A Robust Receding Horizon Scheme for Decentralized Control of Inter-constrained Continuous Nonlinear Systems

KTH Royal Institute of Technology

School of Electrical Engineering

Department of Automatic Control

Student: Alexandros Filotheou (alefil@kth.se)

Supervisor: Alexandros Nikou (anikou@kth.se)

Examiner: Dimos Dimarogonas (dimos@kth.se)

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1 Stabilization in the face of Disturbances

Remark 1.1. For the sake of modularity, separability, and direct comparison to the disturbance-free case, this chapter will assume that the previous disturbance-free analysis has not been given.

In this chapter we are interested in steering each agent $i \in \mathcal{V}$ into a *position* in 3D space, while conforming to the requirements of the problem; that is, all agents should avoid colliding with each other, all obstacles in the workspace, and the workspace boundary itself, while remaining in a non-singular configuration and sustaining the connectivity to their respective neighbours.

1.1 Formalizing the model

We begin by rewriting the system equations (**??**), (**??**) for a generic agent $i \in \mathcal{V}$ in state-space form:

$$\begin{aligned}
\dot{\mathbf{x}}_i(t) &= \boldsymbol{J}_i^{-1}(\mathbf{x}_i)\mathbf{v}_i(t) \\
\dot{\mathbf{v}}_i(t) &= -\boldsymbol{M}_i^{-1}(\mathbf{x}_i)\boldsymbol{C}_i(\mathbf{x}_i, \dot{\mathbf{x}}_i)\mathbf{v}_i(t) - \boldsymbol{M}_i^{-1}(\mathbf{x}_i)\mathbf{g}_i(\mathbf{x}_i) + \boldsymbol{M}_i^{-1}(\mathbf{x}_i)\mathbf{u}_i(t)
\end{aligned}$$

where the inversion of $\boldsymbol{M}_i$ is possible due to it being positive-definite $\forall i \in \mathcal{V}$. Denoting by $\mathbf{z}_i(t)$

$$\mathbf{z}_i(t) = \begin{bmatrix} \mathbf{x}_i(t) \\ \mathbf{v}_i(t) \end{bmatrix}, \ \mathbf{z}_i(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^9 \times \mathbb{T}^3$$

and $\dot{\mathbf{x}}_i(t)$ and $\dot{\mathbf{v}}_i(t)$ by

$$\begin{aligned} \dot{\mathbf{x}}_i(t) &= f_{i,x}(\mathbf{z}_i, \mathbf{u}_i) \\ \dot{\mathbf{v}}_i(t) &= f_{i,v}(\mathbf{z}_i, \mathbf{u}_i) \end{aligned}$$

we get the compact representation of the system's model

$$\dot{\mathbf{z}}_i(t) = \begin{bmatrix} f_{i,x}(\mathbf{z}_i, \mathbf{u}_i) \\ f_{i,v}(\mathbf{z}_i, \mathbf{u}_i) \end{bmatrix} = f_i\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big)$$

The state evolution of agent i is modeled by a system of non-linear continuous-time differential equations of the form

$$\begin{aligned} \dot{\mathbf{z}}_i(t) &= f_i\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big) \\ \mathbf{z}_i(0) &= \mathbf{z}_{i,0} \\ \mathbf{z}_i(t) &\subset \mathbb{R}^9 \times \mathbb{T}^3 \\ \mathbf{u}_i(t) &\subset \mathbb{R}^6 \end{aligned} \tag{3}$$

where state $\mathbf{z}_i$ is directly measurable. It should be noted that equation (3) does not consider model-plant mismatches or external disturbances. We assume that the real system is subject to bounded additive disturbances $\boldsymbol{\delta}_i$ such that $\boldsymbol{\delta}_i \in \Delta_i \subset \mathbb{R}^9 \times \mathbb{T}^3$, where Δ_i is a compact set containing the origin. The real system is described by:

$$\begin{aligned} \dot{\mathbf{z}}_i(t) &= f_i^R\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big) \\ &= f_i\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big) + \boldsymbol{\delta}_i(t) \\ \mathbf{z}_i(0) &= \mathbf{z}_{i,0} \\ \mathbf{z}_i(t) &\subset \mathbb{R}^9 \times \mathbb{T}^3 \\ \mathbf{u}_i(t) &\subset \mathbb{R}^6 \\ \boldsymbol{\delta}_i(t) &\in \Delta_i \subset \mathbb{R}^9 \times \mathbb{T}^3, \ t \in \mathbb{R}_{\geq 0} \\ \sup_{t \in \mathbb{R}_{\geq 0}} \|\boldsymbol{\delta}_i(t)\| &\leq \overline{\delta}_i \end{aligned}$$

We define the set $\mathcal{Z}_i \subset \mathbb{R}^9 \times \mathbb{T}^3$ as the set that captures all the state constraints of the system's dynamics posed by the problem (**??**), for $t \in \mathbb{R}_{\geq 0}$. Therefore $\mathcal{Z}_i$ is such that:

$$\begin{aligned}\mathcal{Z}_i = \big\{\mathbf{z}_i(t) \in \mathbb{R}^9 \times \mathbb{T}^3 : \;& \|\mathbf{p}_i(t) - \mathbf{p}_j(t)\| > \underline{d}_{ij,a}, \forall j \in \mathcal{R}_i(t), \\ & \|\mathbf{p}_i(t) - \mathbf{p}_j(t)\| < d_i, \forall j \in \mathcal{N}_i, \\ & \|\mathbf{p}_i(t) - \mathbf{p}_\ell\| > \underline{d}_{i\ell,o}, \forall \ell \in \mathcal{L}, \\ & \|\mathbf{p}_W - \mathbf{p}_i(t)\| < \overline{d}_{i,W}, \\ & -\frac{\pi}{2} < \theta_i(t) < \frac{\pi}{2}, \\ & \forall t \in \mathbb{R}_{\geq 0}\big\}\end{aligned}$$

1.2 The error model

A feasible desired configuration $\mathbf{z}_{i,des} \in \mathbb{R}^9 \times \mathbb{T}^3$ is associated to each agent $i \in \mathcal{V}$, with the aim of agent i achieving it in steady-state: $\lim\limits_{t\to\infty} \|\mathbf{z}_i(t) - \mathbf{z}_{i,des}\| = 0$. The interior of the norm of this expression denotes the state error of agent i:

$$\mathbf{e}_i(t) = \mathbf{z}_i(t) - \mathbf{z}_{i,des},\; \mathbf{e}_i(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^9 \times \mathbb{T}^3$$

The error dynamics are denoted by $g_i^R(\mathbf{e}_i, \mathbf{u}_i)$:

$$\begin{aligned}\dot{\mathbf{e}}_i(t) = \dot{\mathbf{z}}_i(t) - \dot{\mathbf{z}}_{i,des} = \dot{\mathbf{z}}_i(t) = f_i^R\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big) = f_i\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big) + \boldsymbol{\delta}_i(t) \\ = g_i\big(\mathbf{e}_i(t), \mathbf{u}_i(t)\big) + \boldsymbol{\delta}_i(t) \\ = g_i^R(\mathbf{e}_i(t), \mathbf{u}_i(t)\big) \end{aligned} \tag{4}$$

with $\mathbf{e}_i(0) = \mathbf{z}_i(0) - \mathbf{z}_{i,des}$. In order to translate the constraints that are dictated for the state $\mathbf{z}_i(t)$ into constraints regarding the error state $\mathbf{e}_i(t)$, we define the set $\mathcal{E}_i \subset \mathbb{R}^9 \times \mathbb{T}^3$ as:

$$\mathcal{E}_i = \{\mathbf{e}_i(t) \in \mathbb{R}^9 \times \mathbb{T}^3 : \mathbf{e}_i(t) \in \mathcal{Z}_i \oplus (-\mathbf{z}_{i,des})\}$$

as the set that captures all constraints for the error dynamics (4) dictated by the problem (**??**).

If we design control laws $\mathbf{u}_i \in \mathcal{U}_i$, $\forall i \in \mathcal{V}$ such that the error signal $\mathbf{e}_i(t)$ with dynamics given in (4), constrained under $\mathbf{e}_i(t) \in \mathcal{E}_i$, satisfies $\lim\limits_{t\to\infty} \|\mathbf{e}_i(t)\| = 0$, while all system related signals remain bounded in their respective regions,− if all of the above are achieved, then problem (**??**) has been solved.

In order to achieve this task, we employ a Nonlinear Receding Horizon scheme.

1.3 The optimization problem

Consider a sequence of sampling times $\{t_k\}_{k\geq 0}$, with a constant sampling time h, $0 < h < T_p$, where T_p is the finite time-horizon, such that $t_{k+1} = t_k + h$. In sampling data NMPC, a finite-horizon open-loop optimal control problem (OCP) is solved at discrete sampling time instants t_k based on the then-current state error measurement $\mathbf{e}_i(t_k)$. The solution is an optimal control signal $\overline{\mathbf{u}}_i^\star(t)$, computed over $t \in [t_k, t_k + T_p]$. This signal is applied to the open-loop system in between sampling times t_k and $t_k + h$.

At a generic time t_k, agent i solves the following optimization problem:

Problem 1.1.

Find

$$J_i^\star\big(\mathbf{e}_i(t_k)\big) \triangleq \min_{\overline{\mathbf{u}}_i(\cdot)} \quad J_i\big(\mathbf{e}_i(t_k), \overline{\mathbf{u}}_i(\cdot)\big)$$

where

$$J_i\big(\mathbf{e}_i(t_k), \overline{\mathbf{u}}_i(\cdot)\big) \triangleq \int_{t_k}^{t_k+T_p} F_i\big(\overline{\mathbf{e}}_i(s), \overline{\mathbf{u}}_i(s)\big) ds + V_i\big(\overline{\mathbf{e}}_i(t_k + T_p)\big) \tag{5}$$

subject to:

$$\dot{\overline{\mathbf{e}}}_i(s) = g_i\big(\overline{\mathbf{e}}_i(s), \overline{\mathbf{u}}_i(s)\big),\ \overline{\mathbf{e}}_i(t_k) = \mathbf{e}_i(t_k) \tag{6}$$

$$\overline{\mathbf{u}}_i(s) \in \mathcal{U}_i,\ \overline{\mathbf{e}}_i(s) \in \mathcal{E}_{i, s-t_k},\ s \in [t_k, t_k + T_p]$$

$$\overline{\mathbf{e}}_i(t_k + T_p) \in \Omega_i$$

The notation $\bar{\cdot}$ is used to distinguish predicted states which are internal to the controller, as opposed to their actual values, because, even in the nominal case, the predicted values will not be equal to the actual closed-loop values. This means that $\overline{\mathbf{e}}_i(\cdot)$ is the solution to (6) driven by the control input $\overline{\mathbf{u}}_i(\cdot) : [t_k, t_k + T_p] \to \mathcal{U}_i$ with initial condition $\mathbf{e}_i(t_k)$.

The applied input signal is a portion of the optimal solution to an optimization problem where information on the states of the neighbouring agents of agent i are taken into account only in the constraints considered in the optimization problem. These constraints pertain to the set of its neighbours $\mathcal{N}_i$ and, in total, to the set of all agents within its sensing range $\mathcal{R}_i$. Regarding these, we make the following assumption:

Assumption 1.1. (*Access to Predicted Information from an Inter-agent Perspective*)
Considering the context of Receding Horizon Control, when at time t_k agent i solves a finite

horizon optimization problem, he has access to[a]

1. measurements of the states[b]

    - $\mathbf{z}_j(t_k)$ of all agents $j \in \mathcal{R}_i(t_k)$ within its sensing range at time t_k

    - $\mathbf{z}_{j'}(t_k)$ of all of its neighbouring agents $j' \in \mathcal{N}_i$ at time t_k

2. the *predicted states*

    - $\overline{\mathbf{z}}_j(\tau)$ of all agents $j \in \mathcal{R}_i(t_k)$ within its sensing range

    - $\overline{\mathbf{z}}_{j'}(\tau)$ of all of its neighbouring agents $j' \in \mathcal{N}_i$

    across the entire horizon $\tau \in (t_k, t_k + T_p]$

[a] Although $\mathcal{N}_i \subseteq \mathcal{R}_i$, we make the distinction between the two because all agents $j \in \mathcal{R}_i$ need to avoid collision with agent i, but only agents $j' \in \mathcal{N}_i$ need to remain within the sensing range of agent i.

[b] as per assumption (??)

In other words, each time an agent solves its own individual optimization problem, he knows the error predictions that have been generated by the solution of the optimization problem of all agents within its range at that time, for the next T_p timesteps. This assumption is crucial to satisfying the constraints regarding collision aversion and connectivity maintenance between neighbouring agents. We assume that the above pieces of information are (a) always available and accurate, and (b) exchanged without delay. We encapsulate these pieces of information in four stacked vectors:

$$\mathbf{z}_{\mathcal{R}_i}(t_k) \triangleq col[\mathbf{z}_j(t_k)], \forall j \in \mathcal{R}_i(t_k)$$
$$\mathbf{z}_{\mathcal{N}_i}(t_k) \triangleq col[\mathbf{z}_j(t_k)], \forall j \in \mathcal{N}_i$$
$$\overline{\mathbf{z}}_{\mathcal{R}_i}(\tau) \triangleq col[\overline{\mathbf{z}}_j(\tau)], \forall j \in \mathcal{R}_i(t_k), \tau \in [t_k, t_k + T_p]$$
$$\overline{\mathbf{z}}_{\mathcal{N}_i}(\tau) \triangleq col[\overline{\mathbf{z}}_j(\tau)], \forall j \in \mathcal{N}_i, \tau \in [t_k, t_k + T_p]$$

Remark 1.2. The justification for this assumption is the following: considering that $\mathcal{N}_i \subseteq \mathcal{R}_i$, that the state vectors $\mathbf{z}_j$ are comprised of 12 real numbers that are encoded by 4 bytes, and that sampling occurs with a frequency f for all agents, the overall downstream bandwidth required by each agent is

$$BW_d = 12 \times 32 \text{ [bits]} \times |\mathcal{R}_i| \times \frac{T_p}{h} \times f \text{ [sec}^{-1}\text{]}$$

Given conservative constants $f = 100$ Hz, $\frac{T_p}{h} = 100$, the wireless protocol IEEE 802.11n-2009 (a standard for present-day devices) can accommodate up to

$$|\mathcal{R}_i| = \frac{600 \text{ [Mbit} \cdot \text{sec}^{-1}]}{12 \times 32[\text{bit}] \times 10^4[\text{sec}^{-1}]} \approx 16 \cdot 10^2 \text{ agents}$$

within the range of one agent. We deem this number to be large enough for practical applications for the approach of assuming access to the predicted states of agents within the range of one agent to be legal.

?? more on the actual $\mathcal{E}_i$

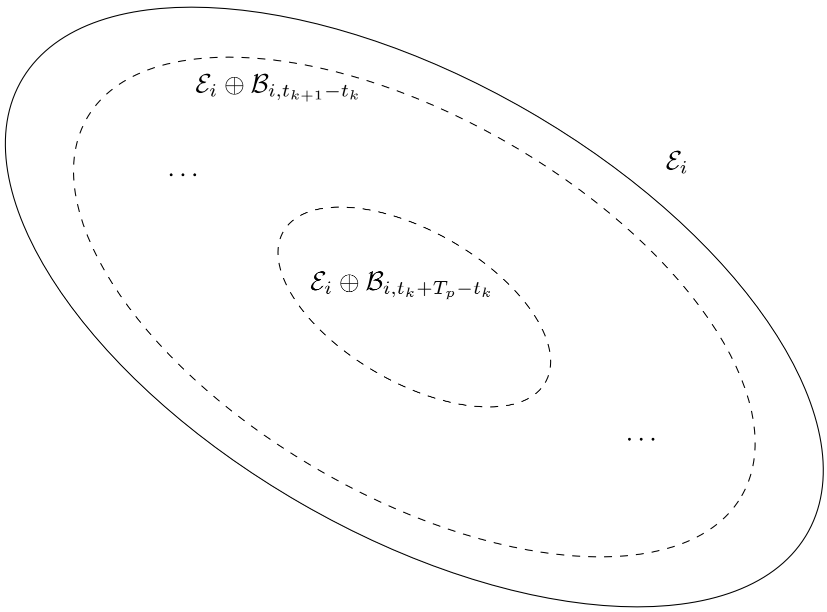

Figure 1: The nominal constraint set $\mathcal{E}_i$ in bold and the consecutive restricted constraint sets $\mathcal{E}_i \oplus \mathcal{B}_{i,s-t_k}$, $s \in [t_k, t_k + T_p]$, dashed.

While in the disturbance-free case the constraint set is $\mathcal{E}_i$, due to the existence of disturbances here, the constraint set is replaced in problem (1.1) by

$$\mathcal{E}_{i,s-t_k} \equiv \mathcal{E}_i \oplus \mathcal{B}_{i,s-t_k} \tag{8}$$

where

$$\mathcal{B}_{i,s-t_k} \equiv \left\{ \mathbf{e}_i \in \mathbb{R}^9 \times \mathbb{T}^3 : \|\mathbf{e}_i(s)\| \leq \frac{\overline{\delta}_i}{L_{g_i}} \left(e^{L_{g_i}(s-t_k)} - 1 \right), \ \forall s \in [t_k, t_k + T_p] \right\} \tag{9}$$

The reason for this substitution lies in the following. Consider that there are no disturbances affecting the states of the plant; the state evolution of the plant and its model considered in the solution to the optimization problem abide both by the state constraints since the two models are identical. Consider now that there are disturbances affecting the states of the plant, disturbances that are unknown to the model considered in the solution to the optimization problem. If the state constraint set was left unchanged during the solution of the optimization problem, the applied input to the plant, coupled with the uncertainty affecting the states of the plant could, without loss of generality[1], force the states of the plant to escape their intended bounds.

If the state constraint set considered in the solution of the optimization problem (1.1) is equal to (8), then the state of the real system, the plant, is guaranteed to abide by the original state constraint set $\mathcal{E}_i$. We formalize this statement in property (1.1).

Property 1.1. For every $s \in [t_k, t_k + T_p]$

$$\overline{\mathbf{e}}_i\big(s;\ \mathbf{u}_i(\cdot,\ \mathbf{e}_i(t_k)), \mathbf{e}_i(t_k)\big) \in \mathcal{E}_i \oplus \mathcal{B}_{i,s-t_k} \Rightarrow \mathbf{e}_i(s) \in \mathcal{E}_i$$

where $\mathcal{B}_{i,s-t_k}$ is given by (9).

Proof of property (1.1)

Let us define for convenience $\boldsymbol{\zeta}_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^9 \times \mathbb{T}^3$: $\boldsymbol{\zeta}_i(s) \triangleq \mathbf{e}_i(s) - \overline{\mathbf{e}}_i(s;\ \mathbf{u}_i(s;\ \mathbf{e}_i(t_k)), \mathbf{e}_i(t_k))$, for $s \in [t_k, t_k + T_p]$.

According to lemma (1.3)

$$\begin{aligned}
\|\mathbf{e}_i(s) - \overline{\mathbf{e}}_i(s;\ \mathbf{u}_i(s;\ \mathbf{e}_i(t_k)), \mathbf{e}_i(t_k))\| &\leq \frac{\overline{\delta}_i}{\mathrm{L}_{g_i}}(e^{L_{g_i}(s-t_k)} - 1) \\
\|\boldsymbol{\zeta}_i(s)\| &\leq \frac{\overline{\delta}_i}{\mathrm{L}_{g_i}}(e^{L_{g_i}(s-t_k)} - 1)
\end{aligned}$$

which means that $\boldsymbol{\zeta}_i(s) \in \mathcal{B}_{i,s-t_k}$. Now let us assume that $\overline{\mathbf{e}}_i\big(s;\ \mathbf{u}_i(\cdot,\ \mathbf{e}_i(t_k)), \mathbf{e}_i(t_k)\big) \in \mathcal{E}_i \oplus \mathcal{B}_{i,s-t_k}$. Then,

$$\begin{aligned}
\overline{\mathbf{e}}_i\big(s;\ \mathbf{u}_i(\cdot,\ \mathbf{e}_i(t_k)), \mathbf{e}_i(t_k)\big) &\in \mathcal{E}_i \oplus \mathcal{B}_{i,s-t_k} \\
\mathbf{e}_i(s) - \boldsymbol{\zeta}_i(s) &\in \mathcal{E}_i \oplus \mathcal{B}_{i,s-t_k}
\end{aligned}$$

If $\boldsymbol{\zeta}_i(s)$ resides in $\mathcal{B}_{i,s-t_k}$, then $-\boldsymbol{\zeta}_i(s)$ also resides in $\mathcal{B}_{i,s-t_k}$, which means that

$$\begin{aligned}
\mathbf{e}_i(s) + \big(-\boldsymbol{\zeta}_i(s)\big) &\in \mathcal{E}_i \oplus \mathcal{B}_{i,s-t_k} \\
\mathbf{e}_i(s) &\in \mathcal{E}_i
\end{aligned}$$

$\blacksquare$

Before defining the terminal set Ω_i it is necessary to state the definition of a positively invariant set:

[1] Receding Horizon Control is inherently robust under certain considerations, see [4] for more.

Definition 1.1. (*Positively Invariant Set*)

Consider a dynamical system $\dot{\mathbf{x}} = f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, and a trajectory $\mathbf{x}(t;\ \mathbf{x}_0)$, where $\mathbf{x}_0$ is the initial condition. The set $S = \{\mathbf{x} \in \mathbb{R}^n : \gamma(\mathbf{x}) = 0\}$, where γ is a valued function, is said to be *positively invariant* if the following holds:

$$\mathbf{x}_0 \in S \Rightarrow \mathbf{x}(t;\ \mathbf{x}_0) \in S,\ \forall t \geq t_0$$

Intuitively, this means that the set S is positively invariant if a trajectory of the system does not exit it once it enters it.

The terminal set $\Omega_i \subseteq \Psi_i$ is a subset of an admissible and positively invariant set Ψ_i, where Ψ_i is defined as

$$\Psi_i \triangleq \left\{\mathbf{e}_i \in \mathcal{E}_i : V_i(\mathbf{e}_i) \leq \varepsilon_{\Psi_i}\right\},\ \varepsilon_{\Psi_i} > 0$$

In particular, the set Ψ_i belongs to the set Φ_i, $\Psi_i \subseteq \Phi_i$, which is the set of states within $\mathcal{E}_{i,T_p}$ for which there is an admissible control input whose form is of linear feedback with regard to the state:

$$\Phi_i \triangleq \left\{\mathbf{e}_i \in \mathcal{E}_{i,T_p} : \mathbf{h}_i(\mathbf{e}_i) \in \mathcal{U}_i\right\}$$

Furthermore, the admissible and positively invariant set Ψ_i is such that

$$\forall \mathbf{e}_i \in \Psi_i \Rightarrow g_i(\mathbf{e}_i, \mathbf{h}_i(\mathbf{e}_i)) \in \Omega_i$$

The terminal set Ω_i is defined as

$$\Omega_i \triangleq \left\{\mathbf{e}_i \in \mathcal{E}_i : V_i(\mathbf{e}_i) \leq \varepsilon_{\Omega_i}\right\}\text{ , where } \varepsilon_{\Omega_i} \in (0, \varepsilon_{\Psi_i})$$

The functions $F_i : \mathcal{E}_i \times \mathcal{U}_i \to \mathbb{R}_{\geq 0}$ and $V_i : \Omega_i \to \mathbb{R}_{\geq 0}$ are defined as

$$F_i\big(\overline{\mathbf{e}}_i(t), \overline{\mathbf{u}}_i(t)\big) \triangleq \overline{\mathbf{e}}_i(t)^\top \boldsymbol{Q}_i \overline{\mathbf{e}}_i(t) + \overline{\mathbf{u}}_i(t)^\top \boldsymbol{R}_i \overline{\mathbf{u}}_i(t)$$
$$V_i\big(\overline{\mathbf{e}}_i(t)\big) \triangleq \overline{\mathbf{e}}_i(t)^\top \boldsymbol{P}_i \overline{\mathbf{e}}_i(t)$$

Matrices $\boldsymbol{R}_i \in \mathbb{R}^{6\times 6}$ are symmetric and positive definite, while matrices $\boldsymbol{Q}_i, \boldsymbol{P}_i \in \mathbb{R}^{12\times 12}$ are symmetric

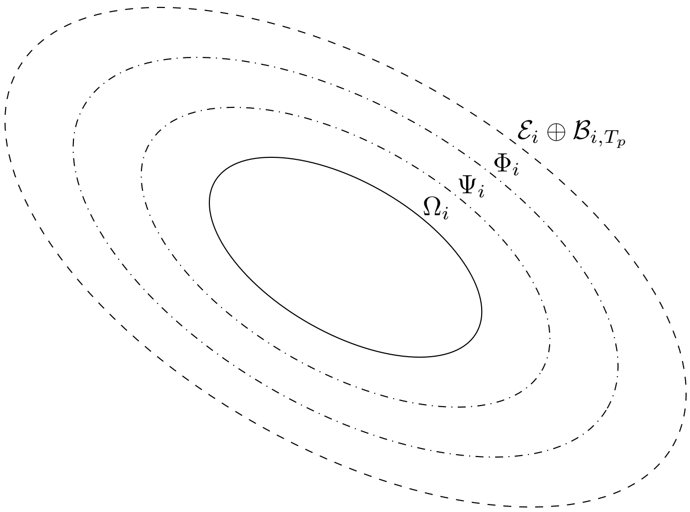

Figure 2: The hierarchy of sets $\Omega_i \subseteq \Psi_i \subseteq \Phi_i \subseteq \mathcal{E}_{i,T_p}$. For every state in Φ_i there is a linear state feedback control $\mathbf{h}_i(\mathbf{e}_i)$ which, when applied to a state $\mathbf{e}_i \in \Psi_i$, causes the trajectory of the state of the system to fall into the terminal set Ω_i.

and positive semi-definite. The running costs F_i are upper- and lower-bounded:

$$\begin{aligned}
\lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\|\mathbf{e}_i(t)\|^2 &\leq \lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\left\| \begin{bmatrix} \mathbf{e}_i(t) \\ \mathbf{u}_i(t) \end{bmatrix} \right\|^2 \\
&\leq F_i\big(\mathbf{e}_i(t), \mathbf{u}_i(t)\big) \\
&\leq \lambda_{max}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\left\| \begin{bmatrix} \mathbf{e}_i(t) \\ \mathbf{u}_i(t) \end{bmatrix} \right\|^2 \leq \lambda_{max}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\|\mathbf{e}_i(t)\|^2
\end{aligned}$$

where $\lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i)$ is the smallest eigenvalue between those of matrices $\boldsymbol{Q}_i$ and $\boldsymbol{R}_i$, and $\lambda_{max}(\boldsymbol{Q}_i, \boldsymbol{R}_i)$ the largest. Since the terms $\lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\|\mathbf{e}_i(t)\|$ and $\lambda_{max}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\|\mathbf{e}_i(t)\|$ are themselves class $\mathcal{K}$ functions according to definition (**??**), F_i is lower- and upper-bounded by class $\mathcal{K}$ functions. As is obvious, $F_i(\mathbf{0}, \mathbf{0}) = 0$.

With regard to the terminal penalty function V_i, the following lemma will prove to be useful in guaranteeing the convergence of the solution to the optimal control problem to the terminal region Ω_i:

Lemma 1.1. (V_i is Lipschitz continuous in $\mathcal{E}_{i,F}$)

The terminal penalty function V_i is Lipschitz continuous in $\mathcal{E}_{i,F}$

$$|V(\mathbf{e}_{1,i}) - V(\mathbf{e}_{2,i})| \leq L_{V_i}\|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\|$$

where $\mathbf{e}_{1,i}, \mathbf{e}_{2,i} \in \mathcal{E}_{i,F}$, with Lipschitz constant $L_{V_i} = 2\varepsilon_F \lambda_{max}(P_i)$

Proof For every $\mathbf{e}_i \in \Omega_i$, it holds that

$$\begin{aligned}
|V(\mathbf{e}_{1,i}) - V(\mathbf{e}_{2,i})| &= |\mathbf{e}_{1,i}^\top \boldsymbol{P}_i \mathbf{e}_{1,i} - \mathbf{e}_{2,i}^\top \boldsymbol{P}_i \mathbf{e}_{2,i}| \\
&= |\mathbf{e}_{1,i}^\top \boldsymbol{P}_i \mathbf{e}_{1,i} - \mathbf{e}_{2,i}^\top \boldsymbol{P}_i \mathbf{e}_{2,i} \pm \mathbf{e}_{1,i}^\top \boldsymbol{P}_i \mathbf{e}_{2,i}| \\
&= |\mathbf{e}_{1,i}^\top \boldsymbol{P}_i (\mathbf{e}_{1,i} - \mathbf{e}_{2,i}) - \mathbf{e}_{2,i}^\top \boldsymbol{P}_i (\mathbf{e}_{1,i} - \mathbf{e}_{2,i})| \\
&\leq |\mathbf{e}_{1,i}^\top \boldsymbol{P}_i (\mathbf{e}_{1,i} - \mathbf{e}_{2,i})| + |\mathbf{e}_{2,i}^\top \boldsymbol{P}_i (\mathbf{e}_{1,i} - \mathbf{e}_{2,i})|
\end{aligned}$$

But for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$|\mathbf{x}^\top \boldsymbol{A} \mathbf{y}| \leq \lambda_{max}(A) \|\mathbf{x}\| \|\mathbf{y}\|$$

where $\lambda_{max}(A)$ denotes the largest eigenvalue of matrix $\boldsymbol{A}$. Hence:

$$\begin{aligned}
|V(\mathbf{e}_{1,i}) - V(\mathbf{e}_{2,i})| &\leq \lambda_{max}(\boldsymbol{P}_i) \|\mathbf{e}_{1,i}\| \|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\| + \lambda_{max}(\boldsymbol{P}_i) \|\mathbf{e}_{2,i}\| \|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\| \\
&= \lambda_{max}(\boldsymbol{P}_i)(\|\mathbf{e}_{1,i}\| + \|\mathbf{e}_{2,i}\|) \|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\| \\
&\leq \lambda_{max}(\boldsymbol{P}_i)(\varepsilon_F + \varepsilon_F) \|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\| \\
&= 2\varepsilon_F \lambda_{max}(\boldsymbol{P}_i) \|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\|
\end{aligned}$$

■

The solution to the optimal control problem (5) at time t_k is an optimal control input, denoted by $\overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t_k))$, which is applied to the open-loop system until the next sampling instant $t_k + h$, with $h \in (0, T_p)$, at which time a new optimal control problem is solved in the same manner:

$$\mathbf{u}_i(t) = \overline{\mathbf{u}}_i^\star\big(t;\ \mathbf{e}_i(t_k)\big),\ t \in [t_k, t_k + h]$$

The control input $\mathbf{u}_i(\cdot)$ is of feedback form, since it is recalculated at each sampling instant based on the then-current state. The solution to equation (4), starting at time t_1, from an initial condition $\mathbf{e}_i(t_1)$, by application of the control input $\mathbf{u}_i : [t_1, t_2] \to \mathcal{U}_i$ is denoted by

$$\mathbf{e}_i\big(t;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(t_1)\big),\ t \in [t_1, t_2]$$

The *predicted* state of the system (4) at time $t_k + \tau$, based on the measurement of the state at time t_k, $\mathbf{e}_i(t_k)$, by application of the control input $\mathbf{u}_i\big(t;\ \mathbf{e}_i(t_k)\big)$, for the time period $t \in [t_k, t_k + \tau]$ is denoted by

$$\overline{\mathbf{e}}_i\big(t_k + \tau;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(t_k)\big)$$

In contrast to the disturbance-free case:

$$\overline{\mathbf{e}}_i\big(\tau_1;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(\tau_0)\big) \neq \mathbf{e}_i\big(\tau_1;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(\tau_0)\big)$$

holds true here because *there are* disturbances acting on the system.

The closed-loop system for which stability is to be guaranteed is

$$\mathbf{e}_i(\tau) = g_i^R\big(\mathbf{e}_i(\tau), \overline{\mathbf{u}}_i^\star(\tau)\big),\ \tau \geq t_0 = 0$$

where $\overline{\mathbf{u}}_i^\star(\tau) = \overline{\mathbf{u}}_i^\star(\tau;\ \mathbf{e}_i(t_k))$, $\tau \in [t_k, t_k + h)$.

We can now give the definition of an *admissible input*:

Definition 1.2. (Admissible input)

A control input $\mathbf{u}_i : [t_k, t_k + T_p] \to \mathbb{R}^6$ for a state $\mathbf{e}_i(t_k)$ is called *admissible* if all the following hold:

1. $\mathbf{u}_i(\cdot)$ is piecewise continuous
2. $\mathbf{u}_i(\tau) \in \mathcal{U}_i,\ \forall \tau \in [t_k, t_k + T_p]$
3. $\mathbf{e}_i\big(\tau;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(t_k)\big) \in \mathcal{E}_i,\ \forall \tau \in [t_k, t_k + T_p]$
4. $\mathbf{e}_i\big(t_k + T_p;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(t_k)\big) \in \Omega_i$

1.4 Feasibility and Convergence

Under these considerations, we can now state the theorem that relates to the guaranteeing of the stability of the compound system of agents $i \in \mathcal{V}$, when each of them is assigned a desired position which results in feasible displacements:

Theorem 1.1. Suppose that

1. the terminal region $\Omega_i \subseteq \mathcal{E}_i$ is closed with $\mathbf{0} \in \Omega_i$
2. a solution to the optimal control problem (**??**) is feasible at time $t = 0$, that is, assumptions (**??**), (**??**), and (**??**) hold at time $t = 0$
3. there exists an admissible control input $\mathbf{u}_{i,f} : [0, h] \to \mathcal{U}_i$ such that for all $\mathbf{e}_i \in \Omega_i$ and $\forall \tau \in [0, h]$:

(a) $\mathbf{e}_i(\tau) \in \Omega_i$

(b) $\dfrac{\partial V_i}{\partial \mathbf{e}_i} g_i\big(\mathbf{e}_i(\tau), \mathbf{u}_{i,f}(\tau)\big) + F_i\big(\mathbf{e}_i(\tau), \mathbf{u}_{i,f}(\tau)\big) \leq 0$

then the closed loop system (**??**) under the control input (**??**) converges to the set Ω_i when $t \to \infty$.

Proof. The proof of the above theorem consists of two parts: in the first, recursive feasibility is established, that is, initial feasibility is shown to imply subsequent feasibility; in the second, and based on the first part, it is shown that the error state $\mathbf{e}_i(t)$ converges to the terminal set Ω_i.

The following proof relies heavily on the Grönwall-Bellman inequality. We state it here for reference purposes.

Lemma 1.2. [1] *Grönwall-Bellman Inequality*

Let $\lambda : [a, b] \to \mathbb{R}$ be continuous and $\mu : [a, b] \to \mathbb{R}$ be continuous and non-negative. If a continuous function $y : [a, b] \to \mathbb{R}$ satisfies

$$y(t) \leq \lambda(t) + \int_a^t \mu(s) y(s) ds$$

for $a \leq t \leq b$, then on the same interval

$$y(t) \leq \lambda(t) + \int_a^t \lambda(s)\mu(s) e^{\int_s^t \mu(\tau)d\tau} ds$$

In particular, if $\lambda(t) \equiv \lambda$ is a constant, then

$$y(t) \leq \lambda e^{\int_a^t \mu(\tau)d\tau} ds$$

If $\lambda(t) \equiv \lambda$ and $\mu(t) \equiv \mu$ are both constants, then

$$y(t) \leq \lambda e^{\mu(t-a)}$$

Remark 1.3. Given that disturbances *are* present, for the predicted and actual states at time

$\tau_1 \geq \tau_0 \in \mathbb{R}_{\geq 0}$ it holds that:

$$\mathbf{e}_i\big(\tau_1;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(\tau_0)\big) = \mathbf{e}_i(\tau_0) + \int_{\tau_0}^{\tau_1} g_i^R\big(\mathbf{e}_i(s;\ \mathbf{e}_i(\tau_0)), \mathbf{u}_i(s)\big)ds$$
$$\overline{\mathbf{e}}_i\big(\tau_1;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(\tau_0)\big) = \mathbf{e}_i(\tau_0) + \int_{\tau_0}^{\tau_1} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(\tau_0)), \mathbf{u}_i(s)\big)ds$$

Lemma 1.3. Suppose that the real system, which is under the existence of bounded additive disturbances, and the model are both at time t at state $\mathbf{e}_i(t)$. Applying at time t a control law $\mathbf{u}(\cdot)$ to the system model deemed "real" and its model will cause at time $t+\tau$, $\tau \geq 0$ a divergence between the states of the real system and its model. The norm of the difference between the state of the real system and the state of the model system is bounded by

$$\Big\|\mathbf{e}_i\big(t+\tau;\ \mathbf{u}(\cdot), \mathbf{e}_i(t)\big) - \overline{\mathbf{e}}_i\big(t+\tau;\ \mathbf{u}(\cdot), \mathbf{e}_i(t)\big)\Big\| \leq \frac{\overline{\delta}_i}{\mathrm{L}_{g_i}}(e^{L_{g_i}\tau} - 1)$$

where $\overline{\delta}_i$ is the upper bound of the disturbance, and L_{g_i} the Lipschitz constant of both models.

Proof of lemma (1.3)

Since there are disturbances present, consulting remark (1.3) and substituting for $\tau_0 = t_k$ and $\tau_1 = t_{k+1}$ yields:

$$\mathbf{e}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) = \mathbf{e}_i(t_k) + \int_{t_k}^{t_{k+1}} g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big)ds + \int_{t_k}^{t_{k+1}} \delta_i(s)ds$$
$$\overline{\mathbf{e}}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) = \mathbf{e}_i(t_k) + \int_{t_k}^{t_{k+1}} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big)ds$$

Subtracting the latter from the former and taking norms on either side yields:

$$\begin{aligned}
&\Big\|\mathbf{e}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) - \overline{\mathbf{e}}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big)\Big\| \\
&= \Big\|\int_{t_k}^{t_{k+1}} g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big)ds - \int_{t_k}^{t_{k+1}} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big)ds + \int_{t_k}^{t_{k+1}} \delta_i(s)ds\Big\| \\
&\leq \Big\|\int_{t_k}^{t_{k+1}} g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big)ds - \int_{t_k}^{t_{k+1}} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big)ds\Big\| + (t_{k+1} - t_k)\overline{\delta}_i \\
&= \int_{t_k}^{t_{k+1}} \Big\|g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big)s - g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big)\Big\|ds + h\overline{\delta}_i \\
&\leq L_{g_i} \int_{t_k}^{t_{k+1}} \Big\|\mathbf{e}_i\big(s;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) - \overline{\mathbf{e}}_i\big(s;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big)\Big\|ds + h\overline{\delta}_i
\end{aligned}$$

since g_i is Lipschitz continuous in $\mathcal{E}_i$ with Lipschitz constant L_{g_i}. Reformulation yields

$$\begin{aligned}
&\Big\|\mathbf{e}_i\big(t_k+h;\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t_k)),\mathbf{e}_i(t_k)\big) - \overline{\mathbf{e}}_i\big(t_k+h;\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t_k)),\mathbf{e}_i(t_k)\big)\Big\| \\
&\leq h\overline{\delta}_i + L_{g_i}\int_0^h \Big\|\mathbf{e}_i\big(t_k+s;\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t_k)),\mathbf{e}_i(t_k)\big) - \overline{\mathbf{e}}_i\big(t_k+s;\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t_k)),\mathbf{e}_i(t_k)\big)\Big\| ds
\end{aligned}$$

By applying the Grönwall-Bellman inequality we get:

$$\begin{aligned}
&\Big\|\mathbf{e}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t_k)),\mathbf{e}_i(t_k)\big) - \overline{\mathbf{e}}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t_k)),\mathbf{e}_i(t_k)\big)\Big\| \\
&\leq h\overline{\delta}_i + L_{g_i}\int_0^h s\overline{\delta}_i e^{L_{g_i}(h-s)} ds \\
&= h\overline{\delta}_i - \overline{\delta}_i\int_0^h s\big(e^{L_{g_i}(h-s)}\big)' ds \\
&= h\overline{\delta}_i - \overline{\delta}_i\bigg(\big[se^{L_{g_i}(h-s)}\big]_0^h - \int_0^h e^{L_{g_i}(h-s)} ds\bigg) \\
&= h\overline{\delta}_i - \overline{\delta}_i\bigg(h + \frac{1}{L_{g_i}}(1-e^{L_{g_i}h})\bigg) \\
&= \frac{\overline{\delta}_i}{L_{g_i}}(e^{L_{g_i}h}-1)
\end{aligned}$$

■

Feasibility analysis Consider a sampling instant t_k for which a solution $\overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big)$ to problem (1.1) exists. Suppose now a time instant t_{k+1} such that[2] $t_k < t_{k+1} < t_k + T_p$, and consider that the optimal control signal calculated at t_k is comprised by the following two portions:

$$\overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big) = \begin{cases} \overline{\mathbf{u}}_i^\star\big(\tau_1;\ \mathbf{e}_i(t_k)\big), & \tau_1 \in [t_k, t_{k+1}] \\ \overline{\mathbf{u}}_i^\star\big(\tau_2;\ \mathbf{e}_i(t_k)\big), & \tau_2 \in [t_{k+1}, t_k + T_p] \end{cases} \tag{10}$$

Both portions are admissible since the calculated optimal control input is admissible, and hence they both conform to the input constraints. As for the resulting predicted states, they satisfy the state constraints, and, crucially: $\overline{\mathbf{e}}_i\big(t_k+T_p;\ \overline{\mathbf{u}}_i^\star(\cdot),\mathbf{e}_i(t_k)\big) \in \Omega_i$. Furthermore, according to assumption (3) of the theorem, there exists an admissible (and certainly not guaranteed optimal) input $\mathbf{u}_{i,f}$ that renders Ω_i invariant over $[t_k+T_p, t_k+T_p+h]$.

Given the above facts, we can construct an admissible input $\widetilde{\mathbf{u}}_i(\cdot)$ for time t_{k+1} by sewing together the second portion of (10) and the input $\mathbf{u}_{i,f}(\cdot)$:

[2]It is not strictly necessary that $t_{k+1} = t_k + h$ here, however it is necessary for the following that $t_{k+1} - t_k \leq h$

$$\widetilde{\mathbf{u}}_i(\tau) = \begin{cases} \overline{\mathbf{u}}_i^{\star}\big(\tau;\ \mathbf{e}_i(t_k)\big), & \tau \in [t_{k+1}, t_k + T_p] \\ \mathbf{u}_{i,f}(\tau - t_k - T_p), & \tau \in (t_k + T_p, t_{k+1} + T_p] \end{cases} \tag{11}$$

Applied at time t_{k+1}, $\widetilde{\mathbf{u}}_i(\tau)$ is an admissible control input as a composition of admissible control inputs, for all $\tau \in [t_{k+1}, t_{k+1} + T_p]$.

Furthermore, $\overline{\mathbf{e}}_i\big(t_{k+1} + T_p;\ \widetilde{\mathbf{u}}_i(\cdot), \mathbf{e}_i(t_{k+1})\big) \in \Omega_i$.

To prove this statement we begin with

$$\begin{aligned}
& V_i\Big(\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^{\star}(\cdot), \mathbf{e}_i(t_{k+1})\big)\Big) - V_i\Big(\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^{\star}(\cdot), \mathbf{e}_i(t_k)\big)\Big) \\
\leq & \left\| V_i\Big(\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^{\star}(\cdot), \mathbf{e}_i(t_{k+1})\big)\Big) - V_i\Big(\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^{\star}(\cdot), \mathbf{e}_i(t_k)\big)\Big) \right\| \\
\leq & L_{V_i} \left\| \overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^{\star}(\cdot), \mathbf{e}_i(t_{k+1})\big) - \overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^{\star}(\cdot), \mathbf{e}_i(t_k)\big) \right\| \text{ (for } V_i \text{ is Lipschitz continuous in } \mathcal{E}_i) \\
= & L_{V_i} \left\| \overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^{\star}(\cdot), \mathbf{e}_i(t_{k+1})\big) - \overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^{\star}(\cdot), \mathbf{e}_i(t_k)\big) \right\|
\end{aligned} \tag{12}$$

Consulting with remark (1.3) we get that the two terms interior to the norm are respectively equal to

$$\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^{\star}(\cdot), \mathbf{e}_i(t_{k+1})\big) = \mathbf{e}_i(t_{k+1}) + \int_{t_{k+1}}^{t_k + T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_{k+1})), \overline{\mathbf{u}}_i^{\star}(s)\big) ds$$

and

$$\begin{aligned}
\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^{\star}(\cdot), \mathbf{e}_i(t_k)\big) &= \mathbf{e}_i(t_k) + \int_{t_k}^{t_k + T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^{\star}(s)\big) ds \\
&= \mathbf{e}_i(t_k) + \int_{t_k}^{t_{k+1}} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^{\star}(s)\big) ds \\
&\quad + \int_{t_{k+1}}^{t_k + T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^{\star}(s)\big) ds \\
&= \overline{\mathbf{e}}_i(t_{k+1}) + \int_{t_{k+1}}^{t_k + T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^{\star}(s)\big) ds
\end{aligned}$$

Subtracting the latter from the former and taking norms on either side we get

$$
\begin{aligned}
&\Big\|\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big) - \overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big)\Big\| \\
&= \Big\|\mathbf{e}_i(t_{k+1}) - \overline{\mathbf{e}}_i(t_{k+1}) \\
&+ \int_{t_{k+1}}^{t_k+T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_{k+1})), \overline{\mathbf{u}}_i^\star(s)\big)ds - \int_{t_{k+1}}^{t_k+T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big)ds\Big\| \\
&\leq \Big\|\mathbf{e}_i(t_{k+1}) - \overline{\mathbf{e}}_i(t_{k+1})\Big\| \\
&+ \Big\|\int_{t_{k+1}}^{t_k+T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_{k+1})), \overline{\mathbf{u}}_i^\star(s)\big)ds - \int_{t_{k+1}}^{t_k+T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big)ds\Big\| \\
&= \Big\|\mathbf{e}_i(t_{k+1}) - \overline{\mathbf{e}}_i(t_{k+1})\Big\| \\
&+ \Big\|\int_{t_{k+1}}^{t_k+T_p} \Big(g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_{k+1})), \overline{\mathbf{u}}_i^\star(s)\big) - g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big)\Big)ds\Big\| \\
&= \Big\|\mathbf{e}_i(t_{k+1}) - \overline{\mathbf{e}}_i(t_{k+1})\Big\| \\
&+ \int_{t_{k+1}}^{t_k+T_p} \Big\|g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_{k+1})), \overline{\mathbf{u}}_i^\star(s)\big) - g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big)\Big\|ds \\
&\leq \Big\|\mathbf{e}_i(t_{k+1}) - \overline{\mathbf{e}}_i(t_{k+1})\Big\| \\
&+ L_{g_i} \int_{t_{k+1}}^{t_k+T_p} \Big\|\overline{\mathbf{e}}_i\big(s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big) - \overline{\mathbf{e}}_i\big(s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big)\Big\|ds, \text{ for } g_i \text{ is Lipschitz continuous in } \mathcal{E}_i \\
&= \Big\|\mathbf{e}_i(t_{k+1}) - \overline{\mathbf{e}}_i(t_{k+1})\Big\| \\
&+ L_{g_i} \int_{h}^{T_p} \Big\|\overline{\mathbf{e}}_i\big(t_k + s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big) - \overline{\mathbf{e}}_i\big(t_k + s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big)\Big\|ds
\end{aligned}
$$

By applying the Grönwall-Bellman inequality we get

$$
\Big\|\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big) - \overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big)\Big\| \leq \Big\|\mathbf{e}_i(t_{k+1}) - \overline{\mathbf{e}}_i(t_{k+1})\Big\| e^{L_{g_i}(T_p - h)}
$$

By applying lemma (1.3) for $\tau = h$ we get

$$
\Big\|\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big) - \overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big)\Big\| \leq \frac{\overline{\delta}_i}{L_{g_i}}(e^{L_{g_i}h} - 1)e^{L_{g_i}(T_p - h)}
$$

Hence (12) becomes

$$
\begin{aligned}
V_i\big(\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big)\big) &- V_i\big(\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \\
&\leq L_{V_i}\Big\|\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big) - \overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big)\Big\| \\
&= L_{V_i}\frac{\overline{\delta}_i}{L_{g_i}}(e^{L_{g_i}h} - 1)e^{L_{g_i}(T_p - h)} \qquad (13)
\end{aligned}
$$

==

This means that feasibility of a solution to the optimization problem at time t_k implies feasibility at time $t_{k+1} > t_k$, and, thus, since at time $t = 0$ a solution is assumed to be feasible, a solution to the optimal control problem is feasible for all $t \geq 0$.

Convergence analysis The second part of the proof involves demonstrating the convergence of the state $\mathbf{e}_i$ to the terminal set Ω_i. In order for this to be proved, it must be shown that a proper value function decreases along the solution trajectories starting at some initial time t_k. We consider the *optimal* cost $J_i^\star\big(\mathbf{e}_i(t)\big)$ as a candidate Lyapunov function:

$$J_i^\star\big(\mathbf{e}_i(t)\big) \triangleq J_i\Big(\mathbf{e}_i(t), \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t)\big)\Big)$$

and, in particular, our goal is to show that that this cost decreases over consecutive sampling instants $t_{k+1} = t_k + h$, i.e. $J_i^\star\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) \leq 0$.

In order not to wreak notational havoc, let us define the following terms:

- $\mathbf{u}_{0,i}(\tau) \triangleq \overline{\mathbf{u}}_i^\star\big(\tau;\ \mathbf{e}_i(t_k)\big)$ as the *optimal* input that results from the solution to problem (**??**) based on the measurement of state $\mathbf{e}_i(t_k)$, applied at time $\tau \geq t_k$
- $\mathbf{e}_{0,i}(\tau) \triangleq \overline{\mathbf{e}}_i\big(\tau;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big)$ as the *predicted* state at time $\tau \geq t_k$, that is, the state that results from the application of the above input $\overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big)$ to the state $\mathbf{e}_i(t_k)$, at time τ
- $\mathbf{u}_{1,i}(\tau) \triangleq \widetilde{\mathbf{u}}_i(\tau)$ as the *admissible* input at $\tau \geq t_{k+1}$ (see eq. (11))
- $\mathbf{e}_{1,i}(\tau) \triangleq \overline{\mathbf{e}}_i\big(\tau;\ \widetilde{\mathbf{u}}_i(\cdot), \mathbf{e}_i(t_{k+1})\big)$ as the *predicted* state at time $\tau \geq t_{k+1}$, that is, the state that results from the application of the above input $\widetilde{\mathbf{u}}_i(\cdot)$ to the state $\mathbf{e}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big)$, at time τ

Before beginning to prove convergence, it is worth noting that while the cost

$$J_i\Big(\mathbf{e}_i(t), \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t)\big)\Big)$$

is optimal (in the sense that it is based on the optimal input, which provides its minimum realization), a cost that is based on a plainly admissible (and thus, without loss of generality, sub-optimal) input

$\mathbf{u}_i \neq \overline{\mathbf{u}}_i^\star$ will result in a configuration where

$$J_i\Big(\mathbf{e}_i(t), \mathbf{u}_i\big(\cdot;\ \mathbf{e}_i(t)\big)\Big) \geq J_i\Big(\mathbf{e}_i(t), \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t)\big)\Big)$$

Let us now begin our investigation on the sign of the difference between the cost that results from the application of the feasible input $\mathbf{u}_{1,i}$, which we shall denote by $\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big)$, and the optimal cost $J_i^\star\big(\mathbf{e}_i(t_k)\big)$, while reminding ourselves that $J_i\big(\mathbf{e}_i(t), \overline{\mathbf{u}}_i(\cdot)\big) = \int_t^{t+T_p} F_i\big(\overline{\mathbf{e}}_i(s), \overline{\mathbf{u}}_i(s)\big)ds + V_i\big(\overline{\mathbf{e}}_i(t+T_p)\big)$:

$$\begin{aligned}\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) = V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) + \int_{t_{k+1}}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds \\ -V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) - \int_{t_k}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds\end{aligned}$$

Considering that $t_k < t_{k+1} < t_k + T_p < t_{k+1} + T_p$, we break down the two integrals above in between these intervals:

$$\begin{aligned}&\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) = \\ &\quad V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) + \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds \\ &\quad -V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) - \int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds - \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds \end{aligned} \tag{14}$$

We begin working on (14) focusing first on the difference between the two intervals over $[t_{k+1}, t_{k+1} + T_p]$:

$$\begin{aligned}&\int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds - \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds \\ &\int_{t_k+h}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds - \int_{t_k+h}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds \\ &\leq \left\| \int_{t_k+h}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds - \int_{t_k+h}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds \right\| \text{ (for } x \leq |x|, x \in \mathbb{R}) \\ &= \left\| \int_{t_k+h}^{t_k+T_p} \Big(F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) - F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)\Big) ds \right\| \\ &= \int_{t_k+h}^{t_k+T_p} \Big\| F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) - F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)\Big\| ds \\ &\leq L_{F_i} \int_{t_k+h}^{t_k+T_p} \Big\| \overline{\mathbf{e}}_i\big(s;\ \mathbf{u}_{1,i}(\cdot), \mathbf{e}_i(t_k+h)\big) - \overline{\mathbf{e}}_i\big(s;\ \mathbf{u}_{0,i}(\cdot), \mathbf{e}_i(t_k)\big)\Big\| ds \text{ (for } F_i \text{ is Lipschitz continuous in } \mathcal{E}_i) \\ &= L_{F_i} \int_h^{T_p} \Big\| \overline{\mathbf{e}}_i\big(t_k+s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k+h)\big) - \overline{\mathbf{e}}_i\big(t_k+s;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big)\Big\| ds \end{aligned} \tag{15}$$

Consulting with remark (1.3) for the two different initial conditions we get

$$\overline{\mathbf{e}}_i\big(t_k + s;\ \mathbf{u}_i^\star(\cdot), \mathbf{e}_i(t_k + h)\big) = \mathbf{e}_i(t_k + h) + \int_{t_k+h}^{t_k+s} g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k + h)), \mathbf{u}_i^\star(\tau)\big)d\tau$$

and

$$\begin{aligned}
\overline{\mathbf{e}}_i\big(t_k + s;\ \mathbf{u}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) &= \mathbf{e}_i(t_k) + \int_{t_k}^{t_k+s} g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k)), \mathbf{u}_i^\star(\tau)\big)d\tau \\
&= \mathbf{e}_i(t_k) + \int_{t_k}^{t_k+h} g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k)), \mathbf{u}_i^\star(\tau)\big)d\tau \\
&\quad + \int_{t_k+h}^{t_k+s} g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k)), \mathbf{u}_i^\star(\tau)\big)d\tau
\end{aligned}$$

Subtracting the latter from the former and taking norms on either side yields

$$\begin{aligned}
&\Big\|\overline{\mathbf{e}}_i\big(t_k + s;\ \mathbf{u}_i^\star(\cdot), \mathbf{e}_i(t_k + h)\big) - \overline{\mathbf{e}}_i\big(t_k + s;\ \mathbf{u}_i^\star(\cdot), \mathbf{e}_i(t_k)\big)\Big\| \\
&= \Big\|\mathbf{e}_i(t_k + h) - \Big(\mathbf{e}_i(t_k) + \int_{t_k}^{t_k+h} g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k)), \mathbf{u}_i^\star(\tau)\big)d\tau\Big) \\
&\quad + \int_{t_k+h}^{t_k+s} g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k + h)), \mathbf{u}_i^\star(\tau)\big)d\tau - \int_{t_k+h}^{t_k+s} g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k)), \mathbf{u}_i^\star(\tau)\big)d\tau\Big\| \\
&= \Big\|\mathbf{e}_i(t_k + h) - \overline{\mathbf{e}}_i(t_k + h) + \int_{t_k+h}^{t_k+s} \Big(g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k + h)), \mathbf{u}_i^\star(\tau)\big) - g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k)), \mathbf{u}_i^\star(\tau)\big)\Big)d\tau\Big\| \\
&\leq \Big\|\mathbf{e}_i(t_k + h) - \overline{\mathbf{e}}_i(t_k + h)\Big\| + \Big\|\int_{t_k+h}^{t_k+s} \Big(g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k + h)), \mathbf{u}_i^\star(\tau)\big) - g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k)), \mathbf{u}_i^\star(\tau)\big)\Big)d\tau\Big\| \\
&= \Big\|\mathbf{e}_i(t_k + h) - \overline{\mathbf{e}}_i(t_k + h)\Big\| + \int_{t_k+h}^{t_k+s} \Big\|g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k + h)), \mathbf{u}_i^\star(\tau)\big) - g_i\big(\overline{\mathbf{e}}_i(\tau;\ \mathbf{e}_i(t_k)), \mathbf{u}_i^\star(\tau)\big)\Big\|d\tau \\
&\leq \Big\|\mathbf{e}_i(t_k + h) - \overline{\mathbf{e}}_i(t_k + h)\Big\| + L_{g_i} \int_{t_k+h}^{t_k+s} \Big\|\overline{\mathbf{e}}_i\big(\tau;\ \mathbf{u}_i^\star(\cdot), \mathbf{e}_i(t_k + h)\big) - \overline{\mathbf{e}}_i\big(\tau;\ \mathbf{u}_i^\star(\cdot), \mathbf{e}_i(t_k)\big)\Big\|d\tau \\
&= \Big\|\mathbf{e}_i(t_k + h) - \overline{\mathbf{e}}_i(t_k + h)\Big\| + L_{g_i} \int_{h}^{s} \Big\|\overline{\mathbf{e}}_i\big(t_k + \tau;\ \mathbf{u}_i^\star(\cdot), \mathbf{e}_i(t_k + h)\big) - \overline{\mathbf{e}}_i\big(t_k + \tau;\ \mathbf{u}_i^\star(\cdot), \mathbf{e}_i(t_k)\big)\Big\|d\tau
\end{aligned} \tag{16}$$

Given that from lemma (1.3) the first term of the sum featured in (16) is a constant, by application of the the Grönwall-Bellman inequality, (16) becomes:

$$\begin{aligned}
&\Big\|\overline{\mathbf{e}}_i\big(t_k + s;\ \mathbf{u}_i^\star(\cdot), \mathbf{e}_i(t_k + h)\big) - \overline{\mathbf{e}}_i\big(t_k + s;\ \mathbf{u}_i^\star(\cdot), \mathbf{e}_i(t_k)\big)\Big\| \\
&\leq \Big\|\mathbf{e}_i(t_k + h) - \overline{\mathbf{e}}_i(t_k + h)\Big\| e^{L_{g_i}(s-h)} \\
&\leq \frac{\overline{\delta}_i}{\mathrm{L}_{g_i}}(e^{L_{g_i}h} - 1)e^{L_{g_i}(s-h)}
\end{aligned}$$

Given the above result, (15) becomes

$$
\begin{aligned}
&\int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds - \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds \\
&\leq L_{F_i} \int_h^{T_p} \frac{\overline{\delta}_i}{\text{Ł}_{g_i}}(e^{L_{g_i}h} - 1)e^{L_{g_i}(s-h)}ds \\
&= L_{F_i} \frac{\overline{\delta}_i}{\text{Ł}_{g_i}}(e^{L_{g_i}h} - 1) \int_h^{T_p} e^{L_{g_i}(s-h)}ds \\
&= L_{F_i} \frac{\overline{\delta}_i}{\text{Ł}_{g_i}}(e^{L_{g_i}h} - 1) \frac{1}{L_{g_i}}(e^{L_{g_i}(T_p-h)} - 1) \\
&= L_{F_i} \frac{\overline{\delta}_i}{\text{Ł}_{g_i}^2}(e^{L_{g_i}h} - 1)(e^{L_{g_i}(T_p-h)} - 1)
\end{aligned}
$$

Hence we discovered that

$$
\begin{aligned}
\int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds - \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds& \\
\leq L_{F_i} \frac{\overline{\delta}_i}{\text{Ł}_{g_i}^2}(e^{L_{g_i}h} - 1)(e^{L_{g_i}(T_p-h)} - 1)&
\end{aligned} \tag{17}
$$

With this partial result established, we turn back to the remaining terms found in (14) and, in particular, we focus on the integral

$$
\int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds
$$

We discern that the range of the above integral has a length[a] equal to the length of the interval where assumption (3b) (??) of theorem (1.1) holds. Integrating the expression found in the assumption over the interval $[t_k + T_p, t_{k+1} + T_p]$, for the controls and states applicable in it we get

$$\int_{t_k+T_p}^{t_{k+1}+T_p} \left(\frac{\partial V_i}{\partial \mathbf{e}_{1,i}} g_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) + F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) \right) ds \leq 0$$

$$\int_{t_k+T_p}^{t_{k+1}+T_p} \frac{d}{ds} V_i\big(\mathbf{e}_{1,i}(s)\big) ds + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds \leq 0$$

$$V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) - V_i\big(\mathbf{e}_{1,i}(t_k+T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds \leq 0$$

$$V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds \leq V_i\big(\mathbf{e}_{1,i}(t_k+T_p)\big)$$

The left-hand side expression is the same as the first two terms in the right-hand side of equality (14). We can introduce the third one by subtracting it from both sides:

$$\begin{aligned}
&V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds - V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) \\
&\leq V_i\big(\mathbf{e}_{1,i}(t_k+T_p)\big) - V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) \\
&\leq \Big\| V_i\big(\mathbf{e}_{1,i}(t_k+T_p)\big) - V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) \Big\|, \text{ for } x \leq |x|, \forall x \in \mathbb{R} \\
&\leq L_{V_i} \Big\| \overline{\mathbf{e}}_i\big(t_k+T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big) - \overline{\mathbf{e}}_i\big(t_k+T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \Big\| \text{ (for } V_i \text{ is Lipschitz continuous in } \mathcal{E}_i) \\
&\leq L_{V_i} \frac{\overline{\delta}_i}{L_{g_i}} (e^{L_{g_i} h} - 1) e^{L_{g_i}(T_p - h)} \text{ (from (13))}
\end{aligned}$$

[a] $(t_{k+1}+T_p) - (t_k+T_p) = t_{k+1} - t_k = h$

Hence, we discovered that

$$\begin{aligned}
&V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds - V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) \\
&\qquad \leq L_{V_i} \frac{\overline{\delta}_i}{L_{g_i}} (e^{L_{g_i} h} - 1) e^{L_{g_i}(T_p - h)}
\end{aligned} \tag{18}$$

Adding the milestone inequalities (17) and (18) yields

$$\begin{aligned}
&\int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds - \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds \\
&+ V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds - V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) \\
&\leq L_{F_i}\frac{\overline{\delta}_i}{L_{g_i}^2}(e^{L_{g_i}h}-1)(e^{L_{g_i}(T_p-h)}-1) + L_{V_i}\frac{\overline{\delta}_i}{L_{g_i}}(e^{L_{g_i}h}-1)e^{L_{g_i}(T_p-h)}
\end{aligned}$$

and therefore (14), by bringing the integral ranging from t_k to t_{k+1} to the left-hand side, becomes

$$\begin{aligned}
&\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) + \int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds \\
&\leq L_{F_i}\frac{\overline{\delta}_i}{L_{g_i}^2}(e^{L_{g_i}h}-1)(e^{L_{g_i}(T_p-h)}-1) + L_{V_i}\frac{\overline{\delta}_i}{L_{g_i}}(e^{L_{g_i}h}-1)e^{L_{g_i}(T_p-h)}
\end{aligned}$$

By rearranging terms, the cost difference becomes bounded by

$$\begin{aligned}
&\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) \\
&\qquad \leq -\int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds + \frac{\overline{\delta}_i}{L_{g_i}}\Big(e^{L_{g_i}h}-1\Big)\Big((L_{V_i}+\frac{L_{F_i}}{L_{g_i}})e^{L_{g_i}(T_p-h)} - \frac{L_{F_i}}{L_{g_i}}\Big) \\
&\qquad = \xi_i - \int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds
\end{aligned}$$

where

$$\xi_i = \frac{\overline{\delta}_i}{L_{g_i}}\Big(e^{L_{g_i}h}-1\Big)\Big((L_{V_i}+\frac{L_{F_i}}{L_{g_i}})e^{L_{g_i}(T_p-h)} - \frac{L_{F_i}}{L_{g_i}}\Big)$$

is the contribution of the bounded additive disturbance $\overline{\delta}_i(t)$ to the nominal cost difference (the case without disturbances).

F_i is a positive-definite function as a sum of a positive-definite $\|\mathbf{u}_i\|_{\boldsymbol{R}_i}^2$ and a positive semi-definite function $\|\mathbf{e}_i\|_{\boldsymbol{Q}_i}^2$. If we denote by $m_i = \lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i) \geq 0$ the minimum eigenvalue between those of matrices $\boldsymbol{R}_i$, $\boldsymbol{Q}_i$, this means that

$$F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) \geq m_i\|\mathbf{e}_{0,i}(s)\|^2$$

By integrating the above between our interval of interest $[t_k, t_{k+1}]$ we get

$$\int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) \geq \int_{t_k}^{t_{k+1}} m_i \|\mathbf{e}_{0,i}(s)\|^2 ds$$

or

$$-\int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) \leq -m_i \int_{t_k}^{t_{k+1}} \|\overline{\mathbf{e}}_i(s;\ \overline{\mathbf{u}}_i^{\star}, \mathbf{e}_i(t_k))\|^2 ds$$

This means that the cost difference is upper-bounded by

$$\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^{\star}\big(\mathbf{e}_i(t_k)\big) \leq \xi_i - m_i \int_{t_k}^{t_{k+1}} \|\overline{\mathbf{e}}_i(s;\ \overline{\mathbf{u}}_i^{\star}(\cdot), \mathbf{e}_i(t_k))\|^2 ds$$

and since the cost $\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big)$ is, in general, sub-optimal: $J_i^{\star}\big(\mathbf{e}_i(t_{k+1})\big) - \overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) \leq 0$:

$$J_i^{\star}\big(\mathbf{e}_i(t_{k+1})\big) - J_i^{\star}\big(\mathbf{e}_i(t_k)\big) \leq \xi_i - m_i \int_{t_k}^{t_{k+1}} \|\overline{\mathbf{e}}_i(s;\ \overline{\mathbf{u}}_i^{\star}(\cdot), \mathbf{e}_i(t_k))\|^2 ds$$

It is easy to verify that both right-hand side terms are class $\mathcal{K}_\infty$ functions, and therefore, according to definition (**??**) and theorem (**??**), the closed-loop system is input-to-state stable. Inevitably then, the closed-loop trajectories converge to the terminal set Ω_i.

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