# A Distributed Nonlinear Model Predictive Control Scheme for Cooperation of Multi-robot Systems Guaranteeing Collision Avoidance and Connectivity Maintenance

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## Part I

# Advocated Solutions

# 1 Disturbance-free Stabililization

Here we are interested in steering each agent $i \in \mathcal{V}$ into a *position* in 3D space, while conforming to the requirements of the problem; that is, all agents should avoid colliding with each other, all obstacles in the workspace, and the workspace boundary itself, while remaining in a non-singular configuration and sustaining the connectivity to their respective neighbours.

## 1.1 Formalizing the model

We begin by rewriting the system equations (**??**), (**??**) for a generic agent $i \in \mathcal{V}$ in state-space form:

$$\dot{\mathbf{x}}_i(t) = \boldsymbol{J}_i^{-1}(\mathbf{x}_i)\mathbf{v}_i(t)$$
$$\dot{\mathbf{v}}_i(t) = -\boldsymbol{M}_i^{-1}(\mathbf{x}_i)\boldsymbol{C}_i(\mathbf{x}_i, \dot{\mathbf{x}}_i)\mathbf{v}_i(t) - \boldsymbol{M}_i^{-1}(\mathbf{x}_i)\mathbf{g}_i(\mathbf{x}_i) + \boldsymbol{M}_i^{-1}(\mathbf{x}_i)\mathbf{u}_i(t)$$

where the inversion of $\boldsymbol{M}_i$ is possible due to it being positive-definite $\forall i \in \mathcal{V}$. Denoting by $\mathbf{z}_i(t)$

$$\mathbf{z}_i(t) = \begin{bmatrix} \mathbf{x}_i(t) \\ \mathbf{v}_i(t) \end{bmatrix}, \ \mathbf{z}_i(t) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^9 \times \mathbb{T}^3$$

and $\dot{\mathbf{x}}_i(t)$ and $\dot{\mathbf{v}}_i(t)$ by

$$\dot{\mathbf{x}}_i(t) = f_{i,x}(\mathbf{z}_i, \mathbf{u}_i)$$
$$\dot{\mathbf{v}}_i(t) = f_{i,v}(\mathbf{z}_i, \mathbf{u}_i)$$

we get the compact representation of the system's model

$$\dot{\mathbf{z}}_i(t) = \begin{bmatrix} f_{i,x}(\mathbf{z}_i, \mathbf{u}_i) \\ f_{i,v}(\mathbf{z}_i, \mathbf{u}_i) \end{bmatrix} = f_i\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big)$$

The state evolution of agent $i$ is modeled by a system of non-linear continuous-time differential equations of the form

$$\dot{\mathbf{z}}_i(t) = f_i\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big) \tag{3}$$
$$\mathbf{z}_i(0) = \mathbf{z}_{i,0}$$
$$\mathbf{z}_i(t) \subset \mathbb{R}^9 \times \mathbb{T}^3$$
$$\mathbf{u}_i(t) \subset \mathbb{R}^6$$

where state $\mathbf{z}_i$ is directly measurable. It should be noted that equation (3) does not consider model-plant mismatches or external disturbances.

We define the set $\mathcal{Z}_i \subset \mathbb{R}^9 \times \mathbb{T}^3$ as the set that captures all the state constraints of the system's dynamics posed by the problem (**??**), for $t \in \mathbb{R}_{\geq 0}$. Therefore $\mathcal{Z}_i$ is such that:

$$\begin{aligned}
\mathcal{Z}_i = \big\{\mathbf{z}_i(t) \in \mathbb{R}^9 \times \mathbb{T}^3 : \;& \|\mathbf{p}_i(t) - \mathbf{p}_j(t)\| > \underline{d}_{ij,a}, \forall j \in \mathcal{R}_i(t), \\
& \|\mathbf{p}_i(t) - \mathbf{p}_j(t)\| < d_i, \forall j \in \mathcal{N}_i, \\
& \|\mathbf{p}_i(t) - \mathbf{p}_\ell\| > \underline{d}_{i\ell,o}, \forall \ell \in \mathcal{L}, \\
& \|\mathbf{p}_W - \mathbf{p}_i(t)\| < \overline{d}_{i,W}, \\
& -\frac{\pi}{2} < \theta_i(t) < \frac{\pi}{2}, \\
& \forall t \in \mathbb{R}_{\geq 0}\big\}
\end{aligned}$$

## 1.2 The error model

A feasible desired configuration $\mathbf{z}_{i,des} \in \mathbb{R}^9 \times \mathbb{T}^3$ is associated to each agent $i \in \mathcal{V}$, with the aim of agent $i$ achieving it in steady-state: $\lim\limits_{t\to\infty} \|\mathbf{z}_i(t) - \mathbf{z}_{i,des}\| = 0$. The interior of the norm of this expression denotes the state error of agent $i$:

$$\mathbf{e}_i(t) = \mathbf{z}_i(t) - \mathbf{z}_{i,des}, \;\; \mathbf{e}_i(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^9 \times \mathbb{T}^3$$

The error dynamics are denoted by $g_i(\mathbf{e}_i, \mathbf{u}_i)$:

$$\dot{\mathbf{e}}_i(t) = \dot{\mathbf{z}}_i(t) - \dot{\mathbf{z}}_{i,des} = \dot{\mathbf{z}}_i(t) = f_i\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big) = g_i\big(\mathbf{e}_i(t), \mathbf{u}_i(t)\big) \tag{4}$$

with $\mathbf{e}_i(0) = \mathbf{z}_i(0) - \mathbf{z}_{i,des}$. In order to translate the constraints that are dictated for the state $\mathbf{z}_i(t)$ into constraints regarding the error state $\mathbf{e}_i(t)$, we define the set $\mathcal{E}_i \subset \mathbb{R}^9 \times \mathbb{T}^3$ as:

$$\mathcal{E}_i = \{\mathbf{e}_i(t) \in \mathbb{R}^9 \times \mathbb{T}^3 : \mathbf{e}_i(t) \in \mathcal{Z}_i \oplus (-\mathbf{z}_{i,des})\}$$

as the set that captures all constraints for the error dynamics (4) dictated by the problem (**??**).

If we design control laws $\mathbf{u}_i \in \mathcal{U}_i$, $\forall i \in \mathcal{V}$ such that the error signal $\mathbf{e}_i(t)$ with dynamics given in (4), constrained under $\mathbf{e}_i(t) \in \mathcal{E}_i$, satisfies $\lim\limits_{t\to\infty} \|\mathbf{e}_i(t)\| = 0$, while all system related signals remain bounded in their respective regions,$-$ if all of the above are achieved, then problem (**??**) has been solved.

In order to achieve this task, we employ a Nonlinear Receding Horizon scheme.

### 1.3 The optimization problem

Consider a sequence of sampling times $\{t_k\}_{k\geq 0}$, with a constant sampling time $h$, $0 < h < T_p$, where $T_p$ is the finite time-horizon, such that $t_{k+1} = t_k + h$. In sampling data NMPC, a finite-horizon open-loop optimal control problem (OCP) is solved at discrete sampling time instants $t_k$ based on the then-current state error measurement $\mathbf{e}_i(t_k)$. The solution is an optimal control signal $\overline{\mathbf{u}}_i^{\star}(t)$, computed over $t \in [t_k, t_k + T_p]$. This signal is applied to the open-loop system in between sampling times $t_k$ and $t_k + h$.

At a generic time $t_k$, agent $i$ solves the following optimization problem:

**Problem 1.1.**

Find

$$J_i^{\star}\big(\mathbf{e}_i(t_k)\big) \triangleq \min_{\overline{\mathbf{u}}_i(\cdot)} \; J_i\big(\mathbf{e}_i(t_k), \overline{\mathbf{u}}_i(\cdot)\big)$$

where

$$J_i\big(\mathbf{e}_i(t_k), \overline{\mathbf{u}}_i(\cdot)\big) \triangleq \int_{t_k}^{t_k+T_p} F_i\big(\overline{\mathbf{e}}_i(s), \overline{\mathbf{u}}_i(s)\big) ds + V_i\big(\overline{\mathbf{e}}_i(t_k+T_p)\big) \tag{5}$$

subject to:

$$\dot{\overline{\mathbf{e}}}_i(s) = g_i\big(\overline{\mathbf{e}}_i(s), \overline{\mathbf{u}}_i(s)\big), \; \overline{\mathbf{e}}_i(t_k) = \mathbf{e}_i(t_k) \tag{6}$$

$$\overline{\mathbf{u}}_i(s) \in \mathcal{U}_i, \; \overline{\mathbf{e}}_i(s) \in \mathcal{E}_i, \; s \in [t_k, t_k + T_p]$$

$$\overline{\mathbf{e}}_i(t_k + T_p) \in \mathcal{E}_{i,f} \subseteq \mathcal{E}_i$$

The notation $\bar{\cdot}$ is used to distinguish predicted states which are internal to the controller, as opposed to their actual values, because, even in the nominal case, the predicted values will not be equal to the actual closed-loop values. This means that $\overline{\mathbf{e}}_i(\cdot)$ is the solution to (6) driven by the control input $\overline{\mathbf{u}}_i(\cdot) : [t_k, t_k + T_p] \to \mathcal{U}_i$ with initial condition $\mathbf{e}_i(t_k)$.

The applied input signal is a portion of the optimal solution to an optimization problem where information on the states of the neighbouring agents of agent $i$ are taken into account only in the constraints considered in the optimization problem. These constraints pertain to the set of its neighbours $\mathcal{N}_i$ and, in total, to the set of all agents within its sensing range $\mathcal{R}_i$. Regarding these, we make the following assumption:

**Assumption 1.1.** (*Access to Predicted Information from an Inter-agent Perspective*)
Considering the context of Receding Horizon Control, when at time $t_k$ agent $i$ solves a finite

horizon optimization problem, he has access to[a]

1. measurements of the states[b]
   - $\mathbf{z}_j(t_k)$ of all agents $j \in \mathcal{R}_i(t_k)$ within its sensing range at time $t_k$
   - $\mathbf{z}_{j'}(t_k)$ of all of its neighbouring agents $j' \in \mathcal{N}_i$ at time $t_k$
2. the *predicted states*
   - $\overline{\mathbf{z}}_j(\tau)$ of all agents $j \in \mathcal{R}_i(t_k)$ within its sensing range
   - $\overline{\mathbf{z}}_{j'}(\tau)$ of all of its neighbouring agents $j' \in \mathcal{N}_i$

   across the entire horizon $\tau \in (t_k, t_k + T_p]$

[a] Although $\mathcal{N}_i \subseteq \mathcal{R}_i$, we make the distinction between the two because all agents $j \in \mathcal{R}_i$ need to avoid collision with agent $i$, but only agents $j' \in \mathcal{N}_i$ need to remain within the sensing range of agent $i$.

[b] as per assumption (**??**)

In other words, each time an agent solves its own individual optimization problem, he knows the error predictions that have been generated by the solution of the optimization problem of all agents within its range at that time, for the next $T_p$ timesteps. This assumption is crucial to satisfying the constraints regarding collision aversion and connectivity maintenance between neighbouring agents. We assume that the above pieces of information are (a) always available and accurate, and (b) exchanged without delay. We encapsulate these pieces of information in four stacked vectors:

$$
\begin{aligned}
\mathbf{z}_{\mathcal{R}_i}(t_k) &\triangleq col[\mathbf{z}_j(t_k)], \forall j \in \mathcal{R}_i(t_k) \\
\mathbf{z}_{\mathcal{N}_i}(t_k) &\triangleq col[\mathbf{z}_j(t_k)], \forall j \in \mathcal{N}_i \\
\overline{\mathbf{z}}_{\mathcal{R}_i}(\tau) &\triangleq col[\overline{\mathbf{z}}_j(\tau)], \forall j \in \mathcal{R}_i(t_k), \tau \in [t_k, t_k + T_p] \\
\overline{\mathbf{z}}_{\mathcal{N}_i}(\tau) &\triangleq col[\overline{\mathbf{z}}_j(\tau)], \forall j \in \mathcal{N}_i, \tau \in [t_k, t_k + T_p]
\end{aligned}
$$

**Remark 1.1.** The justification for this assumption is the following: considering that $\mathcal{N}_i \subseteq \mathcal{R}_i$, that the state vectors $\mathbf{z}_j$ are comprised of 12 real numbers that are encoded by 4 bytes, and that sampling occurs with a frequency $f$ for all agents, the overall downstream bandwidth required by each agent is

$$BW_d = 12 \times 32 \text{ [bits]} \times |\mathcal{R}_i| \times \frac{T_p}{h} \times f \text{ [sec}^{-1}\text{]}$$

Given conservative constants $f = 100$ Hz, $\dfrac{T_p}{h} = 100$, the wireless protocol IEEE 802.11n-2009 (a standard for present-day devices) can accommodate up to

$$|\mathcal{R}_i| = \frac{600 \text{ [Mbit} \cdot \text{sec}^{-1}]}{12 \times 32[\text{bit}] \times 10^4[\text{sec}^{-1}]} \approx 16 \cdot 10^2 \text{ agents}$$

within the range of one agent. We deem this number to be large enough for practical applications for the approach of assuming access to the predicted states of agents within the range of one agent to be legal.

?? more on the actual $\mathcal{E}_i$

The functions $F_i : \mathcal{E}_i \times \mathcal{U}_i \rightarrow \mathbb{R}_{\geq 0}$ and $V_i : \mathcal{E}_{i,f} \rightarrow \mathbb{R}_{\geq 0}$ are defined as

$$F_i\big(\overline{\mathbf{e}}_i(t), \overline{\mathbf{u}}_i(t)\big) \triangleq \overline{\mathbf{e}}_i(t)^\top \boldsymbol{Q}_i \overline{\mathbf{e}}_i(t) + \overline{\mathbf{u}}_i(t)^\top \boldsymbol{R}_i \overline{\mathbf{u}}_i(t)$$
$$V_i\big(\overline{\mathbf{e}}_i(t)\big) \triangleq \overline{\mathbf{e}}_i(t)^\top \boldsymbol{P}_i \overline{\mathbf{e}}_i(t)$$

Matrices $\boldsymbol{R}_i \in \mathbb{R}^{6\times 6}$ are symmetric and positive definite, while matrices $\boldsymbol{Q}_i, \boldsymbol{P}_i \in \mathbb{R}^{12\times 12}$ are symmetric and positive semi-definite. The running costs $F_i$ are upper- and lower-bounded:

$$\begin{aligned}\lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\|\mathbf{e}_i(t)\|^2 &\leq \lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\left\| \begin{bmatrix} \mathbf{e}_i(t) \\ \mathbf{u}_i(t) \end{bmatrix} \right\|^2 \\ &\leq F_i\big(\mathbf{e}_i(t), \mathbf{u}_i(t)\big) \\ &\leq \lambda_{max}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\left\| \begin{bmatrix} \mathbf{e}_i(t) \\ \mathbf{u}_i(t) \end{bmatrix} \right\|^2 \leq \lambda_{max}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\|\mathbf{e}_i(t)\|^2\end{aligned}$$

where $\lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i)$ is the smallest eigenvalue between those of matrices $\boldsymbol{Q}_i$ and $\boldsymbol{R}_i$, and $\lambda_{max}(\boldsymbol{Q}_i, \boldsymbol{R}_i)$ the largest. Since the terms $\lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\|\mathbf{e}_i(t)\|$ and $\lambda_{max}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\|\mathbf{e}_i(t)\|$ are themselves class $\mathcal{K}$ functions according to definition (**??**), $F_i$ is lower- and upper-bounded by class $\mathcal{K}$ functions. As is obvious, $F_i(\mathbf{0}, \mathbf{0}) = 0$.

Before defining the terminal set $\mathcal{E}_{i,f}$ it is necessary to state the definition of a positively invariant set:

**Definition 1.1.** (*Positively Invariant Set*)
Consider a dynamical system $\dot{\mathbf{x}} = f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, and a trajectory $\mathbf{x}(t;\ \mathbf{x}_0)$, where $\mathbf{x}_0$ is the initial condition. The set $S = \{\mathbf{x} \in \mathbb{R}^n : \gamma(\mathbf{x}) = 0\}$, where $\gamma$ is a valued function, is said to be *positively invariant* if the following holds:

$$\mathbf{x}_0 \in S \Rightarrow \mathbf{x}(t;\ \mathbf{x}_0) \in S,\ \forall t \geq t_0$$

Intuitively, this means that the set $S$ is positively invariant if a trajectory of the system does not exit it once it enters it.

The terminal set $\mathcal{E}_{i,f} \subseteq \mathcal{E}_i$ is an admissible positively invariant set for system (4) such that

$$\mathcal{E}_{i,f} = \{\mathbf{e}_i \in \mathcal{E}_i : \|\mathbf{e}_i\| \leq \varepsilon_0\}$$

where $\varepsilon_0$ is an arbitrarily small but fixed positive real scalar.

With regard to the terminal penalty function $V_i$, the following lemma will prove to be useful in guaranteeing the convergence of the solution to the optimal control problem to the terminal region $\mathcal{E}_{i,f}$:

**Lemma 1.1.** ($V_i$ is Lipschitz continuous in $\mathcal{E}_{i,f}$)

The terminal penalty function $V_i$ is Lipschitz continuous in $\mathcal{E}_{i,f}$

$$|V(\mathbf{e}_{1,i}) - V(\mathbf{e}_{2,i})| \leq L_{V_i} \|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\|$$

where $\mathbf{e}_{1,i}, \mathbf{e}_{2,i} \in \mathcal{E}_{i,f}$, with Lipschitz constant $L_{V_i} = 2\varepsilon_0 \lambda_{max}(P_i)$

**Proof** For every $\mathbf{e}_i \in \mathcal{E}_{i,f}$, it holds that

$$\begin{aligned}
|V(\mathbf{e}_{1,i}) - V(\mathbf{e}_{2,i})| &= |\mathbf{e}_{1,i}^\top \boldsymbol{P}_i \mathbf{e}_{1,i} - \mathbf{e}_{2,i}^\top \boldsymbol{P}_i \mathbf{e}_{2,i}| \\
&= |\mathbf{e}_{1,i}^\top \boldsymbol{P}_i \mathbf{e}_{1,i} - \mathbf{e}_{2,i}^\top \boldsymbol{P}_i \mathbf{e}_{2,i} \pm \mathbf{e}_{1,i}^\top \boldsymbol{P}_i \mathbf{e}_{2,i}| \\
&= |\mathbf{e}_{1,i}^\top \boldsymbol{P}_i (\mathbf{e}_{1,i} - \mathbf{e}_{2,i}) - \mathbf{e}_{2,i}^\top \boldsymbol{P}_i (\mathbf{e}_{1,i} - \mathbf{e}_{2,i})| \\
&\leq |\mathbf{e}_{1,i}^\top \boldsymbol{P}_i (\mathbf{e}_{1,i} - \mathbf{e}_{2,i})| + |\mathbf{e}_{2,i}^\top \boldsymbol{P}_i (\mathbf{e}_{1,i} - \mathbf{e}_{2,i})|
\end{aligned}$$

But for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$|\mathbf{x}^\top \boldsymbol{A} \mathbf{y}| \leq \lambda_{max}(A) \|\mathbf{x}\| \|\mathbf{y}\|$$

where $\lambda_{max}(A)$ denotes the largest eigenvalue of matrix $\boldsymbol{A}$. Hence:

$$\begin{aligned}
|V(\mathbf{e}_{1,i}) - V(\mathbf{e}_{2,i})| &\leq \lambda_{max}(\boldsymbol{P}_i) \|\mathbf{e}_{1,i}\| \|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\| + \lambda_{max}(\boldsymbol{P}_i) \|\mathbf{e}_{2,i}\| \|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\| \\
&= \lambda_{max}(\boldsymbol{P}_i)(\|\mathbf{e}_{1,i}\| + \|\mathbf{e}_{2,i}\|) \|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\| \\
&\leq \lambda_{max}(\boldsymbol{P}_i)(\varepsilon_0 + \varepsilon_0) \|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\| \\
&= 2\varepsilon_0 \lambda_{max}(\boldsymbol{P}_i) \|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\|
\end{aligned}$$

■

The solution to the optimal control problem (5) at time $t_k$ is an optimal control input, denoted by $\overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t_k))$, which is applied to the open-loop system until the next sampling instant $t_k + h$, with

$h \in (0, T_p)$, at which time a new optimal control problem is solved in the same manner:

$$\mathbf{u}_i(t) = \overline{\mathbf{u}}_i^{\star}\big(t;\ \mathbf{e}_i(t_k)\big),\ t \in [t_k, t_k + h] \tag{8}$$

The control input $\mathbf{u}_i(\cdot)$ is of feedback form, since it is recalculated at each sampling instant based on the then-current state. The solution to equation (4), starting at time $t_1$, from an initial condition $\mathbf{e}_i(t_1)$, by application of the control input $\mathbf{u}_i : [t_1, t_2] \to \mathcal{U}_i$ is denoted by

$$\mathbf{e}_i\big(t;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(t_1)\big),\ t \in [t_1, t_2]$$

The *predicted* state of the system (4) at time $t_k + \tau$, based on the measurement of the state at time $t_k$, $\mathbf{e}_i(t_k)$, by application of the control input $\mathbf{u}_i\big(t;\ \mathbf{e}_i(t_k)\big)$, for the time period $t \in [t_k, t_k + \tau]$ is denoted by

$$\overline{\mathbf{e}}_i\big(t_k + \tau;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(t_k)\big)$$

As is natural, the equality

$$\overline{\mathbf{e}}_i\big(\tau_1;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(\tau_0)\big) = \mathbf{e}_i\big(\tau_1;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(\tau_0)\big) \tag{9}$$

holds true here because there are no disturbances acting on the system.

The closed-loop system for which stability is to be guaranteed is

$$\mathbf{e}_i(\tau) = g_i\big(\mathbf{e}_i(\tau), \overline{\mathbf{u}}_i^{\star}(\tau)\big),\ \tau \geq t_0 = 0 \tag{10}$$

where $\overline{\mathbf{u}}_i^{\star}(\tau) = \overline{\mathbf{u}}_i^{\star}(\tau;\ \mathbf{e}_i(t_k))$, $\tau \in [t_k, t_k + h)$.

We can now give the definition of an *admissible input*:

**Definition 1.2.** (Admissible input)

A control input $\mathbf{u}_i : [t_k, t_k + T_p] \to \mathbb{R}^6$ for a state $\mathbf{e}_i(t_k)$ is called *admissible* if all the following hold:

1. $\mathbf{u}_i(\cdot)$ is piecewise continuous
2. $\mathbf{u}_i(\tau) \in \mathcal{U}_i,\ \forall \tau \in [t_k, t_k + T_p]$
3. $\mathbf{e}_i\big(\tau;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(t_k)\big) \in \mathcal{E}_i,\ \forall \tau \in [t_k, t_k + T_p]$
4. $\mathbf{e}_i\big(t_k + T_p;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(t_k)\big) \in \mathcal{E}_{i,f}$

## 1.4 Feasibility and Convergence

Under these considerations, we can now state the theorem that relates to the guaranteeing of the stability of the compound system of agents $i \in \mathcal{V}$, when each of them is assigned a desired position which results in feasible displacements:

**Theorem 1.1.** Suppose that

1. the terminal region $\mathcal{E}_{i,f} \subseteq \mathcal{E}_i$ is closed with $\mathbf{0} \in \mathcal{E}_{i,f}$
2. a solution to the optimal control problem (5) is feasible at time $t = 0$, that is, assumptions (**??**), (**??**), and (**??**) hold at time $t = 0$
3. there exists an admissible control input $\mathbf{u}_{i,f} : [0, h] \to \mathcal{U}_i$ such that for all $\mathbf{e}_i \in \mathcal{E}_{i,f}$ and $\forall \tau \in [0, h]$:

   (a) $\mathbf{e}_i(\tau) \in \mathcal{E}_{i,f}$

   (b) $\dfrac{\partial V_i}{\partial \mathbf{e}_i} g_i\big(\mathbf{e}_i(\tau), \mathbf{u}_{i,f}(\tau)\big) + F_i\big(\mathbf{e}_i(\tau), \mathbf{u}_{i,f}(\tau)\big) \leq 0$

then the closed loop system (10) under the control input (8) converges to the set $\mathcal{E}_{i,f}$ when $t \to \infty$.

**Proof**. The proof of the above theorem consists of two parts: in the first, recursive feasibility is established, that is, initial feasibility is shown to imply subsequent feasibility; in the second, and based on the first part, it is shown that the error state $\mathbf{e}_i(t)$ converges to the terminal set $\mathcal{E}_{i,f}$.

**Feasibility analysis** Consider a sampling instant $t_k$ for which a solution $\overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big)$ to (5) exists. Suppose now a time instant $t_{k+1}$ such that[1] $t_k < t_{k+1} < t_k + T_p$, and consider that the optimal control signal calculated at $t_k$ is comprised by the following two portions:

$$\overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big) = \begin{cases} \overline{\mathbf{u}}_i^\star\big(\tau_1;\ \mathbf{e}_i(t_k)\big), & \tau_1 \in [t_k, t_{k+1}] \\ \overline{\mathbf{u}}_i^\star\big(\tau_2;\ \mathbf{e}_i(t_k)\big), & \tau_2 \in [t_{k+1}, t_k + T_p] \end{cases} \tag{11}$$

Both portions are admissible since the calculated optimal control input is admissible, and hence they both conform to the input constraints. As for the resulting predicted states, they satisfy the state constraints, and, crucially: $\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \in \mathcal{E}_{i,f}$. Furthermore, according to assumption (3) of the theorem, there exists an admissible (and certainly not guaranteed optimal) input $\mathbf{u}_{i,f}$ that renders $\mathcal{E}_{i,f}$ invariant over $[t_k + T_p, t_k + T_p + h]$.

Given the above facts, we can construct an admissible input $\widetilde{\mathbf{u}}_i(\cdot)$ for time $t_{k+1}$ by sewing together the second portion of (24) and the input $\mathbf{u}_{i,f}(\cdot)$:

[1] It is not strictly necessary that $t_{k+1} = t_k + h$ here, however it is necessary for the following that $t_{k+1} - t_k \leq h$

$$\widetilde{\mathbf{u}}_i(\tau) = \begin{cases} \overline{\mathbf{u}}_i^{\star}\big(\tau;\ \mathbf{e}_i(t_k)\big), & \tau \in [t_{k+1}, t_k + T_p] \\ \mathbf{u}_{i,f}(\tau - t_k - T_p), & \tau \in (t_k + T_p, t_{k+1} + T_p] \end{cases} \tag{12}$$

Applied at time $t_{k+1}$, $\widetilde{\mathbf{u}}_i(\cdot)$ is an admissible control input as a composition of admissible control inputs.

This means that feasibility of a solution to the optimization problem at time $t_k$ implies feasibility at time $t_{k+1} > t_k$, and, thus, since at time $t = 0$ a solution is assumed to be feasible, a solution to the optimal control problem is feasible for all $t \geq 0$.

**Convergence analysis** The second part of the proof involves demonstrating the convergence of the state $\mathbf{e}_i$ to the terminal set $\mathcal{E}_{i,f}$. In order for this to be proved, it must be shown that a proper value function decreases along the solution trajectories starting at some initial time $t_k$. We consider the *optimal* cost $J_i^{\star}\big(\mathbf{e}_i(t)\big)$ as a candidate Lyapunov function:

$$J_i^{\star}\big(\mathbf{e}_i(t)\big) \triangleq J_i\Big(\mathbf{e}_i(t), \overline{\mathbf{u}}_i^{\star}\big(\cdot;\ \mathbf{e}_i(t)\big)\Big)$$

and, in particular, our goal is to show that that this cost decreases over consecutive sampling instants $t_{k+1} = t_k + h$, i.e. $J_i^{\star}\big(\mathbf{e}_i(t_{k+1})\big) - J_i^{\star}\big(\mathbf{e}_i(t_k)\big) \leq 0$.

In order not to wreak notational havoc, let us define the following terms:

- $\mathbf{u}_{0,i}(\tau) \triangleq \overline{\mathbf{u}}_i^{\star}\big(\tau;\ \mathbf{e}_i(t_k)\big)$ as the *optimal* input that results from the solution to problem (1.1) based on the measurement of state $\mathbf{e}_i(t_k)$, applied at time $\tau \geq t_k$

- $\mathbf{e}_{0,i}(\tau) \triangleq \overline{\mathbf{e}}_i\big(\tau;\ \overline{\mathbf{u}}_i^{\star}\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big)$ as the *predicted* state at time $\tau \geq t_k$, that is, the state that results from the application of the above input $\overline{\mathbf{u}}_i^{\star}\big(\cdot;\ \mathbf{e}_i(t_k)\big)$ to the state $\mathbf{e}_i(t_k)$, at time $\tau$

- $\mathbf{u}_{1,i}(\tau) \triangleq \widetilde{\mathbf{u}}_i(\tau)$ as the *admissible* input at $\tau \geq t_{k+1}$ (see eq. (25))

- $\mathbf{e}_{1,i}(\tau) \triangleq \overline{\mathbf{e}}_i\big(\tau;\ \widetilde{\mathbf{u}}_i(\cdot), \mathbf{e}_i(t_{k+1})\big)$ as the *predicted* state at time $\tau \geq t_{k+1}$, that is, the state that results from the application of the above input $\widetilde{\mathbf{u}}_i(\cdot)$ to the state $\mathbf{e}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^{\star}\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big)$, at time $\tau$

**Remark 1.2.** Given that no model mismatch or disturbances exist, for the predicted and actual states at time $\tau_1 \geq \tau_0 \in \mathbb{R}_{\geq 0}$ it holds that:

$$\mathbf{e}_i\big(\tau_1;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(\tau_0)\big) = \mathbf{e}_i(\tau_0) + \int_{\tau_0}^{\tau_1} g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(\tau_0)), \mathbf{u}_i(s)\big)ds$$
$$\overline{\mathbf{e}}_i\big(\tau_1;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(\tau_0)\big) = \mathbf{e}_i(\tau_0) + \int_{\tau_0}^{\tau_1} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(\tau_0)), \mathbf{u}_i(s)\big)ds$$

Before beginning to prove convergence, it is worth noting that while the cost

$$J_i\Big(\mathbf{e}_i(t), \overline{\mathbf{u}}_i^{\star}\big(\cdot;\ \mathbf{e}_i(t)\big)\Big)$$

is optimal (in the sense that it is based on the optimal input, which provides its minimum realization), a cost that is based on a plainly admissible (and thus, without loss of generality, sub-optimal) input $\mathbf{u}_i \neq \overline{\mathbf{u}}_i^{\star}$ will result in a configuration where

$$J_i\Big(\mathbf{e}_i(t), \mathbf{u}_i\big(\cdot;\ \mathbf{e}_i(t)\big)\Big) \geq J_i\Big(\mathbf{e}_i(t), \overline{\mathbf{u}}_i^{\star}\big(\cdot;\ \mathbf{e}_i(t)\big)\Big)$$

Let us now begin our investigation on the sign of the difference between the cost that results from the application of the feasible input $\mathbf{u}_{1,i}$, which we shall denote by $\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big)$, and the optimal cost $J_i^{\star}\big(\mathbf{e}_i(t_k)\big)$, while reminding ourselves that $J_i\big(\mathbf{e}_i(t), \overline{\mathbf{u}}_i(\cdot)\big) = \int_t^{t+T_p} F_i\big(\overline{\mathbf{e}}_i(s), \overline{\mathbf{u}}_i(s)\big)ds + V_i\big(\overline{\mathbf{e}}_i(t+T_p)\big)$:

$$\begin{aligned}\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^{\star}\big(\mathbf{e}_i(t_k)\big) = V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) + \int_{t_{k+1}}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds \\ -V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) - \int_{t_k}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds\end{aligned}$$

Considering that $t_k < t_{k+1} < t_k + T_p < t_{k+1} + T_p$, we break down the two integrals above in between these intervals:

$$\begin{aligned}&\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^{\star}\big(\mathbf{e}_i(t_k)\big) = \\ &\quad V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) + \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds \\ &\quad -V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) - \int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds - \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds \qquad (13)\end{aligned}$$

Since no model mismatch or disturbances are present, consulting remark (1.2) and substituting for $\tau_0 = t_k$ and $\tau_1 = t_{k+1}$ yields:

$$\mathbf{e}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) = \mathbf{e}_i(t_k) + \int_{t_k}^{t_{k+1}} g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big)ds$$
$$\overline{\mathbf{e}}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) = \mathbf{e}_i(t_k) + \int_{t_k}^{t_{k+1}} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big)ds$$

Subtracting the second expression from the first, we get

$$\begin{aligned}
&\mathbf{e}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) - \overline{\mathbf{e}}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) \\
&= \int_{t_k}^{t_{k+1}} g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big)ds - \int_{t_k}^{t_{k+1}} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big)ds \\
&= \int_{t_k}^{t_{k+1}} \Big(g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big) - g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big)\Big)ds
\end{aligned}$$

Taking norms on either side yields

$$\begin{aligned}
&\Big\|\mathbf{e}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) - \overline{\mathbf{e}}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big)\Big\| \\
&= \Big\|\int_{t_k}^{t_{k+1}} \Big(g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big) - g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big)\Big)ds\Big\| \\
&= \int_{t_k}^{t_{k+1}} \Big\|g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big) - g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big)\Big\|ds \\
&\leq L_{g_i} \int_{t_k}^{t_{k+1}} \Big\|\mathbf{e}_i\big(s;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) - \overline{\mathbf{e}}_i\big(s;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big)\Big\|ds
\end{aligned}$$

since $g_i$ is Lipschitz continuous in $\mathbf{e}_i$ with Lipschitz constant $L_{g_i}$. Reformulation yields

$$\begin{aligned}
&\Big\|\mathbf{e}_i\big(t_k + h;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) - \overline{\mathbf{e}}_i\big(t_k + h;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big)\Big\| \\
&\leq L_{g_i} \int_0^h \Big\|\mathbf{e}_i\big(t_k + s;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) - \overline{\mathbf{e}}_i\big(t_k + s;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big)\Big\|ds
\end{aligned}$$

By applying the Grönwall-Bellman inequality we obtain zero as an upper bound for the norm of the difference between the two states. Since any norm cannot be negative, we conclude that

$$\Big\|\mathbf{e}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) - \overline{\mathbf{e}}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big)\Big\| = 0$$

which means that

$$\mathbf{e}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big),\mathbf{e}_i(t_k)\big) = \overline{\mathbf{e}}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big),\mathbf{e}_i(t_k)\big)$$

In between times $t_{k+1}$ and $t_k + T_p$, the constructed admissible input $\widetilde{\mathbf{u}}_i(\cdot)$ is equal to the optimal input $\overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big)$ (see eq. 25), which means that $\mathbf{u}_{1,i}(\tau) = \mathbf{u}_{0,i}(\tau)$ in the interval $\tau \in [t_{k+1}, t_k + T_p]$. Since the initial conditions at $t = t_{k+1}$ are equal and the control laws are also equal, so will the predicted states over the same interval:

$$\overline{\mathbf{e}}_i\big(\tau;\ \widetilde{\mathbf{u}}_i(\cdot),\mathbf{e}_i(t_{k+1})\big) = \overline{\mathbf{e}}_i\big(\tau;\ \overline{\mathbf{u}}_i^\star(\cdot),\overline{\mathbf{e}}_i(t_{k+1})\big),\ \tau \in [t_{k+1}, t_k + T_p]$$

Using our notation then, in the same interval: $\mathbf{e}_{1,i}(\cdot) = \mathbf{e}_{0,i}(\cdot)$, and therefore the following equality holds over $[t_{k+1}, t_k + T_p]$:

$$F_i\big(\mathbf{e}_{1,i}(s),\mathbf{u}_{1,i}(s)\big) = F_i\big(\mathbf{e}_{0,i}(s),\mathbf{u}_{0,i}(s)\big),\ s \in [t_{k+1}, t_k + T_p]$$

Integrating this equality over the interval where it is valid yields

$$\int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s),\mathbf{u}_{1,i}(s)\big)ds = \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s),\mathbf{u}_{0,i}(s)\big)ds$$

This means that these two integrals with ends over the interval $[t_{k+1}, t_k+T_p]$ featured in the right-hand side of eq. (13) vanish, and thus the cost difference becomes

$$\begin{aligned}\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) = V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) &+ \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s),\mathbf{u}_{1,i}(s)\big)ds \\ - V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) &- \int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s),\mathbf{u}_{0,i}(s)\big)ds \end{aligned} \tag{14}$$

We turn our attention to the first integral in the above expression, and we note that $(t_{k+1}+T_p) - (t_k + T_p) = t_{k+1} - t_k = h$, which is exactly the length of the interval where assumption (3b) of the theorem holds. Hence, we decide to integrate the expression found in the assumption over the interval $[t_k + T_p, t_{k+1} + T_p]$, for the controls and states applicable in it:

$$\int_{t_k+T_p}^{t_{k+1}+T_p} \left( \frac{\partial V_i}{\partial \mathbf{e}_{1,i}} g_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) + F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) \right) ds \leq 0$$

$$\int_{t_k+T_p}^{t_{k+1}+T_p} \frac{d}{ds} V_i\big(\mathbf{e}_{1,i}(s)\big) ds + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds \leq 0$$

$$V_i\big(\mathbf{e}_{1,i}(t_{k+1} + T_p)\big) - V_i\big(\mathbf{e}_{1,i}(t_k + T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds \leq 0$$

$$V_i\big(\mathbf{e}_{1,i}(t_{k+1} + T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds \leq V_i\big(\mathbf{e}_{1,i}(t_k + T_p)\big)$$

The left-hand side expression is the same as the first two terms in the right-hand side of equality (28). We can introduce the third one by subtracting it from both sides:

$$\begin{aligned} &V_i\big(\mathbf{e}_{1,i}(t_{k+1} + T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds - V_i\big(\mathbf{e}_{0,i}(t_k + T_p)\big) \\ &\leq V_i\big(\mathbf{e}_{1,i}(t_k + T_p)\big) - V_i\big(\mathbf{e}_{0,i}(t_k + T_p)\big) \leq \Big| V_i\big(\mathbf{e}_{1,i}(t_k + T_p)\big) - V_i\big(\mathbf{e}_{0,i}(t_k + T_p)\big) \Big| \end{aligned}$$

since $x \leq |x|, \forall x \in \mathbb{R}$.

By revisiting lemma (1.1), the above inequality becomes

$$\begin{aligned} V_i\big(\mathbf{e}_{1,i}(t_{k+1} + T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds - V_i\big(\mathbf{e}_{0,i}(t_k + T_p)\big) \\ \leq L_{V_i} \big\| \mathbf{e}_{1,i}(t_k + T_p) - \mathbf{e}_{0,i}(t_k + T_p) \big\| \end{aligned}$$

However, in the interval $[t_{k+1}, t_k + T_p]$: $\mathbf{e}_{1,i}(\cdot) = \mathbf{e}_{0,i}(\cdot)$, hence the right-hand side of the inequality equals zero:

$$V_i\big(\mathbf{e}_{1,i}(t_{k+1} + T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds - V_i\big(\mathbf{e}_{0,i}(t_k + T_p)\big) \leq 0$$

By subtracting the fourth term needed to complete the right-hand side expression of (28), i.e.

$\int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds$ from both sides we get

$$V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds$$

$$-V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) - \int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds \le -\int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds$$

The left-hand side of this inequality is now equal to the cost difference $\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big)$.

Hence, the cost difference becomes bounded by

$$\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) \le -\int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds$$

$F_i$ is a positive-definite function as a sum of a positive-definite $\|\mathbf{u}_i\|_{\boldsymbol{R}_i}^2$ and a positive semi-definite function $\|\mathbf{e}_i\|_{\boldsymbol{Q}_i}^2$. If we denote by $m = \lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i) \ge 0$ the minimum eigenvalue between those of matrices $\boldsymbol{R}_i$, $\boldsymbol{Q}_i$, this means that

$$F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) \ge m\|\mathbf{e}_{0,i}(s)\|^2$$

By integrating the above between our interval of interest $[t_k, t_{k+1}]$ we get

$$\int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) \ge \int_{t_k}^{t_{k+1}} m\|\mathbf{e}_{0,i}(s)\|^2 ds$$

or

$$-\int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) \le -m\int_{t_k}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds$$

This means that the cost difference is upper-bounded by a class $\mathcal{K}$ function

$$\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) \le -m\int_{t_k}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds \le 0$$

and since the cost $\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big)$ is, in general, sub-optimal: $J_i^\star\big(\mathbf{e}_i(t_{k+1})\big) - \overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) \le 0$:

$$J_i^\star\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) \le -m\int_{t_k}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds \tag{15}$$

With this milestone result established, we need to trace the time $t_k$ back to $t_0 = 0$.

The integral of $\|\mathbf{e}_{0,i}(\tau)\|^2$ over the interval $[t_0, t_{k+1}]$, $t_0 < t_k < t_{k+1}$ can be decomposed into the addition of two integrals with limits ranging from (a) $t_0$ to $t_k$ and (b) $t_k$ to $t_{k+1}$:

$$\int_{t_0}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds = \int_{t_0}^{t_k} \|\mathbf{e}_{0,i}(s)\|^2 ds + \int_{t_k}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds$$

By rearranging terms, this means that

$$\int_{t_k}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds = \int_{t_0}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds - \int_{t_0}^{t_k} \|\mathbf{e}_{0,i}(s)\|^2 ds$$

making the optimal cost difference between the consecutive sampling times $t_k$ and $t_{k+1}$ in (29)

$$J_i^\star\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) \leq -m \int_{t_0}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds + m \int_{t_0}^{t_k} \|\mathbf{e}_{0,i}(s)\|^2 ds$$

Similarly, the optimal cost difference between the sampling times $t_{k-1}$ and $t_k$ is

$$J_i^\star\big(\mathbf{e}_i(t_k)\big) - J_i^\star\big(\mathbf{e}_i(t_{k-1})\big) \leq -m \int_{t_0}^{t_k} \|\mathbf{e}_{0,i}(s)\|^2 ds + m \int_{t_0}^{t_{k-1}} \|\mathbf{e}_{0,i}(s)\|^2 ds$$

and we can apply this rationale all the way back to the cost difference between $t_0$ and $t_1$. Summing all the inequalities between the pairs of consecutive sampling times $(t_0, t_1)$, $(t_1, t_2)$, ..., $(t_{k-1}, t_k)$, we get

$$J_i^\star\big(\mathbf{e}_i(t_k)\big) - J_i^\star\big(\mathbf{e}_i(t_0)\big) \leq -m \int_{t_0}^{t_k} \|\mathbf{e}_{0,i}(s)\|^2 ds$$

Hence, for $t_0 = 0$

$$J_i^\star\big(\mathbf{e}_i(t_k)\big) - J_i^\star\big(\mathbf{e}_i(0)\big) \leq -m \int_0^{t_k} \|\mathbf{e}_{0,i}(s)\|^2 ds \leq 0 \tag{16}$$

which implies that the value function $J_i^\star\big(\mathbf{e}_i(t_k)\big)$ in non-increasing for all sampling times:

$$J_i^\star\big(\mathbf{e}_i(t_k)\big) \leq J_i^\star\big(\mathbf{e}_i(0)\big),\ \forall t_k \in \mathbb{R}_{\geq 0}$$

Let us now define the function $V_i(\mathbf{e}_i(t))$:

$$V_i\big(\mathbf{e}(t)\big) \triangleq J_i^\star\big(\mathbf{e}_i(\tau)\big) \leq J_i^\star\big(\mathbf{e}_i(0)\big),\ t \in \mathbb{R}_{\geq 0}$$

where $\tau = max\{t_k : t_k \leq t\}$. Since $J_i^\star\big(\mathbf{e}_i(0)\big)$ is bounded, this implies that $V_i\big(\mathbf{e}(t)\big)$ is also bounded. The signals $\mathbf{e}_i(t) \in \mathcal{E}_i$ and $\mathbf{u}_i(t) \in \mathcal{U}_i$ are also bounded. According to (4), this means that $\dot{\mathbf{e}}_i(t)$ is bounded as well. From inequality (30) we then have

$$V_i\big(\mathbf{e}_i(t)\big) = J_i^\star\big(\mathbf{e}_i(\tau)\big) \leq J_i^\star\big(\mathbf{e}_i(0)\big) - m\int_0^\tau \|\mathbf{e}_{0,i}(s)\|^2 ds \leq 0$$

which, due to the fact that $\tau \leq t$, is equivalent to

$$V_i\big(\mathbf{e}_i(t)\big) \leq J_i^\star\big(\mathbf{e}_i(0)\big) - m\int_0^t \|\mathbf{e}_{0,i}(s)\|^2 ds \leq 0,\ t \in \mathbb{R}_{t\geq 0}$$

Solving for the integral we get

$$\int_0^t \|\mathbf{e}_{0,i}(s)\|^2 ds \leq \frac{1}{m}\Big(J_i^\star\big(\mathbf{e}_i(0)\big) - V_i\big(\mathbf{e}_i(t)\big)\Big),\ t \in \mathbb{R}_{t\geq 0}$$

Both $J_i^\star\big(\mathbf{e}_i(0)\big)$ and $V_i\big(\mathbf{e}_i(t)\big)$ are bounded, and therefore so is their difference, which means that the integral $\int\limits_0^t \|\mathbf{e}_{0,i}(s)\|^2 ds$ is bounded as well. We make use of the following lemma to show that the error internal to the norm of the integral goes to zero in steady-state:

**Lemma 1.2.** (*A modification of Barbalat's lemma*[3])

Let $f$ be a continuous, positive-definite function, and $\mathbf{x}$ be an absolutely continuous function in $\mathbb{R}$. If the following hold:

- $\|\mathbf{x}(\cdot)\| < \infty, \|\dot{\mathbf{x}}(\cdot)\| < \infty$
- $\lim\limits_{t\to\infty} \int\limits_0^t f\big(\mathbf{x}(s)\big) < \infty$

then $\lim\limits_{t\to\infty} \|\mathbf{x}(t)\| = 0$

Lemma (2.5) assures us that under these conditions for the error and its dynamics, which are fulfilled in our case, the error

$$\begin{aligned}
&\lim_{t\to\infty} \|\mathbf{e}_{0,i}(t)\| = 0 \Leftrightarrow \\
&\lim_{t\to\infty} \Big\|\overline{\mathbf{e}}_i\Big(t;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\Big)\Big\| = 0,\ \forall t_k \in \mathbb{R}_{\geq 0}
\end{aligned}$$

which, given (9) and substituting for $\tau_1 = t$ while dropping the initial condition at $\tau_0 = t_k$, means that

$$\lim_{t\to\infty} \|\mathbf{e}_i(t)\| = 0$$

which implies that

$$\lim_{t\to\infty} \mathbf{e}_i(t) \in \mathcal{E}_{i,f}$$

Therefore, the closed-loop trajectory of the error state $\mathbf{e}_i$ converges to the terminal set $\mathcal{E}_{i,f}$ as $t \to \infty$.

■

# 2 Stabilization in the face of Disturbances

**Remark 2.1.** For the sake of modularity, separability, and direct comparison to the disturbance-free case, this chapter will assume that the previous disturbance-free analysis has not been given.

In this chapter we are interested in steering each agent $i \in \mathcal{V}$ into a *position* in 3D space, while conforming to the requirements of the problem; that is, all agents should avoid colliding with each other, all obstacles in the workspace, and the workspace boundary itself, while remaining in a non-singular configuration and sustaining the connectivity to their respective neighbours.

## 2.1 Formalizing the model

We begin by rewriting the system equations (**??**), (**??**) for a generic agent $i \in \mathcal{V}$ in state-space form:

$$\begin{aligned}
\dot{\mathbf{x}}_i(t) &= \boldsymbol{J}_i^{-1}(\mathbf{x}_i)\mathbf{v}_i(t) \\
\dot{\mathbf{v}}_i(t) &= -\boldsymbol{M}_i^{-1}(\mathbf{x}_i)\boldsymbol{C}_i(\mathbf{x}_i, \dot{\mathbf{x}}_i)\mathbf{v}_i(t) - \boldsymbol{M}_i^{-1}(\mathbf{x}_i)\mathbf{g}_i(\mathbf{x}_i) + \boldsymbol{M}_i^{-1}(\mathbf{x}_i)\mathbf{u}_i(t)
\end{aligned}$$

where the inversion of $\boldsymbol{M}_i$ is possible due to it being positive-definite $\forall i \in \mathcal{V}$. Denoting by $\mathbf{z}_i(t)$

$$\mathbf{z}_i(t) = \begin{bmatrix} \mathbf{x}_i(t) \\ \mathbf{v}_i(t) \end{bmatrix}, \ \mathbf{z}_i(t) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^9 \times \mathbb{T}^3$$

and $\dot{\mathbf{x}}_i(t)$ and $\dot{\mathbf{v}}_i(t)$ by

$$\begin{aligned}
\dot{\mathbf{x}}_i(t) &= f_{i,x}(\mathbf{z}_i, \mathbf{u}_i) \\
\dot{\mathbf{v}}_i(t) &= f_{i,v}(\mathbf{z}_i, \mathbf{u}_i)
\end{aligned}$$

we get the compact representation of the system's model

$$\dot{\mathbf{z}}_i(t) = \begin{bmatrix} f_{i,x}(\mathbf{z}_i, \mathbf{u}_i) \\ f_{i,v}(\mathbf{z}_i, \mathbf{u}_i) \end{bmatrix} = f_i\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big)$$

The state evolution of agent $i$ is modeled by a system of non-linear continuous-time differential equations of the form

$$\begin{aligned}
\dot{\mathbf{z}}_i(t) &= f_i\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big) \\
\mathbf{z}_i(0) &= \mathbf{z}_{i,0} \\
\mathbf{z}_i(t) &\subset \mathbb{R}^9 \times \mathbb{T}^3 \\
\mathbf{u}_i(t) &\subset \mathbb{R}^6
\end{aligned} \tag{19}$$

where state $\mathbf{z}_i$ is directly measurable. It should be noted that equation (19) does not consider model-plant mismatches or external disturbances. We assume that the real system is subject to bounded additive disturbances $\boldsymbol{\delta}_i$ such that $\boldsymbol{\delta}_i \in \Delta_i \subset \mathbb{R}^9 \times \mathbb{T}^3$, where $\Delta_i$ is a compact set containing the origin. The real system is described by:

$$\begin{aligned}
\dot{\mathbf{z}}_i(t) &= f_i^R\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big) \\
&= f_i\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big) + \boldsymbol{\delta}_i(t) \\
\mathbf{z}_i(0) &= \mathbf{z}_{i,0} \\
\mathbf{z}_i(t) &\subset \mathbb{R}^9 \times \mathbb{T}^3 \\
\mathbf{u}_i(t) &\subset \mathbb{R}^6 \\
\boldsymbol{\delta}_i(t) &\in \Delta_i \subset \mathbb{R}^9 \times \mathbb{T}^3, \; t \in \mathbb{R}_{\geq 0} \\
\sup_{t \in \mathbb{R}_{\geq 0}} \|\boldsymbol{\delta}_i(t)\| &\leq \overline{\delta}_i
\end{aligned}$$

We define the set $\mathcal{Z}_i \subset \mathbb{R}^9 \times \mathbb{T}^3$ as the set that captures all the state constraints of the system's dynamics posed by the problem (**??**), for $t \in \mathbb{R}_{\geq 0}$. Therefore $\mathcal{Z}_i$ is such that:

$$\begin{aligned}
\mathcal{Z}_i = \big\{\mathbf{z}_i(t) \in \mathbb{R}^9 \times \mathbb{T}^3 : \; & \|\mathbf{p}_i(t) - \mathbf{p}_j(t)\| > \underline{d}_{ij,a}, \forall j \in \mathcal{R}_i(t), \\
& \|\mathbf{p}_i(t) - \mathbf{p}_j(t)\| < d_i, \forall j \in \mathcal{N}_i, \\
& \|\mathbf{p}_i(t) - \mathbf{p}_\ell\| > \underline{d}_{i\ell,o}, \forall \ell \in \mathcal{L}, \\
& \|\mathbf{p}_W - \mathbf{p}_i(t)\| < \overline{d}_{i,W}, \\
& -\frac{\pi}{2} < \theta_i(t) < \frac{\pi}{2}, \\
& \forall t \in \mathbb{R}_{\geq 0}\big\}
\end{aligned}$$

## 2.2 The error model

A feasible desired configuration $\mathbf{z}_{i,des} \in \mathbb{R}^9 \times \mathbb{T}^3$ is associated to each agent $i \in \mathcal{V}$, with the aim of agent $i$ achieving it in steady-state: $\lim\limits_{t\to\infty} \|\mathbf{z}_i(t) - \mathbf{z}_{i,des}\| = 0$. The interior of the norm of this expression denotes the state error of agent $i$:

$$\mathbf{e}_i(t) = \mathbf{z}_i(t) - \mathbf{z}_{i,des}, \ \mathbf{e}_i(t) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^9 \times \mathbb{T}^3$$

The error dynamics are denoted by $g_i^R(\mathbf{e}_i, \mathbf{u}_i)$:

$$\begin{aligned}\dot{\mathbf{e}}_i(t) &= \dot{\mathbf{z}}_i(t) - \dot{\mathbf{z}}_{i,des} = \dot{\mathbf{z}}_i(t) = f_i^R\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big) = f_i\big(\mathbf{z}_i(t), \mathbf{u}_i(t)\big) + \boldsymbol{\delta}_i(t) \\ &= g_i\big(\mathbf{e}_i(t), \mathbf{u}_i(t)\big) + \boldsymbol{\delta}_i(t) \\ &= g_i^R\big(\mathbf{e}_i(t), \mathbf{u}_i(t)\big) \end{aligned} \tag{20}$$

with $\mathbf{e}_i(0) = \mathbf{z}_i(0) - \mathbf{z}_{i,des}$. In order to translate the constraints that are dictated for the state $\mathbf{z}_i(t)$ into constraints regarding the error state $\mathbf{e}_i(t)$, we define the set $\mathcal{E}_i \subset \mathbb{R}^9 \times \mathbb{T}^3$ as:

$$\mathcal{E}_i = \{\mathbf{e}_i(t) \in \mathbb{R}^9 \times \mathbb{T}^3 : \mathbf{e}_i(t) \in \mathcal{Z}_i \oplus (-\mathbf{z}_{i,des})\}$$

as the set that captures all constraints for the error dynamics (20) dictated by the problem (**??**).

If we design control laws $\mathbf{u}_i \in \mathcal{U}_i$, $\forall i \in \mathcal{V}$ such that the error signal $\mathbf{e}_i(t)$ with dynamics given in (20), constrained under $\mathbf{e}_i(t) \in \mathcal{E}_i$, satisfies $\lim\limits_{t\to\infty} \|\mathbf{e}_i(t)\| = 0$, while all system related signals remain bounded in their respective regions,− if all of the above are achieved, then problem (**??**) has been solved.

In order to achieve this task, we employ a Nonlinear Receding Horizon scheme.

## 2.3 The optimization problem

Consider a sequence of sampling times $\{t_k\}_{k\geq 0}$, with a constant sampling time $h$, $0 < h < T_p$, where $T_p$ is the finite time-horizon, such that $t_{k+1} = t_k + h$. In sampling data NMPC, a finite-horizon open-loop optimal control problem (OCP) is solved at discrete sampling time instants $t_k$ based on the then-current state error measurement $\mathbf{e}_i(t_k)$. The solution is an optimal control signal $\overline{\mathbf{u}}_i^\star(t)$, computed over $t \in [t_k, t_k + T_p]$. This signal is applied to the open-loop system in between sampling times $t_k$ and $t_k + h$.

At a generic time $t_k$, agent $i$ solves the following optimization problem:

**Problem 2.1.**

Find

$$J_i^\star\big(\mathbf{e}_i(t_k)\big) \triangleq \min_{\overline{\mathbf{u}}_i(\cdot)} \;\; J_i\big(\mathbf{e}_i(t_k), \overline{\mathbf{u}}_i(\cdot)\big)$$

where

$$J_i\big(\mathbf{e}_i(t_k), \overline{\mathbf{u}}_i(\cdot)\big) \triangleq \int_{t_k}^{t_k+T_p} F_i\big(\overline{\mathbf{e}}_i(s), \overline{\mathbf{u}}_i(s)\big)ds + V_i\big(\overline{\mathbf{e}}_i(t_k+T_p)\big) \tag{21}$$

subject to:

$$\dot{\overline{\mathbf{e}}}_i(s) = g_i\big(\overline{\mathbf{e}}_i(s), \overline{\mathbf{u}}_i(s)\big),\ \overline{\mathbf{e}}_i(t_k) = \mathbf{e}_i(t_k) \tag{22}$$

$$\overline{\mathbf{u}}_i(s) \in \mathcal{U}_i,\ \overline{\mathbf{e}}_i(s) \in \mathcal{E}_i,\ s \in [t_k, t_k+T_p]$$

$$\overline{\mathbf{e}}_i(t_k+T_p) \in \mathcal{E}_{i,f} \subseteq \mathcal{E}_i$$

The notation $\bar{\cdot}$ is used to distinguish predicted states which are internal to the controller, as opposed to their actual values, because, even in the nominal case, the predicted values will not be equal to the actual closed-loop values. This means that $\overline{\mathbf{e}}_i(\cdot)$ is the solution to (22) driven by the control input $\overline{\mathbf{u}}_i(\cdot) : [t_k, t_k+T_p] \to \mathcal{U}_i$ with initial condition $\mathbf{e}_i(t_k)$.

The applied input signal is a portion of the optimal solution to an optimization problem where information on the states of the neighbouring agents of agent $i$ are taken into account only in the constraints considered in the optimization problem. These constraints pertain to the set of its neighbours $\mathcal{N}_i$ and, in total, to the set of all agents within its sensing range $\mathcal{R}_i$. Regarding these, we make the following assumption:

**Assumption 2.1.** (*Access to Predicted Information from an Inter-agent Perspective*)
Considering the context of Receding Horizon Control, when at time $t_k$ agent $i$ solves a finite horizon optimization problem, he has access to[a]

1. measurements of the states[b]
   - $\mathbf{z}_j(t_k)$ of all agents $j \in \mathcal{R}_i(t_k)$ within its sensing range at time $t_k$
   - $\mathbf{z}_{j'}(t_k)$ of all of its neighbouring agents $j' \in \mathcal{N}_i$ at time $t_k$
2. the *predicted states*
   - $\overline{\mathbf{z}}_j(\tau)$ of all agents $j \in \mathcal{R}_i(t_k)$ within its sensing range
   - $\overline{\mathbf{z}}_{j'}(\tau)$ of all of its neighbouring agents $j' \in \mathcal{N}_i$

across the entire horizon $\tau \in (t_k, t_k + T_p]$

[a] Although $\mathcal{N}_i \subseteq \mathcal{R}_i$, we make the distinction between the two because all agents $j \in \mathcal{R}_i$ need to avoid collision with agent $i$, but only agents $j' \in \mathcal{N}_i$ need to remain within the sensing range of agent $i$.

[b] as per assumption (**??**)

In other words, each time an agent solves its own individual optimization problem, he knows the error predictions that have been generated by the solution of the optimization problem of all agents within its range at that time, for the next $T_p$ timesteps. This assumption is crucial to satisfying the constraints regarding collision aversion and connectivity maintenance between neighbouring agents. We assume that the above pieces of information are (a) always available and accurate, and (b) exchanged without delay. We encapsulate these pieces of information in four stacked vectors:

$$\mathbf{z}_{\mathcal{R}_i}(t_k) \triangleq col[\mathbf{z}_j(t_k)], \forall j \in \mathcal{R}_i(t_k)$$
$$\mathbf{z}_{\mathcal{N}_i}(t_k) \triangleq col[\mathbf{z}_j(t_k)], \forall j \in \mathcal{N}_i$$
$$\overline{\mathbf{z}}_{\mathcal{R}_i}(\tau) \triangleq col[\overline{\mathbf{z}}_j(\tau)], \forall j \in \mathcal{R}_i(t_k), \tau \in [t_k, t_k + T_p]$$
$$\overline{\mathbf{z}}_{\mathcal{N}_i}(\tau) \triangleq col[\overline{\mathbf{z}}_j(\tau)], \forall j \in \mathcal{N}_i, \tau \in [t_k, t_k + T_p]$$

**Remark 2.2.** The justification for this assumption is the following: considering that $\mathcal{N}_i \subseteq \mathcal{R}_i$, that the state vectors $\mathbf{z}_j$ are comprised of 12 real numbers that are encoded by 4 bytes, and that sampling occurs with a frequency $f$ for all agents, the overall downstream bandwidth required by each agent is

$$BW_d = 12 \times 32 \text{ [bits]} \times |\mathcal{R}_i| \times \frac{T_p}{h} \times f \text{ [sec}^{-1}]$$

Given conservative constants $f = 100$ Hz, $\frac{T_p}{h} = 100$, the wireless protocol IEEE 802.11n-2009 (a standard for present-day devices) can accommodate up to

$$|\mathcal{R}_i| = \frac{600 \text{ [Mbit} \cdot \text{sec}^{-1}]}{12 \times 32 \text{[bit]} \times 10^4 \text{[sec}^{-1}]} \approx 16 \cdot 10^2 \text{ agents}$$

within the range of one agent. We deem this number to be large enough for practical applications for the approach of assuming access to the predicted states of agents within the range of one agent to be legal.

?? more on the actual $\mathcal{E}_i$

The functions $F_i : \mathcal{E}_i \times \mathcal{U}_i \rightarrow \mathbb{R}_{\geq 0}$ and $V_i : \mathcal{E}_{i,f} \rightarrow \mathbb{R}_{\geq 0}$ are defined as

$$F_i\big(\overline{\mathbf{e}}_i(t), \overline{\mathbf{u}}_i(t)\big) \triangleq \overline{\mathbf{e}}_i(t)^\top \boldsymbol{Q}_i \overline{\mathbf{e}}_i(t) + \overline{\mathbf{u}}_i(t)^\top \boldsymbol{R}_i \overline{\mathbf{u}}_i(t)$$
$$V_i\big(\overline{\mathbf{e}}_i(t)\big) \triangleq \overline{\mathbf{e}}_i(t)^\top \boldsymbol{P}_i \overline{\mathbf{e}}_i(t)$$

Matrices $\boldsymbol{R}_i \in \mathbb{R}^{6\times 6}$ are symmetric and positive definite, while matrices $\boldsymbol{Q}_i, \boldsymbol{P}_i \in \mathbb{R}^{12\times 12}$ are symmetric and positive semi-definite. The running costs $F_i$ are upper- and lower-bounded:

$$\begin{aligned} \lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\|\mathbf{e}_i(t)\|^2 &\leq \lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\left\| \begin{bmatrix} \mathbf{e}_i(t) \\ \mathbf{u}_i(t) \end{bmatrix} \right\|^2 \\ &\leq F_i\big(\mathbf{e}_i(t), \mathbf{u}_i(t)\big) \\ &\leq \lambda_{max}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\left\| \begin{bmatrix} \mathbf{e}_i(t) \\ \mathbf{u}_i(t) \end{bmatrix} \right\|^2 \leq \lambda_{max}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\|\mathbf{e}_i(t)\|^2 \end{aligned}$$

where $\lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i)$ is the smallest eigenvalue between those of matrices $\boldsymbol{Q}_i$ and $\boldsymbol{R}_i$, and $\lambda_{max}(\boldsymbol{Q}_i, \boldsymbol{R}_i)$ the largest. Since the terms $\lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\|\mathbf{e}_i(t)\|$ and $\lambda_{max}(\boldsymbol{Q}_i, \boldsymbol{R}_i)\|\mathbf{e}_i(t)\|$ are themselves class $\mathcal{K}$ functions according to definition (**??**), $F_i$ is lower- and upper-bounded by class $\mathcal{K}$ functions. As is obvious, $F_i(\mathbf{0}, \mathbf{0}) = 0$.

Before defining the terminal set $\mathcal{E}_{i,f}$ it is necessary to state the definition of a positively invariant set:

**Definition 2.1.** (*Positively Invariant Set*)
Consider a dynamical system $\dot{\mathbf{x}} = f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, and a trajectory $\mathbf{x}(t;\ \mathbf{x}_0)$, where $\mathbf{x}_0$ is the initial condition. The set $S = \{\mathbf{x} \in \mathbb{R}^n : \gamma(\mathbf{x}) = 0\}$, where $\gamma$ is a valued function, is said to be *positively invariant* if the following holds:

$$\mathbf{x}_0 \in S \Rightarrow \mathbf{x}(t;\ \mathbf{x}_0) \in S,\ \ \forall t \geq t_0$$

Intuitively, this means that the set $S$ is positively invariant if a trajectory of the system does not exit it once it enters it.

The terminal set $\mathcal{E}_{i,f} \subseteq \mathcal{E}_i$ is an admissible positively invariant set for system (**??**) such that

$$\mathcal{E}_{i,f} = \{\mathbf{e}_i \in \mathcal{E}_i : \|\mathbf{e}_i\| \leq \varepsilon_0\}$$

where $\varepsilon_0$ is an arbitrarily small but fixed positive real scalar.

With regard to the terminal penalty function $V_i$, the following lemma will prove to be useful in guaranteeing the convergence of the solution to the optimal control problem to the terminal region $\mathcal{E}_{i,f}$:

**Lemma 2.1.** ($V_i$ is Lipschitz continuous in $\mathcal{E}_{i,f}$)

The terminal penalty function $V_i$ is Lipschitz continuous in $\mathcal{E}_{i,f}$

$$|V(\mathbf{e}_{1,i}) - V(\mathbf{e}_{2,i})| \leq L_{V_i}\|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\|$$

where $\mathbf{e}_{1,i}, \mathbf{e}_{2,i} \in \mathcal{E}_{i,f}$, with Lipschitz constant $L_{V_i} = 2\varepsilon_0\lambda_{max}(P_i)$

**Proof** For every $\mathbf{e}_i \in \mathcal{E}_{i,f}$, it holds that

$$\begin{aligned}
|V(\mathbf{e}_{1,i}) - V(\mathbf{e}_{2,i})| &= |\mathbf{e}_{1,i}^\top \boldsymbol{P}_i \mathbf{e}_{1,i} - \mathbf{e}_{2,i}^\top \boldsymbol{P}_i \mathbf{e}_{2,i}| \\
&= |\mathbf{e}_{1,i}^\top \boldsymbol{P}_i \mathbf{e}_{1,i} - \mathbf{e}_{2,i}^\top \boldsymbol{P}_i \mathbf{e}_{2,i} \pm \mathbf{e}_{1,i}^\top \boldsymbol{P}_i \mathbf{e}_{2,i}| \\
&= |\mathbf{e}_{1,i}^\top \boldsymbol{P}_i (\mathbf{e}_{1,i} - \mathbf{e}_{2,i}) - \mathbf{e}_{2,i}^\top \boldsymbol{P}_i (\mathbf{e}_{1,i} - \mathbf{e}_{2,i})| \\
&\leq |\mathbf{e}_{1,i}^\top \boldsymbol{P}_i (\mathbf{e}_{1,i} - \mathbf{e}_{2,i})| + |\mathbf{e}_{2,i}^\top \boldsymbol{P}_i (\mathbf{e}_{1,i} - \mathbf{e}_{2,i})|
\end{aligned}$$

But for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$|\mathbf{x}^\top \boldsymbol{A}\mathbf{y}| \leq \lambda_{max}(A)\|\mathbf{x}\|\|\mathbf{y}\|$$

where $\lambda_{max}(A)$ denotes the largest eigenvalue of matrix $\boldsymbol{A}$. Hence:

$$\begin{aligned}
|V(\mathbf{e}_{1,i}) - V(\mathbf{e}_{2,i})| &\leq \lambda_{max}(\boldsymbol{P}_i)\|\mathbf{e}_{1,i}\|\|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\| + \lambda_{max}(\boldsymbol{P}_i)\|\mathbf{e}_{2,i}\|\|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\| \\
&= \lambda_{max}(\boldsymbol{P}_i)(\|\mathbf{e}_{1,i}\| + \|\mathbf{e}_{2,i}\|)\|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\| \\
&\leq \lambda_{max}(\boldsymbol{P}_i)(\varepsilon_0 + \varepsilon_0)\|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\| \\
&= 2\varepsilon_0\lambda_{max}(\boldsymbol{P}_i)\|\mathbf{e}_{1,i} - \mathbf{e}_{2,i}\|
\end{aligned}$$

■

The solution to the optimal control problem (21) at time $t_k$ is an optimal control input, denoted by $\overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t_k))$, which is applied to the open-loop system until the next sampling instant $t_k + h$, with $h \in (0, T_p)$, at which time a new optimal control problem is solved in the same manner:

$$\mathbf{u}_i(t) = \overline{\mathbf{u}}_i^\star\big(t;\ \mathbf{e}_i(t_k)\big),\ t \in [t_k, t_k + h]$$

The control input $\mathbf{u}_i(\cdot)$ is of feedback form, since it is recalculated at each sampling instant based on the then-current state. The solution to equation (20), starting at time $t_1$, from an initial condition $\mathbf{e}_i(t_1)$, by application of the control input $\mathbf{u}_i : [t_1, t_2] \to \mathcal{U}_i$ is denoted by

$$\mathbf{e}_i\big(t;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(t_1)\big),\ t \in [t_1, t_2]$$

The *predicted* state of the system (20) at time $t_k + \tau$, based on the measurement of the state at time $t_k$, $\mathbf{e}_i(t_k)$, by application of the control input $\mathbf{u}_i\big(t;\ \mathbf{e}_i(t_k)\big)$, for the time period $t \in [t_k, t_k + \tau]$ is denoted by

$$\overline{\mathbf{e}}_i\big(t_k + \tau;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(t_k)\big)$$

In contrast to the disturbance-free case:

$$\overline{\mathbf{e}}_i\big(\tau_1;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(\tau_0)\big) \neq \mathbf{e}_i\big(\tau_1;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(\tau_0)\big)$$

holds true here because *there are* disturbances acting on the system.

The closed-loop system for which stability is to be guaranteed is

$$\mathbf{e}_i(\tau) = g_i^R\big(\mathbf{e}_i(\tau), \overline{\mathbf{u}}_i^\star(\tau)\big),\ \tau \geq t_0 = 0$$

where $\overline{\mathbf{u}}_i^\star(\tau) = \overline{\mathbf{u}}_i^\star(\tau;\ \mathbf{e}_i(t_k))$, $\tau \in [t_k, t_k + h)$.

We can now give the definition of an *admissible input*:

**Definition 2.2.** (Admissible input)

A control input $\mathbf{u}_i : [t_k, t_k + T_p] \to \mathbb{R}^6$ for a state $\mathbf{e}_i(t_k)$ is called *admissible* if all the following hold:

1. $\mathbf{u}_i(\cdot)$ is piecewise continuous
2. $\mathbf{u}_i(\tau) \in \mathcal{U}_i,\ \forall \tau \in [t_k, t_k + T_p]$
3. $\mathbf{e}_i\big(\tau;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(t_k)\big) \in \mathcal{E}_i,\ \forall \tau \in [t_k, t_k + T_p]$
4. $\mathbf{e}_i\big(t_k + T_p;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(t_k)\big) \in \mathcal{E}_{i,f}$

## 2.4 Feasibility and Convergence

Under these considerations, we can now state the theorem that relates to the guaranteeing of the stability of the compound system of agents $i \in \mathcal{V}$, when each of them is assigned a desired position which results in feasible displacements:

**Theorem 2.1.** Suppose that

1. the terminal region $\mathcal{E}_{i,f} \subseteq \mathcal{E}_i$ is closed with $\mathbf{0} \in \mathcal{E}_{i,f}$
2. a solution to the optimal control problem (5) is feasible at time $t = 0$, that is, assumptions (**??**), (**??**), and (**??**) hold at time $t = 0$
3. there exists an admissible control input $\mathbf{u}_{i,f} : [0, h] \to \mathcal{U}_i$ such that for all $\mathbf{e}_i \in \mathcal{E}_{i,f}$ and $\forall \tau \in [0, h]$:
   (a) $\mathbf{e}_i(\tau) \in \mathcal{E}_{i,f}$
   (b) $\dfrac{\partial V_i}{\partial \mathbf{e}_i} g_i\big(\mathbf{e}_i(\tau), \mathbf{u}_{i,f}(\tau)\big) + F_i\big(\mathbf{e}_i(\tau), \mathbf{u}_{i,f}(\tau)\big) \leq 0$

then the closed loop system (10) under the control input (8) converges to the set $\mathcal{E}_{i,f}$ when $t \to \infty$.

**Proof**. The proof of the above theorem consists of two parts: in the first, recursive feasibility is established, that is, initial feasibility is shown to imply subsequent feasibility; in the second, and based on the first part, it is shown that the error state $\mathbf{e}_i(t)$ converges to the terminal set $\mathcal{E}_{i,f}$.

**Feasibility analysis** Consider a sampling instant $t_k$ for which a solution $\overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big)$ to (5) exists. Suppose now a time instant $t_{k+1}$ such that[2] $t_k < t_{k+1} < t_k + T_p$, and consider that the optimal control signal calculated at $t_k$ is comprised by the following two portions:

$$\overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big) = \begin{cases} \overline{\mathbf{u}}_i^\star\big(\tau_1;\ \mathbf{e}_i(t_k)\big), & \tau_1 \in [t_k, t_{k+1}] \\ \overline{\mathbf{u}}_i^\star\big(\tau_2;\ \mathbf{e}_i(t_k)\big), & \tau_2 \in [t_{k+1}, t_k + T_p] \end{cases} \tag{24}$$

Both portions are admissible since the calculated optimal control input is admissible, and hence they both conform to the input constraints. As for the resulting predicted states, they satisfy the state constraints, and, crucially: $\overline{\mathbf{e}}_i\big(t_k + T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \in \mathcal{E}_{i,f}$. Furthermore, according to assumption (3) of the theorem, there exists an admissible (and certainly not guaranteed optimal) input $\mathbf{u}_{i,f}$ that renders $\mathcal{E}_{i,f}$ invariant over $[t_k + T_p, t_k + T_p + h]$.

Given the above facts, we can construct an admissible input $\widetilde{\mathbf{u}}_i(\cdot)$ for time $t_{k+1}$ by sewing together the second portion of (24) and the input $\mathbf{u}_{i,f}(\cdot)$:

[2]It is not strictly necessary that $t_{k+1} = t_k + h$ here, however it is necessary for the following that $t_{k+1} - t_k \leq h$

$$\widetilde{\mathbf{u}}_i(\tau) = \begin{cases} \overline{\mathbf{u}}_i^{\star}\big(\tau;\ \mathbf{e}_i(t_k)\big), & \tau \in [t_{k+1}, t_k + T_p] \\ \mathbf{u}_{i,f}(\tau - t_k - T_p), & \tau \in (t_k + T_p, t_{k+1} + T_p] \end{cases} \tag{25}$$

Applied at time $t_{k+1}$, $\widetilde{\mathbf{u}}_i(\cdot)$ is an admissible control input as a composition of admissible control inputs.

This means that feasibility of a solution to the optimization problem at time $t_k$ implies feasibility at time $t_{k+1} > t_k$, and, thus, since at time $t = 0$ a solution is assumed to be feasible, a solution to the optimal control problem is feasible for all $t \geq 0$.

**Convergence analysis** The second part of the proof involves demonstrating the convergence of the state $\mathbf{e}_i$ to the terminal set $\mathcal{E}_{i,f}$. In order for this to be proved, it must be shown that a proper value function decreases along the solution trajectories starting at some initial time $t_k$. We consider the *optimal* cost $J_i^{\star}\big(\mathbf{e}_i(t)\big)$ as a candidate Lyapunov function:

$$J_i^{\star}\big(\mathbf{e}_i(t)\big) \triangleq J_i\Big(\mathbf{e}_i(t), \overline{\mathbf{u}}_i^{\star}\big(\cdot;\ \mathbf{e}_i(t)\big)\Big)$$

and, in particular, our goal is to show that that this cost decreases over consecutive sampling instants $t_{k+1} = t_k + h$, i.e. $J_i^{\star}\big(\mathbf{e}_i(t_{k+1})\big) - J_i^{\star}\big(\mathbf{e}_i(t_k)\big) \leq 0$.

In order not to wreak notational havoc, let us define the following terms:

- $\mathbf{u}_{0,i}(\tau) \triangleq \overline{\mathbf{u}}_i^{\star}\big(\tau;\ \mathbf{e}_i(t_k)\big)$ as the *optimal* input that results from the solution to problem (1.1) based on the measurement of state $\mathbf{e}_i(t_k)$, applied at time $\tau \geq t_k$

- $\mathbf{e}_{0,i}(\tau) \triangleq \overline{\mathbf{e}}_i\big(\tau;\ \overline{\mathbf{u}}_i^{\star}\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big)$ as the *predicted* state at time $\tau \geq t_k$, that is, the state that results from the application of the above input $\overline{\mathbf{u}}_i^{\star}\big(\cdot;\ \mathbf{e}_i(t_k)\big)$ to the state $\mathbf{e}_i(t_k)$, at time $\tau$

- $\mathbf{u}_{1,i}(\tau) \triangleq \widetilde{\mathbf{u}}_i(\tau)$ as the *admissible* input at $\tau \geq t_{k+1}$ (see eq. (25))

- $\mathbf{e}_{1,i}(\tau) \triangleq \overline{\mathbf{e}}_i\big(\tau;\ \widetilde{\mathbf{u}}_i(\cdot), \mathbf{e}_i(t_{k+1})\big)$ as the *predicted* state at time $\tau \geq t_{k+1}$, that is, the state that results from the application of the above input $\widetilde{\mathbf{u}}_i(\cdot)$ to the state $\mathbf{e}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^{\star}\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big)$, at time $\tau$

**Remark 2.3.** Given that disturbances *are* present, for the predicted and actual states at time

$\tau_1 \geq \tau_0 \in \mathbb{R}_{\geq 0}$ it holds that:

$$\mathbf{e}_i\big(\tau_1;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(\tau_0)\big) = \mathbf{e}_i(\tau_0) + \int_{\tau_0}^{\tau_1} g_i^R\big(\mathbf{e}_i(s;\ \mathbf{e}_i(\tau_0)), \mathbf{u}_i(s)\big)ds$$
$$\overline{\mathbf{e}}_i\big(\tau_1;\ \mathbf{u}_i(\cdot), \mathbf{e}_i(\tau_0)\big) = \mathbf{e}_i(\tau_0) + \int_{\tau_0}^{\tau_1} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(\tau_0)), \mathbf{u}_i(s)\big)ds$$

The following proof of convergence to the terminal set relies heavily on the Grönwall-Bellman inequality. We state it here for reference purposes.

**Lemma 2.2.** (**??**) *Grönwall-Bellman Inequality* Let $\lambda : [a, b] \to \mathbb{R}$ be continuous and $\mu : [a, b] \to \mathbb{R}$ be continuous and non-negative. If a continuous function $y : [a, b] \to \mathbb{R}$ satisfies

$$y(t) \leq \lambda(t) + \int_a^t \mu(s)y(s)ds$$

for $a \leq t \leq b$, then on the same interval

$$y(t) \leq \lambda(t) + \int_a^t \lambda(s)\mu(s)e^{\int_s^t \mu(\tau)d\tau}ds$$

In particular, if $\lambda(t) \equiv \lambda$ is a constant, then

$$y(t) \leq \lambda e^{\int_a^t \mu(\tau)d\tau}ds$$

If $\lambda(t) \equiv \lambda$ and $\mu(t) \equiv \mu$ are both constants, then

$$y(t) \leq \lambda e^{\mu(t-a)}ds$$

Before beginning to prove convergence, it is worth noting that while the cost

$$J_i\Big(\mathbf{e}_i(t), \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t)\big)\Big)$$

is optimal (in the sense that it is based on the optimal input, which provides its minimum realization), a cost that is based on a plainly admissible (and thus, without loss of generality, sub-optimal) input $\mathbf{u}_i \neq \overline{\mathbf{u}}_i^\star$ will result in a configuration where

$$J_i\Big(\mathbf{e}_i(t), \mathbf{u}_i\big(\cdot;\ \mathbf{e}_i(t)\big)\Big) \geq J_i\Big(\mathbf{e}_i(t), \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t)\big)\Big)$$

Let us now begin our investigation on the sign of the difference between the cost that results from the application of the feasible input $\mathbf{u}_{1,i}$, which we shall denote by $\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big)$, and the optimal cost $J_i^\star\big(\mathbf{e}_i(t_k)\big)$, while reminding ourselves that $J_i\big(\mathbf{e}_i(t), \overline{\mathbf{u}}_i(\cdot)\big) = \int_t^{t+T_p} F_i\big(\overline{\mathbf{e}}_i(s), \overline{\mathbf{u}}_i(s)\big)ds + V_i\big(\overline{\mathbf{e}}_i(t+T_p)\big)$:

$$\begin{aligned}\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) = V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) &+ \int_{t_{k+1}}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds \\ -V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) &- \int_{t_k}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds\end{aligned}$$

Considering that $t_k < t_{k+1} < t_k + T_p < t_{k+1} + T_p$, we break down the two integrals above in between these intervals:

$$\begin{aligned}&\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) = \\ &\quad V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) + \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds \\ &\quad -V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) - \int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds - \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds \end{aligned} \tag{26}$$

Since there are disturbances present, consulting remark (2.3) and substituting for $\tau_0 = t_k$ and $\tau_1 = t_{k+1}$ yields:

$$\begin{aligned}\mathbf{e}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) &= \mathbf{e}_i(t_k) + \int_{t_k}^{t_{k+1}} g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big)ds + \int_{t_k}^{t_{k+1}} \delta_i(s)ds \\ \overline{\mathbf{e}}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) &= \mathbf{e}_i(t_k) + \int_{t_k}^{t_{k+1}} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big)ds\end{aligned}$$

Subtracting the latter from the former and taking norms on either side yields:

$$\begin{aligned}&\Big\|\mathbf{e}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) - \overline{\mathbf{e}}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big)\Big\| \\ &= \Big\|\int_{t_k}^{t_{k+1}} g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big)ds - \int_{t_k}^{t_{k+1}} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big)ds + \int_{t_k}^{t_{k+1}} \delta_i(s)ds\Big\| \\ &\leq \Big\|\int_{t_k}^{t_{k+1}} g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big)ds - \int_{t_k}^{t_{k+1}} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big)ds\Big\| + (t_{k+1} - t_k)\overline{\delta}_i \\ &= \int_{t_k}^{t_{k+1}} \Big\|g_i\big(\mathbf{e}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big)s - g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k)), \overline{\mathbf{u}}_i^\star(s)\big)\Big\|ds + h\overline{\delta}_i \\ &\leq L_{g_i} \int_{t_k}^{t_{k+1}} \Big\|\mathbf{e}_i\big(s;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big) - \overline{\mathbf{e}}_i\big(s;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\big)\Big\|ds + h\overline{\delta}_i\end{aligned}$$

since $g_i$ is Lipschitz continuous in $\mathcal{E}_i$ with Lipschitz constant $L_{g_i}$. Reformulation yields

$$\begin{aligned}
&\Big\|\mathbf{e}_i\big(t_k+h;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big),\mathbf{e}_i(t_k)\big)-\overline{\mathbf{e}}_i\big(t_k+h;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big),\mathbf{e}_i(t_k)\big)\Big\| \\
&\leq h\overline{\delta}_i+L_{g_i}\int_0^h\Big\|\mathbf{e}_i\big(t_k+s;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big),\mathbf{e}_i(t_k)\big)-\overline{\mathbf{e}}_i\big(t_k+s;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big),\mathbf{e}_i(t_k)\big)\Big\|ds
\end{aligned}$$

By applying the Grönwall-Bellman inequality we get:

$$\begin{aligned}
&\Big\|\mathbf{e}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big),\mathbf{e}_i(t_k)\big)-\overline{\mathbf{e}}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big),\mathbf{e}_i(t_k)\big)\Big\| \\
&\leq h\overline{\delta}_i+L_{g_i}\int_0^h s\overline{\delta}_i e^{L_{g_i}(h-s)}ds \\
&= h\overline{\delta}_i-\overline{\delta}_i\int_0^h s\big(e^{L_{g_i}(h-s)}\big)'ds \\
&= h\overline{\delta}_i-\overline{\delta}_i\Big(\big[se^{L_{g_i}(h-s)}\big]_0^h-\int_0^h e^{L_{g_i}(h-s)}ds\Big) \\
&= h\overline{\delta}_i-\overline{\delta}_i\Big(h+\frac{1}{L_{g_i}}(1-e^{L_{g_i}h})\Big) \\
&= \frac{\overline{\delta}_i}{L_{g_i}}(e^{L_{g_i}h}-1)
\end{aligned}$$

**Lemma 2.3.** Suppose that the real system, which is under the existence of bounded additive disturbances, and the model are both at time $t_k$ at state $\mathbf{e}_i(t_k)$. Applying at time $t_k$ a control law $\mathbf{u}(\cdot)$ to the system model deemed "real" and its model will cause at time $t_k+\tau$, $\tau\geq 0$ a divergence between the states of the real system and its model. The norm of the difference between the state of the real system and the state of the model system is bounded by

$$\Big\|\mathbf{e}_i\big(t_k+\tau;\ \mathbf{u}(\cdot),\mathbf{e}_i(t_k)\big)-\overline{\mathbf{e}}_i\big(t_k+\tau;\ \mathbf{u}(\cdot),\mathbf{e}_i(t_k)\big)\Big\|\leq\frac{\overline{\delta}_i}{L_{g_i}}(e^{L_{g_i}\tau}-1)$$

where $\overline{\delta}_i$ is the upper bound of the disturbance, and $L_{g_i}$ the Lipschitz constant of both models.

Let us now begin working on (26), focusing first on the difference between the two intervals over

$[t_{k+1}, t_{k+1}+T_p]$:

$$
\begin{aligned}
&\int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds - \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds \\
&\int_{t_k+h}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds - \int_{t_k+h}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds \\
&\leq \Big\| \int_{t_k+h}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds - \int_{t_k+h}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds \Big\| \text{ (for } x \leq |x|, x \in \mathbb{R}) \\
&= \Big\| \int_{t_k+h}^{t_k+T_p} \Big( F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) - F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) \Big) ds \Big\| \\
&= \int_{t_k+h}^{t_k+T_p} \Big\| F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) - F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) \Big\| ds \\
&\leq L_{F_i} \int_{t_k+h}^{t_k+T_p} \Big\| \overline{\mathbf{e}}_i\big(s;\ \mathbf{u}_{1,i}(\cdot), \mathbf{e}_i(t_k+h)\big) - \overline{\mathbf{e}}_i\big(s;\ \mathbf{u}_{0,i}(\cdot), \overline{\mathbf{e}}_i(t_k+h)\big) \Big\| ds \text{ (for } F_i \text{ is Lipschitz continuous in } \mathcal{E}_i)
\end{aligned}
\tag{27}
$$

Consulting with remark (2.3) and substutiting for $\tau_1 = t_k + T_p$ and $\tau_0 = t_k + h$ in the second equation for the two different initial conditions we get

$$
\begin{aligned}
\overline{\mathbf{e}}_i\big(t_k+T_p;\ \mathbf{u}_i^{\star}(\cdot), \mathbf{e}_i(t_k+h)\big) &= \mathbf{e}_i(t_k+h) + \int_{t_k+h}^{t_k+T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k+h)), \mathbf{u}_i^{\star}(s)\big)ds \\
\overline{\mathbf{e}}_i\big(t_k+T_p;\ \mathbf{u}_i^{\star}(\cdot), \overline{\mathbf{e}}_i(t_k+h)\big) &= \overline{\mathbf{e}}_i(t_k+h) + \int_{t_k+h}^{t_k+T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \overline{\mathbf{e}}_i(t_k+h)), \mathbf{u}_i^{\star}(s)\big)ds
\end{aligned}
$$

Subtracting the latter from the former and taking norms on either side yields

$$\begin{aligned}
&\Big\|\overline{\mathbf{e}}_i\big(t_k+T_p;\ \overline{\mathbf{u}}_i^\star(\cdot),\mathbf{e}_i(t_k+h)\big)-\overline{\mathbf{e}}_i\big(t_k+T_p;\ \overline{\mathbf{u}_i^\star}(\cdot),\overline{\mathbf{e}}_i(t_k+h)\big)\Big\| \\
&\leq \Big\|\mathbf{e}_i(t_k+h)-\overline{\mathbf{e}}_i(t_k+h) \\
&+\int_{t_k+h}^{t_k+T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k+h)),\overline{\mathbf{u}}_i^\star(s)\big)ds-\int_{t_k+h}^{t_k+T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \overline{\mathbf{e}}_i(t_k+h)),\overline{\mathbf{u}}_i^\star(s)\big)ds\Big\| \\
&\leq \big\|\mathbf{e}_i(t_k+h)-\overline{\mathbf{e}}_i(t_k+h)\big\| \\
&+\Big\|\int_{t_k+h}^{t_k+T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k+h)),\overline{\mathbf{u}}_i^\star(s)\big)ds-\int_{t_k+h}^{t_k+T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \overline{\mathbf{e}}_i(t_k+h)),\overline{\mathbf{u}}_i^\star(s)\big)ds\Big\| \\
&\leq \big\|\mathbf{e}_i(t_k+h)-\overline{\mathbf{e}}_i(t_k+h)\big\| \\
&+\Big\|\int_{t_k+h}^{t_k+T_p} \Big(g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k+h)),\overline{\mathbf{u}}_i^\star(s)\big)-g_i\big(\overline{\mathbf{e}}_i(s;\ \overline{\mathbf{e}}_i(t_k+h)),\overline{\mathbf{u}}_i^\star(s)\big)\Big)ds\Big\| \\
&= \big\|\mathbf{e}_i(t_k+h)-\overline{\mathbf{e}}_i(t_k+h)\big\| \\
&+\int_{t_k+h}^{t_k+T_p} \Big\|g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_k+h)),\overline{\mathbf{u}}_i^\star(s)\big)-g_i\big(\overline{\mathbf{e}}_i(s;\ \overline{\mathbf{e}}_i(t_k+h)),\overline{\mathbf{u}}_i^\star(s)\big)\Big\|ds \\
&\leq \big\|\mathbf{e}_i(t_k+h)-\overline{\mathbf{e}}_i(t_k+h)\big\| \\
&+L_{g_i}\int_{t_k+h}^{t_k+T_p} \Big\|\overline{\mathbf{e}}_i\big(s;\ \overline{u}_i^\star(\cdot),\mathbf{e}_i(t_k+h)\big)-\overline{\mathbf{e}}_i\big(s;\ \overline{\mathbf{u}}_i^\star(\cdot),\overline{\mathbf{e}}_i(t_k+h)\big)\Big\|ds \\
&= \big\|\mathbf{e}_i(t_k+h)-\overline{\mathbf{e}}_i(t_k+h)\big\| \\
&+L_{g_i}\int_{h}^{T_p} \Big\|\overline{\mathbf{e}}_i\big(t_k+s;\ \overline{\mathbf{u}}_i^\star(\cdot),\mathbf{e}_i(t_k+h)\big)-\overline{\mathbf{e}}_i\big(t_k+s;\ \overline{\mathbf{u}}_i^\star(\cdot),\overline{\mathbf{e}}_i(t_k+h)\big)\Big\|ds
\end{aligned}$$

By applying the Grönwall-Bellman inequality we get:

$$\begin{aligned}
&\Big\|\overline{\mathbf{e}}_i\big(t_k+T_p;\ \overline{\mathbf{u}}_i^\star(\cdot),\mathbf{e}_i(t_k+h)\big)-\overline{\mathbf{e}}_i\big(t_k+T_p;\ \overline{\mathbf{u}}_i^\star(\cdot),\overline{\mathbf{e}}_i(t_k+h)\big)\Big\| \\
&\qquad\qquad \leq \big\|\mathbf{e}_i(t_k+h)-\overline{\mathbf{e}}_i(t_k+h)\big\|e^{L_{g_i}(T_p-h)}
\end{aligned}$$

**Lemma 2.4.** Suppose that the model of the real system (*not* the real model itself), at time $t_k$ is at state $\mathbf{e}_i(t_k)$, and that another identical model is at time $t_k$ at state $\overline{\mathbf{e}}_i(t_k)$. Applying at time $t_k$ a control law $\mathbf{u}(\cdot)$ to both will cause at time $t_k+\tau$, $\tau\geq 0$ a divergence between their states. The norm of the difference between these states is bounded by

$$\Big\|\overline{\mathbf{e}}_i\big(t_k+\tau;\ \mathbf{u}(\cdot),\mathbf{e}_i(t_k)\big)-\overline{\mathbf{e}}_i\big(t_k+\tau;\ \mathbf{u}(\cdot),\overline{\mathbf{e}}_i(t_k)\big)\Big\| \leq \big\|\mathbf{e}_i(t_k)-\overline{\mathbf{e}}_i(t_k)\big\|e^{L_{g_i}\tau}$$

where $\overline{\delta}_i$ is the upper bound of the disturbance, and $L_{g_i}$ the Lipschitz constant of the models.

Given lemma (2.4), (27) becomes

$$
\begin{aligned}
&\int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s),\mathbf{u}_{1,i}(s)\big)ds - \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s),\mathbf{u}_{0,i}(s)\big)ds \\
&\leq L_{F_i}\int_{t_k+h}^{t_k+T_p}\Big\|\overline{\mathbf{e}}_i\big(s;\ \mathbf{u}_{1,i}(\cdot),\mathbf{e}_i(t_k+h)\big)-\overline{\mathbf{e}}_i\big(s;\ \mathbf{u}_{0,i}(\cdot),\overline{\mathbf{e}}_i(t_k+h)\big)\Big\|ds \\
&= L_{F_i}\int_{h}^{T_p}\Big\|\overline{\mathbf{e}}_i\big(t_k+s;\ \mathbf{u}_{1,i}(\cdot),\mathbf{e}_i(t_k+h)\big)-\overline{\mathbf{e}}_i\big(t_k+s;\ \mathbf{u}_{0,i}(\cdot),\overline{\mathbf{e}}_i(t_k+h)\big)\Big\|ds \\
&\leq L_{F_i}\int_{h}^{T_p}\Big\|\mathbf{e}_i(t_k+h)-\overline{\mathbf{e}}_i(t_k+h)\Big\|e^{L_{g_i}(s-h)}ds \\
&\leq L_{F_i}\frac{\overline{\delta}_i}{\text{Ł}_{g_i}}(e^{L_{g_i}h}-1)\int_{h}^{T_p} e^{L_{g_i}(s-h)}ds, \text{ from lemma (2.3) for } \tau = h \\
&= L_{F_i}\frac{\overline{\delta}_i}{\text{Ł}_{g_i}}(e^{L_{g_i}h}-1)\frac{1}{L_{g_i}}(e^{L_{g_i}(T_p-h)}-1)ds \\
&= L_{F_i}\frac{\overline{\delta}_i}{\text{Ł}_{g_i}^2}(e^{L_{g_i}h}-1)(e^{L_{g_i}(T_p-h)}-1)ds
\end{aligned}
$$

Hence we discovered that

$$
\boxed{
\begin{aligned}
&\int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s),\mathbf{u}_{1,i}(s)\big)ds - \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s),\mathbf{u}_{0,i}(s)\big)ds \\
&\qquad\qquad \leq L_{F_i}\frac{\overline{\delta}_i}{\text{Ł}_{g_i}^2}(e^{L_{g_i}h}-1)(e^{L_{g_i}(T_p-h)}-1)ds
\end{aligned}
}
$$

With this partial result established, we turn back to the remaining terms found in (26) and, in particular, we focus on the integral

$$
\int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s),\mathbf{u}_{1,i}(s)\big)ds
$$

We discern that the range of the above integral has a length[a] equal to the length of the interval where assumption (3b) (??) of theorem (2.1) holds. Integrating the expression found in the assumption over the interval $[t_k + T_p, t_{k+1} + T_p]$, for the controls and states applicable in it we get

$$\int_{t_k+T_p}^{t_{k+1}+T_p} \left( \frac{\partial V_i}{\partial \mathbf{e}_{1,i}} g_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) + F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) \right) ds \leq 0$$

$$\int_{t_k+T_p}^{t_{k+1}+T_p} \frac{d}{ds} V_i\big(\mathbf{e}_{1,i}(s)\big) ds + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds \leq 0$$

$$V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) - V_i\big(\mathbf{e}_{1,i}(t_k+T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds \leq 0$$

$$V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds \leq V_i\big(\mathbf{e}_{1,i}(t_k+T_p)\big)$$

The left-hand side expression is the same as the first two terms in the right-hand side of equality (26). We can introduce the third one by subtracting it from both sides:

$$\begin{aligned}
&V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds - V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) \\
&\leq V_i\big(\mathbf{e}_{1,i}(t_k+T_p)\big) - V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) \\
&\leq \Big\| V_i\big(\mathbf{e}_{1,i}(t_k+T_p)\big) - V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) \Big\|, \text{ for } x \leq |x|, \forall x \in \mathbb{R} \\
&\leq L_{V_i} \Big\| \overline{\mathbf{e}}_i\big(t_k+T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big) - \overline{\mathbf{e}}_i\big(t_k+T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_k)\big) \Big\|, \text{ from lemma (1.1)}
\end{aligned}$$

Consulting with remark (2.3) we get that the two terms interior to the norm are respectively equal to

$$\begin{aligned}
\overline{\mathbf{e}}_i\big(t_k+T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big) &= \mathbf{e}_i(t_{k+1}) + \int_{t_{k+1}}^{t_k+T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_{k+1})), \overline{\mathbf{u}}_i^\star(s)\big) ds \\
\overline{\mathbf{e}}_i\big(t_k+T_p;\ \overline{\mathbf{u}}_i^\star(\cdot), \mathbf{e}_i(t_{k+1})\big) &= \mathbf{e}_i(t_{k+1}) + \int_{t_{k+1}}^{t_k+T_p} g_i\big(\overline{\mathbf{e}}_i(s;\ \mathbf{e}_i(t_{k+1})), \overline{\mathbf{u}}_i^\star(s)\big) ds
\end{aligned}$$

[a] $(t_{k+1}+T_p) - (t_k+T_p) = t_{k+1} - t_k = h$

==========================================================

hence, since the right-hand side expressions are equal:

$$\mathbf{e}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t_k)), \mathbf{e}_i(t_k)\big) = \overline{\mathbf{e}}_i\big(t_{k+1};\ \overline{\mathbf{u}}_i^\star(\cdot;\ \mathbf{e}_i(t_k)), \mathbf{e}_i(t_k)\big)$$

In between times $t_{k+1}$ and $t_k + T_p$, the constructed admissible input $\widetilde{\mathbf{u}}_i(\cdot)$ is equal to the optimal input $\overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big)$ (see eq. 25), which means that $\mathbf{u}_{1,i}(\tau) = \mathbf{u}_{0,i}(\tau)$ in the interval $\tau \in [t_{k+1}, t_k + T_p]$. Since the initial conditions at $t = t_{k+1}$ are equal and the control laws are also equal, so will the predicted states over the same interval:

$$\overline{\mathbf{e}}_i\big(\tau;\ \widetilde{\mathbf{u}}_i(\cdot), \mathbf{e}_i(t_{k+1})\big) = \overline{\mathbf{e}}_i\big(\tau;\ \overline{\mathbf{u}}_i^\star(\cdot), \overline{\mathbf{e}}_i(t_{k+1})\big),\ \tau \in [t_{k+1}, t_k + T_p]$$

Using our notation then, in the same interval: $\mathbf{e}_{1,i}(\cdot) = \mathbf{e}_{0,i}(\cdot)$, and therefore the following equality holds over $[t_{k+1}, t_k + T_p]$:

$$F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) = F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big),\ s \in [t_{k+1}, t_k + T_p]$$

Integrating this equality over the interval where it is valid yields

$$\int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds = \int_{t_{k+1}}^{t_k+T_p} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) ds$$

This means that these two integrals with ends over the interval $[t_{k+1}, t_k+T_p]$ featured in the right-hand side of eq. (13) vanish, and thus the cost difference becomes

$$\begin{aligned} \overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) = V_i\big(\mathbf{e}_{1,i}(t_{k+1} + T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds \\ - V_i\big(\mathbf{e}_{0,i}(t_k + T_p)\big) - \int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) ds \end{aligned} \tag{28}$$

We turn our attention to the first integral in the above expression, and we note that $(t_{k+1} + T_p) - (t_k + T_p) = t_{k+1} - t_k = h$, which is exactly the length of the interval where assumption (3b) of the theorem holds. Hence, we decide to integrate the expression found in the assumption over the interval $[t_{k+1}+T_p, t_k+T_p]$, for the controls and states applicable in it:

$$\int_{t_k+T_p}^{t_{k+1}+T_p} \left( \frac{\partial V_i}{\partial \mathbf{e}_{1,i}} g_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) + F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) \right) ds \leq 0$$

$$\int_{t_k+T_p}^{t_{k+1}+T_p} \frac{d}{ds} V_i\big(\mathbf{e}_{1,i}(s)\big) ds + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds \leq 0$$

$$V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) - V_i\big(\mathbf{e}_{1,i}(t_k+T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds \leq 0$$

$$V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds \leq V_i\big(\mathbf{e}_{1,i}(t_k+T_p)\big)$$

The left-hand side expression is the same as the first two terms in the right-hand side of equality (28). We can introduce the third one by subtracting it from both sides:

$$\begin{aligned} &V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds - V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) \\ &\leq V_i\big(\mathbf{e}_{1,i}(t_k+T_p)\big) - V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) \leq \Big| V_i\big(\mathbf{e}_{1,i}(t_k+T_p)\big) - V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) \Big| \end{aligned}$$

since $x \leq |x|, \forall x \in \mathbb{R}$.

By revisiting lemma (1.1), the above inequality becomes

$$\begin{aligned} V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds - V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) \\ \leq L_{V_i} \big\| \mathbf{e}_{1,i}(t_k+T_p) - \mathbf{e}_{0,i}(t_k+T_p) \big\| \end{aligned}$$

However, in the interval $[t_{k+1}, t_k+T_p]$: $\mathbf{e}_{1,i}(\cdot) = \mathbf{e}_{0,i}(\cdot)$, hence the right-hand side of the inequality equals zero:

$$V_i\big(\mathbf{e}_{1,i}(t_{k+1}+T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big) ds - V_i\big(\mathbf{e}_{0,i}(t_k+T_p)\big) \leq 0$$

By subtracting the fourth term needed to complete the right-hand side expression of (28), i.e.

$\int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds$ from both sides we get

$$V_i\big(\mathbf{e}_{1,i}(t_{k+1} + T_p)\big) + \int_{t_k+T_p}^{t_{k+1}+T_p} F_i\big(\mathbf{e}_{1,i}(s), \mathbf{u}_{1,i}(s)\big)ds$$

$$-V_i\big(\mathbf{e}_{0,i}(t_k + T_p)\big) - \int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds \leq -\int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds$$

The left-hand side of this inequality is now equal to the cost difference $\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big)$.

Hence, the cost difference becomes bounded by

$$\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) \leq -\int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big)ds$$

$F_i$ is a positive-definite function as a sum of a positive-definite $\|\mathbf{u}_i\|^2_{\boldsymbol{R}_i}$ and a positive semi-definite function $\|\mathbf{e}_i\|^2_{\boldsymbol{Q}_i}$. If we denote by $m = \lambda_{min}(\boldsymbol{Q}_i, \boldsymbol{R}_i) \geq 0$ the minimum eigenvalue between those of matrices $\boldsymbol{R}_i$, $\boldsymbol{Q}_i$, this means that

$$F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) \geq m\|\mathbf{e}_{0,i}(s)\|^2$$

By integrating the above between our interval of interest $[t_k, t_{k+1}]$ we get

$$\int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) \geq \int_{t_k}^{t_{k+1}} m\|\mathbf{e}_{0,i}(s)\|^2 ds$$

or

$$-\int_{t_k}^{t_{k+1}} F_i\big(\mathbf{e}_{0,i}(s), \mathbf{u}_{0,i}(s)\big) \leq -m\int_{t_k}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds$$

This means that the cost difference is upper-bounded by a class $\mathcal{K}$ function

$$\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) \leq -m\int_{t_k}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds \leq 0$$

and since the cost $\overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big)$ is, in general, sub-optimal: $J_i^\star\big(\mathbf{e}_i(t_{k+1})\big) - \overline{J}_i\big(\mathbf{e}_i(t_{k+1})\big) \leq 0$:

$$J_i^\star\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) \leq -m\int_{t_k}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds \tag{29}$$

With this milestone result established, we need to trace the time $t_k$ back to $t_0 = 0$.

The integral of $\|\mathbf{e}_{0,i}(\tau)\|^2$ over the interval $[t_0, t_{k+1}]$, $t_0 < t_k < t_{k+1}$ can be decomposed into the addition of two integrals with limits ranging from (a) $t_0$ to $t_k$ and (b) $t_k$ to $t_{k+1}$:

$$\int_{t_0}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds = \int_{t_0}^{t_k} \|\mathbf{e}_{0,i}(s)\|^2 ds + \int_{t_k}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds$$

By rearranging terms, this means that

$$\int_{t_k}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds = \int_{t_0}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds - \int_{t_0}^{t_k} \|\mathbf{e}_{0,i}(s)\|^2 ds$$

making the optimal cost difference between the consecutive sampling times $t_k$ and $t_{k+1}$ in (29)

$$J_i^\star\big(\mathbf{e}_i(t_{k+1})\big) - J_i^\star\big(\mathbf{e}_i(t_k)\big) \leq -m \int_{t_0}^{t_{k+1}} \|\mathbf{e}_{0,i}(s)\|^2 ds + m \int_{t_0}^{t_k} \|\mathbf{e}_{0,i}(s)\|^2 ds$$

Similarly, the optimal cost difference between the sampling times $t_{k-1}$ and $t_k$ is

$$J_i^\star\big(\mathbf{e}_i(t_k)\big) - J_i^\star\big(\mathbf{e}_i(t_{k-1})\big) \leq -m \int_{t_0}^{t_k} \|\mathbf{e}_{0,i}(s)\|^2 ds + m \int_{t_0}^{t_{k-1}} \|\mathbf{e}_{0,i}(s)\|^2 ds$$

and we can apply this rationale all the way back to the cost difference between $t_0$ and $t_1$. Summing all the inequalities between the pairs of consecutive sampling times $(t_0, t_1)$, $(t_1, t_2)$, $\ldots$, $(t_{k-1}, t_k)$, we get

$$J_i^\star\big(\mathbf{e}_i(t_k)\big) - J_i^\star\big(\mathbf{e}_i(t_0)\big) \leq -m \int_{t_0}^{t_k} \|\mathbf{e}_{0,i}(s)\|^2 ds$$

Hence, for $t_0 = 0$

$$J_i^\star\big(\mathbf{e}_i(t_k)\big) - J_i^\star\big(\mathbf{e}_i(0)\big) \leq -m \int_{0}^{t_k} \|\mathbf{e}_{0,i}(s)\|^2 ds \leq 0 \tag{30}$$

which implies that the value function $J_i^\star\big(\mathbf{e}_i(t_k)\big)$ in non-increasing for all sampling times:

$$J_i^\star\big(\mathbf{e}_i(t_k)\big) \leq J_i^\star\big(\mathbf{e}_i(0)\big), \ \forall t_k \in \mathbb{R}_{\geq 0}$$

Let us now define the function $V_i(\mathbf{e}_i(t))$:

$$V_i\big(\mathbf{e}(t)\big) \triangleq J_i^\star\big(\mathbf{e}_i(\tau)\big) \leq J_i^\star\big(\mathbf{e}_i(0)\big), \ t \in \mathbb{R}_{\geq 0}$$

where $\tau = max\{t_k : t_k \leq t\}$. Since $J_i^\star\big(\mathbf{e}_i(0)\big)$ is bounded, this implies that $V_i\big(\mathbf{e}(t)\big)$ is also bounded. The signals $\mathbf{e}_i(t) \in \mathcal{E}_i$ and $\mathbf{u}_i(t) \in \mathcal{U}_i$ are also bounded. According to (4), this means that $\dot{\mathbf{e}}_i(t)$ is bounded as well. From inequality (30) we then have

$$V_i\big(\mathbf{e}_i(t)\big) = J_i^\star\big(\mathbf{e}_i(\tau)\big) \le J_i^\star\big(\mathbf{e}_i(0)\big) - m\int_0^\tau \|\mathbf{e}_{0,i}(s)\|^2 ds \le 0$$

which, due to the fact that $\tau \le t$, is equivalent to

$$V_i\big(\mathbf{e}_i(t)\big) \le J_i^\star\big(\mathbf{e}_i(0)\big) - m\int_0^t \|\mathbf{e}_{0,i}(s)\|^2 ds \le 0,\ t \in \mathbb{R}_{t\ge 0}$$

Solving for the integral we get

$$\int_0^t \|\mathbf{e}_{0,i}(s)\|^2 ds \le \frac{1}{m}\Big(J_i^\star\big(\mathbf{e}_i(0)\big) - V_i\big(\mathbf{e}_i(t)\big)\Big),\ t \in \mathbb{R}_{t\ge 0}$$

Both $J_i^\star\big(\mathbf{e}_i(0)\big)$ and $V_i\big(\mathbf{e}_i(t)\big)$ are bounded, and therefore so is their difference, which means that the integral $\int\limits_0^t \|\mathbf{e}_{0,i}(s)\|^2 ds$ is bounded as well. We make use of the following lemma to show that the error internal to the norm of the integral goes to zero in steady-state:

**Lemma 2.5.** (*A modification of Barbalat's lemma*[3])

Let $f$ be a continuous, positive-definite function, and $\mathbf{x}$ be an absolutely continuous function in $\mathbb{R}$. If the following hold:

- $\|\mathbf{x}(\cdot)\| < \infty, \|\dot{\mathbf{x}}(\cdot)\| < \infty$
- $\lim\limits_{t\to\infty} \int\limits_0^t f\big(\mathbf{x}(s)\big) < \infty$

then $\lim\limits_{t\to\infty} \|\mathbf{x}(t)\| = 0$

Lemma (2.5) assures us that under these conditions for the error and its dynamics, which are fulfilled in our case, the error

$$\lim_{t\to\infty} \|\mathbf{e}_{0,i}(t)\| = 0 \Leftrightarrow$$
$$\lim_{t\to\infty} \Big\|\overline{\mathbf{e}}_i\Big(t;\ \overline{\mathbf{u}}_i^\star\big(\cdot;\ \mathbf{e}_i(t_k)\big), \mathbf{e}_i(t_k)\Big)\Big\| = 0,\ \forall t_k \in \mathbb{R}_{\ge 0}$$

which, given (9) and substituting for $\tau_1 = t$ while dropping the initial condition at $\tau_0 = t_k$, means that

$$\lim_{t\to\infty} \|\mathbf{e}_i(t)\| = 0$$

which implies that

$$\lim_{t\to\infty} \mathbf{e}_i(t) \in \mathcal{E}_{i,f}$$

Therefore, the closed-loop trajectory of the error state $\mathbf{e}_i$ converges to the terminal set $\mathcal{E}_{i,f}$ as $t \to \infty$.

■

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