DD2423 - Lab I
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1 Section 1.3 - Basic Functions

1.1 Question 1

What we see here is that:

- The further the non-zero point (p, q) from the origin $O(0, 0)$, the smaller the wavelength of the spatial image (more dense lines in the real and imaginary part of the spatial image),
- The amplitude of all the spatial images is the same,
- The direction of the waveforms in the spatial images is dictated by the position of the non-zero point (p, q) relative to the origin $O(0, 0)$

The output of the `fftwave` function for $(p, q) = (5, 9)$, $(p, q) = (9, 5)$, $(p, q) = (17, 9)$, $(p, q) = (17, 121)$, $(p, q) = (5, 1)$ and $(p, q) = (125, 1)$ is illustrated in figures 1, 2, 3, 4, 5 and 6 respectively.

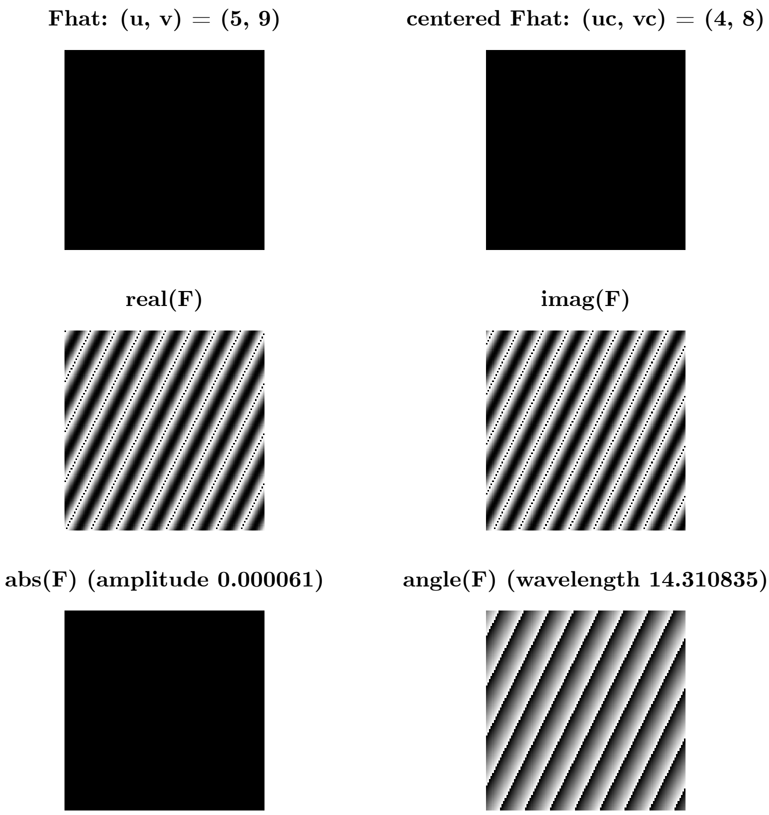

Figure 1: $(p, q) = (5, 9)$

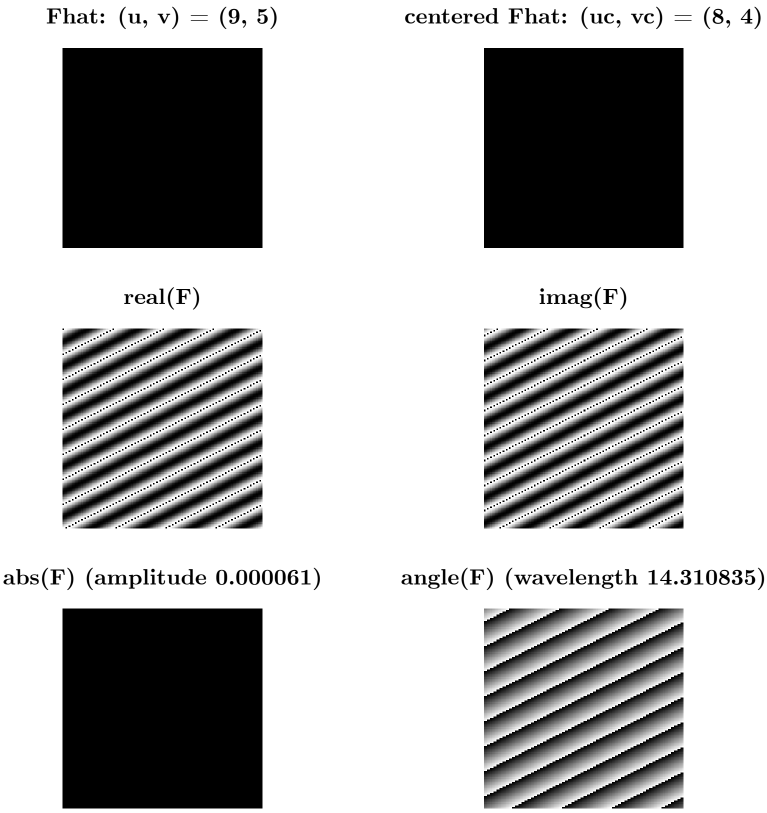

Figure 2: $(p, q) = (9, 5)$

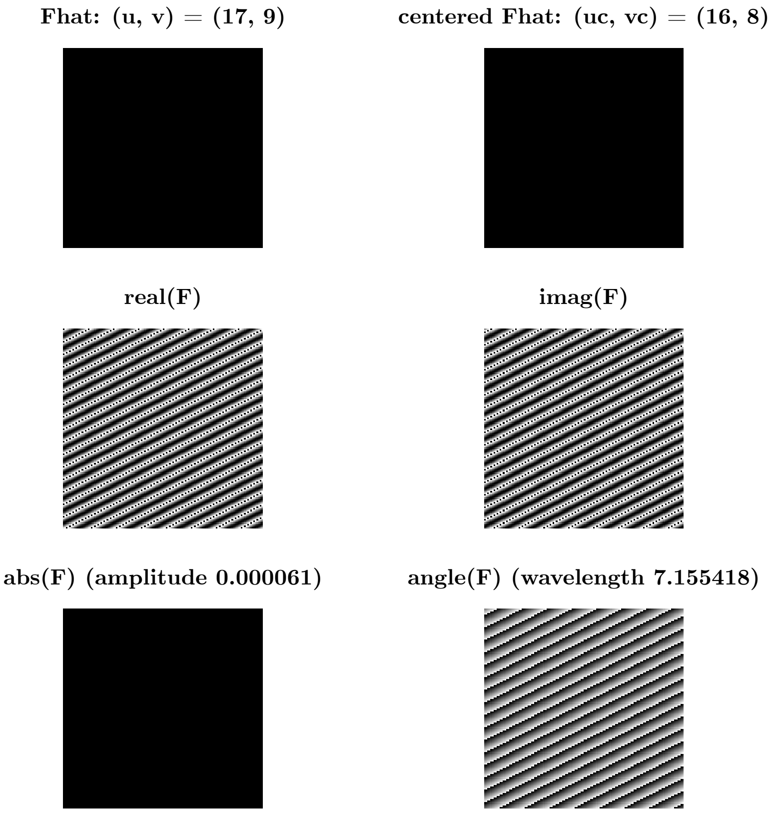

Figure 3: $(p, q) = (17, 9)$

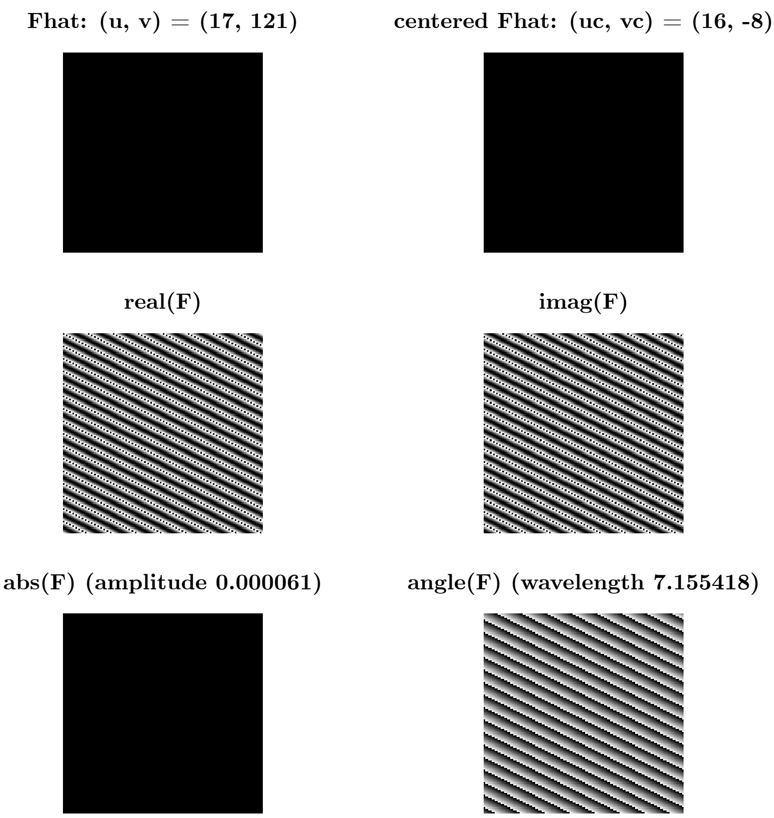

Figure 4: $(p, q) = (17, 121)$

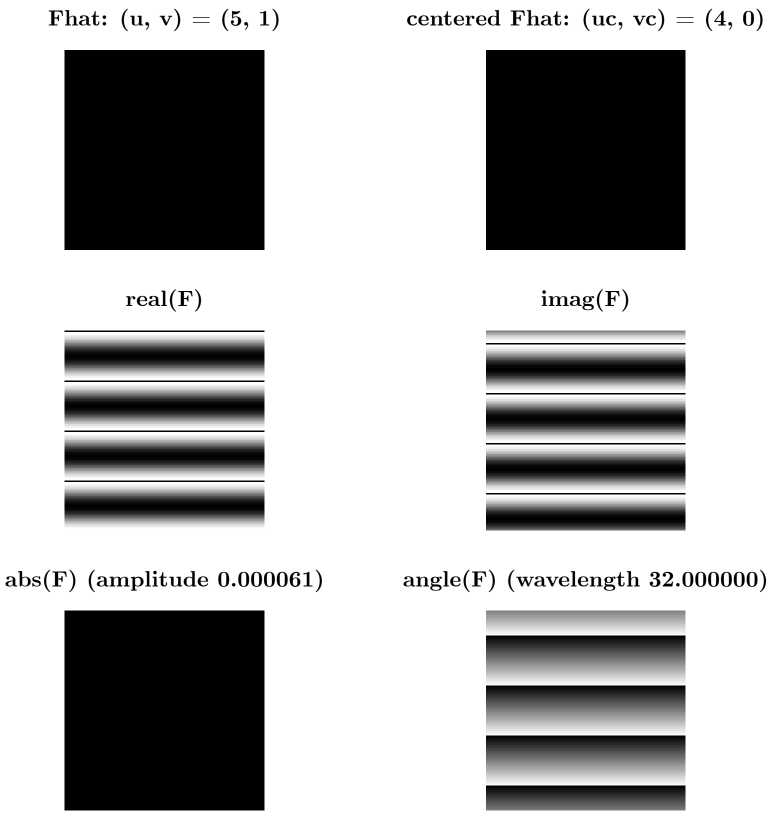

Figure 5: $(p, q) = (5, 1)$

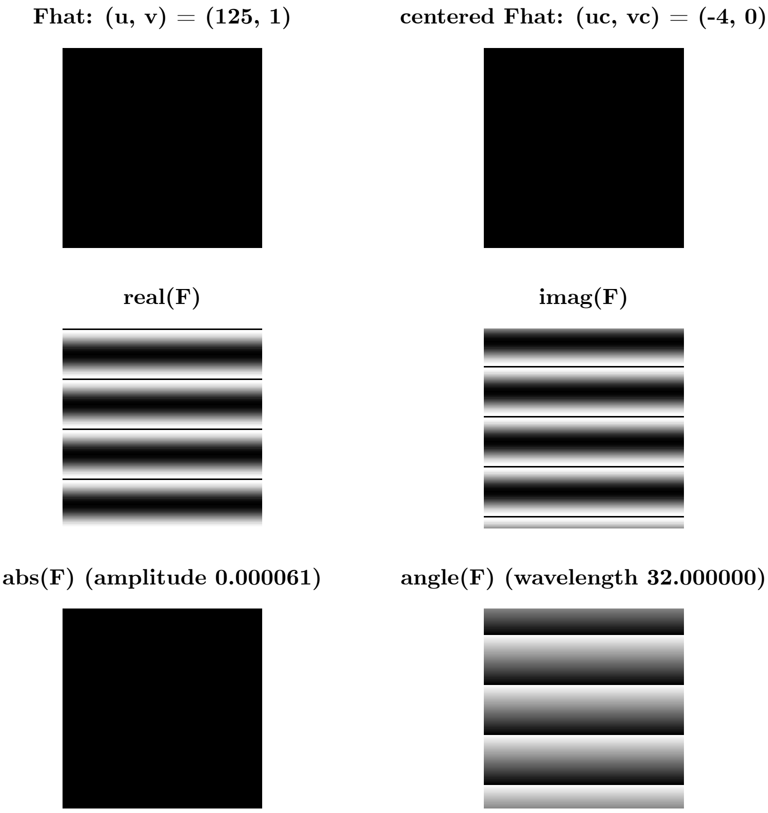

Figure 6: $(p, q) = (125, 1)$

1.2 Question 2

We exploit equation $4.2 - 33$ from [?]

$$\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} s(x, y) \cdot A\delta(x - x_0, y - y_0) = A \cdot s(x_0, y_0) \tag{1}$$

knowing that the output Fourier transform is a delta function at (p, q). Hence, for a quadratic image $M = N$ and in the spatial domain:

$$f(x, y) = \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} \delta(u-p, v-q) \cdot e^{\dfrac{2\pi i \cdot (xu + yv)}{N}} = \frac{1}{N^2} \cdot e^{\dfrac{2\pi i \cdot (px + qy)}{N}} \tag{2}$$

Hence,

$$f(x,y) = \frac{1}{N^2} \cdot (\cos(\frac{2\pi \cdot (px + qy)}{N}) + i \sin(\frac{2\pi \cdot (px + qy)}{N})) \tag{3}$$

The spatial image for $(p,q) = (5,9)$ is plotted in figure 7. The real part of equation 3 is plotted in figure 8.

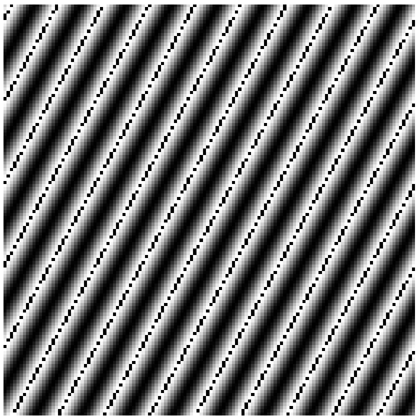

Figure 7: The inverse Fourier transform of an image for $(p,q) = (5,9)$.

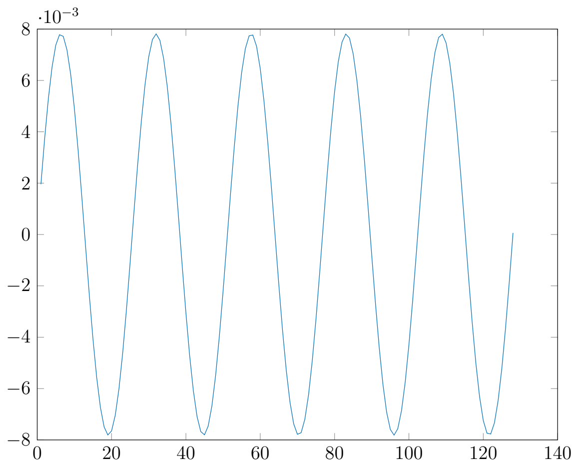

Figure 8: The real part of the spatial image above.

1.3 Question 3

As can be seen in equation 3, the amplitude of the waveform is

$$A = \frac{1}{N^2} \tag{4}$$

1.4 Question 4

As seen in the lecture notes,

$$\lambda = \frac{2\pi}{|\omega|} \tag{5}$$

and

$$\omega = [\frac{2\pi u}{N} \; \frac{2\pi v}{N}]^T \tag{6}$$

Hence, equation 5 for $(u, v) = (p, q)$ becomes

$$\lambda = \frac{1}{\sqrt{p^2 + q^2}} \tag{7}$$

The direction of travel of the waveforms in the spatial images is dictated by the position of the non-zero point (p, q) relative to the origin $O(0, 0)$. Figure 9 illustrates the dependence of the direction of a waveform on $p = \frac{\omega_l N}{2\pi}$ and $q = \frac{\omega_k N}{2\pi}$.

1.5 Question 5

For an quadratic image of size N, the highest number of cycles that can fit in it is $N/2$, that is, stripes of width of 1 pixel. The corresponding (maximum) frequency is $\omega_{max} = 2\pi/2 = \pi$. However,

$$\omega = \frac{2\pi u}{N} \leq \omega_{max} \tag{8}$$

Hence, when either p or q exceed the value of $N/2$, which in our case is $N/2 = 64$, the Nyquist frequency is exceeded and the corresponding waveform in the spatial domain is no longer a sinusoid. Figure 10 illustrates the waveform of the real part of a spatial image whose Fourier transform is an image with a non-zero pixel located at $(p, q) = (69, 120)$.

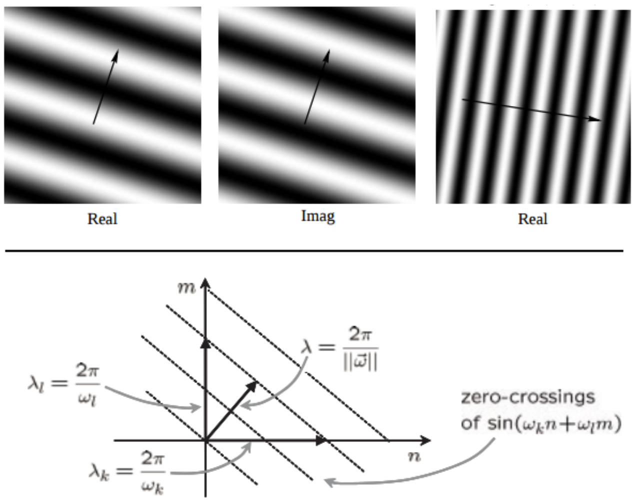

Figure 9: Image taken from Allan$_+$ Jepson's notes (`www.cs.toronto.edu/~jepson/csc320/notes/linearFilters2.pdf`)

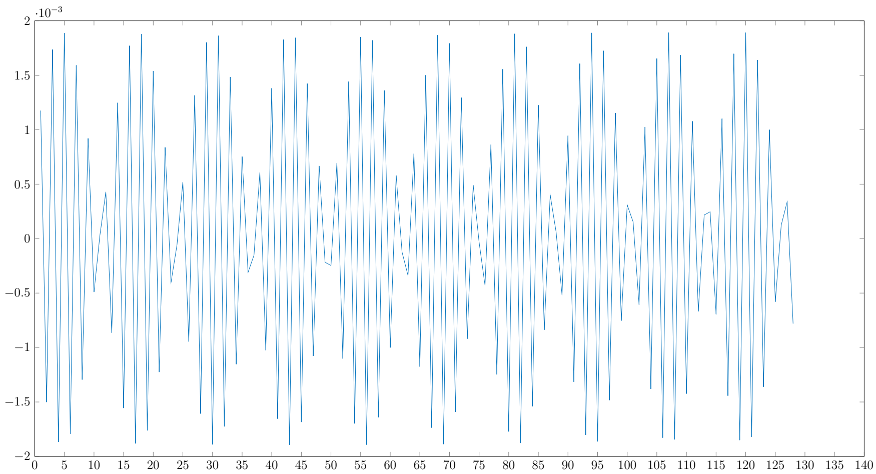

Figure 10: Example waveform in the spatial domain for $(p, q) = (69, 120)$

1.6 Question 6

The purpose of these lines is to visualize the mapping of the angular frequency values ω_x and ω_y inside the interval

$$-\pi \leq \omega_x, \omega_y \leq \pi \tag{9}$$

The exact operation is performed by `fftshift()` function.

2 Section 1.4 - Linearity

Figures 11, 12 and 13 show the spatial F, G and H images. Figures 14, 15 and 16 show their respective Fourier transforms, with the origin being in the upper left corner. Figures 17, 18 and 19 show the Fourier transforms of images F, G and H with the origin in the middle of the image.

Here, the `fftshift()` method is used to shift the origin $O(0,0)$ from the upper left corner to the middle of each image. The `log` command is used to make details in the images in the frequency domain visible. Figures 20, 21 and 22 illustrate the Fourier transforms of images F, G and H with the origin in the middle of the image without the use of the `log` command.

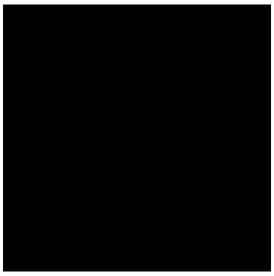

Figure 11: Image `F`

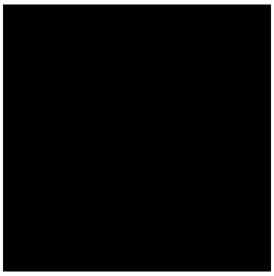

Figure 12: Image `G = F'`

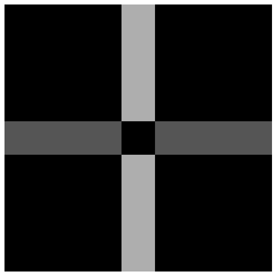

Figure 13: Image `H = F + 2 * G`

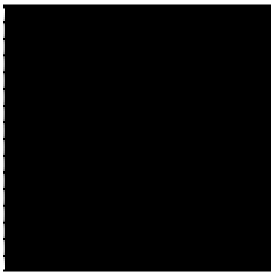

Figure 14: Image $\mathcal{F}(F)$. Origin at the upper left corner.

Figure 15: Image $\mathcal{F}(G)$. Origin at the upper left corner.

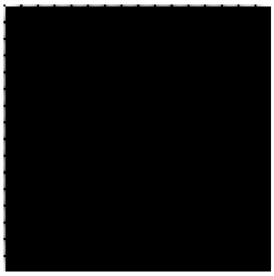

Figure 16: Image $\mathcal{F}(H)$. Origin at the upper left corner.

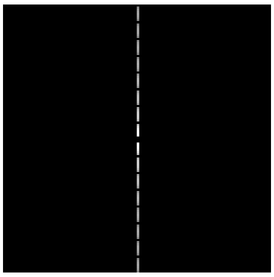

Figure 17: Image $\mathcal{F}(F)$. Origin at the middle.

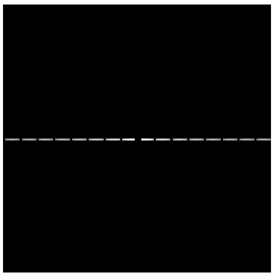

Figure 18: Image $\mathcal{F}(G)$. Origin at the middle.

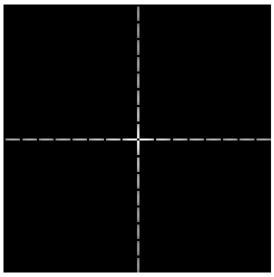

Figure 19: Image $\mathcal{F}(H)$. Origin at the middle.

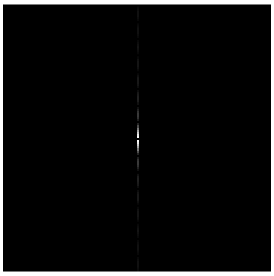

Figure 20: Image $\mathcal{F}(F)$. Origin at the middle. Illustration without the use of the `log` function.

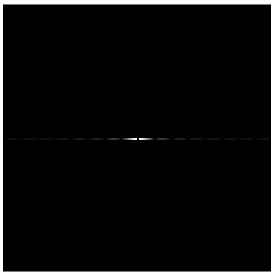

Figure 21: Image $\mathcal{F}(G)$. Origin at the middle. Illustration without the use of the `log` function.

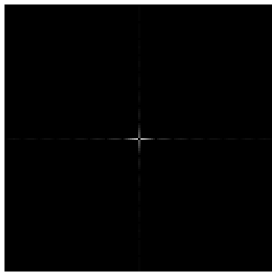

Figure 22: Image $\mathcal{F}(H)$. Origin at the middle. Illustration without the use of the `log` function.

2.1 Question 7

In essence, image F in the spatial domain is a two-dimensional box function:

$$F(x,y) = \begin{cases} 1, & x_1 \leq x \leq x_2 \\ 0, & everywhere\ else \end{cases} \tag{10}$$

Its Fourier transform is:

$$\mathcal{F}(F(x,y)) = \sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\cdot e^{-\dfrac{2\pi i(xu+yv)}{N}} =$$
$$\sum_{x=x_1}^{x_2} e^{-\dfrac{2\pi ixu}{N}} \sum_{y=0}^{N-1} e^{-\dfrac{2\pi iyv}{N}} = \sum_{x=x_1}^{x_2} e^{-\dfrac{2\pi ixu}{N}} \sum_{y=0}^{N-1} \mathbf{1}\cdot e^{-\dfrac{2\pi iyv}{N}} =$$
$$\delta(v)\cdot \sum_{x=x_1}^{x_2} \cdot e^{-\dfrac{2\pi ixu}{N}} \tag{11}$$

where we exploited the identity

$$\mathcal{F}(1) = \delta(v) \tag{12}$$

Since $\delta(v) = 1$ only where $v = 0$, $\mathcal{F}(F(x, y))$ will be non-zero only where $v = 0$. Hence, that is why F's Fourier spectrum is concentrated in the left border. Similarly, the same can be derived for image G, but for transposed axes. As for H, since the Fourier transform possesses the property of linearity and H is a linear combination of F and H, its Fourier transform is the combination of those of F and H.

2.2 Question 8

As stated above, application of a logarithm on an image can reveal details in an image to the human eye. A logarithm transformation is an image enhancement technique used to compress the range of pixel values in an image so that bright regions are finely tuned, but darker ones are tuned coarsely so as for differences between pixels to be visible.

2.3 Question 9

TODO

2.4 Question 10

Images `F .* G` and $\mathcal{F}$(`F .* G`) are shown in figures 23 and 24 respectively.

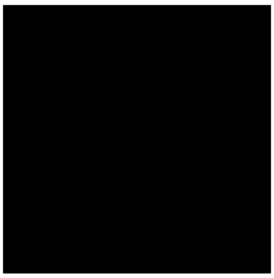

Figure 23: Image `F .* G`.

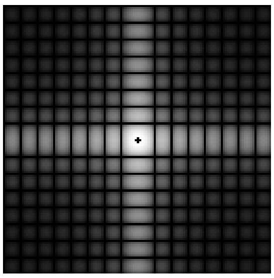

Figure 24: Image $\mathcal{F}$(`F .* G`).

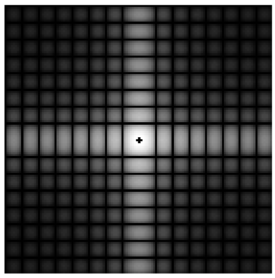

Figure 25: Image $\mathcal{F}(F) * \mathcal{F}(G)$.

Since the multiplication in either the spatial or the frequency domain is translated into a convolution in the other, the same result can be obtained by convolving the Fourier transforms of F and G. The result of this operation is illustrated in figure 25.

2.5 Question 11

Image F and its Fourier transform are illustrated in figures 26 and 27 respectively.

Figure 26: Image F

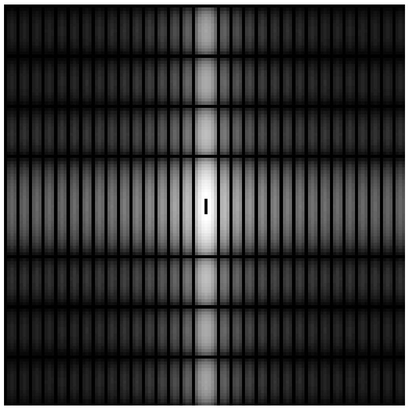

Figure 27: Image $\mathcal{F}(F)$.

Compared to figure 23, the height of the non-zero area in image 26 is cut in half, while its width is increased by a factor of 2. Comparing their respective Fourier transforms verifies the transform's scaling property: a compression in either the spatial or the frequency domain is expressed as an expansion in the other.

2.6 Question 12

Figure 28 illustrates the effect of rotation of the original image to the spectra of the various images. The orientation of each spectrum follows the rotation of each image, i.e. it is rotated by the same angle and towards the same direction. However, because of the rotation, each image loses its original smoothness, due to the limited resolution and the nature of the shape of the pixels. This has a direct effect on the Fourier transform of each image, as is clearly seen by the wave-like patterns in the spectrum of the images rotated by 30 and 60 degrees.

The rotation of the image can be seen as the rotation of a sum of pixels. If a pixel with coordinates $A(x, y)$ with respect to the center of the image is rotated by an angle θ, then its new coordinates can then be expressed as

$$A(x', y') = A(xcos\theta + ysin\theta, -xsin\theta + ycos\theta)$$

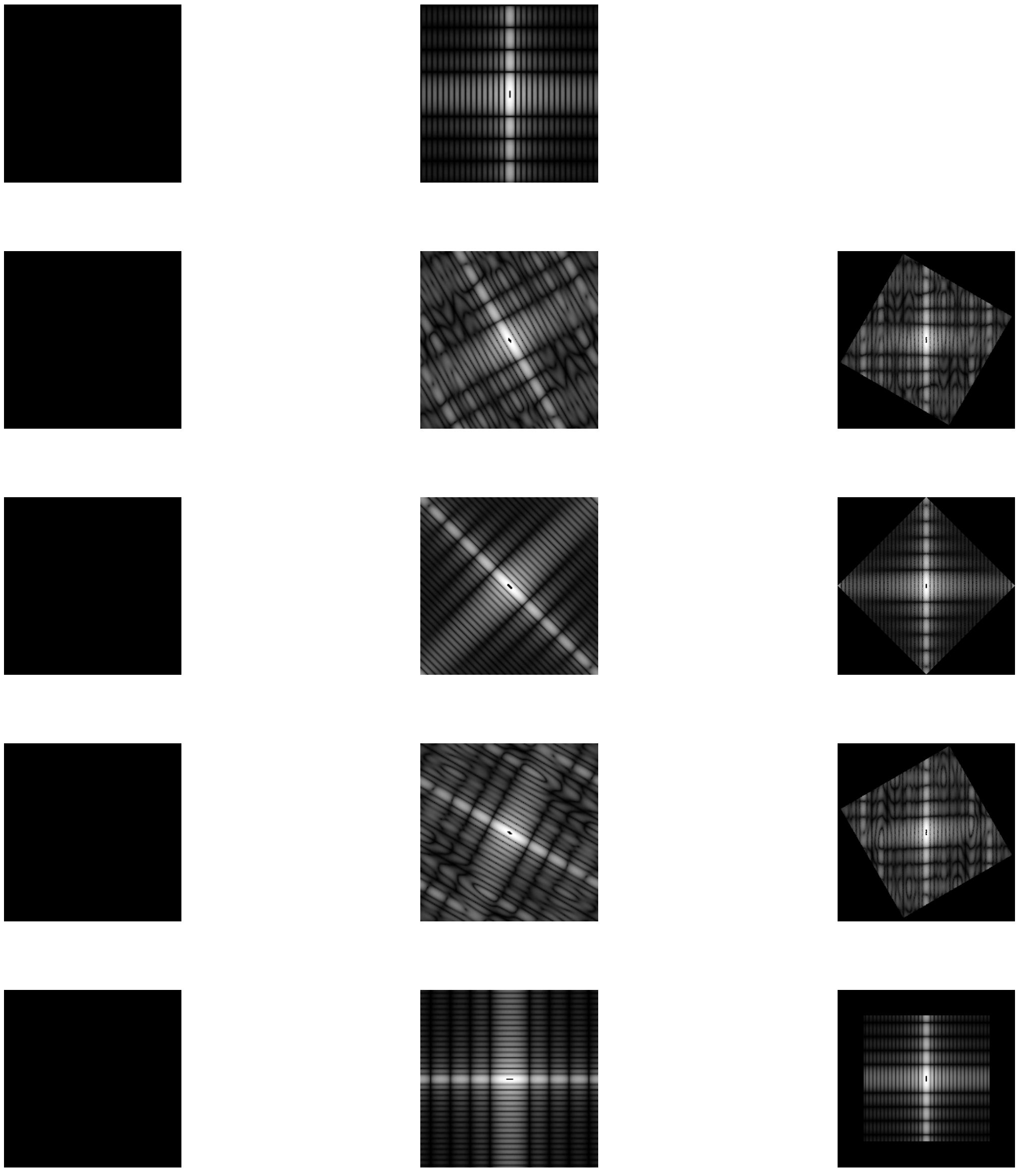

Figure 28: The first row depicts the original F image and its Fourier transform. The next images in the first column represent F rotated by a) 30, b) 45, c) 60 and d) 90 degrees respectively. The second column features their respective Fourier transforms. The third column features the Fourier spectra of the second column rotated, so as to match the orientation of the Fourier spectrum of image F.

Hence,

$$x = x'cos\theta - y'sin\theta$$

and

$$y = x'sin\theta + y'cos\theta$$

If we now take the Fourier transform of the rotated image $f(x', y')$, then

$$\mathcal{F}(f') = \sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x', y') \cdot e^{-\dfrac{2\pi i(xu + yv)}{N}} =$$

$$\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x', y') \cdot e^{-\dfrac{2\pi i((x'cos\theta - y'sin\theta)u + (x'sin\theta + y'cos\theta)v)}{N}} =$$

$$\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x', y') \cdot e^{-\dfrac{2\pi i(x'(ucos\theta + vsin\theta) + y'(vcos\theta - usin\theta))}{N}} =$$

$$\sum_{x'=0}^{N-1}\sum_{y'=0}^{N-1} f(x', y') \cdot e^{-\dfrac{2\pi i(x'(ucos\theta + vsin\theta) + y'(vcos\theta - usin\theta))}{N}} \tag{13}$$

which means that the rotation is propagated in the frequency domain:

$$u' = ucos\theta + vsin\theta$$

and

$$v' = vcos\theta - usin\theta$$