

DD2423 - Lab I

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1 Section 1.3 - Basic Functions

1.1 Question 1

What we see here is that:

- The further the non-zero point (p, q) from the origin $O(0, 0)$, the smaller the wavelength of the spatial image (more dense lines in the real and imaginary part of the spatial image),
- The amplitude of all the spatial images is the same,
- The direction of the waveforms in the spatial images is dictated by the position of the non-zero point (p, q) relative to the origin $O(0, 0)$

The output of the `fftwave` function for $(p, q) = (5, 9)$, $(p, q) = (9, 5)$, $(p, q) = (17, 9)$, $(p, q) = (17, 121)$, $(p, q) = (5, 1)$ and $(p, q) = (125, 1)$ is illustrated in figures 1, 2, 3, 4, 5 and 6 respectively.

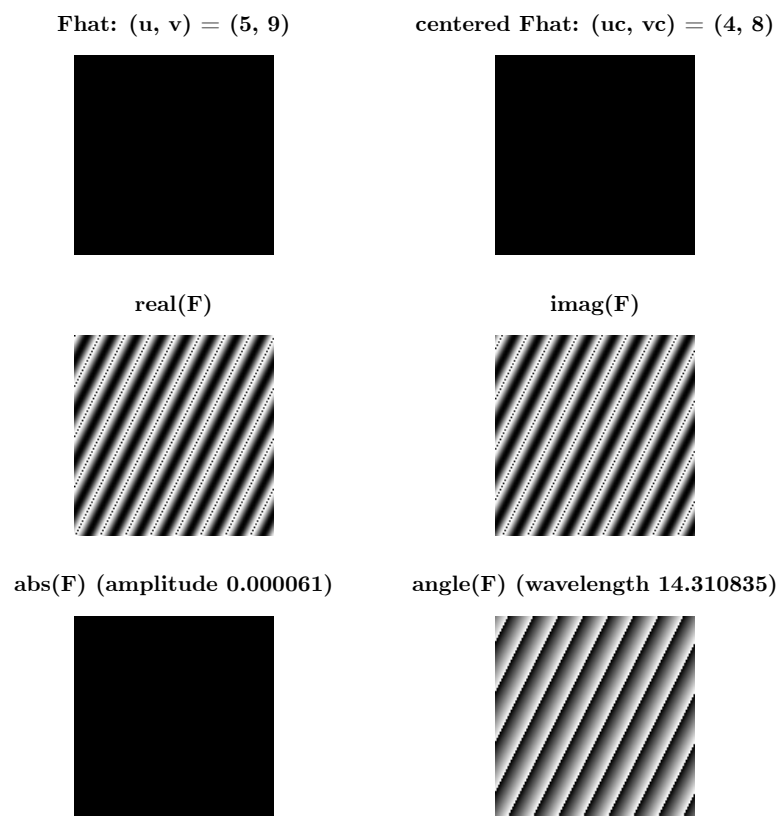


Figure 1: $(p, q) = (5, 9)$

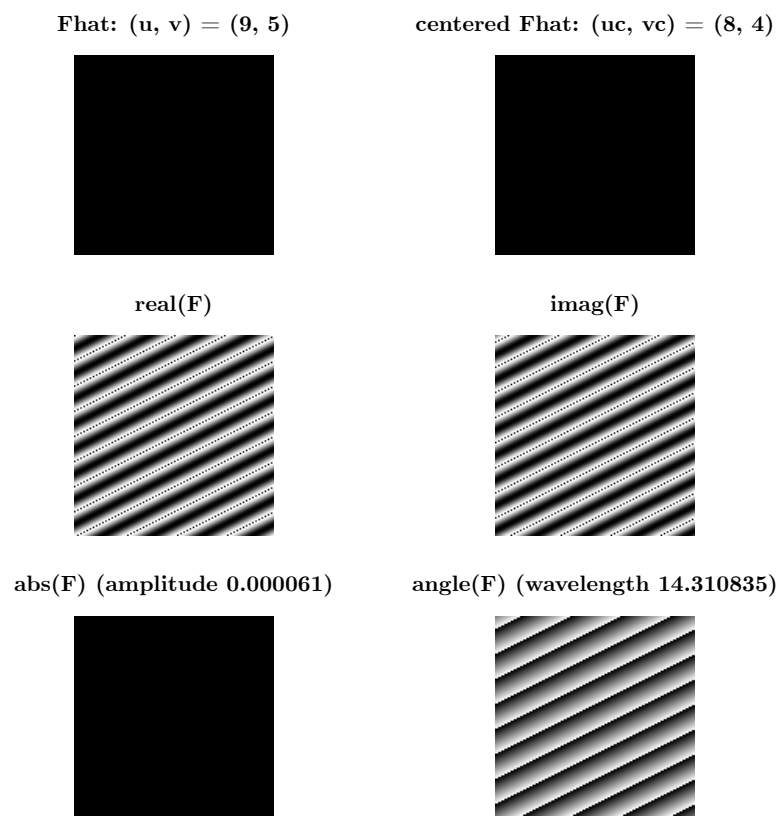
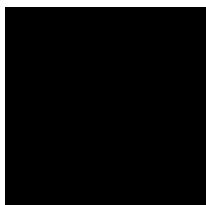
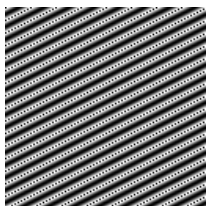


Figure 2: $(p, q) = (9, 5)$

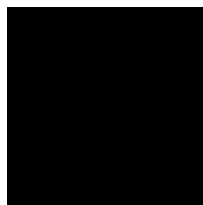
Fhat: $(u, v) = (17, 9)$



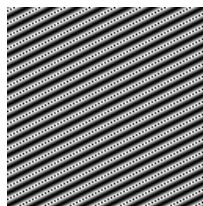
real(F)



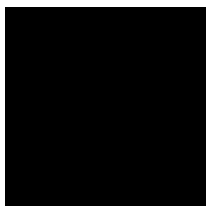
centered Fhat: $(uc, vc) = (16, 8)$



imag(F)



abs(F) (amplitude 0.000061)



angle(F) (wavelength 7.155418)

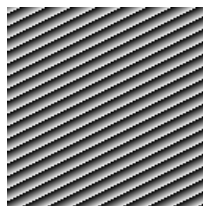
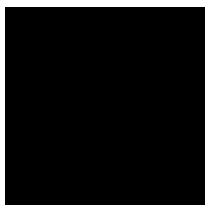
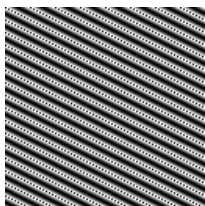


Figure 3: $(p, q) = (17, 9)$

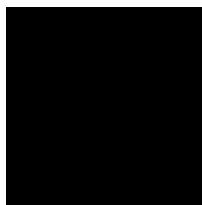
Fhat: $(u, v) = (17, 121)$



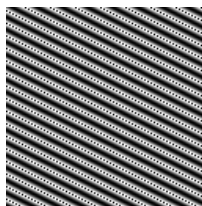
real(F)



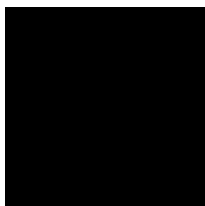
centered Fhat: $(uc, vc) = (16, -8)$



imag(F)



abs(F) (amplitude 0.000061)



angle(F) (wavelength 7.155418)

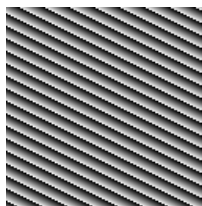


Figure 4: $(p, q) = (17, 121)$

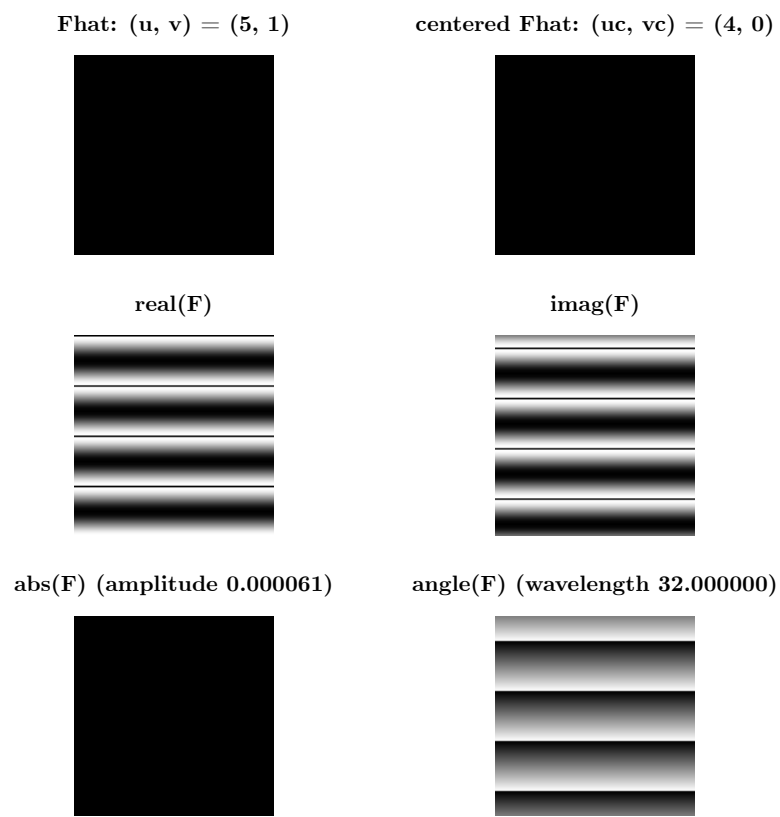


Figure 5: $(p, q) = (5, 1)$

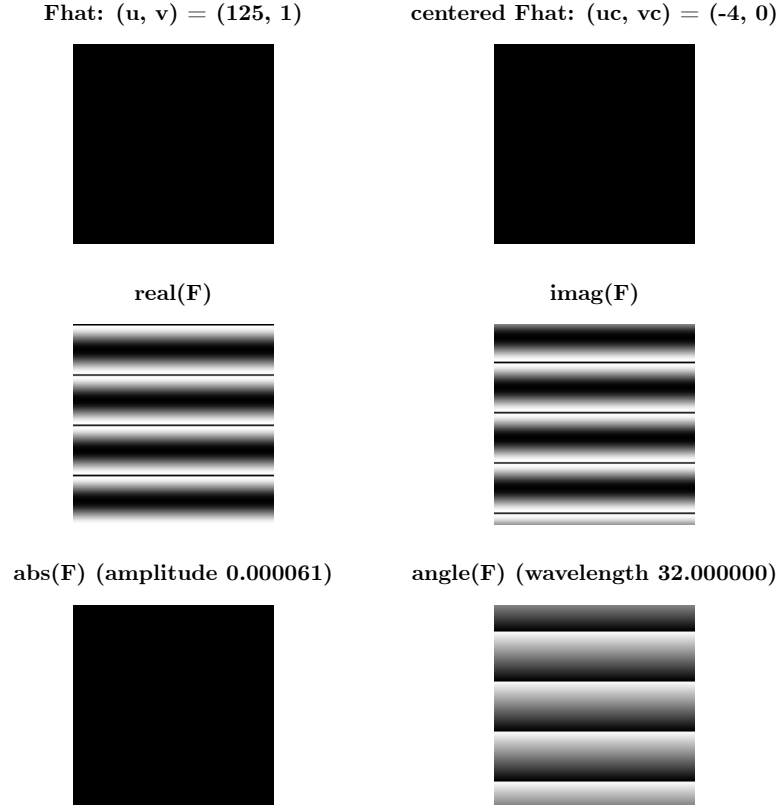


Figure 6: $(p, q) = (125, 1)$

1.2 Question 2

We exploit equation 4.2 – 33 from [?]

$$\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} s(x, y) \cdot A \delta(x - x_0, y - y_0) = A \cdot s(x_0, y_0) \quad (1)$$

knowing that the output Fourier transform is a delta function at (p, q) . Hence, for a quadratic image $M = N$ and in the spatial domain:

$$f(x, y) = \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} \delta(u - p, v - q) \cdot e^{\frac{2\pi i \cdot (xu + yv)}{N}} = \frac{1}{N^2} \cdot e^{\frac{2\pi i \cdot (px + qy)}{N}} \quad (2)$$

Hence,

$$f(x, y) = \frac{1}{N^2} \cdot \left(\cos\left(\frac{2\pi \cdot (px + qy)}{N}\right) + i \sin\left(\frac{2\pi \cdot (px + qy)}{N}\right) \right) \quad (3)$$

The spatial image for $(p, q) = (5, 9)$ is plotted in figure 7. The real part of equation 3 is plotted in figure 8.

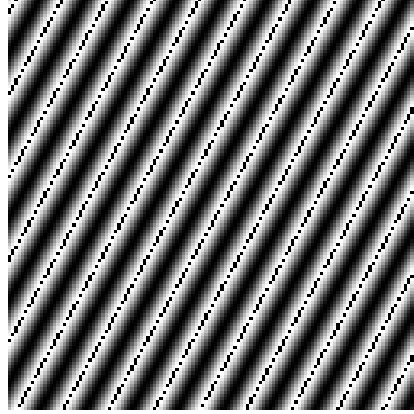


Figure 7: The inverse Fourier transform of an image for $(p, q) = (5, 9)$.

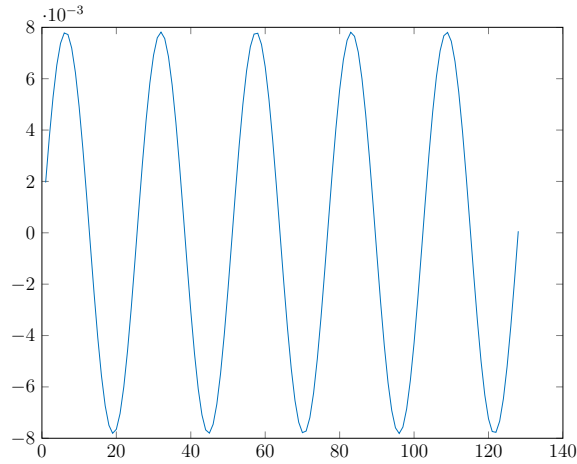


Figure 8: The real part of the spatial image above.

1.3 Question 3

As can be seen in equation 3, the amplitude of the waveform is

$$A = \frac{1}{N^2} \quad (4)$$

1.4 Question 4

As seen in the lecture notes,

$$\lambda = \frac{2\pi}{|\omega|} \quad (5)$$

and

$$\omega = \left[\frac{2\pi u}{N} \quad \frac{2\pi v}{N} \right]^T \quad (6)$$

Hence, equation 5 for $(u, v) = (p, q)$ becomes

$$\lambda = \frac{1}{\sqrt{p^2 + q^2}} \quad (7)$$

The direction of travel of the waveforms in the spatial images is dictated by the position of the non-zero point (p, q) relative to the origin $O(0, 0)$. Figure 9 illustrates the dependence of the direction of a waveform on $p = \frac{\omega_l N}{2\pi}$ and $q = \frac{\omega_k N}{2\pi}$.

1.5 Question 5

For an quadratic image of size N , the highest number of cycles that can fit in it is $N/2$, that is, stripes of width of 1 pixel. The corresponding (maximum) frequency is $\omega_{max} = 2\pi/2 = \pi$. However,

$$\omega = \frac{2\pi u}{N} \leq \omega_{max} \quad (8)$$

Hence, when either p or q exceed the value of $N/2$, which in our case is $N/2 = 64$, the Nyquist frequency is exceeded and the corresponding waveform in the spatial domain is no longer a sinusoid. Figure 10 illustrates the waveform of the real part of a spatial image whose Fourier transform is an image with a non-zero pixel located at $(p, q) = (69, 120)$.

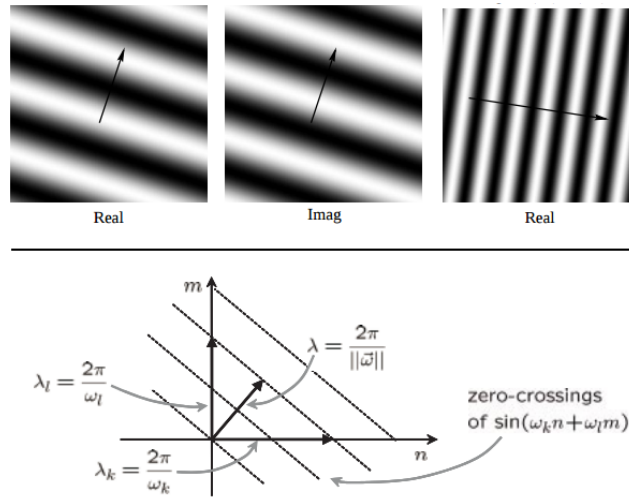


Figure 9: Image taken from Allan+ Jepson's notes (www.cs.toronto.edu/~jepson/csc320/notes/linearFilters2.pdf)

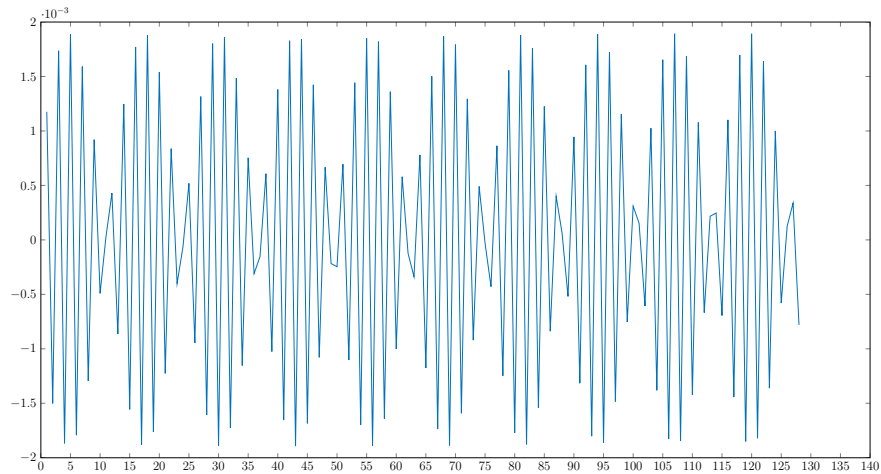


Figure 10: Example waveform in the spatial domain for $(p, q) = (69, 120)$

1.6 Question 6

The purpose of these lines is to visualize the mapping of the angular frequency values ω_x and ω_y inside the interval

$$-\pi \leq \omega_x, \omega_y \leq \pi \quad (9)$$

The exact operation is performed by `fftshift()` function.

2 Section 1.4 - Linearity

Figures 11, 12 and 13 show the spatial F , G and H images. Figures 14, 15 and 16 show their respective Fourier transforms, with the origin being in the upper left corner. Figures 17, 18 and 19 show the Fourier transforms of images F , G and H with the origin in the middle of the image.

Here, the `fftshift()` method is used to shift the origin $O(0,0)$ from the upper left corner to the middle of each image. The `log` command is used to make details in the images in the frequency domain visible. Figures 20, 21 and 22 illustrate the Fourier transforms of images F , G and H with the origin in the middle of the image without the use of the `log` command.

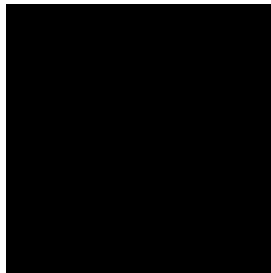


Figure 11: Image F

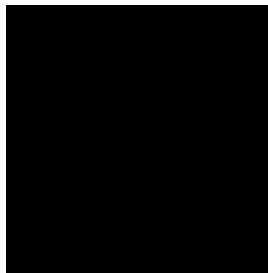


Figure 12: Image G
= F'

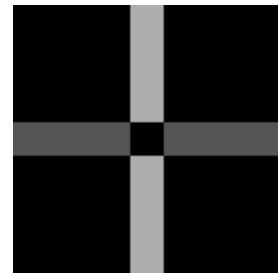


Figure 13: Image H
= $F + 2 * G$

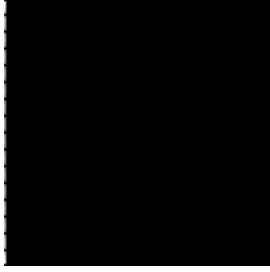


Figure 14: Image $\mathcal{F}(F)$. Origin at the upper left corner.

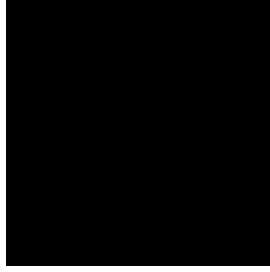


Figure 15: Image $\mathcal{F}(G)$. Origin at the upper left corner.



Figure 16: Image $\mathcal{F}(H)$. Origin at the upper left corner.



Figure 17: Image $\mathcal{F}(F)$. Origin at the middle.

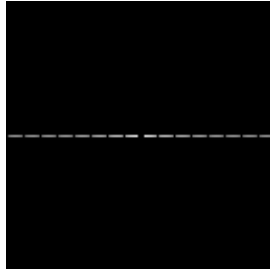


Figure 18: Image $\mathcal{F}(G)$. Origin at the middle.

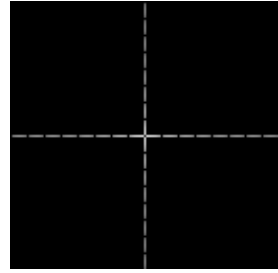


Figure 19: Image $\mathcal{F}(H)$. Origin at the middle.



Figure 20: Image $\mathcal{F}(F)$. Origin at the middle. Illustration without the use of the log function.



Figure 21: Image $\mathcal{F}(G)$. Origin at the middle. Illustration without the use of the log function.



Figure 22: Image $\mathcal{F}(H)$. Origin at the middle. Illustration without the use of the log function.

2.1 Question 7

In essence, image F in the spatial domain is a two-dimensional box function:

$$F(x, y) = \begin{cases} 1, & x_1 \leq x \leq x_2 \\ 0, & \text{everywhere else} \end{cases} \quad (10)$$

Its Fourier transform is:

$$\begin{aligned} \mathcal{F}(F(x, y)) &= \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x, y) \cdot e^{-\frac{2\pi i(xu + yv)}{N}} = \\ &= \sum_{x=x_1}^{x_2} e^{-\frac{2\pi i x u}{N}} \sum_{y=0}^{N-1} e^{-\frac{2\pi i y v}{N}} = \sum_{x=x_1}^{x_2} e^{-\frac{2\pi i x u}{N}} \sum_{y=0}^{N-1} \mathbf{1} \cdot e^{-\frac{2\pi i y v}{N}} = \\ &= \delta(v) \cdot \sum_{x=x_1}^{x_2} e^{-\frac{2\pi i x u}{N}} \end{aligned} \quad (11)$$

where we exploited the identity

$$\mathcal{F}(1) = \delta(v) \quad (12)$$

Since $\delta(v) = 1$ only where $v = 0$, $\mathcal{F}(F(x, y))$ will be non-zero only where $v = 0$. Hence, that is why F 's Fourier spectrum is concentrated in the left border. Similarly, the same can be derived for image G , but for transposed axes. As for H , since the Fourier transform possesses the property of linearity and H is a linear combination of F and G , its Fourier transform is the combination of those of F and G .

2.2 Question 8

As stated above, application of a logarithm on an image can reveal details in an image to the human eye. A logarithm transformation is an image enhancement technique used to compress the range of pixel values in an image so that bright regions are finely tuned, but darker ones are tuned coarsely so as for differences between pixels to be visible.

2.3 Question 9

TODO

2.4 Question 10

Images $F \cdot G$ and $\mathcal{F}(F \cdot G)$ are shown in figures 23 and 24 respectively.

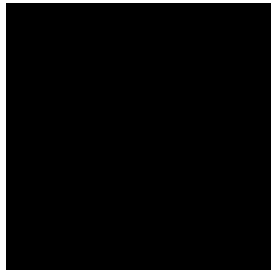


Figure 23: Image $F \cdot G$.

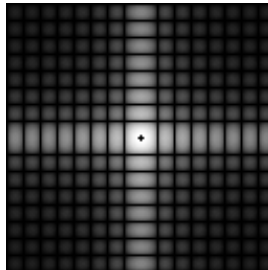


Figure 24: Image $\mathcal{F}(F \cdot G)$.

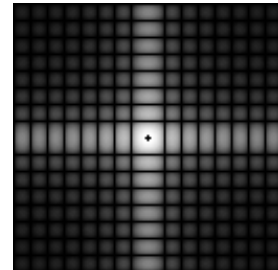


Figure 25: Image $\mathcal{F}(F) * \mathcal{F}(G)$.

Since the multiplication in either the spatial or the frequency domain is translated into a convolution in the other, the same result can be obtained by convolving the Fourier transforms of F and G . The result of this operation is illustrated in figure 25.