B Basic Set Definitions

CONSIDER A collection of elements called a *set*. The plane is a set; the real line is a set; a point is a set; the unit interval is a set. Sets can also be listed as collections of elements, e.g., $S_1 = \{1, 4, 9\}$ and $S_2 = \{\text{cow, chicken, pig}\}$ are both sets. The collection of these sets is also a set, i.e., $\{\mathbb{R}^2, \mathbb{R}, [0, 1]\}$ is a set. Given two sets A and B, A is said to be a *subset* of B (denoted $A \subset B$) if every element of A is also an element of B. Of the two examples above, S_1 is a subset of the set of positive integers and S_2 is a subset of the set of animals.

Given $A \subset B$, the *complement* of A in B (denoted $B \backslash A$) is defined to be all of the elements of B that are not in A, i.e.,

$$B \backslash A = \{x \mid x \in B, x \notin A\} .$$

The *union* of A and B (denoted $A \cup B$) is to be the set of points that is in either A or B, i.e.,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

The *intersection* of A and B (denoted $A \cap B$) is defined to be the set of all points that are in both A and B, i.e.,

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

For the remainder of this appendix we restrict the discussion to sets that are subsets of $\mathbb{R}^n$ for some n. Consider a point $x \in \mathbb{R}^n$, and define an *ϵ-neighborhood* of x to be the set

$$B_\epsilon(x) = \{y \in \mathbb{R}^n \mid d(x, y) < \epsilon\} .$$

The set $B_\epsilon(x)$ is also sometimes called an *open ball* of radius ϵ around the point x. We also sometime use the word *neighborhood* to refer to an ϵ-neighborhood with ϵ arbitrarily small.

A set $A \subset \mathbb{R}^n$ is said to be *open* if, for every point x in A, there is some ϵ so that $B_\epsilon(x)$ is also contained in A. A set A is said to be *closed* if its complement is open. Note that the concept of closure depends on the the ambient space. The set $\mathbb{R}^m$ considered by itself is open. But if $m < n$ and we consider $\mathbb{R}^m$ as a subset of the ambient space $\mathbb{R}^n$, then $\mathbb{R}^m$ is closed since its complement $\mathbb{R}^n \backslash \mathbb{R}^m$ is open. By the same token, when considered as a subset of the plane, the interval (0, 1) is neither closed nor open.

The following definitions derive from open and closed sets for ACS:

DEFINITION B.0.2 (Closure/Interior/Boundary)

- Closure *of A, denoted* $\mathrm{cl}(A)$*, is the intersection of all closed sets containing A.*
- Interior *of A, denoted* $\mathrm{int}(A)$*, is the union of all open sets contained in A.*
- Boundary *of A, denoted ∂A, is* $\mathrm{cl}(A) \bigcap \mathrm{cl}(S\backslash A)$.

EXAMPLE B.0.3 *Consider* $[0, 1]$ *as a subset of* $\mathbb{R}^1$.

$$\begin{aligned}
\mathrm{cl}([0,1]) &= [0,1] \\
\mathrm{int}([0,1]) &= (0,1) \\
\partial[0,1] &= [0,1] \bigcap \mathrm{cl}\left((-\infty,0) \bigcup ((1,\infty))\right) \\
&= [0,1] \bigcap ((-\infty,0] \bigcup [1,\infty)) \\
&= \{0,1\}
\end{aligned}$$

The following demonstrate how union and intersection operate on closures and interiors:

$$\begin{aligned}
A \subset B \quad & \Rightarrow \mathrm{int}(A) \subset \mathrm{int}(B) \text{ and } \mathrm{cl}(A) \subset \mathrm{cl}(B) \\
A \subset S \quad & \Rightarrow S\backslash\mathrm{cl}(A) = \mathrm{int}(S\backslash A),\ S\backslash\mathrm{int}(A) = \mathrm{cl}(S\backslash A). \\
A \subset S \quad & \Rightarrow \mathrm{cl}(\emptyset) = \mathrm{int}(\emptyset) = \emptyset,\ \mathrm{cl}(S) = \mathrm{int}(S) = S \\
& \mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A) \\
& \mathrm{cl}(A \bigcup B) = \mathrm{cl}(A) \bigcup \mathrm{cl}(B),\ \mathrm{int}(A) \bigcup \mathrm{int}(B) \subset \mathrm{int}(A \bigcup B) \\
& \mathrm{cl}(A \bigcap B) \subset \mathrm{cl}(A) \bigcap \mathrm{cl}(B),\ \mathrm{int}(A \bigcap B) = \mathrm{int}(A) \bigcap \mathrm{int}(B).
\end{aligned}$$

A subset A of B is *dense* if $\mathrm{cl}(A) = B$. So $(0, 1)$ is dense in $[0, 1]$ because $\mathrm{cl}(0, 1) = [0, 1]$. Intuitively, a subset A of B is dense if A is "almost as big" as B. The open interval and closed interval both have length 1. The set $[0, 1]\backslash\{.5\}$ is dense in $[0, 1]$. Intuitively, this means that taking away one point from an interval does not affect the size of the interval. The set of rational numbers, i.e., the set of real numbers that can be written as a fraction of two integers, is dense in the real line. A line is *not* dense in the plane. The plane, with a line removed from it, is dense in the plane.

We can also define a notion of subtraction and addition of sets. The *Minkowski sum* of A and B is

$$A \oplus B = \{x \mid x = a + b, a \in A, b \in B\}.$$

The *Minkowski difference* is

$$A \ominus B = \{x \mid x = a - b, a \in A, b \in B\}.$$

Two points in a set A are said to be within *line of sight* of each other if the straight line segment connecting them is completely contained in A. Line of sight is related to convexity. A set A is *convex* if for every $x, y \in A$, the line segment

$$\{tx + (1 - t)y \mid t \in [0, 1]\}$$

is contained in A. The *convex hull* of a set A is denoted as $\mathrm{Co}(A)$ and is defined to be the smallest convex set that contains A. If $A \subset \mathbb{R}^n$ is a finite set with m elements $\{x_1, x_2, \ldots, x_m\}$, we can express

$$\mathrm{Co}(A) = \left\{ y = \sum_{i=1}^{m} a_i x_i \,\middle|\, a_i \geq 0 \text{ for all } i;\ \sum_{i=1}^{m} a_i = 1 \right\}.$$

A set A is said to be *star-shaped* if there exists an $x \in A$ such that for every $y \in A$ the line segment $\{tx + (1 - t)y \mid t \in [0, 1]\}$ is contained in A. In other words, all points in A are within line of sight of at least one common point. All convex sets are star-shaped, but the converse is not true.