A Robust Receding Horizon Scheme for Decentralized Control of Inter-constrained Continuous Nonlinear Systems

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1 Introduction

formation of multi-agent systems, mpc intro etc.

motivation why we need mpc controllers...

In many control problems it is desired to design a stabilizing feedback such that a performance criterion is minimized while satisfying constraints on the controls and the states. Ideally one would look for a closed solution for the feedback law satisfying the constraints while optimizing the performance. However, typically the optimal feedback law cannot be found analytically, even in the unconstrained case, since it involves the solution of the corresponding Hamilton-Jacobi-Bellman partial differential equations. One approach to circumvent this problem is the repeated solution of an open-loop optimal control problem for a given state. The first part of the resulting open-loop input signal is implemented and the whole process is repeated. Control approaches using this strategy are referred to as Model Predictive Control (MPC).

Part I

The problem

2 Notation and Preliminaries

2.1 Notation

The set of positive integers is denoted by $\mathbb{N}$. The real n-coordinate space, with $n \in \mathbb{N}$, is denoted by $\mathbb{R}^n$; $\mathbb{R}^n_{\geq 0}$ and $\mathbb{R}^n_{>0}$ are the sets of real n-vectors with all elements nonnegative and positive, respectively. Given a set S, we denote as $|S|$ its cardinality. The notation $\|\mathbf{x}\|$ is used for the Euclidean norm of a vector $\mathbf{x} \in \mathbb{R}^n$. Given a symmetric matrix $\boldsymbol{A} = \boldsymbol{A}^T$, $\lambda_{\min}(\boldsymbol{A})$ denotes the minimum eigenvalue of $\boldsymbol{A}$, respectively, where $\sigma(\boldsymbol{A})$ is the set of all the eigenvalues of $\boldsymbol{A}$ and $\text{rank}(\boldsymbol{A})$ is its rank. Given two sets S_1 and S_2, the operation $S_1 \oplus S_2$ denotes the Minkowski addition, defined by $S_1 \oplus S_2 = \{s_1 + s_2 : s_1 \in S_1, s_2 \in S_2$. Define by $\mathbf{1}_n \in \mathbb{R}^n$, $\boldsymbol{I_n} \in \mathbb{R}^{n \times n}$, $\boldsymbol{0}_{m \times n} \in \mathbb{R}^{m \times n}$ the column vector with all entries 1, the unit matrix and the $m \times n$ matrix with all entries zeros, respectively. A matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is called skew-symmetric if and only if $\boldsymbol{A}^\top = -\boldsymbol{A}$. $\mathcal{B}(\mathbf{c}, r) \triangleq \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x} - \mathbf{c}\| \leq r\}$ is the 3D sphere of radius $r \in \mathbb{R}_{\geq 0}$ and center $\mathbf{c} \in \mathbb{R}^3$.

The vector expressing the coordinates of the origin of frame $\{j\}$ in frame $\{i\}$ is denoted by $\mathbf{p}_{j \triangleright i}$. When this vector is expressed in 3D space in a third frame, frame $\{k\}$, it is denoted by $\mathbf{p}^k_{j \triangleright i}$. The angular velocity of frame $\{j\}$ with respect to frame $\{i\}$, expressed in frame $\{k\}$ coordinates, is denoted by $\boldsymbol{\omega}^k_{j \triangleright i} \in \mathbb{R}^3$. We also use the notation $\mathbb{M} = \mathbb{R}^3 \times \mathbb{T}^3$. We further denote as $\mathbf{q}_{j \triangleright i} \in \mathbb{T}^3$ the Euler angles representing the orientation of frame $\{j\}$ with respect to frame $\{i\}$, where $\mathbb{T}^3$ is the 3D torus. For notational brevity, when a coordinate frame corresponds to the inertial frame of reference $\{\mathcal{O}\}$, we will omit its explicit notation (e.g., $\mathbf{p}_i = \mathbf{p}_{i \triangleright \mathcal{O}} = \mathbf{p}^{\mathcal{O}}_{i \triangleright \mathcal{O}}$, $\boldsymbol{\omega}_i = \boldsymbol{\omega}_{i \triangleright \mathcal{O}} = \boldsymbol{\omega}^{\mathcal{O}}_{i \triangleright \mathcal{O}}$). All vector and matrix differentiations are derived with respect to the inertial frame $\{\mathcal{O}\}$ unless stated otherwise.

Definition 2.1. [1] (*Class $\mathcal{K}$ function*)
A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if

1. it is strictly increasing
2. $f(0) = 0$

If $a = \infty$ and $\lim_{r \to \infty} a(r) = \infty$, then function α is said to belong to class $\mathcal{K}_\infty$

Definition 2.2. [1] (*Class $\mathcal{KL}$ function*)
A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if

1. for a fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r
2. for a fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s
3. $\lim_{s \to \infty} \beta(r, s) = 0$

Definition 2.3. [2] (*Input to State Stability*)

A nonlinear system $\dot{x} = f(x, u)$ with initial condition $x(t_0)$ is said to be *Input to State Stable (ISS)* if there exist functions $\sigma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ such that

$$\|x(t)\| \leq \beta\big(\|x(t_0)\|, t\big) + \sigma\big(\|u\|_{infty}\big)$$

Definition 2.4. [2] (*ISS Lyapunov function*)

A Lyapunov function $V(x, u)$ for the nonlinear system $\dot{x} = f(x, u)$ with initial condition $x(t_0)$ is said to be a *ISS-Lyapunov function* if there exist functions $\sigma, \alpha \in \mathcal{K}_\infty$ such that

$$\dot{V}(x, u) \leq -\alpha\big(\|x\|\big) + \sigma\big(\|u\|\big),\ \forall x, u$$

Theorem 2.1. [2]

A nonlinear system $\dot{x} = f(x, u)$ with initial condition $x(t_0)$ is said to be *Input to State Stable* if and only if it admits an ISS-Lyapunov function.

2.2 Graph Theory

?? ISTORISOU // EXPAND An *undirected graph* $\mathcal{G}$ is a pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a finite set of nodes, representing a team of agents, and $\mathcal{E} \subseteq \big\{\{i, j\} : i, j \in \mathcal{V}, i \neq j\big\}$, with $M = |\mathcal{E}|$, is the set of edges that model the communication capability between neighboring agents. For each agent, its neighbors' set $\mathcal{N}_i$ is defined as $\mathcal{N}_i = \{i_1, \ldots, i_{N_i}\} = \big\{j \in \mathcal{V} : \{i, j\} \in \mathcal{E}\big\}$, where $i_1, \ldots, i_{N_i}$ is an enumeration of the neighbors of agent i and $N_i = |\mathcal{N}_i|$.

If there is an edge $\{i, j\} \in \mathcal{E}$, then i, j are called *adjacent*. A *path* of length r from vertex i to vertex j is a sequence of $r+1$ distinct vertices, starting with i and ending with j, such that consecutive vertices are adjacent. For $i = j$, the path is called a *cycle*. If there is a path between any two vertices of the graph $\mathcal{G}$, then $\mathcal{G}$ is called *connected*. A connected graph is called a *tree* if it contains no cycles.

2.3 Non-linear Model Predictive Control for continuous-time systems

3 Problem Formulation

3.1 System Model

Consider a set $\mathcal{V}$ of N rigid bodies, $\mathcal{V} = \{1, 2, \ldots, N\}$, $|\mathcal{V}| = N \geq 2$, operating in a workspace $W \subseteq \mathbb{R}^3$. A coordinate frame $\{i\}, i \in \mathcal{V}$ is attached to the center of mass of each body. The workspace is assumed to be modeled as a bounded sphere $\mathcal{B}(\mathbf{p}_W, r_W)$ expressed in an inertial frame $\{\mathcal{O}\}$.

We consider that over time t each agent $i \in \mathcal{V}$ occupies the space of a sphere $\mathcal{B}(\mathbf{p}_i(t), r_i)$, where $\mathbf{p}_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^3$ is the position of the agent's center of mass, and $r_i < r_W$ is the radius of the agent's body. We denote by $\mathbf{q}_i(t) : \mathbb{R}_{\geq 0} \to \mathbb{T}^3$, the Euler angles representing the agents' orientation with respect to the inertial frame $\{\mathcal{O}\}$, with $\mathbf{q}_i \triangleq [\phi_i, \theta_i, \psi_i]^\top$, where $\phi_i, \psi_i \in [-\pi, \pi]$ and $\theta_i \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. We define

$$\mathbf{x}_i(t) \triangleq [\mathbf{p}_i(t)^\top, \mathbf{q}_i(t)^\top]^\top, \mathbf{x}_i(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^3 \times \mathbb{T}^3 \equiv \mathbb{M}$$

$$\mathbf{v}_i(t) \triangleq [\dot{\mathbf{p}}_i(t)^\top, \boldsymbol{\omega}_i(t)^\top]^\top, \mathbf{v}_i(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^3 \times \mathbb{R}^3 \equiv \mathbb{R}^6$$

and model the motion of agent i under second order dynamics:

$$\dot{\mathbf{x}}_i(t) = \boldsymbol{J}_i^{-1}(\mathbf{x}_i)\mathbf{v}_i(t), \tag{1a}$$

$$\mathbf{u}_i(t) = \boldsymbol{M}_i(\mathbf{x}_i)\dot{\mathbf{v}}_i(t) + \boldsymbol{C}_i(\mathbf{x}_i, \dot{\mathbf{x}}_i)\mathbf{v}_i(t) + \mathbf{g}_i(\mathbf{x}_i), \tag{1b}$$

In equation (1a), $\boldsymbol{J}_i : \mathbb{T}^3 \to \mathbb{R}^{6\times 6}$ is a Jacobian matrix that maps the non-orthogonal Euler angle rates to the orthogonal angular velocities $\mathbf{v}_i$:

$$\boldsymbol{J}_i(\mathbf{x}_i) = \begin{bmatrix} \boldsymbol{I}_3 & \boldsymbol{0}_{3\times 3} \\ \boldsymbol{0}_{3\times 3} & \boldsymbol{J}_q(\mathbf{x}_i) \end{bmatrix}, \text{ with } \boldsymbol{J}_q(\mathbf{x}_i) = \begin{bmatrix} 1 & 0 & \sin\theta_i \\ 0 & \cos\phi_i & -\cos\theta_i \sin\phi_i \\ 0 & \sin\phi_i & \cos\phi_i\cos\theta_i \end{bmatrix}$$

The matrix $\boldsymbol{J}_i$ is singular when $det(\boldsymbol{J}_i) = \cos\theta_i = 0 \Leftrightarrow \theta_i = \pm\frac{\pi}{2}$. The control scheme proposed in this thesis guarantees that this is always avoided, and hence equation (1a) is well defined. This gives rise to the following remark:

Remark 3.1. $det(\boldsymbol{J}_i) = \cos\theta_i \leq 1, \forall i \in \mathcal{V}$

In equation (1b), $\boldsymbol{M}_i : \mathbb{M} \to \mathbb{R}^{6\times 6}$ is the symmetric and positive definite *inertia matrix*, $\boldsymbol{C}_i : \mathbb{M} \times \mathbb{R}^6 \to \mathbb{R}^{6\times 6}$ is the *Coriolis matrix* and $\mathbf{g}_i : \mathbb{M} \to \mathbb{R}^6$ is the *gravity vector*. Finally, $\mathbf{u}_i \in \mathbb{R}^6$ is the control input vector representing the 6D generalized *actuation force* acting on the agent.

Remark 3.2. According to [3], the matrices $\dot{\boldsymbol{M}}_i - 2\boldsymbol{C}_i, i \in \mathcal{V}$ are skew-symmetric. The quadratic form of a skew-symmetric matrix is always equal to 0, hence:

$$\mathbf{y}^\top \left[\dot{\boldsymbol{M}}_i - 2\boldsymbol{C}_i\right] \mathbf{y} = 0, \forall \mathbf{y} \in \mathbb{R}^6, i \in \mathcal{V}.$$

However, access to measurements of, or knowledge about these matrices and vectors was not hitherto considered. At this point we make the following assumption:

Assumption 3.1. (*Measurements and Access to Information from an Inter-agent Perspective*)

1. Agent i has access to measurements $\mathbf{p}_i, \mathbf{q}_i, \dot{\mathbf{p}}_i, \boldsymbol{\omega}_i, \forall i \in \mathcal{V}$, that is, vectors $\mathbf{x}_i, \mathbf{v}_i$ pertaining to himself,

2. Agent i has a (upper-bounded) sensing range d_i such that

$$d_i > \max\{r_i + r_j : \forall i, j \in \mathcal{V}, i \neq j\}$$

3. the inertia $\boldsymbol{M}$ and Coriolis $\boldsymbol{C}$ vector fields are bounded and unknown

4. the gravity vectors $\mathbf{g}$ are bounded and known

The consequence of points 1 and 2 of assumption (3.1) is that, by defining the set of agents j that are within the sensing range of agent i at time t as

$$\mathcal{R}_i(t) \triangleq \{j \in \mathcal{V} : \mathbf{p}_j(t) \in \mathcal{B}(\mathbf{p}_i(t), d_i)\}$$

or equivalently

$$\mathcal{R}_i(t) \triangleq \{j \in \mathcal{V} : \|\mathbf{p}_i(t) - \mathbf{p}_j(t)\| \leq d_i\}$$

agent i also knows at each time instant t all

$$\mathbf{p}_{j \triangleright i}(t), \mathbf{q}_{j \triangleright i}(t), \dot{\mathbf{p}}_{j \triangleright i}(t), \boldsymbol{\omega}_{j \triangleright i}(t)$$

Therefore, agent i assumes access to all measurements

$$\mathbf{p}_j(t), \mathbf{q}_j(t), \dot{\mathbf{p}}_j(t), \boldsymbol{\omega}_j(t), \ \forall j \in \mathcal{R}_i(t), t \in \mathbb{R}_{\geq 0}$$

of agent $j \in \mathcal{R}_i(t)$ by virtue of being able to calculate them using knowledge of its own $\mathbf{p}_i(t)$, $\mathbf{q}_i(t)$, $\dot{\mathbf{p}}_i(t)$, $\boldsymbol{\omega}_i(t)$.

In the workspace there is a set $\mathcal{L}$ of L *static obstacles*, $\mathcal{L} = \{1, 2, \ldots, L\}$, $L = |\mathcal{L}|$, also modeled as spheres, with centers at positions $\mathbf{p}_\ell \in \mathbb{R}^3$ with radii $r_\ell \in \mathbb{R}$, $\ell \in \mathcal{L}$. Thus, the obstacles are modeled by spheres $\mathcal{B}(\mathbf{p}_\ell, r_\ell)$, $\ell \in \mathcal{L}$. Their position and size in 3D space is a priori known to each agent. The geometry of two agents i and j as well as an obstacle ℓ in workspace W is depicted in Fig. 1.

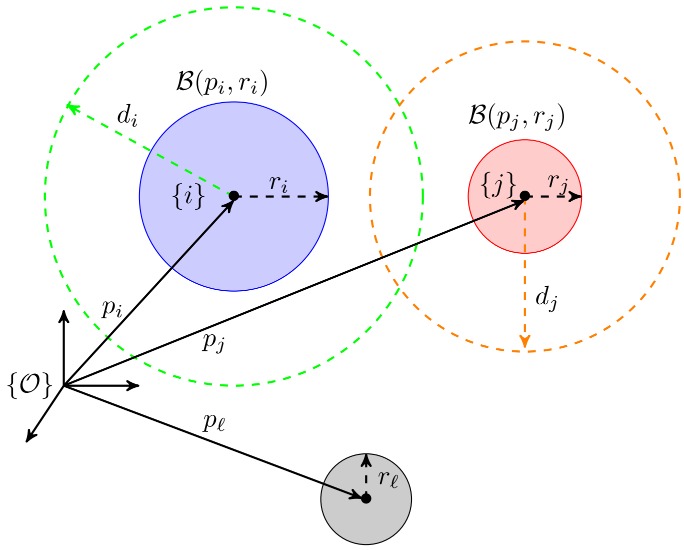

Figure 1: Illustration of two agents $i, j \in \mathcal{V}$ and an static obstacle $\ell \in \mathcal{L}$ in the workspace; $\{\mathcal{O}\}$ is the inertial frame, $\{i\}, \{j\}$ are the frames attached to the agents' center of mass, $\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_\ell \in \mathbb{R}^3$ are the positions of the center of mass of the agents i, j and the obstacle ℓ respectively, expressed in frame $\{\mathcal{O}\}$. r_i, r_j, r_ℓ are the radii of the agents i, j and the obstacle ℓ respectively. d_i, d_j with $d_i > d_j$ are the agents' sensing ranges. In this figure, agents i and j are not neighbours, since the center of mass of agent j is not within the sensing range of agent i and vice versa: $\mathbf{p}_j \notin \mathcal{B}(\mathbf{p}_i(t), d_i)$ and $\mathbf{p}_i \notin \mathcal{B}(\mathbf{p}_j(t), d_j)$.

Let us now define the distance between any two agents i, j at time t as $d_{ij,a}(t)$; that between agent i and obstacle ℓ as $d_{i\ell,o}(t)$; and that between an agent i and the origin of the workspace W as $d_{i,W}(t)$, with $d_{ij,a}, d_{i\ell,o}, d_{i,W} : \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$:

$$
\begin{aligned}
d_{ij,a}(t) &\triangleq \|\mathbf{p}_i(t) - \mathbf{p}_j(t)\|, \\
d_{i\ell,o}(t) &\triangleq \|\mathbf{p}_i(t) - \mathbf{p}_\ell(t)\|, \\
d_{i,W}(t) &\triangleq \|\mathbf{p}_W - \mathbf{p}_i(t)\|
\end{aligned}
$$

as well as constants

$$
\begin{aligned}
\underline{d}_{ij,a} &\triangleq r_i + r_j, \\
\underline{d}_{i\ell,o} &\triangleq r_i + r_\ell \\
\overline{d}_{i,W} &\triangleq r_W - r_i
\end{aligned}
$$

$\forall i, j \in \mathcal{V}$, $i \neq j$, $\ell \in \mathcal{L}$. The latter stand for the minimum distance between two *agents*, the minimum distance between an *agent* and an *obstacle*, and the maximum distance between an *agent* and the origin of the workspace, respectively. They arise spatially as physical limitations and will be utilized in collision-avoidance constraints.

Based on these definitions, we will now define the concept of a *collision-free configuration*:

Definition 3.1. (*Collision-free Configuration*)
A collision-free configuration between

- any two agents $i, j \in \mathcal{V}$, is when $d_{ij,a}(t) > \underline{d}_{ij,a}$
- an agent $i \in \mathcal{V}$ and an obstacle $\ell \in \mathcal{L}$, is when $d_{il,o}(t) > \underline{d}_{il,o}$
- an agent $i \in \mathcal{V}$ and the workspace W boundary, is when $d_{i,W}(t) < \overline{d}_{i,W}$

at a generic time instant t. When all three conditions are met, we will simply refer to the overall configuration as a collision-free configuration.

3.2 Initial Conditions

We assume that at time $t = 0$ *all* agents are in a *collision-free configuration*, i.e.

$$\begin{aligned} d_{ij,a}(0) &> \underline{d}_{ij,a} \\ d_{il,o}(0) &> \underline{d}_{il,o} \\ d_{i,W}(0) &< \overline{d}_{i,W} \end{aligned}$$

$\forall i \in \mathcal{V}, \ell \in \mathcal{L}$. Before declaring further assumptions that relate to the initial conditions of the system's configuration, we give the definition of the *neighbour set* $\mathcal{N}_i$ of a generic agent $i \in \mathcal{V}$:

Definition 3.2. (*Neighbours Set*)
Agents $j \in \mathcal{N}_i$ are defined as the *neighbours* of agent $i \in \mathcal{V}$. The set $\mathcal{N}_i$ is composed of the indices of agents $j \in \mathcal{V}$ which

1. are within the sensing range of agent i at time $t = 0$, i.e. $j \in \mathcal{R}_i(0)$, and
2. are *intended* to be kept within the sensing range of agent i at all times $t \in \mathbb{R}_{>0}$

Therefore, while the composition of the set $\mathcal{R}_i(t)$ evolves and varies through time in general, the set $\mathcal{N}_i$ *should* remain invariant over time[1].

[1] This reason, and the fact that the proposed control scheme guarantees that $\mathcal{N}_i$ *will* remain invariant over time, is why we do not refer to this set as $\mathcal{N}_i(t)$.

It is assumed that $\mathcal{N}_i$ is given at $t = 0$, and that neighbouring relations are reciprocal, i.e. agent i is a neighbour of agent j if and only if j is a neighbour of i:

$$j \in \mathcal{N}_i \Leftrightarrow i \in \mathcal{N}_j, \ \forall i, j \in \mathcal{V}, i \neq j$$

Topologically, this means that the graph $\mathcal{G}$ constructed by the edges connecting neighbouring agents $i \in \mathcal{V}$ is undirected.

Furthermore, it is assumed that, initially, the Jacobians $\boldsymbol{J}_i$ are well-defined $\forall i \in \mathcal{V}$. These assumptions, which concern the initial conditions of the problem, are summarized in assumption 3.2:

Assumption 3.2. (*Initial Conditions Assumption*)

At time $t = 0$

1. the sets $\mathcal{N}_i$ are known for all $i \in \mathcal{V}$ and[a] $\sum\limits_i |\mathcal{N}_i| > 0$
2. the given neighbouring relations are reciprocal
3. all agents are in a collision-free configuration with each other, the obstacles $\ell \in \mathcal{L}$ and the workspace W boundary
4. all agents are in a singularity-free configuration:

$$-\frac{\pi}{2} < \theta_i(0) < \frac{\pi}{2}, \ \forall i \in \mathcal{V}$$

[a] It is not necessary that all agents are assigned a set of agents with whom they should maintain connectivity; however, if *all* agents were neighbour-less, then the concept of *cooperation* between them would be void, since in that case the problem would break down into $|\mathcal{V}|$ individual problems of smaller significance or interest, while weakening the "multi-agent" perspective as well.

3.3 Objective

Given the aforementioned structure of the system, the objective to be pursued is the *stabilization of all agents* $i \in \mathcal{V}$ starting from an initial configuration abiding by assumption (3.2) to a desired feasible configuration $\mathbf{x}_{i,des}, \mathbf{v}_{i,des}$, while satisfying all communication constraints, i.e. sustaining connectivity between neighbouring agents, and avoiding collisions between agents, obstacles, and the workspace boundary.

Definition 3.3. (*Feasible Steady-state Configuration*)

The desired steady-state configuration $\mathbf{x}_{i,des}$ of agents $\forall i \in \mathcal{V}, j \in \mathcal{N}_i$ is *feasible* if and only if

1. it is a collision-free configuration
2. it does not result in violation of the communication constraints between neighbouring agents i, j, i.e. both following inequalities hold true:

$$\|\mathbf{p}_{i,des} - \mathbf{p}_{j,des}\| < d_i$$
$$\|\mathbf{p}_{i,des} - \mathbf{p}_{j,des}\| < d_j$$

Formally then, the objective of each agent i is $\lim_{t\to\infty} \|\mathbf{x}_i(t) - \mathbf{x}_{i,des}\| = 0$.

Before formally stating the problem to be solved, we must address an issue that refers to the feasibility of its solution and relates to the avoidance of collision from an intra-environmental perspective. Namely, we demand that a solution be feasible if and only if the agent with the largest radius is able to pass through the spaces demarcated by (a) the two least distant obstacles, and (b) the obstacle closest to the boundary of the workspace and the boundary of the workspace itself. To this end, we formalize the relevant notions in Definition (3.4):

Definition 3.4. (*Intra-environmental Arrangement*)
Let us define $\underline{d}_{\ell'\ell}$

$$\underline{d}_{\ell'\ell} \triangleq \min\{\|\mathbf{p}_\ell - \mathbf{p}_{\ell'}\| + r_\ell + r_{\ell'} : \ell, \ell' \in \mathcal{L}, \ell \neq \ell'\},$$

as the distance between the two least distant obstacles in the workspace, $\underline{d}_{\ell,W}$

$$\underline{d}_{\ell,W} \triangleq \min\{r_W - (\|\mathbf{p}_W - \mathbf{p}_\ell\| + r_\ell) : \ell \in \mathcal{L}\},$$

as the distance between the least distant obstacle from the boundary of the workspace and the boundary itself, and D

$$D \triangleq \min\{\underline{d}_{\ell'\ell}, \underline{d}_{\ell,W}\}$$

as the least of these two distances.

Given these notions, we can state an assumption on the feasibility of a solution to the problem that this work addresses:

Assumption 3.3. (*Intra-environmental Arrangement of Obstacles*)
All obstacles $\ell \in \mathcal{L}$ are situated inside the workspace W in such a way that

$$D > 2r_i, \ i \in \mathcal{V} : r_i = max\{r_j\} \ \forall j \in \mathcal{V}$$

where D is defined in Definition (3.4).

3.4 Problem Statement

In this section, the problem considered will be stated in a concrete manner.

Due to the fact that the agents are not dimensionless and their communication capabilities are limited, given feasible steady-state configurations $\mathbf{p}_{i,des}, \mathbf{q}_{i,des}$, the control protocol should for all agents $i \in \mathcal{V}$ guarantee that:

1. the desired positions $\mathbf{p}_{i,des}$ are achieved in finite time
2. the desired angles $\mathbf{q}_{i,des}$ are achieved in finite time
3. connectivity between neighbouring agents $j \in \mathcal{N}_i$ is maintained at all times

Furthermore, for all agents $i \in \mathcal{V}$, obstacles $\ell \in \mathcal{L}$ and the workspace boundary W, it should guarantee for all $t \in \mathbb{R}_{\geq 0}$ that:

1. all agents avoid collision with each other
2. all agents avoid collision with all obstacles
3. all agents avoid collision with the workspace boundary
4. singularity of the Jacobian matrices $\boldsymbol{J}_i$ is avoided

Therefore, all neighboring agents of agent i must remain within a distance less than d_i to him, for all $i \in \mathcal{V} : |\mathcal{N}_i| \neq 0$, and all agents $i, j \in \mathcal{V}, i \neq j$ must remain within distance greater than $\underline{d}_{ij,a}$ with one another.

Lastly, the control protocol should provide feasible control inputs $\mathbf{u}_i$ per agent $i \in \mathcal{V}$, that is, inputs that abide by the input constraints $\mathcal{U}_i$

$$\mathbf{u}_i(t) \in \mathcal{U}_i = \{\mathbf{u}_i(t) \in \mathbb{R}^6 : \|\mathbf{u}_i(t)\| \leq \overline{u}_i\}$$

Formally, the control problem under the aforementioned constraints is formulated as follows:

Problem 3.1. Consider N agents modeled as bounded spheres $\mathcal{B}(\mathbf{p}_i, r_i)$, $i \in \mathcal{V}, N = |\mathcal{V}|$, that operate in workspace W that is also modeled as a bounded sphere $\mathcal{B}(\mathbf{p}_W, r_W)$, featuring $|\mathcal{L}|$ spherical obstacles, also modeled as bounded spheres $\mathcal{B}(\mathbf{p}_\ell, r_\ell), \ell \in \mathcal{L}$.

Each agent i is governed by the dynamics (1), and the compound system of agents, obstacles and the workspace is subject to assumptions (3.1), (3.2), (3.3). Given desired *feasible* steady-state agent configurations $\mathbf{x}_{i,des}$, $\forall i \in \mathcal{V}$, design decentralized control laws $\mathbf{u}_i(t)$ such that $\forall i \in \mathcal{V}$ and for all times $t \in \mathbb{R}_{\geq 0}$, the following hold:

1. Position and orientation configuration is achieved in steady-state

$$\lim_{t\to\infty} \|\mathbf{x}_i(t) - \mathbf{x}_{i,des}(t)\| = 0$$

2. Inter-agent collision is avoided

$$\|\mathbf{p}_i(t) - \mathbf{p}_j(t)\| = d_{ij,a}(t) > \underline{d}_{ij,a}, \forall j \in \mathcal{V}\backslash\{i\}$$

3. Inter-agent connectivity loss between neighbouring agents is avoided

$$\|\mathbf{p}_i(t) - \mathbf{p}_j(t)\| = d_{ij,a}(t) < d_i, \forall j \in \mathcal{N}_i, \forall i : |\mathcal{N}_i| \neq 0$$

4. Agent-with-obstacle collision is avoided

$$\|\mathbf{p}_i(t) - \mathbf{p}_\ell(t)\| = d_{i\ell,o}(t) > \underline{d}_{i\ell,o}, \forall \ell \in \mathcal{L}$$

5. Agent-with-workspace-boundary collision is avoided

$$\|\mathbf{p}_W - \mathbf{p}_i(t)\| = d_{i,W}(t) < \overline{d}_{i,W}$$

6. All maps $\boldsymbol{J}_i$ are well defined

$$-\frac{\pi}{2} < \theta_i(t) < \frac{\pi}{2}$$

7. The control laws $\mathbf{u}_i(t)$ abide by their respective input constraints at all times

$$\mathbf{u}_i(t) \in \mathcal{U}_i, \ \forall t \in \mathbb{R}_{\geq 0}$$

Part II

Advocated Solutions

4 Disturbance-free Stabililization

Here we are interested in steering each agent $i \in \mathcal{V}$ into a *position* in 3D space, while conforming to the requirements of the problem; that is, all agents should avoid colliding with each other, all obstacles in the workspace, and the workspace boundary itself, while remaining in a non-singular configuration and sustaining the connectivity to their respective neighbours.

4.1 Formalizing the model

We begin by rewriting the system equations (1a), (1b) for a generic agent $i \in \mathcal{V}$ in state-space form:

$$\dot{\mathbf{x}}_i(t) = \boldsymbol{J}_i^{-1}(\mathbf{x}_i)\mathbf{v}_i(t)$$
$$\dot{\mathbf{v}}_i(t) = -\boldsymbol{M}_i^{-1}(\mathbf{x}_i)\boldsymbol{C}_i(\mathbf{x}_i, \dot{\mathbf{x}}_i)\mathbf{v}_i(t) - \boldsymbol{M}_i^{-1}(\mathbf{x}_i)\mathbf{g}_i(\mathbf{x}_i) + \boldsymbol{M}_i^{-1}(\mathbf{x}_i)\mathbf{u}_i(t)$$

where the inversion of $\boldsymbol{M}_i$ is possible due to it being positive-definite $\forall i \in \mathcal{V}$. Denoting by $\mathbf{z}_i(t)$

$$\mathbf{z}_i(t) = \begin{bmatrix} \mathbf{x}_i(t) \\ \mathbf{v}_i(t) \end{bmatrix}, \ \mathbf{z}_i(t) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^9 \times \mathbb{T}^3$$

and $\dot{\mathbf{x}}_i(t)$ and $\dot{\mathbf{v}}_i(t)$ by

$$\dot{\mathbf{x}}_i(t) = f_{i,x}(\mathbf{z}_i, \mathbf{u}_i)$$
$$\dot{\mathbf{v}}_i(t) = f_{i,v}(\mathbf{z}_i, \mathbf{u}_i)$$

we get the compact representation of the system's model

$$\dot{\mathbf{z}}_i(t) = \begin{bmatrix} f_{i,x}(\mathbf{z}_i, \mathbf{u}_i) \\ f_{i,v}(\mathbf{z}_i, \mathbf{u}_i) \end{bmatrix} = f_i\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big)$$

The state evolution of agent i is modeled by a system of non-linear continuous-time differential equations of the form

$$\dot{\mathbf{z}}_i(t) = f_i\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big) \tag{8}$$
$$\mathbf{z}_i(0) = \mathbf{z}_{i,0}$$
$$\mathbf{z}_i(t) \subset \mathbb{R}^9 \times \mathbb{T}^3$$
$$\mathbf{u}_i(t) \subset \mathbb{R}^6$$

where state $\mathbf{z}_i$ is directly measurable. It should be noted that equation (8) does not consider model-plant mismatches or external disturbances.

We define the set $\mathcal{Z}_i \subset \mathbb{R}^9 \times \mathbb{T}^3$ as the set that captures all the state constraints of the system's dynamics posed by the problem (3.1), for $t \in \mathbb{R}_{\geq 0}$. Therefore $\mathcal{Z}_i$ is such that:

$$\begin{aligned}\mathcal{Z}_i = \big\{\mathbf{z}_i(t) \in \mathbb{R}^9 \times \mathbb{T}^3 : \ & \|\mathbf{p}_i(t) - \mathbf{p}_j(t)\| > \underline{d}_{ij,a}, \forall j \in \mathcal{R}_i(t), \\ & \|\mathbf{p}_i(t) - \mathbf{p}_j(t)\| < d_i, \forall j \in \mathcal{N}_i, \\ & \|\mathbf{p}_i(t) - \mathbf{p}_\ell\| > \underline{d}_{i\ell,o}, \forall \ell \in \mathcal{L}, \\ & \|\mathbf{p}_W - \mathbf{p}_i(t)\| < \overline{d}_{i,W}, \\ & -\frac{\pi}{2} < \theta_i(t) < \frac{\pi}{2}, \\ & \forall t \in \mathbb{R}_{\geq 0}\big\}\end{aligned}$$

4.2 The error model

A feasible desired configuration $\mathbf{z}_{i,des} \in \mathbb{R}^9 \times \mathbb{T}^3$ is associated to each agent $i \in \mathcal{V}$, with the aim of agent i achieving it in steady-state: $\lim\limits_{t\to\infty} \|\mathbf{z}_i(t) - \mathbf{z}_{i,des}\| = 0$. The interior of the norm of this expression denotes the state error of agent i:

$$\mathbf{e}_i(t) = \mathbf{z}_i(t) - \mathbf{z}_{i,des}, \ \mathbf{e}_i(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^9 \times \mathbb{T}^3$$

The error dynamics are denoted by $g_i(\mathbf{e}_i, \mathbf{u}_i)$:

$$\dot{\mathbf{e}}_i(t) = \dot{\mathbf{z}}_i(t) - \dot{\mathbf{z}}_{i,des} = \dot{\mathbf{z}}_i(t) = f_i\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big) = g_i\big(\mathbf{e}_i(t), \mathbf{u}_i(t)\big) \tag{9}$$

with $\mathbf{e}_i(0) = \mathbf{z}_i(0) - \mathbf{z}_{i,des}$. In order to translate the constraints that are dictated for the state $\mathbf{z}_i(t)$ into constraints regarding the error state $\mathbf{e}_i(t)$, we define the set $\mathcal{E}_i \subset \mathbb{R}^9 \times \mathbb{T}^3$ as:

$$\mathcal{E}_i = \{\mathbf{e}_i(t) \in \mathbb{R}^9 \times \mathbb{T}^3 : \mathbf{e}_i(t) \in \mathcal{Z}_i \oplus (-\mathbf{z}_{i,des})\}$$

as the set that captures all constraints for the error dynamics (9) dictated by the problem (3.1).

If we design control laws $\mathbf{u}_i \in \mathcal{U}_i$, $\forall i \in \mathcal{V}$ such that the error signal $\mathbf{e}_i(t)$ with dynamics given in (9), constrained under $\mathbf{e}_i(t) \in \mathcal{E}_i$, satisfies $\lim\limits_{t\to\infty} \|\mathbf{e}_i(t)\| = 0$, while all system related signals remain bounded in their respective regions,− if all of the above are achieved, then problem (3.1) has been solved.

In order to achieve this task, we employ a Nonlinear Receding Horizon scheme.

4.3 The optimization problem

Consider a sequence of sampling times $\{t_k\}_{k\geq 0}$, with a constant sampling time h, $0 < h < T_p$, where T_p is the finite time-horizon, such that $t_{k+1} = t_k + h$. In sampling data NMPC, a finite-horizon open-loop optimal control problem (OCP) is solved at discrete sampling time instants t_k based on the then-current state error measurement $\mathbf{e}_i(t_k)$. The solution is an optimal control signal $\overline{\mathbf{u}}_i^\star(t)$, computed over $t \in [t_k, t_k + T_p]$. This signal is applied to the open-loop system in between sampling times t_k and $t_k + h$.

At a generic time t_k, agent i solves the following optimization problem:

Problem 4.1.

Find

$$J_i^\star\big(\mathbf{e}_i(t_k)\big) \triangleq \min_{\overline{\mathbf{u}}_i(\cdot)} \; J_i\big(\mathbf{e}_i(t_k), \overline{\mathbf{u}}_i(\cdot)\big)$$

where

$$J_i\big(\mathbf{e}_i(t_k), \overline{\mathbf{u}}_i(\cdot)\big) \triangleq \int_{t_k}^{t_k+T_p} F_i\big(\overline{\mathbf{e}}_i(s), \overline{\mathbf{u}}_i(s)\big)ds + V_i\big(\overline{\mathbf{e}}_i(t_k + T_p)\big) \tag{10}$$

subject to:

$$\dot{\overline{\mathbf{e}}}_i(s) = g_i\big(\overline{\mathbf{e}}_i(s), \overline{\mathbf{u}}_i(s)\big), \; \overline{\mathbf{e}}_i(t_k) = \mathbf{e}_i(t_k) \tag{11}$$

$$\overline{\mathbf{u}}_i(s) \in \mathcal{U}_i, \; \overline{\mathbf{e}}_i(s) \in \mathcal{E}_i, \; s \in [t_k, t_k + T_p]$$

$$\overline{\mathbf{e}}_i(t_k + T_p) \in \mathcal{E}_{i,f} \subseteq \mathcal{E}_i$$

The notation $\bar{\cdot}$ is used to distinguish predicted states which are internal to the controller, as opposed to their actual values, because, even in the nominal case, the predicted values will not be equal to the actual closed-loop values. This means that $\overline{\mathbf{e}}_i(\cdot)$ is the solution to (11) driven by the control input $\overline{\mathbf{u}}_i(\cdot) : [t_k, t_k + T_p] \to \mathcal{U}_i$ with initial condition $\mathbf{e}_i(t_k)$.

The applied input signal is a portion of the optimal solution to an optimization problem where information on the states of the neighbouring agents of agent i are taken into account only in the constraints considered in the optimization problem. These constraints pertain to the set of its neighbours $\mathcal{N}_i$ and, in total, to the set of all agents within its sensing range $\mathcal{R}_i$. Regarding these, we make the following assumption:

Assumption 4.1. (*Access to Predicted Information from an Inter-agent Perspective*)
Considering the context of Receding Horizon Control, when at time t_k agent i solves a finite

horizon optimization problem, he has access to[a]

1. measurements of the states[b]
   - $\mathbf{z}_j(t_k)$ of all agents $j \in \mathcal{R}_i(t_k)$ within its sensing range at time t_k
   - $\mathbf{z}_{j'}(t_k)$ of all of its neighbouring agents $j' \in \mathcal{N}_i$ at time t_k
2. the *predicted states*
   - $\overline{\mathbf{z}}_j(\tau)$ of all agents $j \in \mathcal{R}_i(t_k)$ within its sensing range
   - $\overline{\mathbf{z}}_{j'}(\tau)$ of all of its neighbouring agents $j' \in \mathcal{N}_i$

   across the entire horizon $\tau \in (t_k, t_k + T_p]$

[a] Although $\mathcal{N}_i \subseteq \mathcal{R}_i$, we make the distinction between the two because all agents $j \in \mathcal{R}_i$ need to avoid collision with agent i, but only agents $j' \in \mathcal{N}_i$ need to remain within the sensing range of agent i.

[b] as per assumption (3.1)

In other words, each time an agent solves its own individual optimization problem, he knows the error predictions that have been generated by the solution of the optimization problem of all agents within its range at that time, for the next T_p timesteps. This assumption is crucial to satisfying the constraints regarding collision aversion and connectivity maintenance between neighbouring agents. We assume that the above pieces of information are (a) always available and accurate, and (b) exchanged without delay. We encapsulate these pieces of information in four stacked vectors:

$$\begin{aligned}
\mathbf{z}_{\mathcal{R}_i}(t_k) &\triangleq col[\mathbf{z}_j(t_k)], \forall j \in \mathcal{R}_i(t_k) \\
\mathbf{z}_{\mathcal{N}_i}(t_k) &\triangleq col[\mathbf{z}_j(t_k)], \forall j \in \mathcal{N}_i \\
\overline{\mathbf{z}}_{\mathcal{R}_i}(\tau) &\triangleq col[\overline{\mathbf{z}}_j(\tau)], \forall j \in \mathcal{R}_i(t_k), \tau \in [t_k, t_k + T_p] \\
\overline{\mathbf{z}}_{\mathcal{N}_i}(\tau) &\triangleq col[\overline{\mathbf{z}}_j(\tau)], \forall j \in \mathcal{N}_i, \tau \in [t_k, t_k + T_p]
\end{aligned}$$

Remark 4.1. The justification for this assumption is the following: considering that $\mathcal{N}_i \subseteq \mathcal{R}_i$, that the state vectors $\mathbf{z}_j$ are comprised of 12 real numbers that are encoded by 4 bytes, and that sampling occurs with a frequency f for all agents, the overall downstream bandwidth required by each agent is

$$BW_d = 12 \times 32 \text{ [bits]} \times |\mathcal{R}_i| \times \frac{T_p}{h} \times f \text{ [sec}^{-1}\text{]}$$

Given conservative constants $f = 100$ Hz, $\dfrac{T_p}{h} = 100$, the wireless protocol IEEE 802.11n-2009 (a standard for present-day devices) can accommodate up to

$$|\mathcal{R}_i| = \frac{600 \text{ [Mbit} \cdot \text{sec}^{-1}]}{12 \times 32[\text{bit}] \times 10^4[\text{sec}^{-1}]} \approx 16 \cdot 10^2 \text{ agents}$$

within the range of one agent. We deem this number to be large enough for practical applications for the approach of assuming access to the predicted states of agents within the range of one agent to be legal.

?? more on the actual $\mathcal{E}_i$

The functions $F_i : \mathcal{E}_i \times \mathcal{U}_i \to \mathbb{R}_{\geq 0}$ and $V_i : \mathcal{E}_{i,f} \to \mathbb{R}_{\geq 0}$ are defined as

$$F_i\big(\overline{\mathbf{e}}_i(t), \overline{\mathbf{u}}_i(t)\big) \triangleq \overline{\mathbf{e}}_i(t)^\top \boldsymbol{Q}_i \overline{\mathbf{e}}_i(t) + \overline{\mathbf{u}}_i(t)^\top \boldsymbol{R}_i \overline{\mathbf{u}}_i(t)$$

$$V_i\big(\overline{\mathbf{e}}_i(t)\big) \triangleq \overline{\mathbf{e}}_i(t)^\top \boldsymbol{P}_i \overline{\mathbf{e}}_i(t)$$

Matrices $\boldsymbol{R}_i \in \mathbb{R}^{6\times 6}$ are symmetric and positive definite, while matrices $\boldsymbol{Q}_i, \boldsymbol{P}_i \in \mathbb{R}^{12\times 12}$ are symmetric and positive semi-definite. The running costs F_i are upper- and lower-bounded:

$$\begin{aligned}
\lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\|\mathbf{e}_i(t)\|^2 &\leq \lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\left\| \begin{bmatrix} \mathbf{e}_i(t) \\ \mathbf{u}_i(t) \end{bmatrix} \right\|^2 \\
&\leq F_i\big(\mathbf{e}_i(t), \mathbf{u}_i(t)\big) \\
&\leq \lambda_{max}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\left\| \begin{bmatrix} \mathbf{e}_i(t) \\ \mathbf{u}_i(t) \end{bmatrix} \right\|^2 \leq \lambda_{max}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\|\mathbf{e}_i(t)\|^2
\end{aligned}$$

where $\lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i)$ is the smallest eigenvalue between those of matrices $\boldsymbol{Q}_i$ and $\boldsymbol{R}_i$, and $\lambda_{max}(\boldsymbol{Q}_i, \boldsymbol{R}_i)$ the largest. Since the terms $\lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\|\mathbf{e}_i(t)\|$ and $\lambda_{max}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\|\mathbf{e}_i(t)\|$ are themselves class $\mathcal{K}$ functions according to definition (2.1), F_i is lower- and upper-bounded by class $\mathcal{K}$ functions. As is obvious, $F_i(\mathbf{0}, \mathbf{0}) = 0$.

Before defining the terminal set $\mathcal{E}_{i,f}$ it is necessary to state the definition of a positively invariant set:

Definition 4.1. (*Positively Invariant Set*)
Consider a dynamical system $\dot{\mathbf{x}} = f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, and a trajectory $\mathbf{x}(t;\ \mathbf{x}_0)$, where $\mathbf{x}_0$ is the initial condition. The set $S = \{\mathbf{x} \in \mathbb{R}^n : \gamma(\mathbf{x}) = 0\}$, where γ is a valued function, is said to be *positively invariant* if the following holds:

$$\mathbf{x}_0 \in S \Rightarrow \mathbf{x}(t;\ \mathbf{x}_0) \in S,\ \forall t \geq t_0$$

Intuitively, this means that the set S is positively invariant if a trajectory of the system does not exit it once it enters it.

The terminal set $\mathcal{E}_{i,f} \subseteq \mathcal{E}_i$ is an admissible positively invariant set for system (9) such that

$$\mathcal{E}_{i,f} = \{\mathbf{e}_i \in \mathcal{E}_i : \|\mathbf{e}_i\| \leq \varepsilon_0\}$$

where ε_0 is an arbitrarily small but fixed positive real scalar.

With regard to the terminal penalty function V_i, the following lemma will prove to be useful in guaranteeing the convergence of the solution to the optimal control problem to the terminal region $\mathcal{E}_{i,f}$:

Lemma 4.1. (V_i is Lipschitz continuous in $\mathcal{E}_{i,f}$)

The terminal penalty function V_i is Lipschitz continuous in $\mathcal{E}_{i,f}$

$$|V(\mathbf{e}_{1,i}) - V(\mathbf{e}_{2,i})| \leq L_{V_i}\|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\|$$

where $\mathbf{e}_{1,i}, \mathbf{e}_{2,i} \in \mathcal{E}_{i,f}$, with Lipschitz constant $L_{V_i} = 2\varepsilon_0\lambda_{max}(P_i)$

Proof For every $\mathbf{e}_i \in \mathcal{E}_{i,f}$, it holds that

$$\begin{aligned}
|V(\mathbf{e}_{1,i}) - V(\mathbf{e}_{2,i})| &= |\mathbf{e}_{1,i}^\top \boldsymbol{P}_i \mathbf{e}_{1,i} - \mathbf{e}_{2,i}^\top \boldsymbol{P}_i \mathbf{e}_{2,i}| \\
&= |\mathbf{e}_{1,i}^\top \boldsymbol{P}_i \mathbf{e}_{1,i} - \mathbf{e}_{2,i}^\top \boldsymbol{P}_i \mathbf{e}_{2,i} \pm \mathbf{e}_{1,i}^\top \boldsymbol{P}_i \mathbf{e}_{2,i}| \\
&= |\mathbf{e}_{1,i}^\top \boldsymbol{P}_i (\mathbf{e}_{1,i} - \mathbf{e}_{2,i}) - \mathbf{e}_{2,i}^\top \boldsymbol{P}_i (\mathbf{e}_{1,i} - \mathbf{e}_{2,i})| \\
&\leq |\mathbf{e}_{1,i}^\top \boldsymbol{P}_i (\mathbf{e}_{1,i} - \mathbf{e}_{2,i})| + |\mathbf{e}_{2,i}^\top \boldsymbol{P}_i (\mathbf{e}_{1,i} - \mathbf{e}_{2,i})|
\end{aligned}$$

But for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$|\mathbf{x}^\top \boldsymbol{A} \mathbf{y}| \leq \lambda_{max}(A)\|\mathbf{x}\|\|\mathbf{y}\|$$

where $\lambda_{max}(A)$ denotes the largest eigenvalue of matrix $\boldsymbol{A}$. Hence:

$$\begin{aligned}
|V(\mathbf{e}_{1,i}) - V(\mathbf{e}_{2,i})| &\leq \lambda_{max}(\boldsymbol{P}_i)\|\mathbf{e}_{1,i}\|\|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\| + \lambda_{max}(\boldsymbol{P}_i)\|\mathbf{e}_{2,i}\|\|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\| \\
&= \lambda_{max}(\boldsymbol{P}_i)(\|\mathbf{e}_{1,i}\| + \|\mathbf{e}_{2,i}\|)\|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\| \\
&\leq \lambda_{max}(\boldsymbol{P}_i)(\varepsilon_0 + \varepsilon_0)\|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\| \\
&= 2\varepsilon_0\lambda_{max}(\boldsymbol{P}_i)\|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\|
\end{aligned}$$

■

The solution to the optimal control problem (10) at time t_k is an optimal control input, denoted by $\overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t_k))$, which is applied to the open-loop system until the next sampling instant $t_k + h$, with

$h \in (0, T_p)$, at which time a new optimal control problem is solved in the same manner:

$$\mathbf{u}_i(t) = \overline{\mathbf{u}}_i^{\star}\big(t;\ \mathbf{e}_i(t_k)\big),\ t \in [t_k, t_k + h] \tag{13}$$

The control input $\mathbf{u}_i(\cdot)$ is of feedback form, since it is recalculated at each sampling instant based on the then-current state. The solution to equation (9), starting at time t_1, from an initial condition $\mathbf{e}_i(t_1)$, by application of the control input $\mathbf{u}_i : [t_1, t_2] \to \mathcal{U}_i$ is denoted by

$$\mathbf{e}_i\big(t;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(t_1)\big),\ t \in [t_1, t_2]$$

The *predicted* state of the system (9) at time $t_k + \tau$, based on the measurement of the state at time t_k, $\mathbf{e}_i(t_k)$, by application of the control input $\mathbf{u}_i\big(t;\ \mathbf{e}_i(t_k)\big)$, for the time period $t \in [t_k, t_k + \tau]$ is denoted by

$$\overline{\mathbf{e}}_i\big(t_k + \tau;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(t_k)\big)$$

As is natural, the equality

$$\overline{\mathbf{e}}_i\big(\tau_1;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(\tau_0)\big) = \mathbf{e}_i\big(\tau_1;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(\tau_0)\big) \tag{14}$$

holds true here because there are no disturbances acting on the system.

The closed-loop system for which stability is to be guaranteed is

$$\mathbf{e}_i(\tau) = g_i\big(\mathbf{e}_i(\tau), \overline{\mathbf{u}}_i^{\star}(\tau)\big),\ \tau \geq t_0 = 0 \tag{15}$$

where $\overline{\mathbf{u}}_i^{\star}(\tau) = \overline{\mathbf{u}}_i^{\star}(\tau;\ \mathbf{e}_i(t_k))$, $\tau \in [t_k, t_k + h)$.

We can now give the definition of an *admissible input*:

Definition 4.2. (Admissible input)

A control input $\mathbf{u}_i : [t_k, t_k + T_p] \to \mathbb{R}^6$ for a state $\mathbf{e}_i(t_k)$ is called *admissible* if all the following hold:

1. $\mathbf{u}_i(\cdot)$ is piecewise continuous
2. $\mathbf{u}_i(\tau) \in \mathcal{U}_i,\ \forall \tau \in [t_k, t_k + T_p]$
3. $\mathbf{e}_i\big(\tau;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(t_k)\big) \in \mathcal{E}_i,\ \forall \tau \in [t_k, t_k + T_p]$
4. $\mathbf{e}_i\big(t_k + T_p;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(t_k)\big) \in \mathcal{E}_{i,f}$

4.4 Feasibility and Convergence

Under these considerations, we can now state the theorem that relates to the guaranteeing of the stability of the compound system of agents $i \in \mathcal{V}$, when each of them is assigned a desired position which results in feasible displacements:

Theorem 4.1. Suppose that

1. the terminal region $\mathcal{E}_{i,f} \subseteq \mathcal{E}_i$ is closed with $\mathbf{0} \in \mathcal{E}_{i,f}$
2. a solution to the optimal control problem (10) is feasible at time $t = 0$, that is, assumptions (3.1), (3.2), and (3.3) hold at time $t = 0$
3. there exists an admissible control input $\mathbf{u}_{i,f} : [0, h] \rightarrow \mathcal{U}_i$ such that for all $\mathbf{e}_i \in \mathcal{E}_{i,f}$ and $\forall \tau \in [0, h]$:

   (a) $\mathbf{e}_i(\tau) \in \mathcal{E}_{i,f}$

   (b) $\dfrac{\partial V_i}{\partial \mathbf{e}_i} g_i\big(\mathbf{e}_i(\tau), \mathbf{u}_{i,f}(\tau)\big) + F_i\big(\mathbf{e}_i(\tau), \mathbf{u}_{i,f}(\tau)\big) \leq 0$

then the closed loop system (15) under the control input (13) converges to the set $\mathcal{E}_{i,f}$ when $t \rightarrow \infty$.

Proof. The proof of the above theorem consists of two parts: in the first, recursive feasibility is established, that is, initial feasibility is shown to imply subsequent feasibility; in the second, and based on the first part, it is shown that the error state $\mathbf{e}_i(t)$ converges to the terminal set $\mathcal{E}_{i,f}$.

Feasibility analysis Consider a sampling instant t_k for which a solution $\overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big)$ to (10) exists. Suppose now a time instant t_{k+1} such that[2] $t_k < t_{k+1} < t_k + T_p$, and consider that the optimal control signal calculated at t_k is comprised by the following two portions:

$$\overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big) = \begin{cases} \overline{\mathbf{u}}_i^\star\big(\tau_1;\ \mathbf{e}_i(t_k)\big), & \tau_1 \in [t_k, t_{k+1}] \\ \overline{\mathbf{u}}_i^\star\big(\tau_2;\ \mathbf{e}_i(t_k)\big), & \tau_2 \in [t_{k+1}, t_k + T_p] \end{cases} \tag{16}$$

Both portions are admissible since the calculated optimal control input is admissible, and hence they both conform to the input constraints. As for the resulting predicted states, they satisfy the state constraints, and, crucially: $\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \in \mathcal{E}_{i,f}$. Furthermore, according to assumption (3) of the theorem, there exists an admissible (and certainly not guaranteed optimal) input $\mathbf{u}_{i,f}$ that renders $\mathcal{E}_{i,f}$ invariant over $[t_k + T_p, t_k + T_p + h]$.

[2]It is not strictly necessary that $t_{k+1} = t_k + h$ here, however it is necessary for the following that $t_{k+1} - t_k \leq h$

Given the above facts, we can construct an admissible input $\widetilde{\mathbf{u}}_i(\cdot)$ for time t_{k+1} by sewing together the second portion of (29) and the input $\mathbf{u}_{i,f}(\cdot)$:

$$\widetilde{\mathbf{u}}_i(\tau) = \begin{cases} \overline{\mathbf{u}}_i^\star\big(\tau;\ \mathbf{e}_i(t_k)\big), & \tau \in [t_{k+1}, t_k + T_p] \\ \mathbf{u}_{i,f}(\tau - t_k - T_p), & \tau \in (t_k + T_p, t_{k+1} + T_p] \end{cases} \tag{17}$$

Applied at time t_{k+1}, $\widetilde{\mathbf{u}}_i(\cdot)$ is an admissible control input as a composition of admissible control inputs.

This means that feasibility of a solution to the optimization problem at time t_k implies feasibility at time $t_{k+1} > t_k$, and, thus, since at time $t = 0$ a solution is assumed to be feasible, a solution to the optimal control problem is feasible for all $t \geq 0$.

Convergence analysis The second part of the proof involves demonstrating the convergence of the state $\mathbf{e}_i$ to the terminal set $\mathcal{E}_{i,f}$. In order for this to be proved, it must be shown that a proper value function decreases along the solution trajectories starting at some initial time t_k. We consider the *optimal* cost $J_i^\star\big(\mathbf{e}_i(t)\big)$ as a candidate Lyapunov function:

$$J_i^\star\big(\mathbf{e}_i(t)\big) \triangleq J_i\Big(\mathbf{e}_i(t), \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t)\big)\Big)$$

and, in particular, our goal is to show that that this cost decreases over consecutive sampling instants $t_{k+1} = t_k + h$, i.e. $J_i^\star\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) \leq 0$.

In order not to wreak notational havoc, let us define the following terms:

- $\mathbf{u}_{0,i}(\tau) \triangleq \overline{\mathbf{u}}_i^\star\big(\tau;\ \mathbf{e}_i(t_k)\big)$ as the *optimal* input that results from the solution to problem (4.1) based on the measurement of state $\mathbf{e}_i(t_k)$, applied at time $\tau \geq t_k$
- $\mathbf{e}_{0,i}(\tau) \triangleq \overline{\mathbf{e}}_i\big(\tau;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big)$ as the *predicted* state at time $\tau \geq t_k$, that is, the state that results from the application of the above input $\overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big)$ to the state $\mathbf{e}_i(t_k)$, at time τ
- $\mathbf{u}_{1,i}(\tau) \triangleq \widetilde{\mathbf{u}}_i(\tau)$ as the *admissible* input at $\tau \geq t_{k+1}$ (see eq. (30))
- $\mathbf{e}_{1,i}(\tau) \triangleq \overline{\mathbf{e}}_i\big(\tau;\ \widetilde{\mathbf{u}}_i(\cdot), \mathbf{e}_i(t_{k+1})\big)$ as the *predicted* state at time $\tau \geq t_{k+1}$, that is, the state that results from the application of the above input $\widetilde{\mathbf{u}}_i(\cdot)$ to the state $\mathbf{e}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big)$, at time τ

Remark 4.2. Given that no model mismatch or disturbances exist, for the predicted and actual states at time $\tau_1 \geq \tau_0 \in \mathbb{R}_{\geq 0}$ it holds that:

$$\mathbf{e}_i\big(\tau_1;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(\tau_0)\big) = \mathbf{e}_i(\tau_0) + \int_{\tau_0}^{\tau_1} g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(\tau_0)), \mathbf{u}_i(s)\big)ds$$
$$\overline{\mathbf{e}}_i\big(\tau_1;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(\tau_0)\big) = \mathbf{e}_i(\tau_0) + \int_{\tau_0}^{\tau_1} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(\tau_0)), \mathbf{u}_i(s)\big)ds$$

Before beginning to prove convergence, it is worth noting that while the cost

$$J_i\Big(\mathbf{e}_i(t), \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t)\big)\Big)$$

is optimal (in the sense that it is based on the optimal input, which provides its minimum realization), a cost that is based on a plainly admissible (and thus, without loss of generality, sub-optimal) input $\mathbf{u}_i \neq \overline{\mathbf{u}}_i^\star$ will result in a configuration where

$$J_i\Big(\mathbf{e}_i(t), \mathbf{u}_i\big(\cdot;\ \mathbf{e}_i(t)\big)\Big) \geq J_i\Big(\mathbf{e}_i(t), \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t)\big)\Big)$$

Let us now begin our investigation on the sign of the difference between the cost that results from the application of the feasible input $\mathbf{u}_{1,i}$, which we shall denote by $\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big)$, and the optimal cost $J_i^\star\big(\mathbf{e}_i(t_k)\big)$, while reminding ourselves that $J_i\big(\mathbf{e}_i(t), \overline{\mathbf{u}}_i(\cdot)\big) = \int_t^{t+T_p} F_i\big(\overline{\mathbf{e}}_i(s), \overline{\mathbf{u}}_i(s)\big)ds + V_i\big(\overline{\mathbf{e}}_i(t+T_p)\big)$:

$$\begin{aligned}\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) = {} & V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) + \int_{t_{k+1}}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds \\ & -V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) - \int_{t_k}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds\end{aligned}$$

Considering that $t_k < t_{k+1} < t_k + T_p < t_{k+1} + T_p$, we break down the two integrals above in between these intervals:

$$\begin{aligned}&\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) = \\ &\quad V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) + \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds \\ &\quad -V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) - \int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds - \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds \end{aligned} \tag{18}$$

Since no model mismatch or disturbances are present, consulting remark (4.2) and substituting for $\tau_0 = t_k$ and $\tau_1 = t_{k+1}$ yields:

$$\mathbf{e}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) = \mathbf{e}_i(t_k) + \int_{t_k}^{t_{k+1}} g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big) ds$$
$$\overline{\mathbf{e}}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) = \mathbf{e}_i(t_k) + \int_{t_k}^{t_{k+1}} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big) ds$$

Subtracting the second expression from the first, we get

$$\begin{aligned}
&\mathbf{e}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) - \overline{\mathbf{e}}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) \\
&= \int_{t_k}^{t_{k+1}} g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big) ds - \int_{t_k}^{t_{k+1}} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big) ds \\
&= \int_{t_k}^{t_{k+1}} \Big(g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big) - g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big) \Big) ds
\end{aligned}$$

Taking norms on either side yields

$$\begin{aligned}
&\Big\| \mathbf{e}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) - \overline{\mathbf{e}}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) \Big\| \\
&= \Big\| \int_{t_k}^{t_{k+1}} \Big(g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big) - g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big) \Big) ds \Big\| \\
&= \int_{t_k}^{t_{k+1}} \Big\| g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big) - g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big) \Big\| ds \\
&\leq L_{g_i} \int_{t_k}^{t_{k+1}} \Big\| \mathbf{e}_i\big(s;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) - \overline{\mathbf{e}}_i\big(s;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) \Big\| ds
\end{aligned}$$

since g_i is Lipschitz continuous in $\mathbf{e}_i$ with Lipschitz constant L_{g_i}. Reformulation yields

$$\begin{aligned}
&\Big\| \mathbf{e}_i\big(t_k + h;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) - \overline{\mathbf{e}}_i\big(t_k + h;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) \Big\| \\
&\leq L_{g_i} \int_0^h \Big\| \mathbf{e}_i\big(t_k + s;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) - \overline{\mathbf{e}}_i\big(t_k + s;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) \Big\| ds
\end{aligned}$$

By applying the Grönwall-Bellman inequality we obtain zero as an upper bound for the norm of the difference between the two states. Since any norm cannot be negative, we conclude that

$$\left\| \mathbf{e}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^{\star}\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) - \overline{\mathbf{e}}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^{\star}\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) \right\| = 0$$

which means that

$$\mathbf{e}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^{\star}\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) = \overline{\mathbf{e}}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^{\star}\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big)$$

In between times t_{k+1} and $t_k + T_p$, the constructed admissible input $\widetilde{\mathbf{u}}_i(\cdot)$ is equal to the optimal input $\overline{\mathbf{u}}_i^{\star}\big(\cdot;\ \mathbf{e}_i(t_k)\big)$ (see eq. 30), which means that $\mathbf{u}_{1,i}(\tau) = \mathbf{u}_{0,i}(\tau)$ in the interval $\tau \in [t_{k+1}, t_k + T_p]$. Since the initial conditions at $t = t_{k+1}$ are equal and the control laws are also equal, so will the predicted states over the same interval:

$$\overline{\mathbf{e}}_i\big(\tau;\ \widetilde{\mathbf{u}}_i(\cdot), \mathbf{e}_i(t_{k+1})\big) = \overline{\mathbf{e}}_i\big(\tau;\ \overline{\mathbf{u}}_i^{\star}(\cdot), \overline{\mathbf{e}}_i(t_{k+1})\big),\ \tau \in [t_{k+1}, t_k + T_p]$$

Using our notation then, in the same interval: $\mathbf{e}_{1,i}(\cdot) = \mathbf{e}_{0,i}(\cdot)$, and therefore the following equality holds over $[t_{k+1}, t_k + T_p]$:

$$F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) = F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big),\ s \in [t_{k+1}, t_k + T_p]$$

Integrating this equality over the interval where it is valid yields

$$\int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds = \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) ds$$

This means that these two integrals with ends over the interval $[t_{k+1}, t_k + T_p]$ featured in the right-hand side of eq. (18) vanish, and thus the cost difference becomes

$$\begin{aligned} \overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^{\star}\big(\mathbf{e}_i(t_k)\big) = V_i\big(\mathbf{e}_{1,i}(t_{k+1} + T_p)\big) &+ \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds \\ - V_i\big(\mathbf{e}_{0,i}(t_k + T_p)\big) &- \int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) ds \end{aligned} \tag{19}$$

We turn our attention to the first integral in the above expression, and we note that $(t_{k+1} + T_p) - (t_k + T_p) = t_{k+1} - t_k = h$, which is exactly the length of the interval where assumption (3b) of the theorem holds. Hence, we decide to integrate the expression found in the assumption over the interval $[t_k + T_p, t_{k+1} + T_p]$, for the controls and states applicable in it:

$$\int_{t_k+T_p}^{t_{k+1}+T_p} \left(\frac{\partial V_i}{\partial \mathbf{e}_{1,i}} g_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) + F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) \right) ds \leq 0$$

$$\int_{t_k+T_p}^{t_{k+1}+T_p} \frac{d}{ds} V_i\big(\mathbf{e}_{1,i}(s)\big) ds + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds \leq 0$$

$$V_i\big(\mathbf{e}_{1,i}(t_{k+1} + T_p)\big) - V_i\big(\mathbf{e}_{1,i}(t_k + T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds \leq 0$$

$$V_i\big(\mathbf{e}_{1,i}(t_{k+1} + T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds \leq V_i\big(\mathbf{e}_{1,i}(t_k + T_p)\big)$$

The left-hand side expression is the same as the first two terms in the right-hand side of equality (19). We can introduce the third one by subtracting it from both sides:

$$\begin{aligned} &V_i\big(\mathbf{e}_{1,i}(t_{k+1} + T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds - V_i\big(\mathbf{e}_{0,i}(t_k + T_p)\big) \\ &\leq V_i\big(\mathbf{e}_{1,i}(t_k + T_p)\big) - V_i\big(\mathbf{e}_{0,i}(t_k + T_p)\big) \leq \Big| V_i\big(\mathbf{e}_{1,i}(t_k + T_p)\big) - V_i\big(\mathbf{e}_{0,i}(t_k + T_p)\big) \Big| \end{aligned}$$

since $x \leq |x|, \forall x \in \mathbb{R}$.

By revisiting lemma (4.1), the above inequality becomes

$$\begin{aligned} V_i\big(\mathbf{e}_{1,i}(t_{k+1} + T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds - V_i\big(\mathbf{e}_{0,i}(t_k + T_p)\big) \\ \leq L_{V_i} \big\| \mathbf{e}_{1,i}(t_k + T_p) - \mathbf{e}_{0,i}(t_k + T_p) \big\| \end{aligned}$$

However, in the interval $[t_{k+1}, t_k + T_p]$: $\mathbf{e}_{1,i}(\cdot) = \mathbf{e}_{0,i}(\cdot)$, hence the right-hand side of the inequality equals zero:

$$V_i\big(\mathbf{e}_{1,i}(t_{k+1} + T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds - V_i\big(\mathbf{e}_{0,i}(t_k + T_p)\big) \leq 0$$

By subtracting the fourth term needed to complete the right-hand side expression of (19), i.e.

$\int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds$ from both sides we get

$$V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds$$

$$-V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) - \int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds \leq - \int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds$$

The left-hand side of this inequality is now equal to the cost difference $\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big)$.

Hence, the cost difference becomes bounded by

$$\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) \leq - \int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds$$

F_i is a positive-definite function as a sum of a positive-definite $\|\mathbf{u}_i\|^2_{\boldsymbol{R}_i}$ and a positive semi-definite function $\|\mathbf{e}_i\|^2_{\boldsymbol{Q}_i}$. If we denote by $m = \lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i) \geq 0$ the minimum eigenvalue between those of matrices $\boldsymbol{R}_i$, $\boldsymbol{Q}_i$, this means that

$$F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) \geq m\|\mathbf{e}_{0,i}(s)\|^2$$

By integrating the above between our interval of interest $[t_k, t_{k+1}]$ we get

$$\int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) \geq \int_{t_k}^{t_{k+1}} m\|\mathbf{e}_{0,i}(s)\|^2 ds$$

or

$$-\int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) \leq -m\int_{t_k}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds$$

This means that the cost difference is upper-bounded by a class $\mathcal{K}$ function

$$\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) \leq -m\int_{t_k}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds \leq 0$$

and since the cost $\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big)$ is, in general, sub-optimal: $J_i^\star\big(\mathbf{e}_i(t_{k+1})\big) - \overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) \leq 0$:

$$J_i^\star\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) \leq -m\int_{t_k}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds \tag{20}$$

With this milestone result established, we need to trace the time t_k back to $t_0 = 0$.

The integral of $\|\mathbf{e}_{0,i}(\tau)\|^2$ over the interval $[t_0, t_{k+1}]$, $t_0 < t_k < t_{k+1}$ can be decomposed into the addition of two integrals with limits ranging from (a) t_0 to t_k and (b) t_k to t_{k+1}:

$$\int_{t_0}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds = \int_{t_0}^{t_k} \|\mathbf{e}_{0,i}(s)\|^2 ds + \int_{t_k}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds$$

By rearranging terms, this means that

$$\int_{t_k}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds = \int_{t_0}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds - \int_{t_0}^{t_k} \|\mathbf{e}_{0,i}(s)\|^2 ds$$

making the optimal cost difference between the consecutive sampling times t_k and t_{k+1} in (20)

$$J_i^\star\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) \leq -m \int_{t_0}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds + m \int_{t_0}^{t_k} \|\mathbf{e}_{0,i}(s)\|^2 ds$$

Similarly, the optimal cost difference between the sampling times t_{k-1} and t_k is

$$J_i^\star\big(\mathbf{e}_i(t_k)\big) - J_i^\star\big(\mathbf{e}_i(t_{k-1})\big) \leq -m \int_{t_0}^{t_k} \|\mathbf{e}_{0,i}(s)\|^2 ds + m \int_{t_0}^{t_{k-1}} \|\mathbf{e}_{0,i}(s)\|^2 ds$$

and we can apply this rationale all the way back to the cost difference between t_0 and t_1. Summing all the inequalities between the pairs of consecutive sampling times (t_0, t_1), (t_1, t_2), $\ldots$, (t_{k-1}, t_k), we get

$$J_i^\star\big(\mathbf{e}_i(t_k)\big) - J_i^\star\big(\mathbf{e}_i(t_0)\big) \leq -m \int_{t_0}^{t_k} \|\mathbf{e}_{0,i}(s)\|^2 ds$$

Hence, for $t_0 = 0$

$$J_i^\star\big(\mathbf{e}_i(t_k)\big) - J_i^\star\big(\mathbf{e}_i(0)\big) \leq -m \int_{0}^{t_k} \|\mathbf{e}_{0,i}(s)\|^2 ds \leq 0 \tag{21}$$

which implies that the value function $J_i^\star\big(\mathbf{e}_i(t_k)\big)$ in non-increasing for all sampling times:

$$J_i^\star\big(\mathbf{e}_i(t_k)\big) \leq J_i^\star\big(\mathbf{e}_i(0)\big),\ \forall t_k \in \mathbb{R}_{\geq 0}$$

Let us now define the function $V_i(\mathbf{e}_i(t))$:

$$V_i\big(\mathbf{e}(t)\big) \triangleq J_i^\star\big(\mathbf{e}_i(\tau)\big) \leq J_i^\star\big(\mathbf{e}_i(0)\big),\ t \in \mathbb{R}_{\geq 0}$$

where $\tau = max\{t_k : t_k \leq t\}$. Since $J_i^\star\big(\mathbf{e}_i(0)\big)$ is bounded, this implies that $V_i\big(\mathbf{e}(t)\big)$ is also bounded. The signals $\mathbf{e}_i(t) \in \mathcal{E}_i$ and $\mathbf{u}_i(t) \in \mathcal{U}_i$ are also bounded. According to (9), this means that $\dot{\mathbf{e}}_i(t)$ is bounded as well. From inequality (21) we then have

$$V_i\big(\mathbf{e}_i(t)\big) = J_i^\star\big(\mathbf{e}_i(\tau)\big) \leq J_i^\star\big(\mathbf{e}_i(0)\big) - m\int_0^\tau \|\mathbf{e}_{0,i}(s)\|^2 ds \leq 0$$

which, due to the fact that $\tau \leq t$, is equivalent to

$$V_i\big(\mathbf{e}_i(t)\big) \leq J_i^\star\big(\mathbf{e}_i(0)\big) - m\int_0^t \|\mathbf{e}_{0,i}(s)\|^2 ds \leq 0,\ t \in \mathbb{R}_{t\geq 0}$$

Solving for the integral we get

$$\int_0^t \|\mathbf{e}_{0,i}(s)\|^2 ds \leq \frac{1}{m}\Big(J_i^\star\big(\mathbf{e}_i(0)\big) - V_i\big(\mathbf{e}_i(t)\big)\Big),\ t \in \mathbb{R}_{t\geq 0}$$

Both $J_i^\star\big(\mathbf{e}_i(0)\big)$ and $V_i\big(\mathbf{e}_i(t)\big)$ are bounded, and therefore so is their difference, which means that the integral $\int\limits_0^t \|\mathbf{e}_{0,i}(s)\|^2 ds$ is bounded as well. We make use of the following lemma to show that the error internal to the norm of the integral goes to zero in steady-state:

Lemma 4.2. (*A modification of Barbalat's lemma*[4])

Let f be a continuous, positive-definite function, and $\mathbf{x}$ be an absolutely continuous function in $\mathbb{R}$. If the following hold:

- $\|\mathbf{x}(\cdot)\| < \infty, \|\dot{\mathbf{x}}(\cdot)\| < \infty$
- $\lim\limits_{t\to\infty} \int\limits_0^t f\big(\mathbf{x}(s)\big) < \infty$

then $\lim\limits_{t\to\infty} \|\mathbf{x}(t)\| = 0$

Lemma (4.2) assures us that under these conditions for the error and its dynamics, which are fulfilled in our case, the error

$$\lim_{t\to\infty}\|\mathbf{e}_{0,i}(t)\| = 0 \Leftrightarrow$$
$$\lim_{t\to\infty}\Big\|\overline{\mathbf{e}}_i\Big(t;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\Big)\Big\| = 0,\ \forall t_k \in \mathbb{R}_{\geq 0}$$

which, given (14) and substituting for $\tau_1 = t$ while dropping the initial condition at $\tau_0 = t_k$, means that

$$\lim_{t\to\infty} \|\mathbf{e}_i(t)\| = 0$$

which implies that

$$\lim_{t\to\infty} \mathbf{e}_i(t) \in \mathcal{E}_{i,f}$$

Therefore, the closed-loop trajectory of the error state $\mathbf{e}_i$ converges to the terminal set $\mathcal{E}_{i,f}$ as $t \to \infty$.

■

5 Stabilization in the face of Disturbances

Remark 5.1. For the sake of modularity, separability, and direct comparison to the disturbance-free case, this chapter will assume that the previous disturbance-free analysis has not been given.

In this chapter we are interested in steering each agent $i \in \mathcal{V}$ into a *position* in 3D space, while conforming to the requirements of the problem; that is, all agents should avoid colliding with each other, all obstacles in the workspace, and the workspace boundary itself, while remaining in a non-singular configuration and sustaining the connectivity to their respective neighbours.

5.1 Formalizing the model

We begin by rewriting the system equations (1a), (1b) for a generic agent $i \in \mathcal{V}$ in state-space form:

$$\dot{\mathbf{x}}_i(t) = \boldsymbol{J}_i^{-1}(\mathbf{x}_i)\mathbf{v}_i(t)$$
$$\dot{\mathbf{v}}_i(t) = -\boldsymbol{M}_i^{-1}(\mathbf{x}_i)\boldsymbol{C}_i(\mathbf{x}_i, \dot{\mathbf{x}}_i)\mathbf{v}_i(t) - \boldsymbol{M}_i^{-1}(\mathbf{x}_i)\mathbf{g}_i(\mathbf{x}_i) + \boldsymbol{M}_i^{-1}(\mathbf{x}_i)\mathbf{u}_i(t)$$

where the inversion of $\boldsymbol{M}_i$ is possible due to it being positive-definite $\forall i \in \mathcal{V}$. Denoting by $\mathbf{z}_i(t)$

$$\mathbf{z}_i(t) = \begin{bmatrix} \mathbf{x}_i(t) \\ \mathbf{v}_i(t) \end{bmatrix}, \ \mathbf{z}_i(t) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^9 \times \mathbb{T}^3$$

and $\dot{\mathbf{x}}_i(t)$ and $\dot{\mathbf{v}}_i(t)$ by

$$\dot{\mathbf{x}}_i(t) = f_{i,x}(\mathbf{z}_i, \mathbf{u}_i)$$
$$\dot{\mathbf{v}}_i(t) = f_{i,v}(\mathbf{z}_i, \mathbf{u}_i)$$

we get the compact representation of the system's model

$$\dot{\mathbf{z}}_i(t) = \begin{bmatrix} f_{i,x}(\mathbf{z}_i, \mathbf{u}_i) \\ f_{i,v}(\mathbf{z}_i, \mathbf{u}_i) \end{bmatrix} = f_i\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big)$$

The state evolution of agent i is modeled by a system of non-linear continuous-time differential equations of the form

$$
\begin{aligned}
\dot{\mathbf{z}}_i(t) &= f_i\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big) \\
\mathbf{z}_i(0) &= \mathbf{z}_{i,0} \\
\mathbf{z}_i(t) &\subset \mathbb{R}^9 \times \mathbb{T}^3 \\
\mathbf{u}_i(t) &\subset \mathbb{R}^6
\end{aligned} \tag{24}
$$

where state $\mathbf{z}_i$ is directly measurable. It should be noted that equation (24) does not consider model-plant mismatches or external disturbances. We assume that the real system is subject to bounded additive disturbances $\boldsymbol{\delta}_i$ such that $\boldsymbol{\delta}_i \in \Delta_i \subset \mathbb{R}^9 \times \mathbb{T}^3$, where Δ_i is a compact set containing the origin. The real system is described by:

$$
\begin{aligned}
\dot{\mathbf{z}}_i(t) &= f_i^R\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big) \\
&= f_i\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big) + \boldsymbol{\delta}_i(t) \\
\mathbf{z}_i(0) &= \mathbf{z}_{i,0} \\
\mathbf{z}_i(t) &\subset \mathbb{R}^9 \times \mathbb{T}^3 \\
\mathbf{u}_i(t) &\subset \mathbb{R}^6 \\
\boldsymbol{\delta}_i(t) &\in \Delta_i \subset \mathbb{R}^9 \times \mathbb{T}^3,\ t \in \mathbb{R}_{\geq 0} \\
\sup_{t \in \mathbb{R}_{\geq 0}} \|\boldsymbol{\delta}_i(t)\| &\leq \overline{\delta}_i
\end{aligned}
$$

We define the set $\mathcal{Z}_i \subset \mathbb{R}^9 \times \mathbb{T}^3$ as the set that captures all the state constraints of the system's dynamics posed by the problem (3.1), for $t \in \mathbb{R}_{\geq 0}$. Therefore $\mathcal{Z}_i$ is such that:

$$
\begin{aligned}
\mathcal{Z}_i = \big\{\mathbf{z}_i(t) \in \mathbb{R}^9 \times \mathbb{T}^3 : \; & \|\mathbf{p}_i(t) - \mathbf{p}_j(t)\| > \underline{d}_{ij,a}, \forall j \in \mathcal{R}_i(t), \\
& \|\mathbf{p}_i(t) - \mathbf{p}_j(t)\| < d_i, \forall j \in \mathcal{N}_i, \\
& \|\mathbf{p}_i(t) - \mathbf{p}_\ell\| > \underline{d}_{i\ell,o}, \forall \ell \in \mathcal{L}, \\
& \|\mathbf{p}_W - \mathbf{p}_i(t)\| < \overline{d}_{i,W}, \\
& -\frac{\pi}{2} < \theta_i(t) < \frac{\pi}{2}, \\
& \forall t \in \mathbb{R}_{\geq 0}\big\}
\end{aligned}
$$

5.2 The error model

A feasible desired configuration $\mathbf{z}_{i,des} \in \mathbb{R}^9 \times \mathbb{T}^3$ is associated to each agent $i \in \mathcal{V}$, with the aim of agent i achieving it in steady-state: $\lim_{t\to\infty} \|\mathbf{z}_i(t) - \mathbf{z}_{i,des}\| = 0$. The interior of the norm of this expression denotes the state error of agent i:

$$\mathbf{e}_i(t) = \mathbf{z}_i(t) - \mathbf{z}_{i,des}, \;\; \mathbf{e}_i(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^9 \times \mathbb{T}^3$$

The error dynamics are denoted by $g_i^R(\mathbf{e}_i, \mathbf{u}_i)$:

$$\begin{aligned}
\dot{\mathbf{e}}_i(t) &= \dot{\mathbf{z}}_i(t) - \dot{\mathbf{z}}_{i,des} = \dot{\mathbf{z}}_i(t) = f_i^R\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big) = f_i\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big) + \boldsymbol{\delta}_i(t) \\
&= g_i\big(\mathbf{e}_i(t), \mathbf{u}_i(t)\big) + \boldsymbol{\delta}_i(t) \\
&= g_i^R\big(\mathbf{e}_i(t), \mathbf{u}_i(t)\big) \qquad (25)
\end{aligned}$$

with $\mathbf{e}_i(0) = \mathbf{z}_i(0) - \mathbf{z}_{i,des}$. In order to translate the constraints that are dictated for the state $\mathbf{z}_i(t)$ into constraints regarding the error state $\mathbf{e}_i(t)$, we define the set $\mathcal{E}_i \subset \mathbb{R}^9 \times \mathbb{T}^3$ as:

$$\mathcal{E}_i = \{\mathbf{e}_i(t) \in \mathbb{R}^9 \times \mathbb{T}^3 : \mathbf{e}_i(t) \in \mathcal{Z}_i \oplus (-\mathbf{z}_{i,des})\}$$

as the set that captures all constraints for the error dynamics (25) dictated by the problem (3.1).

If we design control laws $\mathbf{u}_i \in \mathcal{U}_i$, $\forall i \in \mathcal{V}$ such that the error signal $\mathbf{e}_i(t)$ with dynamics given in (25), constrained under $\mathbf{e}_i(t) \in \mathcal{E}_i$, satisfies $\lim_{t\to\infty} \|\mathbf{e}_i(t)\| = 0$, while all system related signals remain bounded in their respective regions,− if all of the above are achieved, then problem (3.1) has been solved.

In order to achieve this task, we employ a Nonlinear Receding Horizon scheme.

5.3 The optimization problem

Consider a sequence of sampling times $\{t_k\}_{k\geq 0}$, with a constant sampling time h, $0 < h < T_p$, where T_p is the finite time-horizon, such that $t_{k+1} = t_k + h$. In sampling data NMPC, a finite-horizon open-loop optimal control problem (OCP) is solved at discrete sampling time instants t_k based on the then-current state error measurement $\mathbf{e}_i(t_k)$. The solution is an optimal control signal $\overline{\mathbf{u}}_i^\star(t)$, computed over $t \in [t_k, t_k + T_p]$. This signal is applied to the open-loop system in between sampling times t_k and $t_k + h$.

At a generic time t_k, agent i solves the following optimization problem:

Problem 5.1.

Find

$$J_i^\star\big(\mathbf{e}_i(t_k)\big) \triangleq \min_{\overline{\mathbf{u}}_i(\cdot)} \ J_i\big(\mathbf{e}_i(t_k), \overline{\mathbf{u}}_i(\cdot)\big)$$

where

$$J_i\big(\mathbf{e}_i(t_k), \overline{\mathbf{u}}_i(\cdot)\big) \triangleq \int_{t_k}^{t_k+T_p} F_i\big(\overline{\mathbf{e}}_i(s), \overline{\mathbf{u}}_i(s)\big)ds + V_i\big(\overline{\mathbf{e}}_i(t_k+T_p)\big) \tag{26}$$

subject to:

$$\dot{\overline{\mathbf{e}}}_i(s) = g_i\big(\overline{\mathbf{e}}_i(s), \overline{\mathbf{u}}_i(s)\big),\ \overline{\mathbf{e}}_i(t_k) = \mathbf{e}_i(t_k) \tag{27}$$

$$\overline{\mathbf{u}}_i(s) \in \mathcal{U}_i,\ \overline{\mathbf{e}}_i(s) \in \mathcal{E}_i,\ s \in [t_k, t_k+T_p]$$

$$\overline{\mathbf{e}}_i(t_k+T_p) \in \mathcal{E}_{i,f} \subseteq \mathcal{E}_i$$

The notation $\bar{\cdot}$ is used to distinguish predicted states which are internal to the controller, as opposed to their actual values, because, even in the nominal case, the predicted values will not be equal to the actual closed-loop values. This means that $\overline{\mathbf{e}}_i(\cdot)$ is the solution to (27) driven by the control input $\overline{\mathbf{u}}_i(\cdot) : [t_k, t_k+T_p] \to \mathcal{U}_i$ with initial condition $\mathbf{e}_i(t_k)$.

The applied input signal is a portion of the optimal solution to an optimization problem where information on the states of the neighbouring agents of agent i are taken into account only in the constraints considered in the optimization problem. These constraints pertain to the set of its neighbours $\mathcal{N}_i$ and, in total, to the set of all agents within its sensing range $\mathcal{R}_i$. Regarding these, we make the following assumption:

Assumption 5.1. (*Access to Predicted Information from an Inter-agent Perspective*) Considering the context of Receding Horizon Control, when at time t_k agent i solves a finite horizon optimization problem, he has access to[a]

1. measurements of the states[b]
   - $\mathbf{z}_j(t_k)$ of all agents $j \in \mathcal{R}_i(t_k)$ within its sensing range at time t_k
   - $\mathbf{z}_{j'}(t_k)$ of all of its neighbouring agents $j' \in \mathcal{N}_i$ at time t_k
2. the *predicted states*
   - $\overline{\mathbf{z}}_j(\tau)$ of all agents $j \in \mathcal{R}_i(t_k)$ within its sensing range
   - $\overline{\mathbf{z}}_{j'}(\tau)$ of all of its neighbouring agents $j' \in \mathcal{N}_i$

across the entire horizon $\tau \in (t_k, t_k + T_p]$

[a] Although $\mathcal{N}_i \subseteq \mathcal{R}_i$, we make the distinction between the two because all agents $j \in \mathcal{R}_i$ need to avoid collision with agent i, but only agents $j' \in \mathcal{N}_i$ need to remain within the sensing range of agent i.

[b] as per assumption (**??**)

In other words, each time an agent solves its own individual optimization problem, he knows the error predictions that have been generated by the solution of the optimization problem of all agents within its range at that time, for the next T_p timesteps. This assumption is crucial to satisfying the constraints regarding collision aversion and connectivity maintenance between neighbouring agents. We assume that the above pieces of information are (a) always available and accurate, and (b) exchanged without delay. We encapsulate these pieces of information in four stacked vectors:

$$\mathbf{z}_{\mathcal{R}_i}(t_k) \triangleq col[\mathbf{z}_j(t_k)], \forall j \in \mathcal{R}_i(t_k)$$

$$\mathbf{z}_{\mathcal{N}_i}(t_k) \triangleq col[\mathbf{z}_j(t_k)], \forall j \in \mathcal{N}_i$$

$$\overline{\mathbf{z}}_{\mathcal{R}_i}(\tau) \triangleq col[\overline{\mathbf{z}}_j(\tau)], \forall j \in \mathcal{R}_i(t_k), \tau \in [t_k, t_k + T_p]$$

$$\overline{\mathbf{z}}_{\mathcal{N}_i}(\tau) \triangleq col[\overline{\mathbf{z}}_j(\tau)], \forall j \in \mathcal{N}_i, \tau \in [t_k, t_k + T_p]$$

Remark 5.2. The justification for this assumption is the following: considering that $\mathcal{N}_i \subseteq \mathcal{R}_i$, that the state vectors $\mathbf{z}_j$ are comprised of 12 real numbers that are encoded by 4 bytes, and that sampling occurs with a frequency f for all agents, the overall downstream bandwidth required by each agent is

$$BW_d = 12 \times 32 \text{ [bits]} \times |\mathcal{R}_i| \times \frac{T_p}{h} \times f \text{ [sec}^{-1}]$$

Given conservative constants $f = 100$ Hz, $\frac{T_p}{h} = 100$, the wireless protocol IEEE 802.11n-2009 (a standard for present-day devices) can accommodate up to

$$|\mathcal{R}_i| = \frac{600 \text{ [Mbit} \cdot \text{sec}^{-1}]}{12 \times 32 \text{[bit]} \times 10^4 \text{[sec}^{-1}]} \approx 16 \cdot 10^2 \text{ agents}$$

within the range of one agent. We deem this number to be large enough for practical applications for the approach of assuming access to the predicted states of agents within the range of one agent to be legal.

?? more on the actual $\mathcal{E}_i$

The functions $F_i : \mathcal{E}_i \times \mathcal{U}_i \rightarrow \mathbb{R}_{\geq 0}$ and $V_i : \mathcal{E}_{i,f} \rightarrow \mathbb{R}_{\geq 0}$ are defined as

$$F_i\big(\overline{\mathbf{e}}_i(t), \overline{\mathbf{u}}_i(t)\big) \triangleq \overline{\mathbf{e}}_i(t)^\top \boldsymbol{Q}_i \overline{\mathbf{e}}_i(t) + \overline{\mathbf{u}}_i(t)^\top \boldsymbol{R}_i \overline{\mathbf{u}}_i(t)$$

$$V_i\big(\overline{\mathbf{e}}_i(t)\big) \triangleq \overline{\mathbf{e}}_i(t)^\top \boldsymbol{P}_i \overline{\mathbf{e}}_i(t)$$

Matrices $\boldsymbol{R}_i \in \mathbb{R}^{6\times 6}$ are symmetric and positive definite, while matrices $\boldsymbol{Q}_i, \boldsymbol{P}_i \in \mathbb{R}^{12\times 12}$ are symmetric and positive semi-definite. The running costs F_i are upper- and lower-bounded:

$$\begin{aligned}
\lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\|\mathbf{e}_i(t)\|^2 &\leq \lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\left\| \begin{bmatrix} \mathbf{e}_i(t) \\ \mathbf{u}_i(t) \end{bmatrix} \right\|^2 \\
&\leq F_i\big(\mathbf{e}_i(t), \mathbf{u}_i(t)\big) \\
&\leq \lambda_{max}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\left\| \begin{bmatrix} \mathbf{e}_i(t) \\ \mathbf{u}_i(t) \end{bmatrix} \right\|^2 \leq \lambda_{max}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\|\mathbf{e}_i(t)\|^2
\end{aligned}$$

where $\lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i)$ is the smallest eigenvalue between those of matrices $\boldsymbol{Q}_i$ and $\boldsymbol{R}_i$, and $\lambda_{max}(\boldsymbol{Q}_i, \boldsymbol{R}_i)$ the largest. Since the terms $\lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\|\mathbf{e}_i(t)\|$ and $\lambda_{max}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\|\mathbf{e}_i(t)\|$ are themselves class $\mathcal{K}$ functions according to definition (**??**), F_i is lower- and upper-bounded by class $\mathcal{K}$ functions. As is obvious, $F_i(\mathbf{0}, \mathbf{0}) = 0$.

Before defining the terminal set $\mathcal{E}_{i,f}$ it is necessary to state the definition of a positively invariant set:

Definition 5.1. (*Positively Invariant Set*)
Consider a dynamical system $\dot{\mathbf{x}} = f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, and a trajectory $\mathbf{x}(t;\ \mathbf{x}_0)$, where $\mathbf{x}_0$ is the initial condition. The set $S = \{\mathbf{x} \in \mathbb{R}^n : \gamma(\mathbf{x}) = 0\}$, where γ is a valued function, is said to be *positively invariant* if the following holds:

$$\mathbf{x}_0 \in S \Rightarrow \mathbf{x}(t;\ \mathbf{x}_0) \in S,\ \ \forall t \geq t_0$$

Intuitively, this means that the set S is positively invariant if a trajectory of the system does not exit it once it enters it.

The terminal set $\mathcal{E}_{i,f} \subseteq \mathcal{E}_i$ is an admissible positively invariant set for system (**??**) such that

$$\mathcal{E}_{i,f} = \{\mathbf{e}_i \in \mathcal{E}_i : \|\mathbf{e}_i\| \leq \varepsilon_0\}$$

where ε_0 is an arbitrarily small but fixed positive real scalar.

With regard to the terminal penalty function V_i, the following lemma will prove to be useful in guaranteeing the convergence of the solution to the optimal control problem to the terminal region $\mathcal{E}_{i,f}$:

Lemma 5.1. (V_i is Lipschitz continuous in $\mathcal{E}_{i,f}$)

The terminal penalty function V_i is Lipschitz continuous in $\mathcal{E}_{i,f}$

$$|V(\mathbf{e}_{1,i}) - V(\mathbf{e}_{2,i})| \leq L_{V_i}\|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\|$$

where $\mathbf{e}_{1,i}, \mathbf{e}_{2,i} \in \mathcal{E}_{i,f}$, with Lipschitz constant $L_{V_i} = 2\varepsilon_0\lambda_{max}(P_i)$

Proof For every $\mathbf{e}_i \in \mathcal{E}_{i,f}$, it holds that

$$\begin{aligned}
|V(\mathbf{e}_{1,i}) - V(\mathbf{e}_{2,i})| &= |\mathbf{e}_{1,i}^\top \boldsymbol{P}_i \mathbf{e}_{1,i} - \mathbf{e}_{2,i}^\top \boldsymbol{P}_i \mathbf{e}_{2,i}| \\
&= |\mathbf{e}_{1,i}^\top \boldsymbol{P}_i \mathbf{e}_{1,i} - \mathbf{e}_{2,i}^\top \boldsymbol{P}_i \mathbf{e}_{2,i} \pm \mathbf{e}_{1,i}^\top \boldsymbol{P}_i \mathbf{e}_{2,i}| \\
&= |\mathbf{e}_{1,i}^\top \boldsymbol{P}_i (\mathbf{e}_{1,i} - \mathbf{e}_{2,i}) - \mathbf{e}_{2,i}^\top \boldsymbol{P}_i (\mathbf{e}_{1,i} - \mathbf{e}_{2,i})| \\
&\leq |\mathbf{e}_{1,i}^\top \boldsymbol{P}_i (\mathbf{e}_{1,i} - \mathbf{e}_{2,i})| + |\mathbf{e}_{2,i}^\top \boldsymbol{P}_i (\mathbf{e}_{1,i} - \mathbf{e}_{2,i})|
\end{aligned}$$

But for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$|\mathbf{x}^\top \boldsymbol{A} \mathbf{y}| \leq \lambda_{max}(A)\|\mathbf{x}\|\|\mathbf{y}\|$$

where $\lambda_{max}(A)$ denotes the largest eigenvalue of matrix $\boldsymbol{A}$. Hence:

$$\begin{aligned}
|V(\mathbf{e}_{1,i}) - V(\mathbf{e}_{2,i})| &\leq \lambda_{max}(\boldsymbol{P}_i)\|\mathbf{e}_{1,i}\|\|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\| + \lambda_{max}(\boldsymbol{P}_i)\|\mathbf{e}_{2,i}\|\|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\| \\
&= \lambda_{max}(\boldsymbol{P}_i)(\|\mathbf{e}_{1,i}\| + \|\mathbf{e}_{2,i}\|)\|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\| \\
&\leq \lambda_{max}(\boldsymbol{P}_i)(\varepsilon_0 + \varepsilon_0)\|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\| \\
&= 2\varepsilon_0\lambda_{max}(\boldsymbol{P}_i)\|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\|
\end{aligned}$$

■

The solution to the optimal control problem (26) at time t_k is an optimal control input, denoted by $\overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t_k))$, which is applied to the open-loop system until the next sampling instant $t_k + h$, with $h \in (0, T_p)$, at which time a new optimal control problem is solved in the same manner:

$$\mathbf{u}_i(t) = \overline{\mathbf{u}}_i^\star\big(t;\ \mathbf{e}_i(t_k)\big),\ t \in [t_k, t_k + h]$$

The control input $\mathbf{u}_i(\cdot)$ is of feedback form, since it is recalculated at each sampling instant based on the then-current state. The solution to equation (25), starting at time t_1, from an initial condition $\mathbf{e}_i(t_1)$, by application of the control input $\mathbf{u}_i : [t_1, t_2] \to \mathcal{U}_i$ is denoted by

$$\mathbf{e}_i\big(t;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(t_1)\big),\ t \in [t_1, t_2]$$

The *predicted* state of the system (25) at time $t_k + \tau$, based on the measurement of the state at time t_k, $\mathbf{e}_i(t_k)$, by application of the control input $\mathbf{u}_i\big(t;\ \mathbf{e}_i(t_k)\big)$, for the time period $t \in [t_k, t_k + \tau]$ is denoted by

$$\overline{\mathbf{e}}_i\big(t_k + \tau;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(t_k)\big)$$

In contrast to the disturbance-free case:

$$\overline{\mathbf{e}}_i\big(\tau_1;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(\tau_0)\big) \neq \mathbf{e}_i\big(\tau_1;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(\tau_0)\big)$$

holds true here because *there are* disturbances acting on the system.

The closed-loop system for which stability is to be guaranteed is

$$\mathbf{e}_i(\tau) = g_i^R\big(\mathbf{e}_i(\tau), \overline{\mathbf{u}}_i^\star(\tau)\big),\ \tau \geq t_0 = 0$$

where $\overline{\mathbf{u}}_i^\star(\tau) = \overline{\mathbf{u}}_i^\star(\tau;\ \mathbf{e}_i(t_k))$, $\tau \in [t_k, t_k + h)$.

We can now give the definition of an *admissible input*:

Definition 5.2. (Admissible input)

A control input $\mathbf{u}_i : [t_k, t_k + T_p] \to \mathbb{R}^6$ for a state $\mathbf{e}_i(t_k)$ is called *admissible* if all the following hold:

1. $\mathbf{u}_i(\cdot)$ is piecewise continuous
2. $\mathbf{u}_i(\tau) \in \mathcal{U}_i,\ \forall \tau \in [t_k, t_k + T_p]$
3. $\mathbf{e}_i\big(\tau;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(t_k)\big) \in \mathcal{E}_i,\ \forall \tau \in [t_k, t_k + T_p]$
4. $\mathbf{e}_i\big(t_k + T_p;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(t_k)\big) \in \mathcal{E}_{i,f}$

5.4 Feasibility and Convergence

Under these considerations, we can now state the theorem that relates to the guaranteeing of the stability of the compound system of agents $i \in \mathcal{V}$, when each of them is assigned a desired position which results in feasible displacements:

Theorem 5.1. Suppose that

1. the terminal region $\mathcal{E}_{i,f} \subseteq \mathcal{E}_i$ is closed with $\mathbf{0} \in \mathcal{E}_{i,f}$
2. a solution to the optimal control problem (10) is feasible at time $t = 0$, that is, assumptions (3.1), (3.2), and (3.3) hold at time $t = 0$
3. there exists an admissible control input $\mathbf{u}_{i,f} : [0, h] \to \mathcal{U}_i$ such that for all $\mathbf{e}_i \in \mathcal{E}_{i,f}$ and $\forall \tau \in [0, h]$:
   (a) $\mathbf{e}_i(\tau) \in \mathcal{E}_{i,f}$
   (b) $\dfrac{\partial V_i}{\partial \mathbf{e}_i} g_i\big(\mathbf{e}_i(\tau), \mathbf{u}_{i,f}(\tau)\big) + F_i\big(\mathbf{e}_i(\tau), \mathbf{u}_{i,f}(\tau)\big) \leq 0$

then the closed loop system (15) under the control input (13) converges to the set $\mathcal{E}_{i,f}$ when $t \to \infty$.

Proof. The proof of the above theorem consists of two parts: in the first, recursive feasibility is established, that is, initial feasibility is shown to imply subsequent feasibility; in the second, and based on the first part, it is shown that the error state $\mathbf{e}_i(t)$ converges to the terminal set $\mathcal{E}_{i,f}$.

Feasibility analysis Consider a sampling instant t_k for which a solution $\overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big)$ to (10) exists. Suppose now a time instant t_{k+1} such that[3] $t_k < t_{k+1} < t_k + T_p$, and consider that the optimal control signal calculated at t_k is comprised by the following two portions:

$$\overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big) = \begin{cases} \overline{\mathbf{u}}_i^\star\big(\tau_1;\ \mathbf{e}_i(t_k)\big), & \tau_1 \in [t_k, t_{k+1}] \\ \overline{\mathbf{u}}_i^\star\big(\tau_2;\ \mathbf{e}_i(t_k)\big), & \tau_2 \in [t_{k+1}, t_k + T_p] \end{cases} \tag{29}$$

Both portions are admissible since the calculated optimal control input is admissible, and hence they both conform to the input constraints. As for the resulting predicted states, they satisfy the state constraints, and, crucially: $\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \in \mathcal{E}_{i,f}$. Furthermore, according to assumption (3) of the theorem, there exists an admissible (and certainly not guaranteed optimal) input $\mathbf{u}_{i,f}$ that renders $\mathcal{E}_{i,f}$ invariant over $[t_k + T_p, t_k + T_p + h]$.

Given the above facts, we can construct an admissible input $\widetilde{\mathbf{u}}_i(\cdot)$ for time t_{k+1} by sewing together the second portion of (29) and the input $\mathbf{u}_{i,f}(\cdot)$:

[3] It is not strictly necessary that $t_{k+1} = t_k + h$ here, however it is necessary for the following that $t_{k+1} - t_k \leq h$

$$\widetilde{\mathbf{u}}_i(\tau) = \begin{cases} \overline{\mathbf{u}}_i^\star\big(\tau;\ \mathbf{e}_i(t_k)\big), & \tau \in [t_{k+1}, t_k + T_p] \\ \mathbf{u}_{i,f}(\tau - t_k - T_p), & \tau \in (t_k + T_p, t_{k+1} + T_p] \end{cases} \tag{30}$$

Applied at time t_{k+1}, $\widetilde{\mathbf{u}}_i(\cdot)$ is an admissible control input as a composition of admissible control inputs.

This means that feasibility of a solution to the optimization problem at time t_k implies feasibility at time $t_{k+1} > t_k$, and, thus, since at time $t = 0$ a solution is assumed to be feasible, a solution to the optimal control problem is feasible for all $t \geq 0$.

Convergence analysis The second part of the proof involves demonstrating the convergence of the state $\mathbf{e}_i$ to the terminal set $\mathcal{E}_{i,f}$. In order for this to be proved, it must be shown that a proper value function decreases along the solution trajectories starting at some initial time t_k. We consider the *optimal* cost $J_i^\star\big(\mathbf{e}_i(t)\big)$ as a candidate Lyapunov function:

$$J_i^\star\big(\mathbf{e}_i(t)\big) \triangleq J_i\Big(\mathbf{e}_i(t), \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t)\big)\Big)$$

and, in particular, our goal is to show that that this cost decreases over consecutive sampling instants $t_{k+1} = t_k + h$, i.e. $J_i^\star\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) \leq 0$.

In order not to wreak notational havoc, let us define the following terms:

- $\mathbf{u}_{0,i}(\tau) \triangleq \overline{\mathbf{u}}_i^\star\big(\tau;\ \mathbf{e}_i(t_k)\big)$ as the *optimal* input that results from the solution to problem (4.1) based on the measurement of state $\mathbf{e}_i(t_k)$, applied at time $\tau \geq t_k$
- $\mathbf{e}_{0,i}(\tau) \triangleq \overline{\mathbf{e}}_i\big(\tau;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big)$ as the *predicted* state at time $\tau \geq t_k$, that is, the state that results from the application of the above input $\overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big)$ to the state $\mathbf{e}_i(t_k)$, at time τ
- $\mathbf{u}_{1,i}(\tau) \triangleq \widetilde{\mathbf{u}}_i(\tau)$ as the *admissible* input at $\tau \geq t_{k+1}$ (see eq. (30))
- $\mathbf{e}_{1,i}(\tau) \triangleq \overline{\mathbf{e}}_i\big(\tau;\ \widetilde{\mathbf{u}}_i(\cdot), \mathbf{e}_i(t_{k+1})\big)$ as the *predicted* state at time $\tau \geq t_{k+1}$, that is, the state that results from the application of the above input $\widetilde{\mathbf{u}}_i(\cdot)$ to the state $\mathbf{e}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big)$, at time τ

Remark 5.3. Given that disturbances *are* present, for the predicted and actual states at time

$\tau_1 \geq \tau_0 \in \mathbb{R}_{\geq 0}$ it holds that:

$$\begin{aligned}
\mathbf{e}_i\big(\tau_1;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(\tau_0)\big) &= \mathbf{e}_i(\tau_0) + \int_{\tau_0}^{\tau_1} g_i^R\big(\mathbf{e}_i(s;\ \mathbf{e}_i(\tau_0)), \mathbf{u}_i(s)\big)ds \\
\overline{\mathbf{e}}_i\big(\tau_1;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(\tau_0)\big) &= \mathbf{e}_i(\tau_0) + \int_{\tau_0}^{\tau_1} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(\tau_0)), \mathbf{u}_i(s)\big)ds
\end{aligned}$$

The following proof of convergence to the terminal set relies heavily on the Grönwall-Bellman inequality. We state it here for reference purposes.

Lemma 5.2. [1] *Grönwall-Bellman Inequality*

Let $\lambda : [a, b] \to \mathbb{R}$ be continuous and $\mu : [a, b] \to \mathbb{R}$ be continuous and non-negative. If a continuous function $y : [a, b] \to \mathbb{R}$ satisfies

$$y(t) \leq \lambda(t) + \int_a^t \mu(s)y(s)ds$$

for $a \leq t \leq b$, then on the same interval

$$y(t) \leq \lambda(t) + \int_a^t \lambda(s)\mu(s)e^{\int_s^t \mu(\tau)d\tau}ds$$

In particular, if $\lambda(t) \equiv \lambda$ is a constant, then

$$y(t) \leq \lambda e^{\int_a^t \mu(\tau)d\tau}ds$$

If $\lambda(t) \equiv \lambda$ and $\mu(t) \equiv \mu$ are both constants, then

$$y(t) \leq \lambda e^{\mu(t-a)}$$

Before beginning to prove convergence, it is worth noting that while the cost

$$J_i\Big(\mathbf{e}_i(t), \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t)\big)\Big)$$

is optimal (in the sense that it is based on the optimal input, which provides its minimum realization), a cost that is based on a plainly admissible (and thus, without loss of generality, sub-optimal) input

$\mathbf{u}_i \neq \overline{\mathbf{u}}_i^\star$ will result in a configuration where

$$J_i\Big(\mathbf{e}_i(t), \mathbf{u}_i\big(\cdot;\ \mathbf{e}_i(t)\big)\Big) \geq J_i\Big(\mathbf{e}_i(t), \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t)\big)\Big)$$

Let us now begin our investigation on the sign of the difference between the cost that results from the application of the feasible input $\mathbf{u}_{1,i}$, which we shall denote by $\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big)$, and the optimal cost $J_i^\star\big(\mathbf{e}_i(t_k)\big)$, while reminding ourselves that $J_i\big(\mathbf{e}_i(t), \overline{\mathbf{u}}_i(\cdot)\big) = \int_t^{t+T_p} F_i\big(\overline{\mathbf{e}}_i(s), \overline{\mathbf{u}}_i(s)\big)ds + V_i\big(\overline{\mathbf{e}}_i(t+T_p)\big)$:

$$\begin{aligned}\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) = V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) + \int_{t_{k+1}}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds \\ -V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) - \int_{t_k}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds\end{aligned}$$

Considering that $t_k < t_{k+1} < t_k + T_p < t_{k+1} + T_p$, we break down the two integrals above in between these intervals:

$$\begin{aligned}&\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) = \\ &\quad V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) + \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds \\ &\quad -V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) - \int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds - \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds\end{aligned} \tag{31}$$

We begin working on (31) focusing first on the difference between the two intervals over $[t_{k+1}, t_{k+1} + T_p]$:

$$\begin{aligned}&\int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds - \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds \\ &\int_{t_k+h}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds - \int_{t_k+h}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds \\ &\leq \Big\| \int_{t_k+h}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds - \int_{t_k+h}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds \Big\| \text{ (for } x \leq |x|, x \in \mathbb{R}) \\ &= \Big\| \int_{t_k+h}^{t_k+T_p} \Big(F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) - F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)\Big)ds \Big\| \\ &= \int_{t_k+h}^{t_k+T_p} \Big\| F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) - F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) \Big\| ds \\ &\leq L_{F_i} \int_{t_k+h}^{t_k+T_p} \Big\| \overline{\mathbf{e}}_i\big(s;\ \mathbf{u}_{1,i}(\cdot), \mathbf{e}_i(t_k+h)\big) - \overline{\mathbf{e}}_i\big(s;\ \mathbf{u}_{0,i}(\cdot), \mathbf{e}_i(t_k)\big) \Big\| ds \text{ (for } F_i \text{ is Lipschitz continuous in } \mathcal{E}_i) \\ &= L_{F_i} \int_h^{T_p} \Big\| \overline{\mathbf{e}}_i\big(t_k+s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k+h)\big) - \overline{\mathbf{e}}_i\big(t_k+s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \Big\| ds\end{aligned} \tag{32}$$

Consulting with remark (5.3) for the two different initial conditions we get

$$\overline{\mathbf{e}}_i\big(t_k+s;\ \mathbf{u}_i^\star(\cdot), \mathbf{e}_i(t_k+h)\big) = \mathbf{e}_i(t_k+h) + \int_{t_k+h}^{t_k+s} g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k+h)), \mathbf{u}_i^\star(\tau)\big)d\tau$$

and

$$\begin{aligned}\overline{\mathbf{e}}_i\big(t_k+s;\ \mathbf{u}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) &= \mathbf{e}_i(t_k) + \int_{t_k}^{t_k+s} g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k)), \mathbf{u}_i^\star(\tau)\big)d\tau \\ &= \mathbf{e}_i(t_k) + \int_{t_k}^{t_k+h} g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k)), \mathbf{u}_i^\star(\tau)\big)d\tau \\ &\quad + \int_{t_k+h}^{t_k+s} g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k)), \mathbf{u}_i^\star(\tau)\big)d\tau\end{aligned}$$

Subtracting the latter from the former and taking norms on either side yields

$$\begin{aligned}&\Big\|\overline{\mathbf{e}}_i\big(t_k+s;\ \mathbf{u}_i^\star(\cdot), \mathbf{e}_i(t_k+h)\big) - \overline{\mathbf{e}}_i\big(t_k+s;\ \mathbf{u}_i^\star(\cdot), \mathbf{e}_i(t_k)\big)\Big\| \\ &= \Big\|\mathbf{e}_i(t_k+h) - \Big(\mathbf{e}_i(t_k) + \int_{t_k}^{t_k+h} g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k)), \mathbf{u}_i^\star(\tau)\big)d\tau\Big) \\ &\quad + \int_{t_k+h}^{t_k+s} g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k+h)), \mathbf{u}_i^\star(\tau)\big)d\tau - \int_{t_k+h}^{t_k+s} g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k)), \mathbf{u}_i^\star(\tau)\big)d\tau\Big\| \\ &= \Big\|\mathbf{e}_i(t_k+h) - \overline{\mathbf{e}}_i(t_k+h) + \int_{t_k+h}^{t_k+s}\Big(g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k+h)), \mathbf{u}_i^\star(\tau)\big) - g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k)), \mathbf{u}_i^\star(\tau)\big)\Big)d\tau\Big\| \\ &\leq \Big\|\mathbf{e}_i(t_k+h) - \overline{\mathbf{e}}_i(t_k+h)\Big\| + \Big\|\int_{t_k+h}^{t_k+s}\Big(g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k+h)), \mathbf{u}_i^\star(\tau)\big) - g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k)), \mathbf{u}_i^\star(\tau)\big)\Big)d\tau\Big\| \\ &= \Big\|\mathbf{e}_i(t_k+h) - \overline{\mathbf{e}}_i(t_k+h)\Big\| + \int_{t_k+h}^{t_k+s}\Big\|g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k+h)), \mathbf{u}_i^\star(\tau)\big) - g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k)), \mathbf{u}_i^\star(\tau)\big)\Big\|d\tau \\ &\leq \Big\|\mathbf{e}_i(t_k+h) - \overline{\mathbf{e}}_i(t_k+h)\Big\| + L_{g_i}\int_{t_k+h}^{t_k+s}\Big\|\overline{\mathbf{e}}_i\big(\tau;\ \mathbf{u}_i^\star(\cdot), \mathbf{e}_i(t_k+h)\big) - \overline{\mathbf{e}}_i\big(\tau;\ \mathbf{u}_i^\star(\cdot), \mathbf{e}_i(t_k)\big)\Big\|d\tau \\ &= \Big\|\mathbf{e}_i(t_k+h) - \overline{\mathbf{e}}_i(t_k+h)\Big\| + L_{g_i}\int_{h}^{s}\Big\|\overline{\mathbf{e}}_i\big(t_k+\tau;\ \mathbf{u}_i^\star(\cdot), \mathbf{e}_i(t_k+h)\big) - \overline{\mathbf{e}}_i\big(t_k+\tau;\ \mathbf{u}_i^\star(\cdot), \mathbf{e}_i(t_k)\big)\Big\|d\tau\end{aligned} \tag{33}$$

Before applying the Grönwall-Bellman inequality, we will compute the value of the first term in the above sum.

Since there are disturbances present, consulting remark (5.3) and substituting for $\tau_0 = t_k$ and $\tau_1 = t_{k+1}$ yields:

$$\begin{aligned}\mathbf{e}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) &= \mathbf{e}_i(t_k) + \int_{t_k}^{t_{k+1}} g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big)ds + \int_{t_k}^{t_{k+1}} \delta_i(s)ds \\ \overline{\mathbf{e}}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) &= \mathbf{e}_i(t_k) + \int_{t_k}^{t_{k+1}} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big)ds\end{aligned}$$

Subtracting the latter from the former and taking norms on either side yields:

$$\begin{aligned}
&\Big\|\mathbf{e}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t_k)),\mathbf{e}_i(t_k)\big)-\overline{\mathbf{e}}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t_k)),\mathbf{e}_i(t_k)\big)\Big\|\\
&=\Big\|\int_{t_k}^{t_{k+1}} g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t_k)),\overline{\mathbf{u}}_i^\star(s)\big)ds-\int_{t_k}^{t_{k+1}} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)),\overline{\mathbf{u}}_i^\star(s)\big)ds+\int_{t_k}^{t_{k+1}}\delta_i(s)ds\Big\|\\
&\le\Big\|\int_{t_k}^{t_{k+1}} g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t_k)),\overline{\mathbf{u}}_i^\star(s)\big)ds-\int_{t_k}^{t_{k+1}} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)),\overline{\mathbf{u}}_i^\star(s)\big)ds\Big\|+(t_{k+1}-t_k)\overline{\delta}_i\\
&=\int_{t_k}^{t_{k+1}}\Big\|g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t_k)),\overline{\mathbf{u}}_i^\star(s)\big)s-g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)),\overline{\mathbf{u}}_i^\star(s)\big)\Big\|ds+h\overline{\delta}_i\\
&\le L_{g_i}\int_{t_k}^{t_{k+1}}\Big\|\mathbf{e}_i\big(s;\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t_k)),\mathbf{e}_i(t_k)\big)-\overline{\mathbf{e}}_i\big(s;\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t_k)),\mathbf{e}_i(t_k)\big)\Big\|ds+h\overline{\delta}_i
\end{aligned}$$

since g_i is Lipschitz continuous in $\mathcal{E}_i$ with Lipschitz constant L_{g_i}. Reformulation yields

$$\begin{aligned}
&\Big\|\mathbf{e}_i\big(t_k+h;\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t_k)),\mathbf{e}_i(t_k)\big)-\overline{\mathbf{e}}_i\big(t_k+h;\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t_k)),\mathbf{e}_i(t_k)\big)\Big\|\\
&\le h\overline{\delta}_i+L_{g_i}\int_0^h\Big\|\mathbf{e}_i\big(t_k+s;\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t_k)),\mathbf{e}_i(t_k)\big)-\overline{\mathbf{e}}_i\big(t_k+s;\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t_k)),\mathbf{e}_i(t_k)\big)\Big\|ds
\end{aligned}$$

By applying the Grönwall-Bellman inequality we get:

$$\begin{aligned}
&\Big\|\mathbf{e}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t_k)),\mathbf{e}_i(t_k)\big)-\overline{\mathbf{e}}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t_k)),\mathbf{e}_i(t_k)\big)\Big\|\\
&\le h\overline{\delta}_i+L_{g_i}\int_0^h s\overline{\delta}_i e^{L_{g_i}(h-s)}ds\\
&=h\overline{\delta}_i-\overline{\delta}_i\int_0^h s\big(e^{L_{g_i}(h-s)}\big)'ds\\
&=h\overline{\delta}_i-\overline{\delta}_i\bigg(\big[se^{L_{g_i}(h-s)}\big]_0^h-\int_0^h e^{L_{g_i}(h-s)}ds\bigg)\\
&=h\overline{\delta}_i-\overline{\delta}_i\bigg(h+\frac{1}{L_{g_i}}(1-e^{L_{g_i}h})\bigg)\\
&=\frac{\overline{\delta}_i}{L_{g_i}}(e^{L_{g_i}h}-1)
\end{aligned}$$

Lemma 5.3. Suppose that the real system, which is under the existence of bounded additive disturbances, and the model are both at time t at state $\mathbf{e}_i(t)$. Applying at time t a control law $\mathbf{u}(\cdot)$ to the system model deemed "real" and its model will cause at time $t+\tau$, $\tau \geq 0$ a divergence between the states of the real system and its model. The norm of the difference between the state of the real system and the state of the model system is bounded by

$$\left\| \mathbf{e}_i\big(t+\tau;\ \mathbf{u}(\cdot), \mathbf{e}_i(t)\big) - \overline{\mathbf{e}}_i\big(t+\tau;\ \mathbf{u}(\cdot), \mathbf{e}_i(t)\big) \right\| \leq \frac{\overline{\delta}_i}{\mathrm{L}_{g_i}}(e^{L_{g_i}\tau} - 1)$$

where $\overline{\delta}_i$ is the upper bound of the disturbance, and L_{g_i} the Lipschitz constant of both models.

Given that from lemma (5.3) the first term of the sum featured in (33) is a constant, by application of the the Grönwall-Bellman inequality, (33) becomes:

$$\begin{aligned}\left\| \overline{\mathbf{e}}_i\big(t_k+s;\ \mathbf{u}_i^\star(\cdot), \mathbf{e}_i(t_k+h)\big) - \overline{\mathbf{e}}_i\big(t_k+s;\ \mathbf{u}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \right\| \\ \leq \left\| \mathbf{e}_i(t_k+h) - \overline{\mathbf{e}}_i(t_k+h) \right\| e^{L_{g_i}(s-h)} \\ \leq \frac{\overline{\delta}_i}{\mathrm{L}_{g_i}}(e^{L_{g_i}h} - 1)e^{L_{g_i}(s-h)}\end{aligned}$$

Given the above result, (32) becomes

$$\begin{aligned}&\int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds - \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds \\ &\leq L_{F_i} \int_h^{T_p} \frac{\overline{\delta}_i}{\mathrm{L}_{g_i}}(e^{L_{g_i}h} - 1)e^{L_{g_i}(s-h)}ds \\ &= L_{F_i}\frac{\overline{\delta}_i}{\mathrm{L}_{g_i}}(e^{L_{g_i}h} - 1)\int_h^{T_p} e^{L_{g_i}(s-h)}ds \\ &= L_{F_i}\frac{\overline{\delta}_i}{\mathrm{L}_{g_i}}(e^{L_{g_i}h} - 1)\frac{1}{L_{g_i}}(e^{L_{g_i}(T_p-h)} - 1) \\ &= L_{F_i}\frac{\overline{\delta}_i}{\mathrm{L}_{g_i}^2}(e^{L_{g_i}h} - 1)(e^{L_{g_i}(T_p-h)} - 1)\end{aligned}$$

Hence we discovered that

$$\begin{aligned}\int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds - \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds \\ \leq L_{F_i}\frac{\overline{\delta}_i}{\mathrm{L}_{g_i}^2}(e^{L_{g_i}h} - 1)(e^{L_{g_i}(T_p-h)} - 1)\end{aligned} \tag{34}$$

With this partial result established, we turn back to the remaining terms found in (31) and, in particular, we focus on the integral

$$\int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds$$

We discern that the range of the above integral has a length[a] equal to the length of the interval where assumption (3b) (??) of theorem (5.1) holds. Integrating the expression found in the assumption over the interval $[t_k + T_p, t_{k+1} + T_p]$, for the controls and states applicable in it we get

$$\int_{t_k+T_p}^{t_{k+1}+T_p} \left(\frac{\partial V_i}{\partial \mathbf{e}_{1,i}} g_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) + F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) \right) ds \leq 0$$

$$\int_{t_k+T_p}^{t_{k+1}+T_p} \frac{d}{ds} V_i\big(\mathbf{e}_{1,i}(s)\big) ds + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds \leq 0$$

$$V_i\big(\mathbf{e}_{1,i}(t_{k+1} + T_p)\big) - V_i\big(\mathbf{e}_{1,i}(t_k + T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds \leq 0$$

$$V_i\big(\mathbf{e}_{1,i}(t_{k+1} + T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds \leq V_i\big(\mathbf{e}_{1,i}(t_k + T_p)\big)$$

The left-hand side expression is the same as the first two terms in the right-hand side of equality (31). We can introduce the third one by subtracting it from both sides:

$$\begin{aligned}
&V_i\big(\mathbf{e}_{1,i}(t_{k+1} + T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds - V_i\big(\mathbf{e}_{0,i}(t_k + T_p)\big) \\
&\leq V_i\big(\mathbf{e}_{1,i}(t_k + T_p)\big) - V_i\big(\mathbf{e}_{0,i}(t_k + T_p)\big) \\
&\leq \Big\| V_i\big(\mathbf{e}_{1,i}(t_k + T_p)\big) - V_i\big(\mathbf{e}_{0,i}(t_k + T_p)\big) \Big\|, \text{ for } x \leq |x|, \forall x \in \mathbb{R} \\
&\leq L_{V_i} \Big\| \overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^{\star}(\cdot), \mathbf{e}_i(t_{k+1})\big) - \overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^{\star}(\cdot), \mathbf{e}_i(t_k)\big) \Big\|, \text{ from lemma (4.1)}
\end{aligned} \tag{35}$$

Consulting with remark (5.3) we get that the two terms interior to the norm are respectively equal to

$$\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big) = \mathbf{e}_i(t_{k+1}) + \int_{t_{k+1}}^{t_k+T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_{k+1})), \overline{\mathbf{u}}_i^\star(s)\big) ds$$

and

$$\begin{aligned}
\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) &= \mathbf{e}_i(t_k) \quad + \int_{t_k}^{t_k+T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big) ds \\
&= \mathbf{e}_i(t_k) \quad + \int_{t_k}^{t_{k+1}} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big) ds \\
&\qquad\qquad + \int_{t_{k+1}}^{t_k+T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big) ds \\
&= \overline{\mathbf{e}}_i(t_{k+1}) + \int_{t_{k+1}}^{t_k+T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big) ds
\end{aligned}$$

Subtracting the latter from the former and taking norms on either side we get

$$\begin{aligned}
&\Big\| \overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big) - \overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \Big\| \\
&= \Big\| \mathbf{e}_i(t_{k+1}) - \overline{\mathbf{e}}_i(t_{k+1}) \\
&+ \int_{t_{k+1}}^{t_k+T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_{k+1})), \overline{\mathbf{u}}_i^\star(s)\big) ds - \int_{t_{k+1}}^{t_k+T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big) ds \Big\| \\
&\leq \Big\| \mathbf{e}_i(t_{k+1}) - \overline{\mathbf{e}}_i(t_{k+1}) \Big\| \\
&+ \Big\| \int_{t_{k+1}}^{t_k+T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_{k+1})), \overline{\mathbf{u}}_i^\star(s)\big) ds - \int_{t_{k+1}}^{t_k+T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big) ds \Big\| \\
&= \Big\| \mathbf{e}_i(t_{k+1}) - \overline{\mathbf{e}}_i(t_{k+1}) \Big\| \\
&+ \Big\| \int_{t_{k+1}}^{t_k+T_p} \Big(g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_{k+1})), \overline{\mathbf{u}}_i^\star(s)\big) - g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big) \Big) ds \Big\| \\
&= \Big\| \mathbf{e}_i(t_{k+1}) - \overline{\mathbf{e}}_i(t_{k+1}) \Big\| \\
&+ \int_{t_{k+1}}^{t_k+T_p} \Big\| g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_{k+1})), \overline{\mathbf{u}}_i^\star(s)\big) - g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big) \Big\| ds \\
&\leq \Big\| \mathbf{e}_i(t_{k+1}) - \overline{\mathbf{e}}_i(t_{k+1}) \Big\| \\
&+ L_{g_i} \int_{t_{k+1}}^{t_k+T_p} \Big\| \overline{\mathbf{e}}_i\big(s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big) - \overline{\mathbf{e}}_i\big(s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \Big\| ds, \text{ for } g_i \text{ is Lipschitz continuous in } \mathcal{E}_i \\
&= \Big\| \mathbf{e}_i(t_{k+1}) - \overline{\mathbf{e}}_i(t_{k+1}) \Big\| \\
&+ L_{g_i} \int_{h}^{T_p} \Big\| \overline{\mathbf{e}}_i\big(t_k + s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big) - \overline{\mathbf{e}}_i\big(t_k + s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \Big\| ds
\end{aligned}$$

By applying the Grönwall-Bellman inequality we get

$$\Big\|\overline{\mathbf{e}}_i\big(t_k+T_p;\ \overline{\mathbf{u}}_i^\star(\cdot),\mathbf{e}_i(t_{k+1})\big)-\overline{\mathbf{e}}_i\big(t_k+T_p;\ \overline{\mathbf{u}}_i^\star(\cdot),\mathbf{e}_i(t_k)\big)\Big\| \leq \Big\|\mathbf{e}_i(t_{k+1})-\overline{\mathbf{e}}_i(t_{k+1})\Big\| e^{L_{g_i}(T_p-h)}$$

By applying lemma (5.3) for $\tau = h$ we get

$$\Big\|\overline{\mathbf{e}}_i\big(t_k+T_p;\ \overline{\mathbf{u}}_i^\star(\cdot),\mathbf{e}_i(t_{k+1})\big)-\overline{\mathbf{e}}_i\big(t_k+T_p;\ \overline{\mathbf{u}}_i^\star(\cdot),\mathbf{e}_i(t_k)\big)\Big\| \leq \frac{\overline{\delta}_i}{L_{g_i}}(e^{L_{g_i}h}-1)e^{L_{g_i}(T_p-h)}$$

[a] $(t_{k+1}+T_p)-(t_k+T_p)=t_{k+1}-t_k=h$

With the above result established, we substitute for the norm in (35), getting

$$\boxed{\begin{aligned} V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big)+\int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s),\mathbf{u}_{1,i}(s)\big)ds - V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big)& \\ \leq L_{V_i}\frac{\overline{\delta}_i}{L_{g_i}}(e^{L_{g_i}h}-1)e^{L_{g_i}(T_p-h)}& \qquad (36)\end{aligned}}$$

Adding the milestone inequalities (34) and (36) yields

$$\begin{aligned}
&\int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s),\mathbf{u}_{1,i}(s)\big)ds - \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s),\mathbf{u}_{0,i}(s)\big)ds \\
&+ V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big)+\int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s),\mathbf{u}_{1,i}(s)\big)ds - V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) \\
&\leq L_{F_i}\frac{\overline{\delta}_i}{\text{Ł}_{g_i}^2}(e^{L_{g_i}h}-1)(e^{L_{g_i}(T_p-h)}-1) + L_{V_i}\frac{\overline{\delta}_i}{L_{g_i}}(e^{L_{g_i}h}-1)e^{L_{g_i}(T_p-h)}
\end{aligned}$$

and therefore (31), by bringing the integral ranging from t_k to t_{k+1} to the left-hand side, becomes

$$\begin{aligned}
\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) + \int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s),\mathbf{u}_{0,i}(s)\big)ds& \\
\leq L_{F_i}\frac{\overline{\delta}_i}{\text{Ł}_{g_i}^2}(e^{L_{g_i}h}-1)(e^{L_{g_i}(T_p-h)}-1) + L_{V_i}\frac{\overline{\delta}_i}{L_{g_i}}(e^{L_{g_i}h}-1)e^{L_{g_i}(T_p-h)}&
\end{aligned}$$

By rearranging terms, the cost difference becomes bounded by

$$\begin{aligned}
&\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) \\
&\qquad \leq -\int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s),\mathbf{u}_{0,i}(s)\big)ds + \frac{\overline{\delta}_i}{L_{g_i}}\Big(e^{L_{g_i}h}-1\Big)\Big(\big(L_{V_i}+\frac{L_{F_i}}{\text{Ł}_{g_i}}\big)e^{L_{g_i}(T_p-h)} - \frac{L_{F_i}}{\text{Ł}_{g_i}}\Big) \\
&\qquad = \xi_i - \int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s),\mathbf{u}_{0,i}(s)\big)ds
\end{aligned}$$

where

$$\xi_i = \frac{\overline{\delta}_i}{L_{g_i}}\Big(e^{L_{g_i}h} - 1\Big)\Big(\big(L_{V_i} + \frac{L_{F_i}}{L_{g_i}}\big)e^{L_{g_i}(T_p - h)} - \frac{L_{F_i}}{L_{g_i}}\Big)$$

is the contribution of the bounded additive disturbance $\overline{\delta}_i(t)$ to the nominal cost difference (the case without disturbances).

F_i is a positive-definite function as a sum of a positive-definite $\|\mathbf{u}_i\|^2_{\boldsymbol{R}_i}$ and a positive semi-definite function $\|\mathbf{e}_i\|^2_{\boldsymbol{Q}_i}$. If we denote by $m_i = \lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i) \geq 0$ the minimum eigenvalue between those of matrices $\boldsymbol{R}_i$, $\boldsymbol{Q}_i$, this means that

$$F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) \geq m_i \|\mathbf{e}_{0,i}(s)\|^2$$

By integrating the above between our interval of interest $[t_k, t_{k+1}]$ we get

$$\int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) \geq \int_{t_k}^{t_{k+1}} m_i \|\mathbf{e}_{0,i}(s)\|^2 ds$$

or

$$-\int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) \leq -m_i \int_{t_k}^{t_{k+1}} \|\overline{\mathbf{e}}_i(s;\ \overline{\mathbf{u}}_i^\star, \mathbf{e}_i(t_k))\|^2 ds$$

This means that the cost difference is upper-bounded by

$$\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) \leq \xi_i - m_i \int_{t_k}^{t_{k+1}} \|\overline{\mathbf{e}}_i(s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k))\|^2 ds$$

and since the cost $\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big)$ is, in general, sub-optimal: $J_i^\star\big(\mathbf{e}_i(t_{k+1})\big) - \overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) \leq 0$:

$$J_i^\star\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) \leq \xi_i - m_i \int_{t_k}^{t_{k+1}} \|\overline{\mathbf{e}}_i(s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k))\|^2 ds$$

It is easy to verify that both right-hand side terms are class $\mathcal{K}_\infty$ functions, and therefore, according to definition (2.4) and theorem (2.1), the closed-loop system is Input to State Stable. Inevitably then, the closed-loop trajectories converge to the terminal set $\mathcal{E}_{i,f}$.

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