J Linear Systems and Control

THIS APPENDIX gives a brief review of the theory of linear time invariant (LTI) dynamical systems. Many dynamical systems that appear in science and engineering can be approximated by LTI systems, and linear systems theory provides important tools to control and observe them. We focus on the so-called *state space* formulation of LTI systems because that is the formulation used in the Kalman filter (see chapter 8). In this appendix we present some of the more fundamental concepts of LTI state-space systems, including stability, feedback control, and observability.

J.1 State Space Representation

Consider as an example the mass-spring-damper system depicted in figure J.1, where $z(t)$ denotes the position of the mass m at time t. If we assume that the spring is linear, then the force applied by the spring is given as $F_s = -kz(t)$. Likewise, if we assume that the damper is linear, then the force applied by the damper is proportional to the velocity of the mass, yielding $F_d = -\gamma \frac{dz}{dt}(t) \overset{\triangle}{=} \gamma \dot{z}(t)$. For now we assume the externally applied force $F_{\text{ext}} = 0$. Summing these forces and applying Newton's law (force = mass × acceleration) yields

$$m\ddot{z}(t) = -\gamma \dot{z}(t) - kz(t). \tag{J.1}$$

This second-order ordinary differential equation (ODE) provides a mathematical description of how the position and velocity of mass change with time. Accordingly, we call equation (J.1) a *model* of the mass-spring-damper system. If the position z and

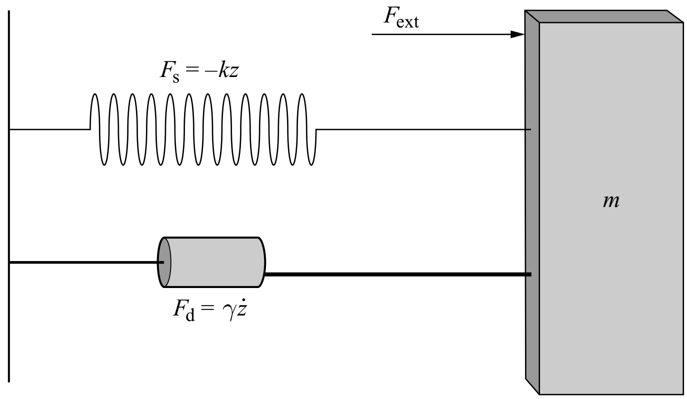

Figure J.1 Mass–spring–damper system.

velocity $\dot{z}$ are known at some instant of time t_0, then the solution to equation (J.1) subject to initial conditions $z(t_0)$ and $\dot{z}(t_0)$ will match the trajectory of the physical system.

Now define the vector

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix}.$$

Equation (J.1) can be rewritten in terms of x as follows:

$$\dot{x}(t) = \begin{bmatrix} \dot{z}(t) \\ \ddot{z}(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -\frac{1}{m}\left(\gamma\dot{z}(t) + kz(t)\right) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -\frac{1}{m}\left(\gamma x_2(t) + kx_1(t)\right) \end{bmatrix},$$

which can finally be summarized as

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{bmatrix} x(t). \tag{J.2}$$

Thus we have taken a second-order scalar ODE and rewritten it as a first-order vector ODE. We call this first-order vector ODE the *state-space representation* of the mass-spring-damper system, and the state vector $x(t)$ is a member of the *state space*. Since the right hand side of equation (J.2) can be written as a constant matrix multiplied by the state vector, this system is both linear and time invariant.

Generally, an LTI state-space system can be written as the vector ODE,

$$\dot{x}(t) = Ax(t); \qquad x(t_0) = x_0, \tag{J.3}$$

where $x(t) \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$. This ODE is sometimes called a *vector field* because it assigns a vector Ax to each point x in the state space. This ODE has a unique solution, and the solution can be written in closed form,

$$x(t) = e^{A(t-t_0)} x_0, \tag{J.4}$$

where the matrix exponential is defined by the Peano-Baker series

$$\begin{aligned} e^{A(t-t_0)} &= \sum_{i=0}^{\infty} \frac{A^i \, (t-t_0)^i}{i\,!} \\ &= I_{n\times n} + A\,(t-t_0) + \frac{A^2 (t-t_0)^2}{2!} + \cdots . \end{aligned}$$

J.2 Stability

Assuming that the matrix A has full rank, then the point $x = 0$ is the only point in the state space that satisfies the equilibrium condition $\dot{x} = 0$. The point $x = 0$ is called an equilibrium point. Note that the state will not move from an equilibrium point. If the initial condition is $x(t_0) = 0$, then $x(t)$ will remain at 0 for all time. In this section we discuss the stability of the origin of an LTI system.

We begin by defining a few notions of stability. An equilibrium point x_e (in the case of LTI systems $x_e = 0$) is said to be *stable* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever the initial condition satisfies $\|x_e - x(t_0)\| < \delta$ the solution $x(t)$ satisfies $\|x_e - x(t)\| < \epsilon$ for all time $t > 0$. In other words, stable means that if the initial condition starts close enough to the equilibrium, then the solution will never drift very far away. x_e is said to be *asymptotically stable* if it is stable and $\|x_e - x(t)\| \to 0$ as $t \to \infty$. Likewise, x_e is said to be *unstable* if it is neither stable nor asymptotically stable.

It is worth noting that for LTI systems, the stability properties are global. If they hold on any open subset of the state space, then they hold everywhere. Stability can be characterized in terms of the eigenvalues of the matrix A, as stated in the following theorem:

THEOREM J.2.1 (LTI stability) *Consider the LTI system stated in equation (J.3), and let λ_i, $i \in \{1, 2, \ldots, n\}$ denote the eigenvalues of A. Let* $\mathrm{re}(\lambda_i)$ *denote the real part of λ_i Then the following holds:*

1. *$x_e = 0$ is stable if and only if* $\mathrm{re}(\lambda_i) \leq 0$ *for all i.*

2. *$x_e = 0$ is asymptotically stable if and only if* $\mathrm{re}(\lambda_i) < 0$ *for all i.*

3. *$x_e = 0$ is unstable if and only if* $\mathrm{re}(\lambda_i) > 0$ *for some i.*

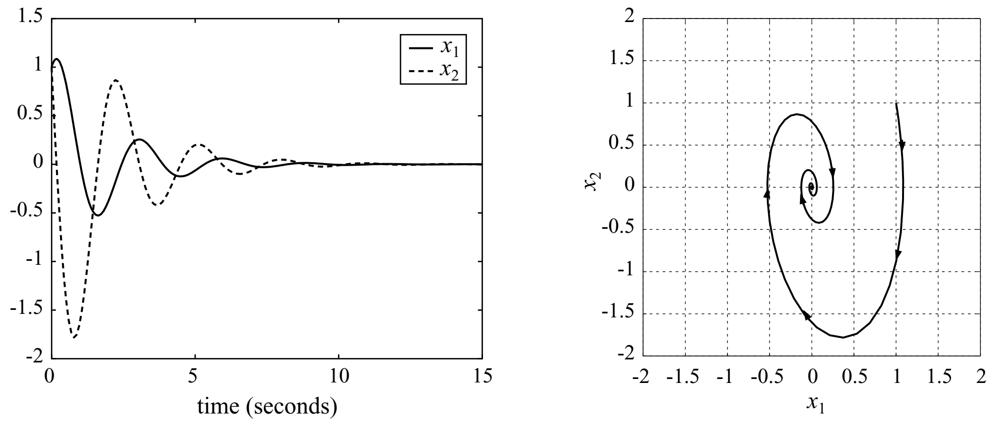

Figure J.2 Asymptotical stability. (Left) The states x_1 and x_2 (z and $\dot{z}$, respectively) plotted as time evolves. (Right) Phase plane plot of x_2 vs. x_1.

Consider the mass-spring-damper example. The eigenvalues of A are

$$\frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}.$$

When the damping term is positive, the real parts of the eigenvalues are negative and the system is asymptotically stable. Figure J.2 shows two different representations of the trajectory of the mass-spring-damper system with $m = 1$, $k = 5$, and $\gamma = 1$. The figure on the left shows the values of x_1 and x_2 plotted as functions of time. As expected for an asymptotically stable system, both converge to zero. The figure on the right shows the trajectory in state space by plotting x_2 vs. x_1. This is sometimes referred to as a "phase plane" plot. The direction in which the trajectory flows is depicted by arrows. Here the trajectory starts at the initial condition and spirals into the origin. When the damping is zero, the system solution is a bounded oscillation and hence is stable but not asymptotically stable. Figure J.3 plots the time and phase plane representations of the stable trajectory that results when $m = 1$, $k = 5$, and $\gamma = 0$. Note that in the phase plane the periodic oscillation becomes a closed loop. When the damping is negative the damping term actually adds energy to the system, creating an oscillation that grows without bound. Time and phase plane plots for the case where $m = 1$, $k = 5$, and $\gamma = -0.4$ are shown in figure J.4.

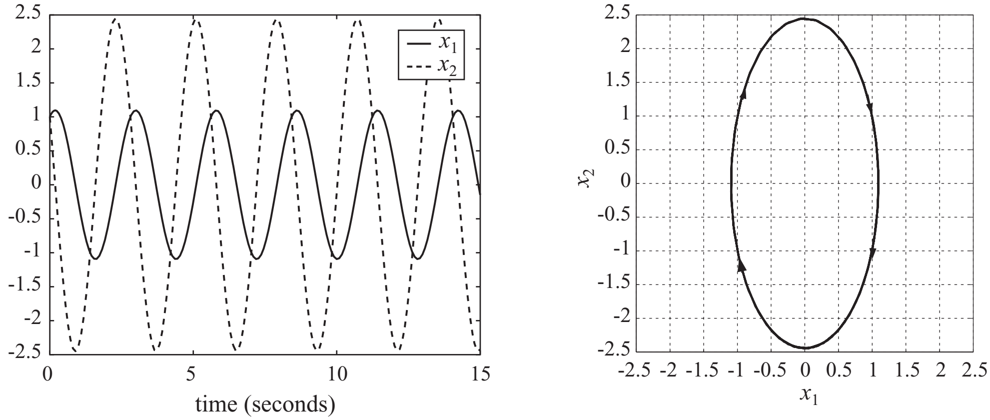

Figure J.3 Stability. (Left) The states x_1 and x_2 (z and $\dot{z}$, respectively) plotted as time evolves. (Right) Phase plane plot of x_2 vs. x_1.

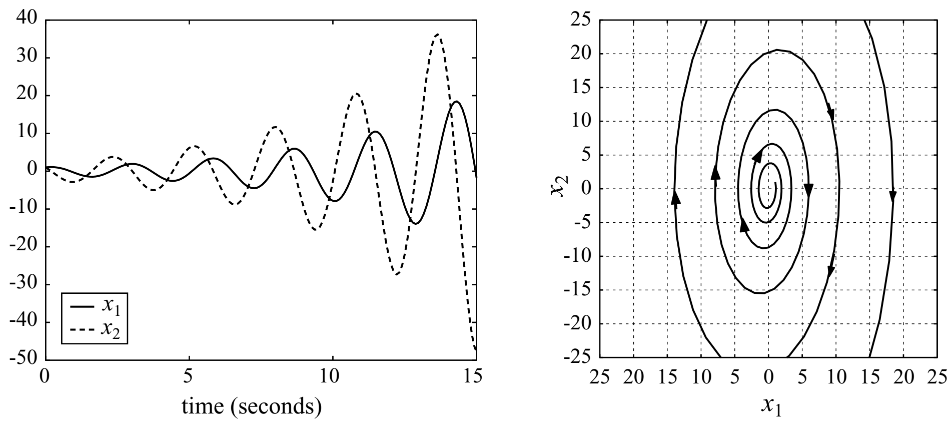

Figure J.4 Instability. (Left) The states x_1 and x_2 (z and $\dot{z}$, respectively) plotted as time evolves. (Right) Phase plane plot of x_2 vs. x_1.

J.3 LTI Control Systems

Often one has the ability to affect the behavior of a dynamical system by applying some sort of external input. For example, in the mass-spring-damper system discussed earlier we can influence the trajectory of the system by applying a time-varying external force $F(t)$ to the mass. This results in the LTI control system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} F(t). \tag{J.5}$$

More generically, we write an LTI control system as

$$\dot{x}(t) = Ax(t) + Bu(t); \qquad x(t_0) = x_0, \tag{J.6}$$

where the state vector $x(t) \in \mathbb{R}^n$ and the external input vector $u(t) \in \mathbb{R}^m$. The matrix $B \in \mathbb{R}^{n\times m}$. The matrix A describes the system dynamics of the unforced system, i.e., A describes how the state would evolve if the input were zero. B describes how the inputs affect the evolution of the state.

The system described in equation (J.3) is said to be *controllable* if for any initial condition $x(t_0)$, there exists a continuous control input $u(t)$ that drives the solution $x(t)$ to the origin, $x = 0$. Note that the origin is an equilibrium point for the unforced system. This definition of controllability is equivalent to the definition of controllability for nonlinear systems presented in chapter 8, section 12.3 where the goal state is restricted to $x_{\text{goal}} = 0$.

THEOREM J.3.1 (LTI Controllability Test) *The LTI control system in equation (J.6) is controllable if and only if the matrix*

$$W_c = [B \ AB \ A^2B \ \cdots \ A^{n-1}B]$$

has rank n.

Because controllability is determined solely by the matrices A and B, we can say that the pair (A, B) is controllable if the system in equation (J.6) is controllable.

One common control objective is to make the origin of a naturally unstable system stable using state feedback. Consider the control input given by the state-dependent control law

$$u(t) = -Kx(t)$$

for some matrix $K \in \mathbb{R}^{m\times n}$. Substituting this into equation (J.6) yields

$$\dot{x}(t) = (A - BK)x(t).$$

As a result, we can examine the stability of this new system in terms of the eigenvalues of the matrix $A - BK$. One of the fundamental properties of real-valued matrices is that their eigenvalues must occur in complex conjugate pairs. If $a+bi$ is an eigenvalue of a matrix, then $a - bi$ must also be an eigenvalue of that matrix. Hence we define a collection of complex numbers $\Lambda = \{\lambda_i \mid i \in \{1, 2, \ldots, n\}\}$ to be *allowable* if for each λ_i that has a nonzero imaginary part there is a corresponding conjugate λ_j. Now we are prepared to state an important result of linear control theory:

THEOREM J.3.2 (Eigenvalue Placement) *Consider the system of equation (J.6) and assume the pair* (A, B) *is controllable and that the matrix B has full column rank. Let* $\Lambda = \{\lambda_i \mid i \in \{1, 2, \ldots, n\}\}$ *be any allowable collection of complex numbers. Then there exists a constant matrix* $K \in \mathbb{R}^{m\times n}$ *such that the set of eigenvalues of* $(A - BK)$ *is equal to* Λ.

Under the assumptions of this theorem, we can place the eigenvalues of the matrix $A - BK$ in any allowable configuration using linear feedback. The task of stabilizing an LTI system is then simply a matter of finding a K so that the corresponding eigenvalues have negative real parts. There are a number of algorithms to perform direct eigenvalue assignment (also sometimes called pole placement). Some of these are implemented in the MATLAB control systems toolbox. Similarly, the famous linear quadratic regulator (LQR) (see e.g., [396]) places the eigenvalues of $A - BK$ to optimize a user-defined cost function.

Consider as an example the mass-spring-damper system with negative damping. As was pointed out earlier, this system is unstable; solutions for initial conditions arbitrarily close to the origin will grow without bound. To use state feedback to stabilize this system, consider the matrix

$$A - BK = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} [k_1 \quad k_2] = \begin{bmatrix} 0 & 1 \\ -\frac{k+k_1}{m} & \frac{\gamma+k_2}{m} \end{bmatrix}.$$

The eigenvalues of $A - BK$ are

$$\frac{(-\gamma - k_2) \pm \sqrt{(-\gamma - k_2)^2 - 4(k - k_1)m}}{2m},$$

so we can ensure that the real part of both eigenvalues is negative by choosing k_2 such that $-\gamma - k_2 < 0$. This is equivalent to adding sufficient positive viscous damping to overcome the energy added by the negative damping term γ.

J.4 Observing LTI Systems

Often it is not possible to directly measure the entire state of an LTI system. Rather, the state must be observed through the use of sensors that provide some lower-dimensional measurement of the current state. If it were possible to measure only velocity in the mass-spring-damper example, then equations of motion together with the output equation for the system would be

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} F(t), \\ y(t) &= [0 \quad 1]\, x(t), \end{aligned} \tag{J.7}$$

where $y(t)$ represents the output signal coming from the sensor. We write a general LTI system with output equation as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t); \qquad x(t_0) = x_0, \\ y(t) &= Cx(t), \end{aligned} \tag{J.8}$$

where the state vector $x(t) \in \mathbb{R}^n$, the control vector $u(t) \in \mathbb{R}^m$, and the output vector $y(t) \in \mathbb{R}^p$. The constant matrix $C \in \mathbb{R}^{p \times n}$. Note that the matrix C may not be invertible (it is usually not even square!), so the state at any instant $x(t)$ cannot be directly observed from the measurement at that instant $y(t)$. We must instead reconstruct the state by measuring the output over some interval of time and using knowledge of the system dynamics. A device that performs such a reconstruction is called an *observer*.

We say that the system of equation (J.8) is *observable* if it is possible to determine the initial state $x(t_0)$ by observing the known signals $y(t)$ and $u(t)$ over some period of time.

THEOREM J.4.1 (LTI Observability Test) *The LTI control system in equation (J.8) is observable if and only if the matrix*

$$W_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has rank n.

As in the case of controllability, we say that the pair (A, C) is observable if the system in equation (J.8) is observable. Note that the pair (A, C) is observable if and only if

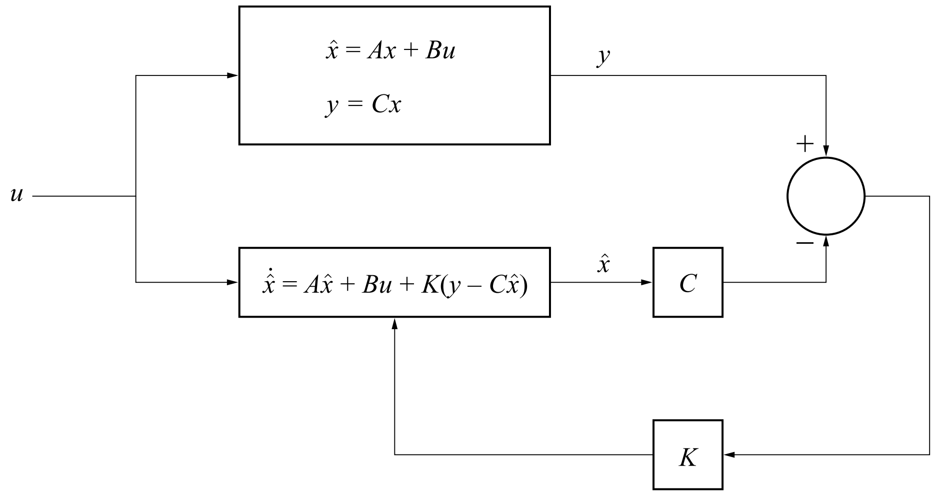

Figure J.5 Block diagram for a linear observer.

the pair (A^T, C^T) is controllable. If the pair (A, B) is controllable and the pair (A, C) is observable, then the system [and the triple (A, B, C)] is said to be *minimal*.

Now consider an observer defined by the ODE

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t)). \tag{J.9}$$

Note that this ODE requires that we know the matrices A, B, and C as well as the input $u(t)$ and output $y(t)$. The vector $\hat{x}(t)$ is called the *state estimate* produced by this observer. As shown in the block diagram in figure J.5, this observer is essentially a copy of the original dynamic system with a correcting term that is a linear function of the difference between the measured output $y(t)$ and the estimated output $C\hat{x}(t)$. The task is then to try to choose K so that the correcting term forces the state estimate to converge to the actual value.

If we define the error signal $e(t) = x(t) - \hat{x}(t)$, we can examine how the error evolves with time:

$$\begin{aligned}
\dot{e}(t) &= \dot{x}(t) - \dot{\hat{x}}(t) \\
&= Ax(t) + Bu(t) - (A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t))) \\
&= A(x(t) - \hat{x}(t)) - K(Cx(t) - C\hat{x}(t)) \\
&= (A - KC)e(t)
\end{aligned}$$

If $e(t) \to 0$, then $\hat{x}(t) \to x(t)$. So the state estimate $\hat{x}(t)$ that results from the observer presented in equation (J.9) converges to the actual state $x(t)$ if K is chosen so that the unforced LTI system $\dot{e}(t) = (A - KC)e(t)$ is asymptotically stable.

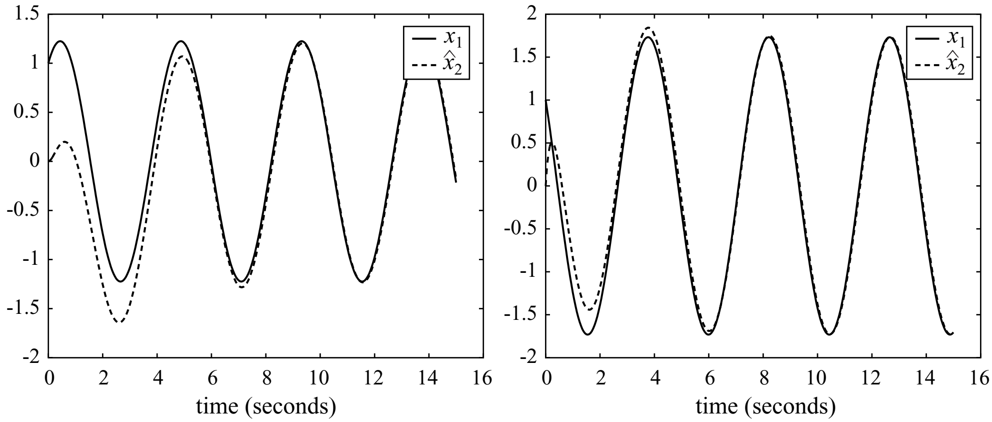

Figure J.6 Solid lines represent the actual state and the dashed line represents the state estimate determined by the observer. The left figure depicts x_1 and the right x_2.

Recall that the eigenvalues of any matrix are equal to the eigenvalues of its transpose, so the eigenvalues of $A - KC$ are identical to the eigenvalues of $A^T - C^T K^T$. According to theorem J.3.2, we can place the eigenvalues of $A^T - C^T K^T$ in any allowable configuration provided that the pair (A^T, C^T) is controllable and the matrix C^T has full column rank. This is equivalent to saying that the eigenvalues of $A - KC$ can be placed in any allowable configuration provided that the pair (A, C) is observable and C has full row rank. Under these conditions, it is possible to chose a K so that the observer estimate $\hat{x}(t)$ converges to $x(t)$.

Consider the mass-spring-damper system of equation (J.7). The matrix

$$A - KC = \begin{bmatrix} 0 & 1 - k_1 \\ -\frac{k}{m} & -\frac{\gamma}{m} - k_2 \end{bmatrix}.$$

The eigenvalues of this matrix are

$$\frac{-(\gamma + mk_2) \pm \sqrt{(-\gamma - mk_2)^2 - 4m(k - k_1)}}{2m},$$

so choosing k_2 such that $-\gamma - mk_2 < 0$ will guarantee that the observer given in equation (J.9) converges, meaning that after some initial transient, estimate $\hat{x}(t)$ will provide a good approximation of the state. For the case where $m = 1$, $k = 2$, and $\gamma = 0$, the choice of $K = [0\ 2]^T$ will provide a convergent observer. Figure J.6 shows how the estimates $\hat{x}_1(t)$ and $\hat{x}_2(t)$ converge to $x_1(t)$ and $x_2(t)$, respectively.

J.5 Discrete Time Systems

The previous sections dealt with an LTI system whose trajectories were continuous in time. In practice, a continuous dynamical system is usually sampled at regular time intervals. The sampled or *discrete time* signal is then fed into a computer as a sequence of numbers. The computer can then use this sequence to calculate a desired control input or to estimate the state. In this section we present an overview of the theory of discrete time LTI systems and their relationship to their continuous time cousins.

Consider the continuous time signal $x(t)$. We define a sequence of vectors using the formula $x_s(k) = x(t_0 + kT)$. The sequence $x_s(k)$ is the *discrete time sampling* of the continuous signal $x(t)$. In the future, we will abuse notation and drop the s subscript on the discrete time sequence. The continuous and discrete signals can be differentiated by the letter used in their argument; $x(k)$ represents an element of the sequence and $x(t)$ denotes the continuous time signal.

Using the first-order derivative approximation

$$\dot{x}(t_0 + kT) \approx \frac{x(k+1) - x(k)}{T}$$

and substituting into the continuous time LTI system of equation (J.8) yields

$$\frac{x(k+1) - x(k)}{T} \approx Ax(k) + Bu(k),$$

which leads to

$$x(k+1) \approx x(k) + TAx(k) + TBu(k).$$

Defining $F = I_{n\times n} + TA$, $G = TB$, and $H = C$, we can then write a discrete time approximation of the continuous system:

$$\begin{aligned} \dot{x}(k+1) &= Fx(k) + Gu(k); \qquad x(0) = x_0 \\ y(k) &= Hx(k) \end{aligned} \tag{J.10}$$

Most of the concepts from continuous LTI systems have direct analogs in discrete time LTI systems. We discuss them briefly here.

J.5.1 Stability

The discrete time notions of stability, asymptotic stability, and instability follow directly from the continuous time definitions. As in the case of continuous systems, the stability of the unforced system $x(k+1) = Fx(k)$ can be evaluated in terms of the eigenvalues of F:

THEOREM J.5.1 (Discrete Time LTI Stability) *Consider the unforced discrete time LTI system described by the equation $x(k+1) = Fx(k)$, and let λ_i, $i \in \{1, 2, \ldots, n\}$ denote the eigenvalues of F. Then the following hold:*

1. *$x_e = 0$ is stable if and only if $|\lambda_i| \leq 1$ for all i.*

2. *$x_e = 0$ is asymptotically stable if and only if $|\lambda_i| < 1$ for all i.*

3. *$x_e = 0$ is unstable if and only if $|\lambda_i| > 1$ for some i.*

J.5.2 Controllability and Observability

The properties of controllability and observability for the discrete time LTI system follow from the properties of the continuous time system. The controllability test is the same for both: the pair (F, G) is controllable if and only if the matrix $[G \ FG \ F^2G \ \cdots \ F^{n-1}G]$ has rank n. The pair (F, H) is observable if and only if the pair (F^T, H^T) is controllable. As in the case of continuous systems, construction of linear state feedback control laws or linear observers results in a pole placement problem which can be solved if the system is controllable or observable, respectively.