E Representations of Orientation

IN CHAPTER 3, we represent orientation by matrices in $SO(3)$, which can be parameterized using three parameters. In this appendix, we describe some of the most popular methods of doing so, including Euler angles and angles with respect to a fixed frame. We also describe how orientation can be described as rotation about an arbitrary axis and by quaternions.

E.1 Euler Angles

Recall that the Euler angles ϕ, θ, ψ in chapter 3 correspond to successive rotations about body Z-Y-Z axes, and that the corresponding rotation matrix is obtained as

$$R = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix} \tag{E.1}$$

in which s_θ and c_θ denote $\sin\theta$ and $\cos\theta$ respectively.

Consider now the problem of using Euler angles to define a chart on some open set $U \subset SO(3)$. It is easy to see that a single chart cannot cover all of $SO(3)$. For example, if $R_{33} = 1$, it must be the case that $\theta = 0$, and the rotation matrix is given by

$$\begin{bmatrix} R_{11} & R_{12} & 0 \\ R_{21} & R_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_{\phi+\psi} & -s_{\phi+\psi} & 0 \\ s_{\phi+\psi} & c_{\phi+\psi} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{E.2}$$

In this case, it is not possible to uniquely define ϕ and ψ, since only their sum is represented in R. A similar case occurs when $R_{33} = -1$.

To define a chart using Euler angles, we begin by defining the open set

$$U = \{R \in SO(3) \mid R_{33} \notin \{-1, 1\}\},$$

and defining the chart Φ such that

$$\Phi(R) \mapsto [\phi(R), \theta(R), \psi(R)]^T \in \mathbb{R}^3.$$

For any $R \in U$, not both of R_{13}, R_{23} are zero. Then the above equations show that $s_\theta \neq 0$. Since not both R_{13} and R_{23} are zero, then $R_{33} \neq \pm 1$, and we have $c_\theta = R_{33}$, $s_\theta = \pm\sqrt{1 - R_{33}^2}$ so

$$\theta = \operatorname{atan2}\left(\sqrt{1 - R_{33}^2}, R_{33}\right) \tag{E.3}$$

or

$$\theta = \operatorname{atan2}\left(-\sqrt{1 - R_{33}^2}, R_{33}\right). \tag{E.4}$$

The function $\theta = \operatorname{atan2}(y, x)$ computes the arc tangent function, where x and y are the cosine and sine, respectively, of the angle θ. This function uses the signs of x and y to select the appropriate quadrant for the angle θ. Note that if both x and y are zero, atan2 is undefined.

If we choose the value for θ given by (E.3), then $s_\theta > 0$, and

$$\phi = \operatorname{atan2}(R_{23}, R_{13}) \tag{E.5}$$

$$\psi = \operatorname{atan2}(R_{32}, -R_{31}). \tag{E.6}$$

If we choose the value for θ given by (E.4), then $s_\theta < 0$, and

$$\phi = \operatorname{atan2}(-R_{23}, -R_{13}) \tag{E.7}$$

$$\psi = \operatorname{atan2}(-R_{32}, R_{31}). \tag{E.8}$$

Thus there are two solutions depending on the sign chosen for θ.

As described above, when $R_{33} = \pm 1$, only the sum $\phi \pm \psi$ can be determined. For $R_{33} = 1$

$$\begin{aligned} \phi + \psi &= \operatorname{atan2}(R_{21}, R_{11}) \\ &= \operatorname{atan2}(-R_{12}, R_{11}). \end{aligned} \tag{E.9}$$

In this case there are infinitely many solutions. We may take $\phi = 0$ by convention. If $R_{33} = -1$, then $c_\theta = -1$ and $s_\theta = 0$, so that $\theta = \pi$. In this case (E.1) becomes

$$\begin{bmatrix} -c_{\phi-\psi} & -s_{\phi-\psi} & 0 \\ s_{\phi-\psi} & c_{\phi-\psi} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & 0 \\ R_{21} & R_{22} & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{E.10}$$

The solution is thus

$$\phi - \psi = \operatorname{atan2}(-R_{12}, -R_{11}) = \operatorname{atan2}(-R_{22}, -R_{21}). \tag{E.11}$$

As before there are infinitely many solutions.

There is nothing special about the choice of axes we used to define Euler angles. We could just as easily have used successive rotations about, say, the x, y, and z axes. In fact, it is easy to see that there are twelve possible ways to define Euler angles: any sequence of three axes, such that no two successive axes are the same, generates a set of Euler angles.

E.2 Roll, Pitch, and Yaw Angles

A rotation matrix R can also be described as a product of successive rotations about the world coordinate axes. These rotations define the *roll*, *pitch*, and *yaw* angles, and they are illustrated in figure E.1. Typically, the order of rotation is taken to be x-y-z: first a yaw about the world x-axis by an angle ψ, then pitch about the world y-axis by an angle θ, and finally a roll about the world z-axis by an angle ϕ[1]. Since the successive rotations are relative to the world coordinate frame, the resulting rotation matrix is given by

$$\begin{aligned} R &= R_{z,\phi} R_{y,\theta} R_{x,\psi} \\ &= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix} \\ &= \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\theta s_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}. \end{aligned} \tag{E.12}$$

1. As with Euler angles, one can choose a different ordering for the rotations to obtain different *fixed axis* representations of orientation. The term *fixed axis* refers to the fact that successive rotations are taken with respect to axes of the fixed coordinate frame.

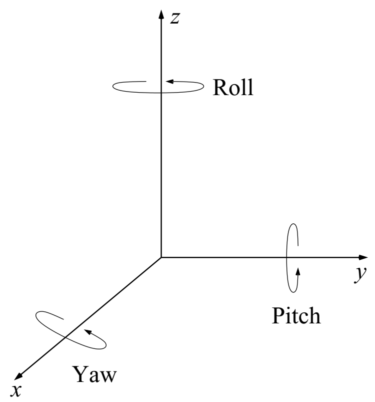

Figure E.1 Roll, pitch, and yaw angles.

The three angles, ϕ, θ, ψ, can be obtained for a given rotation matrix using a method that is similar to that used to derive the Euler angles above.

E.3 Axis-Angle Parameterization

Above we described a rotation matrix by decomposing a rotation into three successive rotations about the coordinate axes. An alternative to this is to specify a rotation matrix in terms of a rotation about an arbitrary axis in space. This provides both a convenient way to describe rotations, and an alternative parameterization for rotation matrices.

Let $k = [k_x, k_y, k_z]^T$ be a unit vector defining an axis expressed in the world frame. To determine the parameterization, we need to derive the rotation matrix $R_{k,\theta}$ representing a rotation of θ degrees about this axis. A simple way to derive this rotation matrix is to rotate the vector k into one of the coordinate axes, say the z-axis, then rotate about this axis by θ, and finally, rotate k back to its original position. As can be seen in figure E.2 we can rotate k into the world z-axis by first rotating about the world z-axis $-\alpha$, then rotating about the world y-axis by $-\beta$. Since all rotations are performed relative to the world frame, the matrix $R_{k,\theta}$ is obtained as

$$R_{k,\theta} = R_{z,\alpha} R_{y,\beta} R_{z,\theta} R_{y,-\beta} R_{z,-\alpha}. \tag{E.13}$$

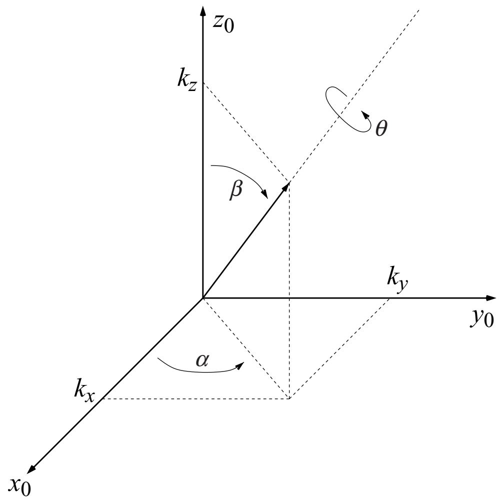

Figure E.2 Rotation about an arbitrary axis.

As can be seen in figure E.2,

$$\sin\alpha = \frac{k_y}{\sqrt{k_x^2 + k_y^2}} \tag{E.14}$$

$$\cos\alpha = \frac{k_x}{\sqrt{k_x^2 + k_y^2}} \tag{E.15}$$

$$\sin\beta = \sqrt{k_x^2 + k_y^2} \tag{E.16}$$

$$\cos\beta = k_z. \tag{E.17}$$

The final two equations follow from the fact that k is a unit vector. Substituting (E.14) through (E.17) into (E.13) we can obtain

$$R_{k,\theta} = \left[\begin{array}{c|c|c} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{array}\right], \tag{E.18}$$

in which $v_\theta = 1 - c_\theta$.

We can use this parameterization to derive a chart on $SO(3)$ as follows. Let R be an arbitrary rotation matrix with components (R_{ij}). Let $U = \{R \mid Tr(R) \neq \pm 1\}$ where

Tr denotes the trace of R. By direct calculation using (E.18) we obtain

$$\theta = \cos^{-1}\left(\frac{R_{11} + R_{22} + R_{33} - 1}{2}\right)$$
$$= \cos^{-1}\left(\frac{Tr(R) - 1}{2}\right), \text{ and}$$
$$k = \frac{1}{2\sin\theta}\begin{bmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{bmatrix}.$$

This representation is not unique since a rotation of $-\theta$ about $-k$ is the same as a rotation of θ about k, that is,

$$R_{k,\theta} = R_{-k,-\theta}. \tag{E.19}$$

We can now define the mapping ϕ using k and θ. Since the axis k is a unit vector, only two of its components are independent. Therefore, only three independent quantities are required in this representation of a rotation. Thus, we can define ϕ as

$$\phi(R) = [\theta k_x, \theta k_y, \theta k_z]^T. \tag{E.20}$$

Using this convention, we can recover k and θ as

$$k = \frac{\phi(R)}{||\phi(R)||} \quad \text{and} \quad \theta = ||\phi(R)||. \tag{E.21}$$

The angle θ is a good distance measure between two elements of $SO(3)$.

E.4 Quaternions

The axis-angle parameterization described above parameterizes a rotation matrix by three parameters (given by (E.21)). Quaternions, which are closely related to the axis-angle parameterization, can be used to define a rotation by four numbers. It is straightforward to use quaternions to define an atlas for $SO(3)$ using only four charts. Furthermore quaternion representations are very convenient for operations such as composition of rotations and coordinate transformations. For these reasons, quaternions are a popular choice for the representation of rotations in three dimensions.

Quaternions are a generalization of the complex numbers to a four-dimensional space. For this reason, we begin with a quick review of how complex numbers can be used to represent orientation in the plane. A first introduction to complex numbers often uses the example of representing orientation in the plane using unit magnitude complex numbers of the form $a + ib$, in which $i = \sqrt{-1}$. In this case, the angle θ from the real axis to the vector $(a + ib) \in \mathbb{C}$ is given by atan2(b, a), and it is easy to see that $\cos\theta = a$ and $\sin\theta = b$. Since $a, b \in \mathbb{R}$, we can consider this as an embedding of S^1 in the plane.

Using this representation, multiplication of two complex numbers corresponds to addition of the corresponding angles. This can be verified by direct calculation as

$$\begin{aligned}(a_1 + ib_1)(a_2 + ib_2) &= a_1a_2 + ib_1a_2 + ia_1b_2 - b_1b_2 \\ &= (a_1a_2 - b_1b_2) + i(b_1a_2 + a_1b_2) \\ &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \\ &\quad +i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2) \\ &= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2).\end{aligned}$$

While a complex number $a + ib$ defines a point in the complex plane, a quaternion defines a point in a four-dimensional complex space, $q_0 + iq_1 + jq_2 + kq_3$. Here, i, j, and k represent independent square roots of negative one. They are independent in the sense that they do not combine using the normal rules of scalar multiplication. In particular, we have

$$-1 = i^2 = j^2 = k^2, \tag{E.22}$$

$$i = jk = -kj, \tag{E.23}$$

$$j = ki = -ik, \tag{E.24}$$

$$k = ij = -ji. \tag{E.25}$$

It is not a coincidence that multiplication of i, j, and k is similar to the vector cross product for the orthogonal unit basis vectors, $i = [1, 0, 0]^T$, $j = [0, 1, 0]^T$, and $k = [0, 0, 1]^T$.

Complex numbers with unit magnitude can be used to represent orientation in the plane simply by using their representation in polar coordinates. Likewise, quaternions can be used to represent rotations in 3D. In particular, for a rotation about an axis $n = [n_x, n_y, n_z]^T$ by angle θ, the corresponding quaternion, Q, is defined as

$$Q = \left(\cos\frac{\theta}{2}, n_x\sin\frac{\theta}{2}, n_y\sin\frac{\theta}{2}, n_z\sin\frac{\theta}{2}\right). \tag{E.26}$$

When we define the axis of rotation to be a unit vector, the corresponding quaternion has unit norm, since

$$\begin{aligned}||Q|| &= \cos^2\frac{\theta}{2} + n_x^2\sin^2\frac{\theta}{2} + n_y^2\sin^2\frac{\theta}{2} + n_z^2\sin^2\frac{\theta}{2} \\ &= \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\left(n_x^2 + n_y^2 + n_z^2\right) \\ &= 1.\end{aligned} \tag{E.27}$$

Quaternions with unit norm are sometimes referred to as rotation quaternions.

It is straightforward to apply the results from section E.3 to determine the rotation matrix $R \in SO(3)$ that corresponds to the rotation represented by a rotation quaternion. For the quaternion $Q = (q_0, q_1, q_2, q_3)$ we have

$$R(Q) = \begin{bmatrix} 2\left(q_0^2 + q_1^2\right) - 1 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_1q_2 + q_0q_3) & 2\left(q_0^2 + q_2^2\right) - 1 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 + q_0q_1) & 2\left(q_0^2 + q_3^2\right) - 1 \end{bmatrix}. \tag{E.28}$$

Quaternions can be used to define an atlas for $SO(3)$ comprising four charts, (U_i, ϕ_i), with $\phi_i : U_i \to \mathbb{R}^3$. This is most easily done by using two steps. First, for a rotation matrix R, we determine the corresponding quaternion Q. Then, we use Q to determine which chart applies (i.e., we implicitly define the neighborhoods U_i in terms of Q), and use the appropriate ϕ_i to define the local coordinates.

Determining the quaternion that correponds to a rotation matrix amounts to solving the inverse of (E.28), and this can be done by a method similar to that given for the axis-angle parameterization of section E.3. In particular, for rotation matrices R such that $Tr(R) \neq \pm 1$ we have

$$q_0 = \frac{1}{2}\sqrt{1 + Tr(R)} \tag{E.29}$$

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \frac{1}{4q_0} \begin{bmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{bmatrix}. \tag{E.30}$$

To define the four charts, we first define the four neighborhoods

$$\begin{aligned} U_0 &= \{Q = (q_0, q_1, q_2, q_3) \mid q_0 \geq q_1, q_2, q_3\} \\ U_1 &= \{Q = (q_0, q_1, q_2, q_3) \mid q_1 \geq q_0, q_2, q_3\} \\ U_2 &= \{Q = (q_0, q_1, q_2, q_3) \mid q_2 \geq q_0, q_1, q_3\} \\ U_3 &= \{Q = (q_0, q_1, q_2, q_3) \mid q_3 \geq q_0, q_1, q_2\}. \end{aligned}$$

These are not actually open sets (due to the nonstrict inequality in the set definitions), but they can be used to define open sets using their interiors. Now we define the coordinate maps ϕ_i as

$$\begin{aligned} \phi_0(q_0, q_1, q_2, q_3) &= \left(\frac{q_1}{|q_0|}, \frac{q_2}{|q_0|}, \frac{q_3}{|q_0|}\right) \\ \phi_1(q_0, q_1, q_2, q_3) &= \left(\frac{q_0}{|q_1|}, \frac{q_2}{|q_1|}, \frac{q_3}{|q_1|}\right) \end{aligned}$$

$$\phi_2(q_0, q_1, q_2, q_3) = \left(\frac{q_0}{|q_2|}, \frac{q_1}{|q_2|}, \frac{q_3}{|q_2|}\right)$$
$$\phi_3(q_0, q_1, q_2, q_3) = \left(\frac{q_0}{|q_3|}, \frac{q_1}{|q_3|}, \frac{q_2}{|q_3|}\right)$$

As we have seen above, $R_i \in SO(3)$ represents a rotation, and the composition of successive rotations, say R_1 and R_2, is represented by the rotation matrix $R = R_1 R_2$. Likewise, multiplication of quaternions corresponds to the composition of successive rotations. In particular, if Q_1 and Q_2 are two quaternions representing a rotation by θ_1 about axis n_1 and a rotation by θ_2 about axis n_2, respectively, then the result of performing these two rotations in succession is represented by the quaternion $Q = Q_1 Q_2$. Using (E.22) through (E.25) it is straightforward to determine the quaternion product. In particular, for two quaternions, X and Y, we compute their product, $Z = XY$, as

$$\begin{aligned} z_0 + iz_1 + jz_2 + kz_3 &= (x_0 + ix_1 + jx_2 + kx_3)(y_0 + iy_1 + yx_2 + yx_3) \\ &= x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3 \\ &\quad + i(x_0 y_1 + x_1 y_0 + x_2 y_3 - x_3 y_2) \\ &\quad + j(x_0 y_2 + x_2 y_0 + x_3 y_1 - x_1 y_3) \\ &\quad + k(x_0 y_3 + x_3 y_0 + x_1 y_2 - x_2 y_1). \end{aligned}$$

By equating the real parts on both sides of the final equality, and by equating the coefficients of i, j, and k on both sides of the final equality, we obtain

$$\begin{aligned} z_0 &= x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3 \\ z_1 &= x_0 y_1 + x_1 y_0 + x_2 y_3 - x_3 y_2 \\ z_2 &= x_0 y_2 + x_2 y_0 + x_3 y_1 - x_1 y_3 \\ z_3 &= x_0 y_3 + x_3 y_0 + x_1 y_2 - x_2 y_1. \end{aligned}$$

The quaternion $Q = (q_0, q_1, q_2, q_3)$ can be thought of as having the scalar component q_0 and the vector component $q = [q_1, q_2, q_3]^T$. Therefore, one often represents a quaternion by a pair, $Q = (q_0, q)$. Using this notation, q_0 represents the real part of Q, and q represents the imaginary part of Q. Using this notation, the quaternion product $Z = XY$ can be represented more compactly as

$$\begin{aligned} z_0 &= x_0 y_0 - x^T y \\ z &= x_0 y + y_0 x + x \times y, \end{aligned}$$

in which $\times$ denotes the vector cross product operator.

For complex numbers, the conjugate of $a + ib$ is defined by $a - ib$. Similarly, for quaternions we denote by Q^* the conjugate of the quaternion Q, and define

$$Q^* = (q_0, -q_1, -q_2, -q_3). \tag{E.31}$$

With regard to rotation, if the quaternion Q represents a rotation by θ about the axis n, then its conjugate Q^* represents a rotation by θ about the axis $-n$. It is easy to see that

$$QQ^* = \left(q_0^2 + ||q||^2, 0, 0, 0\right) \tag{E.32}$$

and that

$$||QQ^*|| = \left|\left|\left(q_0^2 + q_1^2 + q_2^2 + q_3^2, 0, 0, 0\right)\right|\right| = \sum q_i^2 = ||Q||^2. \tag{E.33}$$

A quaternion, Q, with its conjugate, Q^*, can be used to perform coordinate transformations. Let the point p be rigidly attached to a coordinate frame $\mathcal{F}$, with local coordinates (x, y, z). If Q specifies the orientation of $\mathcal{F}$ with respect to the base frame, and T is the vector from the world frame to the origin of $\mathcal{F}$, then the coordinates of p with respect to the world frame are given by

$$Q(0, x, y, z)Q^* + T, \tag{E.34}$$

in which $(0, x, y, z)$ is a quaternion with zero as its real component. Quaternions can also be used to transform vectors. For example, if $n = (n_x, n_y, n_z)$ is the normal vector to the face of a polyhedron, then if the polyhedron is rotated by Q, the new direction of the normal is given by

$$Q(0, n_x, n_y, n_z)Q^*. \tag{E.35}$$