# 10 *Robot Dynamics*

We have seen that a path specifies the set of configurations a robot achieves as it moves from one configuration to another, and thus path planning (e.g., finding collision-free paths) is a kinematic/geometric problem. A path is not a complete description of the motion of a robot system, however, as the timing of the motion is not specified. A trajectory is a path plus a specification of the time at which each configuration is achieved. Trajectory planning is not only a geometric problem, but also a dynamic problem. Finding feasible trajectories of a system obeying dynamics requires knowledge of the masses and inertias of the system, actuator limits, and forces such as gravity and friction. Since we are now dealing with system dynamics, we can pose optimal control problems such as finding minimum-time or minimum-energy motions.

Since trajectory planning requires a full dynamic model of the robotic system, in section 10.1 we review the Lagrangian approach to deriving equations of motion for a mechanical system such as a robot arm. Section 10.2 explores the structure of the equations of motion and gives standard forms for writing them. In section 10.3 we consider systems subject to velocity constraints, such as those imposed when maintaining rolling and sliding contacts. Finally, section 10.4 studies the particular case of a rotating and translating rigid body.

## 10.1 Lagrangian Dynamics

The equations of motion for a mechanical system can be generated in a variety of ways. While all are equivalent (if correctly done!), the number of computations and the size of the resulting expressions may vary. In this chapter we use a Lagrangian formulation, which is based on the kinetic and potential energy of the system. Lagrange's equations provide a straightforward recipe, amenable to computer implementation (using, e.g., Mathematica or Maple), for calculating equations of motion for many robotic systems.

Let $q = [q_1, \ldots, q_{n_Q}]^T \in \mathbb{R}^{n_Q}$ be a vector of *generalized coordinates* representing the configuration of the system on the $n_Q$-dimensional configuration space, and let $u = [u_1, \ldots, u_{n_Q}]^T \in \mathbb{R}^{n_Q}$ be the vector of *generalized forces* acting on the generalized coordinates. For example, for a robot arm, the generalized coordinates would typically be the joint angles for revolute joints and the joint translations for prismatic joints, and the generalized forces would be torques about the joints and forces along the joints, respectively.

The *Lagrangian* $L$ of a mechanical system is written as the kinetic energy minus the potential energy

$$L(q, \dot{q}) = K(q, \dot{q}) - V(q),$$

where $K$ is the kinetic energy, a function of the configuration and the velocity, and $V$ is the potential energy, a function of the configuration only. The Lagrangian equations of motion, also known as the Euler-Lagrange equations, can be written

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = u_i, \quad i = 1 \ldots n_Q,$$

or simply

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = u. \tag{10.1}$$

A derivation of these equations can be found in many dynamics textbooks.

EXAMPLE 10.1.1 *Consider a planar body described by the generalized coordinates $q = [q_1, q_2, q_3]^T \in \mathbb{R}^2 \times S^1$, where $(q_1, q_2) \in \mathbb{R}^2$ specify the location of the center of mass of the body in the plane and $q_3 \in S^1$ specifies the orientation of the body (figure 10.1). The generalized forces $u = [u_1, u_2, u_3]^T \in \mathbb{R}^3$ are the linear forces $(u_1, u_2)$ through the center of mass and the torque $u_3$ about the center of mass. A gravitational acceleration $a_g \geq 0$ acts in the $-q_2$ direction, $[0, -a_g, 0]^T$. The mass of the planar body is $m$ and $I$ is the scalar inertia about an axis through the center of mass and out of the page.*

*The Lagrangian for this system is*

$$L = \frac{1}{2}m\left(\dot{q}_1^2 + \dot{q}_2^2\right) + \frac{1}{2}I\dot{q}_3^2 - m a_g q_2,$$

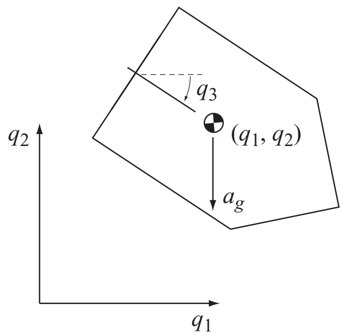


**Figure 10.1** A planar body.

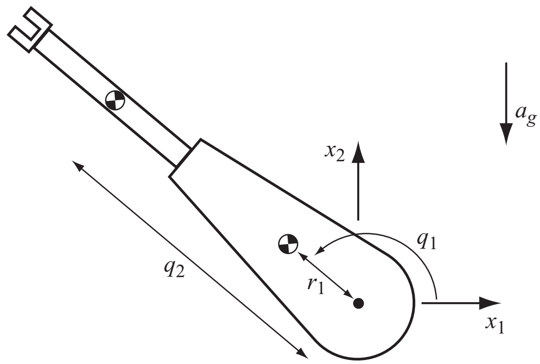


**Figure 10.2** The RP manipulator.

*i.e., the sum of the linear and angular kinetic energies minus the potential energy. Applying (10.1), we get*

$$u_1 = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} = \frac{d}{dt}(m\dot{q}_1) - 0 = m\ddot{q}_1 \tag{10.2}$$

$$u_2 = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} = \frac{d}{dt}(m\dot{q}_2) - ma_g = m\ddot{q}_2 - ma_g \tag{10.3}$$

$$u_3 = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_3} - \frac{\partial L}{\partial q_3} = \frac{d}{dt}(I\dot{q}_3) - 0 = I\ddot{q}_3. \tag{10.4}$$

EXAMPLE 10.1.2 *Figure 10.2 shows a robot arm consisting of one revolute joint and one prismatic (translational) joint. This type of robot is called an RP manipulator.*

*The configuration of the robot is* $[q_1, q_2]^T$*, where* $q_1$ *gives the angle of the first joint from a world frame* $x_1$*-axis, and* $q_2 > 0$ *gives the distance of the center of mass of the second link from the first joint. The center of mass of the first link is a distance* $r_1$ *from the first joint. The first link has mass* $m_1$ *and inertia* $I_1$ *about the center of mass, and the second link has mass* $m_2$ *with inertia* $I_2$ *about the center of mass. A gravitational acceleration* $a_g \geq 0$ *acts in the* $-x_2$ *direction of a world frame. To derive the Lagrangian for this RP arm, we will consider the two links' contributions independently.*

*The kinetic energy of the first link can be expressed as*

$$K_1(q, \dot{q}) = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}I_1\omega_1^2,$$

*where* $v_1$ *and* $\omega_1$ *are the linear velocity of the link center of mass and angular velocity of the link, respectively. We have*

$$v_1 = r_1\dot{q}_1$$
$$\omega_1 = \dot{q}_1$$

*yielding the expression*

$$K_1(q, \dot{q}) = \frac{1}{2}m_1 r_1^2\dot{q}_1^2 + \frac{1}{2}I_1\dot{q}_1^2.$$

*The potential energy of the first link is*

$$V_1(q) = m_1 a_g r_1 \sin q_1.$$

*The kinetic energy of the second link can be expressed as*

$$K_2(q, \dot{q}) = \frac{1}{2}m_2 v_2^2 + \frac{1}{2}I_2\omega_2^2,$$

*where* $v_2$ *and* $\omega_2$ *are the linear velocity of the link center of mass and angular velocity of the link, respectively. We have*

$$v_2 = \sqrt{\dot{q}_2^2 + (q_2\dot{q}_1)^2}$$
$$\omega_2 = \dot{q}_1$$

*yielding*

$$K_2(q, \dot{q}) = \frac{1}{2}m_2\left(\dot{q}_2^2 + (q_2\dot{q}_1)^2\right) + \frac{1}{2}I_2\dot{q}_1^2.$$

*The potential energy of the second link is*

$$V_2(q) = m_2 a_g q_2 \sin q_1.$$

*The Lagrangian for the system is* $L = K_1 + K_2 - V_1 - V_2$*:*

$$L = \frac{1}{2}\left(\left(I_1 + I_2 + m_1 r_1^2 + m_2 q_2^2\right)\dot{q}_1^2 + m_2 \dot{q}_2^2\right) - a_g \sin q_1 (m_1 r_1 + m_2 q_2).$$

*Applying Lagrange's equations, we get*

$$\begin{aligned} u_1 = {} & \left(I_1 + I_2 + m_1 r_1^2 + m_2 q_2^2\right)\ddot{q}_1 + 2 m_2 q_2 \dot{q}_1 \dot{q}_2 \\ & + a_g (m_1 r_1 + m_2 q_2) \cos q_1 \end{aligned} \tag{10.5}$$

$$u_2 = m_2 \ddot{q}_2 - m_2 q_2 \dot{q}_1^2 + a_g m_2 \sin q_1. \tag{10.6}$$

## 10.2 Standard Forms for Dynamics

Let's take a closer look at equations (10.5) and (10.6), since they display much of the interesting structure of many second-order mechanical systems. On the right-hand side of each equation, there is a term depending on the second derivatives of the configuration variables, a term quadratic in the first derivatives of the configuration variables, and a term depending only on the configuration variables. These terms can be collected to write the dynamics in the following standard form:

$$u = M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q), \tag{10.7}$$

where $C(q, \dot{q})\dot{q} \in \mathbb{R}^{n_Q}$ is a vector of velocity product terms with the $n_Q \times n_Q$ matrix $C(q, \dot{q})$ linear in $\dot{q}$, $g(q) \in \mathbb{R}^{n_Q}$ is a vector of gravitational forces, and $M(q)$ is an $n_Q \times n_Q$ symmetric, positive definite *mass* or *inertia matrix*. A matrix $M$ is symmetric if $M_{ij} = M_{ji}$, where $M_{ij}$ is the entry in the $i$th row and $j$th column of $M$. A matrix $M$ is positive definite if $v^T M v > 0$ holds for any nonzero vector $v$. This is true if the determinant and trace (the sum of diagonal elements) of $M$ are positive.

Equations (10.5) and (10.6) for the RP manipulator can be written in the standard form of equation (10.7), where

$$M(q) = \begin{bmatrix} I_1 + I_2 + m_1 r_1^2 + m_2 q_2^2 & 0 \\ 0 & m_2 \end{bmatrix},$$

$$C(q, \dot{q})\dot{q} = \begin{bmatrix} 2 m_2 q_2 \dot{q}_1 \dot{q}_2 \\ -m_2 q_2 \dot{q}_1^2 \end{bmatrix}, \quad g(q) = \begin{bmatrix} a_g (m_1 r_1 + m_2 q_2) \cos q_1 \\ a_g m_2 \sin q_1 \end{bmatrix}.$$

The standard form (10.7) is compact, but the term $C(q, \dot{q})\dot{q}$ masks the fact that these velocity product terms can be derived from the inertia matrix $M(q)$. If we consider the individual components of this vector, $C(q, \dot{q})\dot{q} = [c_1(q, \dot{q}), \ldots, c_{n_Q}(q, \dot{q})]^T$, we find that

$$c_i(q,\dot{q}) = \sum_{j=1}^{n_Q} \left( \sum_{k=1}^{n_Q} \Gamma^i_{jk}(q)\dot{q}_j\dot{q}_k \right) \tag{10.8}$$

where

$$\Gamma^i_{jk}(q) = \frac{1}{2}\left( \frac{\partial M_{ij}(q)}{\partial q_k} + \frac{\partial M_{ik}(q)}{\partial q_j} - \frac{\partial M_{kj}(q)}{\partial q_i} \right). \tag{10.9}$$

The $n_Q^3$ scalars $\Gamma^i_{jk}(q)$ are known as the *Christoffel symbols* of the inertia matrix $M(q)$. In equation (10.8), squared velocity terms (where $j = k$) are known as *centrifugal* terms, and velocity product terms where $j \neq k$ are known as *Coriolis* terms. For example, the centrifugal term $-m_2 q_2 \dot{q}_1^2$ in the RP arm of example 10.1.2 indicates that the linear actuator at the prismatic joint must apply a force to keep the joint stationary as the revolute joint rotates. The Coriolis term $2m_2 q_2 \dot{q}_1 \dot{q}_2$ indicates that the actuator at the revolute joint must apply a torque for the two joints to move at constant velocities. This is because the inertia of the robot about the first joint is changing as the second joint extends or retracts, so the angular momentum is also changing, implying a torque at the first joint.

Although there are many ways to write the *Coriolis matrix* $C(q,\dot{q})$ as a function of the Christoffel symbols, one common choice is

$$C_{ij}(q,\dot{q}) = \sum_{k=1}^{n_Q} \Gamma^k_{ij}(q)\dot{q}_k.$$

Velocity product terms arise due to the noninertial reference frames implicit in the generalized coordinates $q$. The unforced motions (when $u - g(q) = 0$) are not "straight lines" in this choice of coordinates, and the Christoffel symbols carry geometric information on how unforced motions "bend" in this choice of coordinates. For example, if we represent the configuration of a point mass $m$ in the plane by standard Cartesian $(x, y)$ coordinates, unforced motions are straight lines in these inertial coordinates, and the Christoffel symbols are zero. If we represent the configuration by polar coordinates $[q_1, q_2]^T = [r, \theta]^T$, however, unforced motions are not straight lines in this choice of coordinates, and we find $\Gamma^1_{22} = -mq_1$, $\Gamma^2_{12} = \Gamma^2_{21} = mq_1$ (see problem 1). The geometry of the dynamics of mechanical systems is discussed further in chapter 12.

The main point is that the equations of motion (10.7) depend on the choice of coordinates $q$. For this reason, neither $M(q)\ddot{q}$ nor $C(q,\dot{q})\dot{q}$ *individually* should be thought of as a generalized force; only their sum is a force.

When we wish to emphasize the dependence of the velocity product terms on the Christoffel symbols, which in turn are determined by the inertia matrix, we write

$$u = M(q)\ddot{q} + \begin{bmatrix} \dot{q}^T \Gamma^1(q)\dot{q} \\ \vdots \\ \dot{q}^T \Gamma^{n_Q}(q)\dot{q} \end{bmatrix} + g(q),$$

where $\Gamma^i(q)$ is the $n_Q \times n_Q$ symmetric matrix with elements $\Gamma^i_{jk}(q)$, $j, k = 1 \dots n_Q$. We write this more compactly as

$$u = M(q)\ddot{q} + \dot{q}^T \Gamma(q)\dot{q} + g(q). \tag{10.10}$$

Conceptually, $\Gamma(q) \in \mathbb{R}^{n_Q \times n_Q \times n_Q}$ can be viewed as an $n_Q$-dimensional column vector, where each element of the "vector" is a matrix $\Gamma^i(q)$, as shown in the following example.

EXAMPLE 10.2.1 *For the RP arm of Example 10.1.2, there are $n_Q^3 = 2^3 = 8$ Christoffel symbols. The only nonzero Christoffel symbols are $\Gamma^1_{12} = \Gamma^1_{21} = m_2 q_2$ and $\Gamma^2_{11} = -m_2 q_2$. The Coriolis and centrifugal terms $\dot{q}^T \Gamma(q)\dot{q}$ can be calculated as follows:*

$$\begin{aligned} \dot{q}^T \Gamma(q)\dot{q} &= [\dot{q}_1\ \dot{q}_2] \begin{bmatrix} \Gamma^1(q) \\ \Gamma^2(q) \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \\ &= [\dot{q}_1\ \dot{q}_2] \begin{bmatrix} \begin{bmatrix} 0 & m_2 q_2 \\ m_2 q_2 & 0 \end{bmatrix} \\ \begin{bmatrix} -m_2 q_2 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \\ &= \begin{bmatrix} 2 m_2 q_2 \dot{q}_1 \dot{q}_2 \\ -m_2 q_2 \dot{q}_1^2 \end{bmatrix}. \end{aligned}$$

The dynamics described by equation (10.7) are specific to mechanical systems where the actuators act directly on the generalized coordinates. For example, a robot arm typically has an actuator at each joint. A more general form of the dynamics of second-order mechanical systems is

$$T(q)f = M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q), \tag{10.11}$$

where $f$ are the actuator forces and the $n_Q \times n_Q$ matrix $T(q)$ specifies how the actuators act on the generalized coordinates, as a function of the system configuration.

As an example, consider replacing the motors at the joints of our two-joint RP arm of example 10.1.2 with two thrusters attached to the center of mass of the second link. The location of the center of mass of the second link in the world frame is

$x = [x_1, x_2]^T$, and the thrusters provide a force $f = [f_1, f_2]^T$ expressed in the world frame. To use the dynamic equations we have already derived, we would like to express the generalized forces $u$ at the joints as a function of $f$. To do this, let $\phi$ be the forward kinematics (see section 3.8) mapping from $q$ to $x$,

$$x = \phi(q) = [q_2 \cos q_1, q_2 \sin q_1]^T.$$

The velocities are given by

$$\dot{x} = \frac{\partial \phi}{\partial q}\dot{q} = J(q)\dot{q},$$

where $J(q)$ is the manipulator Jacobian at the center of mass at the second link. Then by the analysis in section 4.7, the generalized forces $u$ and $f$ are related by

$$\begin{aligned} u &= T(q)f = J^T(q)f \\ \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} &= \begin{bmatrix} -q_2 \sin q_1 & q_2 \cos q_1 \\ \cos q_1 & \sin q_1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}. \end{aligned} \tag{10.12}$$

In other words, $T(q)$ is simply the transpose of the manipulator Jacobian.

If $T(q)$ is rank $n_Q$, dynamics of the form of equation (10.11) can be put in the form of equation (10.7) by defining "virtual" actuators $u = T(q)f$, and transforming any actuator limits on $f$ to limits on $u$. This is sometimes called a *feedback transformation* since the transformation from $f$ to $u$ depends on $q$.

Finally, mechanical systems are often subject to dissipative forces such as dry Coulomb friction or viscous damping. These can be treated as external forces to be added after deriving the equations of motion using Lagrange's equations. There are many possible models of friction and damping, but in most cases these forces are a function of $\dot{q}$ and possibly $q$, so we write

$$u = M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) + b(q, \dot{q}).$$

## Inertia Matrix

As we have seen, the inertia matrix $M(q)$ determines the equations of motion, except for gravitational and dissipative forces. Another way to see this is by observing that the kinetic energy of a mechanical system is determined by its inertia matrix, and can be written

$$K(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q}. \tag{10.13}$$

The fact that $M(q)$ is positive definite implies that the kinetic energy is positive for any nonzero $\dot{q}$.

Equation (10.13) shows how the kinetic energy depends on the inertia matrix. We can also derive the inertia matrix from the kinetic energy,

$$M_{ij} = \frac{\partial^2 K(q, \dot{q})}{\partial \dot{q}_i \partial \dot{q}_j}. \tag{10.14}$$

In some cases, such as the planar body of example 10.1.1, the inertia matrix can be written independent of the configuration $q$, and the Christoffel symbols are zero. This means that the dynamics are invariant to the configuration—they "look" the same from any configuration. For the planar body, the inertia matrix is

$$M(q) = M = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I \end{bmatrix}.$$

For some robots, such as a mobile manipulator consisting of a robot arm mounted on a cart, the dynamics are invariant to some configuration variables (such as the cart's position and orientation on the floor) but not others (such as the arm's configuration).

## 10.3 Velocity Constraints

Suppose that the mechanical system is subject to a set of $k$ linearly independent constraints linear in velocity, i.e., of the form

$$A(q)\dot{q} = 0, \tag{10.15}$$

where $A(q)$ is a $k \times n_Q$ matrix, and the $k$ row vectors of $A(q)$ are written $a_j(q)$, $j = 1 \ldots k$. Such constraints are called *Pfaffian* constraints. One source of Pfaffian constraints is rolling without slipping, such as in a wheeled mobile robot; a sliding constraint of this form is given in Example 10.3.1.

Since the constraints of equation (10.15) are satisfied throughout motion, we can differentiate the left-hand side and set it equal to zero:

$$A(q)\ddot{q} + \dot{A}(q)\dot{q} = 0.$$

The constrained Lagrange's equations can then be written

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = u + A^T(q)\lambda \tag{10.16}$$

$$A(q)\dot{q} = \dot{A}(q)\dot{q} + A(q)\ddot{q} = 0, \tag{10.17}$$

where $\lambda = [\lambda_1, \ldots, \lambda_k]^T$ are the *Lagrange multipliers*. The generalized force $\lambda_j a_j(q)$ is applied by constraint $j$ to maintain the constraint. The constrained Lagrange's equations yield $n_Q + k$ equations to be solved for the $n_Q + k$ variables $\ddot{q}$ and $\lambda$.

If we are not interested in calculating the $k$ constraint forces, we can use equation (10.17) to eliminate $\lambda$ from equation (10.16). Solving equation (10.16) for $\ddot{q}$ and plugging into equation (10.17), dropping the dependence of $M$, $A$, and $g$ on $q$ and $C$ on $q, \dot{q}$, we get

$$\dot{A}\dot{q} + AM^{-1}(u + A^T\lambda - C\dot{q} - g) = 0.$$

Now solving for $\lambda$, we get

$$\lambda = (AM^{-1}A^T)^{-1}(-\dot{A}\dot{q} + AM^{-1}(C\dot{q} + g - u)).$$

Recognizing that $-\dot{A}\dot{q} = A\ddot{q}$, plugging back into equation (10.16), and manipulating, we get

$$(\mathcal{I} - A^T(AM^{-1}A^T)^{-1}AM^{-1})(M\ddot{q} + C\dot{q} + g - u) = 0,$$

where $\mathcal{I}$ is the identity matrix. If we define

$$P_u = \mathcal{I} - A^T(AM^{-1}A^T)^{-1}AM^{-1},$$

then we get the form

$$P_u(M\ddot{q} + C\dot{q} + g) = P_u u. \tag{10.18}$$

The $n_Q \times n_Q$ matrix $P_u$ is only rank $n_Q - k$, so we cannot premultiply both sides of equation (10.18) by $P_u^{-1}$; if we could, we would be left with the unconstrained dynamics. The projection matrix $P_u$ projects generalized forces to the components that do work on the system. The remaining forces, defined by the projection $(\mathcal{I} - P_u)$, are the components resisted by the constraints. These two sets of forces are orthogonal to each other with respect to the inertia matrix. In other words,

$$(P_u u)^T M^{-1}(\mathcal{I} - P_u)u = 0$$

for any $u$.

Defining the matrix

$$P = M^{-1}P_u M = \mathcal{I} - M^{-1}A^T(AM^{-1}A^T)^{-1}A \tag{10.19}$$

and rearranging equation (10.18), we get the equivalent form

$$P\ddot{q} = PM^{-1}(u - C\dot{q} - g). \tag{10.20}$$

Here the rank $n_Q - k$ matrix $P$ projects general motions to motions satisfying the constraints of equation (10.17). The remaining motions, defined by the projection $(\mathcal{I} - P)$, are the components in the constrained directions. The projections $P$ and $(\mathcal{I} - P)$ are orthogonal by the inertia matrix, i.e.,

$$(P\dot{q})^T M(\mathcal{I} - P)\dot{q} = 0$$

for any $\dot{q}$.

In the discussion above, we used the notion of orthogonality *by the inertia matrix*. This notion of orthogonality is the appropriate one when discussing dynamics; the inertia matrix captures the metric properties of the coordinates used in describing the system, which may mix linear and angular coordinates. This is in contrast to our usual notion of orthogonality, which says two vectors $v_1$ and $v_2$ are orthogonal if $v_1^T \mathcal{I} v_2 = v_1^T v_2 = 0$. The identity matrix $\mathcal{I}$ indicates that space looks the same in every direction. Further discussion of the geometry of mechanical systems is deferred to chapter 12, and further discussion of the projections $P$ and $P_u$ can be found in [60, 293, 308].

From equation (10.19) we know that $MP = P_u M$, so the matrices $P$ and $P_u$ satisfy the identity

$$P = P_u^T.$$

For this reason, we will only refer to the matrices $P$ and $P^T$.

EXAMPLE 10.3.1 *A knife-edge can slide on a horizontal plane in the direction it is pointing or spin about an axis through the contact point and orthogonal to the plane, but it cannot slide perpendicular to its heading direction. Let $q = [q_1, q_2, q_3]^T$ represent the configuration of the knife-edge, where $(q_1, q_2)$ denotes the contact point on the plane and $q_3$ denotes the heading direction (figure 10.3). If the mass of the knife is $m$ and its inertia is $I$ about the axis of rotation, the Lagrangian is the kinetic energy*

$$L = \frac{1}{2} m \left(\dot{q}_1^2 + \dot{q}_2^2\right) + \frac{1}{2} I \dot{q}_3^2.$$

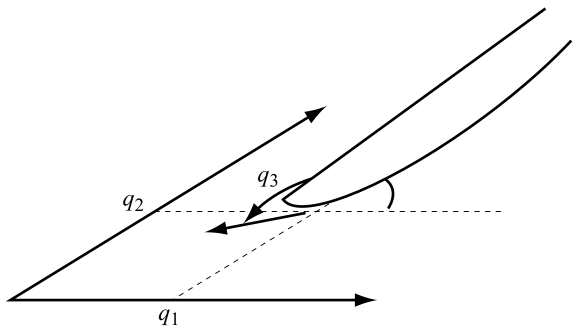


**Figure 10.3** A knife-edge on a plane.

*The single constraint can be written*

$$\dot{q}_1 \sin q_3 - \dot{q}_2 \cos q_3 = 0,$$

*or* $A(q) = a_1(q) = [\sin q_3, -\cos q_3, 0]$. *Differentiating this constraint, we get*

$$\ddot{q}_1 \sin q_3 + \dot{q}_1\dot{q}_3 \cos q_3 - \ddot{q}_2 \cos q_3 + \dot{q}_2\dot{q}_3 \sin q_3 = 0. \tag{10.21}$$

*Applying Lagrange's equations, we get*

$$m\ddot{q}_1 = u_1 + \lambda_1 \sin q_3 \tag{10.22}$$

$$m\ddot{q}_2 = u_2 - \lambda_1 \cos q_3 \tag{10.23}$$

$$I\ddot{q}_3 = u_3. \tag{10.24}$$

*Solving equations (10.21) through (10.24), we get*

$$\ddot{q}_1 = \frac{1}{m}\left(u_1 \cos^2 q_3 + (u_2 - m\dot{q}_1\dot{q}_3) \cos q_3 \sin q_3 - m\dot{q}_2\dot{q}_3 \sin^2 q_3\right) \tag{10.25}$$

$$\ddot{q}_2 = \frac{1}{m}\left(u_2 \sin^2 q_3 + (u_1 + m\dot{q}_2\dot{q}_3) \cos q_3 \sin q_3 + m\dot{q}_1\dot{q}_3 \cos^2 q_3\right) \tag{10.26}$$

$$\ddot{q}_3 = \frac{u_3}{I} \tag{10.27}$$

$$\lambda_1 = (u_2 - m\dot{q}_1\dot{q}_3) \cos q_3 - (u_1 + m\dot{q}_2\dot{q}_3) \sin q_3, \tag{10.28}$$

*where* $(u_1, u_2)$ *are the forces along the* $(q_1, q_2)$*-directions, and* $u_3$ *is the torque about an axis through the contact point and orthogonal to the plane.*

*Alternatively, we could study the projected dynamics. The inertia matrix* $M(q)$ *is*

$$M(q) = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I \end{bmatrix}$$

*and we calculate the projection matrix using (10.19):*

$$P = \begin{bmatrix} \cos^2 q_3 & \sin q_3 \cos q_3 & 0 \\ \sin q_3 \cos q_3 & \sin^2 q_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

*Note that the first two rows (and columns) of* $P$ *are linearly dependent, so* $\operatorname{rank}(P) = n_Q - k = 2$. *In this example,* $C(q, \dot{q})\dot{q}$ *and* $g(q)$ *are zero, so the projected dynamics (10.20) become*

$$P\ddot{q} = PM^{-1}u,$$

*which yields the two independent equations of motion in the unconstrained directions*

$$\ddot{q}_1 \cos q_3 + \ddot{q}_2 \sin q_3 = \frac{1}{m}(u_1 \cos q_3 + u_2 \sin q_3)$$
$$\ddot{q}_3 = \frac{u_3}{I}.$$

*The constraint equation (10.21) completes the system of three equations, and we solve to get equations (10.25), (10.26), and (10.27) above.*

As the previous example shows, even for very simple systems with constraints, the expressions can quickly become unwieldy. When possible, it may be preferable to choose a reduced set of generalized coordinates to eliminate the constraints. For example, if $a_j(q) = \frac{\partial c(q)}{\partial q}$ for some function $c(q) = 0$, then the velocity constraint is actually the time derivative of a configuration constraint $c(q) = 0$, which reduces the dimension of the configuration space by one. Therefore, it is possible to reduce the number of generalized coordinates by one and eliminate the constraint. As an example, the planar motion of a point on a circle centered at the origin can be represented using coordinates $(x, y)$ and the velocity constraint $x\dot{x} + y\dot{y} = 0$. This velocity constraint can be integrated to the configuration constraint $x^2 + y^2 = R$, however, where the radius $R$ is defined by the initial position of the point. This configuration constraint allows an unconstrained description of the configuration using a single angle coordinate $\theta$. When a velocity constraint can be integrated to a configuration constraint, as in this case, the constraint is called *holonomic*. *Nonholonomic* constraints are velocity constraints that cannot be integrated to a configuration constraint. Motion planning with nonholonomic constraints is left to chapter 12.

## 10.4 Dynamics of a Rigid Body

Until now, we have been using freely the concepts of "center of mass" and "inertia about an axis" to get to the use of Lagrange's equations as quickly as possible. We now provide formal definitions of these quantities and apply them to the dynamics of a translating and rotating rigid body.

Let $\mathcal{B}$ be a rigid body occupying a volume $V \subset \mathbb{R}^3$, $r$ be a vector from the origin to a point in $\mathcal{B}$, and $\rho(r)$ be the mass density of $\mathcal{B}$ as a function of the location $r$. Then the mass of $\mathcal{B}$ is the volume integral of the mass density

$$m = \int_V \rho(r) dV,$$

and the *center of mass* is the weighted average of the mass density

$$r_{\mathrm{cm}} = \frac{1}{m}\int_V r\rho(r)dV.$$

When a body moves freely in space, it is convenient to describe the translational position of the body by the Cartesian coordinates $q$ of the center of mass relative to a stationary inertial frame. The translational kinetic energy of the body can be written $K = \frac{1}{2}\dot{q}^T m\dot{q} = \frac{1}{2}m\|\dot{q}\|^2$. Applying Lagrange's equations yields the familiar equation

$$f = m\ddot{q}, \tag{10.29}$$

where $f$ is the linear force applied to the body expressed in the inertial frame.

### 10.4.1 Planar Rotation

When a body moves in a plane, a single configuration variable $q$ can be used to describe its orientation. Such motion occurs, for example, when the body rotates about a fixed axis, or when the body slides freely on a frictionless plane. In the former case, it is convenient to to define a stationary $x$-$y$-$z$ inertial frame with the $z$-axis along the axis of rotation. In the latter case, it will be convenient to define an $x$-$y$-$z$ inertial frame at the center of mass of the body, with the $z$-axis orthogonal to the plane of motion. (Since we are focusing on rotational motion only, the center of mass can be assumed stationary.)

The kinetic energy of a body rotating in the plane is the integral over the body of the differential kinetic energy at each point $r = (x, y, z)^T$:

$$K = \int_V \frac{1}{2}\rho(r)v^2(r)dV,$$

where $q$ is the angle of the body, $\dot{q}$ is the angular velocity, and $v(r) = \dot{q}\sqrt{x^2 + y^2}$ is the linear velocity at $r$. Therefore we can write the kinetic energy in the form of equation (10.13),

$$K = \frac{1}{2}\dot{q}^2 \int_V \rho(r)(x^2 + y^2)dV = \frac{1}{2}\dot{q}^T I_{zz}\dot{q}, \tag{10.30}$$

where

$$I_{zz} = \int_V \rho(r)(x^2 + y^2)dV \tag{10.31}$$

is the inertia of the body about the $z$-axis. If the body is uniform density, equation (10.31) simplifies to

$$I_{zz} = m \int_V (x^2 + y^2) dV,$$

where $m$ is the mass of the body. Applying Lagrange's equations to equation (10.30), we get

$$\tau_z = I_{zz}\ddot{q}, \tag{10.32}$$

where $\tau_z$ is the torque about the $z$-axis.

If we choose a $z$-axis through the center of mass of the body and a parallel $z'$-axis a distance $d$ away, then the scalar inertias $I_{zz}$ and $I_{z'z'}$ are related by the *parallel-axis theorem* for planar rotation:

$$I_{z'z'} = I_{zz} + md^2 \tag{10.33}$$

The proof of this theorem is straightforward and in fact is implicit in our derivation of the equations of motion of the RP arm.

### 10.4.2 Spatial Rotation

This section requires extra mathematical machinery, and can be safely skipped if the reader is not interested in the dynamics of a rotating spatial body.

In our Lagrangian formulation, we first choose a set of coordinates $q$, express the Lagrangian in terms of $q$ and $\dot{q}$, and derive the equations of motion. To do this for a rotating spatial body, we can choose $q$ to be three angles describing the orientation of the body in a world frame. Then we can express the kinetic energy of the body as a function of $q$ and $\dot{q}$ and proceed as before. If we do this, however, the inertia matrix $M(q)$ will be extremely complex for any choice of $q$, providing little insight into the nature of the motion. The equations of motion are rarely written this way. Another problem is that no choice of three orientation variables can provide a smooth, global coordinatization of the space of orientations. In the same way that latitude and longitude coordinates for the Earth "go bad" at the poles, where the longitude changes discontinuously, any choice of three coordinates to represent orientations will have singularities. (For motions away from these bad orientations, however, three coordinates work just fine, so this is not the most serious problem. In fact, we have a similar problem representing a single angle by a real number, which requires the use of mod$2\pi$ arithmetic.)

So we will not begin by choosing angular coordinates, and instead of defining the angular velocity as the time-derivative of coordinates, we define $\omega_s = [\omega_{x_s}, \omega_{y_s}, \omega_{z_s}]^T$ to be the angular velocity of the body about the $x_s$-$y_s$-$z_s$ axes of a stationary inertial frame at the center of mass of the body. The linear velocity at a point $r_s = (x_s, y_s, z_s)^T$

on the body is $\omega_s \times r_s$. The total kinetic energy of the body can be written

$$K = \frac{1}{2}\int_V \rho(r_s)(\omega_s \times r_s)^T(\omega_s \times r_s)dV, \tag{10.34}$$

which can be simplified to

$$K = \frac{1}{2}\omega_s^T\left(\int_V \rho(r_s)\begin{bmatrix} y_s^2+z_s^2 & -x_sy_s & -x_sz_s \\ -x_sy_s & x_s^2+z_s^2 & -y_sz_s \\ -x_sz_s & -y_sz_s & x_s^2+y_s^2 \end{bmatrix} dV\right)\omega_s, \tag{10.35}$$

or

$$K = \frac{1}{2}\omega_s^T \mathbb{I}_s \omega_s. \tag{10.36}$$

The matrix $\mathbb{I}_s$ is the symmetric positive definite inertia matrix for the body written in the inertial frame. Because $\mathbb{I}_s$ is defined in the stationary world frame, it changes as the body rotates.

The *angular momentum* of the body is $P = \mathbb{I}_s\omega_s$, and the torque $\tau_s = [\tau_{xs}, \tau_{ys}, \tau_{zs}]^T$ acting on the body, expressed in the inertial frame, is the rate of change of $P$:

$$\begin{aligned}\tau_s &= \frac{dP}{dt} \\ &= \frac{d\mathbb{I}_s}{dt}\omega_s + \mathbb{I}_s\frac{d\omega_s}{dt}.\end{aligned}$$

The density of the body is not changing as it rotates, so the change of $\mathbb{I}_s$, $d\mathbb{I}_s/dt$, is due only to the motion of the body in the world frame, giving $d\mathbb{I}_s/dt = \omega_s \times \mathbb{I}_s$. Plugging in, we get

$$\tau_s = \omega_s \times \mathbb{I}_s\omega_s + \mathbb{I}_s\dot{\omega}_s. \tag{10.37}$$

This is known as *Euler's equation* in the inertial frame.

To turn equation (10.37) into a matrix equation, we define the *skew-symmetric matrix representation* $\widehat{\omega}_s$ of the vector $\omega_s = [\omega_{x_s}, \omega_{y_s}, \omega_{z_s}]^T$:

$$\widehat{\omega}_s = \begin{bmatrix} 0 & -\omega_{z_s} & \omega_{y_s} \\ \omega_{z_s} & 0 & -\omega_{x_s} \\ -\omega_{y_s} & \omega_{x_s} & 0 \end{bmatrix}.$$

We can now express equation (10.37) as the matrix equation

$$\tau_s = \widehat{\omega}_s\mathbb{I}_s\omega_s + \mathbb{I}_s\dot{\omega}_s. \tag{10.38}$$

We still do not have a representation of the orientation of the body in the world frame, however. We need an equation for the evolution of the body's orientation (the

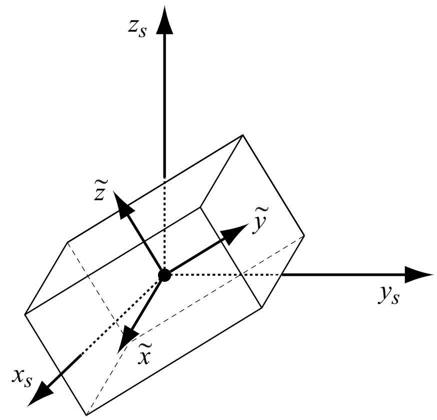


**Figure 10.4** The rotation matrix for a body is obtained by expressing the unit vectors $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ of the body $x$-$y$-$z$ frame in the inertial frame $x_s$-$y_s$-$z_s$.

kinematic equation) to go with equation (10.38) for the evolution of the velocity (the dynamic equation).

To do this, define a frame $x$-$y$-$z$ attached to the body at its center of mass. As described in Chapter 3, our representation of the orientation of the body will be as a $3 \times 3$ rotation matrix

$$R = \begin{bmatrix} \tilde{x}_1 & \tilde{y}_1 & \tilde{z}_1 \\ \tilde{x}_2 & \tilde{y}_2 & \tilde{z}_2 \\ \tilde{x}_3 & \tilde{y}_3 & \tilde{z}_3 \end{bmatrix} \in SO(3),$$

where $\tilde{x} = [\tilde{x}_1, \tilde{x}_2, \tilde{x}_3]^T$ is the unit vector in the body $x$-direction expressed in the inertial coordinate frame. The vectors $\tilde{y}$ and $\tilde{z}$ are defined similarly (figure 10.4).

Each column vector of $R$ moves according to the angular velocity $\omega_s$, so the kinematics of the rotating rigid body can be written

(10.39) $$\dot{R} = \omega_s \times R = \widehat{\omega}_s R.$$

Together, equations (10.39) and (10.38) describe the motion of a rotating rigid body in a spatial frame. The use of the rotation matrix representation of the orientation allows us to write the kinematics in a simple and globally correct fashion, which is not possible with any choice of three coordinates.[1]

1. A globally correct representation of orientation can be achieved using only four numbers with quaternions (see appendix E). We use the matrix representation because it allows the convenient use of matrix multiplications.

One difficulty with the equations of motion in an inertial frame is that $\mathbb{I}_s$ changes as the body rotates. It would be more convenient and intuitive to define the equations of motion in a frame fixed to the body, where a *body inertia matrix* $\mathbb{I}$ is unchanging. To do so, we use the coordinate frame $x$-$y$-$z$ attached to the body at its center of mass. The angular velocity in this frame is written $\omega = [\omega_x, \omega_y, \omega_z]^T$ and the external torque is written $\tau = [\tau_x, \tau_y, \tau_z]^T$. These are related to $\omega_s$ and $\tau_s$ by the following equations:

$$\omega_s = R\omega$$
$$\tau_s = R\tau$$

The inertial frame coordinates $r_s = [x_s, y_s, z_s]^T$ of a point are related to its coordinates in the body frame $r = [x, y, z]^T$ by

$$r_s = Rr.$$

The kinematic equations in the two frames are related by

$$\dot{R} = \widehat{\omega}_s R = R\widehat{\omega}.$$

Plugging these relations into equation (10.34) and simplifying, we find

$$K = \frac{1}{2}\omega^T \mathbb{I}\omega \tag{10.40}$$

$$\mathbb{I} = R^T \mathbb{I}_s R, \tag{10.41}$$

where the symmetric positive definite body-fixed inertia matrix is given by

$$\mathbb{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} = \int_V \rho(r) \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{bmatrix} dV. \tag{10.42}$$

The (possibly non-unique) eigenvectors of $\mathbb{I}$ define orthogonal *principal axes of inertia* of the body. If the body $x$-$y$-$z$ frame is chosen so that the axes are aligned with principal axes of inertia, then all off-diagonal terms of $\mathbb{I}$ are zero, and $I_{xx}$, $I_{yy}$, $I_{zz}$ are the principal moments of inertia. In the general case, one principal axis is the axis of maximum inertia, one principal axis is the axis of minimum inertia, and the third principal axis (the intermediate axis of inertia) is orthogonal. Because of symmetries, however, the inertia about two or three of the principal axes might be identical. Often the principal axes of inertia of a body are evident from symmetries (figure 10.5).

To derive the dynamics in the body frame, we cannot simply take the time-derivative of $\mathbb{I}\omega$, since this is not defined in an inertial frame. (The time-derivative of the momentum $\mathbb{I}_s\omega_s$ is a generalized force, while the time-derivative of $\mathbb{I}\omega$ is not.) Instead,

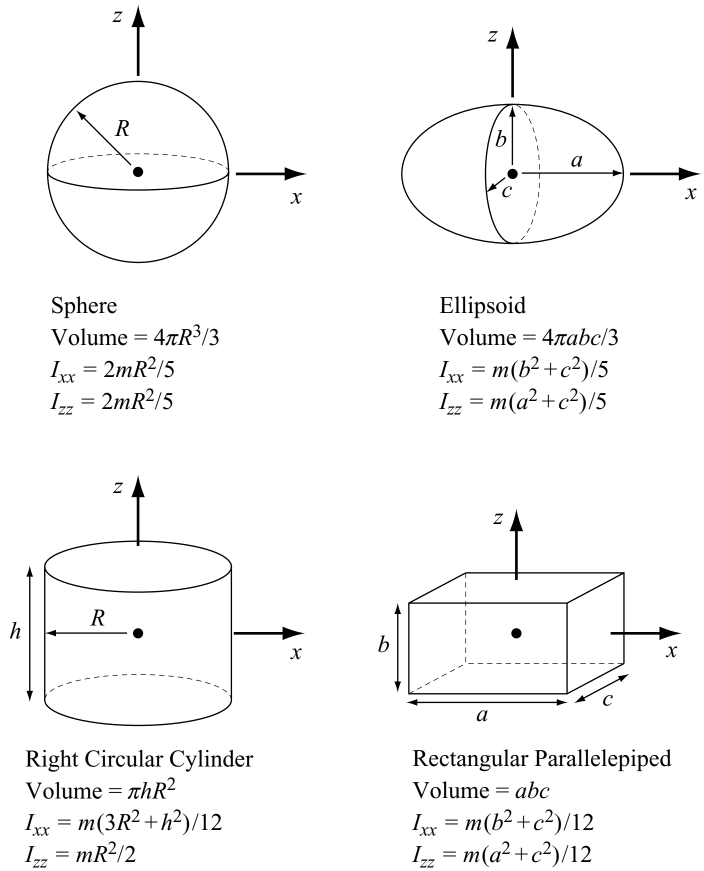


**Figure 10.5** Inertias about principal axes of inertia for four different uniform density volumes. Note that the principal axes of inertia are not unique for the sphere and cylinder.

we begin with equation (10.38),

$$\tau_s = \widehat{\omega}_s \mathbb{I}_s \omega_s + \mathbb{I}_s \dot{\omega}_s,$$

and plug in $\tau_s = R\tau$, $\omega_s = R\omega$, $\mathbb{I}_s = R\mathbb{I}R^T$, and $\widehat{\omega}_s = \dot{R}R^T$ to get

$$\begin{aligned} R\tau &= \dot{R}R^T R\mathbb{I}R^T R\omega + R\mathbb{I}R^T \left(\frac{d}{dt}(R\omega)\right) \\ &= \dot{R}R^T R\mathbb{I}R^T R\omega + R\mathbb{I}R^T (\dot{R}\omega + R\dot{\omega}). \end{aligned}$$

Recognizing from our identities that $\dot{R}\omega = R\widehat{\omega}\omega = 0$, and premultiplying both sides by $R^{-1} = R^T$, we get

$$\tau = R^T \dot{R} R^T R \mathbb{I} R^T R\omega + R^T R \mathbb{I} R^T R\dot{\omega}.$$

Plugging in $\widehat{\omega} = R^T \dot{R}$ and noticing that $R^T R$ is the identity matrix, this simplifies to

$$\tau = \widehat{\omega}\mathbb{I}\omega + \mathbb{I}\dot{\omega}.$$

This is Euler's equation in the body frame. Note that it has the same form as Euler's equation in the spatial frame. Collecting together the kinematics and dynamics in one place, the equations of motion in the body frame are written

$$\dot{R} = R\widehat{\omega} \tag{10.43}$$

$$\tau = \widehat{\omega}\mathbb{I}\omega + \mathbb{I}\dot{\omega}. \tag{10.44}$$

The big advantage of this form over the spatial equations is that $\mathbb{I}$ is constant.

If the body $x$-$y$-$z$ frame is aligned with the principal axes, making all off-diagonal terms of $\mathbb{I}$ zero, equation (10.44) simplifies to

$$\begin{bmatrix} \tau_x \\ \tau_y \\ \tau_z \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{\omega}_x + (I_{zz} - I_{yy})\omega_y\omega_z \\ I_{yy}\dot{\omega}_y + (I_{xx} - I_{zz})\omega_x\omega_z \\ I_{zz}\dot{\omega}_z + (I_{yy} - I_{xx})\omega_x\omega_y \end{bmatrix}. \tag{10.45}$$

One key implication of equations (10.44) and (10.45) is that $\dot{\omega}$ may not be zero even if $\tau$ is zero. Although the angular momentum and kinetic energy of a rotating body are constant when no external torques are applied, the angular velocity of the body may not be constant. For further interpretation of equation (10.45), see the mechanics textbooks by Symon [407] and Marsden and Ratiu [308] or the robotic manipulation textbook by Mason [312].

When it is difficult to solve the integrals of equation (10.42) directly to find $\mathbb{I}$, it may be possible to split the body into simpler components and solve for (or look up in a table) the inertia matrix at the center of mass of each component separately. If we then transform these inertia matrices to a common frame, we can simply add them to get the inertia matrix for the composite body in that common frame. This transformation can be accomplished by a translation followed by a rotation, as outlined below (see figure 10.6).

Let $\mathbb{I}_i$ be the inertia matrix of the $i$th component, expressed in its own local coordinate frame $x_i$-$y_i$-$z_i$ at its center of mass. Let $r_i$ be the vector from the origin of the local frame to the origin of the common frame $x$-$y$-$z$, expressed in the local frame. The inertia $\mathbb{I}_i$ can be expressed in a frame aligned with $x_i$-$y_i$-$z_i$, but located at the

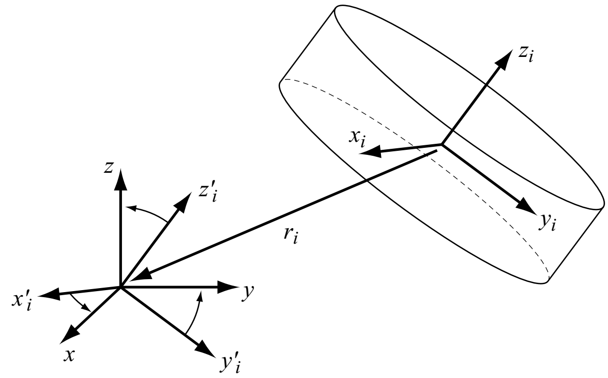


**Figure 10.6** The inertia matrix of body $\mathcal{B}_i$, expressed in a frame $x_i$-$y_i$-$z_i$ at the center of mass of the body, can be expressed in another frame $x$-$y$-$z$ by a translation and rotation.

origin of the common frame, using the *parallel-axis theorem*

$$\mathbb{I}'_i = \mathbb{I}_i + m_i\left(\|r_i\|^2\mathcal{I} - r_i r_i^T\right), \tag{10.46}$$

where $m_i$ is the mass of the $i$th component and $\mathcal{I}$ is the $3 \times 3$ identity matrix. Now let $R_i$ denote the rotation matrix describing the orientation of this translated local frame relative to the common frame. Rotating the inertia matrix into the common frame, we get

$$\mathbb{I}''_i = R_i \mathbb{I}'_i R_i^T.$$

The matrix $\mathbb{I}''_i$ is the inertia of the $i$th component expressed in the common frame. Performing this translation and rotation for all $k$ components of the body, the total inertia of the body in the common frame is $\mathbb{I} = \mathbb{I}''_1 + \cdots + \mathbb{I}''_k$.

### Lagrange's Equations Revisited

We have gone to great lengths to avoid choosing three generalized coordinates and using Lagrange's equations! Now that we have done this work and developed some understanding of the body inertia matrix $\mathbb{I}$, it will be easier to see how Lagrange's equations could be applied.

Figure 10.7 shows a choice of coordinates $q = [q_1, q_2, q_3]^T$ due to Euler. The body $x$-$y$ plane intersects the spatial $x_s$-$y_s$ plane along a line called the *line of nodes*. The coordinate $q_1$ is the angle from the $x_s$-axis to the line of nodes, $q_2$ is the angle from the line of nodes to the body $x$-axis, and $q_3$ is the angle from the $z_s$-axis to the body $z$-axis. With some work, the angular velocity $\omega = [\omega_x, \omega_y, \omega_z]^T$ can be expressed in

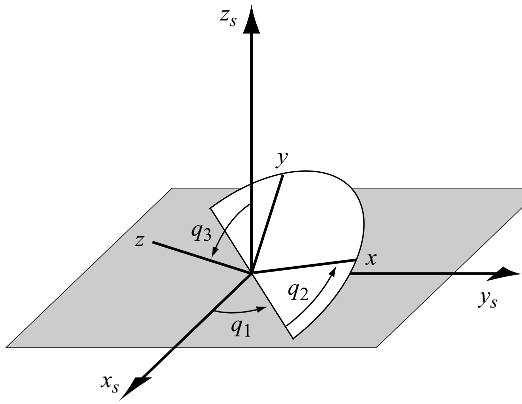


**Figure 10.7** Euler angles for a rotating body.

terms of these coordinates:

$$\omega_x = \dot{q}_3 \cos q_2 + \dot{q}_1 \sin q_2 \sin q_3 \tag{10.47}$$

$$\omega_y = -\dot{q}_3 \sin q_2 + \dot{q}_1 \cos q_2 \sin q_3 \tag{10.48}$$

$$\omega_z = \dot{q}_2 + \dot{q}_1 \cos q_3. \tag{10.49}$$

Plugging these into the kinetic energy $K = \frac{1}{2}\omega^T \mathbb{I}\omega$, we get the kinetic energy in the form $K = \frac{1}{2}\dot{q}^T M(q)\dot{q}$, as we are used to. From there we can apply Lagrange's equations as before to get the dynamic equations of motion for generalized torques acting along the coordinates.

A good thing about this formulation is that we have used the fewest possible numbers to represent the orientation, and the kinematics are trivial. Significant drawbacks are the complexity of the equations, as well as the singularities in the coordinate representation.

## Problems

1. Represent the configuration of a point mass in a plane by polar coordinates $q = [r, \theta]^T$ and use Lagrange's equations to find the equations of motion. Then write the inertia matrix $M(q)$, derive the Christoffel symbols, and show that the dynamics of equation (10.10) are equivalent to the equations you derived using the Lagrangian method.

2. Use Lagrange's equations to derive the equations of motion of a 2R (two revolute joints) robot arm operating in a vertical plane. The first link has length $L_1$, mass $m_1$, and inertia $I_1$ about the center of mass, and the center of mass is a distance $r_1$ from the first joint.

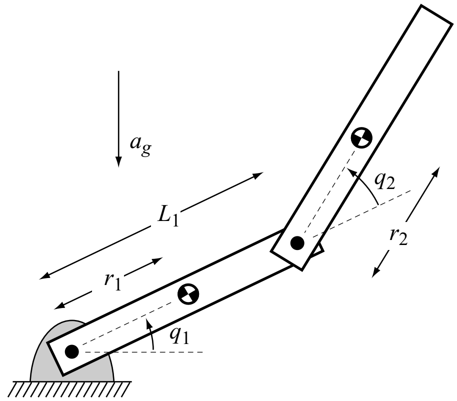


**Figure 10.8** The 2R robot arm.

For the second link, $m_2$, $I_2$, and $r_2$ are defined similarly (figure 10.8). Put the equations of motion in the standard form of equation (10.7).

3. Find the eight Christoffel symbols for the mass matrix of problem 2.

4. Use Lagrange's equations to find the equations of motion of a PR robot arm. Provide your own drawing and parameters and solve with these parameters. Put the equations in the standard form of equation (10.7).

5. Find the inertia matrix of a round tube of length $L$, inner diameter $d_1$, outer diameter $d_2$, and density $\rho$. Choose a frame aligned with the principal axes of inertia. Remember that inertia matrices in a common frame can be added and subtracted.

6. The inertia matrix of a body in a coordinate frame $x_1$-$y_1$-$z_1$ at the center of mass of the body is $\mathbb{I}_1$. The orientation of this coordinate frame is $R_1$ relative to a frame $x$-$y$-$z$. The origin of $x$-$y$-$z$ is at $r_1$ in the frame $x_1$-$y_1$-$z_1$. Transform the inertia matrix $\mathbb{I}_1$ to an inertia matrix $\mathbb{I}$ expressed in the $x$-$y$-$z$ frame, where

$$\mathbb{I}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad r_1 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   Also provide a drawing of the two frames showing their position and orientation relative to each other.

7. Consider a barbell constructed of two spheres of radius 10 cm welded to the ends of a right circular cylinder bar of length 20 cm and radius 2 cm. Each body is a solid volume constructed of steel, with a mass density of 7850 kg/m$^3$. Find the approximate inertia matrix in a principal-axis frame at the center of mass.

8. Prove the parallel-axis theorem [equation (10.46)].

9. Derive equations (10.47), (10.48), and (10.49).

10. Write a program to simulate the tumbling motion of a rigid body in space.

11. A point of mass $m$ moves in three-dimensional space $\mathbb{R}^3$, actuated by three orthogonal thrusters, with equations of motion $u = m\ddot{q}$ (no gravity). Now imagine that the mass (still with three thrusters) is constrained to move on a sphere of radius 1 centered at the origin of the inertial frame. Write the Pfaffian constraint and solve for $\ddot{q}$ and the Lagrange multiplier $\lambda$.

12. Find the projection matrix $P$ for problem 11 and write the constrained equations of motion in the form of equation (10.20).

13. In problem 11, it is possible to reduce the number of generalized coordinates from three to two and eliminate the Lagrange multiplier. Choose latitude ($q_1$) and longitude ($q_2$) coordinates to describe the position of the point on the sphere, and use Lagrange's equations to solve for the dynamics in these coordinates. Explain what the generalized forces are. Give the Christoffel symbols in these coordinates.