Robust Decentralized Control of Inter-constrained Continuous Nonlinear Systems

A Receding Horizon Approach

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Master's Degree Project

Stockholm, Sweden February 2006

TRITA-EE 2006:666

Contents

Part I

Simulations

1

Introduction

The present part illustrates the efficacy of the advocated solutions, as described in chapters **??** and **??**, with regard to stabilization of multiple interconstrained agents, under the absence or presence of additive disturbances in chapters 2 and 3 respectively.

The benefit of solving the problem this thesis addresses, as was formulated in chapter **??** and approached in chapters **??** and **??**, is that the procured solutions can be applied to a general class of likewise problems $-$ the similarity relation relates to/the terms, the dynamic nature of the actors, and the structure of their habitat, but not the spirit of the problem itself.

1.1 The operational model

The simulacrum used for all agents in the following chapters shall be the three-dimensional model of the unicycle; its motion shall be expressed by the nonlinear continuous-time kinematic equations

$$\dot{x} = v \cos\theta$$
$$\dot{y} = v \sin\theta$$
$$\dot{\theta} = \omega$$

with $\mathbf{z} = [x, y, \theta]^\top$ the vector of states, $\mathbf{u} = [v, \omega]^\top$ the vector of inputs, and $\dot{\mathbf{z}} = f(\mathbf{z}, \mathbf{u})$ the (model) system's equation. We consider that $\mathbf{x} \in X$, $\mathbf{y} \in Y$ where $X \equiv Y \equiv \mathbb{R}$, $\theta \in \Theta \equiv (-\pi, \pi]$ and $\mathbf{u} \in \mathcal{U}$. In this 2D spatial environment, the obstacles of the workspace along with the workspace boundary itself assume an appropriately reformed form: that of a circle. The labeled space in which an arbitrary agent i moves, along with the spherical-obstacles-transformed to circles, is depicted in figure (1.1).

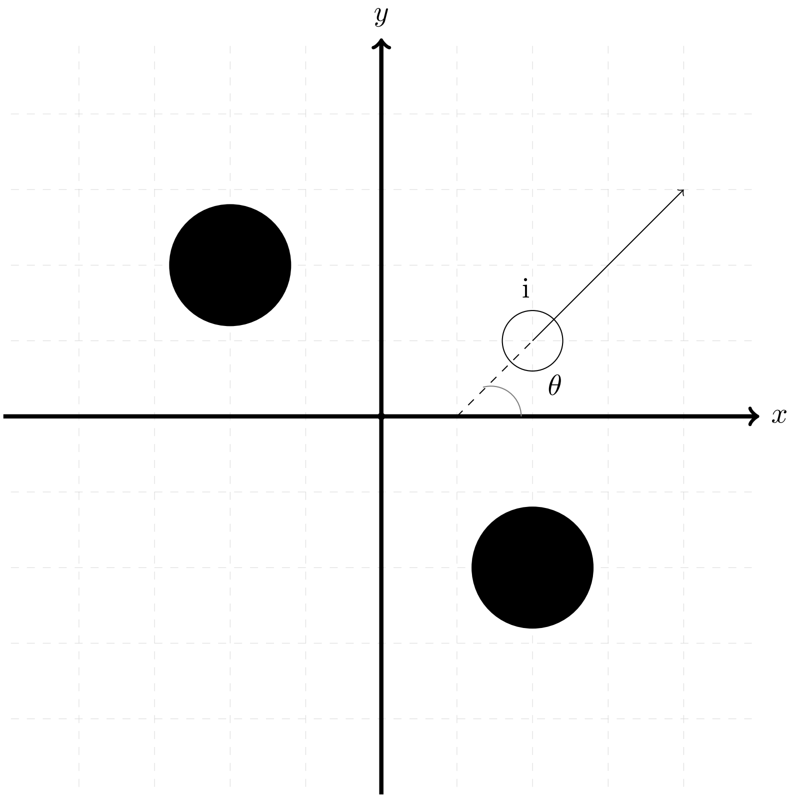

Figure 1.1: The 2D plane, agent i, whose orientation relative to the x axis is θ, and two obstacles.

The desired configuration shall be denoted by $\mathbf{z}_{des}$, the error dynamics by $\dot{\mathbf{e}} = g(\mathbf{e}, \mathbf{u})$ where $\mathbf{e}(t) = \mathbf{z}(t) - \mathbf{z}_{des}$ and $\mathbf{e} \in \mathcal{E} \equiv X \times Y \times \Theta \ominus \mathbf{z}_{des}$.

Lemma 1.1.1. Function g is Lipschitz continuous in $\mathcal{E} \times \mathcal{U}$ with a Lipschitz constant

$$L_g = v\sqrt{Q_{1,1} + Q_{1,2} + Q_{2,1} + Q_{2,2}}$$

where $\boldsymbol{Q} = Q_{\mu\nu}$ is the 3×3 matrix used to weigh the norms involved.

1.2 The problem reformed

Considering the conditions of the motivational problem as stated by problem (**??**), the reformed problem assumes the following form:

Problem 1.2.1. Assuming that

- all agents $i \in \mathcal{V}$ have access to their own and their neighbours' state and input vectors
- all agents $i \in \mathcal{V}$ have a (upper-bounded) sensing range d_i such that

$$d_i > \max\{r_i + r_j : \forall i, j \in \mathcal{V}, i \neq j\}$$

- at time $t = 0$ the sets $\mathcal{N}_i$ are known for all $i \in \mathcal{V}$ and $\sum\limits_i |\mathcal{N}_i| > 0$
- at time $t = 0$ all agents are in a collision-free configuration with each other and the obstacles $\ell \in \mathcal{L}$
- All obstacles $\ell \in \mathcal{L}$ are situated in such a way that the distance between the two least distant obstacles is larger than the diameter of the agent with the largest diameter

the problem lies in procuring feasible controls for each agent $i \in \mathcal{V}$ such that for all agents and for all obstacles $\ell \in \mathcal{L}$ the following hold

1. Position and orientation configuration is achieved in steady-state $\mathbf{z}_{i,des}$

$$\lim_{t \to \infty} \|\mathbf{z}_i(t) - \mathbf{z}_{i,des}\| = 0$$

2. Inter-agent collision is avoided

$$\|\mathbf{p}_i(t) - \mathbf{p}_j(t)\| = d_{ij,a}(t) > \underline{d}_{ij,a}, \forall j \in \mathcal{V} \backslash \{i\}$$

where $\mathbf{p}(t) = [x(t), y(t)]^\top$

3. Inter-agent connectivity loss between neighbouring agents is avoided

$$\|\mathbf{p}_i(t) - \mathbf{p}_j(t)\| = d_{ij,a}(t) < d_i, \forall j \in \mathcal{N}_i, \forall i : |\mathcal{N}_i| \neq 0$$

4. Agent-with-obstacle collision is avoided

$$\|\mathbf{p}_i(t) - \mathbf{p}_\ell(t)\| = d_{i\ell,o}(t) > \underline{d}_{i\ell,o}, \forall \ell \in \mathcal{L}$$

5. The control laws $\mathbf{u}_i(t)$ abide by their respective input constraints

$$\mathbf{u}_i(t) \in \mathcal{U}_i$$

for appropriate choice of constants $r_i, \mathbf{z}_{i,des}, \underline{d}_{ij,a}, d_i, \underline{d}_{i\ell,o}$ and neighbour sets $\mathcal{N}_i$, where $i \in \mathcal{V}$.

The content of chapters 2 and 3 will demonstrate that agents $i \in \mathcal{V}$ can be stabilized when disturbances are absent, as demonstrated in chapter **??**, and, in the case where disturbances are present, that the magnitude of their errors about the equilibrium does not exceed a certain ceiling, as demonstrated in chapter **??**.

1.3 Simulation scenarios

The simulations shall be carried out under four different agents-obstacles configurations:

1. Two agents will have to avoid one obstacle on their way to their steady-state configurations, without colliding with each other and without being separated by the obstacle (we demand that their distance is always smaller than the obstacle's diameter for the sake of cooperation).

2. Two agents will have to pass through the space between two obstacles on their way to their steady-state configurations − again, the maximum allowed distance between the two agents is smaller than the diameter of the obstacle with the smallest diameter.

3. Three agents will have to avoid one obstacle on their way to their steady-state configurations, without colliding with each other and without being separated by the obstacle. In this case, two agents are (independently) neighbours of the third, that is, the third agent should maintain connectivity and avoid collision with both of the other two, but the latter will only have to avoid colliding with each other.

4. Three agents will have to pass through the space between two obstacles on their way to their steady-state configurations. The conditions of this scenario assume those of points 2 and 3.

The four configurations are depicted in figures (1.2), (1.3), (1.4) and (1.5). Agent 1 shall be depicted in blue, agent 2 in red and agent 3 in green. The obstacles shall be depicted in black. Mark X denotes the desired position of an agent and its colour signifies the agent to be stabilized in that position.

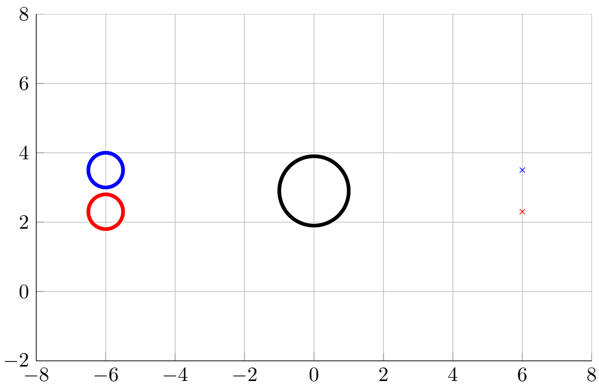

Figure 1.2: Two agents and one obstacle. X's mark the desired position of each agent in steady-state.

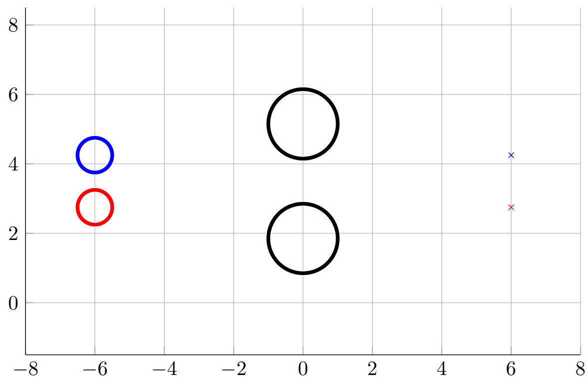

Figure 1.3: Two agents and two obstacles. X's mark the desired position of each agent in steady-state.

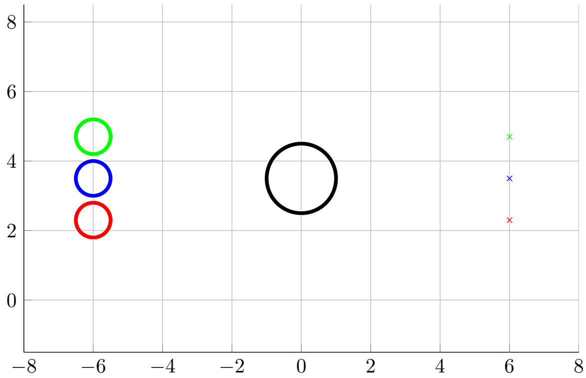

Figure 1.4: Three agents and one obstacle. X's mark the desired position of each agent in steady-state.

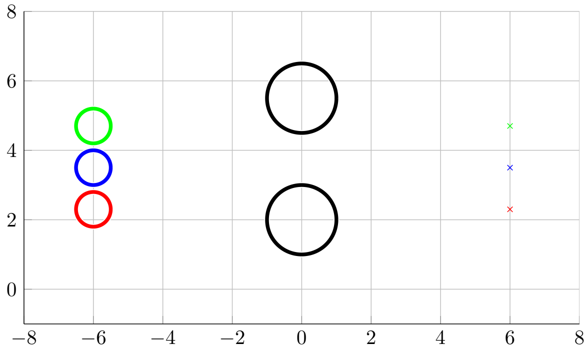

Figure 1.5: Three agents and two obstacles. X's mark the desired position of each agent in steady-state.

All configurations are as follows: the radius of all agents $i, j \in \mathcal{V}$ is $r_i = 0.5$, the radius of all obstacles is $r_\ell = 1.0$, the sensing range of all agents has a radius of $d_i = 4r_i + \epsilon = 2.0 + \epsilon$, the minimum distance between agents is $\underline{d}_{ij,a} = 2r_i + \epsilon = 1 + \epsilon$ and the minimum distance between agents and obstacles is $\underline{d}_{i\ell,o} = r_i + r_\ell + \epsilon = 1.5 + \epsilon$. ϵ was set to $\epsilon = 0.1$ in the disturbance-free cases and $\epsilon = 0.01$ in the cases where disturbances are present.

In the case of two agents, the neighbouring sets are $\mathcal{N}_1 = \{2\}$ and $\mathcal{N}_2 = \{1\}$, while in the case of three agents, $\mathcal{N}_1 = \{2, 3\}$, $\mathcal{N}_2 = \{1\}$ and $\mathcal{N}_3 = \{1\}$.

Under the above configuration regime, all agents are constrained in bypassing the obstacle(s) from the same side, as they are prohibited from overtaking it (them) from different sides by the requirement that their sensing range be lower than the sum of the diameter of one obstacle and the radii of any two agents.

The initial and terminal configurations of each agent shall be reported in the appropriate sections.

2

Simulations of Disturbance-free Stabililization

3

Simulations of Stabilization in the face of Disturbances

Appendices

A

Proofs of lemmas

A.1 Proof of lemma 0

Let $\lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i)$ denote the smallest eigenvalue between those of matrices $\boldsymbol{Q}_i$ and $\boldsymbol{R}_i$, and let $\lambda_{max}(\boldsymbol{Q}_i, \boldsymbol{R}_i)$ denote the largest. Then

$$
\begin{aligned}
\lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\|\mathbf{e}_i(t)\|^2 &\leq \lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\left\| \begin{bmatrix} \mathbf{e}_i(t) \\ \mathbf{u}_i(t) \end{bmatrix} \right\|^2 \\
&\leq F_i\big(\mathbf{e}_i(t), \mathbf{u}_i(t)\big) \\
&\leq \lambda_{max}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\left\| \begin{bmatrix} \mathbf{e}_i(t) \\ \mathbf{u}_i(t) \end{bmatrix} \right\|^2 \leq \lambda_{max}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\|\mathbf{e}_i(t)\|^2
\end{aligned}
$$

Matrices $\boldsymbol{Q}_i, \boldsymbol{R}_i$ are positive definite, hence the functions $\alpha_1\big(\|\mathbf{e}_i\|\big) = \lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\|\mathbf{e}_i\|^2$ and $\alpha_2\big(\|\mathbf{e}_i\|\big) = \lambda_{max}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\|\mathbf{e}_i\|^2$ are class $\mathcal{K}_\infty$ functions according to definition (**??**). Therefore, F_i is lower- and upper-bounded by class $\mathcal{K}_\infty$ functions:

$$\alpha_1\big(\|\mathbf{e}_i\|\big) \leq F_i\big(\mathbf{e}_i, \mathbf{u}_i\big) \leq \alpha_2\big(\|\mathbf{e}_i\|\big)$$

A.2 Proof of lemma 0

For every $\mathbf{e}_1, \mathbf{e}_2 \in \mathcal{E}_i$, and $\mathbf{u}_i \in \mathcal{U}_i$ it holds that

$$\begin{aligned}
\big|F_i(\mathbf{e}_1, \mathbf{u}_i) - F_i(\mathbf{e}_2, \mathbf{u}_i)\big| &= \big|\mathbf{e}_1^\top \boldsymbol{Q}_i \mathbf{e}_1 + \mathbf{u}_i^\top \boldsymbol{R}_i \mathbf{u}_i - \mathbf{e}_2^\top \boldsymbol{Q}_i \mathbf{e}_2 - \mathbf{u}_i^\top \boldsymbol{R}_i \mathbf{u}_i\big| \\
&= \big|\mathbf{e}_1^\top \boldsymbol{Q}_i \mathbf{e}_1 - \mathbf{e}_2^\top \boldsymbol{Q}_i \mathbf{e}_2 \pm \mathbf{e}_1^\top \boldsymbol{Q}_i \mathbf{e}_2\big| \\
&= \big|\mathbf{e}_1^\top \boldsymbol{Q}_i (\mathbf{e}_1 - \mathbf{e}_2) - \mathbf{e}_2^\top \boldsymbol{Q}_i (\mathbf{e}_1 - \mathbf{e}_2)\big| \\
&\leq \big|\mathbf{e}_1^\top \boldsymbol{Q}_i (\mathbf{e}_1 - \mathbf{e}_2)\big| + \big|\mathbf{e}_2^\top \boldsymbol{Q}_i (\mathbf{e}_1 - \mathbf{e}_2)\big|
\end{aligned}$$

But for any $\mathbf{e}_1, \mathbf{e}_2 \in \mathcal{E}_i$

$$|\mathbf{e}_1^\top \boldsymbol{Q}_i \mathbf{e}_2| \leq \sigma_{max}(\boldsymbol{Q}_i)\|\mathbf{e}_1\|\|\mathbf{e}_2\|$$

where $\sigma_{max}(\boldsymbol{Q}_i)$ denotes the largest singular value of matrix $\boldsymbol{Q}_i$. Hence:

$$\begin{aligned}
\big|F_i(\mathbf{e}_1, \mathbf{u}_i) - F_i(\mathbf{e}_2, \mathbf{u}_i)\big| &\leq \sigma_{max}(\boldsymbol{Q}_i)\|\mathbf{e}_1\|\|\mathbf{e}_1 - \mathbf{e}_2\| + \sigma_{max}(\boldsymbol{Q}_i)\|\mathbf{e}_2\|\|\mathbf{e}_1 - \mathbf{e}_2\| \\
&= \sigma_{max}(\boldsymbol{Q}_i)(\|\mathbf{e}_1\| + \|\mathbf{e}_2\|)\|\mathbf{e}_1 - \mathbf{e}_2\| \\
&= \sigma_{max}(\boldsymbol{Q}_i) \sup_{\mathbf{e}_1, \mathbf{e}_2 \in \mathcal{E}_i} \big(\|\mathbf{e}_1\| + \|\mathbf{e}_2\|\big)\|\mathbf{e}_1 - \mathbf{e}_2\| \\
&= 2\sigma_{max}(\boldsymbol{Q}_i) \sup_{\mathbf{e}_i \in \mathcal{E}_i} \big(\|\mathbf{e}_i\|\big)\|\mathbf{e}_1 - \mathbf{e}_2\| \\
&= 2\sigma_{max}(\boldsymbol{Q}_i)\overline{\varepsilon}_i\|\mathbf{e}_1 - \mathbf{e}_2\|
\end{aligned}$$

A.3 Proof of lemma 0

For every $\mathbf{e}_1, \mathbf{e}_2 \in \Omega_i$, it holds that

$$\begin{aligned}\left|V_i(\mathbf{e}_1) - V_i(\mathbf{e}_2)\right| &= \left|\mathbf{e}_1^\top \boldsymbol{P}_i \mathbf{e}_1 - \mathbf{e}_2^\top \boldsymbol{P}_i \mathbf{e}_2\right| \\ &= \left|\mathbf{e}_1^\top \boldsymbol{P}_i \mathbf{e}_1 - \mathbf{e}_2^\top \boldsymbol{P}_i \mathbf{e}_2 \pm \mathbf{e}_1^\top \boldsymbol{P}_i \mathbf{e}_2\right| \\ &= \left|\mathbf{e}_1^\top \boldsymbol{P}_i (\mathbf{e}_1 - \mathbf{e}_2) - \mathbf{e}_2^\top \boldsymbol{P}_i (\mathbf{e}_1 - \mathbf{e}_2)\right| \\ &\leq \left|\mathbf{e}_1^\top \boldsymbol{P}_i (\mathbf{e}_1 - \mathbf{e}_2)\right| + \left|\mathbf{e}_2^\top \boldsymbol{P}_i (\mathbf{e}_1 - \mathbf{e}_2)\right|\end{aligned}$$

But for any $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{R}^n$

$$\left|\mathbf{e}_1^\top \boldsymbol{P}_i \mathbf{e}_2\right| \leq \sigma_{max}(\boldsymbol{P}_i)\|\mathbf{e}_1\|\|\mathbf{e}_2\|$$

where $\sigma_{max}(\boldsymbol{P}_i)$ denotes the largest singular value of matrix $\boldsymbol{P}_i$. Hence:

$$\begin{aligned}\left|V_i(\mathbf{e}_1) - V_i(\mathbf{e}_2)\right| &\leq \sigma_{max}(\boldsymbol{P}_i)\|\mathbf{e}_1\|\|\mathbf{e}_1 - \mathbf{e}_2\| + \sigma_{max}(\boldsymbol{P}_i)\|\mathbf{e}_2\|\|\mathbf{e}_1 - \mathbf{e}_2\| \\ &= \sigma_{max}(\boldsymbol{P}_i)(\|\mathbf{e}_1\| + \|\mathbf{e}_2\|)\|\mathbf{e}_1 - \mathbf{e}_2\| \\ &\leq \sigma_{max}(\boldsymbol{P}_i)(\bar{\varepsilon}_{i,\Omega_i} + \bar{\varepsilon}_{i,\Omega_i})\|\mathbf{e}_1 - \mathbf{e}_2\| \\ &= 2\sigma_{max}(\boldsymbol{P}_i)\bar{\varepsilon}_{i,\Omega_i}\|\mathbf{e}_1 - \mathbf{e}_2\|\end{aligned}$$

A.4 Proof of lemma 0

V_i is defined as $V_i\big(\mathbf{e}_i\big) = \mathbf{e}_i^\top \boldsymbol{P}_i \mathbf{e}_i$. Let us denote the minimum and maximum eigenvalues of matrix $\boldsymbol{P}_i$ by $\lambda_{min}(\boldsymbol{P}_i)$ and $\lambda_{max}(\boldsymbol{P}_i)$ respectively. Then, the following series of inequalities holds:

$$\lambda_{min}(\boldsymbol{P}_i)\|\mathbf{e}_i\|^2 \leq V_i(\mathbf{e}_i) \leq \lambda_{max}(\boldsymbol{P}_i)\|\mathbf{e}_i\|^2$$

Matrix $\boldsymbol{P}_i$ is positive definite, hence the functions $\alpha_1 = \lambda_{min}(\boldsymbol{P}_i)\|\mathbf{e}_i\|^2$ and $\alpha_2 = \lambda_{max}(\boldsymbol{P}_i)\|\mathbf{e}_i\|^2$ are class $\mathcal{K}_\infty$ functions according to definition (**??**). Therefore, V_i is lower- and upper-bounded by class $\mathcal{K}_\infty$ functions:

$$\alpha_1\big(\|\mathbf{e}_i\|\big) \leq V_i(\mathbf{e}_i) \leq \alpha_2\big(\|\mathbf{e}_i\|\big)$$

A.5 Proof of property 0

Let us define for convenience $\boldsymbol{\zeta}_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^9 \times \mathbb{T}^3$: $\boldsymbol{\zeta}_i(s) \triangleq \mathbf{e}_i(s) - \overline{\mathbf{e}}_i(s;\ \mathbf{u}_i(s;\ \mathbf{e}_i(t)), \mathbf{e}_i(t))$, for $s \in [t, t+T_p]$.

According to lemma (**??**)

$$\|\mathbf{e}_i(s) - \overline{\mathbf{e}}_i\big(s;\ \mathbf{u}_i(s;\ \mathbf{e}_i(t)), \mathbf{e}_i(t)\big)\| \leq \frac{\overline{\delta}_i}{\text{Ł}_{g_i}}(e^{L_{g_i}(s-t)} - 1)$$

$$\|\boldsymbol{\zeta}_i(s)\| \leq \frac{\overline{\delta}_i}{\text{Ł}_{g_i}}(e^{L_{g_i}(s-t)} - 1)$$

which means that $\boldsymbol{\zeta}_i(s) \in \mathcal{B}_{i,s-t}$. Now let us assume that $\overline{\mathbf{e}}_i\big(s;\ \mathbf{u}_i(\cdot,\ \mathbf{e}_i(t)), \mathbf{e}_i(t)\big) \in \mathcal{E}_i \ominus \mathcal{B}_{i,s-t}$. Then, we add the two include statements:

$$\overline{\mathbf{e}}_i\big(s;\ \mathbf{u}_i(\cdot,\ \mathbf{e}_i(t)), \mathbf{e}_i(t)\big) \in \mathcal{E}_i \ominus \mathcal{B}_{i,s-t}$$

$$\boldsymbol{\zeta}_i(s) \in \mathcal{B}_{i,s-t}$$

which yields

$$\boldsymbol{\zeta}_i(s) + \overline{\mathbf{e}}_i\big(s;\ \mathbf{u}_i(s;\ \mathbf{e}_i(t)), \mathbf{e}_i(t)\big) \in \big(\mathcal{E}_i \ominus \mathcal{B}_{i,s-t}\big) \oplus \mathcal{B}_{i,s-t}$$

Utilizing Theorem 2.1 (ii) from [6] yields

$$\boldsymbol{\zeta}_i(s) + \overline{\mathbf{e}}_i\big(s;\ \mathbf{u}_i(s;\ \mathbf{e}_i(t)), \mathbf{e}_i(t)\big) \in \mathcal{E}_i$$

$$\mathbf{e}_i(s) \in \mathcal{E}_i$$

A.6 Proof of lemma 0

Since there are disturbances present, consulting remark (**??**) and substituting for $\tau_0 = t$ and $\tau_1 = t+\tau$ yields:

$$\mathbf{e}_i\big(t+\tau;\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t)),\mathbf{e}_i(t)\big) = \mathbf{e}_i(t) + \int_t^{t+\tau} g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t)),\overline{\mathbf{u}}_i^\star(s)\big)ds + \int_t^{t+\tau} \delta_i(s)ds$$

$$\overline{\mathbf{e}}_i\big(t+\tau;\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t)),\mathbf{e}_i(t)\big) = \mathbf{e}_i(t) + \int_t^{t+\tau} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t)),\overline{\mathbf{u}}_i^\star(s)\big)ds$$

Subtracting the latter from the former and taking norms on either side yields:

$$\begin{aligned}
&\Big\|\mathbf{e}_i\big(t+\tau;\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t)),\mathbf{e}_i(t)\big) - \overline{\mathbf{e}}_i\big(t+\tau;\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t)),\mathbf{e}_i(t)\big)\Big\| \\
&\quad= \Big\|\int_t^{t+\tau} g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t)),\overline{\mathbf{u}}_i^\star(s)\big)ds - \int_t^{t+\tau} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t)),\overline{\mathbf{u}}_i^\star(s)\big)ds + \int_t^{t+\tau} \delta_i(s)ds\Big\| \\
&\quad\leq \Big\|\int_t^{t+\tau} g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t)),\overline{\mathbf{u}}_i^\star(s)\big)ds - \int_t^{t+\tau} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t)),\overline{\mathbf{u}}_i^\star(s)\big)ds\Big\| + (t+\tau-t)\overline{\delta}_i \\
&\quad= \int_t^{t+\tau} \Big\|g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t)),\overline{\mathbf{u}}_i^\star(s)\big)s - g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t)),\overline{\mathbf{u}}_i^\star(s)\big)\Big\|ds + \tau\overline{\delta}_i \\
&\quad\leq L_{g_i}\int_t^{t+\tau} \Big\|\mathbf{e}_i\big(s;\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t)),\mathbf{e}_i(t)\big) - \overline{\mathbf{e}}_i\big(s;\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t)),\mathbf{e}_i(t)\big)\Big\|ds + \tau\overline{\delta}_i
\end{aligned}$$

since g_i is Lipschitz continuous in $\mathcal{E}_i$ with Lipschitz constant L_{g_i}. Reformulation yields

$$\begin{aligned}
&\Big\|\mathbf{e}_i\big(t+\tau;\overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t)),\mathbf{e}_i(t)\big) - \overline{\mathbf{e}}_i\big(t+\tau;\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t)),\mathbf{e}_i(t)\big)\Big\| \\
&\qquad\leq \tau\overline{\delta}_i + L_{g_i}\int_0^{\tau} \Big\|\mathbf{e}_i\big(t+s;\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t)),\mathbf{e}_i(t)\big) - \overline{\mathbf{e}}_i\big(t+s;\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t)),\mathbf{e}_i(t)\big)\Big\|ds
\end{aligned}$$

By applying the Grönwall-Bellman inequality we get:

$$\Big\|\mathbf{e}_i\big(t+\tau;\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t)),\mathbf{e}_i(t)\big)-\overline{\mathbf{e}}_i\big(t+\tau;\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t)),\mathbf{e}_i(t)\big)\Big\|$$

$$\begin{aligned}
&\leq \tau\overline{\delta}_i + L_{g_i}\int_0^\tau s\overline{\delta}_i e^{L_{g_i}(\tau-s)}ds \\
&= \tau\overline{\delta}_i - \overline{\delta}_i\int_0^\tau s\big(e^{L_{g_i}(\tau-s)}\big)'ds \\
&= \tau\overline{\delta}_i - \overline{\delta}_i\bigg(\big[se^{L_{g_i}(\tau-s)}\big]_0^\tau - \int_0^\tau e^{L_{g_i}(\tau-s)}ds\bigg) \\
&= \tau\overline{\delta}_i - \overline{\delta}_i\bigg(\tau+\frac{1}{L_{g_i}}(1-e^{L_{g_i}\tau})\bigg) \\
&= \frac{\overline{\delta}_i}{L_{g_i}}(e^{L_{g_i}\tau}-1)
\end{aligned}$$

A.7 Proof of lemma 1.1.1

Let $g(\mathbf{e},\mathbf{u})$ be the differential equation describing the error of the motion of the unicycle with respect to a fixed desired configuration, and $\boldsymbol{Q}=Q_{\mu\nu}$ a 3×3 matrix. Then

$$\begin{aligned}
\|g(\mathbf{e}_1,\mathbf{u})-g(\mathbf{e}_2,\mathbf{u})\|_Q &= \left\|\begin{bmatrix} v\cos\theta_1 - v\cos\theta_2 \\ v\sin\theta_1 - v\sin\theta_2 \\ \omega-\omega\end{bmatrix}\right\|_Q \\
&= \sqrt{v^2\begin{bmatrix}\cos\theta_1-\cos\theta_2 & \sin\theta_1-\sin\theta_2 & 0\end{bmatrix} Q \begin{bmatrix}\cos\theta_1-\cos\theta_2 \\ \sin\theta_1-\sin\theta_2 \\ 0\end{bmatrix}} \\
&= \sqrt{v^2(\cos\theta_1-\cos\theta_2)\Big((Q_{11}+Q_{12})(\cos\theta_1-\cos\theta_2)+(Q_{21}+Q_{22})(\sin\theta_1-\sin\theta_2)\Big)}
\end{aligned}$$

From the mean value theorem we derive the following

$$\cos\theta_1-\cos\theta_2 = -\sin\gamma(\theta_1-\theta_2) \leq |\theta_1-\theta_2|$$

$$\sin\theta_1 - \sin\theta_2 = \cos\gamma(\theta_1 - \theta_2) \leq |\theta_1 - \theta_2|$$

where $\theta_1 \leq \gamma \leq \theta_2$. Hence

$$\begin{aligned}
\|g(\mathbf{e}_1,\mathbf{u}) - g(\mathbf{e}_2,\mathbf{u})\|_Q &\leq \sqrt{v^2|\theta_1-\theta_2|\Big((Q_{11}+Q_{12})|\theta_1-\theta_2| + (Q_{21}+Q_{22})|\theta_1-\theta_2|\Big)} \\
&= \sqrt{v^2|\theta_1-\theta_2|^2(Q_{1,1}+Q_{1,2}+Q_{2,1}+Q_{2,2})} \\
&= v\sqrt{Q_{1,1}+Q_{1,2}+Q_{2,1}+Q_{2,2}}\cdot|\theta_1-\theta_2|
\end{aligned}$$

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