Driver assistance system design A

Linearization

Carlo Novara

Politecnico di Torino

Dip. Elettronica e Telecomunicazioni

Linearization - Introduction

- Nonlinear systems are in general difficult to analyze or control.

- It is often of interest to linearize the nonlinear system around
  - an equilibrium point,
  - a trajectory.

- Linearization can be useful to
  - analyze the system (e.g. stability analysis),
  - control (e.g. stabilization around an equilibrium point or gain scheduling).

Linearization - Preliminaries

- Consider a nonlinear system

$$\dot{x}(t) = f\left[x(t), u(t)\right]$$
$$y(t) = h\left[x(t), u(t)\right]$$

where $x(t) \in \mathrm{R}^{n_x}, \quad u(t) \in \mathrm{R}^{n_u}, \quad y(t) \in \mathrm{R}^{n_y}$

$$f : \mathrm{R}^{n_x + n_u} \to \mathrm{R}^{n_x}, \quad h : \mathrm{R}^{n_x + n_u} \to \mathrm{R}^{n_y}$$

- Let $\bar{u}$ be a constant input and $\bar{x}$ be an equilibrium state corresponding to $\bar{u}$.

- Define

$$\delta x(t) = x(t) - \bar{x}$$
$$\delta u(t) = u(t) - \bar{u}$$
$$\delta y(t) = y(t) - h(\bar{x}, \bar{u})$$

Linearization - Preliminaries

- Consider the Taylor expansions of f and h around the point $(\bar{x}, \bar{u})$ truncated at the first order:

$$\dot{x}(t) \cong f(\bar{x},\bar{u}) + \left[\frac{\partial f}{\partial x}\right]_{(\bar{x},\bar{u})} \delta x + \left[\frac{\partial f}{\partial u}\right]_{(\bar{x},\bar{u})} \delta u$$

$$\delta\dot{x}(t) \cong \left[\frac{\partial f}{\partial x}\right]_{(\bar{x},\bar{u})} \delta x + \left[\frac{\partial f}{\partial u}\right]_{(\bar{x},\bar{u})} \delta u \qquad \Leftarrow \begin{cases} \delta\dot{x}(t) = \dot{x}(t) \\ f(\bar{x},\bar{u}) = 0 \end{cases}$$

$$y(t) \cong h(\bar{x},\bar{u}) + \left[\frac{\partial h}{\partial x}\right]_{(\bar{x},\bar{u})} \delta x + \left[\frac{\partial h}{\partial u}\right]_{(\bar{x},\bar{u})} \delta u$$

$$\delta y(t) = y(t) - h(\bar{x},\bar{u}) = \left[\frac{\partial h}{\partial x}\right]_{(\bar{x},\bar{u})} \delta x + \left[\frac{\partial h}{\partial u}\right]_{(\bar{x},\bar{u})} \delta u$$

Linearization - Definition

$$\delta\dot{x}(t) = A\delta x(t) + B\delta u(t)$$
$$\delta y(t) = C\delta x(t) + D\delta u(t)$$

A, B, C, D: Jacobian matrices.

$$A = \left[\frac{\partial f}{\partial x}\right]_{(\bar{x},\bar{u})} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_{n_x}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{n_x}}{\partial x_1} & \cdots & \frac{\partial f_{n_x}}{\partial x_{n_x}} \end{bmatrix}_{(\bar{x},\bar{u})} \in \mathbb{R}^{n_x \times n_x}, \quad B = \left[\frac{\partial f}{\partial u}\right]_{(\bar{x},\bar{u})} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_{n_u}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{n_x}}{\partial u_1} & \cdots & \frac{\partial f_{n_x}}{\partial u_{n_u}} \end{bmatrix}_{(\bar{x},\bar{u})} \in \mathbb{R}^{n_x \times n_u}$$

$$C = \left[\frac{\partial h}{\partial x}\right]_{(\bar{x},\bar{u})} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_{n_x}} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_{n_y}}{\partial x_1} & \cdots & \frac{\partial h_{n_y}}{\partial x_{n_x}} \end{bmatrix}_{(\bar{x},\bar{u})} \in \mathbb{R}^{n_y \times n_x}, \quad D = \left[\frac{\partial h}{\partial u}\right]_{(\bar{x},\bar{u})} = \begin{bmatrix} \frac{\partial h_1}{\partial u_1} & \cdots & \frac{\partial h_1}{\partial u_{n_u}} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_{n_y}}{\partial u_1} & \cdots & \frac{\partial h_{n_y}}{\partial u_{n_u}} \end{bmatrix}_{(\bar{x},\bar{u})} \in \mathbb{R}^{n_y \times n_u}$$

- The linearized system is an approximation of the nonlinear system holding in a neighborhood of the point $(\bar{x}, \bar{u})$.

Linearization - Equilibrium point stability

- Linearization allows us to study the stability properties of equilibrium points of the nonlinear system.

- An equilibrium point $(\bar{x}, \bar{u})$ of the nonlinear system is

  - Asymptotically stable if the linearized system is asymptotically stable ($Re(\lambda_i) < 0, \forall i$).

  - Unstable if the linearized system is exponentially unstable ($Re(\lambda_i) > 0$ for some i).

  - No indications can be obtained in the intermediate situations.

Linearization – Example: pendulum equilibrium states

- State equations:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k}{J}\sin(x_1) - \frac{\beta}{J}x_2 + \frac{1}{J}u$$

- Suppose that $u(t) = \bar{u} = 0$. The corresponding equilibrium points are the solutions of the algebraic equations

$$\bar{x}_2 = 0$$

$$-\frac{k}{J}\sin(\bar{x}_1) - \frac{\beta}{J}\bar{x}_2 = 0$$

- These solutions are given by

$$\begin{cases} \bar{x}_2 = 0 \\ \sin(\bar{x}_1) = 0 \end{cases} \Rightarrow \begin{cases} \bar{x}_1 = k\pi, \, k = 0,1,\ldots \\ \bar{x}_2 = 0 \end{cases}$$

Linearization – Example: pendulum equilibrium states

- Matrices of the system linearized around $(\bar{x}_1, \bar{x}_2, \bar{u}) = (0,0,0)$:

$$A = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix}_{(0,0,0)} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{J}\cos(x_1) & -\dfrac{\beta}{J} \end{bmatrix}_{(0,0,0)} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{J} & -\dfrac{\beta}{J} \end{bmatrix}$$

$$B = \begin{bmatrix} \dfrac{\partial f_1}{\partial u} \\ \dfrac{\partial f_2}{\partial u} \end{bmatrix}_{(0,0,0)} = \begin{bmatrix} 0 \\ \dfrac{1}{J} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0 \qquad (y = x_1)$$

```
>> k=1; beta=2; J=3;
>> A=[0 1;-k/J -beta/J];
>> eig(A)
-0.33333 +     0.4714i
-0.33333 -     0.4714i
```

$\longrightarrow$ the eq. point $(0,0,0)$ is asymptotically stable

Linearization – Example: pendulum equilibrium states

- Matrices of the system linearized around $(\bar{x}_1, \bar{x}_2, \bar{u}) = (\pi, 0{,}0)$:

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{(\pi,0,0)} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{J}\cos(x_1) & -\frac{\beta}{J} \end{bmatrix}_{(\pi,0,0)} = \begin{bmatrix} 0 & 1 \\ \frac{k}{J} & -\frac{\beta}{J} \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}_{(\pi,0,0)} = \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0 \qquad (y = x_1)$$

```
>> k=1; beta=2; J=3;
>> A=[0 1;k/J -beta/J];
>> eig(A)
0.33333
-1
```

→ the eq. point $(\pi, 0{,}0)$ is unstable

Driver assistance system design A

Laplace transform

Carlo Novara
Politecnico di Torino
Dip. Elettronica e Telecomunicazioni

Laplace transform - Introduction

- The Laplace transform is a generalization of Fourier transform.

- It allows us to:
  - Analyze LTI systems in the frequency domain.
  - Transform LTI differential equations into algebraic equations.
  - Easily solve LTI differential equations.
  - Design control systems for LTI systems.

Remark: The Laplace transform can in general be used only for LTI systems.

Laplace transform - Definition

Definition. Consider a function $f(t): \mathbb{R} \to \mathbb{R}^m$.

The (unilateral) Laplace transform $\mathcal{L}\{f(t)\}$ of $f(t)$ is defined as

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = F(s) \qquad s \in \mathbb{C}$$

- The Laplace transform is an operator which associates, to any function of a real variable, a function of a complex variable:

$$f(t): \mathbb{R} \to \mathbb{R}^m \quad \xrightarrow{\mathcal{L}} \quad F(s): \mathbb{C} \to \mathbb{C}^m$$

- Convergence region: $s : \mathrm{Re}(s) > \sigma_o$

where σ_o is such that $\int_0^\infty \left| f(t) e^{-\sigma t} \right| dt < \infty, \quad \forall \sigma > \sigma_o$

Laplace transform - Properties

- <u>Linearity</u>: $$\mathcal{L}\left\{af\left(t\right)+bg\left(t\right)\right\}=aF\left(s\right)+bG\left(s\right),\quad a,b\in\mathbb{R}$$

- <u>Derivative</u>: $$\mathcal{L}\left\{\dot{f}\left(t\right)\right\}=sF\left(s\right)-f\left(0\right)$$
$$\mathcal{L}\left\{\ddot{f}\left(t\right)\right\}=s^2F\left(s\right)-sf\left(0\right)-\dot{f}\left(0\right)$$

- <u>Integral</u>: $$\mathcal{L}\left\{\int_0^t f\left(\tau\right)d\tau\right\}=\frac{F\left(s\right)}{s}$$

- In general $$\mathcal{L}\left\{f\left(t\right)g\left(t\right)\right\}\neq F\left(s\right)G\left(s\right)$$

Laplace transform - Properties

Final value theorem

Let p_i be the poles of $sF(s)$. If $\mathrm{Re}\left(p_i\right) < 0, \forall i,$ then:

$$\lim_{t\to\infty} f\left(t\right) = \lim_{s\to 0} sF\left(s\right)$$

Initial value theorem

$$\lim_{t\to 0} f\left(t\right) = \lim_{s\to\infty} sF\left(s\right)$$

Laplace transform – Tables of main Laplace transforms

polynomial signals		
	$\boldsymbol{f(t)}$	$\boldsymbol{F(s)}$
impulse	$\delta(t)$	1
step	$\varepsilon(t)$	$\frac{1}{s}$
	$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$

exponential signals	
$\boldsymbol{f(t)}$	$\boldsymbol{F(s)}$
e^{pt}	$\frac{1}{s-p}$
e^{At}	$(sI-A)^{-1}$
$\frac{t^n}{n!}e^{pt}$	$\frac{1}{(s-p)^{n+1}}$

harmonic signals	
$\boldsymbol{f(t)}$	$\boldsymbol{F(s)}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$

exponential harmonic signal	
$\boldsymbol{f(t)}$	$\boldsymbol{F(s)}$
$2\lvert r\rvert e^{Re(p)t}\cos[Im(p)t+\angle r]$	$\frac{r}{s-p}+\frac{r^*}{s-p^*}$

Laplace transform – Tables of Laplace transforms

$f(t) = L^{-1}\{F(s)\}$	$F(s)$	$f(t) = L^{-1}\{F(s)\}$	$F(s)$
$a \quad t \geq 0$	$\frac{a}{s} \quad s > 0$	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$at \quad t \geq 0$	$\frac{a}{s^2}$	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
e^{-at}	$\frac{1}{s+a}$	$\sin(\omega t + \theta)$	$\frac{s \sin\theta + \omega \cos\theta}{s^2 + \omega^2}$
te^{-at}	$\frac{1}{(s+a)^2}$	$\cos(\omega t + \theta)$	$\frac{s \cos\theta - \omega \sin\theta}{s^2 + \omega^2}$
$\frac{1}{2}t^2 e^{-at}$	$\frac{1}{(s+a)^3}$	$t \sin \omega t$	$\frac{2\omega s}{(s^2 + \omega^2)^2}$
$\frac{1}{(n-1)!}t^{n-1}e^{-at}$	$\frac{1}{(s+a)^n}$	$t \cos \omega t$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
e^{at}	$\frac{1}{s-a} \quad s > a$	$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2} \quad s > \lvert\omega\rvert$
te^{at}	$\frac{1}{(s-a)^2}$	$\cosh \omega t$	$\frac{s}{s^2 - \omega^2} \quad s > \lvert\omega\rvert$
$\frac{1}{b-a}(e^{-at} - e^{-bt})$	$\frac{1}{(s+a)(s+b)}$	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$
$\frac{1}{a^2}[1 - e^{-at}(1 + at)]$	$\frac{1}{s(s+a)^2}$	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$
t^n	$\frac{n!}{s^{n+1}} \quad n = 1,2,3..$	$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2 + \omega^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}} \quad s > a$	$e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2 + \omega^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}} \quad s > a$	$1 - e^{-at}$	$\frac{a}{s(s+a)}$
$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$	$\frac{1}{a^2}(at - 1 + e^{-at})$	$\frac{1}{s^2(s+a)}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}} \quad s > 0$	$f(t - t_1)$	$e^{-t_1 s}F(s)$
$g(t) \cdot p(t)$	$G(s) \cdot P(s)$	$f_1(t) \pm f_2(t)$	$F_1(s) \pm F_2(s)$
$\int f(t)dt$	$\frac{F(s)}{s} + \frac{f^{-1}(0)}{s}$	$\delta(t)$ *unit impulse*	$1 \quad$ *all* s
$\frac{df}{dt}$	$sF(s) - f(0)$	$\frac{d^2 f}{df^2}$	$s^2F(s) - sf(0) - f'(0)$
$\frac{d^n f}{dt^n}$	$s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{n-1}(0)$		

Driver assistance system design A

Transfer functions

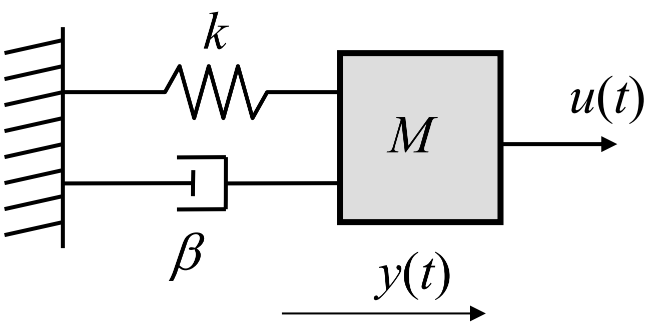
Carlo Novara

Politecnico di Torino

Dip. Elettronica e Telecomunicazioni

Transfer functions – Example: mass-spring-damper system

- Assume zero initial conditions.



$$M\,\ddot{y}(t) = -\beta\,\dot{y}(t) - k\,y(t) + u(t)$$
$$y(0) = 0,\ \dot{y}(0) = 0,$$

$\Downarrow$ Laplace transform

$$Ms^2Y(s) = -\beta sY(s) - kY(s) + U(s)$$
$$\left(Ms^2 + \beta s + k\right)Y(s) = U(s)$$

We obtain $\quad Y(s) = \dfrac{1}{Ms^2 + \beta s + k}U(s) = G(s)U(s)$

$G(s) = \dfrac{1}{Ms^2 + \beta s + k}$ is called the transfer function of the system.

Transfer functions – Definition

- LTI system state equations:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

$$x(t) \in \mathrm{R}^{n_x}$$
$$u(t) \in \mathrm{R}^{n_u}$$
$$y(t) \in \mathrm{R}^{n_y}$$

- Applying the Laplace transform to the first equation:

$$sX(s) - x(0) = AX(s) + BU(s)$$
$$(sI - A)X(s) - x(0) = BU(s)$$

we obtain

$$X(s) = \underbrace{(sI - A)^{-1} x(0)}_{\text{free solution}} + \underbrace{(sI - A)^{-1} BU(s)}_{\text{forced solution}}$$

Transfer functions – Definition

- Applying the Laplace transform to the second equation:

$$Y(s) = CX(s) + DU(s)$$

and considering that $X(s) = (sI - A)^{-1} x(0) + (sI - A)^{-1} BU(s)$

we obtain

$$Y(s) = \underbrace{C(sI - A)^{-1} x(0)}_{\text{free response}} + \underbrace{\left[C(sI - A)^{-1} B + D\right]U(s)}_{\text{forced response}}$$

Transfer functions – Definition

$$Y(s) = \underbrace{C(sI - A)^{-1} x(0)}_{\text{free response}} + \underbrace{\left[C(sI - A)^{-1} B + D\right] U(s)}_{\text{forced response}}$$

Definition. The function

$$G(s) = C(sI - A)^{-1}B + D$$

is called the transfer function (tf) of the system.

- The transfer function describes the forced response ($x(0) = 0$ and $U \neq 0$) of the system:

$$Y(s) = G(s)U(s)$$

Transfer functions – Terminology

- In the SISO case ($n_y, n_u = 1$), $G(s)$ is a rational function, i.e. the ratio between two polynomials in s:

$$G : \mathrm{C} \rightarrow \mathrm{C}, \quad G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_0}$$

  - $m \leq n \leq n_x$ = system order.
  - a_k, b_k: real coefficients.

- In the MIMO case ($n_y, n_u > 1$) $G(s)$ is an n_y x n_u matrix, called transfer matrix, where each entry is a rational function:

$$G_{kl} : \mathrm{C} \rightarrow \mathrm{C},\ k = 1,\ldots,n_y,\ l = 1,\ldots,n_u$$

Transfer functions – Terminology

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0} =$$

⟸ rational representation

$$= K_G \frac{\prod_{i=1}^{m} (s - z_i)}{\prod_{i=1}^{n} (s - p_i)}$$

⟸ zero-pole-gain representation

- $p_1,\ldots,p_n$ = poles of $G(s)$ = roots of its denominator
  $z_1,\ldots,z_m$ = zeros of $G(s)$ = roots of its numerator
  K_G = gain

- Proper tf: $n \geq m$

  Strictly proper tf: $n > m$

  Minimum form tf: $z_i \neq p_l \ \forall i,l$

  Minimum phase zero: $Re(z_i) < 0$

  Minimum phase tf: $G(s) : Re(z_i) < 0, \forall i$

Transfer functions – Matlab

- Example:

$$G(s)=\frac{2s^2-3s+1}{s^3+5s^2+4s}$$

```
s = tf('s');
G = (2*s^2-3*s+1)/(s^3+5*s^2+4*s);
```
or
```
G = tf([2 -3 1],[1 5 4 0]);
```

- Example:

$$G(s)=\frac{5(s+1)}{(s+2)(s+3)}$$

```
s = tf('s');
G = 5*(s+1)/(s+2)/(s+3);
```
or
```
G = zpk([-1],[-2 -3],5);
```

- Changes of representation:

```
sys = ss(A,B,C,D);   →   G = tf(sys);   G = zpk(sys);
G = tf(num,den);     →   sys = ss(G);   G = zpk(G);
G = zpk(z,p,k);      →   sys = ss(G);   G = tf(G);
```

- See `help tf`, `help zpk`, `help ss`.

Driver assistance system design A

Laplace and state space descriptions

Carlo Novara
Politecnico di Torino
Dip. Elettronica e Telecomunicazioni

Laplace and SS descriptions – Discussion

- Description of an LTI system in state space form:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\end{aligned} \quad \Longrightarrow \quad \boxed{S: A, B, C, D}$$

internal description

- Description of an LTI system in transfer function form :

$$Y(s) = G(s)U(s) \quad \Longrightarrow \quad \boxed{S: G(s)}$$

external (input-output) description

Laplace and SS descriptions – Eigenvalues and poles

- The transfer function of an LTI SISO system is given by

$$G(s) = C(sI - A)^{-1} B + D = \frac{N(s)}{P(s)}$$

$$P(s) = \det(sI - A) = (s - \lambda_1)\ldots(s - \lambda_{n_x})$$ = characteristic polynomial OF THE MATRIX A

$\lambda_1, \ldots, \lambda_{n_x}$ = eigenvalues of A

- It may happen that $N(s) = K_G\,(s - z_1)(s - z_2)\ldots(s - \lambda_k)(s - \lambda_l)$

$$G(s) = \frac{K_G (s - z_1)(s - z_2)\ldots\cancel{(s - \lambda_k)}\cancel{(s - \lambda_l)}}{(s - \lambda_1)\ldots\cancel{(s - \lambda_k)}\cancel{(s - \lambda_l)}\ldots(s - \lambda_{n_x})} = \frac{K_G (s - z_1)\ldots(s - z_m)}{(s - p_1)\ldots(s - p_n)}$$

$p_1, \ldots, p_n$: poles of $G(s)$.

Laplace and SS descriptions – Eigenvalues and poles

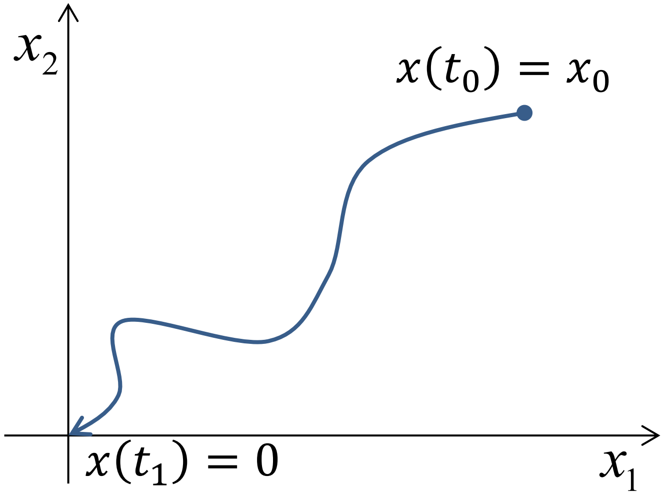
$$G(s) = \frac{K_G(s-z_1)\ldots(s-z_m)}{(s-p_1)\ldots(s-p_n)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0} = \frac{N(s)}{P(s)}$$

$\{p_1,\ldots,p_n\}$ = poles of $G(s)$ = roots of $P(s)$

- In general $\{p_1,\ldots,p_n\} \subseteq \{\lambda_1,\ldots,\lambda_{n_x}\}$.

- If $\{p_1,\ldots,p_n\} = \{\lambda_1,\ldots,\lambda_{n_x}\}$, then the SS system (A,B,C,D) is said in minimum form or a minimal realization of $G(s)$.

- $\{p_1,\ldots,p_n\} = \{\lambda_1,\ldots,\lambda_{n_x}\}$ if and only if the system is (completely) controllable and observable.



Controllability is the ability of an external input to move the state of a system from an initial state to zero in a finite time interval.

Theorem. An LTI system is (completely) controllable if and only if the controllability matrix, defined as

$$M_c = [B \quad AB \quad A^2B \quad \dots \quad A^{n_x-1}B],$$

has full rank: $rank(M_c) = n_x$.

Observability is the possibility of recovering the system state from the measurement of system input and output.

Theorem. An LTI system is (completely) observable if and only if the observability matrix, defined as

$$M_o = [C; \quad CA; \quad CA^2; \quad \dots; \quad CA^{n_x-1}],$$

has full rank: $rank(M_o) = n_x$.

Laplace and SS descriptions – Stability

- Consider an LTI system with transfer function $G(s)$ and null initial conditions. Let $p_1, \ldots, p_n$ be the poles of $G(s)$ and let $k_1, \ldots, k_n$ be their multiplicities.

BIBO stability:

(1) input-output stable iff

$$Re(p_i) < 0, \forall i;$$

(2) input-output unstable iff

$$\exists i: Re(p_i) \geq 0.$$

Internal stability (if the system is completely controllable and observable):

(1) asymptotically (exponentially) stable iff

$$Re(p_i) < 0, \forall i;$$

(2) (simply) stable iff

$$Re(p_i) \leq 0, \forall i \text{ and } k_l = 1 \text{ for } \forall l: Re(p_l) = 0;$$

(3) unstable iff

$$\exists i: Re(p_i) > 0 \text{ or } \exists i: Re(p_i) = 0 \text{ with } k_i > 1.$$

Laplace and SS descriptions – Matlab commands

- Controllability matrix:

```
>> A=[1 -2;-3 4];
>> B=[5;-6];
>> Mc=ctrb(A,B)
Mc =
    5    17
   -6   -39
>> rank(Mc)
ans = 2
```

- Observability matrix:

```
>> A=[1 -2;-3 4];
>> C=[-1 2];
>> Mo=obsv(A,C)
Mo =
   -1     2
   -7    10
>> rank(Mo)
ans = 2
```

- Poles of a transfer function:

```
>> s=tf('s');
>> G=(s+1)/(s^4+2*s^3-3*s+1);
>> pole(G)
ans =
  -1.5815 + 0.9392i
  -1.5815 - 0.9392i
   0.7879
   0.3751
```

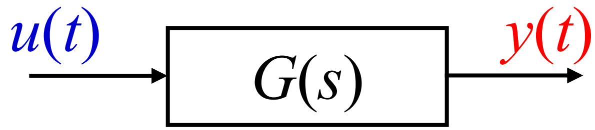
Driver assistance system design A

Frequency response

Carlo Novara
Politecnico di Torino
Dip. Elettronica e Telecomunicazioni

Frequency response – Introduction

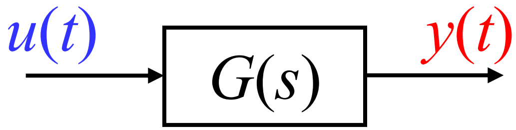
- Consider an asympotically stable LTI SISO system, described by a transfer function $G(s)$:



- Consider the harmonic input signal $u(t)=A_u\sin(\omega t)$.

- It is of interest to study the steady-state behavior of the resulting forced response $y(t)$:
    - Frequency analysis of the system.

Frequency response – Example



$$G = \frac{8}{(1+0.1s)(1+10s)}$$

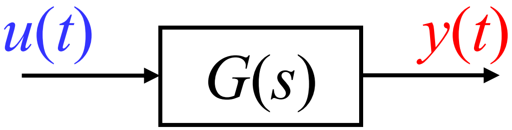
$$u(t) = A_u \sin(\omega t)$$

$$A_u = 1$$

$$\omega = 0.6 \text{ rad/s}$$

$$y_{ss}(t) = ?$$

Frequency response – Example

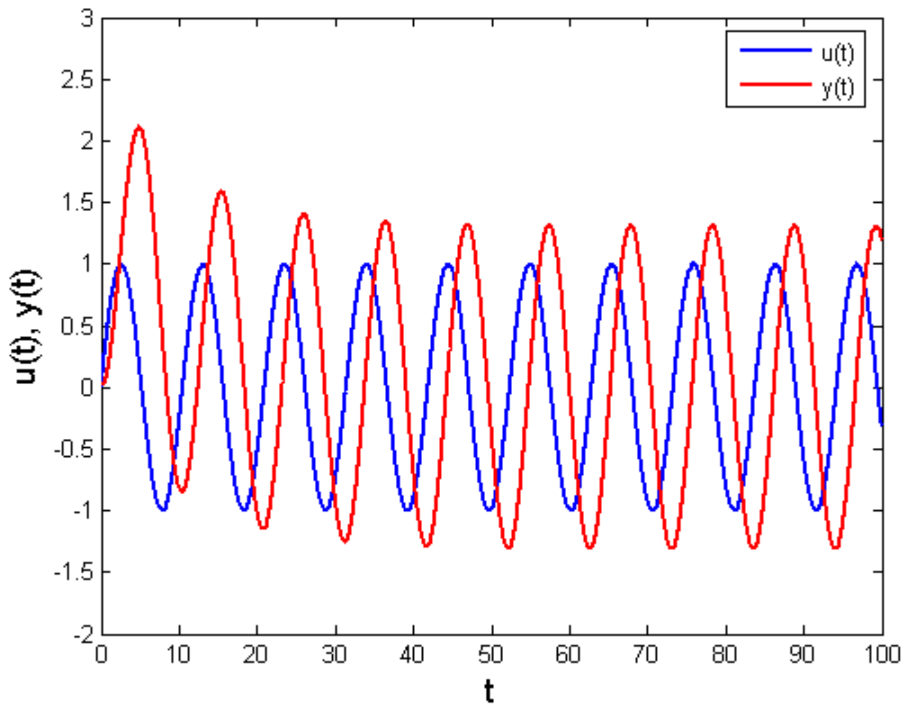


$$G = \frac{8}{(1+0.1s)(1+10s)}$$

$$u(t) = A_u \sin(\omega t)$$

$$A_u = 1$$

$$\omega = 0.6 \text{ rad/s}$$



Frequency response – Main result

- Consider an asymptotically stable LTI SISO system, described by a transfer function $G(s)$:

- Suppose that the following harmonic signal is used as the input of the system:

$$u(t) = A_u \sin(\omega t)\ .$$

Frequency response theorem. The resulting steady-state forced response is given by

$$y_{ss}(t) = A_y \sin(\omega t + \varphi)$$

where $A_y = A_u|G(j\omega)|$ and $\varphi = \angle G(j\omega)$.

Frequency response – Frequency response function

- The steady-state forced response of an LTI system due to an harmonic input signal is still an harmonic signal with
  - the same frequency ω as the input,
  - amplitude multiplied by $|G(j\omega)|$,
  - phase equal to $\angle G(j\omega)$.

- The function $G(j\omega)$ is fundamental to study the frequency behavior of the LTI system.

- The function $G(j\omega)$ is called the frequency response function.

- A signal can be written as the superposition of harmonic functions (Fourier theorem) → $G(j\omega)$ can be used to study the frequency behavior of any signal for which the Fourier theorem holds.

- The frequency response function has real domain (the frequency is a real variable) and complex codomain:

$$G(j\omega)\colon\ \mathbb{R} \to \mathbb{C}$$

- For given ω, $G(j\omega)$ is a complex number.

- Two main approaches are commonly used to represent a complex number:
  - Cartesian representation: real and imaginary part.
  - Polar representation: magnitude and phase.

Frequency response – Frequency response function

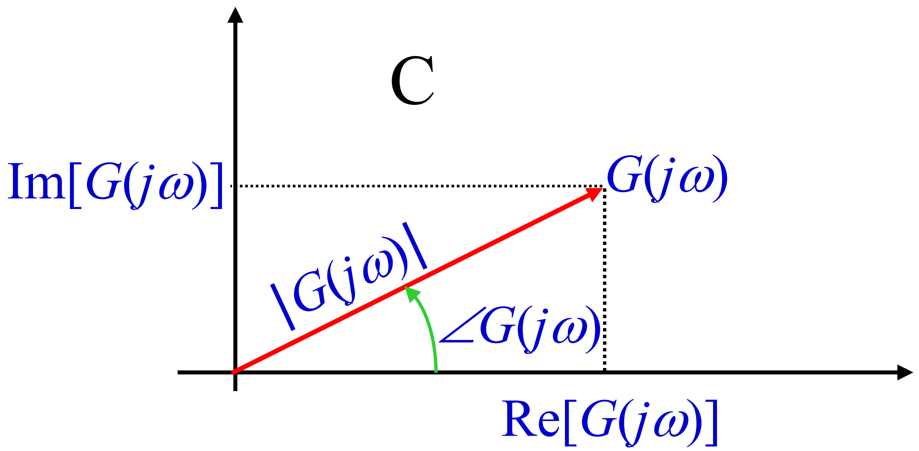
$$G(j\omega): \ \mathbb{R} \to \mathbb{C}, \quad \omega \in \mathbb{R}, \quad G(j\omega) \in \mathbb{C}$$

- Cartesian representation:

$$G(j\omega) = \mathrm{Re}\left[G(j\omega)\right] + j\,\mathrm{Im}\left[G(j\omega)\right]$$

- Polar representation:

$$G(j\omega) = \left|G(j\omega)\right| e^{j\angle G(j\omega)}$$



- The most common graphical representations of the frequency response function are the following:

  – **Bode diagrams**. Two plots:

    1st plot: - x axis: ω (rad/s, log scale)
    - y axis: $|G(j\omega)|_{\mathrm{dB}}$ (linear scale)

    2nd plot: - x axis: ω (rad/s, log scale)
    - y axis: $\angle G(j\omega)$ (degrees, linear scale)

– **Nyquist diagrams**. A single plot:

- x axis: $\mathrm{Re}[G(jw)]$ (linear scale)
- y axis: $\mathrm{Im}[G(jw)]$ (linear scale)

– **Nichols diagrams**. A single plot:

- x axis : $\angle G(j\omega)$ (degrees, linear scale)
- y axis : $|G(j\omega)|_{\mathrm{dB}}$ (linear scale)

- In order to construct the Bode diagrams, let us introduce the Bode representation for a transfer function:

$$G(s)=\frac{K}{s^l}\frac{\prod_i(1+\tau_i's)}{\prod_i(1+\tau_i s)}\frac{\prod_i\left(1+\frac{2\zeta_i's}{\omega_{n,i}'}+\frac{s^2}{\omega_{n,i}'^2}\right)}{\prod_i\left(1+\frac{2\zeta_i s}{\omega_{n,i}}+\frac{s^2}{\omega_{n,i}^2}\right)}=\prod_m H_m(s)$$

where the functions $H_m(s)$ are called the elementary factors (or terms).

- The elementary terms are functions of the following types:

K : constant

s^q : pole (q<0) or zero (q>0) at the origin

$(1+\tau s)^q$: real pole (q<0) or zero (q>0)

$\left(1+\frac{2\zeta_i s}{\omega_n}+\frac{s^2}{\omega_n^2}\right)^q$: couple of compl. conj poles (q<0) or zeros (q>0)

- For the logarithm properties:

$$\left|G(s)\right|_{dB} = \sum_m \left|H_m(s)\right|_{dB}$$

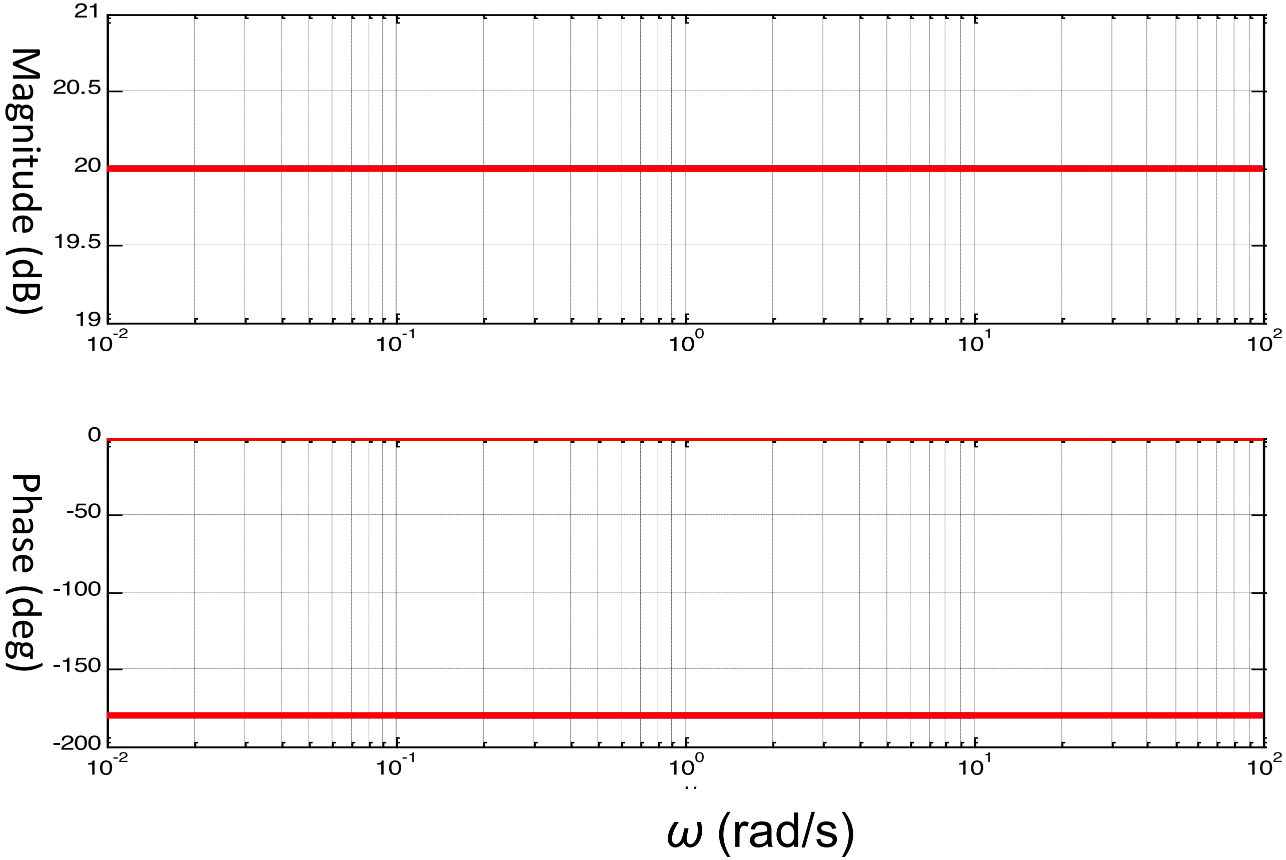
- For the phase properties:

$$\angle G(s) = \sum_m \angle H_m(s)$$

- Thanks to these properties, the Bode diagram of a complicated function can be obtained as the sum of the Bode diagrams of its elementary terms. This allows us to
  - draw the Bode diagrams of complicated functions "by hand";
  - verify the correctness of Bode diagrams drawn using Matlab.

Frequency response – Bode diagrams: elementary terms

Constant term: $G(s) = K$

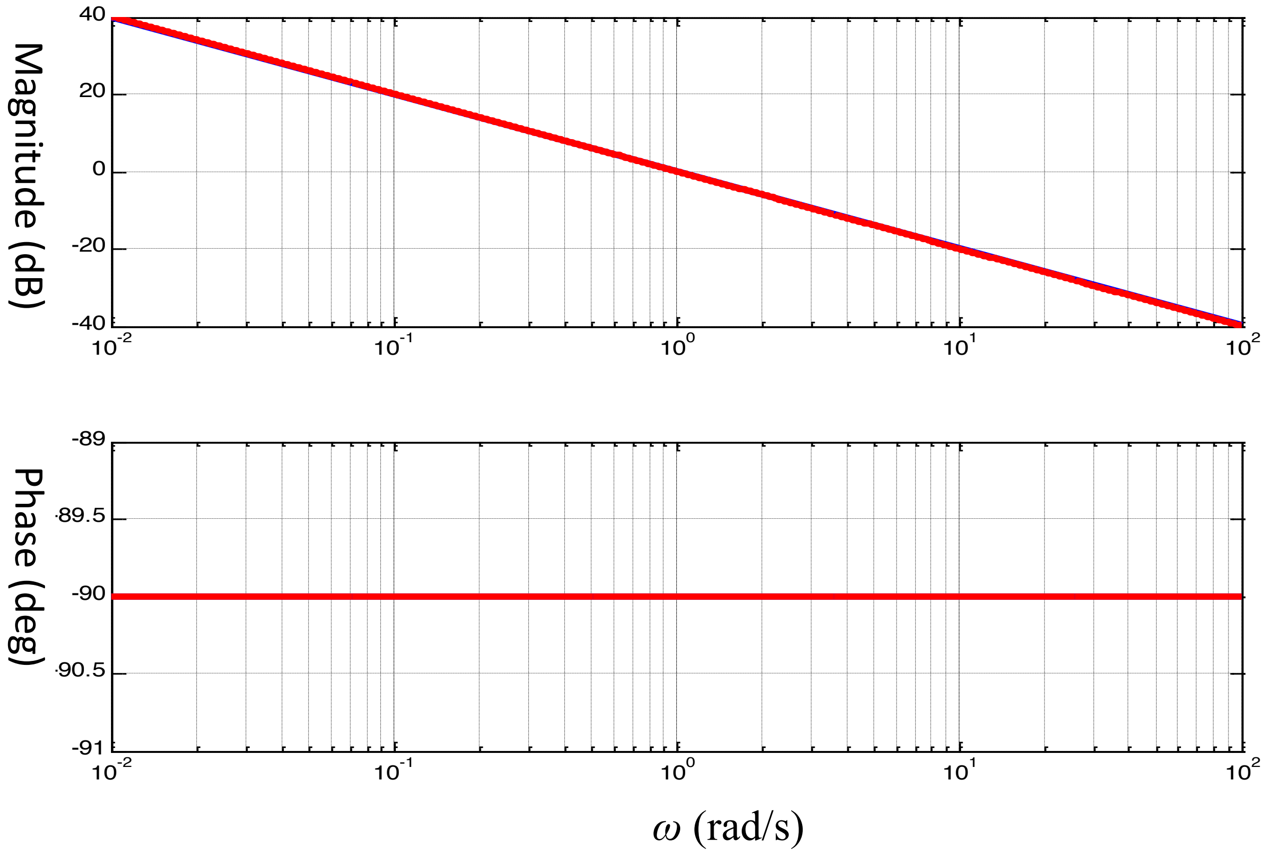


$$\left|G(j\omega)\right|_{dB} = \left|K\right|_{dB}$$

$$\angle G(j\omega) = \begin{cases} 0^{\circ}, K > 0 \\ -180^{\circ}, K < 0 \end{cases}$$

Frequency response – Bode diagrams : elementary terms

Pole at the origin: $G(s)=\dfrac{1}{s}$



$$\left|G(j\omega)\right|_{dB}=20\log_{10}\left(\left|\frac{1}{j\omega}\right|\right)$$

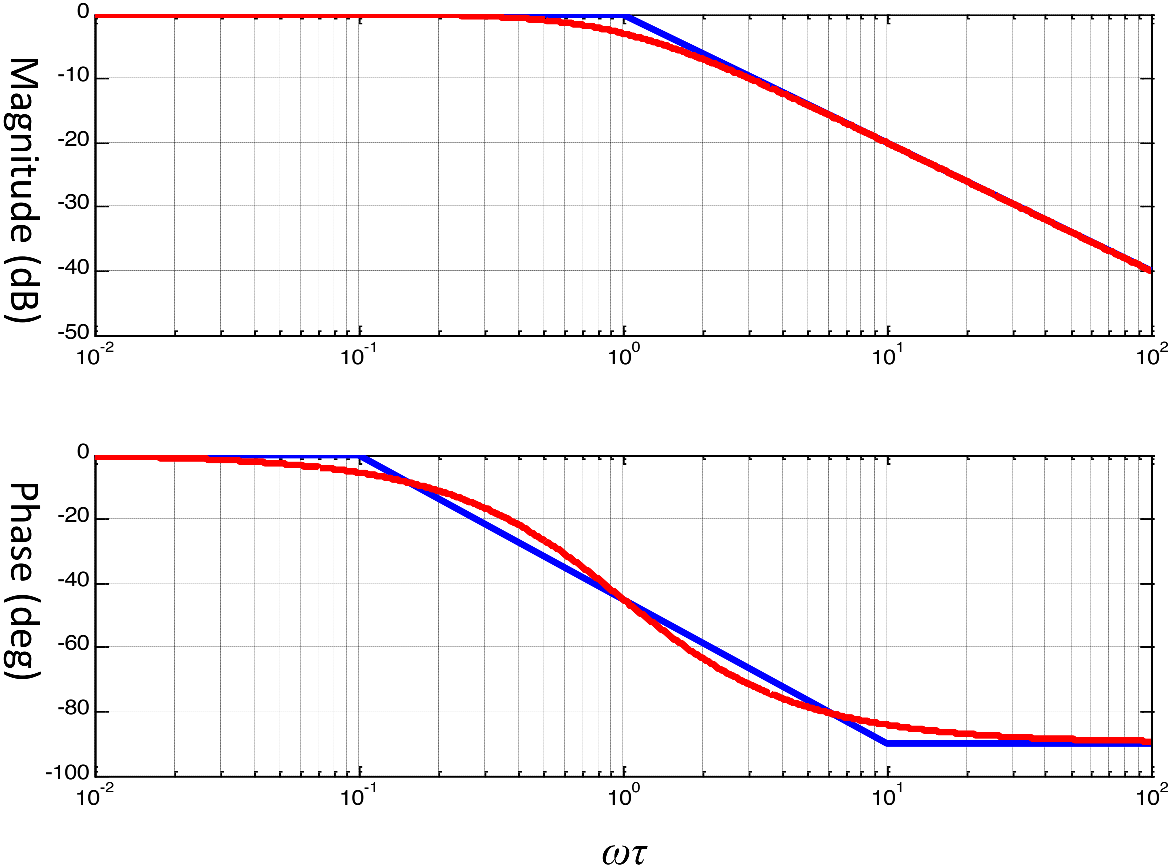
$$=-20\log_{10}(\omega)$$

$$\angle G(j\omega)=\angle\left(\frac{1}{j\omega}\right)=-90^{\circ}$$

$$\frac{1}{j\omega}\cdot\frac{j}{j}=-\frac{j}{\omega}$$

Frequency response – Bode diagrams : elementary terms

Real negative pole: $G(s)=\dfrac{1}{1+\tau s},\quad \tau>0$



$$\left|G(j\omega)\right|_{dB}=20\log_{10}\left(\left|\frac{1}{1+j\omega\tau}\right|\right)$$

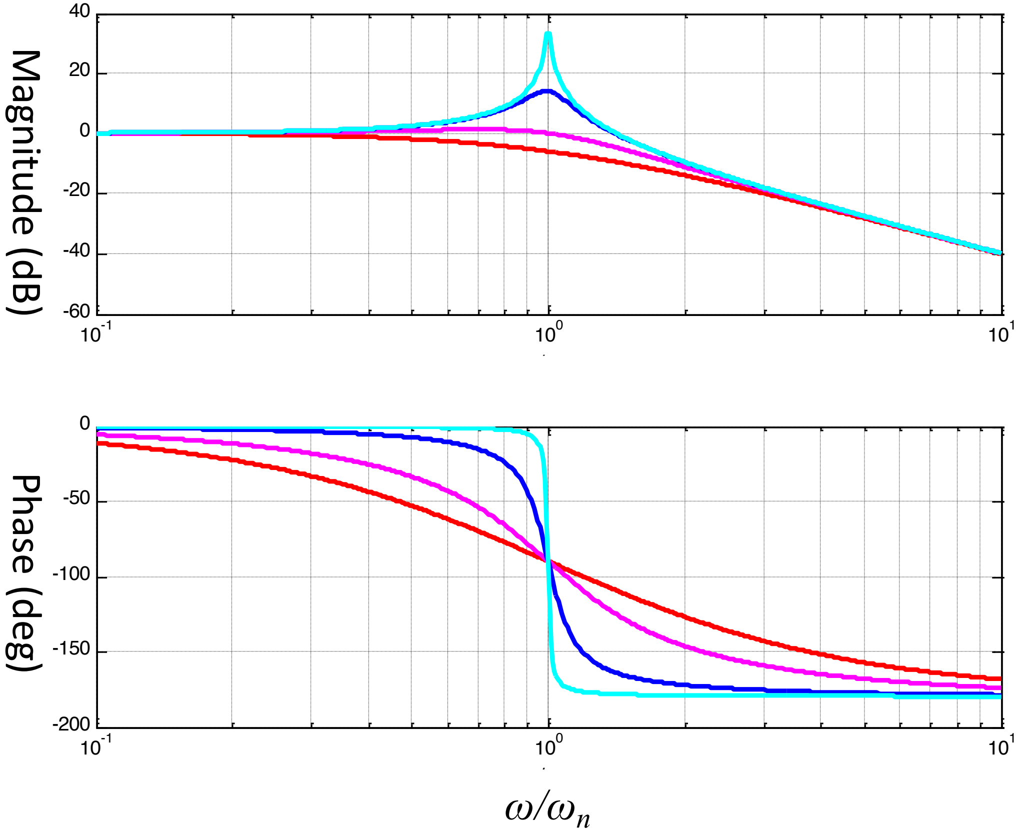
$$=-20\log_{10}\left(\sqrt{1+(\omega\tau)^2}\right)$$

$$\angle G(j\omega)=-\angle\left(1+j\omega\tau\right)=$$

$$=-arctg\left(\omega\tau\right)$$

Frequency response – Bode diagrams : elementary terms

Couple of complex conjugate poles: $G(s) = \dfrac{1}{1 + \dfrac{2\zeta s}{\omega_n} + \dfrac{s^2}{\omega_n^2}}$, $\zeta \geq 0$



For $0 \leq \zeta \leq 1/\sqrt{2}$ we have a resonance peak

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

at the frequency

$$\omega_r = \omega_n\sqrt{1-2\zeta^2}$$

ζ = 0.01 0.1 0.5 1

Frequency response – Bode diagrams: general rules

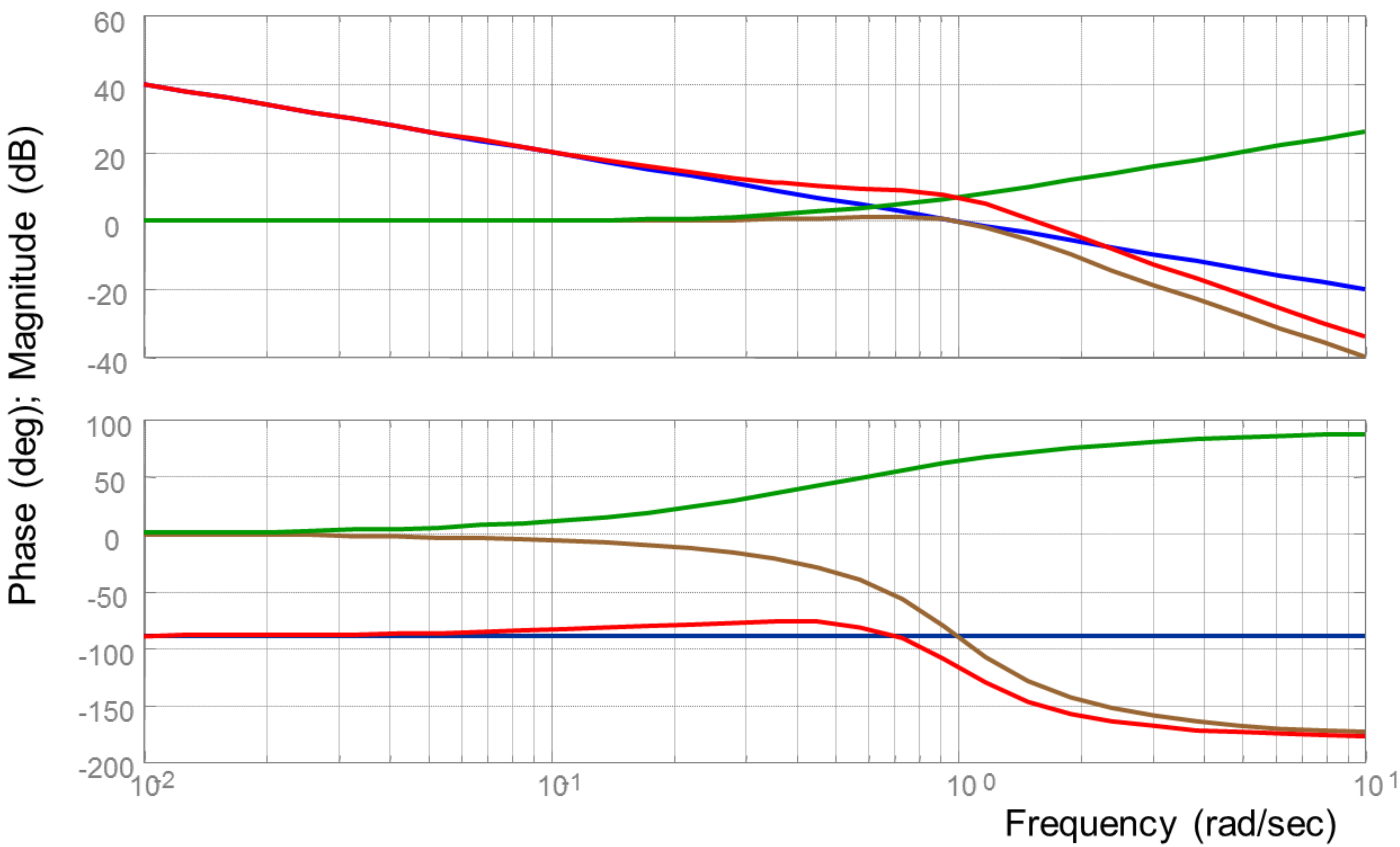
- Zeros:
  - the diagrams of the zeros can be obtained from those of the poles by symmetric reflection with respect to the x axis.

- Poles (zeros) with positive real part:
  - the magnitude diagrams are equal to the magnitude diagrams of the corresponding poles (zeros) with negative real part;
  - the phase diagrams can be obtained from those of the poles by symmetric reflection with respect to the x axis.

- Multiple terms (factors):
  - the diagrams of multiple terms can be obtained from those of the elementary terms by summation.

Frequency response – Bode diagrams: functions with several factors

$$G(s)=(1+2s)/(s(s^2+s+1))$$



Frequency response – Bode diagrams: Matlab commands

- To draw the Bode plot of a transfer function:

```
>> s=tf('s');
>> G=1/(s^2+3*s+2);
>> bode(G)      % magnitude in dB
```

- To compute magnitude and phase for a given frequency **w**:

```
>> w=5;
>> [m,f] = bode(G,w)
m =
    0.0364      % magnitude in linear scale
f =
 -146.8887
```