Nonlinear Pose Tracking Controller for Bar Tethered to Two Aerial Vehicles with Bounded Linear and Angular Accelerations

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Abstract

We consider a system composed of a bar tethered to two aerial vehicles, and develop a controller for pose tracking of the bar, i.e., a controller for position and attitude tracking. The controller is designed by first computing an input and a state transformations, which convert the system vector field into one that highlights the cascaded structure of the problem. We then design a controller for the transformed system by exploring that cascaded structure. There are three main contributions: *i)* we provide bounds on the linear and angular acceleration of the bar that guarantee well-posedness of the controller, and such bounds can be used when selecting the gains and saturations of bounded controllers for both three dimensional and unit vector double integrators; *ii)* the proposed control law includes a degree of freedom which can be used to regulate the relative position between the aerial vehicles; and *iii)* the proposed control law for the throttle guarantees that the cascaded structure of the problem is preserved. Simulations are presented which validate the proposed algorithm.

I. Introduction

Aerial vehicles provide a platform for automating inspection and maintenance of infrastructures [1]. Vertical take off and landing rotorcrafts, with hover capabilities, and in particular quadrotors, form a class of underactuated vehicles whose popularity stems from their ability to be used in small spaces, their reduced mechanical complexity, and inexpensive components [2], [3].

Slung load transportation by aerial vehicles is an important task in the scope of inspection and maintenance of infrastructures, and it forms a class of underactuated systems for which trajectory tracking control strategies are necessary [4]. To be specific, the system we focus on is composed of a one dimensional bar and two quadrotors attached to that bar by cables. Different slung load systems and control strategies have been studied and proposed. Load lifting of a point mass by a single or multiple aerial vehicles has been studied, with focus on exploring differential flatness for the purposes of control and motion planning [5]–[8], on nonlinear control techniques for an extended range of operation [9]–[13], and on adaptive control laws for compensating model uncertainties and disturbances [14]–[16]. Experimental results are also found, with vision being used for measuring the position of the load so as to estimate the cable length [17], and with information exchange between the aerial vehicles being considered [18]. Load lifting of a rigid body by multiple aerial vehicles is also found in [19]–[21]. In particular, in [19], static equilibrium relations for a tethered rigid body are described and analyzed. In [20], [21] a controller is designed for three of more vehicles transporting a rigid body; in particular the throttle control laws are designed by minimizing the error to the desired 3D force requested to each vehicle. In this work, however, the proposed throttle control laws are different, so as to preserve the cascaded structure of the problem. A critical issue in trajectory tracking controllers for VTOL vehicles is the need of a bounded control law for a 3D double integrator for guaranteeing well-posedness of the control law over the whole state space [3], [22]–[24]. One of our main contributions is in realizing that the quadrotors-bar system is an extended VTOL vehicle, in the sense that a bounded control law for a unit vector double integrator is necessary, in addition to a bounded control law for a 3D double integrator.

In this manuscript, we design a control law that guarantees that the lifted bar asymptotically tracks a desired pose trajectory, i.e., a desired position trajectory in the three dimensional space and a desired unit vector attitude trajectory. The design process follows two steps: in the first, we compute an input and a state transformation, which transform the system quadrotors-bar vector field into one where the cascaded structure of the problem becomes explicit; and in the second step, we explore that cascaded structure and design a controller via a backstepping procedure. There are three main contributions, which we emphasize here. First, we provide bounds on the linear and angular acceleration of the bar that guarantee well-posedness of the controller, where these bounds are necessary when selecting the gains and saturations of bounded controllers for three dimensional double integrators and unit vector double integrators. Secondly, the proposed control law includes a degree of freedom which can be used to regulate the relative position between the aerial vehicles, namely to drive the vehicles further or closer together. Finally, another novelty is that the proposed control law for the throttle guarantees the preservation of the cascaded structure of the problem.

The remainder of this paper is structured as follows. In Section III, the model of the quadrotors-bar system is presented. In Section IV the control design process is explained, and in Subsections IV-B and IV-C we provide the input and state transformation described before. In Section V, we design the controller via a backstepping procedure, and, in Section VI, we

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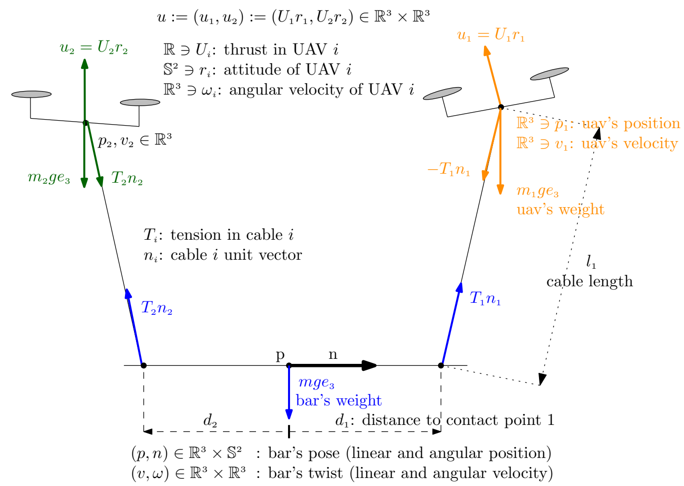


Fig. 1: Modeling of quadrotors-bar system

design the quadrotors throttle and angular velocity which guarantee that the cascaded structure of the problem is preserved. Finally, in Section VII, we present illustrative simulations.

This document is to be read in conjunction with the mathematica notebook files available in the repository.

II. Notation

The map $\mathcal{S} : \mathbb{R}^3 \ni x \mapsto \mathcal{S}(x) \in \mathbb{R}^{3\times 3}$ yields a skew-symmetric matrix and it satisfies $\mathcal{S}(a)\, b := a \times b$, for any $a, b \in \mathbb{R}^3$. $\mathbb{S}^2 := \{x \in \mathbb{R}^3 : x^T x = 1\}$ denotes the set of unit vectors in $\mathbb{R}^3$. The map $\Pi : \mathbb{S}^2 \ni x \mapsto \Pi(x) := I_3 - xx^T \in \mathbb{R}^{3\times 3}$ yields a matrix that represents the orthogonal projection onto the subspace perpendicular to $x \in \mathbb{S}^2$. We denote $A_1 \oplus \cdots \oplus A_n$ as the block diagonal matrix with block diagonal entries A_1 to A_n (square matrices). Given $\bar{\sigma} > 0$, denote $\mathcal{B}(\bar{\sigma}) := \{\xi \in \mathbb{R}^3 : \|\xi\| \leq \bar{\sigma}\}$. We denote by $e_1, \cdots, e_n \in \mathbb{R}^n$ the canonical basis vectors in $\mathbb{R}^n$; when clear from the context, n is omitted. For some set A, id_A denotes the identity map on that set. Given some normed spaces A and B, and a function $f : A \ni a \mapsto f(a) \in B$, $Df : A \ni a \mapsto Df(a) \in \mathcal{L}(A, B)$ denotes the derivative of f. Given a manifold A, $T_a A$ denotes the tangent set of A at some $a \in A$.

III. Modeling and Problem Statement

Consider the system illustrated in Fig. 1, with two aerial vehicles (in particular quadrotors), a one dimensional bar and two cables connecting the aerial vehicles to distinct contact points on the bar. All the physical quantities are those in Fig. 1. Hereafter, and for brevity, we refer to this system as quadrotors-bar system. As a first step in modeling the system, we assume that we have control over $u_1, u_2 \in \mathbb{R}^3$ (and thus over the quadrotors' throttle and attitude), which we assume are inputs to the quadrotors-load system. Later, in Section VI, we let the quadrotors' attitude (i.e., $r_1, r_2 \in \mathbb{S}^2$) be part of the state, and we assume we have control over the quadrotors throttles and angular velocities (i.e., $U_1, U_2 \in \mathbb{R}$ and $\omega_{r_1}, \omega_{r_2} \in \mathbb{R}^3$).

Consider then the state space

$$\begin{aligned}
\mathbb{Z} = \big\{&(p, n, p_1, p_2, v, \omega, v_1, v_2) \in (\mathbb{R}^3)^8 : (p, n, p_1, p_2) \in \mathbb{Z}_k, \qquad (1)\\
&n^T\omega = 0, (v_i - (v + d_i \mathcal{S}(\omega)\, n))^T (p_i - (p + d_i n)) = 0, i \in \{1, 2\}\big\}, \text{ where}\\
\mathbb{Z}_k = \big\{&(p, n, p_1, p_2) \in \times(\mathbb{R}^3)^4 : n^T n = 1, (p_i - (p + d_i n))^T (p_i - (p + d_i n)) = l_i^2, i \in \{1, 2\}\big\}. \qquad (2)
\end{aligned}$$

which encapsulates the constraints illustrated in Fig. 1. We emphasize that the constraints in (1)-(2) imply that the distance between each contact point on the bar and the corresponding quadrotor is constant and equal to the corresponding cable length. We always decompose a $z \in \mathbb{Z}$ and a $u \in \mathbb{R}^6$ in the same way, namely $u \in \mathbb{R}^6 :\Leftrightarrow (u_1, u_2) \in \mathbb{R}^3 \times \mathbb{R}^3$ and

$$z \in \mathbb{Z} :\Leftrightarrow (z_k, z_d) \in \mathbb{Z} :\Leftrightarrow (p, n, p_1, p_2, v, \omega, v_1, v_2) \in \mathbb{Z}, \tag{3}$$

where $z_k := (p, n, p_1, p_2) \in \mathbb{Z}_k$ corresponds to the pose of the bar and positions of the aerial vehicles (with $\mathbb{Z}_k$ as in (2)). Given the state space (1), its tangent set at a $z \in \mathbb{Z}$ is given by

$$\begin{aligned}
T_z\mathbb{Z} := \big\{&(\delta p, \delta n, \delta p_1, \delta p_2, \delta v, \delta\omega, \delta v_2, \delta v_1) \in (\mathbb{R}^3)^8 : (\delta p, \delta n, \delta p_1, \delta p_2) \in T_{z_k}\mathbb{Z}_k\\
&\delta n^T \omega + n^T \delta\omega = 0, i \in \{1, 2\},\\
&(\delta v_i - (\delta v + d_i \mathcal{S}(\delta\omega)\, n + d_i \mathcal{S}(\omega)\, \delta n))^T (p_i - (p + d_i n)) + (v_i - (v + d_i \mathcal{S}(\omega)\, n))^T (\delta p_i - (\delta p + d_i \delta n)) = 0\big\},
\end{aligned}$$

$$T_{z_k}\mathbb{Z}_k := \big\{(\delta p, \delta n, \delta p_1, \delta p_2) \in (\mathbb{R}^3)^4 : \delta n^T n = 0, (p_i - (p + d_i n))^T (\delta p_i - (\delta p + d_i \delta n)) = 0, i \in \{1, 2\}\big\}.$$

The state space definition in (1) also allows for the definition of the cables' unit vectors (see Fig. 1). Specifically, for $i \in \{1,2\}$, we define

$$\mathbb{Z}_k \ni z_k \mapsto n_i(z_k) := \frac{p_i - (p + d_i n)}{\|p_i - (p + d_i n)\|} \overset{(2)}{=} \frac{p_i - (p + d_i n)}{l_i} \in \mathbb{S}^2, \tag{4}$$

where (4) can be visualized in Fig 1.

Given an appropriate $u : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^6$, a system's quadrotors-bar trajectory $z : \mathbb{R}_{\geq 0} \ni t \mapsto z(t) \in \mathbb{Z}$ evolves according to

$$\dot{z}(t) = Z(z(t), u(t)), z(0) \in \mathbb{Z}, \tag{5}$$

where $Z : \mathbb{Z} \times \mathbb{R}^6 \ni (z, u) \mapsto Z(z, u) \in \mathbb{R}^{24}$ is given by

$$Z(z,u) := \begin{bmatrix} Z_k(z_k) z_d \\ Z_d(z,u) \end{bmatrix} \left(= \begin{bmatrix} \dot{z}_k \\ \dot{z}_d \end{bmatrix} \right) \in \mathbb{R}^{24}, \tag{6}$$

$$Z_k(z_k) := (I_3 \oplus -\mathcal{S}(n) \oplus I_3 \oplus I_3) \in \mathbb{R}^{12\times 12}, \tag{7}$$

$$Z_d(z,u) := \begin{bmatrix} \sum \frac{T_i(z,u)}{m} n_i(z_k) - g e_3 \\ \sum \frac{T_i(z,u)}{J} \mathcal{S}(d_i n) n_i(z_k) \\ \frac{u_1}{m_1} - \frac{T_1(z,u)}{m_1} n_1(z_k) - g e_3 \\ \frac{u_2}{m_2} - \frac{T_2(z,u)}{m_2} n_2(z_k) - g e_3 \end{bmatrix} \left(= \begin{bmatrix} \dot{v} \\ \dot{\omega} \\ \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} \right),$$

with g as the acceleration due to gravity; and T_1 and T_2 as the tensions on the cables (see Fig 1). The linear and angular accelerations in (6) are written from the Newton's and Euler's equations of motion, considering the net force and torque on each rigid body: the bar is taken as a rigid body (with net force and torque in blue – see Fig. 1); while the quadrotors are taken as point masses (with net forces in orange and green – see Fig. 1). The tensions T_1 and T_2 constitute internal forces to the quadrotors-bar system, and the Newton's and Euler's equations of motion do not provide any insight into these forces. However, the constraint that the state must remain in the state set $\mathbb{Z}$ in (1), enforces the vector field Z in (6) to be in the tangent set. This constraint uniquely defines the tensions on the cable, i.e., for any $(z,u) \in \mathbb{Z} \times \mathbb{R}^6$, $Z(z,u) \in T_z\mathbb{Z}$ implies that

$$\begin{bmatrix} T_1(z,u) \\ T_2(z,u) \end{bmatrix} = \underbrace{\tilde{M}_T^{-1}(z_k) \Bigg(\begin{bmatrix} \frac{m}{m_1} n_1(z_k)^T & 0_{1\times 3} \\ 0_{1\times 3} & \frac{m}{m_2} n_2(z_k)^T \end{bmatrix}}_{=: M_T(z_k) \in \mathbb{R}^{2\times 6}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + m \begin{bmatrix} \frac{\|v_1 - (v + d_1 \mathcal{S}(\omega) n)\|^2}{l_1} \\ \frac{\|v_2 - (v + d_2 \mathcal{S}(\omega) n)\|^2}{l_2} \end{bmatrix} + m\|\omega\|^2 \begin{bmatrix} d_1 n^T n_1(z_k) \\ d_2 n^T n_2(z_k) \end{bmatrix} \Bigg) \tag{8}$$

$$= M_T(z_k) u + \begin{bmatrix} T_1(z, 0_6) \\ T_2(z, 0_6) \end{bmatrix}, \text{ where} \tag{9}$$

$$\tilde{M}_T(z_k) = \begin{bmatrix} \frac{m}{m_1} & 0 \\ 0 & \frac{m}{m_2} \end{bmatrix} + \begin{bmatrix} 1 & \cos(\theta) \\ \cos(\theta) & 1 \end{bmatrix} + \frac{m d_1 d_2}{J} \begin{bmatrix} \frac{d_1}{d_2}\|a\|^2 & a^T b \\ a^T b & \frac{d_2}{d_1}\|b\|^2 \end{bmatrix} \Big|_{\substack{\cos(\theta) = n_1(z_k)^T n_2(z_k) \\ a = \mathcal{S}(n) n_1(z_k) \\ b = \mathcal{S}(n) n_2(z_k)}}.$$

We can now formulate the problem treated in this paper.

Problem 1: Given the vector field Z in (6) and a desired bar pose, i.e, $(p^\star, n^\star) : \mathbb{R}_{\geq 0} \ni t \mapsto (p^\star(t), n^\star(t)) \in \mathbb{R}^3 \times \mathbb{S}^2$, design a control law $u^{cl} : \mathbb{R}_{\geq 0} \times \mathbb{Z} \mapsto \mathbb{R}^6$ such that $\lim_{t\to\infty}(p(t) - p^\star(t), n(t) - n^\star(t)) = 0_6$ along a solution $\mathbb{R}_{\geq 0} \ni t \mapsto z(t) \in \mathbb{Z}$ of $\dot{z}(t) = Z(z(t), u^{cl}(t, z(t)))$.

We emphasize that the vector field (6) is input affine. To be specific,

$$Z(z,u) = \begin{bmatrix} Z_k(z_k) z_d \\ Z_d(z,u) \end{bmatrix} = A(z) + \begin{bmatrix} 0_{12\times 6} \\ B(z_k) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

where

$$A(z) := Z(z, 0_6) \in \mathbb{R}^{24}$$

and

$$B(z_k) := \begin{bmatrix} \frac{1}{m} \sum_{i\in\{1,2\}} n_i(z_k) e_i^T M_T(z_k) \\ \frac{1}{J} \sum_{i\in\{1,2\}} \mathcal{S}(n) d_i n_i(z_k) e_i^T M_T(z_k) \\ \frac{1}{m_1} ([I_3 \, 0_{3\times 3}] - n_1(z_k) e_1^T M_T(z_k)) \\ \frac{1}{m_2} ([0_{3\times 3} \, I_3] - n_2(z_k) e_2^T M_T(z_k)) \end{bmatrix} \in \mathbb{R}^{12\times 6}, \tag{10}$$

where we emphasize that B in (10) depends exclusively on the kinematic variables, i.e., $z_k \in \mathbb{Z}_k$ (and where $M_T(z_k) \in \mathbb{R}^{2\times 6}$ is that in (9), $e_1 = (1,0) \in \mathbb{S}^1$ and $e_2 = (0,1) \in \mathbb{S}^1$).

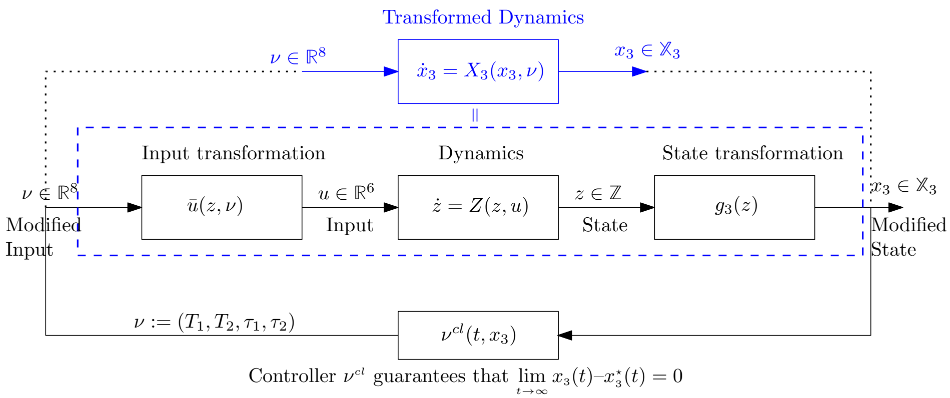


Fig. 2: Control strategy: we provide an input and state transformations which convert the vector field Z into the vector field X_3 which, in turn, highlights the cascaded structure of the problem.

IV. Control Law Design

Let us explain the pursued control strategy, which is illustrated in Fig. 2, and with the vector field Z as in (6).

As a first step, we introduce an input transformation from $\nu := (T_1, T_2, \tau_1, \tau_2) \in \mathbb{R}^8$ to $u \in \mathbb{R}^6$, where T_i and τ_i will stand for the tension and angular acceleration of cable $i \in \{1, 2\}$. As shall be seen later, ν provides a more meaningful input to design, and once it is designed one can map ν to u (the actual input) by means of the function $\bar{u}$ (see Fig. 2). This is done in Section IV-B

In the second step, done in Section IV-C, we provide a coordinate transformation g_3 that maps a $z \in \mathbb{Z}$ into an $x_3 \in \mathbb{X}_3$ (g_3 and $\mathbb{X}_3$ are defined later), and with this coordinate transformation, we obtain a new vector field X_3, which comes from the composition of the vector field Z with the input transformation and the coordinate change. The benefit of this coordinate transformation is that it highlights the cascaded structure of the problem, namely the cascaded nature of X_3.

In the final step, done in Section V, and by exploiting the cascaded nature of X_3, one designs a control law defined as

$$\begin{aligned}
&\nu^{cl} : \mathbb{R}_{\geq 0} \times \mathbb{X}_3 \ni (t, x_3) \mapsto \nu^{cl}(t, x_3) \in \mathbb{R}^8 \\
&\nu^{cl}(t, x_3) := (T_1^{cl}(t, x_2), T_2^{cl}(t, x_2), \tau_1^{cl}(t, x_3), \tau_2^{cl}(t, x_3)) \in \mathbb{R}^{1+1+3+3} \qquad (11)
\end{aligned}$$

that guarantees that Problem 1 is accomplished, and, using the mapping $\bar{u}$, one constructs the control law for the original system, i.e., $\mathbb{R}_{\geq 0} \times \mathbb{Z} \ni (t, z) \mapsto u^{cl}(t, z) \in \mathbb{R}^6$ defined as

$$u^{cl}(t, z) := \bar{u}(z, \nu^{cl}(t, x_3))|_{x_3 = g_3^{-1}(z)}. \qquad (12)$$

A. Angular velocity and acceleration of cables

In order to construct the input and coordinate transformation, illustrated in Fig. 2, we must define the angular velocity and acceleration of the cables. For that purpose, let us define the first and second derivatives of a smooth function on $\mathbb{Z}_k$ and along a solution of (5): consider then a function $f : \mathbb{Z}_k \in z_k \mapsto f(z_k) \in \mathbb{R}^3$, and denote

$$f^{(1)} : \mathbb{Z} \ni (z_k, z_d) \mapsto f^{(1)}(z_k, z_d) := Df(z_k) Z_k(z_k) z_d (= \dot{f}(z_k)) \in \mathbb{R}^3, \qquad (13)$$

$$f^{(2)} : \mathbb{Z} \times \mathbb{R}^6 \ni (z, u) \mapsto f^{(2)}(z, u) (= \ddot{f}(z_k)) \in \mathbb{R}^3, \qquad (14)$$

as the first and second derivatives of f along a solution of (5), where

$$\begin{aligned}
f^{(2)}(z, u) &:= \partial_1 f^{(1)}(z_k, z_d) Z_k(z_k) z_d + \partial_2 f^{(1)}(z_k, z_d) Z_d(z, u) \qquad (15) \\
&\overset{(13)}{=} \partial_1 f^{(1)}(z_k, z_d) Z_k(z_k) z_d + Df(z_k) Z_k(z_k) Z_d(z, u) \\
&\overset{(7)}{=} \underbrace{\partial_1 f^{(1)}(z_k, z_d) z_d + Df(z_k) Z_k(z_k) Z_d(z, 0_6)}_{=: A_f(z) \in \mathbb{R}^3} + \underbrace{Df(z_k) Z_k(z_k) B(z_k)}_{=: B_f(z_k) \in \mathbb{R}^{3\times 6}} u.
\end{aligned}$$

Proposition 1: Let $p, v, a : \mathbb{R} \mapsto \mathbb{R}^3$, where $(\dot{p}(t), \dot{v}(t)) = (v(t), a(t))$ and $\|p(t)\| = d$ for all $t \in \mathbb{R}$ and some $d > 0$. Then, for

$$\begin{aligned}
\mathbb{R}^3 \backslash \{0_3\} \ni p \mapsto & n(p) := \frac{p}{d} \in \mathbb{S}^2, \\
\mathbb{R}^3 \backslash \{0_3\} \times \mathbb{R}^3 \ni (p, v) \mapsto & \omega(p, v) := \mathcal{S}(n(p)) \frac{v}{d} \in T_{n(p)} \mathbb{S}^2, \\
\mathbb{R}^3 \backslash \{0_3\} \times \mathbb{R}^3 \ni (p, a) \mapsto & \tau(p, a) := \mathcal{S}(n(p)) \frac{a}{d} \in T_{n(p)} \mathbb{S}^2,
\end{aligned}$$

it holds that, for all $t \in \mathbb{R}$,

$$\dot{n}(p(t)) = \mathcal{S}\left(\omega(p(t), v(t))\right) n(p(t)),$$
$$\dot{\omega}(p(t), v(t)) = \tau(p(t), a(t)).$$

Proposition 1 follows from straightforward computations.

With Proposition 1 in mind, consider a smooth function $f : \mathbb{Z}_k \in z_k \mapsto f(z_k) \in \mathbb{R}^3$ where f has constant norm pointwise (i.e., $\|f(z_k)\| = d_f$ for some $d_f > 0$ and for all $z_k \in \mathbb{Z}_k$). With (13), (14) and Proposition 1 in mind, we can then define

$$\begin{aligned}
n_f &: \mathbb{Z}_k \ni z_k \mapsto n_f(z_k) := \frac{f(z_k)}{d_f} \in \mathbb{S}^2, \\
\omega_f &: \mathbb{Z} \ni (z_k, z_d) \mapsto \omega_f(z_k, z_d) := \mathcal{S}\left(n_f(z_k)\right) \frac{f^{(1)}(z_k, z_d)}{d_f} \in \mathbb{R}^3, \\
\tau_f &: \mathbb{Z} \times \mathbb{R}^6 \ni (z, u) \mapsto \tau_f(z, u) := \mathcal{S}\left(n_f(z_k)\right) \frac{f^{(2)}(z, u)}{d_f} \in \mathbb{R}^3,
\end{aligned} \tag{16}$$

where n_f is the unit vector associated to the function f, ω_f is the angular velocity of n_f, and τ_f is the angular acceleration of n_f. The angular acceleration in (16) can be equivalently expressed as

$$\tau_f(z, u) \overset{(15)}{=} \underbrace{d_f^{-1} \mathcal{S}\left(n_f(z_k)\right) A_f(z)}_{=:A_{n_f}(z) \in \mathbb{R}^3} + \underbrace{d_f^{-1} \mathcal{S}\left(n_f(z_k)\right) B_f(z_k)}_{=:B_{n_f}(z_k) \in \mathbb{R}^{3\times 6}} u. \tag{17}$$

For our particular purposes, we are interested in the angular acceleration of the cables, thus we consider, for $i \in \{1, 2\}$, the function

$$f_i : \mathbb{Z}_k \ni z_k \mapsto f_i(z_k) := p_i - (p + d_i n) \in \mathbb{R}^3.$$

We know from (2) that $d_{f_i} = l_i$, and therefore we can define the cable unit vector n_{f_i} (this is the same function as in (4)), its angular velocity ω_{f_i} and its torque τ_{f_i}. We remark here that, for any $(z, u) \in \mathbb{Z} \times \mathbb{R}^6$,

$$\tau_{f_1}(z, (u_1, u_2)) = \tau_{f_1}(z, (u_1, \Pi\left(n_2(z_k)\right) u_2)), \tag{18}$$

which means that the angular acceleration of cable 1 does not depend on the input component of vehicle 2 on the space orthogonal to cable 2 (same conclusion if the roles of 1 and 2 are reversed).

B. Input Transformation

Let us now provide the input transformation $\bar{u}$ illustrated in Fig. 2. For that purpose we introduce a function

$$R : \mathbb{Z} \times \mathbb{R}^6 \ni (z, u) \mapsto R(z, u) \in \mathbb{R}^m$$

of $m \in \mathbb{N}$ *physical quantities* we wish to regulate/control. If the function R is invertible w.r.t. its second argument then, given any $\nu \in \mathbb{R}^m$, $u = R^{-1}(z, \nu) :\Leftrightarrow \nu = R(z, u)$ provides a control input to control those $m \in \mathbb{N}$ physical quantities.

In particular, we wish to control 6 quantities: the two tensions on the cables (2 quantities, namely T_1 and T_2), and the angular accelerations on each cable ($2\times 2 = 4$ quantities: note that the angular accelerations τ_{f_1} and τ_{f_2} are three dimensional, but each angular acceleration is always orthogonal to the corresponding cable, which means that we only need to control $2 \times (3 - 1)$ quantities). As such, we define

$$R(z, u) := \begin{bmatrix} T_1(z, u) \\ T_2(z, u) \\ \tau_{f_1}(z, u) \\ \tau_{f_2}(z, u) \end{bmatrix} = \underbrace{\begin{bmatrix} T_1(z, 0_6) \\ T_2(z, 0_6) \\ A_{n_{f_1}}(z) \\ A_{n_{f_2}}(z) \end{bmatrix}}_{=:A_R(z) \in \mathbb{R}^8} + \underbrace{\begin{bmatrix} e_1^T M_T(z_k) \\ e_2^T M_T(z_k) \\ B_{n_{f_1}}(z_k) \\ B_{n_{f_2}}(z_k) \end{bmatrix}}_{=:B_R(z_k) \in \mathbb{R}^{8\times 6}} u =: A_R(z) + B_R(z_k) u, \tag{19}$$

where we made use of (9) and of (17). Notice that given any $z \in \mathbb{Z}$ and any $\nu \in \mathbb{R}^8$, there does not necessarily exist a $u \in \mathbb{R}^6$ such that $\nu = R(z, u)$. However, given any $(z, \nu) \in \mathcal{V} := \{(z, \nu) \in \mathbb{Z} \times \mathbb{R}^8 : \nu := (T_1, T_2, \tau_1, \tau_2) \in \mathbb{R}^{2+6}, \tau_1^T n_1(z) = 0, \tau_2^T n_1(z) = 0\}$ it follows that there exists a unique $\bar{u} : \mathcal{V} \ni (z, \nu) \mapsto \bar{u}(z, \nu) \in \mathbb{R}^6$ such that $\nu = R(z, \bar{u}(z, \nu))$ and it is given by

$$\bar{u}(z, \nu) := (B_R(z_k)^T B_R(z_k))^{-1} B_R(z_k)^T (\nu - A_R(z)). \tag{20}$$

Notice that there exists an inverse on (20), which depends exclusively on the kinematic configuration, and it can be shown that that inverse exists for any $z_k \in \mathbb{Z}_k$. This can be shown, once one realizes that the tensions depend exclusively on $n_1^T u_1$ and $n_2^T u_2$ (see (8)), while the angular acceleration on the cable $i \in \{1, 2\}$ depends on the tensions and $\Pi\left(n_i\right) u_i$ (see (18)). There is therefore a cascaded structure, which means one can first control the tensions, and, in the next step, one controls the angular accelerations. With the above in mind, one can verify that the input transformation $\bar{u}$ in (20) is equivalently written as

$$\bar{u}(z,\nu)|_{\nu=(T_1,T_2,\tau_1,\tau_2)} = \begin{bmatrix} u_1 n_1 - m_1 \Pi(n_1)\left(l_1 \mathcal{S}(n_1)\tau_1 + d_1\|\omega\|^2 n - \frac{1}{m}T_2 n_2 - \frac{d_1}{J}\Pi(n)\sum_{i\in\{1,2\}} d_i T_i n_i\right) \\ u_2 n_2 - m_2 \Pi(n_2)\left(l_2 \mathcal{S}(n_2)\tau_2 + d_2\|\omega\|^2 n - \frac{1}{m}T_1 n_1 - \frac{d_2}{J}\Pi(n)\sum_{i\in\{1,2\}} d_i T_i n_i\right)\end{bmatrix}\Big|_{\substack{n_1=n_1(z_k),\\ n_2=n_2(z_k)}}$$

where $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{m_1}{m} & 0 \\ 0 & \frac{m_2}{m}\end{bmatrix} M_T(z_k)\left(\begin{bmatrix} T_1 \\ T_2\end{bmatrix} - T(z, 0_6)\right)$

where the function T is that given in (8) (i.e., the tension functions).

C. State Transformation

Let us now provide the coordinate transformation illustrated in Fig. 2, which will emphasize the cascaded structure of the problem. Due to this cascaded structure, let us make the following cascaded set definitions, namely,

$$\mathbb{X}_1 := \{x_1 := (p, v, n, \omega) \in (\mathbb{R}^3)^4 : n \in \mathbb{S}^2, \omega^T n = 0\}, \tag{21}$$
$$\mathbb{X}_2 := \{x_2 := (x_1, n_1, n_2) \in \mathbb{X}_1 \times (\mathbb{S}^2)^2\},$$
$$\mathbb{X}_3 := \{x_3 := (x_2, \omega_1, \omega_2) \in \mathbb{X}_2 \times (\mathbb{R}^3)^2 : \omega_1^T n_1 = 0, \omega_2^T n_2 = 0\}. \tag{22}$$

We always decompose an $x_1 \in \mathbb{X}_1$, an $x_2 \in \mathbb{X}_2$ and an $x_3 \in \mathbb{X}_3$ as decomposed in (21)–(22). Consider then the mappings

$$\mathbb{Z} \ni z \mapsto g_1(z) := (p, v, n, \omega) \in \mathbb{X}_1, \tag{23}$$
$$\mathbb{Z} \ni z \mapsto g_2(z) := (g_1(z), n_{f_1}(z_k), n_{f_2}(z_k)) \in \mathbb{X}_2,$$
$$\mathbb{Z} \ni z \mapsto g_3(z) := (g_2(z), \omega_{f_1}(z), \omega_{f_2}(z)) \in \mathbb{X}_3, \tag{24}$$

where g_1 isolates the pose of the bar, and the twist of the bar; while g_2 and g_3 map also to the cables' unit vectors and the cables' angular velocities (see Section IV-A for those definitions). Consider also the mapping $g_3^{-1} : X_3 \ni x_3 \mapsto g_3^{-1}(x_3) \in \mathbb{Z}$ defined as

$$g_3^{-1}(x_3) := (p, n, p + d_1 n + l_1 n_1, p + d_2 n + l_2 n_2, v + d_1 \mathcal{S}(\omega) n + l_1 \mathcal{S}(\omega_1) n_1, v + d_2 \mathcal{S}(\omega) n + l_2 \mathcal{S}(\omega_2) n_2) \in \mathbb{Z}$$

where we emphasize that, indeed, $g_3 \circ g_3^{-1} = \mathrm{id}_{\mathbb{X}_3}$ and that $g_3^{-1} \circ g_3 = \mathrm{id}_{\mathbb{Z}}$ (thus the mappings are inverses of each other).

It then follows that (denote $\nu := (T_1, T_2, \tau_1, \tau_2) \in \mathbb{R}^{2+6}$)

$$\begin{aligned} X_3(x_3, \nu) :=& Dg_3(z) Z(z, \bar{u}(z,\nu))|_{z=g_3^{-1}(x_3)} \in T_{x_3}\mathbb{X}_3. \\ =& \begin{bmatrix} X_2(x_2, (T_1, T_2, \omega_1, \omega_2)) \\ \Pi(n_1)\tau_1 \\ \Pi(n_2)\tau_2 \end{bmatrix} \left(= \begin{bmatrix} \dot{x}_2 \\ \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} = \dot{x}_3 \right), \end{aligned} \tag{25}$$

where (denote $\nu_2 := (T_1, T_2, \omega_1, \omega_2) \in \mathbb{R}^{2+6}$)

$$X_2(x_2, \nu_2) := \begin{bmatrix} X_1(x_1, (n_1, n_2), (T_1, T_2)) \\ \mathcal{S}(\omega_1) n_1 \\ \mathcal{S}(\omega_2) n_2 \end{bmatrix} \left(= \begin{bmatrix} \dot{x}_1 \\ \dot{n}_1 \\ \dot{n}_2 \end{bmatrix} \right), \tag{26}$$

and where (denote $\nu_1 := (T_1, T_2) \in \mathbb{R}^2$)

$$X_1(x_1, (n_1, n_2), \nu_1) := \begin{bmatrix} v \\ \frac{T_1 n_1 + T_2 n_2}{m} - g e_3 \\ \mathcal{S}(\omega) n \\ \mathcal{S}(n) \frac{d_1 T_1 n_1 + d_2 T_2 n_2}{J} \end{bmatrix} \left(= \begin{bmatrix} \dot{p} \\ \dot{v} \\ \dot{n} \\ \dot{\omega} \end{bmatrix} \right). \tag{27}$$

The choice of the mappings (23)–(24) and of $\bar{u}$ in (20) is now clear: it induces a cascaded structure with three layers (25)-(27), which can be explored in the control design process.

V. Backstepping control

The next three subsections are dedicated to each of the three layers, and for the control design, we apply a backstepping procedure, similar to the one found in [24].

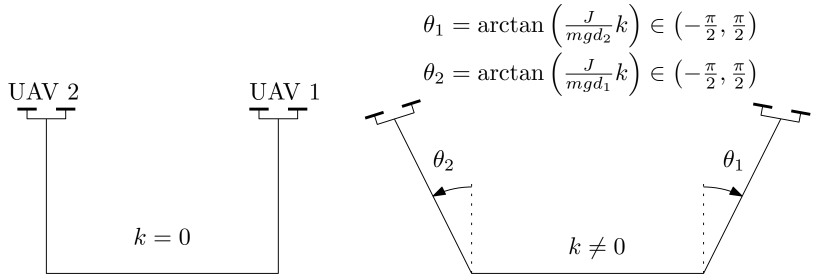


Fig. 3: Effect of degree of freedom $k \in \mathbb{R}$ in (30)

A. Step 1

We focus first on the first layer, and thus on the vector field (27). Consider then the following similar vector field (denote $\mathcal{T} := (\mathcal{T}_1, \mathcal{T}_2) \in \mathbb{R}^{3+3}$)

$$\mathbb{X}_1 \times \mathbb{R}^6 \ni (x_1, \mathcal{T}) \mapsto \mathcal{X}_1(x_1, \mathcal{T}) := \begin{bmatrix} v \\ \frac{\mathcal{T}_1 + \mathcal{T}_2}{m} - g e_3 \\ \mathcal{S}(\omega)\, n \\ \mathcal{S}(n)\, \frac{d_1 \mathcal{T}_1 + d_2 \mathcal{T}_2}{J} \end{bmatrix}, \tag{28}$$

where we observe that, as long as $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2) \in (\mathbb{R}^3 \backslash \{0_3\})^2$

$$X_1\left(x_1, \left(\frac{\mathcal{T}_1}{\|\mathcal{T}_1\|}, \frac{\mathcal{T}_2}{\|\mathcal{T}_2\|}\right), (\|\mathcal{T}_1\|, \|\mathcal{T}_2\|)\right) = \mathcal{X}_1(x_1, \mathcal{T}). \tag{29}$$

The idea in the first step is then to find a control law for $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2)$ in (28) such that Problem 1 is accomplished. Immediately after, and based on observation (29), we define the desired cable $i \in \{1, 2\}$ direction as the unit vector given by $\mathcal{T}_i$. The critical issue, an one of our contributions, is in guaranteeing that $\mathcal{T}_i$ does not vanish, which requires a bounded controller for a 3D double integrator and a unit vector double integrator.

Given $r > 0$, denote $\mathcal{B}(r) := \{\xi \in \mathbb{R}^3 : \|\xi\| \leq r\}$. Given $\bar{a}, \bar{\tau} > 0$, consider then

$$\mathbb{X}_1 \times \mathcal{B}(\bar{a}) \times \mathcal{B}(\bar{\tau}) \ni (x_1, a, \tau) \mapsto \bar{\mathcal{T}}^{cl}(x_1, a, \tau) := (\bar{\mathcal{T}}_1^{cl}(x_1, a, \tau), \bar{\mathcal{T}}_2^{cl}(x_1, a, \tau)) \in \mathbb{R}^{3+3}$$

defined as (for a $k \in \mathbb{R}$)

$$\bar{\mathcal{T}}^{cl}(x_1, a, \tau) = \begin{bmatrix} \bar{\mathcal{T}}_1^{cl}(x_1, a, \tau) \\ \bar{\mathcal{T}}_2^{cl}(x_1, a, \tau) \end{bmatrix} := \frac{1}{d_1 - d_2} \begin{bmatrix} -d_2 m I_3 & J I_3 \\ d_1 m I_3 & -J I_3 \end{bmatrix} \begin{bmatrix} a + g e_3 \\ -\mathcal{S}(n)\,\tau + k n \end{bmatrix}, \tag{30}$$

and, it follows that composing (28) with (30) yields

$$\mathcal{X}_1(x_1, \bar{\mathcal{T}}^{cl}(x, a, \tau)) := \begin{bmatrix} v \\ a \\ \mathcal{S}(\omega)\, n \\ \Pi(n)\, \tau \end{bmatrix} \left(= \begin{bmatrix} \dot{p} \\ \dot{v} \\ \dot{n} \\ \dot{\omega} \end{bmatrix} \right). \tag{31}$$

As such, a and τ in (30) can be understood as the acceleration and torques inputs, while k is a degree of freedom that can be explored for positioning the cables in different configurations, as illustrated in Fig 3. Importantly, it follows from (30) that if

$$\bar{a} + \frac{J}{m \min(|d_1|, |d_2|)} (\bar{\tau} + k) < g \tag{32}$$

is satisfied (where $\bar{a}$ and $\bar{\tau}$ are those in the domain of (30)), then

$$\mathbb{X}_1 \times \mathcal{B}(\bar{a}) \times \mathcal{B}(\bar{\tau}) \overset{(32)}{\ni} (x_1, a, \tau) \mapsto \bar{\mathcal{T}}^{cl}(x_1, a, \tau) := (\bar{\mathcal{T}}_1^{cl}(x_1, a, \tau), \bar{\mathcal{T}}_2^{cl}(x_1, a, \tau)) \in (\mathbb{R}^3 \backslash \{0_3\})^2,$$

which means neither $\bar{\mathcal{T}}_1^{cl}$ nor $\bar{\mathcal{T}}_2^{cl}$ vanish, and therefore their direction is well defined. Also, notice that the control law (30) transforms the vector field (28) into two decoupled vector fields in (31): one three dimensional double integrator related to the bar position (and controlled with $a \in \mathcal{B}(\bar{a})$), and one unit vector double integrator related with the bar attitude (and controlled with $\tau \in \mathcal{B}(\bar{\tau})$). In order to satisfy (32), the need for bounded controllers for the double integrator arises.

Remark 2: Hierarchical controllers for position trajectory tracking of VTOL vehicles require $\bar{a} < g$ for well posedness [3], [22]–[24]. The quadrotors-bar system is in essence an extended VTOL vehicle, which apart from position control also requires attitude control, giving rise to the more restrictive condition (32) for well-posedness.

Recall now Problem 1, and denote $\mathbb{P} := \mathbb{R}_{\geq 0} \times \mathbb{R}^6$ and $\mathbb{O} := \mathbb{R}_{\geq 0} \times \{(n, \omega) \in \mathbb{R}^6 : n \in \mathbb{S}^2, n^T \omega = 0\}$. Suppose then that we are given: *i)* a bounded control law law $a^{cl} : \mathbb{P} \ni (t, p, v) \mapsto a^{cl}(t, p, v) \in \mathbb{B}(\bar{a}) \subset \mathbb{R}^3$ which guarantees that $\lim_{t \to \infty}(p(t) -$

$p^\star(t)) = 0_3$ with a companion Lyapunov function and its non-positive derivative $\mathbb{P} \ni (t, p, v) \mapsto V_\xi(t, p, v), W_\xi(t, p, v)$; and *ii)* a bounded control law $\tau^{cl} : \Theta \ni (t, n, \omega) \mapsto \tau^{cl}(t, n, \omega) \in \mathbb{B}(\bar{\tau}) \subset \mathbb{R}^3$ which guarantees that $\lim_{t\to\infty}(n(t) \pm n^\star(t)) = 0_3$ with a companion Lyapunov function and its non-positive derivative $\Theta \ni (t, p, v) \mapsto V_\theta(t, n, \omega), W_\theta(t, n, \omega)$. Note that the bounds on a^{cl} and τ^{cl} depend on the desired pose trajectory, so (32) imposes constraints on the desired pose trajectory: i.e., there are desired pose trajectories for which the desired cable directions may not be well defined (free falling is one example). The bounds on a^{cl} and τ^{cl} also depend on saturations and gains, which need to be chosen such that (32) is satisfied. See Appendix IX for some possible such functions and their bounds $\bar{a}$ and $\bar{\tau}$ (in (47) and (48)). Given such control laws and (30), we compose these as in

$$\begin{aligned}\mathbb{R}_{\geq 0} \times \mathbb{X}_1 \ni (t, x_1) \mapsto & \mathcal{T}^{cl}(t, x_1) := \bar{\mathcal{T}}^{cl}(x_1, a^{cl}(t, p, v), \tau^{cl}(t, n, \omega)) \\ & \begin{bmatrix}\mathcal{T}_1^{cl}(t, x_1) \\ \mathcal{T}_2^{cl}(t, x_1)\end{bmatrix} := \begin{bmatrix}\bar{\mathcal{T}}_1^{cl}(x_1, a^{cl}(t, p, v), \tau^{cl}(t, n, \omega)) \\ \bar{\mathcal{T}}_2^{cl}(x_1, a^{cl}(t, p, v), \tau^{cl}(t, n, \omega))\end{bmatrix}\end{aligned} \tag{33}$$

and, for $i \in \{1, 2\}$, we define

$$\mathbb{R}_{\geq 0} \times \mathbb{X}_1 \ni (t, x_1) \mapsto n_i^{cl}(t, x_1) := \frac{\mathcal{T}_i^{cl}(t, x_1)}{\|\mathcal{T}_i^{cl}(t, x_1)\|} \in \mathbb{S}^2 \tag{34}$$

as the desired cable i direction, and which is well defined for all $(t, x_1) \in \mathbb{R}_{\geq 0} \times \mathbb{X}_1$ provided that (32) is satisfied. Finally, notice that, given positive k_ξ and k_θ, and given

$$\mathbb{R}_{\geq 0} \times \mathbb{X}_1 \ni (t, x_1) \mapsto V_1(t, x_1) := k_\xi V_\xi(t, p, v) + k_\theta V_\theta(t, n, \omega) \geq 0, \tag{35}$$

it follows that

$$\begin{aligned}\mathbb{R}_{\geq 0} \times \mathbb{X}_1 \ni (t, x_1) \mapsto W_1(t, x_1) :=& \partial_1 V_1(t, x_1) + \partial_2 V_1(t, x_1)\mathcal{X}(x_1, \mathcal{T}^{cl}(t, x_1)) \\ =& k_\xi W_\xi(t, p, v) + k_\theta W_\theta(t, n, \omega) \leq 0.\end{aligned}$$

We are in position to design (T_1, T_2) as they appear in the vector field (27). Based on (33), consider then the control law for the tensions as

$$\mathbb{R}_{\geq 0} \times \mathbb{X}_2 \ni (t, x_2) \mapsto T_i^{cl}(t, x_2) := n_i^T \mathcal{T}_i^{cl}(t, x_1), i \in \{1, 2\}. \tag{36}$$

Since $a^T bb = a - \Pi(b)\, a$ for any $a \in \mathbb{R}^3$ and $b \in \mathbb{S}^2$, it follows, from composing (27) with (36), that

$$X_1(x_1, (n_1, n_2), (T_1^{cl}(t, x_2), T_2^{cl}(t, x_2))) = \mathcal{X}_1(x_1, \mathcal{T}^{cl}(t, x_1)) + e_1(t, x_2), \text{ with} \tag{37}$$

$$e_1(t, x_2) := -\sum_{i\in\{1,2\}} \begin{bmatrix} 0_3 \\ \frac{1}{m}\Pi(n_i)\, \mathcal{T}_i^{cl}(t, x_1) \\ 0_3 \\ \frac{d_i}{J}\mathcal{S}(n)\,\Pi(n_i)\,\mathcal{T}_i^{cl}(t, x_1)\end{bmatrix}, \tag{38}$$

where e_1 can be understood as the error remaining after step 1 is finished. Given the vector field (37), one may compute the angular velocity of the desired cable direction in (34), specifically

$$\begin{aligned}&\omega_{n_i^{cl}} : \mathbb{R}_{\geq 0} \times X_2 \ni (t, x_2) \mapsto \omega_{n_i^{cl}}(t, x_2) \in \mathbb{R}^3 \\ &\omega_{n_i^{cl}}(t, x_2) := \mathcal{S}\left(n_i^{cl}(t, x_1)\right)\left(\partial_1 n_i^{cl}(t, x_1) + \partial_2 n_i^{cl}(t, x_1)(\mathcal{X}_1(x_1, \mathcal{T}^{cl}(t, x_1)) + e_1(t, x_2))\right)\end{aligned}$$

and, finally, one may also compute the derivative of V_1 in (35) along the same vector field (37), which yields

$$\begin{aligned}&\mathbb{R}_{\geq 0} \times X_2 \ni (t, x_2) \mapsto W_1(t, x_1) + \sum_{i\in\{1,2\}} e_{\gamma,i}(t, x_2)^T \delta_i(t, x_2), \\ &e_{\gamma,i}(t, x_2) := \mathcal{S}\left(n_i^{cl}(t, x_1)\right) n_i \\ &\delta_i(t, x_2) := \|\mathcal{T}_i^{cl}(t, x_1)\| \mathcal{S}(n_i)\left(\frac{k_\xi}{m}\partial_v V_\xi(t, p, v) + \frac{k_\theta d_i}{J}\partial_\omega V_\theta(t, n, \omega)\right).\end{aligned} \tag{39}$$

B. Step 2

Let us now proceed to the second step, and thus focus on vector field (26). Given the chosen controls laws in (36), it follows that

$$X_2(x_2, (T_1^{cl}(t, x_2), T_2^{cl}(t, x_2), \omega_1, \omega_2)) = \begin{bmatrix}\mathcal{X}_1(x_1, \mathcal{T}^{cl}(t, x_1)) \\ 0_3 \\ 0_3\end{bmatrix} + \begin{bmatrix} e_1(t, x_2) \\ \mathcal{S}(\omega_1)\, n_1 \\ \mathcal{S}(\omega_2)\, n_2\end{bmatrix}, \tag{40}$$

with e_1 as the error from step 1, defined in (38). Due to the cascaded structure, we can now design control laws for the angular velocities such that the error e_1 is steered to zero. For that purpose, consider the desired cables' direction defined in (34), and define, for some positive k_{V_γ},

$$\mathbb{R}_{\geq 0} \times \mathbb{X}_2 \ni (t, x_2) \mapsto V_2(t, x_2) := V_1(t, x_1) + \sum_{i \in \{1,2\}} k_{V_\gamma}(1 - n_i^T n_i^{cl}(t, x_1)) \geq 0.$$

With the help of (34) and (39), it follows that the derivative of V_2 along the vector field (42) yields

$$W_1(t, x_1) + \partial_2 V_1(t, x_1) e_1(t, x_2) + \sum_{i \in \{1,2\}} k_{V_\gamma} e_{\gamma,i}(t, x_2)^T \left(\omega_i - \omega_{n_i^{cl}}(t, x_2)\right)$$

and thus, if we pick the control law for the cable $i \in \{1, 2\}$ angular velocity as

$$\begin{aligned} &\omega_i^{cl} : (t, \mathbb{X}_2) \ni (t, x_2) \mapsto \omega_i^{cl}(t, x_2) \in \mathbb{R}^3 \\ &\omega_i^{cl}(t, x_2) := \omega_{n_i^{cl}}(t, x_2) - k_\gamma \mathcal{S}\left(n_i^{cl}(t, x_1)\right) n_i - k_{V_\gamma}^{-1} \delta_i(t, x_2), \end{aligned} \tag{41}$$

it follows that

$$\mathbb{R}_{\geq 0} \times \mathbb{X}_2 \ni (t, x_2) \mapsto W_2(t, x_2) := \partial_1 V_2(t, x_2) + \partial_2 V_2(t, x_2) X_2(x_2, (T_1^{cl}(t, x_2), T_2^{cl}(t, x_2), \omega_1^{cl}(t, x_2), \omega_2^{cl}(t, x_2)))$$

simplifies as

$$W_2(t, x_2) = W_1(t, x_1) - \sum_{i \in \{1,2\}} k_{V_\gamma} k_\gamma \|\mathcal{S}\left(n_i^{cl}(t, x_1)\right) n_i)\|^2 \leq 0,$$

i.e. V_2 along the vector field X_2 with the control laws (36) and (41) is non-positive.

C. Step 3

The third and final step follows the same procedure as in Step 2, by exploring the cascaded structure of the problem. In the second step, we assumed we had control over the cables' angular velocities, which is not the case. This introduces a second error (an error e_2, similar to the error e_1 in step 1), and we design control laws for the angular accelerations such that that error is steered to zero.

Indeed, given the chosen controls laws in (36) and (41), it follows that

$$X_3(x_3, (T_1^{cl}(t, x_2), T_2^{cl}(t, x_2), \tau_1, \tau_2)) = \begin{bmatrix} X_2(x_1, (T_1^{cl}(t, x_2), T_2^{cl}(t, x_2), \omega_1^{cl}(t, x_2), \omega_2^{cl}(t, x_2))) \\ 0_3 \\ 0_3 \end{bmatrix} + \begin{bmatrix} e_2(t, x_3) \\ \mathcal{S}\left(n_1\right) \tau_1 \\ \mathcal{S}\left(n_2\right) \tau_2 \end{bmatrix}, \tag{42}$$

where $e_2(t, x_3) := \begin{bmatrix} 0_{12} \\ \mathcal{S}\left(\omega_1 - \omega_1^{cl}(t, x_2)\right) n_1 \\ \mathcal{S}\left(\omega_2 - \omega_2^{cl}(t, x_2)\right) n_2 \end{bmatrix}$,

with e_2 as the error from step 2. Due to the cascaded structure, we can now design control laws for the angular accelerations (τ_1 and τ_2) such that the error e_2 is steered to zero.

Given a positive k_{V_ω}, consider then the Lypuanov function

$$\mathbb{R}_{\geq 0} \times \mathbb{X}_3 \ni (t, x_3) \mapsto V_3(t, x_3) := V_2(t, x_2) + \sum_{i \in \{1,2\}} \frac{k_{V_\omega}}{2} \|\mathcal{S}\left(n_i\right)\left(\omega_i - \omega_i^{cl}(t, x_2)\right)\|^2.$$

Given some positive k_ω, one can then construct a control law for the cables angular accelerations

$$\begin{aligned} &\tau_i^{cl} : \mathbb{R}_{\geq 0} \times \mathbb{X}_3 \ni (t, x_3) \mapsto \tau_i^{cl}(t, x_3) \in T_{n_i}\mathbb{S}^2, i \in \{1, 2\}, \\ &\tau_i^{cl} := \Pi\left(n_i\right)\left(\tau_{\omega_i^{cl}} + (n_i^T \omega_i^{cl}(t, x_2)) \mathcal{S}\left(n_i\right) \omega_i^{cl}(t, x_2)\right) - \frac{k_{V_\gamma}}{k_{V_\omega}} \mathcal{S}\left(n_i\right) n_i^{cl}(t, x_2) - k_\omega \Pi\left(n_i\right)\left(\omega_i - \omega_i^{cl}(t, x_2)\right), \\ &\quad \text{where } \tau_{\omega_i^{cl}} = \partial_1 \omega_i^{cl}(t, x_2) + \partial_2 \omega_i^{cl}(t, x_2) X_2(x_1, (T_1^{cl}(t, x_2), T_2^{cl}(t, x_2), \omega_1, \omega_2)), \end{aligned} \tag{43}$$

such that

$$\begin{aligned} \mathbb{R}_{\geq 0} \times \mathbb{X}_3 \ni (t, x_3) \mapsto W_3(t, x_3) :=& \partial_1 V_3(t, x_3) + \partial_2 V_3(t, x_3) X_3(x_3, (T_1^{cl}(x_2), T_2^{cl}(x_2), \tau_1^{cl}(t, x_3), \tau_2^{cl}(t, x_3))) \\ =& W_2(t, x_2) - \sum_{i \in \{1,2\}} k_{V_\omega} k_\omega \|\mathcal{S}\left(n_i\right)\left(\omega_i - \omega_i^{cl}(t, x_2)\right)\|^2 \leq 0, \end{aligned}$$

i.e., such that V_3 along the closed loop vector field is non-positive.

At this point, the control design is finished, and one can combine (36) and (43) to construct the control law in (11) and therefore the control law in (12).

Theorem 3: Consider the vector field (6), and the control law (12), with double integrator control laws whose bounds satisfy (32). Then $\lim_{t\to\infty}(p(t)-p^\star(t)) = 0_3$ and $\lim_{t\to\infty}(n(t) \pm n^\star(t)) = 0_3$ along a solution of $\dot{z}(t) = Z(z(t), u^{cl}(t, z(t)))$.

Proof: If the double integrator control laws' bounds satisfy (32), then the control law (11) is always well defined, and consequently so is (12). Moreover, given the control law (11), and owing to the backstepping control design, we know that for any solution $t \mapsto x_3(t) \in \mathbb{X}_3$ of $\dot{x}_3(t) = X_3(x_3(t), \nu^{cl}(t, x_3(t)))$ it follows that $\lim_{t\to\infty}(p(t) - p^\star(t)) = 0_3$ and that $\lim_{t\to\infty}(n(t) \pm n^\star(t)) = 0_3$. That same conclusion extends to $t \mapsto z(t) = g_3^{-1}(x_3(t)) \in \mathbb{Z}$ which is a solution of $\dot{z}(t) = Z(z(t), u^{cl}(t, z(t)))$. ■

VI. Attitude Control

Previously, in Section III, we assumed that the quadrotrors were fully actuated, which is not the case in a real aerial vehicle. Consider then the augmented states

$$\bar{z} = (z, r_1, r_2) \in \mathbb{Z} \times \mathbb{S}^2 \times \mathbb{S}^2 =: \bar{\mathbb{Z}}$$
$$x_4 = (x_3, r_1, r_2) \in \mathbb{X}_3 \times \mathbb{S}^2 \times \mathbb{S}^2 =: \mathbb{X}_4$$

where $r_1, r_2 \in \mathbb{S}^2$ stand for the quadrotor's direction where throttle is provided (see Fig. 1); and with z as in (3) and x_3 as in (22). The state $\bar{z} : \mathbb{R}_{\geq 0} \ni t \mapsto \bar{z}(t) \in \bar{\mathbb{Z}}$ evolves according to $\dot{\bar{z}}(t) = \bar{Z}(\bar{z}(t), u(t)), \bar{z}(0) \in \bar{\mathbb{Z}}$, where (denote $u := (U_1, U_2, \omega_{r_2}, \omega_{r_1}) \in \mathbb{R}^{2+6}$)

$$\bar{Z} : \bar{\mathbb{Z}} \times \mathbb{R}^8 \ni (\bar{z}, u) \mapsto \bar{Z}(\bar{z}, u) \in T_{\bar{z}}\bar{\mathbb{Z}}$$
$$\bar{Z}(\bar{z}, u) := \begin{bmatrix} Z(z, (U_1 r_1, U_2 r_2)) \\ \mathcal{S}(\omega_{r_1}) r_1 \\ \mathcal{S}(\omega_{r_2}) r_2 \end{bmatrix} \left(= \begin{bmatrix} \dot{z} \\ \dot{r}_1 \\ \dot{r}_2 \end{bmatrix} \right), \tag{44}$$

with the vector field Z as in (6) (in Section III, we took $u_i = U_i r_i \in \mathbb{R}^3$ as an input, which meant we had immediate control over the vehicles' attitude, namely $r_i \in \mathbb{S}^2$ – see Fig. 1). One must now design control laws for the throttles (U_1 and U_2) and angular velocities (ω_{r_1} and ω_{r_2}) for each quadrotor.

Given the control law (12), one may be tempted to choose $\mathbb{R}_{\geq 0} \times \bar{\mathbb{Z}} \ni (t, \bar{z}) \mapsto U_i^{cl}(t, \bar{z}) := r_i^T u_i^{cl}(t, z) \in \mathbb{R}$, as it minimizes the error to the desired input, i.e., $U_i^{cl}(t, \bar{z}) = \inf_{U_i \in \mathbb{R}} \|U_i r_i - u_i^{cl}(t, z)\|$ [21]. However, if that choice is made, the cascaded structure in Section IV-C is lost.

Notice that given any $u \in \mathbb{R}^3$ and $n, r \in \mathbb{S}^2$ with $n^T r \neq 0$, the equality $\frac{n^T u}{n^T r} r = u + \frac{1}{n^T r}\mathcal{S}(n)\mathcal{S}(r) u$ holds. With that in mind, consider then $\bar{\bar{\mathbb{Z}}} := \{\bar{z} \in \bar{\mathbb{Z}} : r_i^T n_i(z_k) > 0, i \in \{1, 2\}\}$ and the throttle control law

$$\mathbb{R}_{\geq 0} \times \bar{\bar{\mathbb{Z}}} \ni (t, \bar{z}) \mapsto U_i^{cl}(t, \bar{z}) := \frac{n_i(z_k)^T u_i^{cl}(t, z)}{n_i(z_k)^T r_i} \in \mathbb{R}, \tag{45}$$

and thus

$$U_i^{cl}(t, \bar{z}) r_i = u_i^{cl}(t, z) + \frac{\mathcal{S}(n_i(z_k))\mathcal{S}(r_i) u_i^{cl}(t, z)}{r_i^T n_i(z_k)} =: u_i^{cl}(t, z) + e_{4,i}(t, \bar{z}), \tag{46}$$

where the error $e_{4,i}$ is orthogonal to the cable i (i.e., to n_i). Recall from (8) that the tensions depend on $n_1^T u_1$ and $n_2^T u_2$, and thus, it follows from (46) that with the choice in (45) the tensions remain unchanged, i.e., it follows from composing (19) with (46) that

$$R(z, (U_1^{cl}(t, \bar{z}) r_1, U_2^{cl}(t, \bar{z}) r_2)) = R(z, u^{cl}(t, z)) + \begin{bmatrix} 0_{2\times 6} \\ B_{n_{f_1}}(z_k) \\ B_{n_{f_2}}(z_k) \end{bmatrix} \begin{bmatrix} e_{4,1}(t, \bar{z}) \\ e_{4,2}(t, \bar{z}) \end{bmatrix}.$$

Given the mappings $g_4 : \bar{\mathbb{Z}} \ni \bar{z} \mapsto g_4(\bar{z}) := (g_3(z), r_1, r_2) \in \mathbb{X}_4$ and $g_4^{-1} : \mathbb{X}_4 \ni x_4 \mapsto (g_3^{-1}(x_3), r_1, r_2) \in \bar{\mathbb{Z}}$ it follows that

$$Dg_4(\bar{z})\bar{Z}(\bar{z}, (U_1^{cl}(t, \bar{z}), U_2^{cl}(t, \bar{z}), \omega_{r_1}, \omega_{r_2}))|_{\bar{z} = g_4^{-1}(x_4)} = \begin{bmatrix} X_3(x_3, \nu^{cl}(t, x_3)) \\ \\ 0_3 \\ 0_3 \end{bmatrix} + \left[\begin{matrix} \begin{bmatrix} 0_{18\times 6} \\ B_{n_{f_1}}(z_k) \\ B_{n_{f_2}}(z_k) \end{bmatrix} \begin{bmatrix} e_{4,1}(t, \bar{z}) \\ e_{4,2}(t, \bar{z}) \end{bmatrix} \\ \mathcal{S}(\omega_{r_1}) r_1 \\ \mathcal{S}(\omega_{r_2}) r_2 \end{matrix} \right] |_{\bar{z} = g_4^{-1}(x_4)},$$
$$=: X_4(t, x_4, \omega_{r_1}, \omega_{r_2})$$

where the cascaded structure becomes clear, and where we have now to pursue a fourth step, in addition the other three (X_3 is the vector field in (25), and ν^{cl} the control law designed at the end of step 3 in Section V-C). The choice in (45) is what ensures the cascaded structure is preserved, and it is one of the main contributions of the paper. Recall (12), and, for brevity, denote

$$\mathbb{Y}_3 \equiv \{(t, x_3) \in \mathbb{R}_{\geq 0} \times \mathbb{X}_3 : n_i^T r_i \neq 0, u_i^{cl}(t, g_3^{-1}(x_3)) \neq 0_3, i \in \{1, 2\}\},$$

and $\mathbb{Y}_4 \equiv \mathbb{Y}_3 \times \mathbb{S}^2 \times \mathbb{S}^2$. Similarly as in Section V-B, one can define the desired quadrotor $i \in \{1,2\}$ attitude

$$\mathbb{Y}_3 \ni (t, x_3) \mapsto r_i^{cl}(t, x_3) := \frac{u_i^{cl}(t, g_3^{-1}(x_3))}{\|u_i^{cl}(t, g_3^{-1}(x_3))\|} \in \mathbb{S}^2,$$

its angular velocity

$$\begin{aligned}&\omega_{r_i^{cl}} : \mathbb{Y}_4 \ni (t, x_4) \mapsto \omega_{r_i^{cl}}(t, x_4) \in \mathbb{R}^3\\ &\omega_{r_i^{cl}}(t, x_4) := \mathcal{S}\left(r_i^{cl}(t, x_3)\right)\left(\partial_1 r_i^{cl}(t, x_3) + \partial_2 r_i^{cl}(t, x_3)(Dg_3(z)Z(z, (U_1^{cl}(t, \bar{z})r_1, U_2^{cl}(t, \bar{z})r_2)))\right)|_{\bar{z}=g_4^{-1}(x_4)}\end{aligned}$$

and construct a control law (given positive gains k_r and k_{V_r})

$$\begin{aligned}&\omega_{r_i}^{cl} : \mathbb{Y}_4 \ni (t, x_4) \mapsto \omega^{cl}(t, x_4) \in \mathbb{R}^3\\ &\omega_{r_i}^{cl}(t, x_4) := \omega_{r_i^{cl}}(t, x_4) - k_r \mathcal{S}\left(r_i^{cl}(t, x_3)\right) r_i - k_{V_r}^{-1} \mathcal{S}\left(n_i\right)\left(\frac{\|u_i^{cl}(t, z)\|}{r_i^T n_i} P_i \sum_{j \in \{1,2\}} \left(B_{n_{f_j}}(z_k)^T \partial_{\omega_j} V_3(t, x_3)\right)\right)|_{z=g_3^{-1}(x_3)}\end{aligned}$$

where $P_1 = \begin{bmatrix}0_{3\times3} & I_3\end{bmatrix}$ and $P_2 = \begin{bmatrix}I_3 & 0_{3\times3}\end{bmatrix}$,

(notice similarity with (41)); for which it follows that the Lyapunov function

$$\mathbb{Y}_4 \ni (t, x_4) \mapsto V_4(t, x_4) := V_3(t, x_3) + \sum k_{V_r}(1 - r_i^T r_i^{cl}(t, x_3)) > 0$$

has a non-positive derivative along the closed loop vector field, namely

$$\begin{aligned}\mathbb{Y}_4 \ni (t, x_4) \mapsto W_4(t, x_4) &= \partial_1 V_4(t, x_4) + \partial_2 V_4(t, x_4) X_4(t, x_4, \omega_{r_1}^{cl}(t, x_4), \omega_{r_2}^{cl}(t, x_4))\\ &= W_3(t, x_3) - \sum_{i \in \{1,2\}} k_{V_r} k_r \|\mathcal{S}\left(r_i\right) r_i^{cl}(t, x_3)\|^2 \leq 0,\end{aligned}$$

where V_3 and W_3 are those from step 3.

Remark 4: The vector field (44) could now be extended to the case where we have control over the torques on the vehicles, rather than their angular velocities. This would lead to a fifth step, which would be similar to the third step in Section V-C.

Remark 5: The yaw motion of the quadrotor $i \in \{1,2\}$ can be controlled by appropriately designing a component of ω_{r_i} aligned with r_i. However, since the yaw motion of the quadrotor does not affect the motion of the bar, we omit this design step in the paper. See also [3, page 71].

VII. Simulations

Consider the system quadrotors-bar with parameters $m = 0.5$kg, $J = 0.04$kg m^2, $m_1 = 1.2$kg, $m_2 = 1.5$kg, $d_1 = 0.45$m, $d_2 = -0.55$m, $l_1 = 0.5$m and $l_2 = 0.65$m. The desired position trajectory is (see Problem 1)

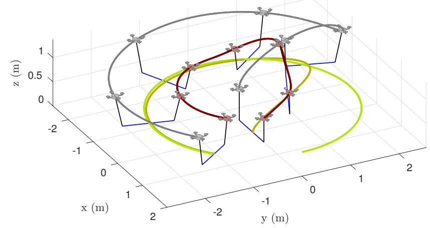
$$\mathbb{R}_{\geq 0} \ni t \mapsto p^\star(t) := r(\cos(\omega t), \sin(\omega t), 0) + (0, 0, 0.5) \in \mathbb{R}^3,$$

with $r = 2$m and $\omega = 2\pi/15\text{s}^{-1}$, and the desired attitude trajectory is

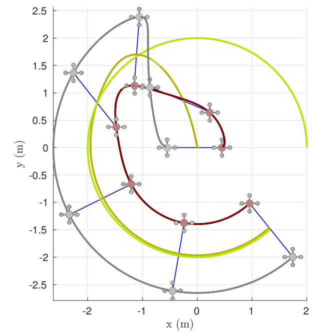
$$\mathbb{R}_{\geq 0} \ni t \mapsto n^\star(t) := \frac{\mathcal{S}\left(e_3\right) p^{\star(1)}(t)}{\|\mathcal{S}\left(e_3\right) p^{\star(1)}(t)\|} \in \mathbb{S}^2.$$

(As such, we want the bar to be perpendicular to the tangent to the path being described). The control law (12) is implemented with $k = 0.25 mgd_1/J$ (see (30)); the other gains are found in the mathematica file, and we only emphasize that the choice of gains and saturations is such that the constraint (32) is satisfied.

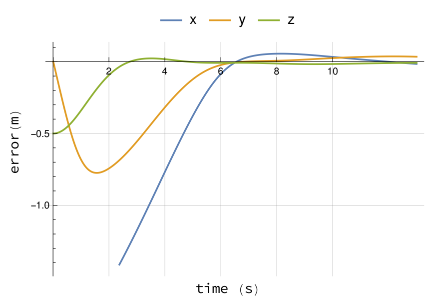
We provide a simulation in Fig. 4, as a solution $t \mapsto \bar{z}(t) \in \bar{\mathbb{Z}}$ of (44) composed with the proposed control law and with $\bar{z}(0) = (0_3, e_1, l_1 e_3, l_2 e_3, 0_{12}, e_3, e_3) \in \bar{Z}$. In Figs. 4(a)-4(b), one can visualize the quadrotors-bar trajectory, and a visual inspection indicates position and attitude tracking. Note that, because $k > 0$, the cables are tilted away from the vertical direction, as illustrated in Fig. 3. In Figs. 4(c) and 4(d), the position and attitude errors are shown, and one verifies indeed that the error position ($t \mapsto p(t) - p^\star(t)$) and error attitude ($t \mapsto \theta(t) := \arccos(n(t)^T n^\star(t))$ are steered to zero. In Fig 4(e), the input computed from (12) is shown, which is used in the control (45), and for computing the desired attitude for the quadrotors; and in Figs 4(f) and 4(g), the UAVs throttle (see (45)) and angular velocity inputs are shown. In Fig. 4(d), one verifies that the quadrotors attitude converges to the desired attitude, i.e., that $t \mapsto \gamma_i(t) := \arccos(r_i(t)^T r_i^\star(t))$ for $i \in \{1,2\}$ is steered to zero (where $r_i^\star$ is the equilibrium attitude). Finally, in Fig. 4(h) one visualizes the tensions in the cables $t \mapsto T_i(z(t), (U_1^{cl}(t, \bar{z}(t)) r_1(t), U_1^{cl}(t, \bar{z}(t)) r_2(t)))$ with U_i^{cl} as in (45) for $i \in \{1,2\}$, and one verifies that those tensions are always positive, which means the cables are always taut. Deriving precise conditions on the initial state that guarantee that the tensions remain positive is a topic of future research.



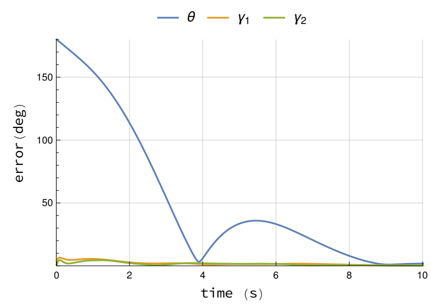
(a) Trajectory



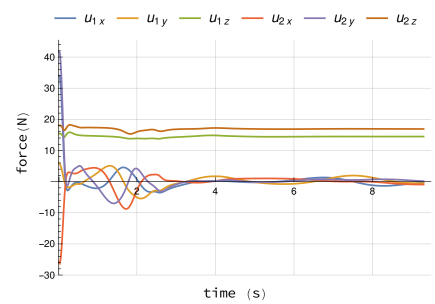
(b) Trajectory (top perspective)



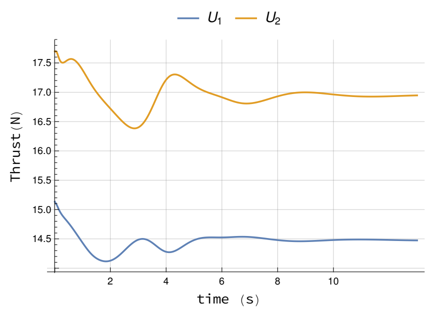
(c) Bar error position



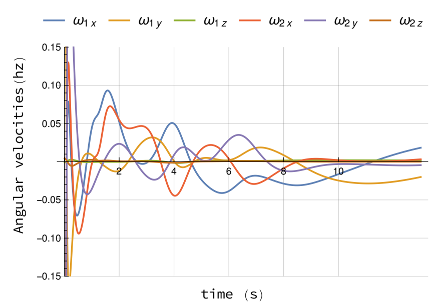
(d) Bar and UAVs attitude error



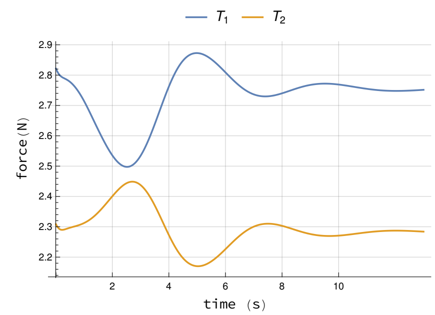
(e) Input as computed from (12)



(f) UAVs input throttles



(g) UAVs input angular velocities



(h) Tensions in cables

Fig. 4: Trajectory for vector field (44).

VIII. Conclusions

We considered a system composed of a bar tethered to two aerial vehicles, and developed a controller for pose tracking of the bar. The controller was designed by first computing an input and a state transformation, which converted the system vector field into one with a cascaded structure, and then exploiting that cascaded structure in a backstepping procedure. We showed that the quadrotors-bar system is an extended VTOL vehicle, in the sense that it requires bounded linear and angular accelerations of the bar in order to guarantee well posedness of the controller. Moreover, a degree of freedom was included in the proposed control law, which allows for regulating the relative position between the aerial vehicles. Finally, the proposed control law for the throttle was chosen so as to preserve the cascaded structure of the problem.

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IX. Bounded control laws

Given $\bar{\sigma} > 0$, let $\sigma(\cdot, \bar{\sigma}) : \mathbb{R}^3 \ni x \mapsto \sigma(x, \bar{\sigma}) := \bar{\sigma}\frac{x}{\sqrt{\bar{\sigma}^2 + x^T x}} \in \mathcal{B}(\bar{\sigma})$. A possible bounded control law for a three dimensional double integrator is given by

$$\begin{aligned} a^{cl}(t, p, v) &:= p^{\star(2)}(t) - k_p \sigma(e_p, \sigma_p) - k_v \sigma(e_v, \sigma_v), \\ e_p &:= p - p^{\star(0)}(t), e_v := v - p^{\star(1)}(t), \end{aligned}$$

and for a unit vector double integrator is given by

$$\begin{aligned} \tau^{cl}(t, p, v) &:= R^\star(t)\left(\tau^{ff} - k_\theta \mathcal{S}\left(e_1\right) e_n - k_\omega \sigma(e_\omega, \sigma_\omega)\right), \\ \tau^{ff} &:= \Pi\left(e_n\right) \dot{\omega}^\star(t) + e_n^T \omega^\star(t) \mathcal{S}\left(e_n\right) \omega^\star(t), \\ e_n &:= R^\star(t) n, e_\omega := \Pi\left(R^\star(t) n\right)\left(R^\star(t)\omega - \omega^\star(t)\right), \end{aligned}$$

where $n^\star(t) := R^\star(t) e_1$. It follows that (denote $\bar{\tau}^\star := \sup_t(\|\Pi\left(n^\star(t)\right)\omega^{\star(1)}(t)\| + 0.5\|\omega^{\star(0)}\|^2)$, it follows that

$$\bar{a} := \sup_{(t,p,v)\in\mathbb{P}} \|a^{cl}(t, p, v)\| = \sup_t \|p^{\star(2)}(t)\| + k_p \sigma_p + k_v \sigma_v, \tag{47}$$

$$\bar{\tau} := \sup_{(t,n,\omega)\in\Theta} \|\tau^{cl}(t, n, \omega)\| = \bar{\tau}^\star + k_\theta + k_\omega \sigma_\omega. \tag{48}$$

We do not provide V_ξ, W_ξ, V_θ and W_θ due to space constraints.