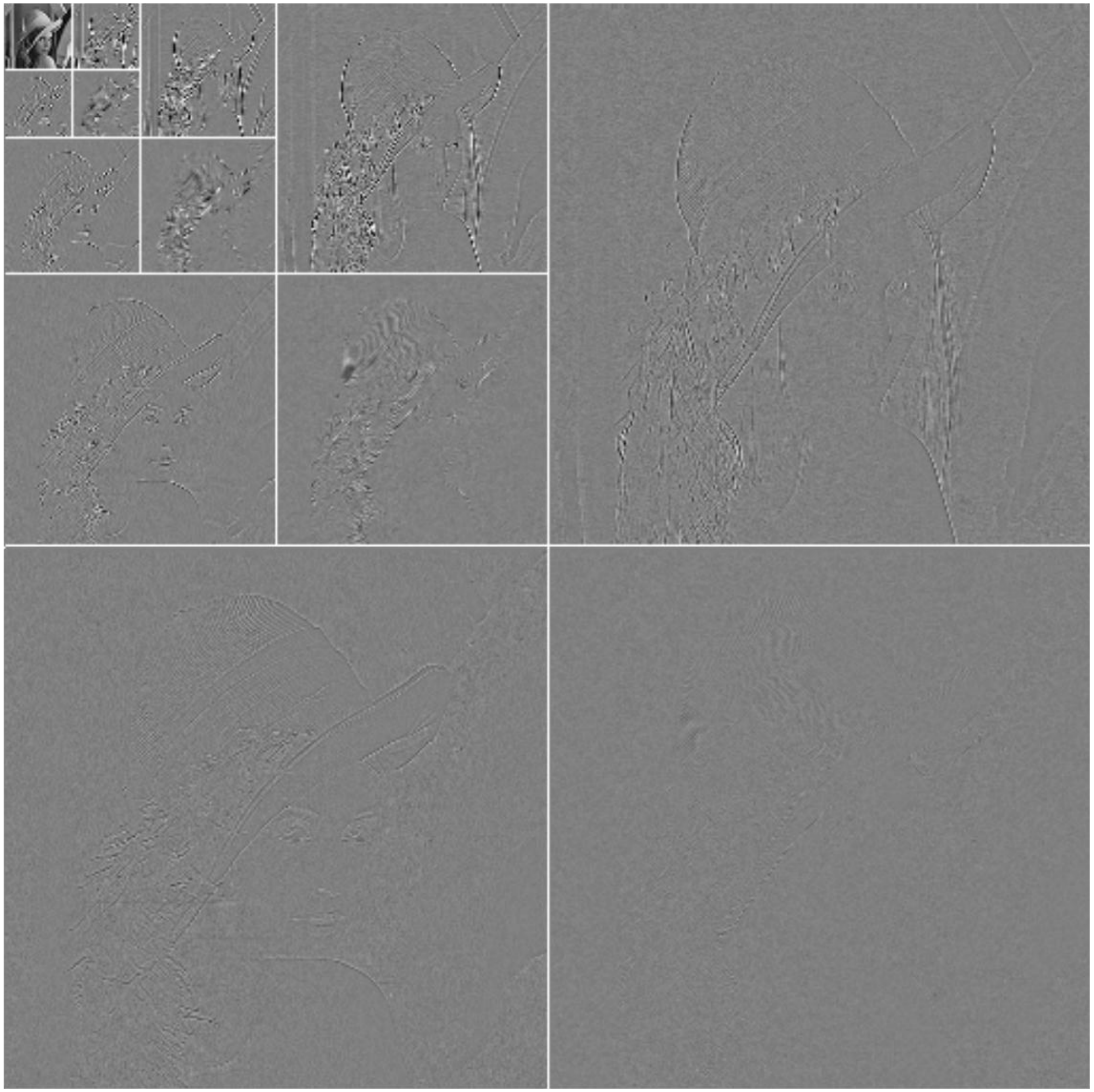
2-d Discrete Wavelet Transform example



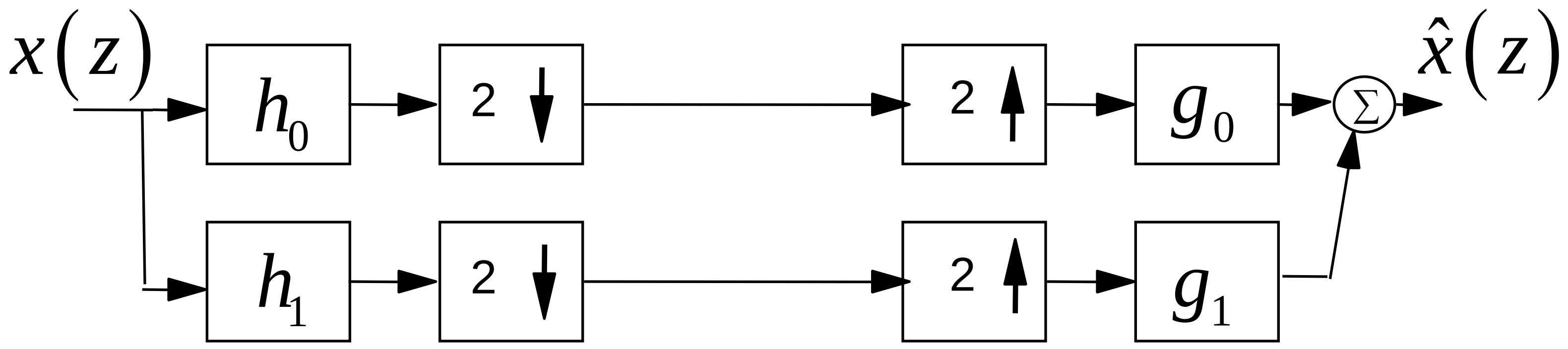
2-d Discrete Wavelet Transform example



2-d Discrete Wavelet Transform example



Two-channel filterbank



$$\hat{x}(z) = \frac{1}{2}\left[h_0(z)g_0(z) + h_1(z)g_1(z)\right]x(z)$$
$$+\frac{1}{2}\left[h_0(-z)g_0(z) + h_1(-z)g_1(z)\right]x(-z)$$

Aliasing

- Aliasing cancellation if :

$$g_0(z) = h_1(-z)$$
$$-g_1(z) = h_0(-z)$$

Example: two-channel filter bank with perfect reconstruction

- Impulse responses, analysis filters:

  Lowpass

  $$\left(\frac{-1}{4},\frac{1}{2},\frac{3}{2},\frac{1}{2},\frac{-1}{4}\right)$$

  Highpass

  $$\left(\frac{1}{4},\frac{-1}{2},\frac{1}{4}\right)$$

- Impulse responses, synthesis filters

  Lowpass

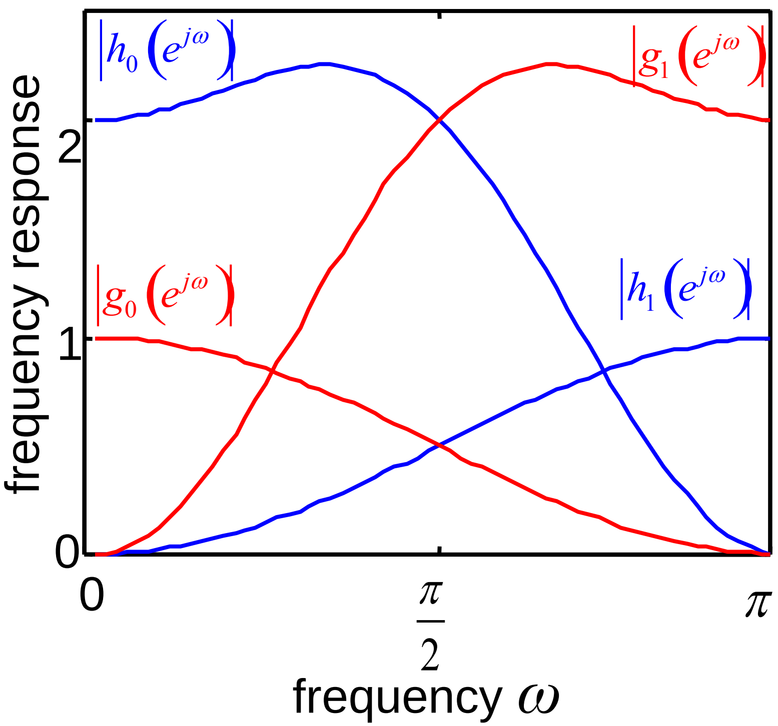
  $$\left(\frac{1}{4},\frac{1}{2},\frac{1}{4}\right)$$

  Highpass

  $$\left(\frac{1}{4},\frac{1}{2},\frac{-3}{2},\frac{1}{2},\frac{1}{4}\right)$$

"Biorthogonal 5/3 filters"
"LeGall-Tabatabai filters"
[LeGall, Tabatabai, 1988]

- Mandatory in JPEG2000
- Frequency responses:



Quadrature Mirror Filters (QMF)

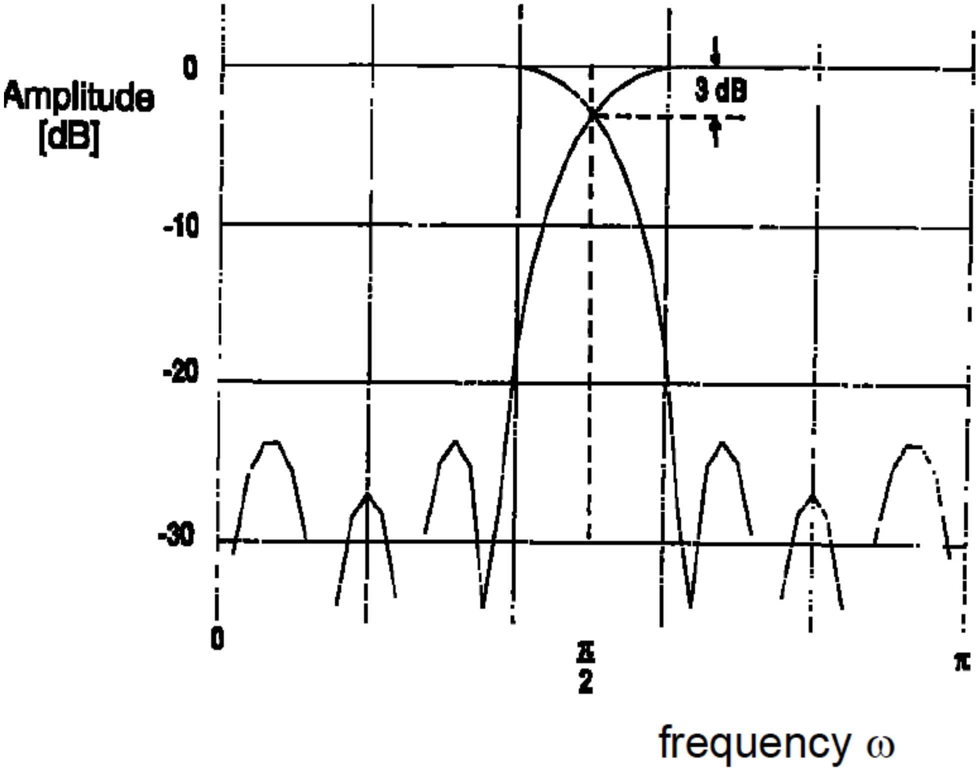
- QMFs achieve aliasing cancellation by choosing

$$\boxed{\begin{aligned} h_1(z) &= h_0(-z) \\ &= -g_1(z) = g_0(-z) \end{aligned}}$$

[Croisier, Esteban, Galand, 1976]

- Highpass band is the mirror image of the lowpass band in the frequency domain
- Need to design only one prototype filter

<u>Example:</u>
<u>16-tap QMF filterbank</u>



Conjugate quadrature filters

- Achieve aliasing cancelation by

Prototype filter

$$h_0(z) = g_0(z^{-1}) \equiv f(z)$$
$$h_1(z) = g_1(z^{-1}) = zf\left(-z^{-1}\right)$$

[Smith, Barnwell, 1986]

- Impulse responses

$$h_0[k] = g_0[-k] = f[k]$$
$$h_1[k] = g_1[-k] = (-1)^{k+1} f\left[-(k+1)\right]$$

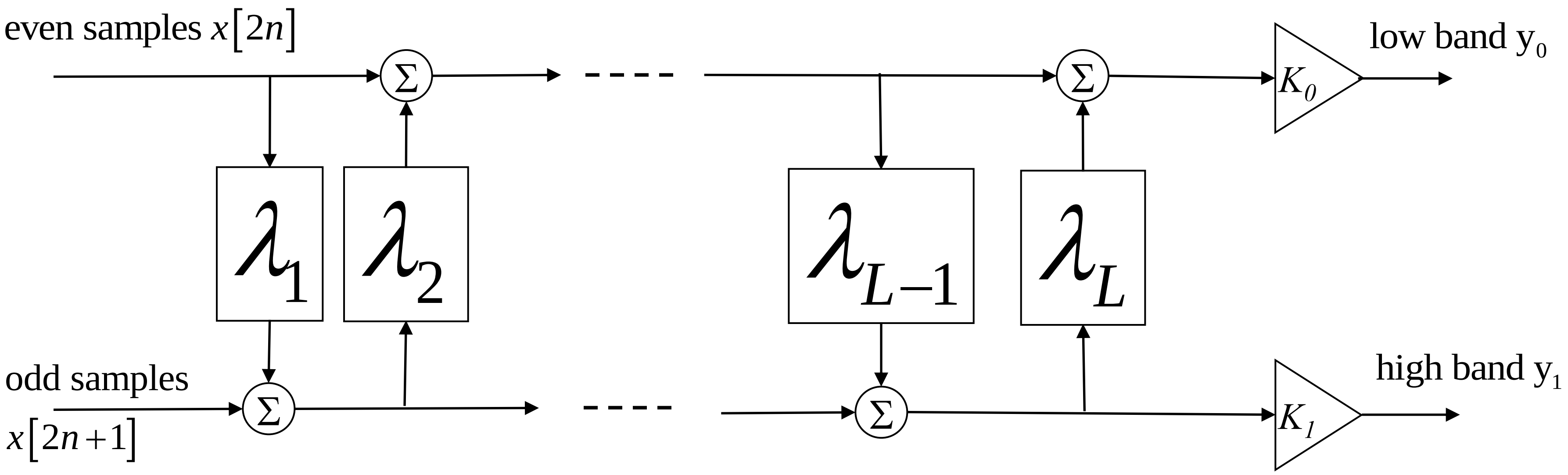
- Perfect reconstruction: find power complementary prototype filter

$$\left|F(\omega)\right|^2 + \left|F(\omega \pm \pi)\right|^2 = 2$$

- Orthonormal subband transform!

Lifting

- Analysis filters

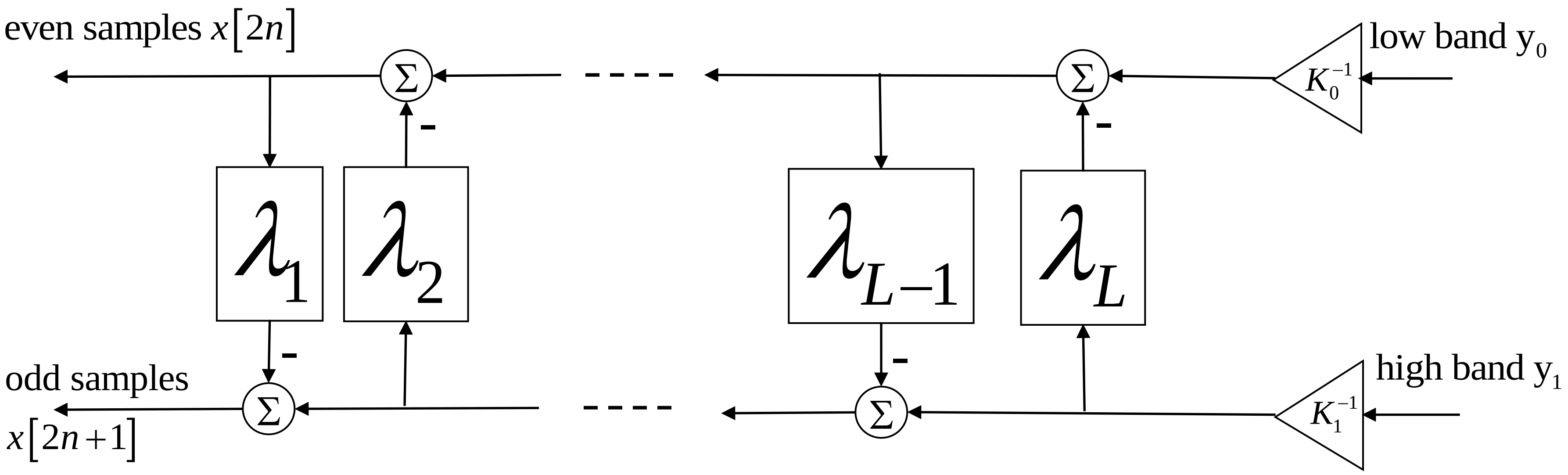


- L "lifting steps"
- First step can be interpreted as prediction of odd samples from the even samples

[Sweldens 1996]

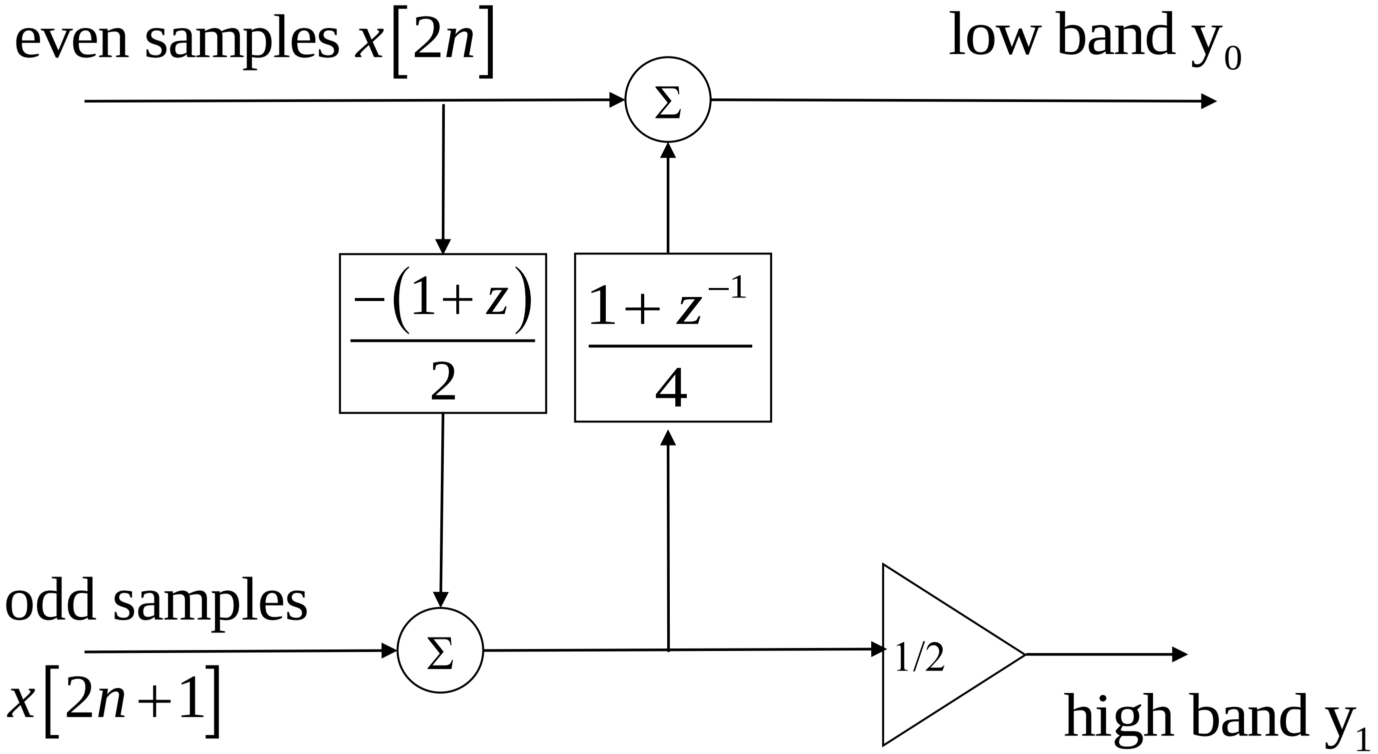
Lifting (cont.)

- Synthesis filters



- Perfect reconstruction (biorthogonality) is directly built into lifting structure
- Powerful for both implementation and filter/wavelet design

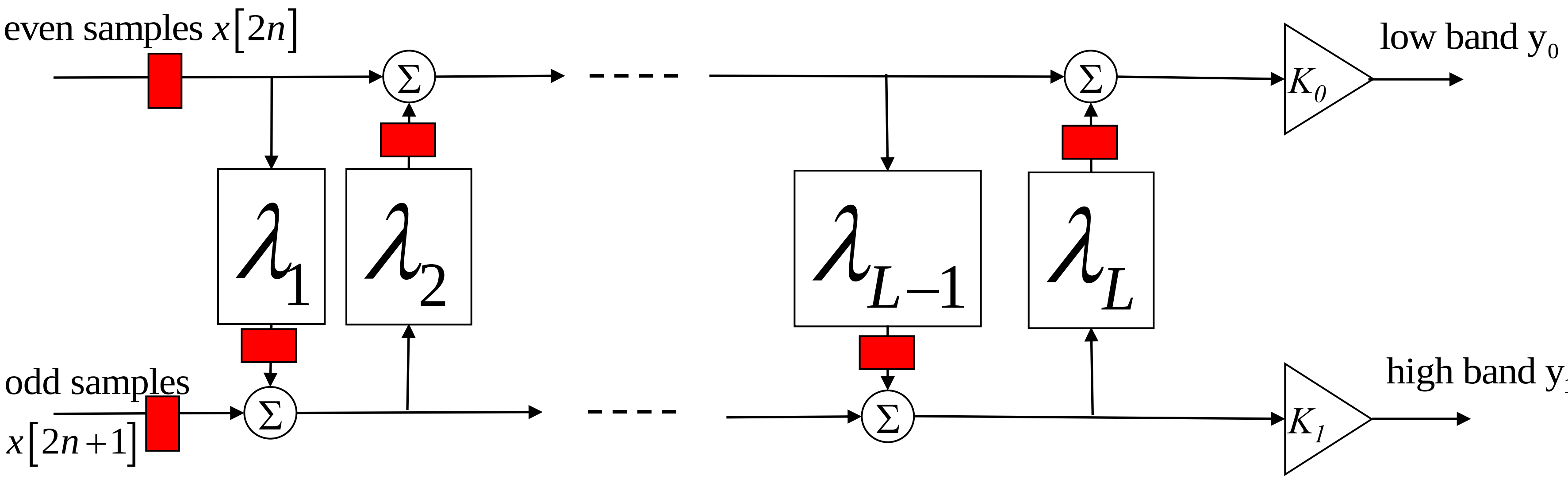
Example: lifting implementation of 5/3 filters



Verify by considering response to unit impulse in even and odd input channel.

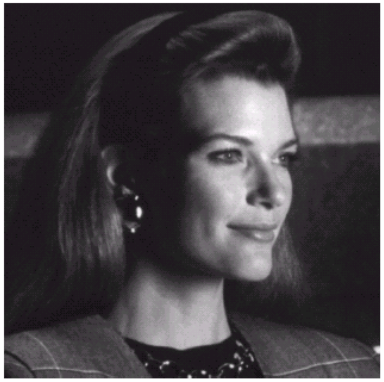
Reversible subband transform

- Observation: lifting operators can be nonlinear.
- Incorporate the necessary rounding into lifting operator:



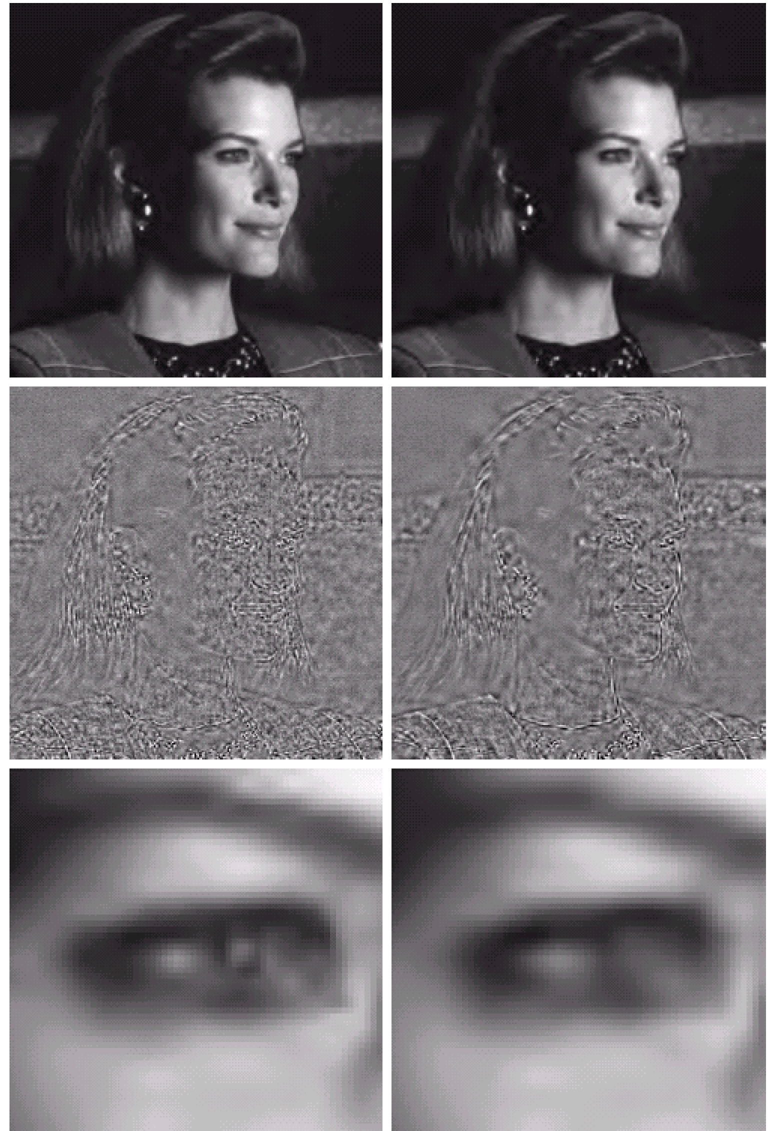
- Used in JPEG2000 as part of 5/3 biorthogonal wavelet transform

Wavelet compression results



Original
512x512
8bpp

0.074 bpp



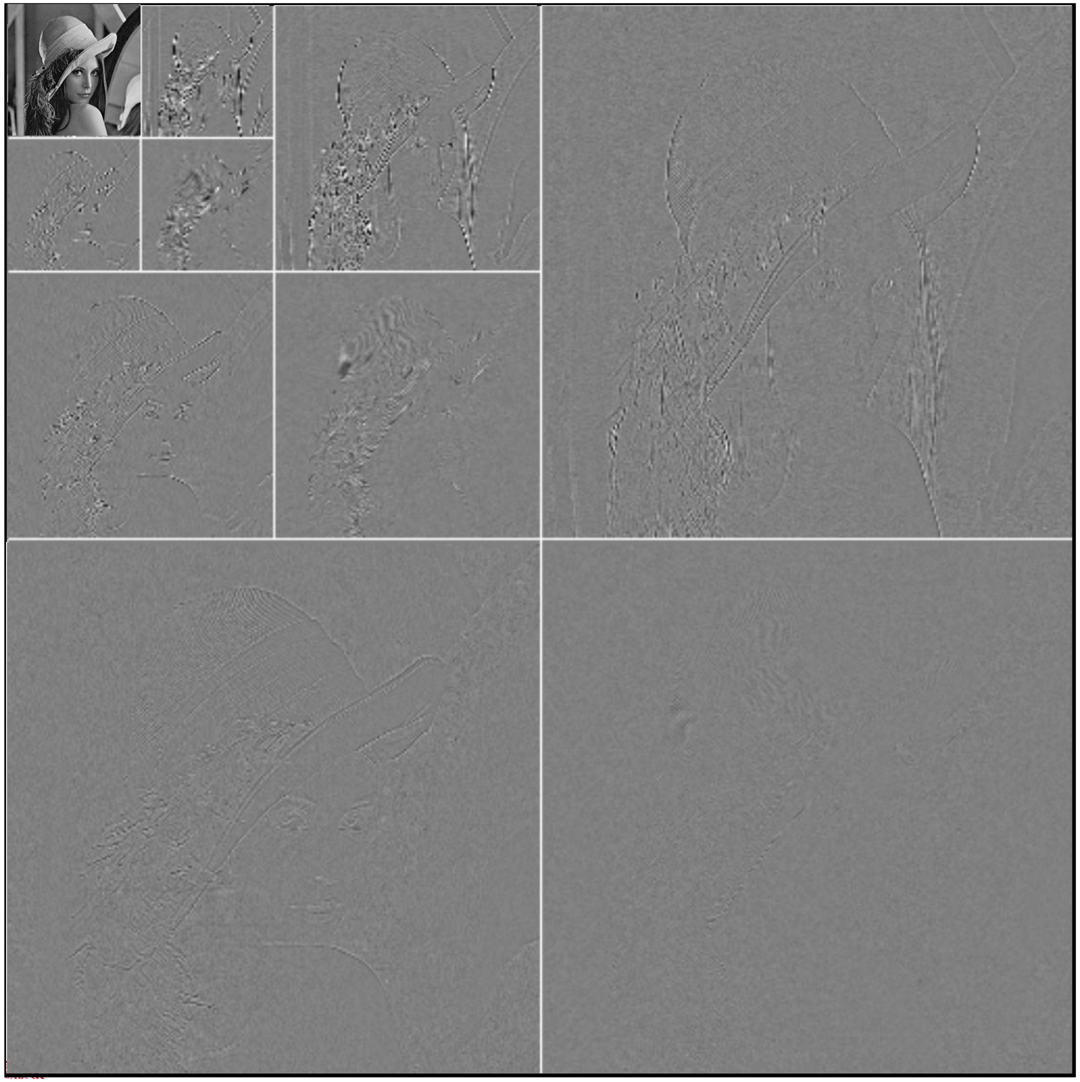
0.048 bpp

Error images

enlarged

[Gonzalez, Woods, 2001]

Embedded zero-tree wavelet algorithm



- Idea: Conditional coding of all descendants (incl. children)
- Coefficient magnitude > threshold: significant coefficients
- Four cases
  1. ZTR: zero-tree, coefficient and all descendants are **not** significant
  2. IZ: coefficient is not significant, but some descendants are significant
  3. POS: positive significant
  4. NEG: negative significant
- For the highest bands, ZTR and IZ symbols are merged into one symbol Z

Embedded zero-tree wavelet algorithm (cont.)

- Successive approximation quantization and encoding
  - Initial „dominant" pass
    - Set initial threshold θ, determine significant coefficients
    - Arithmetic coding of symbols ZTR, IZ, POS, NEG
  - Subordinate pass
    - Refine magnitude of **all** coefficients **found significant so far** by one bit (subdivide magnitude bin by two)
    - Arithmetic coding of sequence of zeros and ones.
  - Repeat dominant pass
    - Omit previously found significant coefficients
    - Decrease threshold θ by factor of 2, determine new significant coefficients
    - Arithmetic coding of symbols ZTR, IZ, POS, NEG
  - Repeat subordinate and dominant passes, until bit budget is exhausted.

Embedded zero-tree wavelet algorithm (cont.)

- Decoding: bitstream can be truncated to yield a coarser approximation: „embedded“ representation
- Further details: *J. M. Shapiro, „Embedded image coding using zerotrees of wavelet coefficients,“ IEEE Transactions on Signal Processing, vol. 41, no. 12, pp. 3445-3462, December 1993.*
- Enhancement SPIHT coder: *A. Said, A., W. A. Pearlman, „A new, fast, and efficient image codec based on set partitioning in hierarchical trees,“ IEEE Transactions on Circuits and Systems for Video Technology, vol. 63 , pp. 243-250, June 1996.*

Conclusions: subband and wavelet coding

- Overlapping basis functions can reduce blocking artifacts
- Biorthogonal subband transforms = perfect reconstruction
- DCT can be understood as a (poor) filter bank
- Discrete Wavelet Transform = cascaded dyadic subband splits
- Quadrature mirror filters and conjugate quadrature filters: aliasing cancellation
- Lifting: powerful for implementation and wavelet construction
- Lifting allows reversible (i.e., lossless) wavelet transform
- Zero-trees: exploit statistical dependencies across subbands

Reading

- Taubman, Marcellin, Sections 4.2, 6.1-6.4, 7
- H. S. Malvar, D. H. Staelin, "The LOT: transform coding without blocking effects," IEEE Transactions on Acoustics, Speech, and Signal Processing, vol. 37, no. 4, pp. 553-559, April 1989.
- D. Le Gall, A. Tabatabai, "Sub-band coding of digital images using symmetric short kernel filters and arithmetic coding techniques," Proc. ICASSP-88, vol. 2, pp. 761-764, April 1988.
- J. M. Shapiro, "Embedded image coding using zerotrees of wavelet coefficients," IEEE Transactions on Signal Processing," vol. 41, no. 12, pp. 3445-3462, Dec. 1993.