Driver assistance system design A

State Feedback Control for LTI systems

Carlo Novara

Politecnico di Torino
Dip. Elettronica e Telecomunicazioni

Outline

State Feedback Control

- State feedback is a fundamental principle for control of linear and nonlinear systems.
- It can be applied to a large class of systems:
  - Nonlinear (without using linearization)
  - Time-varying
  - MIMO.
- It accounts for the connection between the internal and external descriptions.
- It allows to introduce optimality concepts.
- It can deal with constraints on states, input and output.
- It allows a more effective stabilization of unstable complicated systems.

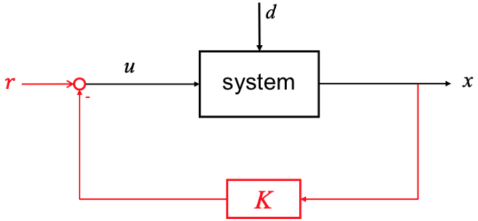
State Feedback Control for LTI systems

- We focus on linear time invariant (LTI) systems in state-space form:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the input, $y \in \mathbb{R}^{n_y}$ is the output, and $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$, $C \in \mathbb{R}^{n_y \times n_x}$, $D \in \mathbb{R}^{n_y \times n_u}$ are constant matrices.

- State feedback is based on the assumption that the state vector x can be measured (or estimated using a suitable observer/filter).



State Feedback Control for LTI systems

- The basic state feedback control law for LTI systems is the following:

$$u = -Kx + r$$

  where $K \in \mathbb{R}^{n_u \times n_x}$ is a matrix/vector and r is a reference variable.

- The closed-loop system resulting from this law is

$$\dot{x} = Ax + Bu = Ax - BKx + Br = (A - BK)x + Br$$

$$\dot{x} = Fx + Br, \qquad F \doteq A - BK.$$

- We have changed the system matrix from A to F. This matrix is fundamental since it determines stability and dynamic behavior.

- **Eigenvalue (pole) placement:** Design K such that the closed-loop matrix F has the desired eigenvalues.

Example: stabilization of the undamped pendulum

- Linearized undamped pendulum state equations:

$$\dot{x} = Ax + Bu, \quad x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Open-loop characteristic equation: $\det(\lambda I - A) = 0$

where $\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -1 \\ \omega_n^2 & \lambda \end{bmatrix} = \lambda^2 + \omega_n^2$

Then, $\mathrm{eig}(A) = \{\lambda_1, \lambda_2\} = \{i\omega_n, -i\omega_n\}$

- **Problem:** place the pendulum poles in $\{-2\omega_n, -2\omega_n\}$

- This corresponds to transforming the undamped pendulum into a pendulum with damping 1 and natural frequency $2\omega_n$.

Example: stabilization of the undamped pendulum

- Control law: $u(t) = -Kx(t), \quad K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$

- Choose K such that $\quad \text{eig}(F) = \{\gamma_1, \gamma_2\} = \{-2\omega_n, -2\omega_n\}$

- Consider the closed-loop characteristic polynomial $\det(\gamma I - F)$:

$$F = A - BK = A - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \end{bmatrix} = A - \begin{bmatrix} 0 & 0 \\ K_1 & K_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 - K_1 & -K_2 \end{bmatrix}$$

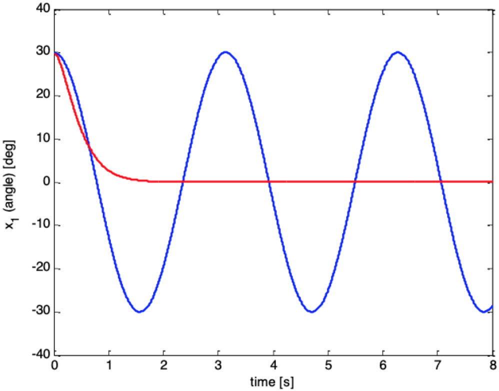
$$\det(\gamma I - F) = \det \begin{bmatrix} \gamma & -1 \\ \omega_n^2 + K_1 & \gamma + K_2 \end{bmatrix} = \gamma^2 + K_2\gamma + \omega_n^2 + K_1$$

- K can be found by comparing the closed-loop polynomial with the desired polynomial (the one with roots $\{-2\omega_n, -2\omega_n\}$): $\gamma^2 + 4\omega_n\gamma + 4\omega_n^2$

- Comparing the coefficients of the two polynomials, we find the controller $\quad K = \begin{bmatrix} 3\omega_n^2 & 4\omega_n \end{bmatrix}$

Example: stabilization of the undamped pendulum

- Chosen natural frequency: $\omega_n = 2$ rad/s
- Initial condition: $x(0)=[\pi/6;0]$



Blue: undamped pendulum. Red: controlled pendulum.

State Feedback Control for LTI systems

- Ackermann formula for pole placement:

$$K = \left[\text{zeros}(1, n_x - 1) \quad 1\right] M_c^{-1} \alpha_c(A)$$

where $\alpha_c(A) = A^{n_x} + \alpha_1 A^{n_x - 1} + \alpha_2 A^{n_x - 2} + \ldots + \alpha_{n_x} I$,

α_i : coefficients of the desired characteristic polynomial.

- **Matlab**:

```
K = acker(A,B,Pc_des);
K = place(A,B,Pc_des);
```

where **`Pc_des`** is the vector with the desired eigevalues.

- The command **`place`** is numerically more robust than **`acker`** and is more suitable for large state dimensions. The command **`place`** does not allow us to place coincident poles.

State Feedback Control for LTI systems

Design criteria

- General criteria for the choice of the closed-loop eigenvalues γ_i, $i = 1, \ldots, n_x$ are the following:
  - The closed-loop system is asymptotically stable if $Re(\gamma_i) < 0$, $\forall i$.
  - Decreasing $Re(\gamma_i)$ leads to
    - increasing the response speed, reducing the rise time;
    - increasing the command activity.

    Trade-off between performance and command activity; critical in the case of input saturation.
  - Non-null imaginary parts give oscillations.
- An optimal approach for designing the controller K is the linear quadratic regulator (LQR) approach.
  - The LQR approach allows an optimal eigenvalue placement.
- To introduce the LQR approach, we need to recall basic notions about signal norms and optimization.

Signal norms

- A signal is a function of time: $f \equiv f(t)$, where $t \in \mathbb{R}^+ \doteq [0, \infty]$.
- Consider a signal $f : \mathbb{R}^+ \to \mathbb{R}$.

$$
\begin{aligned}
L_2 \text{ norm :} \quad & \|f\|_2 \doteq \sqrt{\int_0^\infty f(t)^2 dt} \\
L_1 \text{ norm :} \quad & \|f\|_1 \doteq \int_0^\infty |f(t)|\, dt \\
L_\infty \text{ norm :} \quad & \|f\|_\infty \doteq \sup_{t \in \mathbb{R}^+} |f(t)|\,.
\end{aligned}
$$

- **Interpretation:**
  - $\|f\|_2^2$: (generalized) energy of the signal f.
  - $\|f\|_\infty$: amplitude of the signal f.

Signal norms

- Consider a signal $f : \mathbb{R}^+ \to \mathbb{R}^n$.

$$L_2 \text{ norm}: \quad \|f\|_2 \doteq \sqrt{\int_0^\infty \|f(t)\|_2^2\, dt} = \sqrt{\int_0^\infty f(t)^\top f(t) dt}$$

$$L_2 \text{ weighted norm}: \quad \|f\|_{Q,2} \doteq \sqrt{\int_0^\infty f(t)^\top Q f(t) dt}$$

$$L_1 \text{ norm}: \quad \|f\|_1 \doteq \int_0^\infty \|f(t)\|_1\, dt$$

$$L_\infty \text{ norm}: \quad \|f\|_\infty \doteq \sup_{t \in \mathbb{R}^+} \|f(t)\|_\infty$$

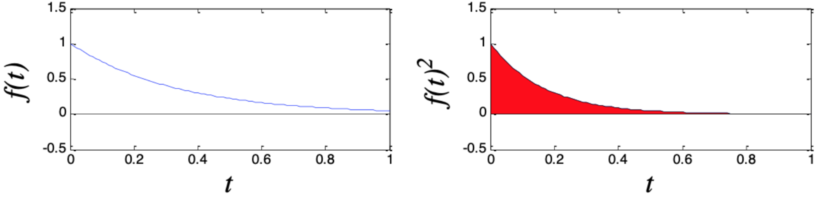
where Q is a square diagonal matrix with non-negative entries.

- **Interpretation:**
  - $\|f\|_2^2$: (generalized) energy of the signal f.
  - $\|f\|_\infty$: amplitude of the signal f.

Signal norms

Example

- Consider the function $f(t) = e^{-3t}$.

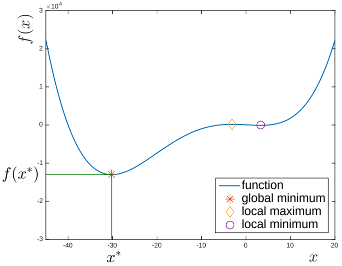


$$\|f\|_2^2 = \int_0^\infty f(t)^2 dt = \int_0^\infty e^{-6t} dt = \left[-\frac{1}{6} e^{-6t} \right]_0^\infty = \frac{1}{6}$$

$$\|f\|_\infty = \max_{t \in R^+} |f(t)| = \max_{t \in R^+} \left| e^{-3t} \right| = 1$$

Optimization

Univariate function $f : \mathbb{R} \to \mathbb{R}$; unique global minimum.



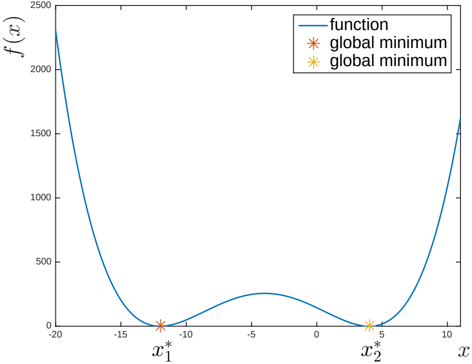
$$x^* = \arg\min_{x \in \mathbb{R}} f(x) = -30$$

$$f(x^*) = \min_{x \in \mathbb{R}} f(x) = -1.3 \times 10^4$$

$f(\cdot)$ is said the *objective function* or *cost function*,
x is said the *decision variable*, x^* the *minimizer*.

Optimization

Univariate function $f : \mathbb{R} \to \mathbb{R}$; several global minima.

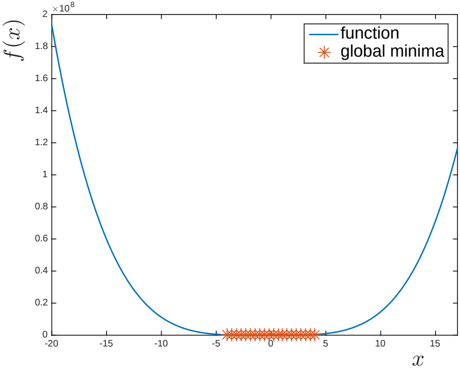


$$\{x_1^*, x_2^*\} = \arg\min_{x \in \mathbb{R}} f(x) = \{-12, 4.5\}$$

$$f(x_1^*) = f(x_2^*) = \min_{x \in \mathbb{R}} f(x) = 0$$

Optimization

Univariate function $f : \mathbb{R} \to \mathbb{R}$; infinite number of global minima.

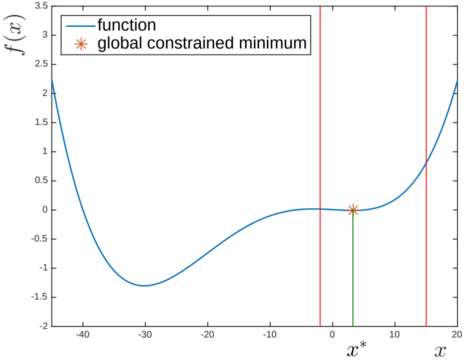


$$X_m = \arg\min_{x \in \mathbb{R}} f(x) = \{x : |x| \leq 4\} \subset \mathbb{R}$$

$$f(x) = \min_{x \in \mathbb{R}} f(x) = 0, \; \forall x \in X_m$$

Optimization

Univariate function $f : \mathbb{R} \to \mathbb{R}$; constrained optimization.

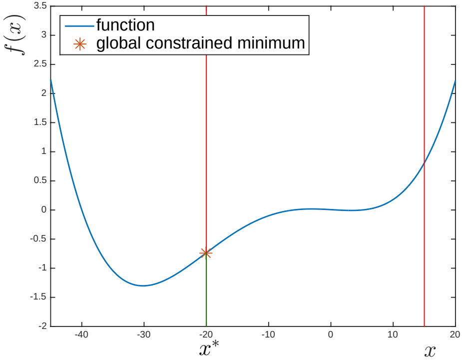


$$x^* = \arg \min_{-2 \le x \le 15} f(x) = 3$$

$$f(x^*) = \min_{-2 \le x \le 15} f(x) = -0.008$$

Optimization

Univariate function $f : \mathbb{R} \to \mathbb{R}$; constrained optimization.

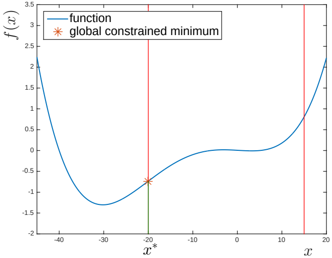


$$x^* = \arg \min_{-20 \leq x \leq 15} f(x) = -20$$

$$f(x^*) = \min_{-20 \leq x \leq 15} f(x) = -0.73$$

Optimization

Univariate function $f : \mathbb{R} \to \mathbb{R}$; constrained optimization.



$$x^* = \arg \min_{-20 \leq x \leq 15} f(x) = -20$$

Alternative notation:

$$x^* = \arg\min f(x) = -20$$

$$\text{subject to:} \quad -20 \leq x \leq 15$$

Optimization

A simple univariate example

- Consider the function

$$f(x) = \frac{x^4}{4} - \frac{5\,x^3}{3} - 17\,x^2 + 80\,x.$$

- To find its extrema, we compute the derivative:

$$\begin{aligned}\tfrac{df(x)}{dx} &= x^3 - 5\,x^2 - 34\,x + 80\\ &= (x-2)\,(x+5)\,(x-8)\,.\end{aligned}$$

- The extrema of f are thus located at $X_e = \{-5, 2, 8\}$, and

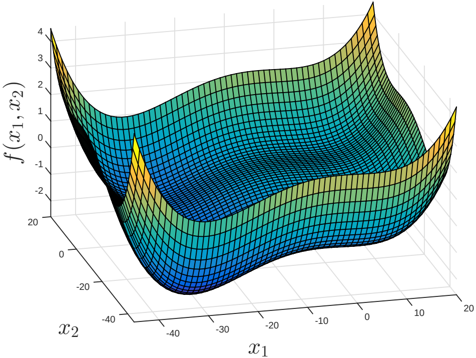
$$f(-5) = -460, \quad f(2) = 83, \quad f(8) = -277.$$

- It follows that

$$\begin{aligned}x^* &= \arg\min_{x\in\mathbb{R}} f(x) = -5\\ f(x^*) &= \min_{x\in\mathbb{R}} f(x) = -460.\end{aligned}$$

Optimization

Multivariate function $f : \mathbb{R}^n \to \mathbb{R}$; constrained optimization.



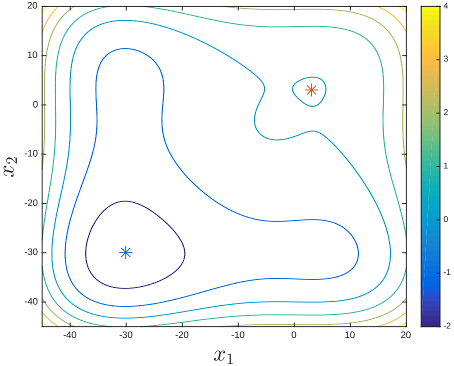
$$X = \left\{ x = [x_1, x_2]^\top : -45 \le x_1, x_2 \le 20 \right\} \subset \mathbb{R}^2$$

$$x^* = \arg\min_{x \in X} f(x) = ?$$

$$f(x^*) = \min_{x \in X} f(x) = ?$$

Optimization

Multivariate function $f : \mathbb{R}^n \to \mathbb{R}$; constrained optimization.



$$X = \left\{ x = [x_1, x_2]^\top : -45 \leq x_1, x_2 \leq 20 \right\} \subset \mathbb{R}^2$$

$$x^* = \arg\min_{x \in X} f(x) = \begin{bmatrix} -30 \\ -30 \end{bmatrix}$$

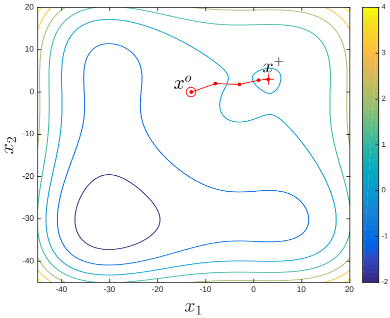
$$f(x^*) = \min_{x \in X} f(x) = -2$$

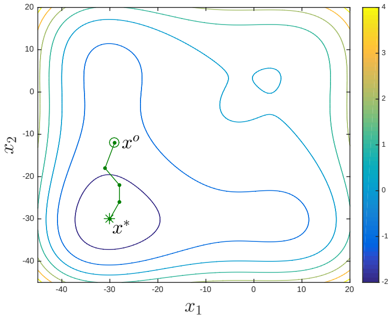
Optimization

- In the above univariate polynomial example, we were able to compute analytically the function minimum.
- In many situations, it is not possible to find an analytical solution:
  - it could not be possible to find the zeros of the derivatives;
  - a large number of variables could be involved.
- In these situations, a numerical solution can be found:
  - Iterative procedures (initial point x^o; other points sequentially visited; stopping conditions). $\Rightarrow$ only find *local minima*.
  - Gridding approaches. $\Rightarrow$ computational complexity *exponential* in n.
- **Problem**: in general, numerical algorithms can only find *local minima* (which depend on the starting point) in a reasonable time.

Optimization

A simple bivariate example





$\uparrow$

local minimum found

$\uparrow$

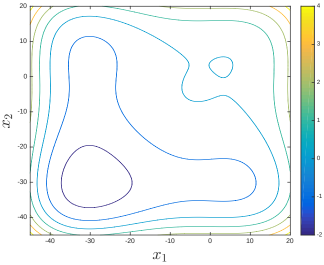
global minimum found

Optimization

- Important is the class of **convex functions**:
  - for these functions *every local minimum is a global minimum*;
    - numerical algorithms can find a global minimum.

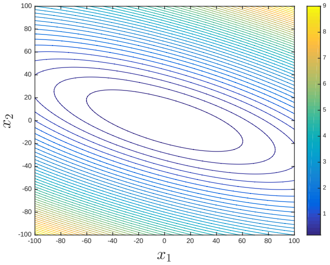
Definition

A function is convex if its level curves define convex sets.



$\uparrow$

non-convex function



$\uparrow$

convex function

Linear Quadratic Regulator

- State feedback is a fundamental principle for control of linear and nonlinear systems.
- We focus on linear time invariant (LTI) systems in state-space form:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

  where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the input, $y \in \mathbb{R}^{n_y}$ is the output, and $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$, $C \in \mathbb{R}^{n_y \times n_x}$, $D \in \mathbb{R}^{n_y \times n_u}$ are constant matrices.

- In the following, we discuss the linear quadratic regulator (LQR) approach to state feedback control.

Linear Quadratic Regulator

- The basic idea of the LQR approach is to design a controller which minimizes the following quantities:

energy of the state signal: $\|x\|_{Q,2}^2 \doteq \int_0^\infty x(t)^\top Q x(t) dt$

energy of the command signal: $\|u\|_{R,2}^2 \doteq \int_0^\infty u(t)^\top R u(t) dt.$

- Motivations:
  - Minimization of $\|x\|_{Q,2}^2 \rightarrow$ increasing the performance in terms of convergence speed, rise time and reduced oscillations.
  - Minimization of $\|u\|_{u,2}^2 \rightarrow$ reducing the command energy and amplitude $\rightarrow$ reducing energy consumption, meeting input constraints.
- The (constant) weight matrices Q and R allow us to manage the trade-off between performance and command activity.

Linear Quadratic Regulator

- We define the objective function

$$J(u, x) \doteq \|u\|^2_{R,2} + \|x\|^2_{Q,2} .$$

  - $u(\cdot)$ and $x(\cdot)$ are signals, i.e., functions of time.
    Therefore, $J(u(\cdot), x(\cdot))$ is a function of other functions.
    Such a mathematical object is often called a *functional*.
  - This functional associates to a couple of functions $(u(\cdot), x(\cdot))$ a non-negative real number: $(u(\cdot), x(\cdot)) \to J \in \mathbb{R}^+$.

- The state signal $x(\cdot)$ depends on the command signal $u(\cdot)$ and the initial conditions through the state equations: $x(\cdot) \equiv x(x_0, u(\cdot))$.

- For a given x_0, J is a function of $u(\cdot)$ only:

$$J(u(\cdot), x(\cdot)) \equiv J(u, x(x_0, u(\cdot))) \equiv J(u(\cdot)).$$

Linear Quadratic Regulator

- The LQR approach is based on the following optimization problem:

$$\begin{aligned} u^*(\cdot) &= \arg \min_{u(\cdot),x(\cdot)} J(u(\cdot), x(\cdot)) \\ &\text{subject to:} \\ &\dot{x} = Ax + Bu, \quad x(0) = x_0. \end{aligned} \tag{1}$$

  The objective function is minimized with the constraint that u and x satisfy the state equation.

- Being $J(u, x) \doteq \|u\|^2_{R,2} + \|x\|^2_{Q,2}$, we aim to enhance the performance and reduce the command activity.
  - These are contrasting criteria.
  - The matrices Q and R allow us to manage the trade-off between performance and command activity.

Linear Quadratic Regulator

Theorem

Assume that the pair (A, B) is controllable. For any initial condition x_0, the solution of the optimization problem (1) is

$$u^* = -Kx$$

where $K = R^{-1}B^\top P$ and P is the solution of the Riccati equation

$$A^\top P + PA + Q - PBR^{-1}B^\top P = 0.$$

Proof. See, e.g., the book Linear Optimal Control Systems[1].

- **Remarks.** The following result are found (not imposed a-priori):
  - state feedback law;
  - closed-loop stability.
- Matlab: `K=lqr(A,B,Q,R)` or `K=lqr(sys,Q,R)`.

[1] Kwakernaak & Sivan, Linear Optimal Control Systems, Wiley, 1972.

Linear Quadratic Regulator

Design criteria

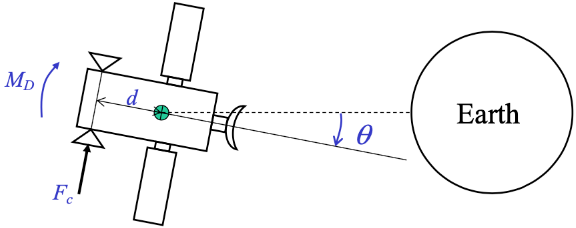
1. **Initial choice:** Supposing that all the variables have similar ranges of variation, Q and R can be chosen diagonal non-negative with
   - $Q_{ii} \begin{cases} > 0 & \text{in the presence of requirements on } x_i \\ \cong 0 & \text{otherwise} \end{cases}$
   - $R_{ii} \begin{cases} > 0 & \text{in the presence of requirements on } u_i \\ \cong 0 & \text{otherwise.} \end{cases}$

   Choose $Q_{ii}, R_{ii} \neq 0$ taking into account the orders of magnitude of the related variables.

2. **Trial and error (in simulation):** Change the values of Q_{ii} and R_{ii} until the requirements are satisfied.

increasing Q_{ii}	$\Rightarrow$	decreasing the energy of x_i	$\Rightarrow$	reducing oscillations and convergence time
increasing R_{ii}	$\Rightarrow$	decreasing the energy of u_i	$\Rightarrow$	reducing command effort and “energy consumption”

Example: feedback control of satellite attitude



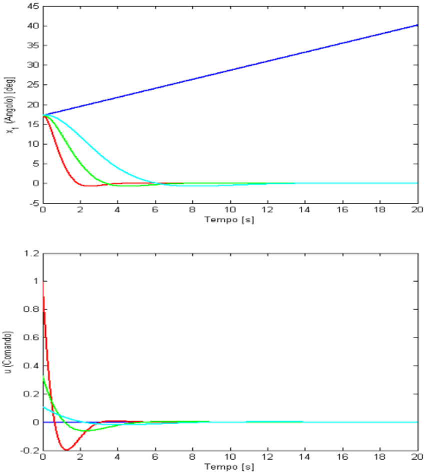
- Differential equation: $J\ddot{\theta}(t) = dF_c(t) + M_D(t)$ $\qquad x = \begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix},\ u = \frac{dF_c}{J}$

$$\dot{x} = Ax + Bu + B\frac{M_D}{J}, \qquad A = \begin{bmatrix}0 & 1 \\ 0 & 0\end{bmatrix},\ B = \begin{bmatrix}0 \\ 1\end{bmatrix}.$$

- Initial conditions: $x(0)=[0.3;0.02]$, disturbance: $M_D=0$.
- LQR control law: $u^*(t) = \arg\min_{u,x} J(u,x) = -Kx(t)$

Example: feedback control of satellite attitude

- **LQR control law**: $u^*(t) = -Kx(t)$



No control: $K = [0 \quad 0]$

$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, R = 1 \Rightarrow K = [1 \quad 1.41]$

$Q = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}, R = 1 \quad \Rightarrow K = [3.16 \quad 2.51]$

$Q = \begin{bmatrix} 1/10 & 0 \\ 0 & 0 \end{bmatrix}, R = 1 \Rightarrow K = [0.32 \quad 0.79]$

Linear Quadratic Regulator

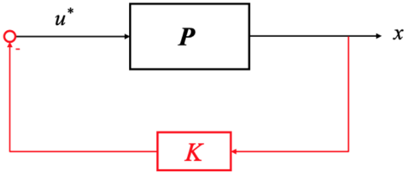
Variants available in Matlab[2]

- `K=lqr(A,B,Q,R)` or `K=lqr(sys,Q,R)`: Standard LQR design, as presented above.
- `K=lqry(sys,Q,R)`: LQR design with output weighting. The objective function is
$$J(u,y) = \|u\|_{R,2}^2 + \|y\|_{Q,2}^2 \,.$$
The same concepts and design criteria discussed above hold. The dimension of Q is different.
- `K=lqi(sys,Q,R)`: LQR design with integrator in the loop (see the control architecture in slide 41). Properties:
  - Null steady-state tracking error for step references.
  - Rejection of constant output disturbances.

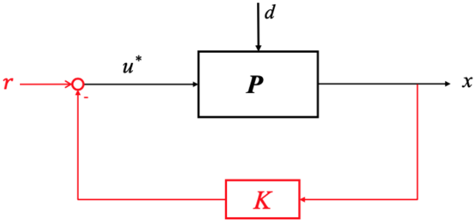
  The same concepts and design criteria discussed above hold. The dimension of K increases, due to the integrator state.

[2] See also the Matlab help.

Feedback control architectures

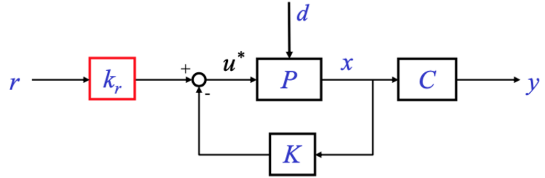


Pure state feedback.



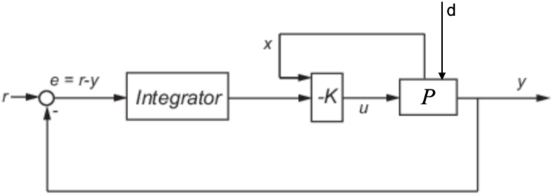
State feedback with reference and disturbance.

Feedback control architectures



State feedback with reference and disturbance.

A gain on the reference can be used to have a “small” tracking error.



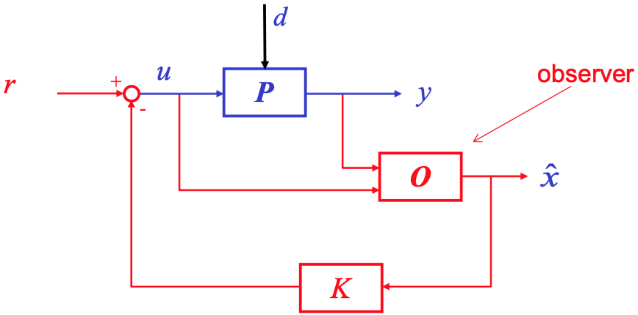
State feedback with reference and integrator in the loop:

- Null steady-state tracking error for step references.
- Rejection of constant disturbances.

Easy design with the Matlab function lqi.

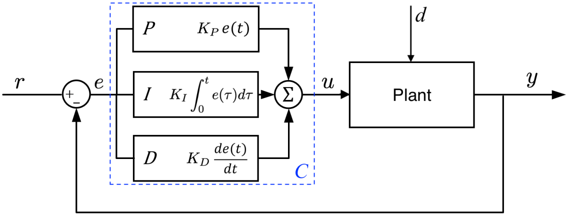
Feedback control architectures

- The above architectures rely on the assumption that all the state variables can be measured.
- If this assumption does not hold an observer/filter must be used allowing state estimation from input-output measurements.



Feedback control architectures

- PID control is not based on state feedback but on output feedback.



- In PID, the controller is dynamic.
- In state feedback:
  - the controller is static
  - the observer/filter is dynamic.
- Measuring (or estimating) the state and using a static controller is "equivalent" to using a dynamic controller (of sufficiently high order).

Discussion

- State feedback control for LTI systems has been discussed and the LQR/LQI approach has been introduced.
- Possible extension to nonlinear systems:
  - gain-scheduling based on state feedback
  - model predictive control
  - feedback linearization
  - others ...
- State feedback is based on the assumption that the state vector can be measured or estimated using a suitable observer/filter.