a distributed nonlinear model predictive control scheme for cooperation of multi-robot systems guaranteeing collision avoidance and connectivity maintenance

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Abstract

This paper addresses the problem of cooperation within a multi-robot system composed of second-order nonlinear multi-agent systems that operate in 3D space. Specifically, we propose a Distributed Nonlinear Model Predictive Control scheme in which each agent solves a nonlinear optimization problem using only local information from its neighbors to calculate its own control signal. Additionally, by introducing certain inter-agent distance constraints, we guarantee collision avoidance both between among the agents and between the agents and possible obstacles of the workspace. Connectivity maintenance between agents that are initially connected is also achieved by the proposed controller scheme. Finally, simulation results verify the performance of the proposed control scheme.

Contents

1 Introduction

formation of multi-agent systems, mpc intro etc.

motivation why we need mpc controllers...

In many control problems it is desired to design a stabilizing feedback such that a performance criterion is minimized while satisfying constraints on the controls and the states. Ideally one would look for a closed solution for the feedback law satisfying the constraints while optimizing the performance. However, typically the optimal feedback law cannot be found analytically, even in the unconstrained case, since it involves the solution of the corresponding Hamilton-Jacobi-Bellman partial differential equations. One approach to circumvent this problem is the repeated solution of an open-loop optimal control problem for a given state. The first part of the resulting open-loop input signal is implemented and the whole process is repeated. Control approaches using this strategy are referred to as Model Predictive Control (MPC).

Part I
The problem

2 Notation and Preliminaries

2.1 Notation

The set of positive integers is denoted by $\mathbb{N}$. The real n-coordinate space, with $n \in \mathbb{N}$, is denoted by $\mathbb{R}^n$; $\mathbb{R}^n_{\geq 0}$ and $\mathbb{R}^n_{>0}$ are the sets of real n-vectors with all elements nonnegative and positive, respectively. Given a set S, we denote as $|S|$ its cardinality. The notation $\|\mathbf{x}\|$ is used for the Euclidean norm of a vector $\mathbf{x} \in \mathbb{R}^n$. Given a symmetric matrix $\boldsymbol{A} = \boldsymbol{A}^T, \lambda_{\min}(\boldsymbol{A})$ denotes the minimum eigenvalue of $\boldsymbol{A}$, respectively, where $\sigma(\boldsymbol{A})$ is the set of all the eigenvalues of $\boldsymbol{A}$ and $\text{rank}(\boldsymbol{A})$ is its rank; $\boldsymbol{A} \otimes \boldsymbol{B}$ denotes the Kronecker product of matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m \times n}$, as was introduced in [?]. Define by $\mathbf{1}_n \in \mathbb{R}^n, \boldsymbol{I_n} \in \mathbb{R}^{n \times n}, \boldsymbol{0}_{m \times n} \in \mathbb{R}^{m \times n}$ the column vector with all entries 1, the unit matrix and the $m \times n$ matrix with all entries zeros, respectively. A matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is called skew-symmetric if and only if $\boldsymbol{A}^\top = -\boldsymbol{A}$. $\mathcal{B}(\mathbf{c}, r) \triangleq \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x} - \mathbf{c}\| \leq r\}$ is the 3D sphere of radius $r \in \mathbb{R}_{\geq 0}$ and center $\mathbf{c} \in \mathbb{R}^3$.

The vector expressing the coordinates of the origin of frame $\{j\}$ in frame $\{i\}$ is denoted by $\mathbf{p}_{j \triangleright i}$. When this vector is expressed in 3D space in a third frame, frame $\{k\}$, it is denoted by $\mathbf{p}^k_{j \triangleright i}$. The angular velocity of frame $\{j\}$ with respect to frame $\{i\}$, expressed in frame $\{k\}$ coordinates, is denoted by $\boldsymbol{\omega}^k_{j \triangleright i} \in \mathbb{R}^3$. We also use the notation $\mathbb{M} = \mathbb{R}^3 \times \mathbb{T}^3$. We further denote as $\mathbf{q}_{j \triangleright i} \in \mathbb{T}^3$ the Euler angles representing the orientation of frame $\{j\}$ with respect to frame $\{i\}$, where $\mathbb{T}^3$ is the 3D torus. For notational brevity, when a coordinate frame corresponds to the inertial frame of reference $\{\mathcal{O}\}$, we will omit its explicit notation (e.g., $\mathbf{p}_i = \mathbf{p}_{i \triangleright \mathcal{O}} = \mathbf{p}^{\mathcal{O}}_{i \triangleright \mathcal{O}}$, $\boldsymbol{\omega}_i = \boldsymbol{\omega}_{i \triangleright \mathcal{O}} = \boldsymbol{\omega}^{\mathcal{O}}_{i \triangleright \mathcal{O}}$). All vector and matrix differentiations are derived with respect to the inertial frame $\{\mathcal{O}\}$ unless stated otherwise.

2.2 Graph Theory

?? ISTORISOU // EXPAND An *undirected graph* $\mathcal{G}$ is a pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a finite set of nodes, representing a team of agents, and $\mathcal{E} \subseteq \big\{\{i,j\} : i,j \in \mathcal{V}, i \neq j\big\}$, with $M = |\mathcal{E}|$, is the set of edges that model the communication capability between neighboring agents. For each agent, its neighbors' set $\mathcal{N}_i$ is defined as $\mathcal{N}_i = \{i_1, \ldots, i_{N_i}\} = \big\{j \in \mathcal{V} : \{i,j\} \in \mathcal{E}\big\}$, where $i_1, \ldots, i_{N_i}$ is an enumeration of the neighbors of agent i and $N_i = |\mathcal{N}_i|$.

If there is an edge $\{i,j\} \in \mathcal{E}$, then i, j are called *adjacent*. A *path* of length r from vertex i to vertex j is a sequence of $r+1$ distinct vertices, starting with i and ending with j, such that consecutive vertices are adjacent. For $i = j$, the path is called a *cycle*. If there is a path between any two vertices of the graph $\mathcal{G}$, then $\mathcal{G}$ is called *connected*. A connected graph is called a *tree* if it contains no cycles.

2.3 Non-linear Model Predictive Control for continuous-time systems

3 Problem Formulation

3.1 System Model

Consider a set of N rigid bodies, with $\mathcal{V} = \{1, 2, \ldots, N\}$, $N \geq 2$, operating in a workspace $W \subseteq \mathbb{R}^3$. A coordinate frame $\{i\}, i \in \mathcal{V}$ is attached to each body's center of mass. The workspace is assumed to be modeled as a bounded sphere $\mathcal{B}(\mathbf{p}_W, r_W)$ expressed in an inertial frame $\{\mathcal{O}\}$.

We consider that over time t each agent i occupies the space of a sphere $\mathcal{B}(\mathbf{p}_i(t), r_i)$, where $\mathbf{p}_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^3$ is the position of the agent's center of mass, and $r_i < r_W$ is the radius of the agent's body. We denote $\mathbf{q}_i(t) : \mathbb{R}_{\geq 0} \to \mathbb{T}^3, i \in \mathcal{V}$, the Euler angles representing the agents' orientation with respect to the inertial frame $\{\mathcal{O}\}$, with $\mathbf{q}_i \triangleq [\phi_i, \theta_i, \psi_i]^\top$. We define

$$\mathbf{x}_i \triangleq [\mathbf{p}_i^\top, \mathbf{q}_i^\top]^\top, \mathbf{x}_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^3 \times \mathbb{T}^3 \equiv \mathbb{M}$$
$$\mathbf{v}_i \triangleq [\dot{\mathbf{p}}_i^\top, \boldsymbol{\omega}_i^\top]^\top, \mathbf{v}_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^3 \times \mathbb{R}^3 \equiv \mathbb{R}^6$$

and model the motion of agent i under second order dynamics:

$$\dot{\mathbf{x}}_i(t) = \boldsymbol{J}_i^{-1}(\mathbf{x}_i)\mathbf{v}_i(t), \tag{1a}$$
$$\mathbf{u}_i = \boldsymbol{M}_i(\mathbf{x}_i)\dot{\mathbf{v}}_i(t) + \boldsymbol{C}_i(\mathbf{x}_i, \dot{\mathbf{x}}_i)\mathbf{v}_i(t) + \mathbf{g}_i(\mathbf{x}_i), \tag{1b}$$

In equation (1a), $\boldsymbol{J}_i : \mathbb{T}^3 \to \mathbb{R}^{6\times 6}$ is a Jacobian matrix that maps the non-orthogonal Euler angle rates to the orthogonal angular velocities $\mathbf{v}_i$:

$$\boldsymbol{J}_i(\mathbf{x}_i) = \begin{bmatrix} \boldsymbol{I}_3 & \boldsymbol{0}_{3\times 3} \\ \boldsymbol{0}_{3\times 3} & \boldsymbol{J}_q(\mathbf{x}_i) \end{bmatrix}, \text{ with } \boldsymbol{J}_q(\mathbf{x}_i) = \begin{bmatrix} 1 & 0 & -\sin\theta_i \\ 0 & \cos\phi_i & \cos\theta_i \sin\phi_i \\ 0 & -\sin\phi_i & \cos\phi_i \cos\theta_i \end{bmatrix}$$

The matrix $\boldsymbol{J}_i$ is singular when $det(\boldsymbol{J}_i) = \cos\theta_i = 0 \Leftrightarrow \theta_i = \pm\frac{\pi}{2}$. The control scheme proposed in this thesis guarantees that this is always avoided, and hence equation (1a) is well defined. This gives rise to the following remark:

Remark 3.1. $det(\boldsymbol{J}_i) = \cos\theta_i \leq 1$, $\forall i \in \mathcal{V}$

In equation (1b), $\boldsymbol{M}_i : \mathbb{M} \to \mathbb{R}^{6\times 6}$ is the symmetric and positive definite *inertia matrix*, $\boldsymbol{C}_i : \mathbb{M} \times \mathbb{R}^6 \to \mathbb{R}^{6\times 6}$ is the *Coriolis matrix* and $\mathbf{g}_i : \mathbb{M} \to \mathbb{R}^6$ is the *gravity vector*. Finally, $\mathbf{u}_i \in \mathbb{R}^6$ is the control input vector representing the 6D generalized *actuation force* acting on the agent.

Remark 3.2. According to [?], the matrices $\dot{\boldsymbol{M}}_i - 2\boldsymbol{C}_i, i \in \mathcal{V}$ are skew-symmetric. The quadratic form of a skew-symmetric matrix is always equal to 0 [?], hence:

$$\mathbf{y}^\top \left[\dot{\boldsymbol{M}}_i - 2\boldsymbol{C}_i\right] \mathbf{y} = 0, \forall \mathbf{y} \in \mathbb{R}^n, i \in \mathcal{V}. \tag{2}$$

However, access to measurements of, or knowledge about these matrices and vectors was not considered up until now. At this point we make the following assumption:

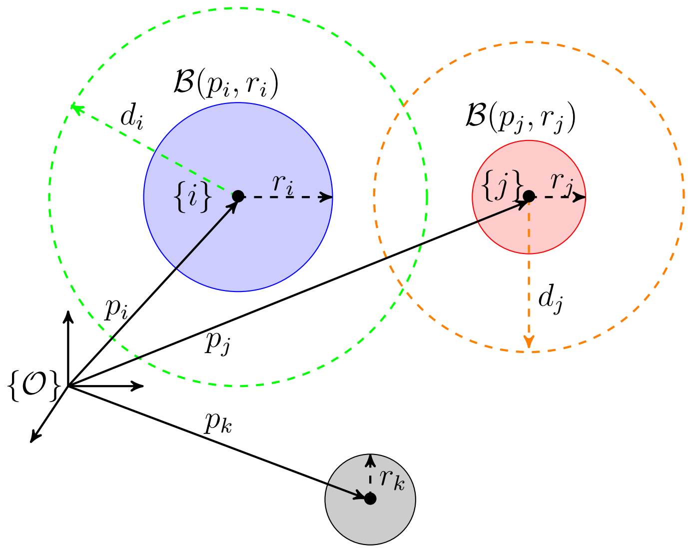

Figure 1: Illustration of two agents $i, j \in \mathcal{V}$ and an static obstacle $\ell \in \mathcal{L}$ in the workspace; $\{\mathcal{O}\}$ is the inertial frame, $\{i\}, \{j\}$ are the frames attached to the agents' center of mass, $\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_\ell \in \mathbb{R}^3$ are the positions of the center of mass of the agents i, j and the obstacle ℓ respectively, expressed in frame $\{\mathcal{O}\}$. r_i, r_j, r_ℓ are the radii of the agents i, j and the obstacle ℓ respectively. d_i, d_j with $d_i > d_j$ are the agents' sensing ranges. In this figure, agents i and j are not neighbours, since the center of mass of agent j is not within the sensing range of agent i and vice versa: $\mathbf{p}_j \notin \mathcal{B}(\mathbf{p}_i(t), d_i)$ and $\mathbf{p}_i \notin \mathcal{B}(\mathbf{p}_j(t), d_j)$.

Assumption 3.1. (Measurements and Access to Information Assumption From an Inter-agent Perspective)

1. Agent i has access to measurements $\mathbf{p}_i, \mathbf{q}_i, \dot{\mathbf{p}}_i, \boldsymbol{\omega}_i, i \in \mathcal{V}$, that is, vectors $\mathbf{x}_i, \mathbf{v}_i$ pertaining to himself,
2. Agent i has a (upper-bounded) sensing range d_i such that
$$d_i > \max\{r_i + r_j : \forall i, j \in \mathcal{V}, i \neq j\}$$
3. the inertia $\boldsymbol{M}$ and Coriolis $\boldsymbol{C}$ vector fields are bounded and unknown
4. the gravity vectors $\mathbf{g}$ are bounded and known

The consequence of points 1 and 2 is that, by defining the set of agents j that are within the sensing range of agent i at time t as

$$\mathcal{R}_i(t) \triangleq \{j \in \mathcal{V} : \mathbf{p}_j(t) \in \mathcal{B}(\mathbf{p}_i(t), d_i)\}$$

or equivalently

$$\mathcal{R}_i(t) \triangleq \{j \in \mathcal{V} : \|\mathbf{p}_i(t) - \mathbf{p}_j(t)\| \leq d_i\}$$

agent i also knows at each time instant t all

$$\mathbf{p}_{j \triangleright i}(t), \mathbf{q}_{j \triangleright i}(t), \dot{\mathbf{p}}_{j \triangleright i}(t), \boldsymbol{\omega}_{j \triangleright i}(t)$$

Therefore, agent i assumes access to all measurements

$$\mathbf{p}_j(t), \mathbf{q}_j(t), \dot{\mathbf{p}}_j(t), \boldsymbol{\omega}_j(t), \forall j \in \mathcal{R}_i(t), t \in \mathbb{R}_{\geq 0}$$

of agent j by virtue of being able to calculate them using knowledge of its own $\mathbf{p}_i(t)$, $\mathbf{q}_i(t)$, $\dot{\mathbf{p}}_i(t)$, $\boldsymbol{\omega}_i(t)$.

In the workspace there are $|\mathcal{L}|$ *static obstacles*, modeled as spheres with centers at positions $\mathbf{p}_\ell$ with radii $r_\ell \in \mathbb{R}^3, \ell \in \mathcal{L} = \{1, \dots, |\ell|\}$. Thus, the obstacles are modeled by the spheres $\mathcal{B}(\mathbf{p}_\ell, r_\ell), \ell \in \{1, \dots, |\ell|\}$. The geometry in workspace W of two agents i and j as well as an obstacle ℓ is depicted in Fig. 1.

Let us also define the distance between any two agents i, j as $d_{ij,a} : \mathbb{R}^6 \to \mathbb{R}_{\geq 0}$, and that between agent i and obstacle ℓ as $d_{i\ell,o} : \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$:

$$d_{ij,a}(t) \triangleq \|\mathbf{p}_i(t) - \mathbf{p}_j(t)\|, \tag{3a}$$

$$d_{i\ell,o}(t) \triangleq \|\mathbf{p}_i(t) - \mathbf{p}_\ell(t)\| \tag{3b}$$

$\forall i, j \in \mathcal{V}, i \neq j, \ell \in \mathcal{L}$, as well as constants $\underline{d}_{ij,a} \triangleq r_i + r_j, \underline{d}_{i\ell,o} \triangleq r_i + r_{o_\ell}$. The latter stand for the minimum distance between two agents, and the minimum distance between an agent and an obstacle. These arise spatially as physical limitations and will be utilized as collision-avoidance constraints.

Agents $j \in \mathcal{N}_i$ are defined as the *neighbours* of agent i. They are the set of indices of agents $j \in \mathcal{V}$ which

1. are within the sensing range of agent i at time $t = 0$, i.e. $j \in \mathcal{R}_i(0)$, and

2. are intended to be kept within the sensing range of agent i at all times $t \in \mathbb{R}_{>0}$

Therefore, while the composition of the set $\mathcal{R}_i(t)$ evolves and varies through time in general, the set $\mathcal{N}_i$ should remain constant through time.

Definition (**??**) implies that the graph is undirected, i.e.,

$$d_{ij,a}(0) < d_i, \forall i \in \mathcal{V}, j \in \mathcal{N}_i \tag{4}$$

$$d_{ji,a}(0) < d_j, \forall j \in \mathcal{V}, i \in \mathcal{N}_j \tag{5}$$

It is also assumed that at $t = 0$ the neighboring agents are in a *collision-free configuration*, i.e.

$$\underline{d}_{ij,a} < d_{ij,a}(0), \forall i, j \in \mathcal{V}, i \neq j \tag{6}$$

Furthermore, it is assumed that, initially, the Jacobians $\boldsymbol{J}_i$ are well-defined $\forall i \in \mathcal{V}$. These four assumptions, which concern the initial conditions of the problem, are summarized in assumption 3.2:

Assumption 3.2. (Initial Conditions Assumption)

At time $t = 0$

1. the communication graph $\mathcal{G}$ is connected,
2. all agents are in a collision-free configuration:

$$\underline{d}_{ij,a} < d_{ij,a}(0) < d_i$$

3. all agents are in a singularity-free configuration:

$$\theta_i(0) \neq \pm\frac{\pi}{2}$$

$\forall i \in \mathcal{V}$ and all agents j within the sensing range of i.

As for $t \geq 0$, agent i is in a collision-and-singularity-free configuration, and is connected to the neighbours it had at time $t = 0$, when

$$\sum_{j \in \mathcal{N}_i} \|\mathbf{x}_i(t) - \mathbf{x}_j(t) - \mathbf{x}_{ji,des}\| < \mu_i, \forall i \in \mathcal{V}$$

where μ_i is an arbitrarily small positive constant. Intuitively, this means that when the agents reach very close to their desired formation, they will be sufficiently away from collisions, connectivity losses and singular configurations. In particular, this concerns collisions between an agent and an obstacle or the workspace boundary (inter-agent collisions and connectivity losses are avoided by choosing feasible formation displacements $p_{ij,des}$), and singular configurations. In other words, the multi-agent formation should be performed sufficiently away from obstacles and the workspace boundary.

The desired steady-state displacements $\mathbf{x}_{ij,des} = [\mathbf{p}_{ij,des}^\top, \mathbf{q}_{ij,des}^\top]^\top$ are themselves *feasible* if $\forall i \in \mathcal{V}$, $j \in \mathcal{N}_i$,

$$\bigcap \{(\mathbf{x}_i, \mathbf{x}_j) \in W^{N_i+1} : \|\mathbf{x}_i - \mathbf{x}_j - \mathbf{x}_{ij,des}\| = 0\} \neq \varnothing$$

At this point, let us define d_o

$$d_o \triangleq \min\{\|\mathbf{p}_\ell - \mathbf{p}_{\ell'}\| : \ell, \ell' \in \mathcal{L}, \ell \neq \ell'\},$$

as the distance between the two least distant obstacles in the workspace, $d_{o,W}$

$$d_{o,W} \triangleq \min\{r_W - (r_\ell + \|\mathbf{p}_\ell\|) : \ell \in \mathcal{K}\},$$

as the distance between the least distant obstacle from the boundary of the workspace and the boundary itself, D

$$D \triangleq \min\{d_o, d_{o,W}\}$$

as the least of these two distances, and Δ

$$\begin{aligned}\Delta \triangleq \max \Big\{ & d_{ij,a} + r_i + r_j : i, j \in \mathcal{V}, i \neq j, \\ & \|\mathbf{p}_k - \mathbf{p}_\ell\| = \mathbf{p}_{k\ell,\mathrm{des}}, \\ & \|\mathbf{q}_k - \mathbf{q}_\ell\| = \mathbf{q}_{k\ell,\mathrm{des}}, \\ & k \in \mathcal{V}, \ell \in \mathcal{N}_k \Big\}\end{aligned}$$

as the *diameter of formation*. Δ is the distance between the two most distant agents when formation is achieved. Given these notions, we can state an assumption on the feasibility of a solution to the problem that this thesis addresses:

Assumption 3.3. (After-formation Geometric Assumption)

- When the multi-agent system reaches the desired formation, it should be able to pass between two of the obstacles and between an obstacle and the boundary of the workspace. Thus, it is required that $D > \Delta$.
- As a consequence, at the very least, all agents should be able to pass between any two obstacles and between all obstacles and the boundary of the workspace, without, simultaneously, any of them colliding with each other or with the obstacles or the boundary of the workspace. Thus, it is required that $D > \sum_{i \in \mathcal{V}} 2r_i$.

These geometric assumptions can be summarized in the following inequality:

$$D > \max\left\{\Delta, \sum_{i \in \mathcal{V}} 2r_i\right\} \tag{7}$$

3.2 Problem Statement

Due to the fact that the agents are not dimensionless and their communication capabilities are limited, the control protocol should for all neighboring agents $i \in \mathcal{V}, j \in \mathcal{N}_i$ make sure that:

1. the desired position formation $\mathbf{p}_{ij,\text{des}}$ is achieved in finite time
2. the desired formation angles $\mathbf{q}_{ij,\text{des}}$ are achieved in finite time
3. connectivity between initially connected agents (neighbours) is maintained at all times, i.e. all edges of $\mathcal{G}(t = 0)$ are maintained

Furthermore, for all agents $i \in \mathcal{V}$, obstacles $\ell \in \mathcal{L}$ and the workspace boundary W, it should guarantee for all $t \in \mathbb{R}_{\geq 0}$ that:

1. all agents avoid collision with each other
2. all agents avoid collision with all obstacles
3. all agents avoid collision with the workspace boundary
4. singularity of the Jacobian matrices $\boldsymbol{J}_i$ is avoided

Therefore, all neighboring agents of agent i must remain within a distance less than d_i from him, for all $i \in \mathcal{V}$, and all agents $i, j \in \mathcal{V}, i \neq j$ must remain within distance greater than $\underline{d}_{ij,a}$ with one another.

Formally, the control problem under the aforementioned constraints is formulated as follows:

Problem 3.1. Consider N agents modeled as bounded spheres $\mathcal{B}(\mathbf{p}_i, r_i)$, $i \in \mathcal{V}, |V| = N$ that operate in workspace W that is also modeled as a bounded sphere $\mathcal{B}(\mathbf{p}_W, r_W)$, featuring $|\mathcal{L}|$ spherical obstacles, also modeled as bounded spheres $\mathcal{B}(p_\ell, r_\ell), \ell \in \mathcal{L}$. Each agent i is governed by the dynamics (9), under assumptions 3.1, 3.2, 3.3. Given desired *feasible* inter-agent displacements $\mathbf{p}_{ij,des}$, $\mathbf{q}_{ij,des}$, $\forall i \in \mathcal{V}, j \in \mathcal{N}_i$ such that

$$\underline{d}_{ij,a} < d_{ij,a} < d_i, \forall (i,j) \in \{(i,j) \in \mathcal{V} \times \mathcal{V} : \|\mathbf{p}_i - \mathbf{p}_j - \mathbf{p}_{ij,des}\| = 0\}$$

design decentralized control laws $\mathbf{u}_i \in \mathbb{R}^6$ such that $\forall i \in \mathcal{V}$ and $t \in \mathbb{R}_{\geq 0}$ the following hold:

1. Position and orientation formation is achieved in steady-state

$$\lim_{t \to \infty} \|\mathbf{x}_i(t) - \mathbf{x}_j(t) - \mathbf{x}_{ij,des}(t)\| < \mu_i, \forall j \in \mathcal{N}_i$$

2. Inter-agent collision is avoided

$$\|\mathbf{p}_i(t) - \mathbf{p}_j(t)\| > \underline{d}_{ij,a}, \forall j \in \mathcal{V} \backslash \{i\}$$

3. Agent-with-obstacle collision is avoided

$$\|\mathbf{p}_i(t) - \mathbf{p}_\ell(t)\| > \underline{d}_{i\ell,o}, \forall \ell \in \mathcal{L}$$

4. Agent-with-workspace-boundary collision is avoided

$$\|\mathbf{p}_i(t)\| + r_i < r_W$$

5. Inter-agent connectivity loss is avoided

$$\|\mathbf{p}_i(t) - \mathbf{p}_j(t)\| < d_i, \forall j \in \mathcal{N}_i$$

6. All maps $\boldsymbol{J}_i$ are well defined

$$\theta_i(t) \neq \pm \frac{\pi}{2}$$

Part II
Advocated Solutions

4 Formalizing the model

We begin by rewriting the system equations (1a), (1b) for a generic agent i in state-space form:

$$\dot{\mathbf{x}}_i(t) = \boldsymbol{J}_i^{-1}(\mathbf{x}_i)\mathbf{v}_i(t) \tag{8}$$

$$\dot{\mathbf{v}}_i(t) = -\boldsymbol{M}_i^{-1}(\mathbf{x}_i)\boldsymbol{C}_i(\mathbf{x}_i, \dot{\mathbf{x}}_i)\mathbf{v}_i(t) - \boldsymbol{M}_i^{-1}(\mathbf{x}_i)\mathbf{g}_i(\mathbf{x}_i) + \boldsymbol{M}_i^{-1}(\mathbf{x}_i)\mathbf{u}_i \tag{9}$$

Denoting $\mathbf{z}_i(t)$ by

$$\mathbf{z}_i(t) = \begin{bmatrix} \mathbf{x}_i(t) \\ \mathbf{v}_i(t) \end{bmatrix} \tag{10}$$

and $\dot{\mathbf{x}}_i(t)$ and $\dot{\mathbf{v}}_i(t)$ by

$$\dot{\mathbf{x}}_i(t) = f_{i,x}(\mathbf{z}_i, \mathbf{u}_i) \tag{11}$$

$$\dot{\mathbf{v}}_i(t) = f_{i,v}(\mathbf{z}_i, \mathbf{u}_i) \tag{12}$$

we get the compact representation of the system's model

$$\dot{\mathbf{z}}_i(t) = \begin{bmatrix} f_{i,x}(\mathbf{z}_i, \mathbf{u}_i) \\ f_{i,v}(\mathbf{z}_i, \mathbf{u}_i) \end{bmatrix} = f_i\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big) \tag{13}$$

The state evolution of agent i is modeled by a system of non-linear continuous-time differential equations of the form

$$\dot{\mathbf{z}}_i(t) = f_i\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big) \tag{14}$$

$$\mathbf{z}_i(0) = \mathbf{z}_{i,0} \tag{15}$$

$$\mathbf{z}_i(t) \in \mathcal{Z}_i \subset \mathbb{R}^9 \times \mathbb{T}^3 \tag{16}$$

$$\mathbf{u}_i(t) \in \mathcal{U}_i \subset \mathbb{R}^6 \tag{17}$$

where state $\mathbf{z}_i$ is directly measurable, and sets $\mathcal{Z}_i$, $\mathcal{U}_i$ are compact and contain the origin. Equation 14 does not consider model-plant mismatches or external disturbances. The applied input $\mathbf{u}_i$ is a portion of the optimal solution to an optimization problem where information on the states of the neighbouring agents of agent i are taken into account, either in the cost function (in the displacement-based formation approach), or only in the constraints considered in the optimization problem (in the position-based approach). These constraints pertain to the set of its neighbours $\mathcal{N}_i$ and, in total, to the set of all agents within its sensing range $\mathcal{R}_i$.

Specifically, at time t, agent i has access to[1]

1. measurements of the states of
   - all agents within its sensing range at time t
   - its neighbouring agents at time t
2. the last applied inputs to

[1] Although $\mathcal{N}_i \subseteq \mathcal{R}_i$, we make the distinction between the two because all agents $j \in \mathcal{R}_i$ need to avoid collision with agent i, but only agents $j' \in \mathcal{N}_i$ need to remain within the sensing range of agent i.

- all agents within its sensing range
- its neighbouring agents

We assume that these pieces of information are (a) always available and accurate, and (b) exchanged without delay. We encapsulate these pieces of information in four stacked vectors:

$$\mathbf{z}_{\mathcal{R}_i}(t) \triangleq col[\mathbf{z}_j(t)], \forall j \in \mathcal{R}_i(t) \tag{18}$$

$$\mathbf{z}_{\mathcal{N}_i}(t) \triangleq col[\mathbf{z}_j(t)], \forall j \in \mathcal{N}_i \tag{19}$$

$$\mathbf{u}_{\mathcal{R}_i}(t) \triangleq col[\mathbf{u}_j(t)], \forall j \in \mathcal{R}_i(t) \tag{20}$$

$$\mathbf{u}_{\mathcal{N}_i}(t) \triangleq col[\mathbf{u}_j(t)], \forall j \in \mathcal{N}_i \tag{21}$$

5 Position-based formation

Here we are interested in steering each agent $i \in \mathcal{V}$ into resting at a *position* in 3D space, while conforming to the requirements of the problem; that is, all agents should avoid colliding with each other, all obstacles in the workspace, and the workspace boundary itself, while remaining in a non-singular configuration and sustaining the connectivity to their respective neighbours.

5.1 The error model

A desired configuration $\mathbf{z}_{i,des} \in \mathbb{R}^9 \times \mathbb{T}^3$ is associated to each agent $i \in \mathcal{V}$, with the aim of agent i achieving it in steady-state: $\lim\limits_{t\to\infty} \|\mathbf{z}_i(t) - \mathbf{z}_{i,des}\| = 0$. The interior of the norm of this expression denotes the state error of agent i:

$$\mathbf{e}_i(t) = \mathbf{z}_i(t) - \mathbf{z}_{i,des}, \mathbf{e}_i(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^9 \times \mathbb{T}^3$$

The error dynamics are denoted by $g_i(\mathbf{e}_i, \mathbf{u}_i)$:

$$\dot{\mathbf{e}}_i(t) = \dot{\mathbf{z}}_i(t) - \dot{\mathbf{z}}_{i,des} = \dot{\mathbf{z}}_i(t) = f_i\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big) = g_i\big(\mathbf{e}_i(t), \mathbf{u}_i(t)\big) \tag{22}$$

with $\mathbf{e}_i(0) = \mathbf{z}_i(0) - \mathbf{z}_{i,des}$

5.2 The optimization problem

At a generic time t_0, agent i solves the following optimization problem:

$$\min_{\overline{\mathbf{u}}_i(\cdot)} J_i\big(\overline{\mathbf{u}}_i(\cdot);\ \mathbf{e}_i(t_0)\big) \triangleq \int_{t_0}^{t_0+T_p} F_i\big(\overline{\mathbf{e}}_i(\tau), \overline{\mathbf{u}}_i(\tau)\big)d\tau + V_i\big(\overline{\mathbf{e}}_i(t_0+T_p)\big) \tag{23}$$

subject to:

$$\dot{\overline{\mathbf{e}}}_i(t) = g_i\big(\overline{\mathbf{e}}_i(t), \overline{\mathbf{u}}_i(t)\big) \tag{24}$$

$$\overline{\mathbf{e}}_i(t_0) = \mathbf{e}_i(t_0) \tag{25}$$

$$\overline{\mathbf{u}}_i(t) \in \mathcal{U}_i, t \in [t_0, t_0+T_p) \tag{26}$$

$$\overline{\mathbf{e}}_i(t) \in \mathcal{E}_i,\ t \in [t_0, t_0+T_p] \tag{27}$$

$$\overline{\mathbf{e}}_i(t_0+T_p) \in \mathcal{E}_i^f \tag{28}$$

and $\forall t \in [t_0, t_0+T_p]$:

$$\|\overline{\mathbf{p}}_i(t) - \overline{\mathbf{p}}_j(t)\| > \underline{d}_{ij,a}, \forall j \in \mathcal{R}_i(t) \tag{29}$$

$$\|\overline{\mathbf{p}}_i(t) - \mathbf{p}_\ell(t)\| > \underline{d}_{i\ell,o}, \forall \ell \in \mathcal{L} \tag{30}$$

$$\|\overline{\mathbf{p}}_i(t)\| + r_i < r_W \tag{31}$$

$$\|\overline{\mathbf{p}}_i(t) - \overline{\mathbf{p}}_j(t)\| < d_i, \forall j \in \mathcal{N}_i \tag{32}$$

$$\overline{\theta}_i(t) \neq \pm\frac{\pi}{2} \tag{33}$$

Constraints 29-33 explicitly address the requirements posed by the problem (3.1), while the rest exist to ensure that formation is achieved under constrained states and input signals.

The functions $F_i : \mathcal{E}_i \times \mathcal{U}_i \rightarrow \mathbb{R}_{\geq 0}$ and $V_i : \mathcal{E}_i^f \rightarrow \mathbb{R}_{\geq 0}$ are defined as

$$F_i(\mathbf{e}_i, \mathbf{u}_i) \triangleq \|\mathbf{e}_i\|^2_{\boldsymbol{Q}_i} + \|\mathbf{u}_i\|^2_{\boldsymbol{R}_i} \tag{34}$$

$$V_i(\mathbf{e}_i) \triangleq \|\mathbf{e}_i\|^2_{\boldsymbol{P}_i} \tag{35}$$

Matrices $\boldsymbol{R}_i \in \mathbb{R}^{6\times 6}$ are symmetric and positive definite, while matrices $\boldsymbol{Q}_i, \boldsymbol{P}_i \in \mathbb{R}^{12\times 12}$ are symmetric and positive semi-definite.

The set $\mathcal{E}_i$ is such that

$$\mathcal{E}_i = \{\mathbf{e}_i \in \mathbb{R}^{12} : \mathbf{e}_i \in \mathcal{Z}_i \oplus (-z_{i,des})\}$$

The terminal set $\mathcal{E}_i^f \subseteq \mathcal{E}_i$ is an admissible positively invariant set ?? define it for system (22) such that

$$\mathcal{E}_i^f = \{\mathbf{e}_i \in \mathcal{E}_i : \|\mathbf{e}_i\| \leq \epsilon_0\} \tag{36}$$

where ϵ_0 is an arbitrarily small but fixed positive real scalar.

With regard to the terminal penalty function V_i, the following lemma will prove to be useful in guaranteeing the convergence of the solution to the optimal control problem to the terminal region $\mathcal{E}_i^f$:

Lemma 5.1. (V_i is Lipschitz continuous in $\mathcal{E}_i^f$)
The terminal penalty function V_i is Lipschitz continuous in $\mathcal{E}_i^f$

$$|V(\mathbf{e}_{1,i}) - V(\mathbf{e}_{2,i})| \leq L_{V_i}\|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\|$$

where $\mathbf{e}_{1,i}, \mathbf{e}_{2,i} \in \mathcal{E}_i^f$, with Lipschitz constant $L_{V_i} = 2\epsilon_0\lambda_{max}(P_i)$

Proof For every $\mathbf{e}_i \in \mathcal{E}_i^f$, it holds that

$$\begin{aligned}
|V(\mathbf{e}_{1,i}) - V(\mathbf{e}_{2,i})| &= |\mathbf{e}_{1,i}^\top \boldsymbol{P}_i \mathbf{e}_{1,i} - \mathbf{e}_{2,i}^\top \boldsymbol{P}_i \mathbf{e}_{2,i}| && (37)\\
&= |\mathbf{e}_{1,i}^\top \boldsymbol{P}_i \mathbf{e}_{1,i} - \mathbf{e}_{2,i}^\top \boldsymbol{P}_i \mathbf{e}_{2,i} \pm \mathbf{e}_{1,i}^\top \boldsymbol{P}_i \mathbf{e}_{2,i}| && (38)\\
&= |\mathbf{e}_{1,i}^\top \boldsymbol{P}_i (\mathbf{e}_{1,i} - \mathbf{e}_{2,i}) - \mathbf{e}_{2,i}^\top \boldsymbol{P}_i (\mathbf{e}_{1,i} - \mathbf{e}_{2,i})| && (39)\\
&\leq |\mathbf{e}_{1,i}^\top \boldsymbol{P}_i (\mathbf{e}_{1,i} - \mathbf{e}_{2,i})| + |\mathbf{e}_{2,i}^\top \boldsymbol{P}_i (\mathbf{e}_{1,i} - \mathbf{e}_{2,i})| && (40)
\end{aligned}$$

But for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$|\mathbf{x}^\top \boldsymbol{A} \mathbf{y}| \leq \lambda_{max}(A)\|\mathbf{x}\|\|\mathbf{y}\|$$

where $\lambda_{max}(A)$ denotes the largest eigenvalue of matrix $\boldsymbol{A}$. Hence:

$$\begin{aligned}
|V(\mathbf{e}_{1,i}) - V(\mathbf{e}_{2,i})| &\leq \lambda_{max}(\boldsymbol{P}_i)\|\mathbf{e}_{1,i}\|\|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\| + \lambda_{max}(\boldsymbol{P}_i)\|\mathbf{e}_{2,i}\|\|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\| && (41)\\
&= \lambda_{max}(\boldsymbol{P}_i)(\|\mathbf{e}_{1,i}\| + \|\mathbf{e}_{2,i}\|)\|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\| && (42)\\
&\leq \lambda_{max}(\boldsymbol{P}_i)(\epsilon_0 + \epsilon_0)\|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\| && (43)\\
&= 2\epsilon_0\lambda_{max}(\boldsymbol{P}_i)\|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\| && (44)
\end{aligned}$$

■

The solution to the optimal control problem (23) - (33) at time t_0 is an optimal control input $\mathbf{u}_i^\star(\cdot;\ \mathbf{e}_i(t_0))$ which is applied to the open-loop system until the next sampling instant $t_0 + h$, at which time a new optimal control problem is solved in the same manner:

$$\begin{aligned}
\mathbf{u}_i\big(t;\ \mathbf{e}_i(t_0)\big) &= \mathbf{u}_i^\star\big(t;\ \mathbf{e}_i(t_0)\big) \\
t &\in [t_0, t_0 + h) \\
0 &< h < T_p
\end{aligned} \tag{45}$$

The control input $\mathbf{u}_i(\cdot)$ is a feedback, since it is recalculated at each sampling instant based on the then-current state. The solution to equation (22), starting at time t_0, from an initial condition $\mathbf{e}_i(t_0)$, by application of the control input $\mathbf{u}_i : [t_0, t_1] \to \mathcal{U}_i$ is denoted by

$$\mathbf{e}_i\big(t;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(t_0)\big)$$

with $t \in [t_0, t_1]$.

The *predicted* state of the system (22) at time $t_0 + \tau$, based on the measurement of the state at time t_0, $\mathbf{e}_i(t_0)$, by application of the control input $\mathbf{u}_i\big(t;\ \mathbf{e}_i(t_0)\big)$, for the time period $t \in [t_0, t_1]$ is denoted by

$$\overline{\mathbf{e}}_i\big(t_0+\tau;\ \mathbf{u}_i(\cdot),\mathbf{e}_i(t_0)\big) \tag{46}$$

As is natural, $\mathbf{e}_i(t_0) = \overline{\mathbf{e}}_i\big(t_0;\ \mathbf{u}_i(\cdot),\mathbf{e}_i(t_0)\big)$.

We can now give the definition of an *admissible input*:

Definition 5.1. (Admissible input)

A control input $\mathbf{u}_i : [t_0, t_0+T_p] \to \mathbb{R}^6$ for a state $\mathbf{e}_i(t_0)$ is called *admissible* if all the following hold:

1. $\mathbf{u}_i(\cdot)$ is piecewise continuous
2. $\mathbf{u}_i(\tau) \in \mathcal{U}_i,\ \forall \tau \in [t_0, t_0+T_p]$
3. $\mathbf{e}_i\big(\tau;\ \mathbf{u}_i(\cdot),\mathbf{e}_i(t_0)\big) \in \mathcal{E}_i,\ \forall \tau \in [t_0, t_0+T_p]$
4. $\mathbf{e}_i\big(t_0+T_p;\ \mathbf{u}_i(\cdot),\mathbf{e}_i(t_0)\big) \in \mathcal{E}_i^f$

5.3 Feasibility and Convergence

Under these considerations, we can now state the theorem that relates to the guaranteeing of the stability of the compound system of agents $i \in \mathcal{V}$, when each of them is assigned a desired position which results in feasible displacements:

Theorem 5.1. Suppose that

1. the terminal region $\mathcal{E}_i^f \subseteq \mathcal{E}_i$ is closed with $0 \in \mathcal{E}_i^f$
2. a solution to the optimal control problem (23) - (33) is feasible at time $t = 0$, that is, assumptions (3.1), (3.2), and (3.3) hold at time $t = 0$
3. there exists an admissible control input $\mathbf{u}_i^f : [0, h] \to \mathcal{U}_i$ such that for all $\mathbf{e}_i \in \mathcal{E}_i^f$ and $\tau \in [0, h]$:
   (a) $\mathbf{e}_i(\tau) \in \mathcal{E}_i^f$
   (b) $\dfrac{\partial V_i}{\partial \mathbf{e}_i} g_i\big(\mathbf{e}_i(\tau), \mathbf{u}_i^f(\tau)\big) + F_i\big(\mathbf{e}_i(\tau), \mathbf{u}_i^f(\tau)\big) \leq 0$

then the closed loop system (22) under the control input (45) converges to the set $\mathcal{E}_i^f$ when $t \to \infty$.

Proof. The proof of the above theorem consists of two parts: in the first, recursive feasibility is established, that is, initial feasibility is shown to imply subsequent feasibility; in the second, and based on the first part, it is shown that the error $\mathbf{e}_i(t)$ converges to the terminal set $\mathcal{E}_i^f$.

Feasibility analysis Consider a sampling instant t_0 for which a solution $\mathbf{u}_i^{\star}\big(\cdot;\ \mathbf{e}_i(t_0)\big)$ to (23) exists. Suppose now a time instant t_1 such that[2] $t_0 < t_1 < t_0 + T_p$, and consider that the optimal control signal calculated at t_0 contains the following two portions:

$$\mathbf{u}_i^{\star}\big(\cdot;\ \mathbf{e}_i(t_0)\big) = \begin{cases} \mathbf{u}_i^{\star}\big(\tau_1;\ \mathbf{e}_i(t_0)\big), & \tau_1 \in [t_0, t_1] \\ \mathbf{u}_i^{\star}\big(\tau_2;\ \mathbf{e}_i(t_0)\big), & \tau_2 \in [t_1, t_0 + T_p] \end{cases} \tag{47}$$

Both portions are admissible since the calculated optimal control input is admissible, and hence they both conform to the input constraints. As for the resulting predicted states, they satisfy the state constraints, and, crucially: $\overline{\mathbf{e}}_i\big(t_0 + T_p;\ \mathbf{u}_i^{\star}(\cdot), \mathbf{e}_i(t_0)\big) \in \mathcal{E}_i^f$. Furthermore, according to assumption (3) of the theorem, there exists an admissible (and certainly not guaranteed optimal) input $\mathbf{u}_i^f$ that renders $\mathcal{E}_i^f$ invariant over $[t_0 + T_p, t_0 + T_p + h]$.

Given the above facts, we can construct an admissible input for time t_1 by sewing together the second portion of (47) and the input $\mathbf{u}_i^f(\cdot)$:

$$\tilde{\mathbf{u}}_i\big(\tau;\ \mathbf{e}_i(t_1)\big) = \begin{cases} \mathbf{u}_i^{\star}\big(\tau;\ \mathbf{e}_i(t_0)\big), & \tau \in [t_1, t_0 + T_p] \\ \mathbf{u}_i^f(\tau - t_0 - T_p), & \tau \in (t_0 + T_p, t_1 + T_p] \end{cases} \tag{48}$$

Applied at time t_1, $\tilde{\mathbf{u}}_i\big(\cdot;\ \mathbf{e}_i(t_1)\big)$ is an admissible control input as a composition of admissible control inputs.

This means that feasibility of a solution to the optimization problem at time t_0 implies feasibility at time $t_1 > t_0$, and, thus, since at time $t = 0$ a solution is assumed to be feasible, a solution to the optimal control problem is feasible for all $t \geq 0$.

Convergence analysis The second part of the proof involves demonstrating the convergence of the state $\mathbf{e}_i$ to the terminal set $\mathcal{E}_i^f$. In order for this to be proved, it must be shown that a proper value function decreases along the solution trajectories starting at some initial time t_0. We consider the *optimal* cost $J_i^{\star}\big(\mathbf{e}_i(t)\big)$ as a candidate Lyapunov function:

$$J_i^{\star}\big(\mathbf{e}_i(t)\big) \triangleq J_i\Big(\mathbf{e}_i(t), \mathbf{u}_i^{\star}\big(\cdot;\ \mathbf{e}_i(t)\big)\Big)$$

and, in particular, our goal is to show that that this cost decreases over consecutive sampling instants $t_1 = t_0 + h$, i.e. $J_i^{\star}\big(\mathbf{e}_i(t_1)\big) - J_i^{\star}\big(\mathbf{e}_i(t_0)\big) \leq 0$.

In order not to wreak notational havoc, let us define the following terms:

[2] It is not strictly necessary that $t_1 = t_0 + h$ here, however it is necessary for the following that $t_1 - t_0 \leq h$

- $\mathbf{u}_{0,i}(\tau) \triangleq \mathbf{u}_i^\star\big(\tau;\ \mathbf{e}_i(t_0)\big)$ as the *optimal* input based on the measurement of state $\mathbf{e}_i(t_0)$, applied at time $\tau \geq t_0$
- $\mathbf{e}_{0,i}(\tau) \triangleq \overline{\mathbf{e}}_i\big(\tau;\ \mathbf{u}_{0,i}(\tau), \mathbf{e}_i(t_0)\big)$ as the *predicted* state at time $\tau \geq t_0$, that is, the state that results from the application of the above input $\mathbf{u}_i^\star\big(\tau;\ \mathbf{e}_i(t_0)\big)$ at time τ
- $\mathbf{u}_{1,i}(\tau) \triangleq \tilde{\mathbf{u}}_i\big(\tau;\ \mathbf{e}_i(t_1)\big)$ as the *feasible* input applied at $\tau \geq t_1$ (see eq. (48) above)
- $\mathbf{e}_{1,i}(\tau) \triangleq \overline{\mathbf{e}}_i\big(\tau;\ \mathbf{u}_{1,i}(\tau), \mathbf{e}_i(t_1)\big)$ as the *predicted* state at time $\tau \geq t_1$, that is, the state that results from the application of the above input $\tilde{\mathbf{u}}_i\big(\tau;\ \mathbf{e}_i(t_1)\big)$ at time τ

Before beginning to prove convergence, it is worth noting that while the cost

$$J_i\Big(\mathbf{e}_i(t), \mathbf{u}_i^\star\big(\cdot;\ \mathbf{e}_i(t)\big)\Big)$$

is optimal (in the sense that it is based on the optimal input, which provides its minimum realization), a cost that is based on a plainly feasible (and thus, without loss of generality, sub-optimal) input $\mathbf{u}_i \neq \mathbf{u}_i^\star$ will result in a configuration where

$$J_i\Big(\mathbf{e}_i(t), \mathbf{u}_i\big(\cdot;\ \mathbf{e}_i(t)\big)\Big) \geq J_i\Big(\mathbf{e}_i(t), \mathbf{u}_i^\star\big(\cdot;\ \mathbf{e}_i(t)\big)\Big) \tag{49}$$

Let us now begin our investigation on the sign of the difference between the cost that results from the application of the feasible input $\mathbf{u}_{1,i}$, which we shall denote by $\overline{J}_i\big(\mathbf{e}_i(t_1)\big)$, and the optimal cost $J_i^\star\big(\mathbf{e}_i(t_0)\big)$, while reminding ourselves that $J_i\big(\overline{\mathbf{u}}_i(\cdot);\ \mathbf{e}_i(t)\big) = \int_t^{t+T_p} F_i\big(\overline{\mathbf{e}}_i(\tau), \overline{\mathbf{u}}_i(\tau)\big)d\tau + V_i\big(\overline{\mathbf{e}}_i(t+T_p)\big)$:

$$\overline{J}_i\big(\mathbf{e}_i(t_1)\big) - J_i^\star\big(\mathbf{e}_i(t_0)\big) = \tag{50}$$

$$V_i\big(\mathbf{e}_{1,i}(t_1+T_p)\big) + \int_{t_1}^{t_1+T_p} F_i\big(\mathbf{e}_{1,i}(\tau), \mathbf{u}_{1,i}(\tau)\big)d\tau \tag{51}$$

$$-V_i\big(\mathbf{e}_{0,i}(t_0+T_p)\big) - \int_{t_0}^{t_0+T_p} F_i\big(\mathbf{e}_{0,i}(\tau), \mathbf{u}_{0,i}(\tau)\big)d\tau \tag{52}$$

Considering that $t_0 < t_1 < t_0 + T_p < t_1 + T_p$, we break down the two integrals above in between these intervals:

$$\overline{J}_i\big(\mathbf{e}_i(t_1)\big) - J_i^\star\big(\mathbf{e}_i(t_0)\big) = \tag{53}$$

$$V_i\big(\mathbf{e}_{1,i}(t_1+T_p)\big) + \int_{t_1}^{t_0+T_p} F_i\big(\mathbf{e}_{1,i}(\tau), \mathbf{u}_{1,i}(\tau)\big)d\tau + \int_{t_0+T_p}^{t_1+T_p} F_i\big(\mathbf{e}_{1,i}(\tau), \mathbf{u}_{1,i}(\tau)\big)d\tau \tag{54}$$

$$-V_i\big(\mathbf{e}_{0,i}(t_0+T_p)\big) - \int_{t_0}^{t_1} F_i\big(\mathbf{e}_{0,i}(\tau), \mathbf{u}_{0,i}(\tau)\big)d\tau - \int_{t_1}^{t_0+T_p} F_i\big(\mathbf{e}_{0,i}(\tau), \mathbf{u}_{0,i}(\tau)\big)d\tau \tag{55}$$

In between the times t_1 and $t_0 + T_p$, the constructed feasible input $\tilde{\mathbf{u}}_i\big(\cdot;\ \mathbf{e}_i(t_1)\big)$ is equal to the optimal input $\mathbf{u}_i^{\star}\big(\cdot;\ \mathbf{e}_i(t_0)\big)$ (see eq. 48), which means that $\mathbf{u}_{1,i}(\cdot) = \mathbf{u}_{0,i}(\cdot)$ in the interval $[t_1, t_0+T_p]$. Furthermore, this means that the predicted states according to these inputs will be also equal in this interval: $\mathbf{e}_{1,i}(\cdot) = \mathbf{e}_{0,i}(\cdot)$. Hence, the following equality holds over $[t_1, t_0 + T_p]$:

$$F_i\big(\mathbf{e}_{1,i}(\tau), \mathbf{u}_{1,i}(\tau)\big) = F_i\big(\mathbf{e}_{0,i}(\tau), \mathbf{u}_{0,i}(\tau)\big),\ \ \tau \in [t_1, t_0 + T_p] \tag{56}$$

Integrating this equality over the interval where it is valid yields

$$\int_{t_1}^{t_0+T_p} F_i\big(\mathbf{e}_{1,i}(\tau), \mathbf{u}_{1,i}(\tau)\big) d\tau = \int_{t_1}^{t_0+T_p} F_i\big(\mathbf{e}_{0,i}(\tau), \mathbf{u}_{0,i}(\tau)\big) d\tau \tag{57}$$

This means that these two integrals featured in the right-hand side of eq. (55) vanish, and thus the cost difference becomes

$$\overline{J}_i\big(\mathbf{e}_i(t_1)\big) - J_i^{\star}\big(\mathbf{e}_i(t_0)\big) = \tag{58}$$

$$V_i\big(\mathbf{e}_{1,i}(t_1 + T_p)\big) + \int_{t_0+T_p}^{t_1+T_p} F_i\big(\mathbf{e}_{1,i}(\tau), \mathbf{u}_{1,i}(\tau)\big) d\tau \tag{59}$$

$$-V_i\big(\mathbf{e}_{0,i}(t_0 + T_p)\big) - \int_{t_0}^{t_1} F_i\big(\mathbf{e}_{0,i}(\tau), \mathbf{u}_{0,i}(\tau)\big) d\tau \tag{60}$$

We turn our attention to the first integral in the above expression, and we note that $(t_1+T_p)-(t_0+T_p)=t_1-t_0=h$, which is exactly the length of the interval where assumption (3b) of the theorem holds. Hence, we decide to integrate the expression found in the assumption over the interval $[t_1+T_p, t_0+T_p]$, for the controls and states that are applicable in it:

$$\int_{t_0+T_p}^{t_1+T_p} \left(\frac{\partial V_i}{\partial \mathbf{e}_{1,i}} g_i\big(\mathbf{e}_{1,i}(\tau), \mathbf{u}_{1,i}(\tau)\big) + F_i\big(\mathbf{e}_{1,i}(\tau), \mathbf{u}_{1,i}(\tau)\big) \right) d\tau \leq 0 \tag{61}$$

$$\int_{t_0+T_p}^{t_1+T_p} \frac{d}{d\tau} V_i\big(\mathbf{e}_{1,i}(\tau)\big) d\tau + \int_{t_0+T_p}^{t_1+T_p} F_i\big(\mathbf{e}_{1,i}(\tau), \mathbf{u}_{1,i}(\tau)\big) d\tau \leq 0 \tag{62}$$

$$V_i\big(\mathbf{e}_{1,i}(t_1+T_p)\big) - V_i\big(\mathbf{e}_{1,i}(t_0+T_p)\big) + \int_{t_0+T_p}^{t_1+T_p} F_i\big(\mathbf{e}_{1,i}(\tau), \mathbf{u}_{1,i}(\tau)\big) d\tau \leq 0 \tag{63}$$

$$V_i\big(\mathbf{e}_{1,i}(t_1+T_p)\big) + \int_{t_0+T_p}^{t_1+T_p} F_i\big(\mathbf{e}_{1,i}(\tau), \mathbf{u}_{1,i}(\tau)\big) d\tau \leq V_i\big(\mathbf{e}_{1,i}(t_0+T_p)\big) \tag{64}$$

The left-hand side expression is the same as the first two terms in the right-hand side of equality (60). We can introduce the third one by subtracting it from both sides of the above expression:

$$V_i\big(\mathbf{e}_{1,i}(t_1+T_p)\big) + \int_{t_0+T_p}^{t_1+T_p} F_i\big(\mathbf{e}_{1,i}(\tau), \mathbf{u}_{1,i}(\tau)\big) d\tau - V_i\big(\mathbf{e}_{0,i}(t_0+T_p)\big) \tag{65}$$

$$\leq V_i\big(\mathbf{e}_{1,i}(t_0+T_p)\big) - V_i\big(\mathbf{e}_{0,i}(t_0+T_p)\big) \tag{66}$$

$$\leq \Big| V_i\big(\mathbf{e}_{1,i}(t_0+T_p)\big) - V_i\big(\mathbf{e}_{0,i}(t_0+T_p)\big) \Big| \tag{67}$$

since $x \leq |x|, \forall x \in \mathbb{R}$.
By revisiting lemma (5.1), the above inequality becomes

$$V_i\big(\mathbf{e}_{1,i}(t_1+T_p)\big) + \int_{t_0+T_p}^{t_1+T_p} F_i\big(\mathbf{e}_{1,i}(\tau), \mathbf{u}_{1,i}(\tau)\big) d\tau - V_i\big(\mathbf{e}_{0,i}(t_0+T_p)\big) \tag{68}$$

$$\leq L_{V_i} \big\| \mathbf{e}_{1,i}(t_0+T_p) - \mathbf{e}_{0,i}(t_0+T_p) \big\| \tag{69}$$

But in the interval $[t_1, t_0+T_p]$: $\mathbf{e}_{1,i}(\cdot) = \mathbf{e}_{0,i}(\cdot)$, hence the right-hand side of the inequality equals zero

$$V_i\big(\mathbf{e}_{1,i}(t_1+T_p)\big) + \int_{t_0+T_p}^{t_1+T_p} F_i\big(\mathbf{e}_{1,i}(\tau), \mathbf{u}_{1,i}(\tau)\big) d\tau - V_i\big(\mathbf{e}_{0,i}(t_0+T_p)\big) \leq 0 \tag{70}$$

By subtracting the term $-\int_{t_0}^{t_1} F_i\big(\mathbf{e}_{0,i}(\tau), \mathbf{u}_{0,i}(\tau)\big) d\tau$ from both sides we get

$$V_i\big(\mathbf{e}_{1,i}(t_1+T_p)\big) + \int_{t_0+T_p}^{t_1+T_p} F_i\big(\mathbf{e}_{1,i}(\tau), \mathbf{u}_{1,i}(\tau)\big) d\tau \tag{71}$$

$$-V_i\big(\mathbf{e}_{0,i}(t_0+T_p)\big) - \int_{t_0}^{t_1} F_i\big(\mathbf{e}_{0,i}(\tau), \mathbf{u}_{0,i}(\tau)\big) d\tau \leq -\int_{t_0}^{t_1} F_i\big(\mathbf{e}_{0,i}(\tau), \mathbf{u}_{0,i}(\tau)\big) d\tau \tag{72}$$

The left-hand side of the inequality is equal to the cost difference.

Hence, the cost difference becomes

$$\overline{J}_i\big(\mathbf{e}_i(t_1)\big) - J_i^\star\big(\mathbf{e}_i(t_0)\big) \leq -\int_{t_0}^{t_1} F_i\big(\mathbf{e}_{0,i}(\tau), \mathbf{u}_{0,i}(\tau)\big) d\tau \tag{73}$$

F_i is a positive-definite function as a sum of a positive-definite $\|\mathbf{u}_i\|^2_{\boldsymbol{R}_i}$ and a positive semi-definite function $\|\mathbf{e}_i\|^2_{\boldsymbol{Q}_i}$. If we denote by $m \geq 0$ the minimum eigenvalue between those of matrices $\boldsymbol{R}_i$, $\boldsymbol{Q}_i$, this means that

$$F_i\big(\mathbf{e}_{0,i}(\tau), \mathbf{u}_{0,i}(\tau)\big) \geq m\|\mathbf{e}_{0,i}(\tau)\|^2$$

By integrating the above between our interval of interest $[t_0, t_1]$ we get

$$\int_{t_0}^{t_1} F_i\big(\mathbf{e}_{0,i}(\tau), \mathbf{u}_{0,i}(\tau)\big) \geq \int_{t_0}^{t_1} m\|\mathbf{e}_{0,i}(\tau)\|^2 d\tau \tag{74}$$

$$-\int_{t_0}^{t_1} F_i\big(\mathbf{e}_{0,i}(\tau), \mathbf{u}_{0,i}(\tau)\big) \leq -m\int_{t_0}^{t_1} \|\mathbf{e}_{0,i}(\tau)\|^2 d\tau \tag{75}$$

Which means that the cost difference

$$\overline{J}_i\big(\mathbf{e}_i(t_1)\big) - J_i^\star\big(\mathbf{e}_i(t_0)\big) \leq -m\int_{t_0}^{t_1} \|\mathbf{e}_{0,i}(\tau)\|^2 d\tau \tag{76}$$

$$\overline{J}_i\big(\mathbf{e}_i(t_1)\big) - J_i^\star\big(\mathbf{e}_i(t_0)\big) \leq 0 \tag{77}$$

And since the cost $\overline{J}_i\big(\mathbf{e}_i(t_1)\big)$ is, in general, sub-optimal: $J_i^\star\big(\mathbf{e}_i(t_1)\big) \leq \overline{J}_i\big(\mathbf{e}_i(t_1)\big)$, the Lyapunov function $J_i^\star(\cdot)$ is decreasing along consecutive sampling times:

$$J_i^\star\big(\mathbf{e}_i(t_1)\big) \leq J_i^\star\big(\mathbf{e}_i(t_0)\big) \tag{78}$$

Therefore, the closed-loop trajectory of $\mathbf{e}_i$ converges to the terminal set $\mathcal{E}_i^f$ as $t \to \infty$.

■

References