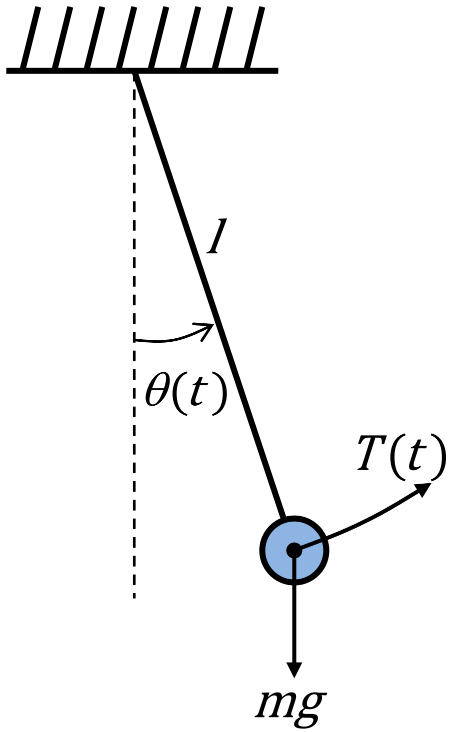
Dynamic systems - A classical example: pendulum



- Variables:
  - $\theta(t)$: angular position
  - $\omega(t) = \dot{\theta}(t)$: angular velocity
  - $T(t)$: applied torque.

- Parameters:
  - m : mass
  - l : length
  - g : gravity acceleration.

- Input: $u(t) = T(t)$.
- Output: we can choose $y(t) = \theta(t)$.

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- According to the 2nd principle of dynamics (Newton's law):

$$J\ddot{\theta}(t) = -K\sin[\theta(t)] - \beta\dot{\theta}(t) + T(t)$$

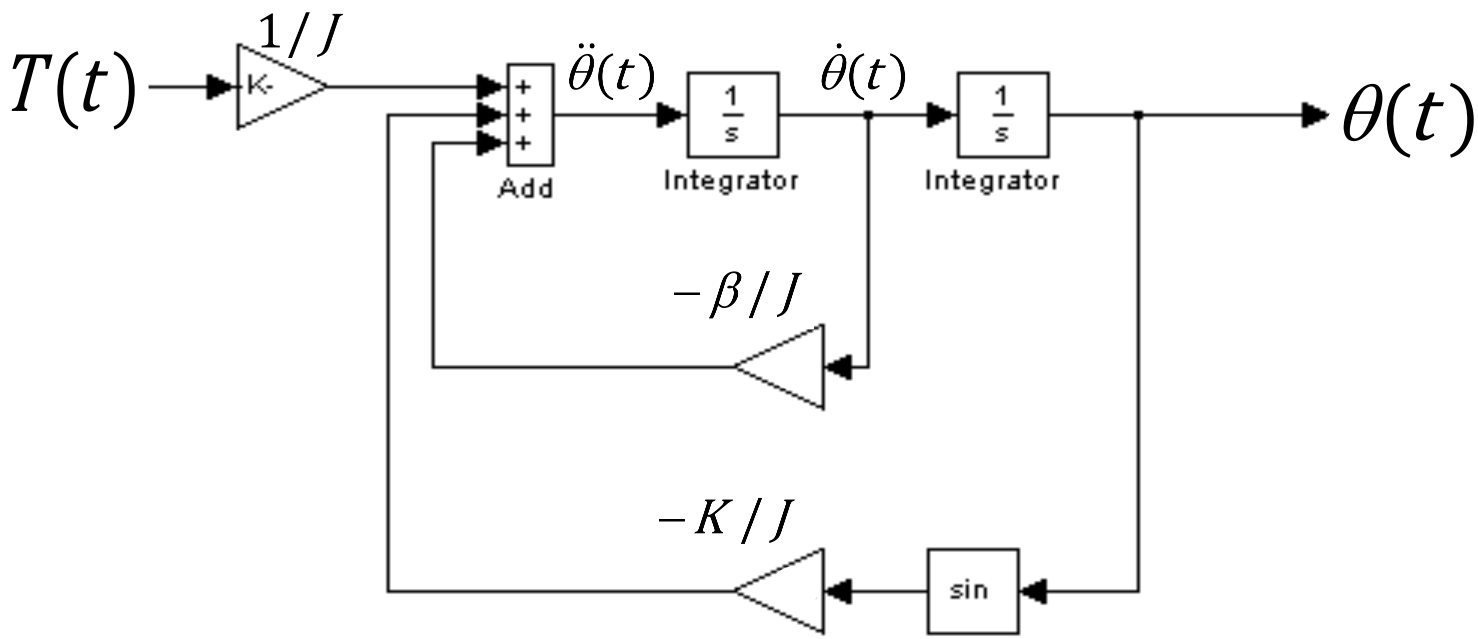
where $J = ml^2$: moment of inertia
$K = gml$: elastic constant
β : friction coefficient.

- The time evolution of the system is described by differential equations.
- The behavior of the system (more precisely, of its variables) can be predicted by integration of the differential equations.
- Integration can be performed
  - analytically (possible in particular cases)
  - numerically (always possible).

Dynamic systems - A classical example: pendulum

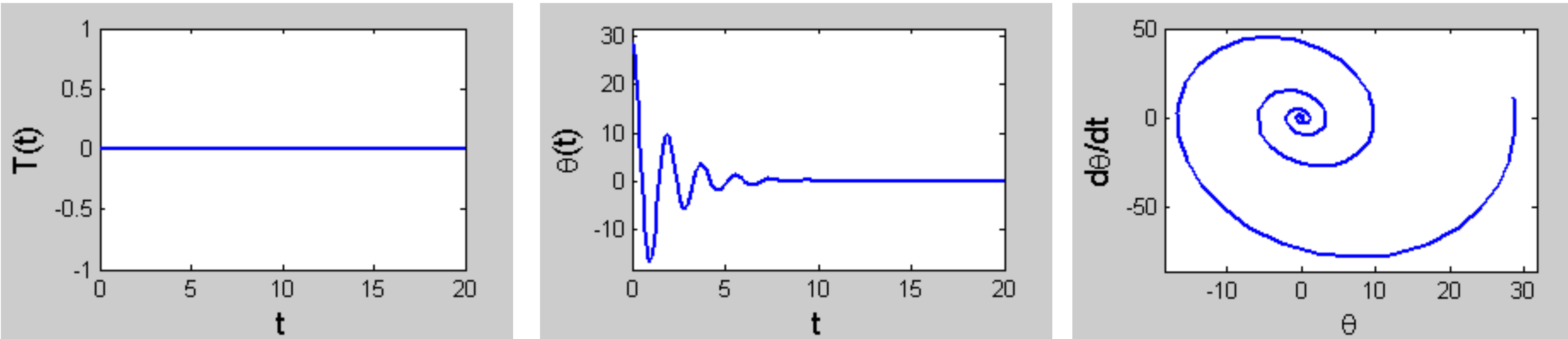
- Pendulum equation: $\ddot{\theta}(t) = -\frac{K}{J}\sin[\theta(t)] - \frac{\beta}{J}\dot{\theta}(t) + \frac{1}{J}T(t)$

- Numerical integration using Simulink:

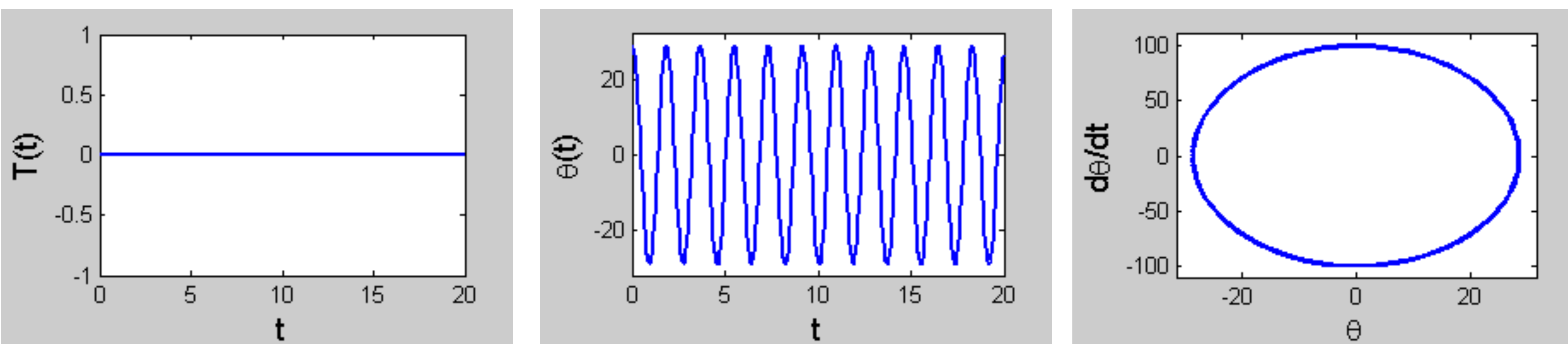


Dynamic systems - A classical example: pendulum

Damped pendulum ($\beta > 0$), non-null initial conditions, free evolution ($T(t) = 0$).

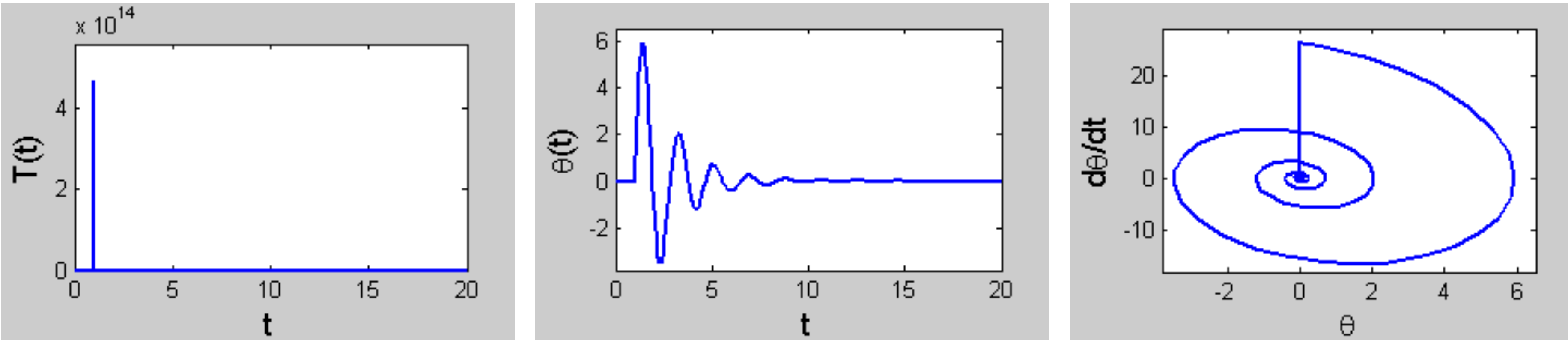


Non-damped pendulum ($\beta = 0$), non-null initial conditions, free evolution ($T(t) = 0$).

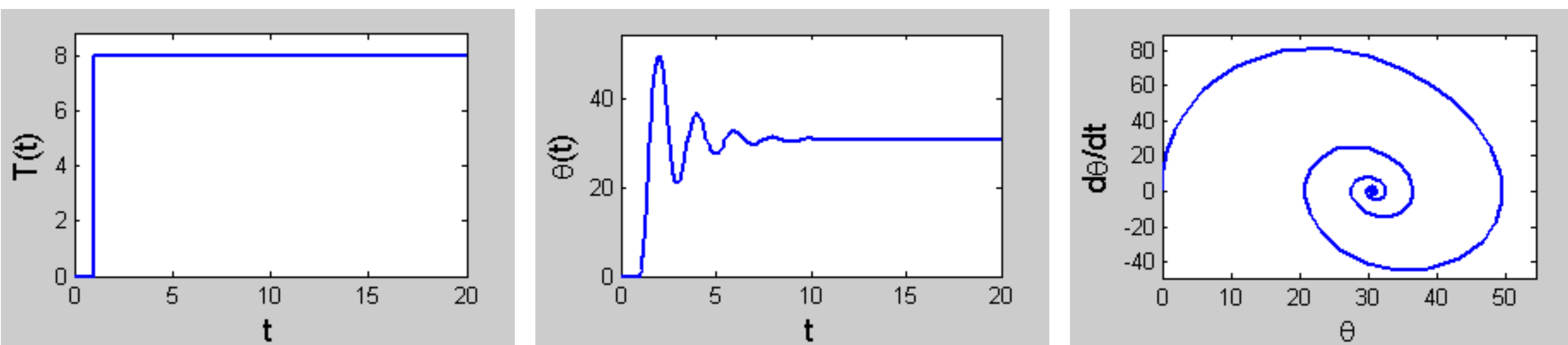


Dynamic systems - A classical example: pendulum

Damped pendulum ($\beta > 0$), null initial conditions, impulse response.



Damped pendulum ($\beta > 0$), null initial conditions, step response.



State equations – Example: pendulum

- Newton's equation: $$J\ddot{\theta}(t) = -k\sin\left[\theta(t)\right] - \beta\dot{\theta}(t) + T(t)$$

  The input is $u(t) = T(t) \in \mathbb{R}$. Suppose that $y(t) = \theta(t) \in \mathbb{R}$. Choosing

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix} \in \mathbb{R}^2$$

we have

$$\boxed{\begin{aligned} &\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\dfrac{k}{J}\sin\left[x_1(t)\right] - \dfrac{\beta}{J}x_2(t) + \dfrac{1}{J}u(t) \end{cases} \\ &y(t) = x_1(t) \end{aligned}}$$

state equations

Note: $$f(x,u;t) = f(x,u) = \begin{bmatrix} x_2 \\ -\dfrac{k}{J}\sin(x_1) - \dfrac{\beta}{J}x_2 + \dfrac{1}{J}u \end{bmatrix}, \quad f : \mathbb{R}^3 \to \mathbb{R}^2$$

$$h(x,u;t) = h(x_1) = x_1, \quad h : \mathbb{R} \to \mathbb{R}$$