Driver assistance system design A

Dynamic systems and models

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Outline

Introduction

- In this lecture, fundamental concepts about dynamic systems are discussed:
  - dynamic systems
  - differential equations
  - state equations
  - solutions/trajectories of dynamic systems
  - linearization
  - nonlinear/linear systems
  - stability notions
  - transfer functions.
- These concepts are illustrated by means of examples/simulations involving physical systems.

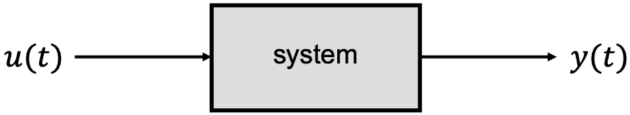
Dynamic systems

A dynamic system can be (roughly) defined as a set of interacting objects which evolve over time.

Examples:

- vehicles
- mechanical systems
- electrical circuits
- aircrafts
- spacecrafts, satellites
- stock market
- animal population
- atmosphere
- planet systems
- and so on...

Dynamic systems

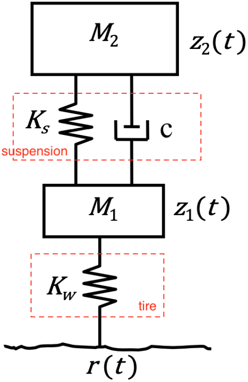


- Fundamental variables:
  - Input $u(t)$: variables which influence the time evolution of the system (causes).
  - Output $y(t)$: measured.

- Input types:
  - Command inputs: their behavior can be chosen by the human user.
  - Disturbances: their behavior is independent on the human user; they cannot be chosen.

Example. Quarter-car model

- Quarter-car models are widely used to describe the vehicle vertical dynamics and to design/test suspension control systems.



- Variables:
  $r(t)$: road profile
  $z_1(t)$: wheel vertical position
  $z_2(t)$: quarter-chassis vertical position;
  positions are expressed wrt equilibrium.
- Parameters:
  M_1: wheel mass
  M_2: quarter-chassis mass
  c: suspension damping coefficient
  K_S: suspension spring coefficient
  K_W: tire elastic coefficient.

Example. Quarter-car model

- To describe the time evolution of the quarter-car system, we write the Newton's equations for the two masses.
  - Wheel equation:
$$M_1 \ddot{z}_1 = -c\,(\dot{z}_1 - \dot{z}_2) - K_S\,(z_1 - z_2) - K_W\,(z_1 - r)\,.$$
  - Quarter-chassis equation:
$$M_2 \ddot{z}_2 = c\,(\dot{z}_1 - \dot{z}_2) + K_S\,(z_1 - z_2)\,.$$
- Comments:
  - r is the input of the system (in general time-varying);
    r is a disturbance.
  - $z_1, z_2, \dot{z}_1, \dot{z}_2, \ddot{z}_1, \ddot{z}_2$ are variables that change in time depending on r.
  - The two equations are coupled.
  - The gravity force is accounted for by a proper definition of the equilibrium positions.

Models of dynamic systems

- We have constructed a model of a quarter-car system.
- Models are often written in the form of differential equations (continuous- time systems) or finite-difference equations (discrete-time systems).
- They describe the time evolution of a system.
- They are fundamental to
  - predict the system's future behavior
  - simulate its behavior in different conditions
  - theoretically analize its properties
  - design a control system
  - define a decision-making process
  - etc.

Models of dynamic systems

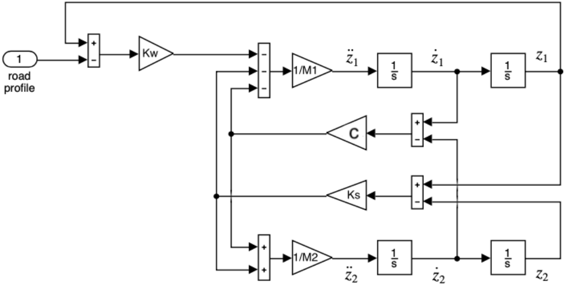
- Models are useful for prediction and simulation, that are fundamental operations in most fields of science and technology (and not only).
- Prediction/simulation requires integration of the differential equations.
  - Analytical integration:
    - A closed-form mathematical expression is obtained for the solution of the differential equations.
    - Possible only in particular cases (e.g., LTI system and "simple" input).
  - Numerical integration:
    - The system is discretized. The solution is then computed numerically by iterating the discretized equations.
    - Only the numerical values of the solution are obtained. No closed-form analytical expression is available.
    - Always possible.
- Integration requires the initial conditions and input signal.
- Matlab/Simulink offers efficient routines for numerical integration.

Example. Quarter-car model

- Consider the quarter-car model equations

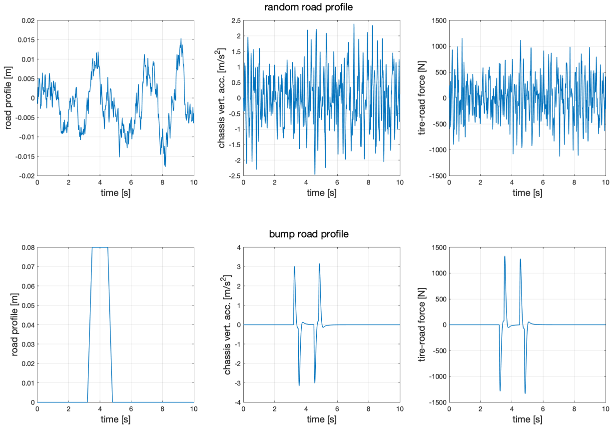
$$
\begin{aligned}
M_1 \ddot{z}_1 &= -c\left(\dot{z}_1 - \dot{z}_2\right) - K_S\left(z_1 - z_2\right) - K_W\left(z_1 - r\right) \\
M_2 \ddot{z}_2 &= c\left(\dot{z}_1 - \dot{z}_2\right) + K_S\left(z_1 - z_2\right)
\end{aligned}
$$

with $M_1 = 40\,\mathrm{kg}$, $M_2 = 390\,\mathrm{kg}$, $c = 6e3\,\mathrm{Ns/m}$, $K_W = 2e5\,\mathrm{N/m}$, $K_S = 17e3\,\mathrm{N/m}$. The model has been implemented in Simulink.



Example. Quarter-car model

- The Simulink model was simulated for two different road profiles. Simulation was performed via numerical integration of the differential equations.



Models of dynamic systems

- In most real-world applications, a model is an approximation of the real system of interest (the plant).
- Two kinds of model uncertainty:
  - parametric: due to approximate parameter values;
  - dynamic: due, e.g., to neglecting some term or some differential equation.
- Models must be validated by means of experimental data.
- If poor physical information is available, models need to be identified from data.

State equations

- State equations provide a standard representation of dynamic systems with finite dimension. They are particularly suitable for
  - theoretical analysis about system properties
  - controller and observer design.
- State equations are first-order differential equations of the form

$$\dot{x} = f(x, u, t)$$
$$y = h(x, u, t)$$

$t \in \mathbb{R}$: time
$u \in \mathbb{R}^{n_u}$: input
$y \in \mathbb{R}^{n_y}$: output
$x \in \mathbb{R}^{n_x}$: state
n_x: order of the system.

- Under very general conditions, a set of differential equations of any finite order can be written in the state equation form.

State equations

- State equations:

$$\dot{x} = f(x, u, t)$$
$$y = h(x, u, t).$$

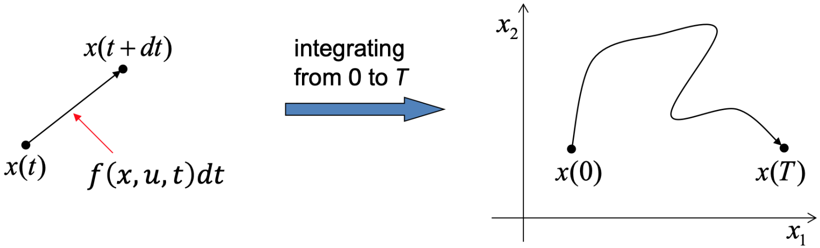
- The state x of a system is a set of variables that represent the system at any given time.
- If the state of a system is known at a given time, then any future value of any system variable can be computed in function of this state and the input signal.
- The number n_x of state variables used to represent a given system is called the order of the system.
- The state is a set of internal variables: It is usually not measured. The measured variable (the output y) can be a function of x.
  - Observers/filters can be used to estimate the state.

State equations

- Consider that $\dot{x} = \frac{x(t+dt)-x(t)}{dt}$ for $dt \to 0$.
- The state equation $\dot{x} = f\,(x, u, t)$ can be written as

$$x(t + dt) = x(t) + f\,(x, u, t)\,dt.$$

The state equation is dynamic: The future state $x(t + dt)$ is obtained from the current state $x(t)$ by adding the increment $f\,(x, u, t)\,dt$.



- The output equation $y = h\,(x, u, t)$ is static, i.e., a relation between variables at the same time t.

State equations for discrete-time systems

- The state equation $\dot{x} = f(x, u, t)$ can be written as

$$x(t + dt) = x(t) + f(x, u, t)\, dt.$$

- The equation can be easily discretized via the the *Forward Euler* method. Suppose that dt is a "small" (but finite) time interval.

- We can discretize the time as $t = k\, dt$, where $k = 0, 1, \ldots$ Then,

$$x((k + 1)dt) = x(k\, dt) + f(x, u, k\, dt)\, dt.$$

- Defining $f_D(x, u, k) \doteq x(k\, dt) + f(x, u, k\, dt)\, dt$ and omitting dt for simplicity, we obtain the discrete-time (DT) equation

$$x(k + 1) = f_D(x(k), u(k), k).$$

- Remark: While $f\, dt$ gives the increment from $x(t)$ to $x(t + dt)$, f_D directly provides $x(k + 1)$.

State equations for discrete-time systems

- The DT equation

$$x(k+1) = f_D\left(x(k), u(k), k\right)$$

  can be used for numerical integration of the corresponding differential equation:
  - Given the initial state $x(0)$ and the input sequence $u(k)$, the DT equation can be used to compute $x(k)$, for all $k = 1, 2, \ldots$
- Such a numerical integration can be carried out by iteration of the DT equation (using a suitable programming language).
- Remark: The Simulink algorithms behind the block diagrams are based on a similar principle: discretization and iteration.

Example (continuous-time). Quarter-car model

- Define the following state vector: $x \doteq [z_1 \ z_2 \ \dot{z}_1 \ \dot{z}_2]^\top$.
- The state equations of the quarter-car model are as follows:

$$
\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= -\frac{c}{M_1}(x_3 - x_4) - \frac{K_S}{M_1}(x_1 - x_2) - \frac{K_W}{M_1}(x_1 - r) \\
\dot{x}_4 &= \frac{c}{M_2}(x_3 - x_4) + \frac{K_S}{M_2}(x_1 - x_2).
\end{aligned}
$$

- The following outputs are of interest:

$$
\begin{aligned}
y_1 &= \frac{c}{M_2}(x_3 - x_4) + \frac{K_S}{M_2}(x_1 - x_2) && \text{chassis acceleration} \\
y_2 &= K_W(x_1 - r) && \text{tire-road force.}
\end{aligned}
$$

Example. Quarter-car model

- The system is linear time invariant (LTI). Hence, the state equations can be written in matrix form:

$$\dot{x} = Ax + B_r r$$
$$y = Cx + D_r r$$

$$A \doteq \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{K_S+K_W}{M_1} & \frac{K_S}{M_1} & -\frac{c}{M_1} & \frac{c}{M_1} \\ \frac{K_S}{M_2} & -\frac{K_S}{M_2} & \frac{c}{M_2} & -\frac{c}{M_2} \end{bmatrix}, \quad B_r \doteq \begin{bmatrix} 0 \\ 0 \\ \frac{K_W}{M_1} \\ 0 \end{bmatrix},$$

$$C \doteq \begin{bmatrix} \frac{K_S}{M_2} & -\frac{K_S}{M_2} & \frac{c}{M_2} & -\frac{c}{M_2} \\ K_W & 0 & 0 & 0 \end{bmatrix}, \qquad D_r \doteq \begin{bmatrix} 0 \\ -K_W \end{bmatrix}.$$

Simulation of dynamic systems

- In general, simulation can be performed considering different approaches:
  - Simulink.
  - Iteration of a DT equation $x(k+1) = f_D\left(x(k), u(k), k\right)$.
  - LTI systems only: a state-space object can be defined: $\mathtt{P} = \mathtt{ss}(\mathtt{A}, \mathtt{Br}, \mathtt{C}, \mathtt{Dr})$. Then, simulation can be performed
    - using the Matlab command `lsim` or
    - using the Simulink block "LTI system".

- In general, numerical integration is based on two fundamental operations: discretization and iteration.

- Approaches based on Euler discretization may not be robust/accurate from a numerical point of view. In critical situations, more sophisticated techniques may be needed (available in Matlab and Simulink).

Classification of dynamic systems

Definitions

- A system is *linear* if the equations are linear in x, u and y.
- A system is *time invariant* if the time does not appear explicitly in the equations.

continuous-time systems		
	nonlinear	linear
time varying	$\dot{x} = f(x, u, t)$ $y = h(x, u, t)$	$\dot{x} = A(t)x + B(t)u$ $y = C(t)x + D(t)u$
time invariant	$\dot{x} = f(x, u)$ $y = h(x, u)$	$\dot{x} = Ax + Bu$ $y = Cx + Du$

Classification of dynamic systems

- For DT systems, the definitions of linear system and of time invariant system are the same. The time is represented by the index $k \in \mathbb{Z}$.

discrete-time systems		
	nonlinear	linear
time varying	$x(k+1) = f\left(x(k), u(k), k\right)$ $y(k) = h\left(x(k), u(k), k\right)$	$x(k+1) = A(k)x(k) + B(k)u(k)$ $y(k) = C(k)x(k) + D(k)u(k)$
time invariant	$x(k+1) = f\left(x(k), u(k)\right)$ $y(k) = h\left(x(k), u(k)\right)$	$x(k+1) = Ax(k) + Bu(k)$ $y(k) = Cx(k) + Du(k)$

Stability of LTI systems

- Consider a LTI system:
  - Continuous-time (CT): $\dot{x} = Ax + Bu$
  - Discrete-time (DT): $x(k+1) = Ax(k) + Bu(k)$
- Let $\lambda_1, \ldots, \lambda_i$ be the eigenvalues of A and let $\mathrm{mult}(\cdot)$ denote the "auxiliary" geometric multiplicity (see next slide).

CT LTI systems

- Asymptotic stability:
  $\mathrm{Re}(\lambda_i) < 0, \ \forall i$
- Marginal stability:
  $\mathrm{Re}(\lambda_i) \leq 0, \ \forall i$
  $\mathrm{mult}(\lambda_i) = 1 \text{ if } \mathrm{Re}(\lambda_i) = 0$
- Instability: otherwise

DT LTI systems

- Asymptotic stability:
  $|\lambda_i| < 1, \ \forall i$
- Marginal stability:
  $|\lambda_i| \leq 1, \ \forall i$
  $\mathrm{mult}(\lambda_i) = 1 \text{ if } |\lambda_i| = 1$
- Instability: otherwise

Linear algebra notions

- **Eigenvectors, eigenvalues and eigenvalue equation**. An eigenvector $x \in \mathbb{R}^n$ of a matrix $A \in \mathbb{R}^{n\times n}$ is a non-zero vector that does not change direction when the matrix is applied to it. In other words, an eigenvector is a vector $x \in \mathbb{R}^n$ satisfying the equation

$$Ax = \lambda x \tag{1}$$

  for some $\lambda \in \mathbb{R}$. The scalar λ is said an eigenvalue of A. The equation (1) is called the eigenvalue equation of A, which can also be written as

$$(\lambda I - A)x = 0 \tag{2}$$

  where I is the identity matrix of dimension n.

- **Eigenvalues and characteristic equation**. Equation (2) has a non-zero solution if and only if the determinant of $\lambda I - A$ is null. Hence, the eigenvalues are the solutions of the characteristic equation

$$\det(\lambda I - A) = 0.$$

  The quantity $P(\lambda) \doteq \det(\lambda I - A)$ is a polynomial in λ, called the characteristic polynomial of A. Hence, the eigenvalues are the roots of the characteristic polynomial.

- **Eigenvectors**. Given a particular eigenvalue λ, the vectors x that satisfy (2) are the eigenvectors of A associated with λ. The set of all eigenvectors associated with λ, defined as

$$E = \{x \in \mathbb{R}^n : (\lambda I - A)x = 0\}$$

  is called the eigenspace of A associated with λ. In other words, E is the kernel (or null-space) of the matrix $\lambda I - A$.

- **Geometric multiplicity**. The dimension d of E (i.e., the number d of linearly independent eigenvectors associated with λ) is the geometric multiplicity of λ.

- **Algebraic multiplicity**. The number of times m that an eigenvalue appears as a root of the characteristic polynomial is the algebraic multiplicity.

- **"Auxiliary" geometric multiplicity**: $k \doteq m - d + 1$.

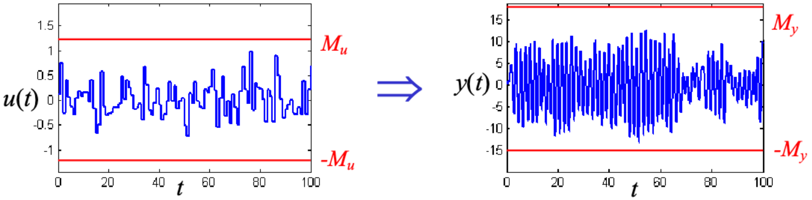
Behavior of LTI systems

	free motion ($u(\cdot) = 0$) with bounded $x(0)$	forced motion with bounded $x(0)$ and $u(\cdot)$
asymptotic stability	x converges to 0	x bounded
marginal stability	x bounded	x may diverge to ∞
instability	x may diverge to ∞	x may diverge to ∞

Stability of LTI systems

- Asymptotic stability implies bounded-input-bounded-output (BIBO) stability:

  For any bounded initial condition and any bounded input signal the resulting output is bounded.



- In the case of marginal stability or instability, divergent system's solutions may occur, even for bounded initial conditions and input signals.

Stability of LTI systems

LTI systems: The following implications can be proven by means of mode analysis.

Internal stability:		External stability:
asymptotic stability	$\Longrightarrow$	BIBO stability
(simple) stability or instability	$\Longleftarrow$	BIBO instability

Note: The inverse implications do not hold.

Example. Quarter-car model

- Quarter-car state equation:

$$\dot{x} = Ax + B_r r$$

$$A \doteq \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{K_S+K_W}{M_1} & \frac{K_S}{M_1} & -\frac{c}{M_1} & \frac{c}{M_1} \\ \frac{K_S}{M_2} & -\frac{K_S}{M_2} & \frac{c}{M_2} & -\frac{c}{M_2} \end{bmatrix}, \quad B_r \doteq \begin{bmatrix} 0 \\ 0 \\ \frac{K_W}{M_1} \\ 0 \end{bmatrix},$$

- It can be seen that:
  - $c > 0$: All the eigenvalues of A have negative real part, implying that the system is asymptotically stable.
  - $c = 0$: Some eigenvalues have null real part and multiplicity 1, implying that the system is marginally stable.
  - $c < 0$: The system becomes unstable. Note that a negative damping coefficient has not a relevant physical meaning.

Example. Mass-spring-damper system

- State equations:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\frac{k}{M}x_1(t) - \frac{c}{M}x_2(t) + \frac{1}{M}u(t) \end{cases}$$

```
>> k=1;    c=2; M=3;
>> A=[0 1;-k/M   -c/M];
>> eig(A)
-0.33333+0.4714i
-0.33333-0.4714i
```

the system is
asymptotically stable

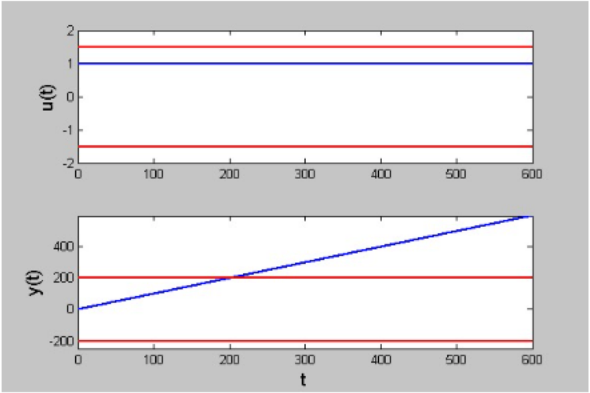
```
>> k=1;    c=0; M=3;
>> A=[0 1;-k/M   -c/M];
>> eig(A)
0+0.57735i
0-0.57735i
```

the system is
(simply) stable

Example. Integrator

$$\dot{x}(t) = u(t)$$
$$y(t) = x(t)$$

$$x(t), u(t), y(t) \in \mathrm{R}$$

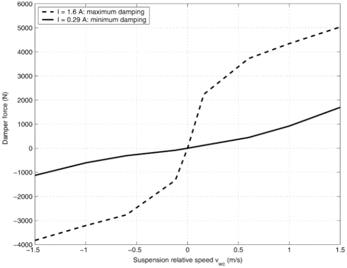


The system is (simply) stable but not asymptotically stable and

not BIBO stable.

Example. Quarter-car with semi-active suspension

- Classification of suspensions:
  - Passive: the damping coefficient c is constant.
  - Semi-active: the damping force can be changed by means of a current (or a tension).
  - Active: the suspension is able to exert a desired force.
- A passive suspension has been considered so far. In the following, we consider a semi-active suspension. The suspension characteristic is shown in the figure.



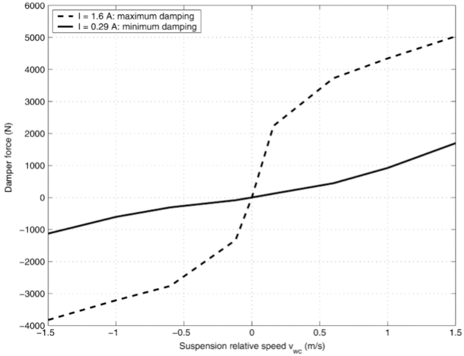
Example. Quarter-car with semi-active suspension

- The bounds in the figure are called *passivity constraints*.
- No active forces can be provided by the suspension. The damper force u can only take values between the passivity constraints.
- The passivity constraints depend on the control current I and the suspension relative speed v_{wc}.
- With this suspension, the quarter-car model becomes nonlinear.

Passivity constraints:

$$\text{if } v_{\mathrm{wc}} \geq 0 \begin{cases} u \leq 1480\, v_{\mathrm{wc}} + 2852 \\ u \leq 3400\, v_{\mathrm{wc}} + 1700 \\ u \leq 13500\, v_{\mathrm{wc}} \\ u \geq 1500\, v_{\mathrm{wc}} - 540 \\ u \geq 600\, v_{\mathrm{wc}} \end{cases}$$

$$\text{if } v_{\mathrm{wc}} < 0 \begin{cases} u \leq 600\, v_{\mathrm{wc}} \\ u \leq 1000\, v_{\mathrm{wc}} + 400 \\ u \geq 13500\, v_{\mathrm{wc}} \\ u \geq 3000\, v_{\mathrm{wc}} - 970 \\ u \geq 1200\, v_{\mathrm{wc}} + 2050 \end{cases}$$

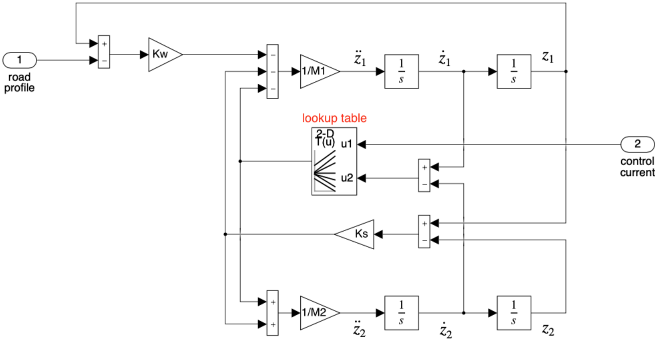


Example. Quarter-car with semi-active suspension

- With the semi-active suspension, the quarter-car equations are

$$
\begin{aligned}
M_1 \ddot{z}_1 &= -u\left(I, v_{wc}\right) - K_S\left(z_1 - z_2\right) - K_W\left(z_1 - r\right) \\
M_2 \ddot{z}_2 &= u\left(I, v_{wc}\right) + K_S\left(z_1 - z_2\right)
\end{aligned}
$$

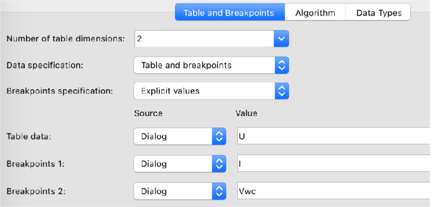
where I is the control current, $v_{wc} \doteq \dot{z}_1 - \dot{z}_2$ and $u(\cdot,\cdot)$ is the function described above. They have been implemented in Matlab/Simulink.

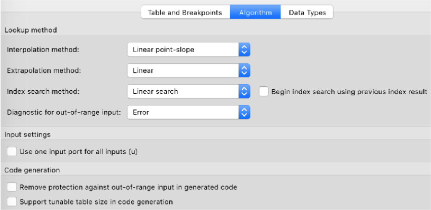


Example. Quarter-car with semi-active suspension

Lookup table details

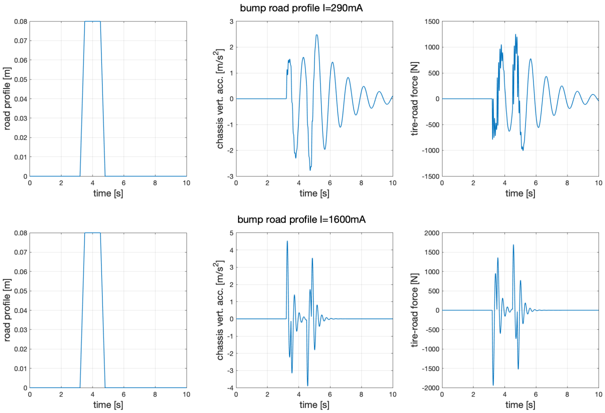
```
I=[0.29 1.6]';

Vwc=[-1.5 -0.6 -0.1 0 0.1 0.6 1.5];

U=[-1050 -250 -50 0 50 400 1750
    -3900 -2800 -1300 0 2200 3700 5000];
```





Example. Quarter-car with semi-active suspension

- The Matlab/Simulink model was simulated for two values of the control current: $I = 0.29\,\mathrm{A}$ (soft $\rightarrow$ better comfort), $I = 1.6\,\mathrm{A}$ (hard $\rightarrow$ better handling).



- The control current may also be used for feedback control.

Example. Quarter-car with semi-active suspension

- Quarter-car with semi-active suspension equations:

$$M_1 \ddot{z}_1 = -u\left(I, v_{wc}\right) - K_S\left(z_1 - z_2\right) - K_W\left(z_1 - r\right)$$
$$M_2 \ddot{z}_2 = u\left(I, v_{wc}\right) + K_S\left(z_1 - z_2\right)$$

  where I is the control current, $v_{wc} \doteq \dot{z}_1 - \dot{z}_2$ and $u(\cdot,\cdot)$ is the function described above.

- The stability study is difficult, since the system is nonlinear. In particular, it is impossible to define a constant matrix A.
- With $I = const$ and small v_{wc}, the system can be linearized. However, the conclusions which can be obtained about stability are local.
- In general, the stability properties and behavior of nonlinear systems can be significantly different from that of linear systems.
- The trajectories in the state domain of a nonlinear system can be significantly more complex.

Nonlinear systems

- Consider a system $\dot{x} = f(x, u, t)$.
- Suppose that
  - the initial state is $x(0)$,
  - the input signal applied to the system is $u(t)$, $t \in [0, \infty)$.
- A signal is a function of time. For simplicity, the notation $u(\cdot), x(\cdot)$ is used for signals.

Definitions

• The signal $x(\cdot)$ resulting from integration of $\dot{x} = f(x, u, t)$ is called the *solution* of the system.

• The set of points in the state domain generated by the solution is called the trajectory of the system.

- **Remark**: The solution (trajectory) is a function of two quantities: the initial condition and the input signal: $x(\cdot) \equiv x(\cdot, x(0), u(\cdot))$.

Nonlinear systems

Equilibrium points

- Consider a system $\dot{x} = f(x,u)$
- Let $\bar{u}$ be a constant input.

Definition. $\bar{x}$ is an equilibrium state (or equilibrium point) corresponding to the constant input $\bar{u}$ if it is solution of the static equation

$$f(\bar{x},\bar{u}) = 0$$

Rationale:

$$f(\bar{x},\bar{u}) = 0 \quad \Leftrightarrow \quad \dot{x} = 0$$
$$\Leftrightarrow \quad \textit{null variation over time}$$
$$\Leftrightarrow \quad \textit{the state remains constant over time}$$
$$\Leftrightarrow \quad \textit{EQUILIBRIUM}$$

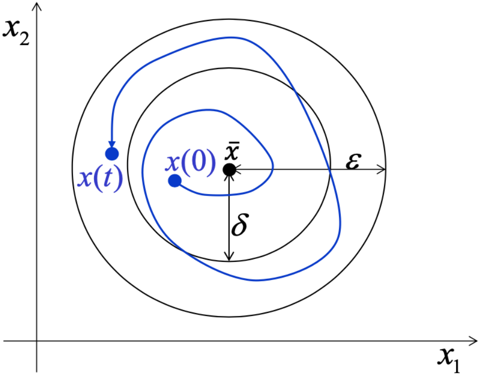
Stability of nonlinear systems

Equilibrium point stability

Definition. The equilibrium state $\bar{x}$ is (simply or marginally) stable if

$$\forall \varepsilon > 0, \quad \exists \delta > 0:$$

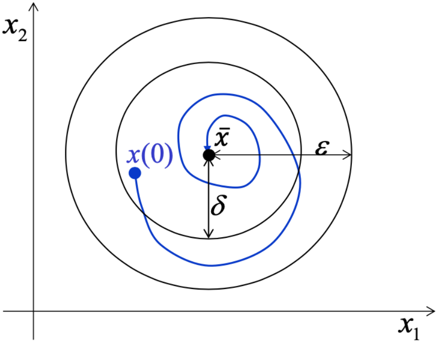
$$\forall x(0):\ \|x(0) - \bar{x}\| < \delta \implies \|x(t) - \bar{x}\| < \varepsilon, \quad \forall t \geq 0$$



Stability of nonlinear systems

Equilibrium point stability

Definition. The equilibrium state $\bar{x}$ is asymptotically stable if it is stable and $\lim_{t\to\infty} \|x(t) - \bar{x}\| = 0$.

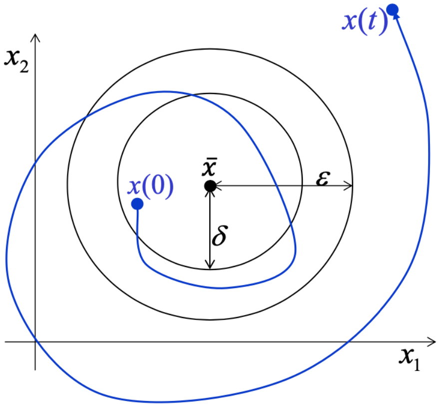


Definition: The equilibrium state $\bar{x}$ is exponentially stable if it is asymptotically stable and the convergence is exponential.

Stability of nonlinear systems

Equilibrium point instability

Definition. The equilibrium state $\bar{x}$ is unstable if it is not stable.



Stability of nonlinear systems

Solution stability

- Consider a system $\dot{x} = f(x, u, t)$ and two its solutions:
  - Nominal solution: $x(\cdot) \equiv x(\cdot, x(0), u(\cdot))$.
  - Perturbed solution: $x^p(\cdot) \equiv x^p(\cdot, x^p(0), u(\cdot))$, where

$$\|x^p(0) - x(0)\| < \delta, \quad \forall t \geq 0. \tag{1}$$

Definitions

• The nominal solution is *(marginally or simply) stable* if, for any $\epsilon > 0$, a sufficiently small δ exists, such that $\|x^p(t) - x(t)\| < \epsilon$ for all $t \geq 0$ and for all $x^p(0)$ that satisfy (1).

• The nominal solution is *asymptotically stable* if it is marginally stable and $\lim_{t\to\infty} x^p(t) = x(t)$.

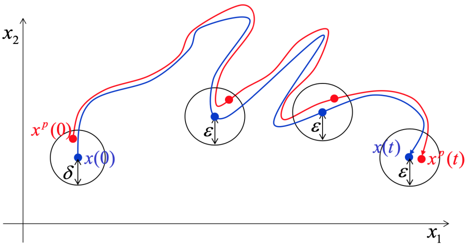
• The nominal solution is *unstable* if not marginally stable.

- Note that unstable solutions can be either bounded or unbounded.

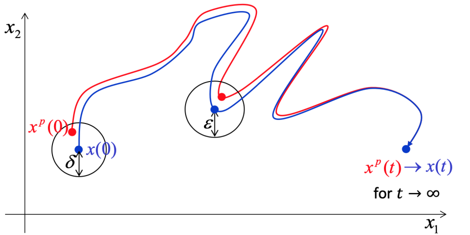
Stability of nonlinear systems

Solution stability

solution
marginal
stability
$\Longrightarrow$

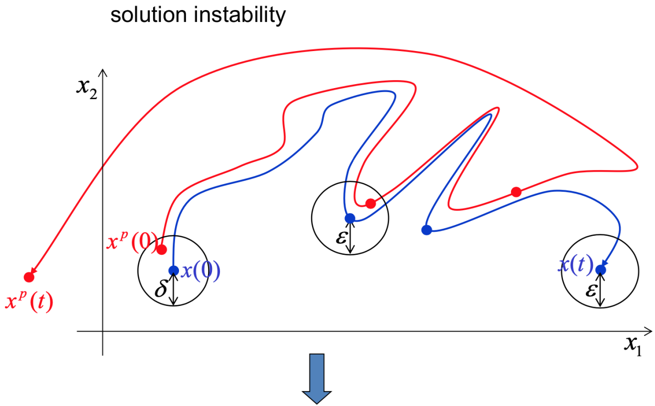


solution
asymptotic
stability
$\Longrightarrow$



Stability of nonlinear systems

Solution instability



High sensitivity to initial conditions:

$$x^p(0) \neq x(0),\ x^p(0) \cong x(0) \Rightarrow\ x^p(t)\ \underline{\text{far away from}}\ x(t)$$

Behavior of nonlinear systems

- Instability of solutions gives rise to the phenomenon of high sensitivity to initial conditions/perturbations (IC/P):
  - A small difference in initial conditions or a small input perturbation may cause a large difference in the system solution.
  - The system output is unpredictable: even if an exact model is available, the long-term system output cannot be predicted, due to errors on the initial conditions and/or to numerical errors.

  $\upuparrows$ These are some of the main features of *chaos*.

- Nonlinear systems: A system may have stable, as. stable and unstable solutions, depending on $x(0)$ and $u(\cdot)$.
- LTI systems:
  - Asymptotically stable:
    - ★ All solutions are as. stable $\Rightarrow$ no high sensitivity to IC/P.
    - ★ BIBO (Bounded Input Bonded Output) stable: the output is bounded for any bounded input.
  - Unstable: Most solutions are unstable and unbounded.

Behavior of nonlinear systems

- In general, the behavior of nonlinear systems can be significantly different from that of linear systems.
- The trajectories in the state domain of a nonlinear system can be much more complex.
- Phenomena typical of nonlinear systems, not occurring for linear systems, are:
  - multiple isolated equilibrium points
  - limit cycles
  - bifurcations
  - finite escape time
  - chaos
  - subharmonics
  - jump resonance
  - etc.
- We'll not discuss all these phenomena but the following example illustrates some of them.

Example. Duffing oscillator

- The Duffing oscillator is a second-order model of an oscillator with a nonlinear spring stiffness (not obeying Hooke's law):

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -k_1x_1 - k_2x_1^3 - cx_2 + u\end{aligned}$$

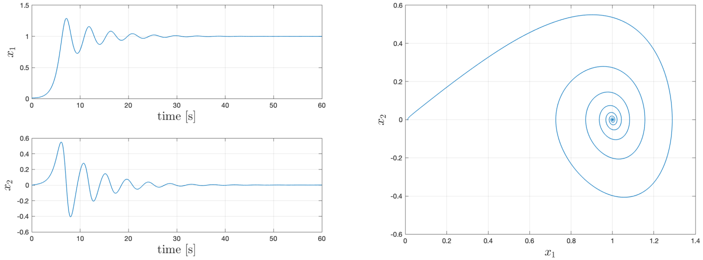
  where $x = (x_1, x_2) \in \mathbb{R}^2$ is the state, $u \in \mathbb{R}$ is the input and $k_1, k_2, c \in \mathbb{R}$ are constant parameters.
- Nonlinear model $\Rightarrow$ in general, no analytical solution can be found.
- The equilibrium states for $u = 0$ are the solutions of the equations

$$\bar{x}_2 = 0, \quad \bar{x}_1(k_1 + k_2\bar{x}_1^2) = 0.$$

- Cubic equation $\Rightarrow$ 3 equilibrium points.
  - Example: $k_1 = -1$, $k_2 = 1$, $c = 0.3 \Rightarrow \bar{x} \in \{(-1, 0), (0, 0), (1, 0)\}$. $(0, 0)$ is unstable, the other two are stable.
- **Note:** An LTI system cannot have multiple isolated equilibria.

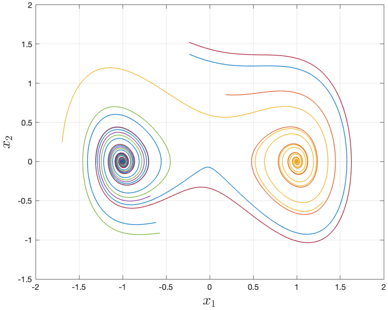
Example. Duffing oscillator

- Suppose $k_1 = -1$, $k_2 = 1$, $c = 0.3$; $u = 0$.
- According to the stability analysis, the system trajectory, starting from an initial condition $x(0) \cong (0,0)$, $x(0) \neq (0,0)$, is expected to
  - diverge from $(0,0)$ and converge to one of the other two eq. points or
  - diverge to infinity.
- A simulation has been carried out using the Matlab command `ode45`. The result is shown in the figures.



Example. Duffing oscillator

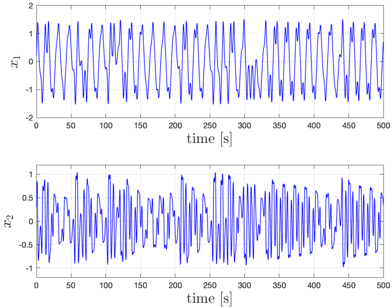
- Starting from random initial conditions, the trajectory always converges to one of the stable equilibria.

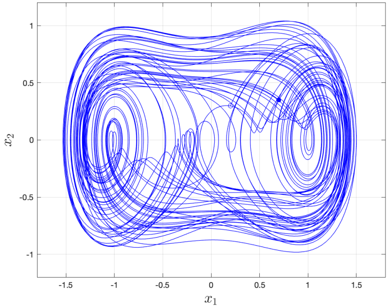


- This kind of picture is called the *phase portrait* of the system. A generalization to higher system orders is given by *Poincaré maps*.

Example. Duffing oscillator

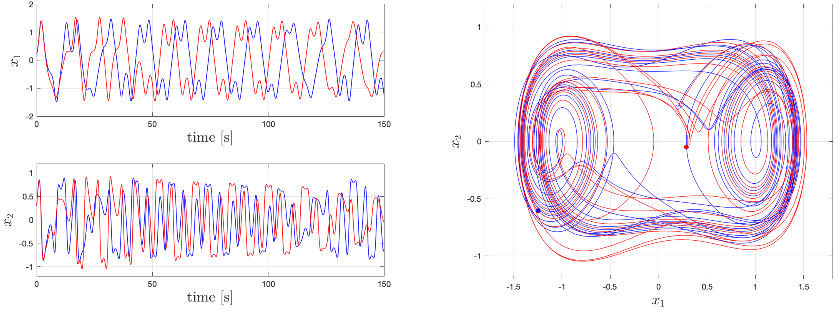
- Suppose $k_1 = -1$, $k_2 = 1$, $c = 0.3$; $u = 0.5\cos(1.2t)$; random initial conditions.
- The trajectory has a fractal shape (fractals are "strange" geometric objects).





Example. Duffing oscillator

- Suppose $k_1 = -1$, $k_2 = 1$, $c = 0.3$; $u = 0.5\cos(1.2t)$.
- Two solutions: same input but slightly different initial conditions.
- The corresponding trajectories diverge from each other (without tending to ∞).
- This phenomenon is called high sensitivity to initial conditions and is typical of *chaotic systems*.



Discussion

- Basic concepts of system theory have been presented.
- These concepts have been illustrated by means of examples involving physical systems:
  - Quarter-car model.
  - Duffing oscillator.
- The behavior of nonlinear systems can be significantly different from that of linear systems.
- Given a real physical plant, different models of it can be written, whose complexity depends on the utilization. Typically:
  - Detailed high-fidelity models $\leftarrow$ simulations, tests, analysis.
  - Simplified models $\leftarrow$ control design, preliminary simulations and tests.
- The complexity of a model should be increased only if the added information is reliable.
  - The "equation" detailed model $=$ accurate model is not true in general.
- Models of real physical plants must be validated by means of experimental data if possible.