D Curve Tracing

MANY sensor-based techniques, such as those in sections 2.3.3 and 5.2.5, are essentially curve-tracing algorithms. In both cases, the robot is, in a sense, "determining" the curve as it is being traced. Such techniques relied on two fundamental principles: the curve being traced exists and under the "right conditions," the curve can be traced with simple predictor-corrector techniques. These principles rested on two theorems, the implicit function theorem, and the Newton-Raphson convergence theorem, described below.

D.1 Implicit Function Theorem

Consider a smooth function of multiple variables, $f(x, y)$, and consider the surface that is defined by the equation $f(x, y) = z_0$ for some fixed z_0. Under certain conditions, this surface can be used to write a new function that defines the y variables in terms of the x variables, i.e., $y = g(x, z_0)$. The theorem that states these conditions is called the *implicit function theorem*.

THEOREM D.1.1 (Implicit Function Theorem) *Let $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ be a smooth vector-valued function, $f(x, y)$. Assume that $D_y f(x_0, y_0)$ is invertible for some $x_0 \in \mathbb{R}^m$, $y_0 \in \mathbb{R}^n$. Then there exist neighborhoods X_0 of x_0 and Z_0 of $f(x_0, y_0)$ and a unique, smooth map $g : X_0 \times Z_0 \to \mathbb{R}^n$ such that*

$$f(x, g(x, z)) = z$$

for all $x \in X_0$, $z \in Z_0$.

D.2 Newton-Raphson Convergence Theorem

By numerically following the set of points where $f(x, y) = 0$, we can locally construct a curve. While there are a number of curve tracing techniques [232], consider an adaptation of a common predictor-corrector scheme. Moving in the tangent direction can serve as a prediction. However, if there is curvature, then the tangent prediction is not correct. Therefore, a correction method is used. The correction procedure occurs on a plane orthogonal to the tangent; this plane is called a correcting plane. The correction step finds the location where the curve being traced intersects the correcting plane and is an application of the Newton Convergence Theorem [232].

THEOREM D.2.1 (Newton-Raphson Convergence Theorem) *Let* $f : \mathbb{R}^n \to \mathbb{R}^n$ *and* $f(y^*) = 0$*. For some* $\rho > 0$*, let* f *satisfy*

- $Df(y^*)$ *is nonsingular with bounded inverse, i.e.,* $\|(Df(y^*))^{-1}\| \leq \beta$
- $\|Df(x) - Df(y)\| \leq \gamma\|x - y\|$ *for all* $x, y \in B_\rho(y^*)$*, where* $\gamma \leq \frac{2}{\rho\beta}$

Now consider the sequence $\{y^h\}$ *defined by*

$$y^{h+1} = y^h - (Df(y^h))^{-1} f(y^h),$$

for any $y^0 \in B_\rho(y^*)$*. Then* $y^h \in B_\rho(y^*)$ *for all* $h > 0$*, and the sequence* $\{y^h\}$ *quadratically converges onto* y^**, i.e.,*

$$\|y^{h+1} - y^*\| \leq a\|y^h - y^*\|^2$$

where $a = \frac{\beta\gamma}{2(1-\rho\beta\gamma)} < \frac{1}{\rho}$.