Robust Decentralized Control of Inter-constrained Continuous Nonlinear Systems

A Receding Horizon Approach

ALEXANDROS FILOTHEOU

KTH Electrical Engineering

Master's Degree Project

Stockholm, Sweden February 2006

TRITA-EE 2006:666

Contents

0.1 Introduction 3

I The problem **5**

1 Notation and Preliminaries **7**

1.1 Notation 7

1.2 Auxiliary Prerequisites 8

1.3 Model Predictive Control for non-linear continuous-time systems 12

2 Problem Formulation **13**

2.1 System Model 13

2.2 Initial Conditions 17

2.3 Objective 18

2.4 Problem Statement 21

II Advocated Solutions **25**

3 Disturbance-free Stabililization **27**

3.1 Formalizing the system's model . . . 28
3.2 The error model . . . 29
3.3 The optimization problem . . . 30
3.4 Stabilization: Feasibility and Convergence . . . 37

4 Stabilization in the face of Disturbances **49**

4.1 The perturbed model . . . 50
4.2 The error model . . . 51
4.3 The optimization problem . . . 52
4.4 Stabilization: Feasibility and Convergence . . . 57

Appendices **77**

A Proofs of lemmas **79**

A.1 Proof of lemma 3.3.1 . . . 79
A.2 Proof of lemma 3.3.2 . . . 80
A.3 Proof of lemma 3.3.3 . . . 81
A.4 Proof of lemma 3.3.4 . . . 81
A.5 Proof of property 4.3.1 . . . 82
A.6 Proof of lemma 4.4.1 . . . 83

0.1 Introduction

formation of multi-agent systems, mpc intro etc.

motivation why we need mpc controllers...

In many control problems it is desired to design a stabilizing feedback such that a performance criterion is minimized while satisfying constraints on the controls and the states. Ideally one would look for a closed solution for the feedback law satisfying the constraints while optimizing the performance. However, typically the optimal feedback law cannot be found analytically, even in the unconstrained case, since it involves the solution of the corresponding Hamilton-Jacobi-Bellman partial differential equations. One approach to circumvent this problem is the repeated solution of an open-loop optimal control problem for a given state. The first part of the resulting open-loop input signal is implemented and the whole process is repeated. Control approaches using this strategy are referred to as Model Predictive Control (MPC).

Part I

The problem

1

Notation and Preliminaries

1.1 Notation

The set of positive integers is denoted by $\mathbb{N}$. The real n-coordinate space, $n \in \mathbb{N}$, is denoted by $\mathbb{R}^n$; $\mathbb{R}^n_{\geq 0}$ and $\mathbb{R}^n_{>0}$ are the sets of real n-vectors with all elements nonnegative and positive, respectively. Given a set S, we denote by $|S|$ its cardinality. The notation $\|\mathbf{x}\|$ is used for the Euclidean norm of a vector $\mathbf{x} \in \mathbb{R}^n$. Given matrix $\boldsymbol{A}$, $\lambda_{\min}(\boldsymbol{A})$ and $\lambda_{\max}(\boldsymbol{A})$ denote the minimum and maximum eigenvalues of $\boldsymbol{A}$, respectively. Its minimum and maximum singular values are denoted by $\sigma_{\min}(\boldsymbol{A})$ and $\sigma_{\max}(\boldsymbol{A})$ respectively. Given two sets A and B, the operation $A \oplus B$ denotes the Minkowski addition, defined by $A \oplus B = \{\mathbf{a}+\mathbf{b},\ \mathbf{a} \in A, \mathbf{b} \in B\}$. Similarly, the Minkowski – or Pontryagin difference is defined by $A \ominus B = \{\mathbf{a} - \mathbf{b},\ \mathbf{a} \in A, \mathbf{b} \in B\}$. $\boldsymbol{I_n} \in \mathbb{R}^{n \times n}$ and $\boldsymbol{0}_{m \times n} \in \mathbb{R}^{m \times n}$ are the unit matrix and the $m \times n$ matrix with all entries zeros respectively. A matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is called skew-symmetric if and only if $\boldsymbol{A}^\top = -\boldsymbol{A}$. The notation $\mathcal{B}(\mathbf{c}, r) \triangleq \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x} - \mathbf{c}\| \leq r\}$ is reserved for the 3D sphere of radius $r \in \mathbb{R}_{\geq 0}$ and center located at $\mathbf{c} \in \mathbb{R}^3$.

The vector expressing the coordinates of the origin of frame $\{j\}$ in frame $\{i\}$ is denoted by $\mathbf{p}_{j\triangleright i}$. When this vector is expressed in 3D space in a third frame, frame $\{k\}$, it is denoted by $\mathbf{p}^k_{j\triangleright i}$. The angular velocity of frame $\{j\}$ with respect to frame $\{i\}$, expressed in frame $\{k\}$ coordinates, is denoted by $\boldsymbol{\omega}^k_{j\triangleright i} \in \mathbb{R}^3$. We further denote by $\mathbf{q}_{j\triangleright i} \in \mathbb{T}^3$ the Euler angles representing the orientation of frame $\{j\}$ with respect to frame $\{i\}$, where $\mathbb{T}^3$ is the 3D torus. We also use the notation $\mathbb{M} = \mathbb{R}^3 \times \mathbb{T}^3$. For notational brevity, when a coordinate frame corresponds to the inertial frame of reference $\{\mathcal{O}\}$, we will omit its explicit notation (e.g., $\mathbf{p}_i = \mathbf{p}_{i\triangleright\mathcal{O}} = \mathbf{p}^{\mathcal{O}}_{i\triangleright\mathcal{O}}$, and $\boldsymbol{\omega}_i = \boldsymbol{\omega}_{i\triangleright\mathcal{O}} = \boldsymbol{\omega}^{\mathcal{O}}_{i\triangleright\mathcal{O}}$).

1.2 Auxiliary Prerequisites

This section features auxiliary and useful theorems, lemmas and definitions needed to support the advocated solutions in part II.

Lemma 1.2.1. [1] *The Grönwall-Bellman Inequality*

Let $\lambda : [a, b] \to \mathbb{R}$ be continuous and $\mu : [a, b] \to \mathbb{R}$ be continuous and non-negative. If a continuous function $y : [a, b] \to \mathbb{R}$ satisfies

$$y(t) \leq \lambda(t) + \int_a^t \mu(s)y(s)ds$$

for $a \leq t \leq b$, then on the same interval

$$y(t) \leq \lambda(t) + \int_a^t \lambda(s)\mu(s)e^{\int_s^t \mu(\tau)d\tau}ds$$

In particular, if $\lambda(t) \equiv \lambda$ is a constant, then

$$y(t) \leq \lambda e^{\int_a^t \mu(\tau)d\tau}ds$$

If $\lambda(t) \equiv \lambda$ and $\mu(t) \equiv \mu$ are both constants, then

$$y(t) \leq \lambda e^{\mu(t-a)}$$

Definition 1.2.1. [1] (*Class $\mathcal{K}$ function*)

A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if

1. it is strictly increasing
2. $\alpha(0) = 0$

If $a = \infty$ and $\lim\limits_{r \to \infty} \alpha(r) = \infty$, then function α is said to belong to class $\mathcal{K}_\infty$

Definition 1.2.2. [1] (*Class $\mathcal{KL}$ function*)

A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if

1. for a fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r
2. for a fixed r, the mapping $\beta(r, s)$ decreases with respect to s
3. $\lim\limits_{s \to \infty} \beta(r, s) = 0$

Lemma 1.2.2. [2] (*A modification of Barbalat's lemma*)

Let f be a continuous, positive-definite function, and $\mathbf{x}$ be an absolutely continuous function in $\mathbb{R}$. If the following hold:

- $\|\mathbf{x}(\cdot)\| < \infty$
- $\|\dot{\mathbf{x}}(\cdot)\| < \infty$
- $\lim\limits_{t \to \infty} \int\limits_0^t f\big(\mathbf{x}(s)\big) ds < \infty$

then $\lim_{t\to\infty} \|\mathbf{x}(t)\| = 0$

Definition 1.2.3. [3] (*Input-to-State Stability*)

A nonlinear system $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$, $\mathbf{x} \in X$, $\mathbf{u} \in U$ with initial condition $\mathbf{x}(t_0)$ is said to be *locally Input-to-State Stable (ISS)* if there exist functions $\sigma \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ and constants $k_1, k_2 \in \mathbb{R}_{>0}$ such that

$$\|\mathbf{x}(t)\| \leq \beta\big(\|\mathbf{x}(t_0)\|, t\big) + \sigma\big(\|\mathbf{u}\|_\infty\big),\ \forall t \geq 0$$

for all $\mathbf{x}(t_0) \in X$ and $\mathbf{u} \in U$ satisfying $\|\mathbf{x}(t_0)\| \leq k_1$ and $\sup_{t>0} \|\mathbf{u}(t)\| = \|\mathbf{u}\|_\infty \leq k_2$.

Remark 1.2.1. [4] A nonlinear system $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$, $\mathbf{x} \in X$, $\mathbf{u} \in U$ which is input-to-output stable, is asymptotically stable in the absence of disturbances $\mathbf{u}$, or if the disturbance is decaying. If the disturbance is merely bounded, then the evolution of the system is *ultimately bounded* in a set whose size depends on the bound of the disturbance.

Definition 1.2.4. [3] (*ISS Lyapunov function*)

A continuous function $V(\mathbf{x}) : \Psi \to \mathbb{R}_{\geq 0}$ for the nonlinear system $\dot{\mathbf{x}} = f(\mathbf{x}, \boldsymbol{\delta})$ is said to be a *ISS Lyapunov function* in Ψ if there are class $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3$, and a class $\mathcal{K}$ function σ such that

$$\alpha_1\big(\|\mathbf{x}\|\big) \leq V\big(\mathbf{x}\big) \leq \alpha_2\big(\|\mathbf{x}\|\big),\ \forall \mathbf{x} \in \Psi$$

and

$$\frac{d}{dt}V\big(\mathbf{x}\big) \leq \sigma\big(\|\boldsymbol{\delta}\|\big) - \alpha_3\big(\|\mathbf{x}\|\big),\ \forall \mathbf{x} \in \Psi, \boldsymbol{\delta} \in \Delta$$

Remark 1.2.2. With regard to definition (1.2.4), the statement

$$\frac{d}{dt}V\big(\mathbf{x}\big) \leq \sigma\big(\|\boldsymbol{\delta}\|\big) - \alpha_3\big(\|\mathbf{x}\|\big),\ \forall \mathbf{x} \in \Psi, \boldsymbol{\delta} \in \Delta$$

is equivalent to

$$V\big(\mathbf{x}(t_1)\big) - V\big(\mathbf{x}(t_0)\big) \leq \int_{t_0}^{t_1} \Big(\sigma\big(\|\boldsymbol{\delta}(t)\|\big) - \alpha_3\big(\|\mathbf{x}(t)\|\big)\Big)dt,\ \forall \mathbf{x} \in \Psi,\ \boldsymbol{\delta} \in \Delta,\ t \in [t_0, t_1]$$

Theorem 1.2.1. [3]

A nonlinear system $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$ is said to be *Input-to-State Stable* in Ψ if and only if it admits an ISS Lyapunov function in Ψ.

Definition 1.2.5. (*Positively Invariant Set*)

Consider a dynamical system $\dot{\mathbf{x}} = f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, and a trajectory $\mathbf{x}(t;\ \mathbf{x}_0)$, where $\mathbf{x}_0$ is the initial condition. The set $S = \{\mathbf{x} \in \mathbb{R}^n : \gamma(\mathbf{x}) = 0\}$, where γ is a valued function, is said to be *positively invariant* if the following holds:

$$\mathbf{x}_0 \in S \Rightarrow \mathbf{x}(t;\ \mathbf{x}_0) \in S,\ \forall t \geq t_0$$

Intuitively, this means that the set S is positively invariant if a trajectory of the system does not exit it once it enters it.

Definition 1.2.6. [5] (*Robust positively-invariant set*)

A set $\Psi \in \mathbb{R}^n$ is a robust positively invariant set for the nonlinear system $\dot{\mathbf{x}} = f(\mathbf{x}, \boldsymbol{\delta})$ if $f(\mathbf{x}, \boldsymbol{\delta}) \in \Psi$, for all $\mathbf{x} \in \Psi$ and for all $\boldsymbol{\delta} \in \Delta$.

1.3 Model Predictive Control for non-linear continuous-time systems

2

Problem Formulation

2.1 System Model

Consider a set $\mathcal{V}$ of N rigid bodies, $\mathcal{V} = \{1, 2, \ldots, N\}$, $|\mathcal{V}| = N \geq 2$, operating in a workspace $W \subseteq \mathbb{R}^3$. A coordinate frame $\{i\}, i \in \mathcal{V}$ is attached to the center of mass of each body. The workspace is assumed to be modeled as a bounded sphere $\mathcal{B}\big(\mathbf{p}_W, r_W\big)$ expressed in an inertial frame $\{\mathcal{O}\}$.

We consider that over time t each agent $i \in \mathcal{V}$ occupies the space of a sphere $\mathcal{B}\big(\mathbf{p}_i(t), r_i\big)$, where $\mathbf{p}_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^3$ is the position of the agent's center of mass, and $r_i < r_W$ is the radius of the agent's body. We denote by $\mathbf{q}_i(t) : \mathbb{R}_{\geq 0} \to \mathbb{T}^3$, the Euler angles representing the agents' orientation with respect to the inertial frame $\{\mathcal{O}\}$, with $\mathbf{q}_i \triangleq [\phi_i, \theta_i, \psi_i]^\top$, where $\phi_i, \psi_i \in [-\pi, \pi]$ and $\theta_i \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. We define

$$\mathbf{x}_i(t) \triangleq [\mathbf{p}_i(t)^\top, \mathbf{q}_i(t)^\top]^\top, \mathbf{x}_i(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^3 \times \mathbb{T}^3 \equiv \mathbb{M}$$

$$\mathbf{v}_i(t) \triangleq [\dot{\mathbf{p}}_i(t)^\top, \boldsymbol{\omega}_i(t)^\top]^\top, \mathbf{v}_i(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^3 \times \mathbb{R}^3 \equiv \mathbb{R}^6$$

and model the motion of agent i under continuous second order dynamics:

$$\dot{\mathbf{x}}_i(t) = \boldsymbol{J}_i^{-1}(\mathbf{x}_i)\mathbf{v}_i(t), \tag{2.1a}$$

$$\mathbf{u}_i(t) = \boldsymbol{M}_i(\mathbf{x}_i)\dot{\mathbf{v}}_i(t) + \boldsymbol{C}_i(\mathbf{x}_i, \dot{\mathbf{x}}_i)\mathbf{v}_i(t) + \mathbf{g}_i(\mathbf{x}_i) \tag{2.1b}$$

In equation (2.1a), $\boldsymbol{J}_i : \mathbb{T}^3 \to \mathbb{R}^{6\times 6}$ is a Jacobian matrix that maps the non-orthogonal Euler angle rates to the orthogonal angular velocities $\mathbf{v}_i$:

$$\boldsymbol{J}_i(\mathbf{x}_i) = \begin{bmatrix} \boldsymbol{I}_3 & \boldsymbol{0}_{3\times 3} \\ \boldsymbol{0}_{3\times 3} & \boldsymbol{J}_q(\mathbf{x}_i) \end{bmatrix}, \text{ where } \boldsymbol{J}_q(\mathbf{x}_i) = \begin{bmatrix} 1 & 0 & \sin\theta_i \\ 0 & \cos\phi_i & -\cos\theta_i \sin\phi_i \\ 0 & \sin\phi_i & \cos\phi_i\cos\theta_i \end{bmatrix}$$

Matrix $\boldsymbol{J}_i$ is singular when $det(\boldsymbol{J}_i) = \cos\theta_i = 0 \Leftrightarrow \theta_i = \pm\frac{\pi}{2}$. The control scheme proposed in this thesis guarantees that this is always avoided, and hence equation (2.1a) is well defined.

In equation (2.1b), $\boldsymbol{M}_i : \mathbb{M} \to \mathbb{R}^{6\times 6}$ is the symmetric and positive definite *inertia matrix*, $\boldsymbol{C}_i : \mathbb{M} \times \mathbb{R}^6 \to \mathbb{R}^{6\times 6}$ is the *Coriolis matrix* and $\mathbf{g}_i : \mathbb{M} \to \mathbb{R}^6$ is the *gravity vector*. Finally, $\mathbf{u}_i \in \mathbb{R}^6$ is the control input vector representing the 6D generalized *actuation force* acting on the agent.

However, access to measurements of, or knowledge about these matrices and vectors was not hitherto considered. At this point we make the following assumption:

Assumption 2.1.1. (*Measurements and Access to Information from an Inter-agent Perspective*)

1. Agent i has access to measurements $\mathbf{p}_i, \mathbf{q}_i, \dot{\mathbf{p}}_i, \boldsymbol{\omega}_i, \forall i \in \mathcal{V}$, that is, vectors $\mathbf{x}_i, \mathbf{v}_i$ pertaining to himself,

2. Agent i has a (upper-bounded) sensing range d_i such that

$$d_i > \max\{r_i + r_j : \forall i, j \in \mathcal{V}, i \neq j\}$$

3. the inertia $\boldsymbol{M}_i$ and Coriolis $\boldsymbol{C}_i$ vector fields are bounded and unknown for all $i \in \mathcal{V}$

4. the gravity vectors $\mathbf{g}$ are bounded and known for all $i \in \mathcal{V}$

The consequence of points 1 and 2 of assumption (2.1.1) is that by defining the set of agents j that are within the sensing range of agent i at time t as

$$\mathcal{R}_i(t) \triangleq \{j \in \mathcal{V} : \mathbf{p}_j(t) \in \mathcal{B}\big(\mathbf{p}_i(t), d_i\big)\}$$

or equivalently

$$\mathcal{R}_i(t) \triangleq \{j \in \mathcal{V} : \|\mathbf{p}_i(t) - \mathbf{p}_j(t)\| \leq d_i\}$$

agent i also knows at each time instant t all

$$\mathbf{p}_{j \triangleright i}(t), \mathbf{q}_{j \triangleright i}(t), \dot{\mathbf{p}}_{j \triangleright i}(t), \boldsymbol{\omega}_{j \triangleright i}(t)$$

Therefore, agent i assumes access to all measurements

$$\mathbf{p}_j(t), \mathbf{q}_j(t), \dot{\mathbf{p}}_j(t), \boldsymbol{\omega}_j(t),\ \forall j \in \mathcal{R}_i(t), t \in \mathbb{R}_{\geq 0}$$

of all agents $j \in \mathcal{R}_i(t)$ by virtue of being able to calculate them using knowledge of its own $\mathbf{p}_i(t)$, $\mathbf{q}_i(t)$, $\dot{\mathbf{p}}_i(t)$, $\boldsymbol{\omega}_i(t)$.

In the workspace there is a set $\mathcal{L}$ of L *static obstacles*, $\mathcal{L} = \{1, 2, \dots, L\}$, $L = |\mathcal{L}|$, also modeled as spheres, with centers at positions $\mathbf{p}_\ell \in \mathbb{R}^3$ with radii $r_\ell \in \mathbb{R}$, $\ell \in \mathcal{L}$. Thus, the obstacles are modeled by spheres $\mathcal{B}\big(\mathbf{p}_\ell, r_\ell\big)$, $\ell \in \mathcal{L}$. Their position and size in 3D space is assumed to be known a priori to each agent. The geometry of two agents i and j as well as an obstacle ℓ in workspace W is depicted in Fig. 2.1.

Let us now define the distance between any two agents i, j at time t as $d_{ij,a}(t)$; that between agent i and obstacle ℓ as $d_{i\ell,o}(t)$; and that between an agent i and the origin of the workspace W as $d_{i,W}(t)$, with $d_{ij,a}, d_{i\ell,o}, d_{i,W} : \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$:

$$d_{ij,a}(t) \triangleq \|\mathbf{p}_i(t) - \mathbf{p}_j(t)\|$$

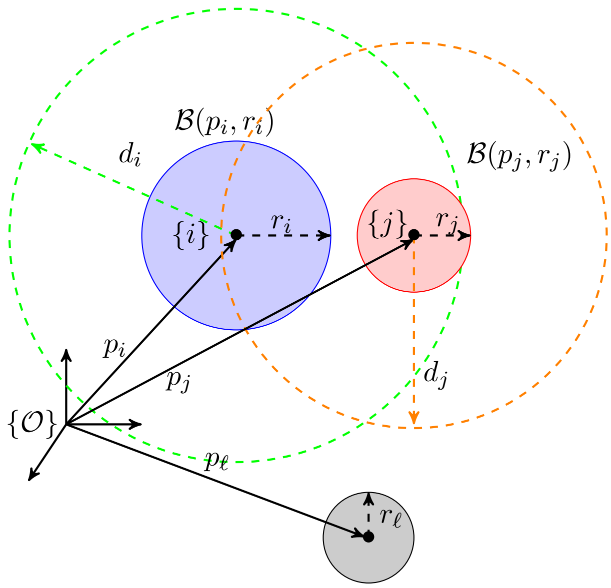

Figure 2.1: Illustration of two agents $i, j \in \mathcal{V}$ and a static obstacle $\ell \in \mathcal{L}$ in the workspace; $\{\mathcal{O}\}$ is the inertial frame, $\{i\}, \{j\}$ are the frames attached to the agents' center of mass, $\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_\ell \in \mathbb{R}^3$ are the positions of the centers of mass of agents i, j and obstacle ℓ respectively, expressed in frame $\{\mathcal{O}\}$. r_i, r_j, r_ℓ are the radii of the agents i, j and the obstacle ℓ respectively. d_i, d_j with $d_i > d_j$ are the agents' sensing ranges. In this figure, agents i and j are neighbours, since the center of mass of agent j is within the sensing range of agent i and vice versa: $\mathbf{p}_j \in \mathcal{B}\big(\mathbf{p}_i(t), d_i\big)$ and $\mathbf{p}_i \in \mathcal{B}\big(\mathbf{p}_j(t), d_j\big)$. Furthermore, the configuration between the two agents and the obstacle is a collision-free configuration.

$$d_{i\ell,o}(t) \triangleq \|\mathbf{p}_i(t) - \mathbf{p}_\ell(t)\|$$

$$d_{i,W}(t) \triangleq \|\mathbf{p}_W - \mathbf{p}_i(t)\|$$

as well as constants

$$\underline{d}_{ij,a} \triangleq r_i + r_j$$

$$\underline{d}_{i\ell,o} \triangleq r_i + r_\ell$$

$$\overline{d}_{i,W} \triangleq r_W - r_i$$

$\forall i, j \in \mathcal{V}$, $i \neq j$, $\ell \in \mathcal{L}$. The latter stand for the minimum distance between two *agents*, the minimum distance between an *agent* and an *obstacle*, and the maximum distance between an *agent* and the origin of the workspace, respectively. They arise spatially as physical limitations and will be utilized in forming collision-avoidance constraints.

Based on these definitions, we will now define the concept of a *collision-free configuration*:

Definition 2.1.1. (*Collision-free Configuration*) A collision-free configuration between

- any two agents $i, j \in \mathcal{V}$ is when $d_{ij,a}(t) > \underline{d}_{ij,a}$
- an agent $i \in \mathcal{V}$ and an obstacle $\ell \in \mathcal{L}$ is when $d_{il,o}(t) > \underline{d}_{il,o}$
- an agent $i \in \mathcal{V}$ and the workspace W boundary, is when $d_{i,W}(t) < \overline{d}_{i,W}$

at a generic time instant $t \in \mathbb{R}_{\geq 0}$. When all three conditions are met, we will simply refer to the overall configuration as a collision-free configuration.

2.2 Initial Conditions

We assume that at time $t = 0$ *all* agents are in a *collision-free configuration*, i.e.

$$d_{ij,a}(0) > \underline{d}_{ij,a}$$

$$d_{il,o}(0) > \underline{d}_{il,o}$$

$$d_{i,W}(0) < \overline{d}_{i,W}$$

$\forall i \in \mathcal{V}, \ell \in \mathcal{L}$. Before declaring further assumptions that relate to the initial conditions of the system's configuration, we give the definition of the *neighbour set* $\mathcal{N}_i$ of a generic agent $i \in \mathcal{V}$:

Definition 2.2.1. (*Neighbours Set*)
Agents $j \in \mathcal{N}_i$ are defined as the *neighbours* of agent $i \in \mathcal{V}$. The set $\mathcal{N}_i$ is composed of the indices of agents $j \in \mathcal{V}$ which

1. are within the sensing range of agent i at time $t = 0$, i.e. $j \in \mathcal{R}_i(0)$, *and*
2. are *intended* to be kept within the sensing range of agent i at all times $t \in \mathbb{R}_{>0}$

Therefore, while the composition of the set $\mathcal{R}_i(t)$ evolves and varies through time in general, the set $\mathcal{N}_i$ *should* remain invariant over time[1].

[1] This reason, and the fact that the proposed control scheme guarantees that $\mathcal{N}_i$ *will* remain invariant over time, is why we do not refer to this set as $\mathcal{N}_i(t)$.

It is not necessary that all agents are assigned a set of agents with whom they should maintain connectivity; however, if *all* agents were neighbour-less, then the concept of *cooperation* between them would be void, since in that case the problem would break down into $|\mathcal{V}|$ individual problems of smaller significance or interest, while weakening the "multi-agent" perspective as well. Therefore, we assume that $\sum_i |\mathcal{N}_i| > 0$.

It is further assumed that $\mathcal{N}_i$ is given at $t = 0$, and that neighbouring relations are reciprocal, i.e. agent i is a neighbour of agent j if and only if j is a neighbour of i:

$$j \in \mathcal{N}_i \Leftrightarrow i \in \mathcal{N}_j, \ \forall i, j \in \mathcal{V}, i \neq j$$

Furthermore, it is assumed that at time $t = 0$ the Jacobians $\boldsymbol{J}_i$ are well-defined $\forall i \in \mathcal{V}$, and that the system (2.1) enjoys a continuous solution for all initial conditions. The assumptions which concern the initial conditions of the problem are formally summarized in assumption 2.2.1:

Assumption 2.2.1. (*Initial Conditions Assumption*)

At time $t = 0$

1. the sets $\mathcal{N}_i$ are known for all $i \in \mathcal{V}$ and $\sum_i |\mathcal{N}_i| > 0$
2. all agents are in a collision-free configuration with each other, the obstacles $\ell \in \mathcal{L}$ and the workspace W boundary
3. all agents are in a singularity-free configuration:

$$-\frac{\pi}{2} < \theta_i(0) < \frac{\pi}{2}, \ \forall i \in \mathcal{V}$$

2.3 Objective

Given the aforementioned structure of the system, the objective to be pursued is the *stabilization of all agents* $i \in \mathcal{V}$ starting from an initial configuration abiding by assumption (2.2.1) to a desired feasible configuration $\mathbf{x}_{i,des}, \mathbf{v}_{i,des}$, while satisfying all communication constraints, i.e. sustaining connectivity between neighbouring agents, and avoiding collisions between agents, obstacles, and the workspace boundary. The concept of a *desired feasible configuration* or *feasible steady-state configuration* is given in definition (2.3.1).

Definition 2.3.1. (*Feasible Steady-state Configuration*)
The desired steady-state configuration $\mathbf{x}_{i,des}$ of agents $\forall i \in \mathcal{V}, j \in \mathcal{N}_i$ is *feasible* if and only if

1. it is a collision-free configuration according to definition (2.1.1)
2. it does not result in violation of the communication constraints between neighbouring agents i, j, i.e. the following inequalities hold true simultaneously:

$$\|\mathbf{p}_{i,des} - \mathbf{p}_{j,des}\| < d_i$$

$$\|\mathbf{p}_{i,des} - \mathbf{p}_{j,des}\| < d_j$$

At this point we must address an issue that refers to the feasibility of its solution and relates to the avoidance of collisions from an intra-environmental perspective. Namely, we demand that a solution be feasible if and only if the agent with the largest radius is able to pass through the spaces demarcated by (a) the two least distant obstacles, and (b) the obstacle closest to the boundary of the workspace and the boundary of the workspace itself. To this end, we formalize the relevant notions in Definition (2.3.2).

Definition 2.3.2. (*Intra-environmental Arrangement*)
Let us define $\underline{d}_{\ell'\ell}$

$$\underline{d}_{\ell'\ell} \triangleq \min\{\|\mathbf{p}_\ell - \mathbf{p}_{\ell'}\| + r_\ell + r_{\ell'} : \ell, \ell' \in \mathcal{L}, \ell \neq \ell'\},$$

as the distance between the two least distant obstacles in the workspace, $\underline{d}_{\ell,W}$

$$\underline{d}_{\ell,W} \triangleq \min\{r_W - (\|\mathbf{p}_W - \mathbf{p}_\ell\| + r_\ell) : \ell \in \mathcal{L}\},$$

as the distance between the least distant obstacle from the boundary of the workspace and the boundary itself, and D

$$D \triangleq \min\{\underline{d}_{\ell'\ell}, \underline{d}_{\ell,W}\}$$

as the least of these two distances.

Given these notions, we can state an assumption on the feasibility of a solution to the problem that this work addresses:

Assumption 2.3.1. (*Intra-environmental Arrangement of Obstacles*)
All obstacles $\ell \in \mathcal{L}$ are situated inside the workspace W in such a way that

$$D > 2r_i, \ i \in \mathcal{V} : r_i = max\{r_j\} \ \forall j \in \mathcal{V}$$

where D is defined in Definition (2.3.2).

The designed desirable control scheme should provide feasible control inputs $\mathbf{u}_i$ per agent $i \in \mathcal{V}$, that is, inputs that abide by the input constraints $\mathcal{U}_i$:

Definition 2.3.3. (*Feasible Control Input*) The control input $\mathbf{u}_i(\cdot)$ is called feasible when it abides by its respective constraint:

$$\mathbf{u}_i(t) \in \mathcal{U}_i = \{\mathbf{u}_i(t) \in \mathbb{R}^6 : \|\mathbf{u}_i(t)\| \leq \overline{u}_i\}$$

Overall then, the objective of each agent i is for $\lim_{t\to\infty} \|\mathbf{x}_i(t) - \mathbf{x}_{i,des}\| = 0$. If we design feasible control inputs $\mathbf{u}_i \in \mathcal{U}_i$, $\forall i \in \mathcal{V}$ such that the signal $\lim_{t\to\infty} \mathbf{x}_i(t) = \mathbf{x}_{i,des}$ with dynamics given in (2.1), constrained under assumptions (2.1.1), (2.2.1), and (2.3.1) satisfies $\lim_{t\to\infty} \|\mathbf{x}_i(t) - \mathbf{x}_{i,des}\| = 0$, while all system related signals remain bounded in their respective regions, $-$ if all of the above are achieved, then problem (2.4) has been solved.

2.4 Problem Statement

Due to the fact that the agents are not dimensionless and their communication capabilities are limited, given feasible steady-state configurations $\mathbf{p}_{i,des}, \mathbf{q}_{i,des}$, the control protocol should for all agents $i \in \mathcal{V}$ guarantee that:

1. the desired positions $\mathbf{p}_{i,des}$ are achieved in finite time
2. the desired angles $\mathbf{q}_{i,des}$ are achieved in finite time
3. connectivity between neighbouring agents $j \in \mathcal{N}_i$ is maintained at all times

Furthermore, for all agents $i \in \mathcal{V}$, obstacles $\ell \in \mathcal{L}$ and the workspace boundary W, it should guarantee for all $t \in \mathbb{R}_{\geq 0}$ that:

1. all agents avoid collision with each other
2. all agents avoid collision with all obstacles
3. all agents avoid collision with the workspace boundary
4. singularity of the Jacobian matrices $\boldsymbol{J}_i$ is avoided
5. all input control signals are bounded in their respective regions

Therefore, all neighboring agents of agent i must remain within a distance less than d_i to him, for all $i \in \mathcal{V} : |\mathcal{N}_i| \neq 0$, and all agents $i, j \in \mathcal{V}, i \neq j$ must remain within distance greater than $\underline{d}_{ij,a}$ with one another.

Formally, the control problem under the aforementioned constraints is formulated as follows:

Problem 2.4.1. Consider N agents modeled as bounded spheres $\mathcal{B}(\mathbf{p}_i, r_i)$, $i \in \mathcal{V}, N = |\mathcal{V}|$, that operate in workspace W that is also modeled as a bounded sphere $\mathcal{B}(\mathbf{p}_W, r_W)$. W features $|\mathcal{L}|$ spherical obstacles in its interior, also modeled as bounded spheres $\mathcal{B}(\mathbf{p}_\ell, r_\ell), \ell \in \mathcal{L}$.

All agents $i \in \mathcal{V}$ are governed by the dynamics (2.1), and the compound system of agents, obstacles and the workspace is subject to assumptions (2.1.1), (2.2.1), (2.3.1). Given desired *feasible* steady-state agent configurations $\mathbf{x}_{i,des}$ according to definition (2.3.1), $\forall i \in \mathcal{V}$, design feasible decentralized control laws $\mathbf{u}_i(t)$ according to definition (2.3.3), such that $\forall i \in \mathcal{V}$ and for all times $t \in \mathbb{R}_{\geq 0}$, the following hold:

1. Position and orientation configuration is achieved in steady-state

$$\lim_{t \to \infty} \|\mathbf{x}_i(t) - \mathbf{x}_{i,des}(t)\| = 0$$

2. Inter-agent collision is avoided

$$\|\mathbf{p}_i(t) - \mathbf{p}_j(t)\| = d_{ij,a}(t) > \underline{d}_{ij,a}, \forall j \in \mathcal{V} \backslash \{i\}$$

3. Inter-agent connectivity loss between neighbouring agents is avoided

$$\|\mathbf{p}_i(t) - \mathbf{p}_j(t)\| = d_{ij,a}(t) < d_i, \forall j \in \mathcal{N}_i, \forall i : |\mathcal{N}_i| \neq 0$$

4. Agent-with-obstacle collision is avoided

$$\|\mathbf{p}_i(t) - \mathbf{p}_\ell(t)\| = d_{i\ell,o}(t) > \underline{d}_{i\ell,o}, \forall \ell \in \mathcal{L}$$

5. Agent-with-workspace-boundary collision is avoided

$$\|\mathbf{p}_W - \mathbf{p}_i(t)\| = d_{i,W}(t) < \overline{d}_{i,W}$$

6. All maps $\boldsymbol{J}_i$ are well defined

$$-\frac{\pi}{2} < \theta_i(t) < \frac{\pi}{2}$$

7. The control laws $\mathbf{u}_i(t)$ abide by their respective input constraints

$$\mathbf{u}_i(t) \in \mathcal{U}_i$$

Part II

Advocated Solutions

3

Disturbance-free Stabililization

The purpose to be saught in this chapter is the steering of each agent $i \in \mathcal{V}$ into a *position* in 3D space while conforming to the requirements posed by the problem. Here, the real system and its model are equivalent: no model-reality mismatches exist, and no disturbances act on the real system. At first, the model of system (2.1) will be formalized. Next, the error model and *its* constraints will be expressed. Then, the optimization problem to be solved periodically will be posed; it will equip us with the optimum feasible input that steers the system towards achieving its intended collision-free steady-state configuration, while avoiding the pitfall of violating its constraints. This will lead to the proof of stability of the compound closed-loop system of agents $i \in \mathcal{V}$ under the proposed control regime.

3.1 Formalizing the system's model

We begin by rewriting the system equations (2.1a), (2.1b) for a generic agent $i \in \mathcal{V}$ in state-space form:

$$\dot{\mathbf{x}}_i(t) = \boldsymbol{J}_i^{-1}(\mathbf{x}_i)\mathbf{v}_i(t)$$

$$\dot{\mathbf{v}}_i(t) = -\boldsymbol{M}_i^{-1}(\mathbf{x}_i)\boldsymbol{C}_i(\mathbf{x}_i, \dot{\mathbf{x}}_i)\mathbf{v}_i(t) - \boldsymbol{M}_i^{-1}(\mathbf{x}_i)\mathbf{g}_i(\mathbf{x}_i) + \boldsymbol{M}_i^{-1}(\mathbf{x}_i)\mathbf{u}_i(t)$$

where the inversion of $\boldsymbol{M}_i$ is possible due to it being positive-definite $\forall i \in \mathcal{V}$. Denoting by $\mathbf{z}_i(t)$

$$\mathbf{z}_i(t) \triangleq \begin{bmatrix} \mathbf{x}_i(t) \\ \mathbf{v}_i(t) \end{bmatrix}, \ \mathbf{z}_i(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^9 \times \mathbb{T}^3$$

and $\dot{\mathbf{x}}_i(t)$ and $\dot{\mathbf{v}}_i(t)$ by

$$\dot{\mathbf{x}}_i(t) = f_{i,x}(\mathbf{z}_i, \mathbf{u}_i)$$

$$\dot{\mathbf{v}}_i(t) = f_{i,v}(\mathbf{z}_i, \mathbf{u}_i)$$

we get the compact representation of the system's model

$$\dot{\mathbf{z}}_i(t) = \begin{bmatrix} f_{i,x}(\mathbf{z}_i, \mathbf{u}_i) \\ f_{i,v}(\mathbf{z}_i, \mathbf{u}_i) \end{bmatrix} = f_i\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big)$$

The state evolution of agent i is modeled by a system of non-linear continuous-time differential equations of the form

$$\dot{\mathbf{z}}_i(t) = f_i\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big)$$

$$\mathbf{z}_i(0) = \mathbf{z}_{i,0}$$

$$\mathbf{z}_i(t) \subset \mathbb{R}^9 \times \mathbb{T}^3$$

$$\mathbf{u}_i(t) \subset \mathbb{R}^6 \tag{3.3}$$

where state $\mathbf{z}_i$ is directly measurable as per assumption (2.1.1).

We define the set $\mathcal{Z}_{i,t} \subset \mathbb{R}^9 \times \mathbb{T}^3$ as the set that captures all the *state* constraints on the system posed by the problem (2.4) at $t \in \mathbb{R}_{\geq 0}$. Therefore $\mathcal{Z}_{i,t}$ is such that:

$$\begin{aligned}
\mathcal{Z}_{i,t} \triangleq \big\{\mathbf{z}_i(t) \in \mathbb{R}^9 \times \mathbb{T}^3 : \; & \|\mathbf{p}_i(t) - \mathbf{p}_j(t)\| > \underline{d}_{ij,a}, \forall j \in \mathcal{R}_i(t), \\
& \|\mathbf{p}_i(t) - \mathbf{p}_j(t)\| < d_i, \forall j \in \mathcal{N}_i, \\
& \|\mathbf{p}_i(t) - \mathbf{p}_\ell\| > \underline{d}_{i\ell,o}, \forall \ell \in \mathcal{L}, \\
& \|\mathbf{p}_W - \mathbf{p}_i(t)\| < \overline{d}_{i,W}, \\
& -\frac{\pi}{2} < \theta_i(t) < \frac{\pi}{2}, \\
& \forall t \in \mathbb{R}_{\geq 0}\big\}
\end{aligned}$$

3.2 The error model

A feasible desired configuration $\mathbf{z}_{i,des} \in \mathbb{R}^9 \times \mathbb{T}^3$ is associated to each agent $i \in \mathcal{V}$, with the aim of agent i achieving it in steady-state: $\lim\limits_{t\to\infty} \|\mathbf{z}_i(t) - \mathbf{z}_{i,des}\| = 0$. The interior of the norm of this expression denotes the state error of agent i:

$$\mathbf{e}_i(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^9 \times \mathbb{T}^3, \; \mathbf{e}_i(t) = \mathbf{z}_i(t) - \mathbf{z}_{i,des}$$

The error dynamics are denoted by $g_i(\mathbf{e}_i, \mathbf{u}_i)$:

$$\dot{\mathbf{e}}_i(t) = \dot{\mathbf{z}}_i(t) - \dot{\mathbf{z}}_{i,des} = \dot{\mathbf{z}}_i(t) = f_i\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big) = g_i\big(\mathbf{e}_i(t), \mathbf{u}_i(t)\big) \tag{3.4}$$

with $\mathbf{e}_i(0) = \mathbf{z}_i(0) - \mathbf{z}_{i,des}$.

In order to translate the constraints that are dictated for the state $\mathbf{z}_i(t)$ into constraints regarding the error state $\mathbf{e}_i(t)$, we define the set $\mathcal{E}_{i,t} \subset \mathbb{R}^9 \times \mathbb{T}^3$ as:

$$\mathcal{E}_{i,t} \triangleq \big\{\mathbf{e}_i(t) \in \mathbb{R}^9 \times \mathbb{T}^3 : \mathbf{e}_i(t) \in \mathcal{Z}_{i,t} \ominus \mathbf{z}_{i,des}\big\}$$

as the set that captures all constraints on the error state with dynamics (3.4) dictated by problem (2.4).

On functions g_i we make the following assumption:

Assumption 3.2.1. *(g_i is Lipschitz continuous in $\mathcal{E}_{i,t} \times \mathcal{U}_i$)*
Suppose that $\mathbf{e}_1, \mathbf{e}_2 \in \mathcal{E}_{i,t}$ and $\mathbf{u} \in \mathcal{U}_i$. Functions g_i are Lipschitz continuous in $\mathcal{E}_{i,t} \times \mathcal{U}_i$ with Lipschitz constants L_{g_i}:

$$\big\|g_i\big(\mathbf{e}_1, \mathbf{u}\big) - g_i\big(\mathbf{e}_2, \mathbf{u}\big)\big\| \leq L_{g_i}\big\|\mathbf{e}_1 - \mathbf{e}_2\big\|$$

If we design control laws $\mathbf{u}_i \in \mathcal{U}_i$, $\forall i \in \mathcal{V}$ such that the error signal $\mathbf{e}_i(t)$ with dynamics given in (3.4), constrained under $\mathbf{e}_i(t) \in \mathcal{E}_{i,t}$, satisfies $\lim\limits_{t\to\infty} \|\mathbf{e}_i(t)\| = 0$, while all system related signals remain bounded in their respective regions,− if all of the above are achieved, then problem (2.4) has been solved.

In order to achieve this task, we employ a Nonlinear Receding Horizon scheme.

3.3 The optimization problem

Consider a sequence of sampling times $\{t_k\}_{k\geq 0}$, with a constant sampling time h, $0 < h < T_p$, where T_p is the finite time-horizon, such that $t_{k+1} = t_k + h$. In sampling data NMPC, a finite-horizon open-loop optimal control problem (FHOCP) is solved at discrete sampling time instants t_k based on the then-current state error measurement $\mathbf{e}_i(t_k)$. The solution is an optimal control signal $\overline{\mathbf{u}}_i^{\star}(t)$, computed over $t \in [t_k, t_k + T_p]$. This signal is applied to the open-loop system in between sampling times t_k and t_{k+1}.

At a generic time t_k then, agent i solves the following optimization problem:

Problem 3.3.1.

Find

$$J_i^\star\big(\mathbf{e}_i(t_k)\big) \triangleq \min_{\overline{\mathbf{u}}_i(\cdot)} \ J_i\big(\mathbf{e}_i(t_k), \overline{\mathbf{u}}_i(\cdot)\big) \tag{3.5}$$

where

$$J_i\big(\mathbf{e}_i(t_k), \overline{\mathbf{u}}_i(\cdot)\big) \triangleq \int_{t_k}^{t_k+T_p} F_i\big(\overline{\mathbf{e}}_i(s), \overline{\mathbf{u}}_i(s)\big)ds + V_i\big(\overline{\mathbf{e}}_i(t_k+T_p)\big)$$

subject to:

$$\dot{\overline{\mathbf{e}}}_i(s) = g_i\big(\overline{\mathbf{e}}_i(s), \overline{\mathbf{u}}_i(s)\big), \ \overline{\mathbf{e}}_i(t_k) = \mathbf{e}_i(t_k) \tag{3.6}$$

$$\overline{\mathbf{u}}_i(s) \in \mathcal{U}_i, \ \overline{\mathbf{e}}_i(s) \in \mathcal{E}_{i,s}, \ s \in [t_k, t_k+T_p]$$

$$\overline{\mathbf{e}}_i(t_k+T_p) \in \Omega_i$$

The notation $\bar{\cdot}$ is used to distinguish predicted states which are internal to the controller, as opposed to their actual values, because the predicted values will not be equal to the actual closed-loop values. This means that $\overline{\mathbf{e}}_i(\cdot)$ is the solution to (3.6) driven by the control input $\overline{\mathbf{u}}_i(\cdot) : [t_k, t_k+T_p] \rightarrow \mathcal{U}_i$ with initial condition $\mathbf{e}_i(t_k)$.

The applied input signal is a portion of the optimal solution to an optimization problem where information on the states of the neighbouring agents of agent i is taken into account only in the constraints considered in the optimization problem. These constraints pertain to the set of its neighbours $\mathcal{N}_i$ and, in total, to the set of all agents within its sensing range $\mathcal{R}_i$. Regarding these, we make the following assumption:

Assumption 3.3.1. (*Access to Predicted Information from an Inter-agent Perspective*)

Considering the context of Receding Horizon Control, when at time t_k agent i solves a finite horizon optimization problem, he has access to[a]

1. measurements of the states[b]

    - $\mathbf{z}_j(t_k)$ of all agents $j \in \mathcal{R}_i(t_k)$ within its sensing range at time t_k
    - $\mathbf{z}_{j'}(t_k)$ of all of its neighbouring agents $j' \in \mathcal{N}_i$ at time t_k

2. the *predicted states*

   - $\overline{\mathbf{z}}_j(\tau)$ of all agents $j \in \mathcal{R}_i(t_k)$ within its sensing range
   - $\overline{\mathbf{z}}_{j'}(\tau)$ of all of its neighbouring agents $j' \in \mathcal{N}_i$

   across the entire horizon $\tau \in (t_k, t_k + T_p]$

[a] Although $\mathcal{N}_i \subseteq \mathcal{R}_i$, we make the distinction between the two because all agents $j \in \mathcal{R}_i$ need to avoid collision with agent i, but only agents $j' \in \mathcal{N}_i$ need to remain within the sensing range of agent i.

[b] as per assumption (2.1.1)

Remark 3.3.1. The justification for this assumption is the following: considering that $\mathcal{N}_i \subseteq \mathcal{R}_i$, that the state vectors $\mathbf{z}_j$ are comprised of 12 real numbers that are encoded by 4 bytes, and that sampling occurs with a frequency f for all agents, the overall downstream bandwidth required by each agent is

$$BW_d = 12 \times 32 \text{ [bits]} \times |\mathcal{R}_i| \times \frac{T_p}{h} \times f \text{ [sec}^{-1}\text{]}$$

Given a conservative sampling time $f = 100$ Hz and a horizon of $\frac{T_p}{h} = 100$ timesteps, the wireless protocol IEEE 802.11n-2009 (a standard for present-day devices) can accommodate up to

$$|\mathcal{R}_i| = \frac{600 \text{ [Mbit} \cdot \text{sec}^{-1}\text{]}}{12 \times 32\text{[bit]} \times 10^4\text{[sec}^{-1}\text{]}} \approx 16 \cdot 10^2 \text{ agents}$$

within the range of one agent. We deem this number to be large enough for practical applications for the approach of assuming access to the predicted states of agents within the range of one agent to be legal.

In other words, each time an agent solves its own individual optimization problem, he knows the state predictions that have been generated by the solution of the optimization problem of all agents within its range at that time, for the next T_p timesteps. This assumption is crucial to satisfying the constraints regarding collision aversion and connectivity maintenance between neighbouring agents. We assume that the above pieces of information are (a) always available and accurate, and (b) exchanged without delay. We encapsulate these pieces of information in four stacked vectors:

$$\mathbf{z}_{\mathcal{R}_i}(t_k) \triangleq col[\mathbf{z}_j(t_k)], \forall j \in \mathcal{R}_i(t_k)$$

$$\mathbf{z}_{\mathcal{N}_i}(t_k) \triangleq col[\mathbf{z}_j(t_k)], \forall j \in \mathcal{N}_i$$

$$\overline{\mathbf{z}}_{\mathcal{R}_i}(\tau) \triangleq col[\overline{\mathbf{z}}_j(\tau)], \forall j \in \mathcal{R}_i(\tau), \tau \in [t_k, t_k + T_p]$$

$$\overline{\mathbf{z}}_{\mathcal{N}_i}(\tau) \triangleq col[\overline{\mathbf{z}}_j(\tau)], \forall j \in \mathcal{N}_i, \tau \in [t_k, t_k + T_p]$$

These are taken into consideration during the solution to the optimization problem as follows: when agent i solves his own optimization problem, his predicted configuration at times $\tau \in [t_k, t_k + T_p]$ is constrained by the predicted configuration of its neighbouring or perceivable[1] agents at the same time instant τ. The form of the constraint regime is necessary to be such, as each agent is constrained not by constant values, but by the trajectories of its associated agents, which are time-varying in nature.

Formally then, $\mathcal{E}_{i,s} = \{\mathbf{e}_i(s) : \mathbf{e}_i(s) \in \mathcal{Z}_{i,s} \ominus \mathbf{z}_{i,des}\}$, for $s \in [t_k, t_k + T_p]$, where, for $s = t_k$:

$$\begin{aligned}\mathcal{Z}_{i,t_k} = \big\{\mathbf{z}_i(t_k) \in \mathbb{R}^9 \times \mathbb{T}^3 : \; & \|\mathbf{p}_i(t_k) - \mathbf{p}_{\mathcal{R}_i}(t_k)\| > \underline{d}_{ij,a}, \forall j \in \mathcal{R}_i(t_k) \\ & \|\mathbf{p}_i(t_k) - \mathbf{p}_{\mathcal{N}_i}(t_k)\| < d_i, \forall j \in \mathcal{N}_i, \\ & \|\mathbf{p}_i(t_k) - \mathbf{p}_\ell\| > \underline{d}_{i\ell,o}, \forall \ell \in \mathcal{L}, \\ & \|\mathbf{p}_W - \mathbf{p}_i(t_k)\| < \overline{d}_{i,W}, \\ & -\frac{\pi}{2} < \theta_i(t_k) < \frac{\pi}{2}\}\end{aligned}$$

and, for $s \in (t_k, t_k + T_p]$:

$$\begin{aligned}\mathcal{Z}_{i,s} = \big\{\mathbf{z}_i(s) \in \mathbb{R}^9 \times \mathbb{T}^3 : \; & \|\overline{\mathbf{p}}_i(s) - \overline{\mathbf{p}}_{\mathcal{R}_i}(s)\| > \underline{d}_{ij,a}, \forall j \in \mathcal{R}_i(s) \\ & \|\overline{\mathbf{p}}_i(s) - \overline{\mathbf{p}}_{\mathcal{N}_i}(s)\| < d_i, \forall j \in \mathcal{N}_i,\end{aligned}$$

[1] agents within its sensing range

$$\|\overline{\mathbf{p}}_i(s) - \mathbf{p}_\ell\| > \underline{d}_{i\ell,o}, \forall \ell \in \mathcal{L},$$

$$\|\mathbf{p}_W - \overline{\mathbf{p}}_i(s)\| < \overline{d}_{i,W},$$

$$-\frac{\pi}{2} < \overline{\theta}_i(s) < \frac{\pi}{2}\}$$

The denotations $\mathbf{p}_{\mathcal{R}_i}(t_k)$, $\mathbf{p}_{\mathcal{N}_i}(t_k)$, $\overline{\mathbf{p}}_{\mathcal{R}_i}(s)$, and $\overline{\mathbf{p}}_{\mathcal{N}_i}(s)$ serve as to point to the column vectors $\mathbf{z}_{\mathcal{R}_i}(t_k)$, $\mathbf{z}_{\mathcal{N}_i}(t_k)$, $\overline{\mathbf{z}}_{\mathcal{R}_i}(s)$, and $\overline{\mathbf{z}}_{\mathcal{N}_i}(s)$ respectively, of which they are components, and whose notation they abide by.

The functions $F_i : \mathcal{E}_{i,s} \times \mathcal{U}_i \to \mathbb{R}_{\geq 0}$ and $V_i : \Omega_i \to \mathbb{R}_{\geq 0}$ are defined as

$$F_i\big(\overline{\mathbf{e}}_i(t), \overline{\mathbf{u}}_i(t)\big) \triangleq \overline{\mathbf{e}}_i(t)^\top \boldsymbol{Q}_i \overline{\mathbf{e}}_i(t) + \overline{\mathbf{u}}_i(t)^\top \boldsymbol{R}_i \overline{\mathbf{u}}_i(t) \tag{3.8}$$

$$V_i\big(\overline{\mathbf{e}}_i(t)\big) \triangleq \overline{\mathbf{e}}_i(t)^\top \boldsymbol{P}_i \overline{\mathbf{e}}_i(t) \tag{3.9}$$

Matrices $\boldsymbol{R}_i \in \mathbb{R}^{6\times 6}$ and $\boldsymbol{Q}_i, \boldsymbol{P}_i \in \mathbb{R}^{12\times 12}$ are symmetric and positive definite. The running costs F_i are upper- and lower-bounded by class $\mathcal{K}_\infty$ functions:

Lemma 3.3.1. (*F_i is lower- and upper-bounded by class $\mathcal{K}_\infty$ functions*)
Let functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and F_i be defined by (3.8). Then, for all $\mathbf{e}_i \in \mathcal{E}_{i,s}$

$$\alpha_1\big(\|\mathbf{e}_i\|\big) \leq F_i\big(\mathbf{e}_i, \mathbf{u}_i\big) \leq \alpha_2\big(\|\mathbf{e}_i\|\big)$$

Lemma 3.3.2. (F_i is Lipschitz continuous in $\mathcal{E}_{i,s} \times \mathcal{U}_i$)
Suppose that $\mathbf{e}_1, \mathbf{e}_2 \in \mathcal{E}_{i,s}$, $\mathbf{u}_i \in \mathcal{U}_i$ and that F_i is defined by (3.8). The running costs F_i are Lipschitz continuous in $\mathcal{E}_{i,s} \times \mathcal{U}_i$:

$$\big|F_i(\mathbf{e}_1, \mathbf{u}_i) - F_i(\mathbf{e}_2, \mathbf{u}_i)\big| \leq L_{F_i} \|\mathbf{e}_1 - \mathbf{e}_2\|$$

with Lipschitz constant $L_{F_i} = 2\sigma_{max}(\boldsymbol{Q}_i)\overline{\varepsilon}_i$, where $\overline{\varepsilon}_i = \sup\limits_{\mathbf{e}_i \in \mathcal{E}_{i,s}} \|\mathbf{e}_i\|$

The terminal set $\Omega_i \subseteq \mathcal{E}_{i,s}$ is an admissible positively invariant set according to definition (1.2.5) for system (3.4) such that

$$\Omega_i = \{\mathbf{e}_i \in \mathcal{E}_{i,s} : V_i(\mathbf{e}_i) \leq \varepsilon_{\Omega_i}\}$$

where ε_{Ω_i} is an arbitrarily small but fixed positive real scalar.

With regard to the terminal penalty function V_i, the following lemma will prove to be useful in guaranteeing the convergence of the solution to the optimal control problem to the terminal region Ω_i:

Lemma 3.3.3. (V_i is Lipschitz continuous in Ω_i)

Suppose that $\mathbf{e}_1, \mathbf{e}_2 \in \Omega_i$, and that V_i is defined by (3.9). The terminal penalty function V_i is Lipschitz continuous in Ω_i

$$\big|V_i(\mathbf{e}_1) - V_i(\mathbf{e}_2)\big| \leq L_{V_i}\|\mathbf{e}_1 - \mathbf{e}_2\|$$

with Lipschitz constant $L_{V_i} = 2\sigma_{max}(\boldsymbol{P}_i)\varepsilon_{\Omega_i}$

Furthermore, V_i is lower- and upper-bounded by class $\mathcal{K}_\infty$ functions.

Lemma 3.3.4. (V_i *is lower- and upper-bounded by class* $\mathcal{K}_\infty$ *functions in* Ω_i)

Let $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\mathbf{e}_i \in \Omega_i$ and let V_i be defined by (3.9). Then

$$\alpha_1\big(\|\mathbf{e}_i\|\big) \leq V_i(\mathbf{e}_i) \leq \alpha_2\big(\|\mathbf{e}_i\|\big)$$

The solution to the optimal control problem (3.5) at time t_k is an optimal control input, denoted by $\overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t_k))$, which is applied to the open-loop system until the next sampling instant $t_k + h$, with $h \in (0, T_p)$:

$$\mathbf{u}_i(t) = \overline{\mathbf{u}}_i^\star\big(t;\ \mathbf{e}_i(t_k)\big),\ t \in [t_k, t_k + h] \tag{3.10}$$

At time t_{k+1} a new finite horizon optimal control problem is solved in the same manner, leading to a receding horizon approach.

The control input $\mathbf{u}_i(\cdot)$ is of feedback form, since it is recalculated at each sampling instant based on the then-current state. The solution to equation (3.4) − the model of the real system, starting at time t_1, from an initial condition $\mathbf{e}_i(t_1) = \overline{\mathbf{e}}_i(t_1)$, by application of the control input $\mathbf{u}_i : [t_1, t_2] \to \mathcal{U}_i$ is denoted by

$$\mathbf{e}_i\big(t;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(t_1)\big),\ t \in [t_1, t_2]$$

On the existence of solutions to (3.4) we assume the following:

Assumption 3.3.2. The system (3.4) has a *continuous solution* for any $\mathbf{e}_i(0) \in \mathcal{E}_{i,s}$ and any *piecewise continuous* input $\mathbf{u}_i(\cdot) : [0, T_p] \to \mathcal{U}_i$.

The states of the open-loop system (3.6) − the predicted states obey the following notation:

Remark 3.3.2. The *predicted* state of the system (3.4) at time $\tau \geq t_k$, based on the measurement of the state at time t_k, $\mathbf{e}_i(t_k)$, by application of the control input (3.10), is denoted by

$$\overline{\mathbf{e}}_i\big(\tau;\ \mathbf{u}_i(\tau), \mathbf{e}_i(t_k)\big)$$

The closed-loop system for which stability is to be guaranteed is

$$\mathbf{e}_i(\tau) = g_i\big(\mathbf{e}_i(\tau), \overline{\mathbf{u}}_i^{\star}(\tau)\big),\ \tau \geq t_0 = 0 \tag{3.11}$$

where $\overline{\mathbf{u}}_i^{\star}(\tau) = \overline{\mathbf{u}}_i^{\star}(\tau;\ \mathbf{e}_i(t_k))$, $\tau \in [t_k, t_k + h)$ and $t_0 = 0$.

We can now give the definition of an *admissible input* for the FHOCP (3.3.1):

Definition 3.3.1. (*Admissible input for the FHOCP* (3.3.1))

A control input $\mathbf{u}_i : [t_k, t_k + T_p] \to \mathbb{R}^6$ for a state $\mathbf{e}_i(t_k)$ is called *admissible* for the problem (3.3.1) if all the following hold:

1. $\mathbf{u}_i(\cdot)$ is piecewise continuous

2. $\mathbf{u}_i(\tau) \in \mathcal{U}_i,\ \forall \tau \in [t_k, t_k + T_p]$

3. $\overline{\mathbf{e}}_i\big(\tau;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(t_k)\big) \in \mathcal{E}_{i,s},\ \forall \tau \in [t_k, t_k + T_p]$

4. $\overline{\mathbf{e}}_i\big(t_k + T_p;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(t_k)\big) \in \Omega_i$

In other words, $\mathbf{u}_i$ is admissible if it conforms to the constraints on the input and its application yields states that conform to the prescribed state constraints of problem (3.3.1) along the entire horizon $[t_k, t_k + T_p]$, and the terminal predicted state conforms to the terminal constraint.

3.4 Stabilization: Feasibility and Convergence

Under these considerations, we can now state the theorem that relates to the guaranteeing of the stability of the compound system of agents $i \in \mathcal{V}$, when each of them is assigned a desired position which results in feasible displacements:

Theorem 3.4.1. Suppose that

1. the terminal region $\Omega_i \subseteq \mathcal{E}_{i,s}$ is closed with $\mathbf{0} \in \Omega_i$
2. a solution to the optimal control problem (3.5) is feasible at time $t = 0$, that is, assumptions (2.1.1), (2.2.1), and (2.3.1) hold at time $t = 0$
3. assumptions (3.2.1), (3.3.1), and (3.3.2) hold
4. there exists an admissible control input $h_i(\mathbf{e}_i) : [t_k + T_p, t_{k+1} + T_p] \rightarrow \mathcal{U}_i$ such that for all $\mathbf{e}_i \in \Omega_i$ and $\forall \tau \in [t_k + T_p, t_{k+1} + T_p]$:

   (a) $\mathbf{e}_i(\tau) \in \Omega_i$

   (b) $\dfrac{\partial V_i}{\partial \mathbf{e}_i} g_i\big(\mathbf{e}_i(\tau), h_i\big(\mathbf{e}_i(\tau)\big) + F_i\big(\mathbf{e}_i(\tau), h_i\big(\mathbf{e}_i(\tau)\big)\big) \leq 0$

then the closed loop system (3.11) under the control input (3.10) converges to the set Ω_i when $t \rightarrow \infty$.

Proof. The proof of the above theorem consists of two parts: in the first, recursive feasibility is established, that is, initial feasibility is shown to imply subsequent feasibility; in the second, and based on the first part, it is shown that the error state $\mathbf{e}_i(t)$ converges to the terminal set Ω_i.

Feasibility analysis Consider a sampling instant t_k for which a solution $\overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big)$ to (3.5) exists. Suppose now a time instant t_{k+1} such that[2] $t_k < t_{k+1} < t_k + T_p$, and consider that the optimal control signal calculated at t_k is comprised by the following two portions:

$$\overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big) = \begin{cases} \overline{\mathbf{u}}_i^\star\big(\tau_1;\ \mathbf{e}_i(t_k)\big), & \tau_1 \in [t_k, t_{k+1}] \\ \\ \overline{\mathbf{u}}_i^\star\big(\tau_2;\ \mathbf{e}_i(t_k)\big), & \tau_2 \in [t_{k+1}, t_k + T_p] \end{cases} \tag{3.12}$$

Both portions are admissible since the calculated optimal control input is admissible, and hence they both conform to the input constraints. As for the resulting predicted states, they satisfy the state constraints, and, crucially: $\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \in \Omega_i$. Furthermore, according to assumption (3) of the theorem, there exists an admissible (and certainly not guaranteed optimal) input $h_i(\mathbf{e}_i)$ that renders Ω_i invariant over $[t_k + T_p, t_{k+1} + T_p]$.

Given the above facts, we can construct an admissible input $\widetilde{\mathbf{u}}_i(\cdot)$ starting at time t_{k+1} by sewing together the second portion of (3.12) and the input $h_i(\mathbf{e}_i)$:

$$\widetilde{\mathbf{u}}_i(\tau) = \begin{cases} \overline{\mathbf{u}}_i^\star\big(\tau;\ \mathbf{e}_i(t_k)\big), & \tau \in [t_{k+1}, t_k + T_p] \\ \\ h_i\big(\mathbf{e}_i(\tau)\big), & \tau \in (t_k + T_p, t_{k+1} + T_p] \end{cases} \tag{3.13}$$

The control input $\widetilde{\mathbf{u}}_i(\cdot)$ is admissible as a composition of admissible control inputs. This means that feasibility of a solution to the optimization problem at time t_k implies feasibility at time $t_{k+1} > t_k$, and, thus, since at time $t = 0$ a solution is assumed to be feasible, a solution to the optimal control problem is feasible for all $t \geq 0$.

[2] It is not strictly necessary that $t_{k+1} = t_k + h$ here, however it is necessary for the following that $t_{k+1} - t_k \leq h$

Convergence analysis The second part of the proof involves demonstrating the convergence of the state $\mathbf{e}_i$ to the terminal set Ω_i. In order for this to be proved, it must be shown that a proper value function decreases along closed-loop trajectories starting at some initial time t_k. We consider the *optimal* cost $J_i^\star\big(\mathbf{e}_i(t)\big)$ as a candidate Lyapunov function:

$$J_i^\star\big(\mathbf{e}_i(t)\big) \triangleq J_i\Big(\mathbf{e}_i(t), \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t)\big)\Big)$$

and, in particular, our goal is to show that that this cost decreases over consecutive sampling instants $t_{k+1} = t_k + h$, i.e. $J_i^\star\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) \leq 0$.

In order not to wreak notational havoc, let us define the following terms:

- $\mathbf{u}_{0,i}(\tau) \triangleq \overline{\mathbf{u}}_i^\star\big(\tau;\ \mathbf{e}_i(t_k)\big)$ as the *optimal* input that results from the solution to problem (3.3.1) based on the measurement of state $\mathbf{e}_i(t_k)$, applied at time $\tau \geq t_k$

- $\mathbf{e}_{0,i}(\tau) \triangleq \overline{\mathbf{e}}_i\big(\tau;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big)$ as the *predicted* state at time $\tau \geq t_k$, that is, the state that results from the application of the above input $\overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big)$ to the state $\mathbf{e}_i(t_k)$, at time τ

- $\mathbf{u}_{1,i}(\tau) \triangleq \widetilde{\mathbf{u}}_i(\tau)$ as the *admissible* input at $\tau \geq t_{k+1}$ (see eq. (3.13))

- $\mathbf{e}_{1,i}(\tau) \triangleq \overline{\mathbf{e}}_i\big(\tau;\ \widetilde{\mathbf{u}}_i(\cdot), \mathbf{e}_i(t_{k+1})\big)$ as the *predicted* state at time $\tau \geq t_{k+1}$, that is, the state that results from the application of the above input $\widetilde{\mathbf{u}}_i(\cdot)$ to the state $\mathbf{e}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big)$, at time τ

Remark 3.4.1. Given that no model mismatch or disturbances exist, for the predicted and actual states at time $\tau_1 \geq \tau_0 \in \mathbb{R}_{\geq 0}$ it holds that:

$$\mathbf{e}_i\big(\tau_1;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(\tau_0)\big) = \mathbf{e}_i(\tau_0) + \int_{\tau_0}^{\tau_1} g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(\tau_0)), \mathbf{u}_i(s)\big)ds$$

$$\overline{\mathbf{e}}_i\big(\tau_1;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(\tau_0)\big) = \mathbf{e}_i(\tau_0) + \int_{\tau_0}^{\tau_1} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(\tau_0)), \mathbf{u}_i(s)\big)ds$$

Before beginning to prove convergence, it is worth noting that while the cost

$$J_i\Big(\mathbf{e}_i(t), \overline{\mathbf{u}}_i^{\star}\big(\cdot;\ \mathbf{e}_i(t)\big)\Big)$$

is optimal (in the sense that it is based on the optimal input, which provides its minimum realization), a cost that is based on a plainly admissible (and thus, without loss of generality, sub-optimal) input $\mathbf{u}_i \neq \overline{\mathbf{u}}_i^{\star}$ will result in a configuration where

$$J_i\Big(\mathbf{e}_i(t), \mathbf{u}_i\big(\cdot;\ \mathbf{e}_i(t)\big)\Big) \geq J_i\Big(\mathbf{e}_i(t), \overline{\mathbf{u}}_i^{\star}\big(\cdot;\ \mathbf{e}_i(t)\big)\Big)$$

Let us now begin our investigation on the difference between the cost that results from the application of the feasible input $\mathbf{u}_{1,i}$, which we shall denote by $\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big)$, and the optimal cost $J_i^{\star}\big(\mathbf{e}_i(t_k)\big)$. We remind ourselves that $J_i\big(\mathbf{e}_i(t), \overline{\mathbf{u}}_i(\cdot)\big) = \int_t^{t+T_p} F_i\big(\overline{\mathbf{e}}_i(s), \overline{\mathbf{u}}_i(s)\big)ds$ $+\ V_i\big(\overline{\mathbf{e}}_i(t+T_p)\big)$:

$$\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^{\star}\big(\mathbf{e}_i(t_k)\big) = V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) + \int_{t_{k+1}}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds$$
$$-V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) - \int_{t_k}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds$$

Considering that $t_k < t_{k+1} < t_k + T_p < t_{k+1} + T_p$, we break down the two integrals above in between these intervals:

$$\begin{aligned}
&\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^{\star}\big(\mathbf{e}_i(t_k)\big) = \\
&\quad V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) + \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds \\
&\quad -V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) - \int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds - \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds
\end{aligned} \tag{3.14}$$

Since no model mismatch or disturbances are present, consulting with remark (3.4.1) and substituting for $\tau_0 = t_k$ and $\tau_1 = t_{k+1}$ yields:

$$\mathbf{e}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big),\mathbf{e}_i(t_k)\big) = \mathbf{e}_i(t_k) + \int_{t_k}^{t_{k+1}} g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t_k)),\overline{\mathbf{u}}_i^\star(s)\big)ds$$

$$\overline{\mathbf{e}}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big),\mathbf{e}_i(t_k)\big) = \mathbf{e}_i(t_k) + \int_{t_k}^{t_{k+1}} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)),\overline{\mathbf{u}}_i^\star(s)\big)ds$$

Subtracting the second expression from the first, we get

$$\begin{aligned}
&\mathbf{e}_i\big(t_{k+1};\overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big),\mathbf{e}_i(t_k)\big) - \overline{\mathbf{e}}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big),\mathbf{e}_i(t_k)\big)\\
&\qquad = \int_{t_k}^{t_{k+1}} g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t_k)),\overline{\mathbf{u}}_i^\star(s)\big)ds - \int_{t_k}^{t_{k+1}} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)),\overline{\mathbf{u}}_i^\star(s)\big)ds\\
&\qquad = \int_{t_k}^{t_{k+1}} \Big(g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t_k)),\overline{\mathbf{u}}_i^\star(s)\big) - g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)),\overline{\mathbf{u}}_i^\star(s)\big)\Big)ds
\end{aligned}$$

Taking norms on either side yields

$$\begin{aligned}
&\Big\|\mathbf{e}_i\big(t_{k+1};\overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big),\mathbf{e}_i(t_k)\big) - \overline{\mathbf{e}}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big),\mathbf{e}_i(t_k)\big)\Big\|\\
&\qquad = \Big\|\int_{t_k}^{t_{k+1}} \Big(g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t_k)),\overline{\mathbf{u}}_i^\star(s)\big) - g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)),\overline{\mathbf{u}}_i^\star(s)\big)\Big)ds\Big\|\\
&\qquad = \int_{t_k}^{t_{k+1}} \Big\|g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t_k)),\overline{\mathbf{u}}_i^\star(s)\big) - g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)),\overline{\mathbf{u}}_i^\star(s)\big)\Big\|ds\\
&\qquad \leq L_{g_i}\int_{t_k}^{t_{k+1}} \Big\|\mathbf{e}_i\big(s;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big),\mathbf{e}_i(t_k)\big) - \overline{\mathbf{e}}_i\big(s;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big),\mathbf{e}_i(t_k)\big)\Big\|ds
\end{aligned}$$

since g_i is Lipschitz continuous in $\mathcal{E}_{i,s}$ with Lipschitz constant L_{g_i}. Reformulation yields

$$\Big\|\mathbf{e}_i\big(t_k + h;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big),\mathbf{e}_i(t_k)\big) - \overline{\mathbf{e}}_i\big(t_k + h;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big),\mathbf{e}_i(t_k)\big)\Big\|$$

$$\leq L_{g_i} \int_0^h \Big\| \mathbf{e}_i\big(t_k + s;\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t_k)), \mathbf{e}_i(t_k)\big) - \overline{\mathbf{e}}_i\big(t_k + s;\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t_k)), \mathbf{e}_i(t_k)\big) \Big\| ds$$

By applying the Grönwall-Bellman inequality we obtain zero as an upper bound for the norm of the difference between the two states. Since any norm cannot be negative, we conclude that

$$\Big\| \mathbf{e}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t_k)), \mathbf{e}_i(t_k)\big) - \overline{\mathbf{e}}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t_k)), \mathbf{e}_i(t_k)\big) \Big\| = 0$$

which means that

$$\mathbf{e}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t_k)), \mathbf{e}_i(t_k)\big) = \overline{\mathbf{e}}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t_k)), \mathbf{e}_i(t_k)\big)$$

In between times t_{k+1} and $t_k + T_p$, the constructed admissible input $\widetilde{\mathbf{u}}_i(\cdot)$ is equal to the optimal input $\overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big)$ (see eq. 3.13), which means that $\mathbf{u}_{1,i}(\tau) = \mathbf{u}_{0,i}(\tau)$ in the interval $\tau \in [t_{k+1}, t_k + T_p]$. Since the initial conditions at $t = t_{k+1}$ are equal and the control laws are also equal, so will the predicted states over the same interval:

$$\overline{\mathbf{e}}_i\big(\tau;\ \widetilde{\mathbf{u}}_i(\cdot), \mathbf{e}_i(t_{k+1})\big) = \overline{\mathbf{e}}_i\big(\tau;\ \overline{\mathbf{u}}_i^\star(\cdot), \overline{\mathbf{e}}_i(t_{k+1})\big),\ \tau \in [t_{k+1}, t_k + T_p] \tag{3.15}$$

Using our notation then, in the same interval: $\mathbf{e}_{1,i}(\cdot) = \mathbf{e}_{0,i}(\cdot)$. Coupled with the fact that in the same interval $\mathbf{u}_{1,i}(\tau) = \mathbf{u}_{0,i}(\tau)$, the following equality holds over $[t_{k+1}, t_k + T_p]$:

$$F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) = F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big),\ s \in [t_{k+1}, t_k + T_p]$$

Integrating this equality over the interval where it is valid yields

$$\int_{t_{k+1}}^{t_k + T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds = \int_{t_{k+1}}^{t_k + T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) ds$$

This means that these two integrals with ends over the interval $[t_{k+1}, t_k + T_p]$ featured in the right-hand side of eq. (3.14) vanish, and thus the cost difference becomes

$$\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) = V_i\big(\mathbf{e}_{1,i}(t_{k+1} + T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds$$

$$- V_i\big(\mathbf{e}_{0,i}(t_k + T_p)\big) - \int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds \tag{3.16}$$

During the course of arriving at the above result, we have concluded that, in the absence of disturbances, the remark (3.4.2) holds.

Remark 3.4.2. In the absence of disturbances, the following equality holds over the entire horizon $t \in [t_k, t_k + T_p]$:

$$\mathbf{e}_i\big(t;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) = \overline{\mathbf{e}}_i\big(t;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big)$$

We turn our attention to the first integral in the above expression, and we note that $[t_{k+1} + T_p, t_k + T_p]$, is exactly the the interval where assumption (3b) of the theorem holds. Hence, we decide to integrate the expression found in the assumption over the interval $[t_k + T_p, t_{k+1} + T_p]$, for the controls and states applicable in it:

$$\int_{t_k+T_p}^{t_{k+1}+T_p} \left(\frac{\partial V_i}{\partial \mathbf{e}_{1,i}} g_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) + F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) \right) ds \leq 0$$

$$\int_{t_k+T_p}^{t_{k+1}+T_p} \frac{d}{ds} V_i\big(\mathbf{e}_{1,i}(s)\big)ds + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds \leq 0$$

$$V_i\big(\mathbf{e}_{1,i}(t_{k+1} + T_p)\big) - V_i\big(\mathbf{e}_{1,i}(t_k + T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds \leq 0$$

$$V_i\big(\mathbf{e}_{1,i}(t_{k+1} + T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds \leq V_i\big(\mathbf{e}_{1,i}(t_k + T_p)\big)$$

The left-hand side expression is the same as the first two terms in the right-hand side of equality (3.16). We can introduce the third one by subtracting it from both sides:

$$V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big)+\int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s),\mathbf{u}_{1,i}(s)\big)ds - V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big)$$

$$\leq V_i\big(\mathbf{e}_{1,i}(t_k+T_p)\big) - V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big)$$

$$\leq \Big|V_i\big(\mathbf{e}_{1,i}(t_k+T_p)\big) - V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big)\Big|$$

since $x \leq |x|, \forall x \in \mathbb{R}$.

By revisiting lemma (3.3.3), the above inequality becomes

$$V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big)+\int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s),\mathbf{u}_{1,i}(s)\big)ds - V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big)$$

$$\leq L_{V_i}\big\|\mathbf{e}_{1,i}(t_k+T_p) - \mathbf{e}_{0,i}(t_k+T_p)\big\|$$

However, as we witnessed in (3.15), in the interval $[t_{k+1}, t_k+T_p]$: $\mathbf{e}_{1,i}(\cdot) = \mathbf{e}_{0,i}(\cdot)$, hence the right-hand side of the inequality equals zero:

$$V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big)+\int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s),\mathbf{u}_{1,i}(s)\big)ds - V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) \leq 0$$

By subtracting the fourth term needed to complete the right-hand side expression of (3.16), i.e. $\int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s),\mathbf{u}_{0,i}(s)\big)ds$ from both sides we get

$$V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big)+\int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s),\mathbf{u}_{1,i}(s)\big)ds$$

$$-V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) - \int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s),\mathbf{u}_{0,i}(s)\big)ds \leq -\int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s),\mathbf{u}_{0,i}(s)\big)ds$$

The left-hand side of this inequality is now equal to the cost difference $\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^{\star}\big(\mathbf{e}_i(t_k)\big)$.

Hence, the cost difference becomes bounded by

$$\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) \leq -\int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) ds$$

F_i is a positive-definite function as a sum of a positive-definite $\|\mathbf{u}_i\|^2_{\boldsymbol{R}_i}$ and a positive semi-definite function $\|\mathbf{e}_i\|^2_{\boldsymbol{Q}_i}$. If we denote by $m_i = \lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i) \geq 0$ the minimum eigenvalue between those of matrices $\boldsymbol{R}_i$, $\boldsymbol{Q}_i$, this means that

$$F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) \geq m_i \|\mathbf{e}_{0,i}(s)\|^2$$

By integrating the above between our interval of interest $[t_k, t_{k+1}]$ we get

$$\int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) \geq \int_{t_k}^{t_{k+1}} m_i \|\mathbf{e}_{0,i}(s)\|^2 ds$$

or

$$-\int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) \leq -m_i \int_{t_k}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds$$

This means that the cost difference is upper-bounded by a class $\mathcal{K}$ function

$$\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) \leq -m_i \int_{t_k}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds \leq 0$$

and since the cost $\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big)$ is, in general, sub-optimal: $J_i^\star\big(\mathbf{e}_i(t_{k+1})\big) - \overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) \leq 0$:

$$J_i^\star\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) \leq -m_i \int_{t_k}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds \tag{3.17}$$

With this milestone result established, we need to trace the time t_k back to $t_0 = 0$ in order to prove that the closed-loop system is stable at all times $t \in \mathbb{R}_{\geq 0}$.

The integral of $\|\mathbf{e}_{0,i}(\tau)\|^2$ over the interval $[t_0, t_{k+1}]$, $t_0 < t_k < t_{k+1}$ can be decomposed into the addition of two integrals with limits ranging from (a) t_0 to t_k and (b) t_k to t_{k+1}:

$$\int_{t_0}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds = \int_{t_0}^{t_k} \|\mathbf{e}_{0,i}(s)\|^2 ds + \int_{t_k}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds$$

By rearranging terms, this means that

$$\int_{t_k}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds = \int_{t_0}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds - \int_{t_0}^{t_k} \|\mathbf{e}_{0,i}(s)\|^2 ds$$

making the optimal cost difference between the consecutive sampling times t_k and t_{k+1} in (3.17)

$$J_i^\star\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) \leq -m_i \int_{t_0}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds + m_i \int_{t_0}^{t_k} \|\mathbf{e}_{0,i}(s)\|^2 ds$$

Similarly, the optimal cost difference between the sampling times t_{k-1} and t_k is

$$J_i^\star\big(\mathbf{e}_i(t_k)\big) - J_i^\star\big(\mathbf{e}_i(t_{k-1})\big) \leq -m_i \int_{t_0}^{t_k} \|\mathbf{e}_{0,i}(s)\|^2 ds + m_i \int_{t_0}^{t_{k-1}} \|\mathbf{e}_{0,i}(s)\|^2 ds$$

and we can apply this rationale all the way back to the cost difference between t_0 and t_1. Summing all the inequalities between the pairs of consecutive sampling times (t_0, t_1), (t_1, t_2), $\dots$, (t_{k-1}, t_k), we get

$$J_i^\star\big(\mathbf{e}_i(t_k)\big) - J_i^\star\big(\mathbf{e}_i(t_0)\big) \leq -m_i \int_{t_0}^{t_k} \|\mathbf{e}_{0,i}(s)\|^2 ds$$

Hence, for $t_0 = 0$

$$J_i^\star\big(\mathbf{e}_i(t_k)\big) - J_i^\star\big(\mathbf{e}_i(0)\big) \leq -m_i \int_{0}^{t_k} \|\mathbf{e}_{0,i}(s)\|^2 ds \leq 0 \tag{3.18}$$

which implies that the value function $J_i^\star\big(\mathbf{e}_i(t_k)\big)$ in non-increasing for all sampling times:

$$J_i^\star\big(\mathbf{e}_i(t_k)\big) \leq J_i^\star\big(\mathbf{e}_i(0)\big), \ \forall t_k \in \mathbb{R}_{\geq 0}$$

Let us now define the function $\Xi_i\big(\mathbf{e}_i(t)\big)$:

$$\Xi_i\big(\mathbf{e}(t)\big) \triangleq J_i^\star\big(\mathbf{e}_i(\tau)\big),\ t \in \mathbb{R}_{\geq 0}$$

where $\tau = max\{t_k : t_k \leq t\}$ – i.e. the immediately previous to t sampling time. Then the above inequality reforms into

$$\Xi_i\big(\mathbf{e}(t)\big) \leq \Xi_i\big(\mathbf{e}_i(0)\big),\ t \in \mathbb{R}_{\geq 0}$$

Since $\Xi_i\big(\mathbf{e}_i(0)\big)$ is bounded (as composition of bounded parts), this implies that $\Xi_i\big(\mathbf{e}(t)\big)$ is also bounded. The signals $\mathbf{e}_i(t) \in \mathcal{E}_{i,s}$ and $\mathbf{u}_i(t) \in \mathcal{U}_i$ are also bounded. According to (3.4), this means that $\dot{\mathbf{e}}_i(t)$ is bounded as well. From inequality (3.18) we then have

$$\Xi_i\big(\mathbf{e}_i(t)\big) \leq \Xi_i\big(\mathbf{e}_i(0)\big) - m_i \int_0^\tau \|\mathbf{e}_{0,i}(s)\|^2 ds \leq 0$$

which, due to the fact that $\tau \leq t$, is equivalent to

$$\Xi_i\big(\mathbf{e}_i(t)\big) \leq \Xi_i\big(\mathbf{e}_i(0)\big) - m_i \int_0^t \|\mathbf{e}_{0,i}(s)\|^2 ds \leq 0,\ t \in \mathbb{R}_{\geq 0}$$

Solving for the integral we get

$$\int_0^t \|\mathbf{e}_{0,i}(s)\|^2 ds \leq \frac{1}{m_i}\Big(\Xi_i\big(\mathbf{e}_i(0)\big) - \Xi_i\big(\mathbf{e}_i(t)\big)\Big),\ t \in \mathbb{R}_{\geq 0}$$

Both $\Xi_i\big(\mathbf{e}_i(0)\big)$ and $\Xi_i\big(\mathbf{e}_i(t)\big)$ are bounded, and therefore so is their difference, which means that the integral $\int\limits_0^t \|\mathbf{e}_{0,i}(s)\|^2 ds$ and its limit when $t \to \infty$ are bounded as well. Lemma (1.2.2) assures us that under these conditions for the error and its dynamics, which are fulfilled in our case, the error

$$\lim_{t\to\infty} \|\mathbf{e}_{0,i}(t)\| = 0 \Leftrightarrow$$

$$\lim_{t\to\infty} \Big\|\overline{\mathbf{e}}_i\Big(t;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\Big)\Big\| = 0,\ \forall t_k \in \mathbb{R}_{\geq 0}$$

which, given remark (3.4.2), and dropping the initial condition, means that

$$\lim_{t \to \infty} \|\mathbf{e}_i(t)\| = 0$$

which implies that

$$\lim_{t \to \infty} \mathbf{e}_i(t) \in \Omega_i$$

Therefore, the closed-loop trajectory of the error state $\mathbf{e}_i$ converges to the terminal set Ω_i as $t \to \infty$.

In turn, this means that the system (3.3) converges to $\mathbf{z}_{i,des}$ while simultaneously conforming to all constraints $\mathcal{Z}_i$, as $t \to \infty$. This conclusion holds for all $i \in \mathcal{V}$, and hence, the compound system of agents $\mathcal{V}$ is stable in $\mathcal{Z}_i$. ■

4

Stabilization in the face of Disturbances

We are interested in steering each agent $i \in \mathcal{V}$ into a *position* in 3D space, while conforming to the requirements posed by the problem. Here, the real system and its model are *not* equivalent: we consider that additive disturbances act on the real system. At first, the model of the perturbed system (2.1) will be formalized. Next, the error model and *its* constraints will be expressed. We will then pose the optimization problem to be solved periodically: it will equip us with the optimum feasible input that steers the system towards achieving its intended configuration regardless of the introduced uncertainty, provided that it is bounded by a certain value. This will lead to the proof of this statement, i.e. that the compound closed-loop system of agents $i \in \mathcal{V}$ is stable under the proposed control regime, provided that the disturbance is bounded.

4.1 The perturbed model

In the following, we assume that the real system is subject to bounded additive disturbances $\boldsymbol{\delta}_i$ such that $\boldsymbol{\delta}_i \in \Delta_i \subset \mathbb{R}^9 \times \mathbb{T}^3$, where Δ_i is a compact set containing the origin. The real system is described by:

$$\begin{aligned}
\dot{\mathbf{z}}_i(t) &= f_i^R\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big) \\
&= f_i\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big) + \boldsymbol{\delta}_i(t) \\
\mathbf{z}_i(0) &= \mathbf{z}_{i,0} \\
\mathbf{z}_i(t) &\subset \mathbb{R}^9 \times \mathbb{T}^3 \\
\mathbf{u}_i(t) &\subset \mathbb{R}^6 \\
\boldsymbol{\delta}_i(t) &\in \Delta_i \subset \mathbb{R}^9 \times \mathbb{T}^3, \ t \in \mathbb{R}_{\geq 0} \\
\sup_{t \in \mathbb{R}_{\geq 0}} \|\boldsymbol{\delta}_i(t)\| &\leq \overline{\delta}_i
\end{aligned} \tag{4.1}$$

where state $\mathbf{z}_i$ is directly measurable as per assumption (2.1.1).

The constraint set $\mathcal{Z}_i \subset \mathbb{R}^9 \times \mathbb{T}^3$ is unchanged: it is the set that captures all the state constraints of the system's dynamics posed by the problem (2.4), for $t \in \mathbb{R}_{\geq 0}$. We include it again here for reference purposes.

$$\begin{aligned}
\mathcal{Z}_i = \big\{\mathbf{z}_i(t) \in \mathbb{R}^9 \times \mathbb{T}^3 : \ & \|\mathbf{p}_i(t) - \mathbf{p}_j(t)\| > \underline{d}_{ij,a}, \forall j \in \mathcal{R}_i(t), \\
& \|\mathbf{p}_i(t) - \mathbf{p}_j(t)\| < d_i, \forall j \in \mathcal{N}_i, \\
& \|\mathbf{p}_i(t) - \mathbf{p}_\ell\| > \underline{d}_{i\ell,o}, \forall \ell \in \mathcal{L}, \\
& \|\mathbf{p}_W - \mathbf{p}_i(t)\| < \overline{d}_{i,W}, \\
& -\frac{\pi}{2} < \theta_i(t) < \frac{\pi}{2},
\end{aligned}$$

$\forall t \in \mathbb{R}_{\geq 0}\big\}$

4.2 The error model

A feasible desired configuration $\mathbf{z}_{i,des} \in \mathbb{R}^9 \times \mathbb{T}^3$ is associated to each agent $i \in \mathcal{V}$, with the aim of agent i achieving it in steady-state: $\lim_{t\to\infty} \|\mathbf{z}_i(t) - \mathbf{z}_{i,des}\| = 0$. The interior of the norm of this expression denotes the state error of agent i:

$$\mathbf{e}_i(t) = \mathbf{z}_i(t) - \mathbf{z}_{i,des}, \;\; \mathbf{e}_i(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^9 \times \mathbb{T}^3$$

The error dynamics are equally affected by the additive uncertainty; they are denoted by $g_i^R(\mathbf{e}_i, \mathbf{u}_i)$:

$$\begin{aligned}\dot{\mathbf{e}}_i(t) = \dot{\mathbf{z}}_i(t) - \dot{\mathbf{z}}_{i,des} = \dot{\mathbf{z}}_i(t) = f_i^R\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big) &= f_i\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big) + \boldsymbol{\delta}_i(t) \\ &= g_i\big(\mathbf{e}_i(t), \mathbf{u}_i(t)\big) + \boldsymbol{\delta}_i(t) \\ &= g_i^R\big(\mathbf{e}_i(t), \mathbf{u}_i(t)\big) \end{aligned} \tag{4.2}$$

with $\mathbf{e}_i(0) = \mathbf{z}_i(0) - \mathbf{z}_{i,des}$. The error-wise translated constraints that are required on the state $\mathbf{z}_i(t)$ are unchanged:

$$\mathcal{E}_i = \big\{\mathbf{e}_i(t) \in \mathbb{R}^9 \times \mathbb{T}^3 : \mathbf{e}_i(t) \in \mathcal{Z}_i \ominus \mathbf{z}_{i,des}\big\}$$

Again, $\mathcal{E}_i$ is the set that captures all constraints for the error dynamics (4.2) dictated by the problem (2.4).

On functions g_i, g_i^R we make the following assumption:

Assumption 4.2.1. Functions g_i, g_i^R are Lipschitz continuous in $\mathcal{E}_i$ with Lipschitz constants L_{g_i}.

If we design control laws $\mathbf{u}_i \in \mathcal{U}_i$, $\forall i \in \mathcal{V}$ such that the error signal $\mathbf{e}_i(t)$ with dynamics given in (4.2), constrained under $\mathbf{e}_i(t) \in \mathcal{E}_i$, satisfies $\lim_{t\to\infty} \|\mathbf{e}_i(t)\| = 0$, while all system related signals remain bounded in their respective regions,– if all of the above are achieved, then problem (2.4) has been solved.

In order to achieve this task, we employ a Nonlinear Receding Horizon scheme.

4.3 The optimization problem

Consider a sequence of sampling times $\{t_k\}_{k\geq 0}$, with a constant sampling time h, $0 < h < T_p$, where T_p is the finite time-horizon, such that $t_{k+1} = t_k + h$. In sampling data NMPC, a finite-horizon open-loop optimal control problem (FHOCP) is solved at discrete sampling time instants t_k based on the then-current state error measurement $\mathbf{e}_i(t_k)$. The solution is an optimal control signal $\overline{\mathbf{u}}_i^\star(t)$, computed over $t \in [t_k, t_k + T_p]$. This signal is applied to the open-loop system in between sampling times t_k and $t_k + h$.

At a generic time t_k then, agent i solves the following optimization problem:

Problem 4.3.1.

Find

$$J_i^\star\big(\mathbf{e}_i(t_k)\big) \triangleq \min_{\overline{\mathbf{u}}_i(\cdot)} \quad J_i\big(\mathbf{e}_i(t_k), \overline{\mathbf{u}}_i(\cdot)\big) \tag{4.3}$$

where

$$J_i\big(\mathbf{e}_i(t_k), \overline{\mathbf{u}}_i(\cdot)\big) \triangleq \int_{t_k}^{t_k+T_p} F_i\big(\overline{\mathbf{e}}_i(s), \overline{\mathbf{u}}_i(s)\big) ds + V_i\big(\overline{\mathbf{e}}_i(t_k + T_p)\big)$$

subject to:

$$\dot{\overline{\mathbf{e}}}_i(s) = g_i\big(\overline{\mathbf{e}}_i(s), \overline{\mathbf{u}}_i(s)\big),\ \overline{\mathbf{e}}_i(t_k) = \mathbf{e}_i(t_k) \tag{4.4}$$

$$\overline{\mathbf{u}}_i(s) \in \mathcal{U}_i,\ \overline{\mathbf{e}}_i(s) \in \mathcal{E}_{i,s-t_k},\ s \in [t_k, t_k + T_p]$$

$$\overline{\mathbf{e}}_i(t_k + T_p) \in \Omega_i$$

The notation $\bar{\cdot}$ is used to distinguish predicted states which are internal to the controller, as opposed to their actual values, because, even in the nominal case, the predicted values will not be equal to the actual closed-loop values. This means that $\overline{\mathbf{e}}_i(\cdot)$ is the solution to (4.4) driven by the control input $\overline{\mathbf{u}}_i(\cdot) : [t_k, t_k + T_p] \rightarrow \mathcal{U}_i$ with initial condition $\mathbf{e}_i(t_k)$.

The applied input signal is a portion of the optimal solution to an optimization problem where information on the states of the neighbouring agents of agent i are taken into account only in the constraints considered in the optimization problem. These constraints pertain to the set of its neighbours $\mathcal{N}_i$ and, in total, to the set of all agents within its sensing range $\mathcal{R}_i$. Regarding these, we assume assumption (3.3.1), i.e. at time t_k when agent i solves the optimization problem, he has access to the measurements of the states and the values of the predicted states for all agents within its sensing range.

?? more on the actual $\mathcal{E}_i$

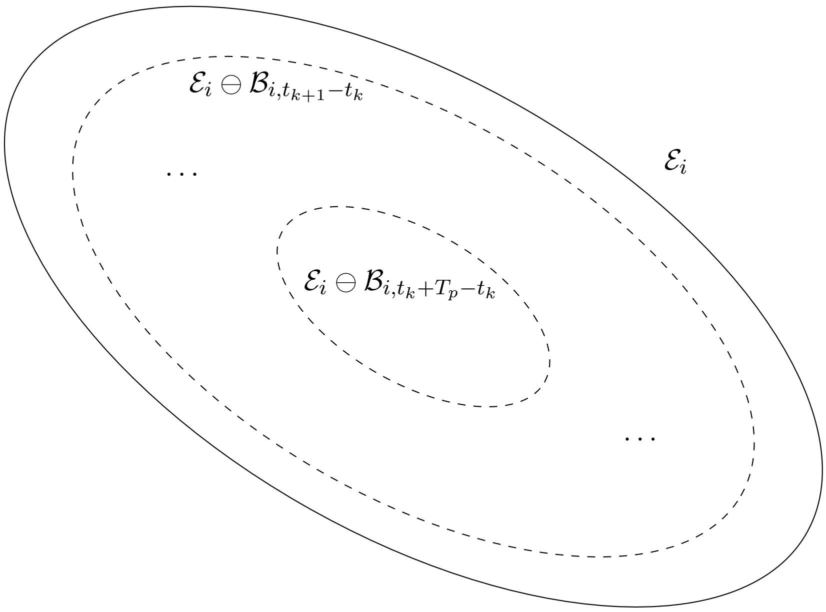

Figure 4.1: The nominal constraint set $\mathcal{E}_i$ in bold and the consecutive restricted constraint sets $\mathcal{E}_i \ominus \mathcal{B}_{i,s-t_k}$, $s \in [t_k, t_k + T_p]$, dashed.

While in the disturbance-free case the constraint set is $\mathcal{E}_i$, due to the existence of disturbances here, the constraint set is replaced in problem (4.3.1) by

$$\mathcal{E}_{i,s-t_k} \equiv \mathcal{E}_i \ominus \mathcal{B}_{i,s-t_k} \tag{4.5}$$

where

$$\mathcal{B}_{i,s-t} \equiv \left\{ \mathbf{e}_i \in \mathbb{R}^9 \times \mathbb{T}^3 : \|\mathbf{e}_i(s)\| \leq \frac{\bar{\delta}_i}{L_{g_i}} \left(e^{L_{g_i}(s-t)} - 1 \right), \ \forall s \in [t, t+T_p] \right\} \tag{4.6}$$

The reason for this substitution lies in the following. Consider that there are no disturbances affecting the states of the plant; the state evolution of the plant and its model considered in the solution to the optimization problem abide both by the state constraints since the two models are identical. Consider now that there are disturbances affecting the states of the plant − disturbances that are unknown to the model considered in the solution to the optimization problem. If the state constraint set was left unchanged during the solution of the optimization problem, the applied input to the plant, coupled with the uncertainty affecting the states of the plant could, without loss of generality[1], force the states of the plant to escape their intended bounds.

If the state constraint set considered in the solution of the optimization problem (4.3.1) is equal to (4.5), then the state of the real system, the plant, is guaranteed to abide by the original state constraint set $\mathcal{E}_i$. We formalize this statement in property (4.3.1).

Property 4.3.1. For every $s \in [t, t + T_p]$

$$\overline{\mathbf{e}}_i\big(s;\ \mathbf{u}_i(\cdot,\ \mathbf{e}_i(t)), \mathbf{e}_i(t)\big) \in \mathcal{E}_i \ominus \mathcal{B}_{i,s-t} \Rightarrow \mathbf{e}_i(s) \in \mathcal{E}_i$$

where $\mathcal{B}_{i,s-t}$ is given by (4.6).

Assumption 4.3.1. The terminal set $\Omega_i \subseteq \Psi_i$ is a subset of an admissible and positively invariant set Ψ_i as per definition (1.2.5), where Ψ_i is defined as

$$\Psi_i \triangleq \big\{\mathbf{e}_i \in \mathcal{E}_i : V_i(\mathbf{e}_i) \leq \varepsilon_{\Psi_i}\big\},\ \varepsilon_{\Psi_i} > 0$$

Assumption 4.3.2. The set Ψ_i belongs to the set Φ_i, $\Psi_i \subseteq \Phi_i$, which is the set of states within $\mathcal{E}_{i,T_p}$ for which there is an admissible control input whose form is of linear feedback with regard to the state:

$$\Phi_i \triangleq \big\{\mathbf{e}_i \in \mathcal{E}_{i,T_p} : h_i(\mathbf{e}_i) \in \mathcal{U}_i\big\}$$

[1]Receding Horizon Control is inherently robust under certain considerations, see [2] for more.

Assumption 4.3.3. The admissible and positively invariant set Ψ_i is such that

$$\forall \mathbf{e}_i(t) \in \Psi_i \Rightarrow \overline{\mathbf{e}}_i\big(t+\tau;\ h_i(\mathbf{e}_i(t)), \mathbf{e}_i(t)\big) \in \Omega_i \subseteq \Psi_i$$

for some $\tau \in [0, h]$

Assumption 4.3.4. The terminal set Ω_i is closed, includes the origin, and is defined by

$$\Omega_i \triangleq \big\{\mathbf{e}_i \in \mathcal{E}_i : V_i(\mathbf{e}_i) \leq \varepsilon_{\Omega_i}\big\}\text{ , where } \varepsilon_{\Omega_i} \in (0, \varepsilon_{\Psi_i})$$

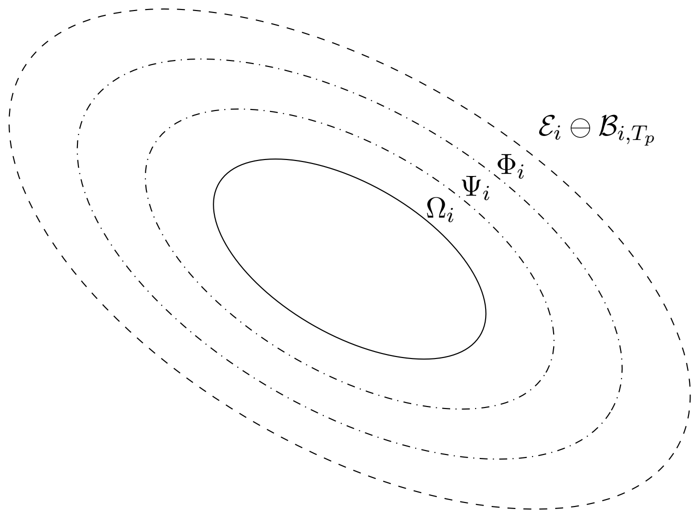

Figure 4.2: The hierarchy of sets $\Omega_i \subseteq \Psi_i \subseteq \Phi_i \subseteq \mathcal{E}_{i,T_p}$, in bold, dash-dotted, dash-dotted, and dashed, respectively. For every state in Φ_i there is a linear state feedback control $h_i(\mathbf{e}_i)$ which, when applied to a state $\mathbf{e}_i \in \Psi_i$, causes the trajectory of the state of the system to fall into the terminal set Ω_i.

Functions $F_i : \mathcal{E}_i \times \mathcal{U}_i \rightarrow \mathbb{R}_{\geq 0}$ and $V_i : \Psi_i \rightarrow \mathbb{R}_{\geq 0}$ are defined by (3.8) and (3.9) respectively, as in the disturbance-free case. Matrices $\boldsymbol{R}_i \in \mathbb{R}^{6\times 6}$, $\boldsymbol{Q}_i, \boldsymbol{P}_i \in \mathbb{R}^{12\times 12}$ are positive definite. Consequently, lemmas (3.3.1), (3.3.2), (3.3.3) and (3.3.4) hold true here, as in the disturbance-free case: the running costs F_i are Lipschitz continuous in $\mathcal{E}_i \times \mathcal{U}_i$ with Lipschitz constant L_{F_i} and they are lower- and upper-bounded by class $\mathcal{K}_\infty$ functions; the terminal penalty functions V_i are Lipschitz continuous in Ψ_i with Lipschitz constant L_{V_i}, and they are lower- and upper-bounded by class $\mathcal{K}_\infty$ functions.

The solution to the optimal control problem (4.3) at time t_k is an optimal control input, denoted by $\overline{\mathbf{u}}_i^{\star}(\cdot;\ \mathbf{e}_i(t_k))$, which is applied to the open-loop system until the next sampling instant $t_k + h$, with $h \in (0, T_p)$.

$$\mathbf{u}_i(t) = \overline{\mathbf{u}}_i^{\star}\big(t;\ \mathbf{e}_i(t_k)\big),\ t \in [t_k, t_k + h] \tag{4.7}$$

At time t_{k+1} a new finite horizon optimal control problem is solved in the same manner, leading to a receding horizon approach.

The control input $\mathbf{u}_i(\cdot)$ is of feedback form, since it is recalculated at each sampling instant based on the then-current state. The solution to equation (4.2), starting at time t_1, from an initial condition $\mathbf{e}_i(t_1) = \overline{\mathbf{e}}_i(t_1)$, by application of the control input $\mathbf{u}_i : [t_1, t_2] \to \mathcal{U}_i$ is denoted by

$$\mathbf{e}_i\big(t;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(t_1)\big),\ t \in [t_1, t_2]$$

As before, the *predicted* state of the system (4.2) at time $t_k + \tau$, based on the measurement of the state at time t_k, $\mathbf{e}_i(t_k)$, by application of the control input $\mathbf{u}_i\big(t;\ \mathbf{e}_i(t_k)\big)$, for the time period $t \in [t_k, t_k + \tau]$ is denoted by

$$\overline{\mathbf{e}}_i\big(t_k + \tau;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(t_k)\big)$$

On the existence of solutions to (4.2) we assume the following:

Assumption 4.3.5. The system (4.2) has a *continuous solution* for any $\mathbf{e}_i(0) \in \mathcal{E}_i$, any *piecewise continuous* input $\mathbf{u}_i(\cdot) : [0, T_p] \to \mathcal{U}_i$, and any *exogenous disturbance* $\delta_i(\cdot) : [0, T_p] \to \Delta_i$.

In contrast to the disturbance-free case where the predicted state coincided with the state of the actual system, due to the existence of disturbances this equality is void.

Remark 4.3.1. The following holds true here because *there are* disturbances acting on the system.

$$\overline{\mathbf{e}}_i\big(\tau_1;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(\tau_0)\big) \neq \mathbf{e}_i\big(\tau_1;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(\tau_0)\big)$$

The closed-loop system for which stability is to be guaranteed is

$$\mathbf{e}_i(\tau) = g_i^R\big(\mathbf{e}_i(\tau), \overline{\mathbf{u}}_i^{\star}(\tau)\big),\ \tau \geq t_0 = 0 \tag{4.8}$$

where $\overline{\mathbf{u}}_i^{\star}(\tau) = \overline{\mathbf{u}}_i^{\star}(\tau;\ \mathbf{e}_i(t_k))$, $\tau \in [t_k, t_k + h)$.

We can now give the definition of an *admissible input* for the FHOCP (4.3.1):

Definition 4.3.1. (*Admissible input for the FHOCP* (4.3.1))
A control input $\mathbf{u}_i : [t_k, t_k + T_p] \rightarrow \mathbb{R}^6$ for a state $\mathbf{e}_i(t_k)$ is called *admissible* for the problem (4.3.1) if all the following hold:

1. $\mathbf{u}_i(\cdot)$ is piecewise continuous
2. $\mathbf{u}_i(\tau) \in \mathcal{U}_i,\ \forall \tau \in [t_k, t_k + T_p]$
3. $\overline{\mathbf{e}}_i\big(t_k + \tau;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(t_k)\big) \in \mathcal{E}_i \ominus \mathcal{B}_{i,\tau},\ \forall \tau \in [0, T_p]$
4. $\overline{\mathbf{e}}_i\big(t_k + T_p;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(t_k)\big) \in \Omega_i$

In other words, $\mathbf{u}_i$ is admissible if it conforms to the constraints on the input and its application yields states that conform to the prescribed state constraints of problem (4.3.1) along the entire horizon $[t_k, t_k + T_p]$, and the terminal predicted state conforms to the terminal constraint.

4.4 Stabilization: Feasibility and Convergence

Under these considerations, we can now state the theorem that relates to the guaranteeing of the stability of the compound system of agents $i \in \mathcal{V}$, when each of them is assigned a desired position which results in feasible displacements:

Theorem 4.4.1. Suppose that

1. a solution to the optimal control problem (4.3) is feasible at time $t = 0$, that is, assumptions (2.1.1), (2.2.1), and (2.3.1) hold at time $t = 0$

2. assumptions (3.3.1), (4.2.1) $-$(4.3.5) hold true

3. there exists an admissible control input of linear feedback form $h_i(\mathbf{e}_i) : [t_k + T_p, t_{k+1} + T_p] \rightarrow \mathcal{U}_i$ such that for all $\mathbf{e}_i \in \Psi_i$ and $\forall \tau \in [t_k + T_p, t_{k+1} + T_p]$:

$$\frac{\partial V_i}{\partial \mathbf{e}_i} g_i\big(\mathbf{e}_i(\tau), h_i(\mathbf{e}_i(\tau))\big) + F_i\big(\mathbf{e}_i(\tau), h_i(\mathbf{e}_i(\tau))\big) \leq 0$$

4. the upper bound $\overline{\delta}_i$ of the disturbance $\boldsymbol{\delta}_i(t)$, $\overline{\delta}_i = \sup_{t \geq 0} \|\boldsymbol{\delta}_i(t)\| = \|\boldsymbol{\delta}_i\|_\infty$ is in turn bounded by

$$\overline{\delta}_i \leq \frac{\varepsilon_{\Psi_i} - \varepsilon_{\Omega_i}}{\dfrac{L_{V_i}}{L_{g_i}}(e^{L_{g_i} h} - 1) e^{L_{g_i}(T_p - h)}}$$

for all $t \in \mathbb{R}_{\geq 0}$

then the closed loop system (4.8) under the control input (4.7) converges to the set Ω_i and is ultimately bounded there.

Proof. The proof of the above theorem consists of two parts: in the first, recursive feasibility is established, that is, initial feasibility is shown to imply subsequent feasibility; in the second, and based on the first part, it is shown that the error state $\mathbf{e}_i(t)$ reaches the terminal set Ω_i and is trapped there.

Remark 4.4.1. Given that disturbances *are* present, for the predicted and actual states at time $\tau_1 \geq \tau_0 \in \mathbb{R}_{\geq 0}$ it holds that:

$$\mathbf{e}_i\big(\tau_1;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(\tau_0)\big) = \mathbf{e}_i(\tau_0) + \int_{\tau_0}^{\tau_1} g_i^R\big(\mathbf{e}_i(s;\ \mathbf{e}_i(\tau_0)), \mathbf{u}_i(s)\big) ds$$

$$\overline{\mathbf{e}}_i\big(\tau_1;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(\tau_0)\big) = \mathbf{e}_i(\tau_0) + \int_{\tau_0}^{\tau_1} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(\tau_0)), \mathbf{u}_i(s)\big)ds$$

Lemma 4.4.1. Suppose that the real system, which is under the existence of bounded additive disturbances, and the model are both at time t at state $\mathbf{e}_i(t)$. Applying at time t a control law $\mathbf{u}(\cdot)$ to the system model deemed "real" and its model will cause at time $t+\tau$, $\tau \geq 0$ a divergence between the states of the real system and its model. The norm of the difference between the state of the real system and the state of the model system is bounded by

$$\Big\|\mathbf{e}_i\big(t+\tau;\ \mathbf{u}(\cdot), \mathbf{e}_i(t)\big) - \overline{\mathbf{e}}_i\big(t+\tau;\ \mathbf{u}(\cdot), \mathbf{e}_i(t)\big)\Big\| \leq \frac{\overline{\delta}_i}{\mathrm{L}_{g_i}}(e^{L_{g_i}\tau} - 1)$$

where $\overline{\delta}_i$ is the upper bound of the disturbance, and L_{g_i} the Lipschitz constant of both models.

Feasibility analysis In this section we will show that there can be constructed an admissible but not necessarily optimal control input according to definition (4.3.1).

Consider a sampling instant t_k for which a solution $\overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big)$ to problem (4.3.1) exists. Suppose now a time instant t_{k+1} such that[2] $t_k < t_{k+1} < t_k + T_p$, and consider that the optimal control signal calculated at t_k is comprised by the following two portions:

$$\overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big) = \begin{cases} \overline{\mathbf{u}}_i^\star\big(\tau_1;\ \mathbf{e}_i(t_k)\big), & \tau_1 \in [t_k, t_{k+1}] \\ \\ \overline{\mathbf{u}}_i^\star\big(\tau_2;\ \mathbf{e}_i(t_k)\big), & \tau_2 \in [t_{k+1}, t_k + T_p] \end{cases} \tag{4.9}$$

Both portions are admissible since the calculated optimal control input is admissible, and hence they both conform to the input constraints. As for the resulting predicted states, they satisfy the state constraints, and, crucially:

$$\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \in \Omega_i \tag{4.10}$$

[2]It is not strictly necessary that $t_{k+1} = t_k + h$ here, however it is necessary for the following that $t_{k+1} - t_k \leq h$

Furthermore, according to assumption (3) of the theorem, there exists an admissible (and certainly not guaranteed optimal) input $h_i \in \mathcal{U}_i$ that renders Ψ_i (and consequently Ω_i) invariant over $[t_k + T_p, t_k + T_p + h]$.

Given the above facts, we can construct an admissible input $\widetilde{\mathbf{u}}_i(\cdot)$ for time t_{k+1} by sewing together the second portion of (4.9) and the admissible input $h_i(\cdot)$:

$$\widetilde{\mathbf{u}}_i(\tau) = \begin{cases} \overline{\mathbf{u}}_i^\star\big(\tau;\ \mathbf{e}_i(t_k)\big), & \tau \in [t_{k+1}, t_k + T_p] \\ \\ h_i\big(\overline{\mathbf{e}}_i\big(\tau;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big)\big), & \tau \in (t_k + T_p, t_{k+1} + T_p] \end{cases} \tag{4.11}$$

Applied at time t_{k+1}, $\widetilde{\mathbf{u}}_i(\tau)$ is an admissible control input with regard to the input constraints as a composition of admissible control inputs, for all $\tau \in [t_{k+1}, t_{k+1} + T_p]$.

Furthermore, $\overline{\mathbf{e}}_i\big(t_{k+1} + s;\ \widetilde{\mathbf{u}}_i(\cdot), \mathbf{e}_i(t_{k+1})\big) \in \mathcal{E}_i \ominus \mathcal{B}_s$, for all $s \in [0, T_p]$

By applying lemma (4.4.1) for $t = t_{k+1} + s$ and $\tau = t_k$ we get

$$\Big\| \mathbf{e}_i\big(t_{k+1} + s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) - \overline{\mathbf{e}}_i\big(t_{k+1} + s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \Big\| \leq \frac{\overline{\delta}_i}{L_{g_i}}\big(e^{L_{g_i}(h+s)} - 1\big)$$

or, in set language

$$\mathbf{e}_i\big(t_{k+1} + s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) - \overline{\mathbf{e}}_i\big(t_{k+1} + s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \in \mathcal{B}_{i,h+s}$$

By applying a reasoning identical to the proof of lemma (4.4.1) for $t = t_{k+1}$ (in the model equation) and $t = t_k$ (in the real model equation), and $\tau = s$ we get

$$\Big\| \mathbf{e}_i\big(t_{k+1} + s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) - \overline{\mathbf{e}}_i\big(t_{k+1} + s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big) \Big\| \leq \frac{\overline{\delta}_i}{L_{g_i}}\big(e^{L_{g_i} s} - 1\big)$$

which translates to

$$\mathbf{e}_i\big(t_{k+1} + s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) - \overline{\mathbf{e}}_i\big(t_{k+1} + s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big) \in \mathcal{B}_{i,s}$$

Furthermore, we know that the solution to the optimization problem is feasible at time t_k, which means that

$$\overline{\mathbf{e}}_i\big(t_{k+1}+s;\ \overline{\mathbf{u}}_i^{\star}(\cdot),\mathbf{e}_i(t_k)\big) \in \mathcal{E}_i \ominus \mathcal{B}_{i,h+s}$$

Let us for sake of readability set

$$\mathbf{e}_0 = \mathbf{e}_i\big(t_{k+1}+s;\ \overline{\mathbf{u}}_i^{\star}(\cdot),\mathbf{e}_i(t_k)\big)$$

$$\overline{\mathbf{e}}_0 = \overline{\mathbf{e}}_i\big(t_{k+1}+s;\ \overline{\mathbf{u}}_i^{\star}(\cdot),\mathbf{e}_i(t_k)\big)$$

$$\overline{\mathbf{e}}_1 = \overline{\mathbf{e}}_i\big(t_{k+1}+s;\ \overline{\mathbf{u}}_i^{\star}(\cdot),\mathbf{e}_i(t_{k+1})\big)$$

and translate the above system of include statements to:

$$\mathbf{e}_{i,0} - \overline{\mathbf{e}}_{i,0} \in \mathcal{B}_{i,h+s}$$

$$\mathbf{e}_{i,0} - \overline{\mathbf{e}}_{i,1} \in \mathcal{B}_{i,s}$$

$$\overline{\mathbf{e}}_{i,0} \in \mathcal{E}_i \ominus \mathcal{B}_{i,h+s}$$

First we will focus on the two first statements, and we will derive a result that will combine with the third statement so as to prove that the predicted state will be feasible from t_{k+1} to $t_{k+1}+T_p$. Subtracting the second from the first yields

$$\overline{\mathbf{e}}_{i,1} - \overline{\mathbf{e}}_{i,0} \in \mathcal{B}_{i,h+s} \ominus \mathcal{B}_{i,s}$$

Now we introduce the third statement

$$\overline{\mathbf{e}}_{i,0} \in \mathcal{E}_i \ominus \mathcal{B}_{i,h+s}$$

$$\overline{\mathbf{e}}_{i,1} - \overline{\mathbf{e}}_{i,0} \in \mathcal{B}_{i,h+s} \ominus \mathcal{B}_{i,s}$$

Adding the latter to the former yields

$$\overline{\mathbf{e}}_{i,1} \in \big(\mathcal{E}_i \ominus \mathcal{B}_{i,h+s}\big) \oplus \big(\mathcal{B}_{i,h+s} \ominus \mathcal{B}_{i,s}\big)$$

But[a] $(A \ominus B) \oplus (B \ominus C) = (A \oplus B) \ominus (B \oplus C)$ for arbitrary sets A, B, C. Hence

$$\overline{\mathbf{e}}_{i,1} \in \big(\mathcal{E}_i \oplus \mathcal{B}_{i,h+s}\big) \ominus \big(\mathcal{B}_{i,h+s} \oplus \mathcal{B}_{i,s}\big)$$

Using implication[b] (v) of theorem 2.1 from [6] yields

$$\overline{\mathbf{e}}_{i,1} \in \Bigg(\big(\mathcal{E}_i \oplus \mathcal{B}_{i,h+s}\big) \ominus \mathcal{B}_{i,h+s}\Bigg) \ominus \mathcal{B}_{i,s}$$

Using implication[c] (3.1.11) from [7] yields

$$\overline{\mathbf{e}}_{i,1} \in \mathcal{E}_i \ominus \mathcal{B}_{i,s}$$

Translating back to our native language

$$\overline{\mathbf{e}}_i\big(t_{k+1} + s;\ \overline{\mathbf{u}}_i^{\star}(\cdot), \mathbf{e}_i(t_{k+1})\big) \in \mathcal{E}_i \ominus \mathcal{B}_{i,s},\ \forall s \in [0, T_p] \tag{4.12}$$

Remark 4.4.2. Consulting with property (4.3.1), this means that the state of the "true" system does not violate the constraints $\mathcal{E}_i$ over the horizon $[t_{k+1}, t_{k+1} + T_p]$:

$$\overline{\mathbf{e}}_i\big(t_{k+1} + s;\ \overline{\mathbf{u}}_i^{\star}(\cdot), \mathbf{e}_i(t_{k+1})\big) \in \mathcal{E}_i \ominus \mathcal{B}_{i,s} \ \Rightarrow\ \mathbf{e}_i\big(t_{k+1} + s;\ \overline{\mathbf{u}}_i^{\star}(\cdot), \mathbf{e}_i(t_{k+1})\big) \in \mathcal{E}_i,\ \forall s \in [0, T_p]$$

■

[a] Suppose sets A, B, C and vectors $\mathbf{a} \in A, \mathbf{b} \in B, \mathbf{c} \in C$, where $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$. Then

$$A \ominus B = \{\mathbf{a} - \mathbf{b},\ \mathbf{a} \in A, \mathbf{b} \in B\}$$

$$B \ominus C = \{\mathbf{b} - \mathbf{c},\ \mathbf{b} \in B, \mathbf{c} \in C\}$$

Adding the latter to the former yields

$$\begin{aligned}(A \ominus B) \oplus (B \ominus C) &= \{\mathbf{a} - \mathbf{b} + \mathbf{b} - \mathbf{c},\ \mathbf{a} \in A, \mathbf{b} \in B, \mathbf{c} \in C\} \\ &= \{\mathbf{a} - \mathbf{c},\ \mathbf{a} \in A, \mathbf{c} \in C\}\end{aligned}$$

On the other hand

$$A \oplus B = \{\mathbf{a} + \mathbf{b},\ \mathbf{a} \in A, \mathbf{b} \in B\}$$

$$B \oplus C = \{\mathbf{b} + \mathbf{c},\ \mathbf{b} \in B, \mathbf{c} \in C\}$$

Subtracting the latter from the former yields

$$(A \oplus B) \ominus (B \oplus C) = \{\mathbf{a} + \mathbf{b} - \mathbf{b} - \mathbf{c},\ \mathbf{a} \in A, \mathbf{b} \in B, \mathbf{c} \in C\}$$

$$= \{\mathbf{a} - \mathbf{c},\ \mathbf{a} \in A, \mathbf{c} \in C\}$$

Therefore

$$(A \ominus B) \oplus (B \ominus C) = (A \oplus B) \ominus (B \oplus C)$$

[b] $A = B_1 \oplus B_2 \Rightarrow A \ominus B = (A \ominus B_1) \ominus B_2$
[c] $(A \oplus B) \ominus B \subset A$

Finally, $\overline{\mathbf{e}}_i\big(t_{k+1} + T_p;\ \widetilde{\mathbf{u}}_i(\cdot), \mathbf{e}_i(t_{k+1})\big) \in \Omega_i$.

To prove this statement we begin with

$$\begin{aligned}
& V_i\Big(\overline{\mathbf{e}}_i\big(t_k + T_p;\, \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big)\Big) - V_i\Big(\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big)\Big) \\
&\qquad \leq \bigg| V_i\Big(\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big)\Big) - V_i\Big(\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big)\Big) \bigg| \\
&\qquad \leq L_{V_i} \Big\| \overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big) - \overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \Big\| \\
&\qquad = L_{V_i} \Big\| \overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big) - \overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \Big\| \qquad (4.13)
\end{aligned}$$

Consulting with remark (4.4.1) we get that the two terms interior to the norm are respectively equal to

$$\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big) = \mathbf{e}_i(t_{k+1}) + \int_{t_{k+1}}^{t_k + T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_{k+1})), \overline{\mathbf{u}}_i^\star(s)\big) ds$$

and

$$\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) = \mathbf{e}_i(t_k) \quad + \int_{t_k}^{t_k + T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big) ds$$

$$
\begin{aligned}
&= \mathbf{e}_i(t_k) \quad + \int_{t_k}^{t_{k+1}} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big) ds \\
&\qquad\qquad + \int_{t_{k+1}}^{t_k+T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big) ds \\
&= \overline{\mathbf{e}}_i(t_{k+1}) + \int_{t_{k+1}}^{t_k+T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big) ds
\end{aligned}
$$

Subtracting the latter from the former and taking norms on either side we get

$$
\begin{aligned}
&\Big\| \overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big) - \overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \Big\| \\
&\quad = \Big\| \mathbf{e}_i(t_{k+1}) - \overline{\mathbf{e}}_i(t_{k+1}) \\
&\quad + \int_{t_{k+1}}^{t_k+T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_{k+1})), \overline{\mathbf{u}}_i^\star(s)\big) ds - \int_{t_{k+1}}^{t_k+T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big) ds \Big\| \\
&\quad \leq \Big\| \mathbf{e}_i(t_{k+1}) - \overline{\mathbf{e}}_i(t_{k+1}) \Big\| \\
&\quad + \Big\| \int_{t_{k+1}}^{t_k+T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_{k+1})), \overline{\mathbf{u}}_i^\star(s)\big) ds - \int_{t_{k+1}}^{t_k+T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big) ds \Big\| \\
&\quad = \Big\| \mathbf{e}_i(t_{k+1}) - \overline{\mathbf{e}}_i(t_{k+1}) \Big\| \\
&\quad + \Big\| \int_{t_{k+1}}^{t_k+T_p} \Big(g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_{k+1})), \overline{\mathbf{u}}_i^\star(s)\big) - g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big) \Big) ds \Big\| \\
&\quad = \Big\| \mathbf{e}_i(t_{k+1}) - \overline{\mathbf{e}}_i(t_{k+1}) \Big\| \\
&\quad + \int_{t_{k+1}}^{t_k+T_p} \Big\| g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_{k+1})), \overline{\mathbf{u}}_i^\star(s)\big) - g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big) \Big\| ds \\
&\quad \leq \Big\| \mathbf{e}_i(t_{k+1}) - \overline{\mathbf{e}}_i(t_{k+1}) \Big\|
\end{aligned}
$$

$$+ L_{g_i} \int_{t_{k+1}}^{t_k+T_p} \Big\| \overline{\mathbf{e}}_i\big(s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big) - \overline{\mathbf{e}}_i\big(s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \Big\| ds$$

$$= \Big\| \mathbf{e}_i(t_{k+1}) - \overline{\mathbf{e}}_i(t_{k+1}) \Big\|$$

$$+ L_{g_i} \int_{h}^{T_p} \Big\| \overline{\mathbf{e}}_i\big(t_k + s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big) - \overline{\mathbf{e}}_i\big(t_k + s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \Big\| ds$$

By applying the Grönwall-Bellman inequality we get

$$\Big\| \overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big) - \overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \Big\|$$

$$\leq \Big\| \mathbf{e}_i(t_{k+1}) - \overline{\mathbf{e}}_i(t_{k+1}) \Big\| e^{L_{g_i}(T_p - h)}$$

By applying lemma (4.4.1) for $t = t_k$ and $\tau = h$ we get

$$\Big\| \overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big) - \overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \Big\| \leq \frac{\overline{\delta}_i}{L_{g_i}} (e^{L_{g_i} h} - 1) e^{L_{g_i}(T_p - h)}$$

Hence (4.13) becomes

$$V_i\Big(\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big)\Big) - V_i\Big(\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big)$$

$$\leq L_{V_i} \Big\| \overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big) - \overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \Big\|$$

$$= L_{V_i} \frac{\overline{\delta}_i}{L_{g_i}} (e^{L_{g_i} h} - 1) e^{L_{g_i}(T_p - h)} \tag{4.14}$$

Since the solution to the optimization problem is assumed to be feasible at time t_k, all states abide by their respective constraints, and in particular, from (4.10), the predicted state $\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \in \Omega_i$. This means that $V_i\Big(\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \leq \varepsilon_{\Omega_i}$. Hence (4.14) becomes

$$V_i\Big(\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big)\Big)$$

$$\leq V_i\big(\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) + L_{V_i}\frac{\overline{\delta}_i}{L_{g_i}}(e^{L_{g_i}h} - 1)e^{L_{g_i}(T_p - h)}$$

$$\leq \varepsilon_{\Omega_i} + L_{V_i}\frac{\overline{\delta}_i}{L_{g_i}}(e^{L_{g_i}h} - 1)e^{L_{g_i}(T_p - h)}$$

From assumption 4 of theorem (4.4.1), the upper bound of the disturbance is in turn bounded by

$$\overline{\delta}_i \leq \frac{\varepsilon_{\Psi_i} - \varepsilon_{\Omega_i}}{\frac{L_{V_i}}{L_{g_i}}(e^{L_{g_i}h} - 1)e^{L_{g_i}(T_p - h)}}$$

Therefore

$$V_i\big(\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big)\big) \leq \varepsilon_{\Omega_i} - \varepsilon_{\Omega_i} + \varepsilon_{\Psi_i} = \varepsilon_{\Psi_i}$$

or, expressing the above in terms of t_{k+1}

$$V_i\big(\overline{\mathbf{e}}_i\big(t_{k+1} + T_p - h;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big)\big) \leq \varepsilon_{\Psi_i}$$

This means that the state $\overline{\mathbf{e}}_i\big(t_{k+1} + T_p - h;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big) \in \Psi_i$. From assumption (4.3.3), and since $\Psi_i \subseteq \Phi_i$, there is an admissible control signal $h_i\big(\overline{\mathbf{e}}_i\big(t_{k+1} + T_p - h;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big)\big)$ such that

$$\overline{\mathbf{e}}_i\big(t_{k+1} + T_p;\ h_i(\cdot), \overline{\mathbf{e}}_i\big(t_{k+1} + T_p - h;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big)\big) \in \Omega_i$$

Therefore, overall

$$\overline{\mathbf{e}}_i\big(t_{k+1} + T_p;\ \widetilde{\mathbf{u}}_i(\cdot), \mathbf{e}_i(t_{k+1})\big) \in \Omega_i \tag{4.15}$$

■

Piecing the admissibility of $\widetilde{\mathbf{u}}_i(\cdot)$ from (4.11) together with conclusions (4.12) and (4.15), we conclude that the application of the control input $\widetilde{\mathbf{u}}_i(\cdot)$ at time t_{k+1} results in the states of the real system abiding by their intended state and terminal constraints across the entire horizon $[t_{k+1}, t_{k+1}+T_p]$. Therefore, overall, the (sub-optimal) control input $\widetilde{\mathbf{u}}_i(\cdot)$ is admissible at time t_{k+1} according to definition (4.3.1), which means that feasibility of a solution to the optimization problem at time t_k implies feasibility at time $t_{k+1} > t_k$. Thus, since at time $t = 0$ a solution is assumed to be feasible, a solution to the optimal control problem is feasible for all $t \geq 0$.

Convergence analysis The second part of the proof involves demonstrating that the state $\mathbf{e}_i$ is ultimately bounded in Ω_i. We will show that the *optimal* cost $J_i^\star\big(\mathbf{e}_i(t)\big)$ is an ISS Lyapunov function for the closed loop system (4.8) according to definition (1.2.4), with

$$J_i^\star\big(\mathbf{e}_i(t)\big) \triangleq J_i\Big(\mathbf{e}_i(t), \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t)\big)\Big)$$

by way of remark (1.2.2).

In order not to wreak notational havoc, let us as before define the following terms:

- $\mathbf{u}_{0,i}(\tau) \triangleq \overline{\mathbf{u}}_i^\star\big(\tau;\ \mathbf{e}_i(t_k)\big)$ as the *optimal* input that results from the solution to problem (3.3.1) based on the measurement of state $\mathbf{e}_i(t_k)$, applied at time $\tau \geq t_k$

- $\mathbf{e}_{0,i}(\tau) \triangleq \overline{\mathbf{e}}_i\big(\tau;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big)$ as the *predicted* state at time $\tau \geq t_k$, that is, the state that results from the application of the above input $\overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big)$ to the state $\mathbf{e}_i(t_k)$, at time τ

- $\mathbf{u}_{1,i}(\tau) \triangleq \widetilde{\mathbf{u}}_i(\tau)$ as the *admissible* input at $\tau \geq t_{k+1}$ (see eq. (4.11))

- $\mathbf{e}_{1,i}(\tau) \triangleq \overline{\mathbf{e}}_i\big(\tau;\ \widetilde{\mathbf{u}}_i(\cdot), \mathbf{e}_i(t_{k+1})\big)$ as the *predicted* state at time $\tau \geq t_{k+1}$, that is, the state that results from the application of the above input $\widetilde{\mathbf{u}}_i(\cdot)$ to the state $\mathbf{e}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big)$, at time τ

Before beginning to prove convergence, it is worth noting that while the cost

$$J_i\Big(\mathbf{e}_i(t), \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t)\big)\Big)$$

is optimal (in the sense that it is based on the optimal input, which provides its minimum realization), a cost that is based on a plainly admissible (and thus, without loss of generality, sub-optimal) input $\mathbf{u}_i \neq \overline{\mathbf{u}}_i^\star$ will result in a configuration where

$$J_i\Big(\mathbf{e}_i(t), \mathbf{u}_i\big(\cdot;\ \mathbf{e}_i(t)\big)\Big) \geq J_i\Big(\mathbf{e}_i(t), \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t)\big)\Big)$$

Let us now begin our investigation on the sign of the difference between the cost that results from the application of the feasible input $\mathbf{u}_{1,i}$, which we shall denote by $\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big)$, and the optimal cost $J_i^\star\big(\mathbf{e}_i(t_k)\big)$, while reminding ourselves that $J_i\big(\mathbf{e}_i(t), \overline{\mathbf{u}}_i(\cdot)\big) = \int_t^{t+T_p} F_i\big(\overline{\mathbf{e}}_i(s), \overline{\mathbf{u}}_i(s)\big)ds$ $+\ V_i\big(\overline{\mathbf{e}}_i(t+T_p)\big)$:

$$\begin{aligned}\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) = V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) + \int_{t_{k+1}}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds \\ -V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) - \int_{t_k}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds\end{aligned}$$

Considering that $t_k < t_{k+1} < t_k + T_p < t_{k+1} + T_p$, we break down the two integrals above in between these intervals:

$$\begin{aligned}&\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) = \\ &\quad V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) + \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds \\ &\quad -V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) - \int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds - \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds\end{aligned} \tag{4.16}$$

We begin working on (4.16) focusing first on the difference between the two intervals over $[t_{k+1}, t_{k+1}+T_p]$:

$$\begin{aligned}&\int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds - \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds \\ &\qquad = \int_{t_k+h}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds - \int_{t_k+h}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds\end{aligned}$$

$$\leq \Big| \int_{t_k+h}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds - \int_{t_k+h}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) ds \Big|$$

$$= \Big| \int_{t_k+h}^{t_k+T_p} \Big(F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) - F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) \Big) ds \Big|$$

$$= \int_{t_k+h}^{t_k+T_p} \Big| F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) - F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) \Big| ds$$

$$\leq L_{F_i} \int_{t_k+h}^{t_k+T_p} \Big\| \overline{\mathbf{e}}_i\big(s;\ \mathbf{u}_{1,i}(\cdot), \mathbf{e}_i(t_k+h)\big) - \overline{\mathbf{e}}_i\big(s;\ \mathbf{u}_{0,i}(\cdot), \mathbf{e}_i(t_k)\big) \Big\| ds$$

$$= L_{F_i} \int_{h}^{T_p} \Big\| \overline{\mathbf{e}}_i\big(t_k+s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k+h)\big) - \overline{\mathbf{e}}_i\big(t_k+s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \Big\| ds \tag{4.17}$$

Consulting with remark (4.4.1) for the two different initial conditions we get

$$\overline{\mathbf{e}}_i\big(t_k+s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k+h)\big) = \mathbf{e}_i(t_k+h) + \int_{t_k+h}^{t_k+s} g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k+h)), \overline{\mathbf{u}}_i^\star(\tau)\big) d\tau$$

and

$$\begin{aligned}
\overline{\mathbf{e}}_i\big(t_k+s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) &= \mathbf{e}_i(t_k) + \int_{t_k}^{t_k+s} g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(\tau)\big) d\tau \\
&= \mathbf{e}_i(t_k) + \int_{t_k}^{t_k+h} g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(\tau)\big) d\tau \\
&\quad + \int_{t_k+h}^{t_k+s} g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(\tau)\big) d\tau
\end{aligned}$$

Subtracting the latter from the former and taking norms on either side yields

$$\begin{aligned}
&\Big\| \overline{\mathbf{e}}_i\big(t_k+s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k+h)\big) - \overline{\mathbf{e}}_i\big(t_k+s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \Big\| \\
&= \Big\| \mathbf{e}_i(t_k+h) - \Big(\mathbf{e}_i(t_k) + \int_{t_k}^{t_k+h} g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(\tau)\big) d\tau \Big) \\
&+ \int_{t_k+h}^{t_k+s} g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k+h)), \overline{\mathbf{u}}_i^\star(\tau)\big) d\tau - \int_{t_k+h}^{t_k+s} g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(\tau)\big) d\tau \Big\|
\end{aligned}$$

$$
\begin{aligned}
&= \Bigg\| \mathbf{e}_i(t_k+h) - \overline{\mathbf{e}}_i(t_k+h) \\
&+ \int_{t_k+h}^{t_k+s} \Bigg(g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k+h)), \overline{\mathbf{u}}_i^\star(\tau)\big) - g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(\tau)\big) \Bigg) d\tau \Bigg\| \\
&\leq \Bigg\| \mathbf{e}_i(t_k+h) - \overline{\mathbf{e}}_i(t_k+h) \Bigg\| \\
&+ \Bigg\| \int_{t_k+h}^{t_k+s} \Bigg(g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k+h)), \overline{\mathbf{u}}_i^\star(\tau)\big) - g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(\tau)\big) \Bigg) d\tau \Bigg\| \\
&= \Bigg\| \mathbf{e}_i(t_k+h) - \overline{\mathbf{e}}_i(t_k+h) \Bigg\| \\
&+ \int_{t_k+h}^{t_k+s} \Bigg\| g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k+h)), \overline{\mathbf{u}}_i^\star(\tau)\big) - g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(\tau)\big) \Bigg\| d\tau \\
&\leq \Bigg\| \mathbf{e}_i(t_k+h) - \overline{\mathbf{e}}_i(t_k+h) \Bigg\| \\
&+ L_{g_i} \int_{t_k+h}^{t_k+s} \Bigg\| \overline{\mathbf{e}}_i\big(\tau;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k+h)\big) - \overline{\mathbf{e}}_i\big(\tau;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \Bigg\| d\tau \\
&= \Bigg\| \mathbf{e}_i(t_k+h) - \overline{\mathbf{e}}_i(t_k+h) \Bigg\| \\
&+ L_{g_i} \int_{h}^{s} \Bigg\| \overline{\mathbf{e}}_i\big(t_k+\tau;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k+h)\big) - \overline{\mathbf{e}}_i\big(t_k+\tau;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \Bigg\| d\tau
\end{aligned}
\tag{4.18}
$$

Given that from lemma (4.4.1) the first term of the sum featured in (4.18) is a constant, by application of the the Grönwall-Bellman inequality, (4.18) becomes:

$$
\begin{aligned}
\Bigg\| \overline{\mathbf{e}}_i\big(t_k+s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k+h)\big) - \overline{\mathbf{e}}_i\big(t_k+s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \Bigg\| \\
\leq \Bigg\| \mathbf{e}_i(t_k+h) - \overline{\mathbf{e}}_i(t_k+h) \Bigg\| e^{L_{g_i}(s-h)} \\
\leq \frac{\overline{\delta}_i}{L_{g_i}} (e^{L_{g_i}h} - 1) e^{L_{g_i}(s-h)}
\end{aligned}
$$

Given the above result, (4.17) becomes

$$\begin{aligned}\int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s),\mathbf{u}_{1,i}(s)\big)ds - \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s),\mathbf{u}_{0,i}(s)\big)ds \\ \leq L_{F_i}\int_h^{T_p}\frac{\overline{\delta}_i}{L_{g_i}}(e^{L_{g_i}h}-1)e^{L_{g_i}(s-h)}ds \\ = L_{F_i}\frac{\overline{\delta}_i}{L_{g_i}}(e^{L_{g_i}h}-1)\int_h^{T_p} e^{L_{g_i}(s-h)}ds \\ = L_{F_i}\frac{\overline{\delta}_i}{L_{g_i}}(e^{L_{g_i}h}-1)\frac{1}{L_{g_i}}(e^{L_{g_i}(T_p-h)}-1) \\ = L_{F_i}\frac{\overline{\delta}_i}{L_{g_i}^2}(e^{L_{g_i}h}-1)(e^{L_{g_i}(T_p-h)}-1)\end{aligned}$$

Hence we discovered that

$$\begin{aligned}\int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s),\mathbf{u}_{1,i}(s)\big)ds - \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s),\mathbf{u}_{0,i}(s)\big)ds \\ \leq L_{F_i}\frac{\overline{\delta}_i}{L_{g_i}^2}(e^{L_{g_i}h}-1)(e^{L_{g_i}(T_p-h)}-1) \qquad (4.19)\end{aligned}$$

With this partial result established, we turn back to the remaining terms found in (4.16) and, in particular, we focus on the integral

$$\int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s),\mathbf{u}_{1,i}(s)\big)ds$$

We discern that the range of the above integral has a length[a] equal to the length of the interval where assumption 2 of theorem (4.4.1) holds. Integrating the expression found in the assumption over the interval $[t_k+T_p, t_{k+1}+T_p]$, for the controls and states applicable in it we get

$$\int_{t_k+T_p}^{t_{k+1}+T_p}\left(\frac{\partial V_i}{\partial \mathbf{e}_{1,i}}g_i\big(\mathbf{e}_{1,i}(s),\mathbf{u}_{1,i}(s)\big)+F_i\big(\mathbf{e}_{1,i}(s),\mathbf{u}_{1,i}(s)\big)\right)ds \leq 0$$

$$\int_{t_k+T_p}^{t_{k+1}+T_p} \frac{d}{ds} V_i\big(\mathbf{e}_{1,i}(s)\big)ds + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds \leq 0$$

$$V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) - V_i\big(\mathbf{e}_{1,i}(t_k+T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds \leq 0$$

$$V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds \leq V_i\big(\mathbf{e}_{1,i}(t_k+T_p)\big)$$

The left-hand side expression is the same as the first two terms in the right-hand side of equality (4.16). We can introduce the third one by subtracting it from both sides:

$$\begin{aligned} V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) &+ \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds - V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) \\ &\leq V_i\big(\mathbf{e}_{1,i}(t_k+T_p)\big) - V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) \\ &\leq \Big|V_i\big(\mathbf{e}_{1,i}(t_k+T_p)\big) - V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big)\Big| \\ &\leq L_{V_i}\Big\|\overline{\mathbf{e}}_i\big(t_k+T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big) - \overline{\mathbf{e}}_i\big(t_k+T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big)\Big\| \\ &\leq L_{V_i}\frac{\overline{\delta}_i}{L_{g_i}}(e^{L_{g_i}h}-1)e^{L_{g_i}(T_p-h)} \text{ (from (4.14))} \end{aligned}$$

[a] $(t_{k+1}+T_p)-(t_k+T_p) = t_{k+1}-t_k = h$

Hence, we discovered that

$$\boxed{\begin{aligned} V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds - V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) \\ \leq L_{V_i}\frac{\overline{\delta}_i}{L_{g_i}}(e^{L_{g_i}h}-1)e^{L_{g_i}(T_p-h)} \end{aligned}} \tag{4.20}$$

Adding the milestone inequalities (4.19) and (4.20) yields

$$\int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds - \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds$$

$$+ V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds - V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big)$$

$$\leq L_{F_i}\frac{\overline{\delta}_i}{\text{Ł}_{g_i}^2}(e^{L_{g_i}h}-1)(e^{L_{g_i}(T_p-h)}-1) + L_{V_i}\frac{\overline{\delta}_i}{L_{g_i}}(e^{L_{g_i}h}-1)e^{L_{g_i}(T_p-h)}$$

and therefore (4.16), by bringing the integral ranging from t_k to t_{k+1} to the left-hand side, becomes

$$\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) + \int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds$$

$$\leq L_{F_i}\frac{\overline{\delta}_i}{\text{Ł}_{g_i}^2}(e^{L_{g_i}h}-1)(e^{L_{g_i}(T_p-h)}-1) + L_{V_i}\frac{\overline{\delta}_i}{L_{g_i}}(e^{L_{g_i}h}-1)e^{L_{g_i}(T_p-h)}$$

By rearranging terms, the cost difference becomes bounded by

$$\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) = \xi_i\overline{\delta}_i - \int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds$$

where

$$\xi_i = \frac{1}{L_{g_i}}\Big(e^{L_{g_i}h}-1\Big)\Big(\big(L_{V_i} + \frac{L_{F_i}}{\text{Ł}_{g_i}}\big)\big(e^{L_{g_i}(T_p-h)}-1\big) + L_{V_i}\Big) > 0$$

and $\xi_i\overline{\delta}_i$ is the contribution of the bounded additive disturbance $\boldsymbol{\delta}_i(t)$ to the nominal cost difference (i.e. the case without disturbances).

F_i is a positive-definite function as a sum of a positive-definite $\|\mathbf{u}_i\|_{\boldsymbol{R}_i}^2$ and a positive semi-definite function $\|\mathbf{e}_i\|_{\boldsymbol{Q}_i}^2$. If we denote by $m_i = \lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i) \geq 0$ the minimum eigenvalue between those of matrices $\boldsymbol{R}_i$, $\boldsymbol{Q}_i$, this means that

$$F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) \geq m_i\|\mathbf{e}_{0,i}(s)\|^2$$

By integrating the above between our interval of interest $[t_k, t_{k+1}]$ we get

$$\int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) \geq \int_{t_k}^{t_{k+1}} m_i\|\mathbf{e}_{0,i}(s)\|^2 ds$$

or

$$-\int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) \leq -m_i \int_{t_k}^{t_{k+1}} \|\overline{\mathbf{e}}_i(s;\ \overline{\mathbf{u}}_i^\star, \mathbf{e}_i(t_k))\|^2 ds$$

This means that the cost difference is upper-bounded by

$$\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) \leq \xi_i \overline{\delta}_i - m_i \int_{t_k}^{t_{k+1}} \|\overline{\mathbf{e}}_i(s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k))\|^2 ds$$

and since the cost $\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big)$ is, in general, sub-optimal: $J_i^\star\big(\mathbf{e}_i(t_{k+1})\big) - \overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) \leq 0$:

$$J_i^\star\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) \leq \xi_i \overline{\delta}_i - m_i \int_{t_k}^{t_{k+1}} \|\overline{\mathbf{e}}_i(s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k))\|^2 ds$$

Let $\Xi_i(\mathbf{e}_i) \triangleq J_i^\star(\mathbf{e}_i)$. Then, between consecutive times t_k and t_{k+1} when the FHOCP is solved, the above inequality reforms into

$$\begin{aligned}
\Xi_i\big(\mathbf{e}_i(t_{k+1})\big) - \Xi_i\big(\mathbf{e}_i(t_k)\big) &\leq \xi_i \overline{\delta}_i - m_i \int_{t_k}^{t_{k+1}} \|\overline{\mathbf{e}}_i(s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k))\|^2 ds \\
&= \xi_i \|\boldsymbol{\delta}_i\|_\infty - \int_{t_k}^{t_{k+1}} m_i \|\overline{\mathbf{e}}_i(s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k))\|^2 ds \\
&\leq \int_{t_k}^{t_{k+1}} \bigg(\frac{\xi_i}{h} \|\boldsymbol{\delta}_i(s)\| - m_i \|\overline{\mathbf{e}}_i(s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k))\|^2 \bigg) ds \qquad (4.21)
\end{aligned}$$

The functions $\sigma\big(\|\boldsymbol{\delta}_i\|\big) = \dfrac{\xi_i}{h}\|\boldsymbol{\delta}_i\|$ and $\alpha_3\big(\|\mathbf{e}_i\|\big) = m_i\|\mathbf{e}_i\|^2$ are class $\mathcal{K}$ functions according to definition (1.2.1), and therefore, according to lemma (3.3.4), remark (1.2.2) and definition (1.2.4), $\Xi_i\big(\mathbf{e}_i\big)$ is an ISS Lyapunov function in Ω_i.

Given this fact, theorem (1.2.1), implies that the closed-loop system is input-to-state stable in Ω_i. Inevitably then, given assumptions (4.3.2) and (4.3.3), the closed-loop trajectories for the error state of agent $i \in \mathcal{V}$ reach the terminal set Ω_i regardless of all $\boldsymbol{\delta}_i(t) : \|\boldsymbol{\delta}_i(t)\| \leq \overline{\delta}_i$, at some point[3] $t = t^\star \geq 0$. Once inside Ω_i, the trajectory is trapped there because of the implications[4] of (4.21) and assumption (4.3.3).

[3]The difference with the scheme designed in [5] is that in sections III, IV, the equivalent of assumption (4.3.3) does not exist.

[4]For more details, refer to the discussion after the declaration of theorem 7.6 in [3].

In turn, this means that the system (4.1) converges to $\mathbf{z}_{i,des}$ while simultaneously conforming to all constraints $\mathcal{Z}_i$, as $t \rightarrow \infty$ in general. This conclusion holds for all $i \in \mathcal{V}$, and hence, the compound system of agents $\mathcal{V}$ is stable. ■

Appendices

A

Proofs of lemmas

A.1 Proof of lemma 3.3.1

Let $\lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i)$ denote the smallest eigenvalue between those of matrices $\boldsymbol{Q}_i$ and $\boldsymbol{R}_i$, and let $\lambda_{max}(\boldsymbol{Q}_i, \boldsymbol{R}_i)$ denote the largest. Then

$$
\begin{aligned}
\lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\|\mathbf{e}_i(t)\|^2 &\le \lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\left\| \begin{bmatrix} \mathbf{e}_i(t) \\ \mathbf{u}_i(t) \end{bmatrix} \right\|^2 \\
&\le F_i\big(\mathbf{e}_i(t), \mathbf{u}_i(t)\big) \\
&\le \lambda_{max}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\left\| \begin{bmatrix} \mathbf{e}_i(t) \\ \mathbf{u}_i(t) \end{bmatrix} \right\|^2 \le \lambda_{max}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\|\mathbf{e}_i(t)\|^2
\end{aligned}
$$

Matrices $\boldsymbol{Q}_i, \boldsymbol{R}_i$ are positive definite, hence the functions $\alpha_1\big(\|\mathbf{e}_i\|\big) = \lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\|\mathbf{e}_i\|^2$ and $\alpha_2\big(\|\mathbf{e}_i\|\big) = \lambda_{max}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\|\mathbf{e}_i\|^2$ are class $\mathcal{K}_\infty$ functions according to definition (1.2.1). Therefore, F_i is lower- and upper-bounded by class $\mathcal{K}_\infty$ functions:

$$\alpha_1\big(\|\mathbf{e}_i\|\big) \leq F_i\big(\mathbf{e}_i, \mathbf{u}_i\big) \leq \alpha_2\big(\|\mathbf{e}_i\|\big)$$

A.2 Proof of lemma 3.3.2

For every $\mathbf{e}_1, \mathbf{e}_2 \in \mathcal{E}_i$, and $\mathbf{u}_i \in \mathcal{U}_i$ it holds that

$$\begin{aligned}
\big|F_i(\mathbf{e}_1, \mathbf{u}_i) - F_i(\mathbf{e}_2, \mathbf{u}_i)\big| &= \big|\mathbf{e}_1^\top \boldsymbol{Q}_i \mathbf{e}_1 + \mathbf{u}_i^\top \boldsymbol{R}_i \mathbf{u}_i - \mathbf{e}_2^\top \boldsymbol{Q}_i \mathbf{e}_2 - \mathbf{u}_i^\top \boldsymbol{R}_i \mathbf{u}_i\big| \\
&= \big|\mathbf{e}_1^\top \boldsymbol{Q}_i \mathbf{e}_1 - \mathbf{e}_2^\top \boldsymbol{Q}_i \mathbf{e}_2 \pm \mathbf{e}_1^\top \boldsymbol{Q}_i \mathbf{e}_2\big| \\
&= \big|\mathbf{e}_1^\top \boldsymbol{Q}_i (\mathbf{e}_1 - \mathbf{e}_2) - \mathbf{e}_2^\top \boldsymbol{Q}_i (\mathbf{e}_1 - \mathbf{e}_2)\big| \\
&\leq \big|\mathbf{e}_1^\top \boldsymbol{Q}_i (\mathbf{e}_1 - \mathbf{e}_2)\big| + \big|\mathbf{e}_2^\top \boldsymbol{Q}_i (\mathbf{e}_1 - \mathbf{e}_2)\big|
\end{aligned}$$

But for any $\mathbf{e}_1, \mathbf{e}_2 \in \mathcal{E}_i$

$$|\mathbf{e}_1^\top \boldsymbol{Q}_i \mathbf{e}_2| \leq \sigma_{max}(\boldsymbol{Q}_i)\|\mathbf{e}_1\|\|\mathbf{e}_2\|$$

where $\sigma_{max}(\boldsymbol{Q}_i)$ denotes the largest singular value of matrix $\boldsymbol{Q}_i$. Hence:

$$\begin{aligned}
\big|F_i(\mathbf{e}_1, \mathbf{u}_i) - F_i(\mathbf{e}_2, \mathbf{u}_i)\big| &\leq \sigma_{max}(\boldsymbol{Q}_i)\|\mathbf{e}_1\|\|\mathbf{e}_1 - \mathbf{e}_2\| + \sigma_{max}(\boldsymbol{Q}_i)\|\mathbf{e}_2\|\|\mathbf{e}_1 - \mathbf{e}_2\| \\
&= \sigma_{max}(\boldsymbol{Q}_i)(\|\mathbf{e}_1\| + \|\mathbf{e}_2\|)\|\mathbf{e}_1 - \mathbf{e}_2\| \\
&= \sigma_{max}(\boldsymbol{Q}_i) \sup_{\mathbf{e}_1, \mathbf{e}_2 \in \mathcal{E}_i} \big(\|\mathbf{e}_1\| + \|\mathbf{e}_2\|\big)\|\mathbf{e}_1 - \mathbf{e}_2\| \\
&= 2\sigma_{max}(\boldsymbol{Q}_i) \sup_{\mathbf{e}_i \in \mathcal{E}_i} \big(\|\mathbf{e}_i\|\big)\|\mathbf{e}_1 - \mathbf{e}_2\| \\
&= 2\sigma_{max}(\boldsymbol{Q}_i)\overline{\varepsilon}_i\|\mathbf{e}_1 - \mathbf{e}_2\|
\end{aligned}$$

A.3 Proof of lemma 3.3.3

For every $\mathbf{e}_1, \mathbf{e}_2 \in \Omega_i$, it holds that

$$\begin{aligned}
\big|V_i(\mathbf{e}_1) - V_i(\mathbf{e}_2)\big| &= \big|\mathbf{e}_1^\top \boldsymbol{P}_i \mathbf{e}_1 - \mathbf{e}_2^\top \boldsymbol{P}_i \mathbf{e}_2\big| \\
&= \big|\mathbf{e}_1^\top \boldsymbol{P}_i \mathbf{e}_1 - \mathbf{e}_2^\top \boldsymbol{P}_i \mathbf{e}_2 \pm \mathbf{e}_1^\top \boldsymbol{P}_i \mathbf{e}_2\big| \\
&= \big|\mathbf{e}_1^\top \boldsymbol{P}_i (\mathbf{e}_1 - \mathbf{e}_2) - \mathbf{e}_2^\top \boldsymbol{P}_i (\mathbf{e}_1 - \mathbf{e}_2)\big| \\
&\leq \big|\mathbf{e}_1^\top \boldsymbol{P}_i (\mathbf{e}_1 - \mathbf{e}_2)\big| + \big|\mathbf{e}_2^\top \boldsymbol{P}_i (\mathbf{e}_1 - \mathbf{e}_2)\big|
\end{aligned}$$

But for any $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{R}^n$

$$\big|\mathbf{e}_1^\top \boldsymbol{P}_i \mathbf{e}_2\big| \leq \sigma_{max}(\boldsymbol{P}_i)\|\mathbf{e}_1\|\|\mathbf{e}_2\|$$

where $\sigma_{max}(\boldsymbol{P}_i)$ denotes the largest singular value of matrix $\boldsymbol{P}_i$. Hence:

$$\begin{aligned}
\big|V_i(\mathbf{e}_1) - V_i(\mathbf{e}_2)\big| &\leq \sigma_{max}(\boldsymbol{P}_i)\|\mathbf{e}_1\|\|\mathbf{e}_1 - \mathbf{e}_2\| + \sigma_{max}(\boldsymbol{P}_i)\|\mathbf{e}_2\|\|\mathbf{e}_1 - \mathbf{e}_2\| \\
&= \sigma_{max}(\boldsymbol{P}_i)(\|\mathbf{e}_1\| + \|\mathbf{e}_2\|)\|\mathbf{e}_1 - \mathbf{e}_2\| \\
&\leq \sigma_{max}(\boldsymbol{P}_i)(\varepsilon_{\Omega_i} + \varepsilon_{\Omega_i})\|\mathbf{e}_1 - \mathbf{e}_2\| \\
&= 2\sigma_{max}(\boldsymbol{P}_i)\varepsilon_{\Omega_i}\|\mathbf{e}_1 - \mathbf{e}_2\|
\end{aligned}$$

A.4 Proof of lemma 3.3.4

V_i is defined as $V_i\big(\mathbf{e}_i\big) = \mathbf{e}_i^\top \boldsymbol{P}_i \mathbf{e}_i$. Let us denote the minimum and maximum eigenvalues of matrix $\boldsymbol{P}_i$ by $\lambda_{min}(\boldsymbol{P}_i)$ and $\lambda_{max}(\boldsymbol{P}_i)$ respectively. Then, the following series of inequalities holds:

$$\lambda_{min}(\boldsymbol{P}_i)\|\mathbf{e}_i\|^2 \leq V_i(\mathbf{e}_i) \leq \lambda_{max}(\boldsymbol{P}_i)\|\mathbf{e}_i\|^2$$

Matrix $\boldsymbol{P}_i$ is positive definite, hence the functions $\alpha_1 = \lambda_{min}(\boldsymbol{P}_i)\|\mathbf{e}_i\|^2$ and $\alpha_2 = \lambda_{max}(\boldsymbol{P}_i)\|\mathbf{e}_i\|^2$ are class $\mathcal{K}_\infty$ functions according to definition (1.2.1). Therefore, V_i is lower- and upper-bounded by class $\mathcal{K}_\infty$ functions:

$$\alpha_1\big(\|\mathbf{e}_i\|\big) \leq V_i(\mathbf{e}_i) \leq \alpha_2\big(\|\mathbf{e}_i\|\big)$$

A.5 Proof of property 4.3.1

Let us define for convenience $\boldsymbol{\zeta}_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^9 \times \mathbb{T}^3$: $\boldsymbol{\zeta}_i(s) \triangleq \mathbf{e}_i(s) - \overline{\mathbf{e}}_i(s;\ \mathbf{u}_i(s;\ \mathbf{e}_i(t)), \mathbf{e}_i(t))$, for $s \in [t, t + T_p]$.

According to lemma (4.4.1)

$$\|\mathbf{e}_i(s) - \overline{\mathbf{e}}_i\big(s;\ \mathbf{u}_i(s;\ \mathbf{e}_i(t)), \mathbf{e}_i(t)\big)\| \leq \frac{\overline{\delta}_i}{\text{Ł}_{g_i}}(e^{L_{g_i}(s-t)} - 1)$$

$$\|\boldsymbol{\zeta}_i(s)\| \leq \frac{\overline{\delta}_i}{\text{Ł}_{g_i}}(e^{L_{g_i}(s-t)} - 1)$$

which means that $\boldsymbol{\zeta}_i(s) \in \mathcal{B}_{i,s-t}$. Now let us assume that $\overline{\mathbf{e}}_i\big(s;\ \mathbf{u}_i(\cdot,\ \mathbf{e}_i(t)), \mathbf{e}_i(t)\big) \in \mathcal{E}_i \ominus \mathcal{B}_{i,s-t}$. Then, we add the two include statements:

$$\overline{\mathbf{e}}_i\big(s;\ \mathbf{u}_i(\cdot,\ \mathbf{e}_i(t)), \mathbf{e}_i(t)\big) \in \mathcal{E}_i \ominus \mathcal{B}_{i,s-t}$$

$$\boldsymbol{\zeta}_i(s) \in \mathcal{B}_{i,s-t}$$

which yields

$$\boldsymbol{\zeta}_i(s) + \overline{\mathbf{e}}_i\big(s;\ \mathbf{u}_i(s;\ \mathbf{e}_i(t)), \mathbf{e}_i(t)\big) \in \big(\mathcal{E}_i \ominus \mathcal{B}_{i,s-t}\big) \oplus \mathcal{B}_{i,s-t}$$

Utilizing Theorem 2.1 (ii) from [6] yields

$$\boldsymbol{\zeta}_i(s) + \overline{\mathbf{e}}_i\big(s;\ \mathbf{u}_i(s;\ \mathbf{e}_i(t)), \mathbf{e}_i(t)\big) \in \mathcal{E}_i$$

$$\mathbf{e}_i(s) \in \mathcal{E}_i$$

A.6 Proof of lemma 4.4.1

Since there are disturbances present, consulting remark (4.4.1) and substituting for $\tau_0 = t$ and $\tau_1 = t + \tau$ yields:

$$\mathbf{e}_i\big(t+\tau;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t)\big), \mathbf{e}_i(t)\big) = \mathbf{e}_i(t) + \int_t^{t+\tau} g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t)), \overline{\mathbf{u}}_i^\star(s)\big)ds + \int_t^{t+\tau} \delta_i(s)ds$$

$$\overline{\mathbf{e}}_i\big(t+\tau;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t)\big), \mathbf{e}_i(t)\big) = \mathbf{e}_i(t) + \int_t^{t+\tau} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t)), \overline{\mathbf{u}}_i^\star(s)\big)ds$$

Subtracting the latter from the former and taking norms on either side yields:

$$\begin{aligned}
&\Big\|\mathbf{e}_i\big(t+\tau;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t)\big), \mathbf{e}_i(t)\big) - \overline{\mathbf{e}}_i\big(t+\tau;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t)\big), \mathbf{e}_i(t)\big)\Big\| \\
&\quad = \Big\|\int_t^{t+\tau} g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t)), \overline{\mathbf{u}}_i^\star(s)\big)ds - \int_t^{t+\tau} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t)), \overline{\mathbf{u}}_i^\star(s)\big)ds + \int_t^{t+\tau} \delta_i(s)ds\Big\| \\
&\quad \leq \Big\|\int_t^{t+\tau} g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t)), \overline{\mathbf{u}}_i^\star(s)\big)ds - \int_t^{t+\tau} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t)), \overline{\mathbf{u}}_i^\star(s)\big)ds\Big\| + (t+\tau-t)\overline{\delta}_i \\
&\quad = \int_t^{t+\tau} \Big\|g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t)), \overline{\mathbf{u}}_i^\star(s)\big)s - g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t)), \overline{\mathbf{u}}_i^\star(s)\big)\Big\|ds + \tau\overline{\delta}_i \\
&\quad \leq L_{g_i}\int_t^{t+\tau} \Big\|\mathbf{e}_i\big(s;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t)\big), \mathbf{e}_i(t)\big) - \overline{\mathbf{e}}_i\big(s;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t)\big), \mathbf{e}_i(t)\big)\Big\|ds + \tau\overline{\delta}_i
\end{aligned}$$

since g_i is Lipschitz continuous in $\mathcal{E}_i$ with Lipschitz constant L_{g_i}. Reformulation yields

$$\begin{aligned}
&\Big\|\mathbf{e}_i\big(t+\tau;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t)\big), \mathbf{e}_i(t)\big) - \overline{\mathbf{e}}_i\big(t+\tau;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t)\big), \mathbf{e}_i(t)\big)\Big\| \\
&\qquad \leq \tau\overline{\delta}_i + L_{g_i}\int_0^{\tau} \Big\|\mathbf{e}_i\big(t+s;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t)\big), \mathbf{e}_i(t)\big) - \overline{\mathbf{e}}_i\big(t+s;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t)\big), \mathbf{e}_i(t)\big)\Big\|ds
\end{aligned}$$

By applying the Grönwall-Bellman inequality we get:

$$
\begin{aligned}
&\Big\| \mathbf{e}_i\big(t+\tau;\ \overline{\mathbf{u}}_i^{\star}\big(\cdot;\ \mathbf{e}_i(t)\big), \mathbf{e}_i(t)\big) - \overline{\mathbf{e}}_i\big(t+\tau;\ \overline{\mathbf{u}}_i^{\star}\big(\cdot;\ \mathbf{e}_i(t)\big), \mathbf{e}_i(t)\big) \Big\| \\
&\qquad \leq \tau\overline{\delta}_i + L_{g_i} \int_0^{\tau} s\overline{\delta}_i e^{L_{g_i}(\tau-s)} ds \\
&\qquad = \tau\overline{\delta}_i - \overline{\delta}_i \int_0^{\tau} s\big(e^{L_{g_i}(\tau-s)}\big)' ds \\
&\qquad = \tau\overline{\delta}_i - \overline{\delta}_i \bigg(\big[s e^{L_{g_i}(\tau-s)}\big]_0^{\tau} - \int_0^{\tau} e^{L_{g_i}(\tau-s)} ds \bigg) \\
&\qquad = \tau\overline{\delta}_i - \overline{\delta}_i \bigg(\tau + \frac{1}{L_{g_i}}(1 - e^{L_{g_i}\tau}) \bigg) \\
&\qquad = \frac{\overline{\delta}_i}{L_{g_i}}(e^{L_{g_i}\tau} - 1)
\end{aligned}
$$

Bibliography

[1] H. Khalil, *Noninear Systems.* Prentice-Hall, New Jersey, 1996.

[2] F. A. C. C. Fontes, L. Magni, and E. Gyurkovics, *Sampled-Data Model Predictive Control for Nonlinear Time-Varying Systems: Stability and Robustness*, pp. 115–129. Berlin, Heidelberg: Springer Berlin Heidelberg, 2007.

[3] H. Márquez, *Nonlinear Control Systems: Analysis and Design.* Wiley, 2003.

[4] D. L. Marruedo, T. Alamo, and E. F. Camacho, "Input-to-state stable mpc for constrained discrete-time nonlinear systems with bounded additive uncertainties," in *Proceedings of the 41st IEEE Conference on Decision and Control, 2002.*, vol. 4, pp. 4619–4624 vol.4, Dec 2002.

[5] L. Magni, D. M. Raimondo, and R. Scattolini, "Input-to-state stability for nonlinear model predictive control," in *Proceedings of the 45th IEEE Conference on Decision and Control*, pp. 4836–4841, Dec 2006.

[6] I. Kolmanovsky and E. G. Gilbert, "Theory and computation of disturbance invariant sets for discrete-time linear systems," *Mathematical Problems in Engineering*, vol. 4, no. 4, pp. 317–367, 1998.

[7] R. Schneider, *Minkowski addition*, pp. 139–207. Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2013.