EL2450 – Assignment I

1 Question 3

The reference signal is a step of 10 units from time 100 seconds, with an offset of 40 units, as seen in figure 1.

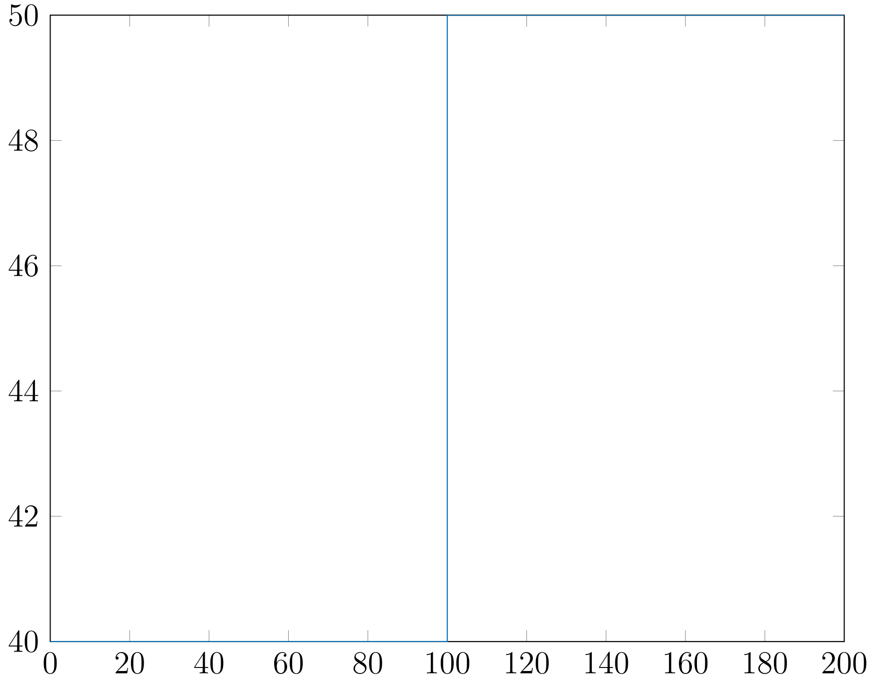

Figure 1: The reference signal.

2 Question 4

Table 1 illustrates the corresponding values of the K, T_I, T_D and N coefficients for set χ, ζ and ω_0.

χ	ζ	ω_0	K	T_I	T_D	M
0.5	0.7	0.1	2.6062	14.4445	5.5143	0.9791
0.5	0.7	0.2	5.9243	9.3823	3.1938	1.1191
0.5	0.8	0.2	6.3325	10.3873	3.1523	1.1591

Table 1: Coefficients of the PID controller per set χ, ζ and ω_0 values.

3 Question 5

Table 2 illustrates the rise time, overshoot and settling time for set values of χ, ζ and ω_0.

χ	ζ	ω_0	T_r	M	T_s
0.5	0.7	0.1	8.2	14.40	39.0
0.5	0.7	0.2	5.0	34.67	23.7
0.5	0.8	0.2	4.95	31.72	24.25

Table 2: Rise time (T_r) in seconds, overshoot (M) as a percentage of the output's steady state value, and settling time T_s in seconds for set values of χ, ζ and ω_0.

Due to our step response requirements, the best control performance is given by the third set of (χ, ζ, ω_0) parameters. All three requirements are fulfilled, as opposed to the case of the first set, and in comparison to the case of the second set, the rise time and overshoot are less, while their settling times are comparable.

Figures 2, 3 and 4 depict the step response for the three sets of (χ, ζ, ω_0) parameters.

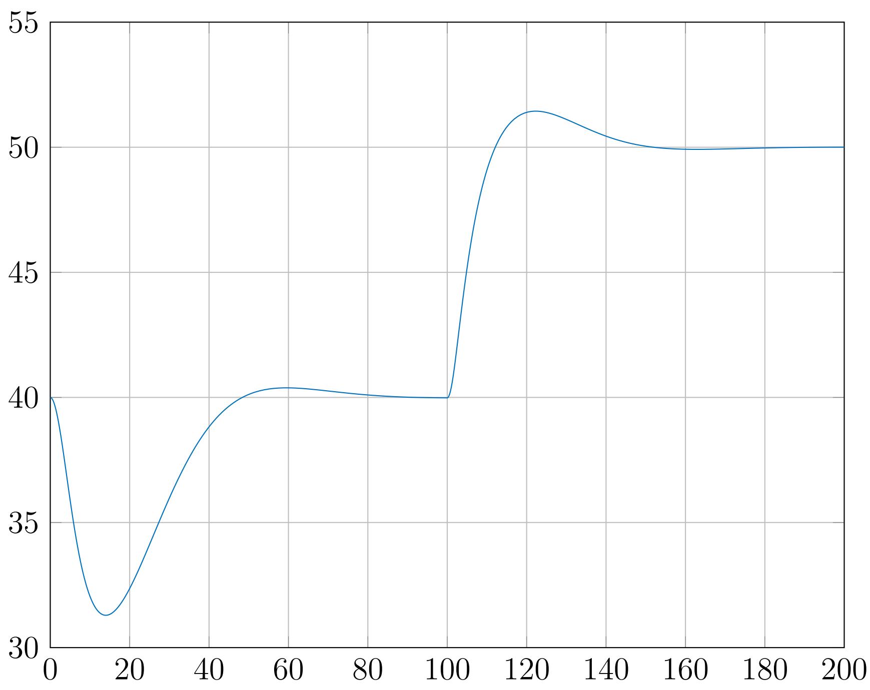

Figure 2: Step response for $(\chi, \zeta, \omega_0) \equiv (0.5, 0.7, 0.1)$.

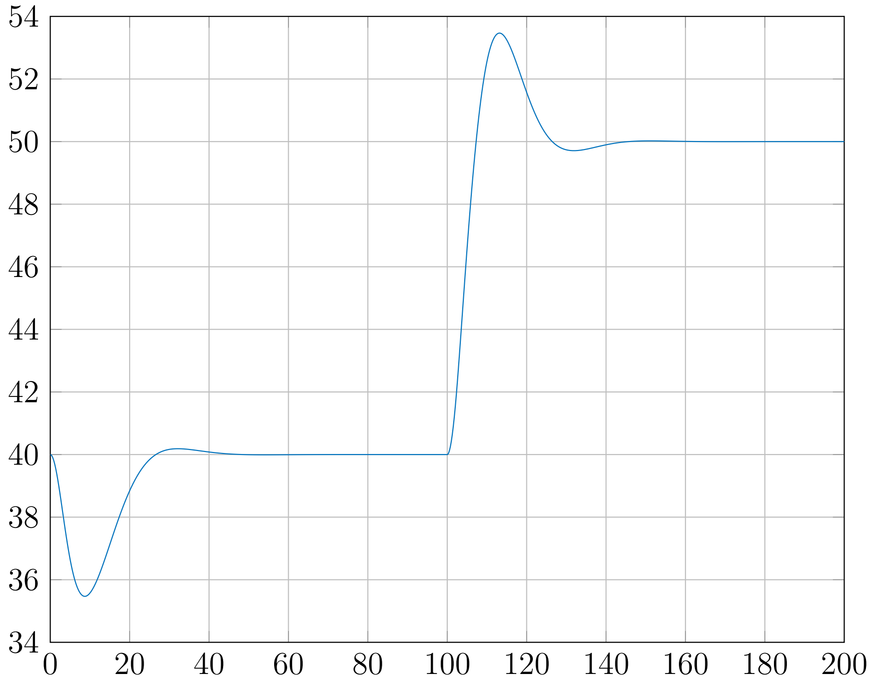

Figure 3: Step response for $(\chi, \zeta, \omega_0) \equiv (0.5, 0.7, 0.2)$.

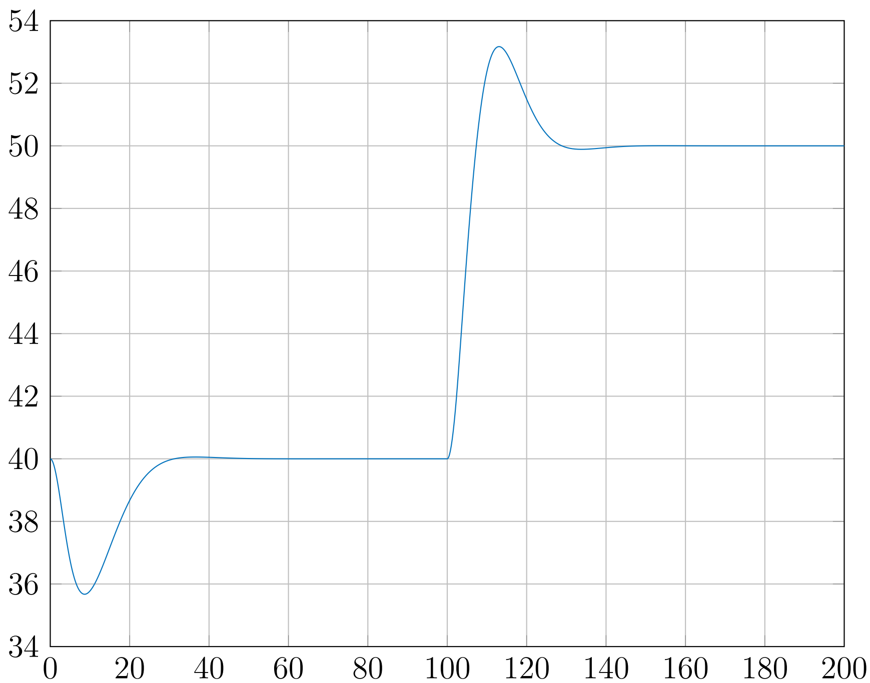

Figure 4: Step response for $(\chi, \zeta, \omega_0) \equiv (0.5, 0.8, 0.2)$.

4 Question 6

The open-loop transfer function is equal to the product of the transfer function of the controller $F(s)$ and that of the process $G(s)$. The crossover frequency ω_c is the frequency at which the magnitute of $F(j\omega)G(j\omega)$ is 1.0.

In practice, we were able to derive the crossover frequency by using MATLAB's `margin()` function, with argument the open-loop transfer function.

Table 3 illustrates the crossover frequencies in rad/s for set values of the χ, ζ and ω_0 parameters.

χ	ζ	ω_0	ω_c
0.5	0.7	0.1	0.2239
0.5	0.7	0.2	0.3426
0.5	0.8	0.2	0.3619

Table 3: Crossover frequencies depending on the set values of χ, ζ and ω_0.

5 Question 7

Figure 5 shows the root locus of the open-loop transfer function without the use of a zero-order hold, while figure 6 shows exacly the same, but with the addition of a zero-order hold between the controller and the process, with sampling time of $h = 1$ sec. It is apparent that without the zero-order hold, the system is stable, since all poles have negative real values. However, the time delay and the pole at zero that the zero-order hold introduces deliver a reduction in the degree of stability.

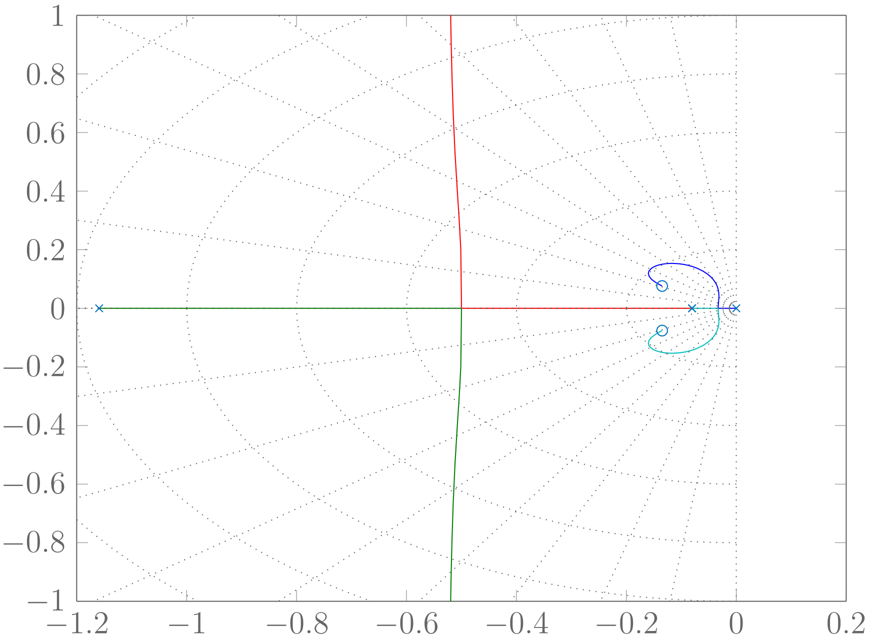

Figure 5: Root locus of the open-loop system without the use of a zero-order hold.

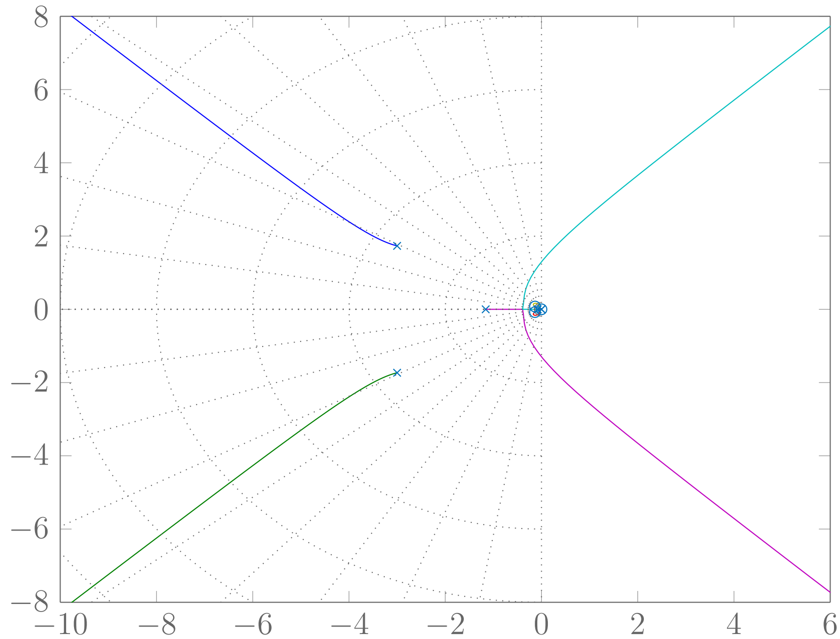

Figure 6: Root locus of the open-loop system with the use of a zero-order hold.

Figures 7-14 illustrate the step response when between the continuous controller and the plant a zero-order hold has been inserted, for sampling time varying between 1 and 8 seconds. The values of the χ, ζ and ω_0 parameters were chosen to be the ones giving the best performance among the three sets, hence $(\chi, \zeta, \omega_0) \equiv (0.5, 0.8, 0.2)$.

Here, up until $h = 5$ sec, as the sampling time increases, so do the rise time, settling time and overshoot. However, increasing h beyond 7 sec make the system critically stable, since two conjugate poles are approaching 0.

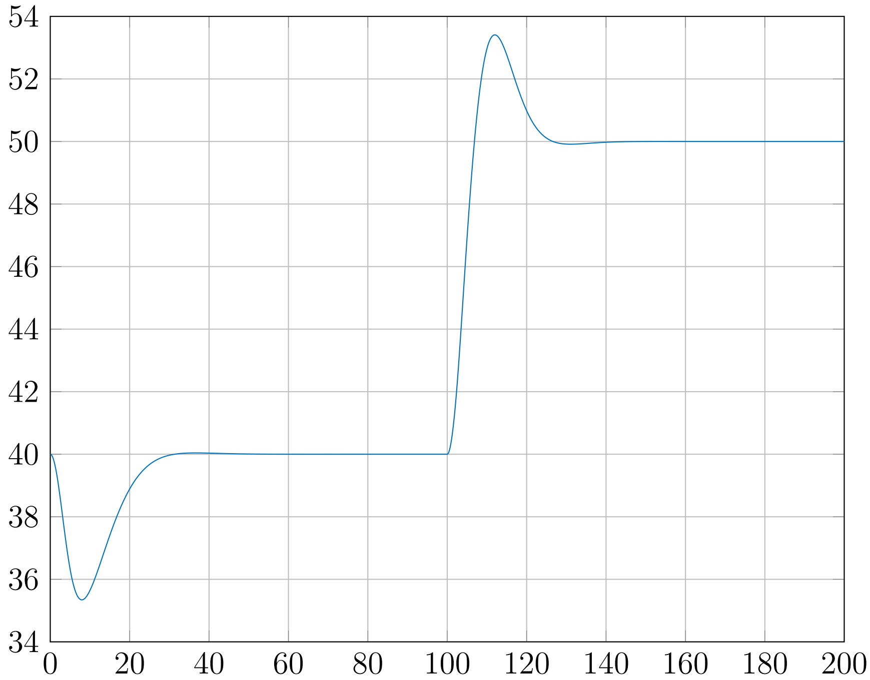

Figure 7: Step response using a zero-order hold of sample time 1 sec.

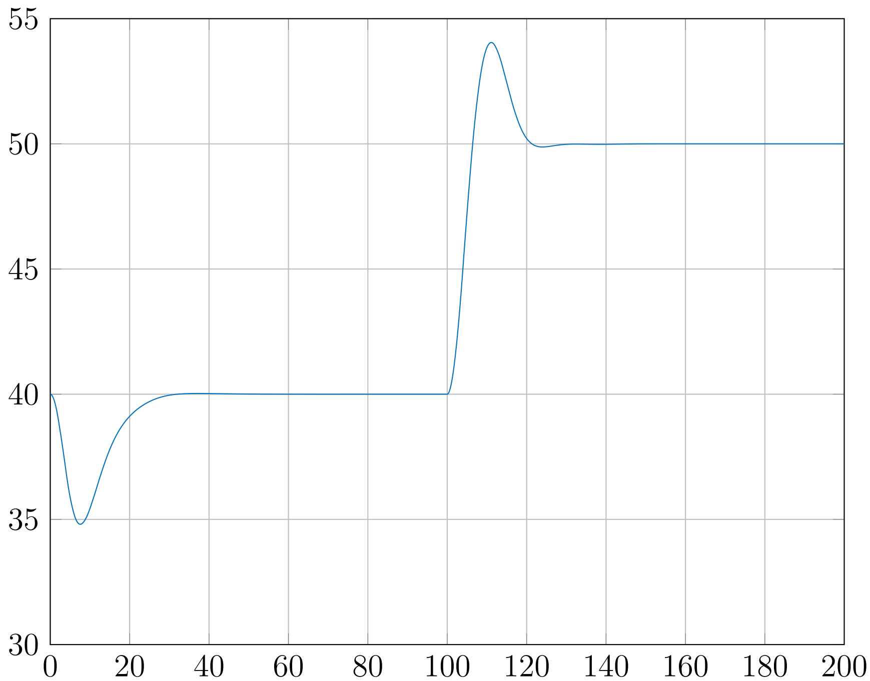

Figure 8: Step response using a zero-order hold of sample time 2 sec.

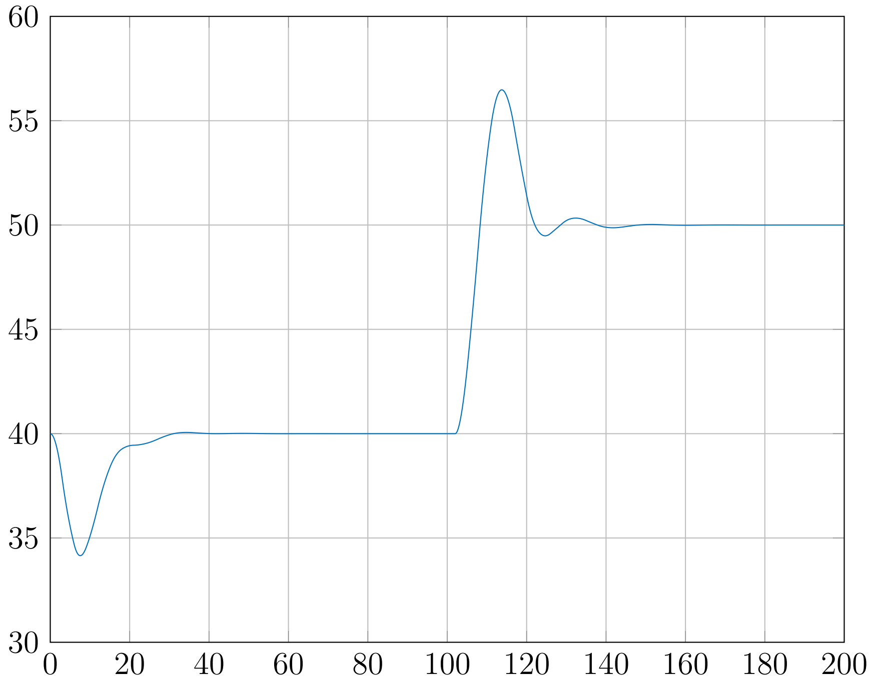

Figure 9: Step response using a zero-order hold of sample time 3 sec.

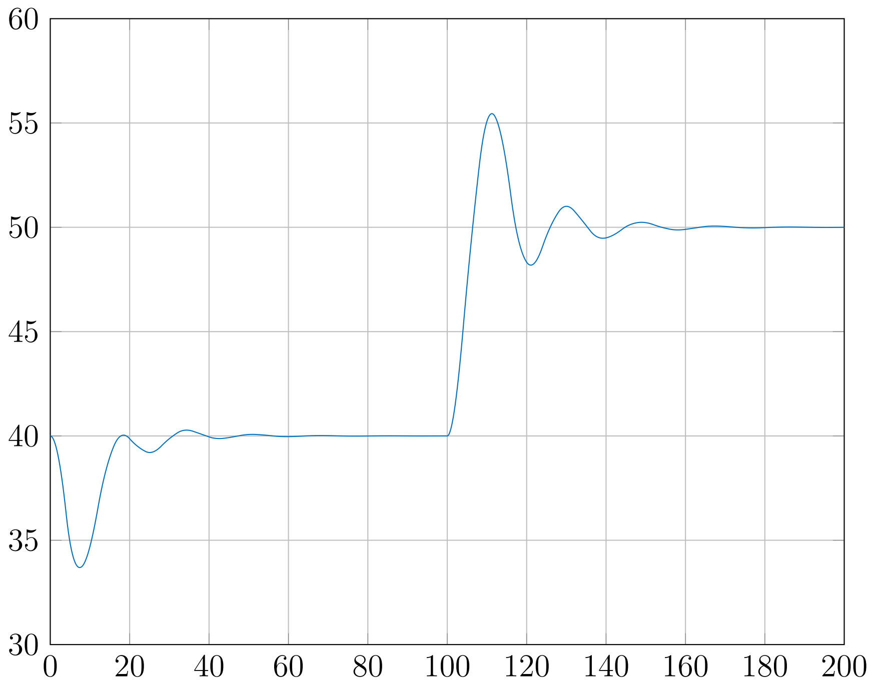

Figure 10: Step response using a zero-order hold of sample time 4 sec.

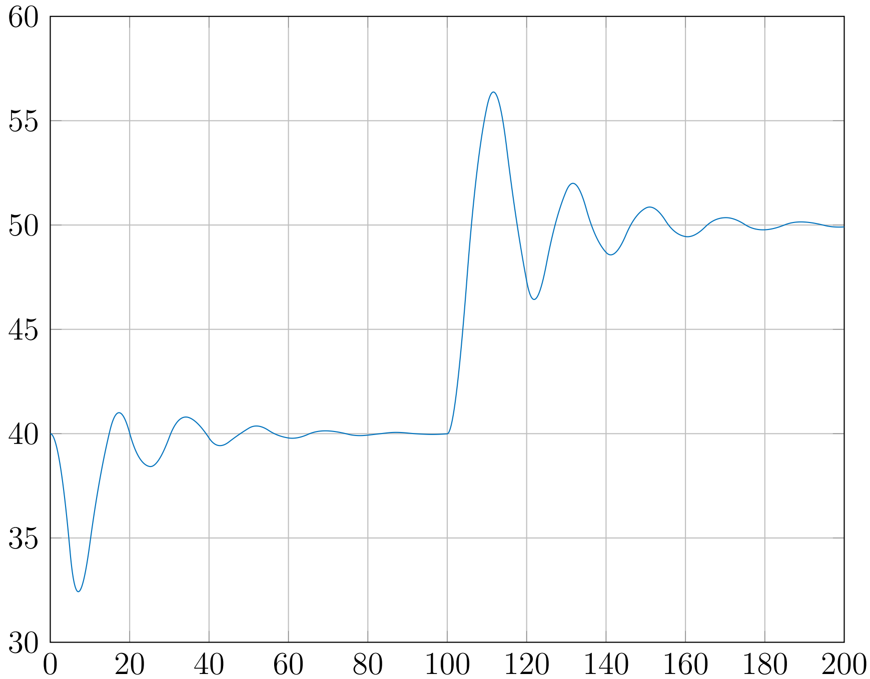

Figure 11: Step response using a zero-order hold of sample time 5 sec.

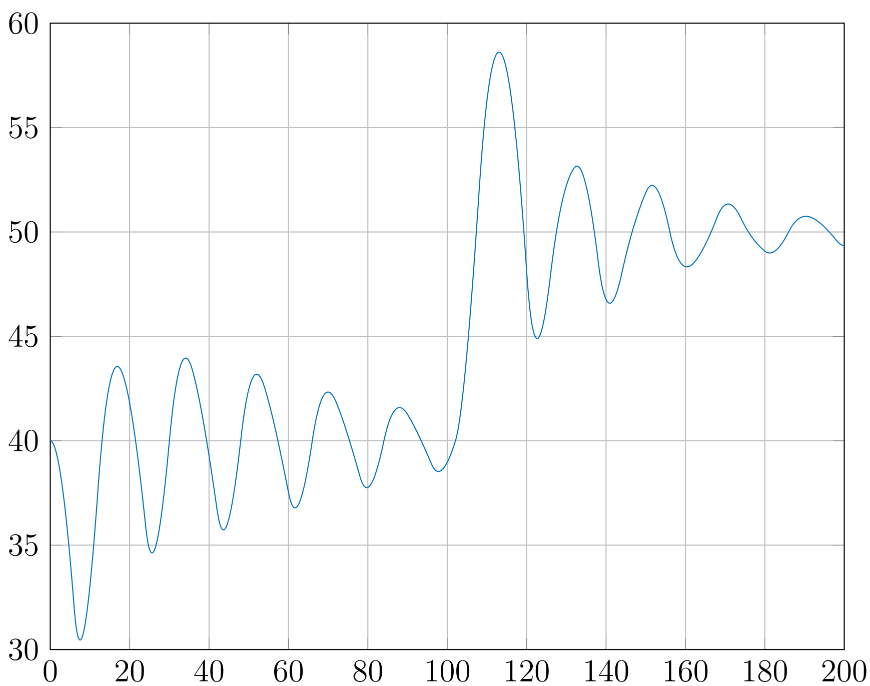

Figure 12: Step response using a zero-order hold of sample time 6 sec.

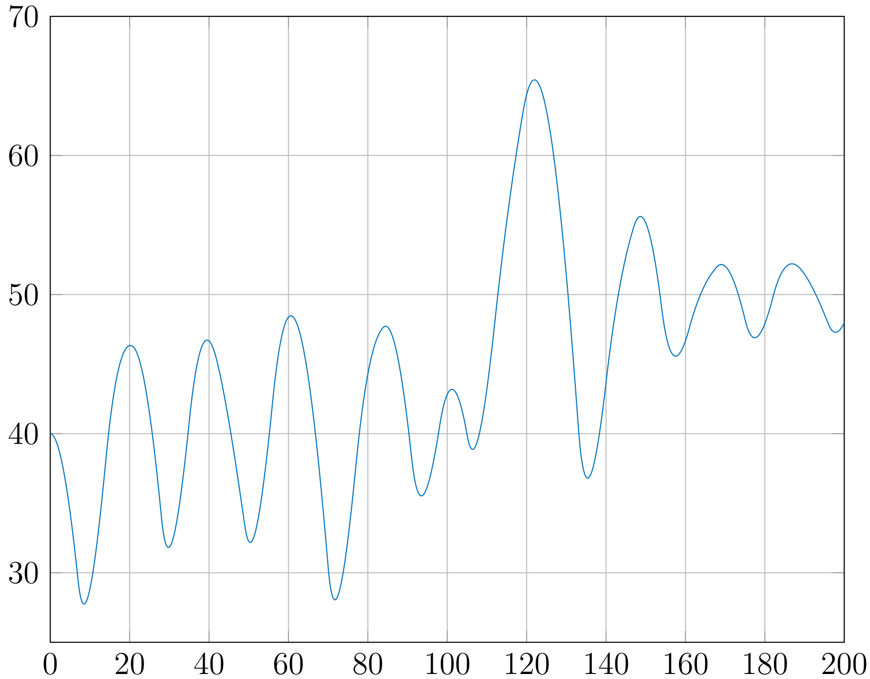

Figure 13: Step response using a zero-order hold of sample time 7 sec.

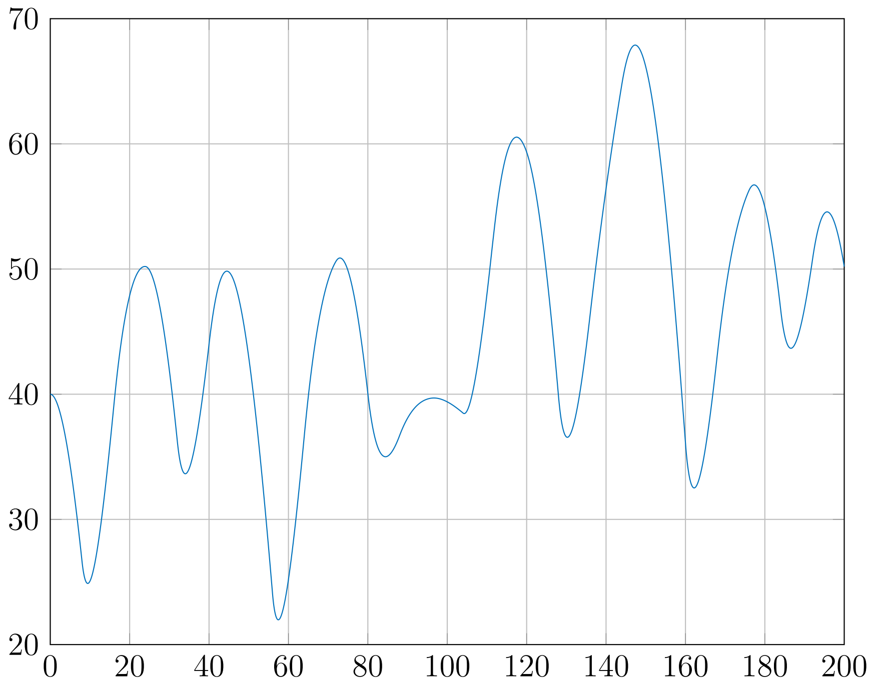

Figure 14: Step response using a zero-order hold of sample time 8 sec.

6 Question 8

Figures 15-27 show the step response for the case of a discrete and a continuous controller with a zero-order hold, marked in red and blue respectively, for varying values of the sampling time. It is evident that the lower the sampling rate, the less stable the system becomes, and the more the responses of the two controllers differ from each other.

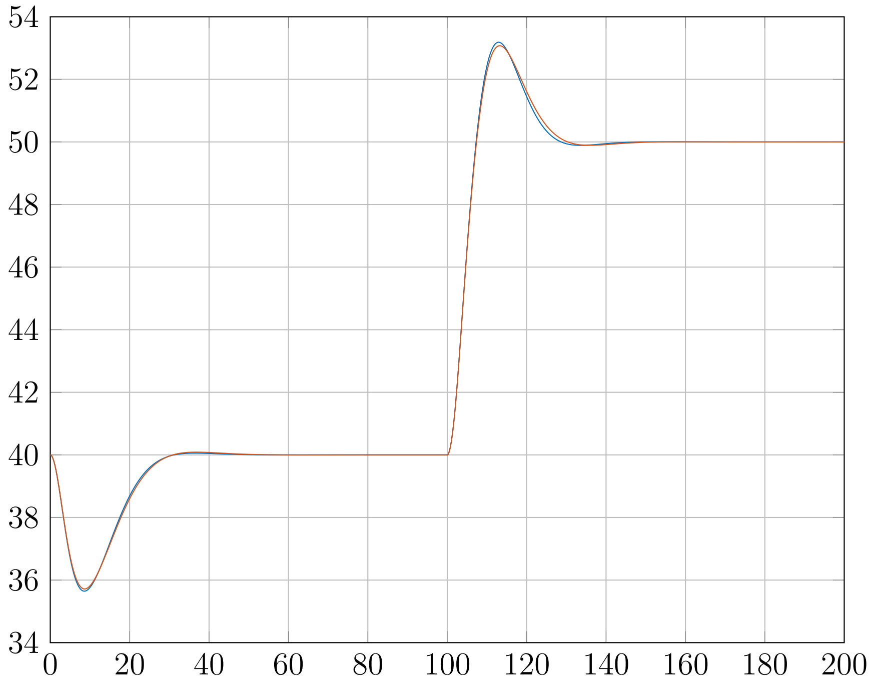

Figure 15: Step response using a discrete controller (red) with sampling time $T_s = 0.1$ sec and a continuous one with a zero-order hold (blue) with sampling time $h = 0.1$.

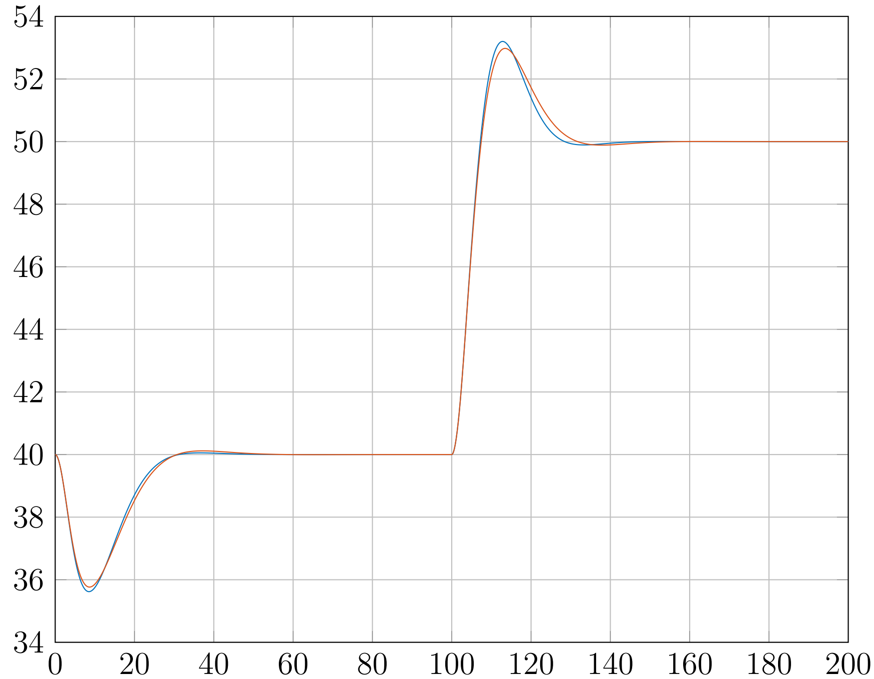

Figure 16: Step response using a discrete controller (red) with sampling time $T_s = 0.2$ sec and a continuous one with a zero-order hold (blue) with sampling time $h = 0.2$.

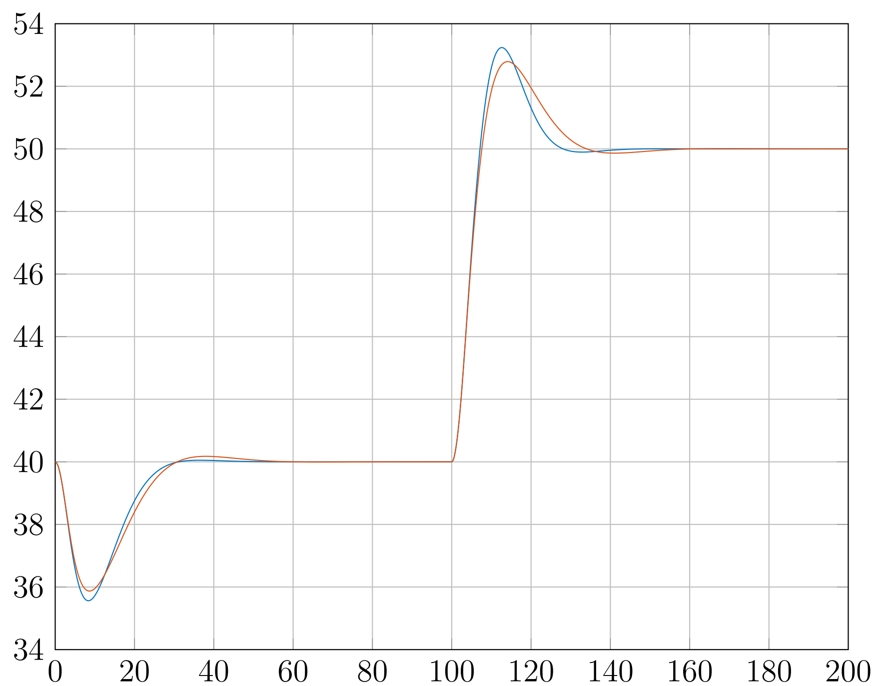

Figure 17: Step response using a discrete controller (red) with sampling time $T_s = 0.4$ sec and a continuous one with a zero-order hold (blue) with sampling time $h = 0.4$.

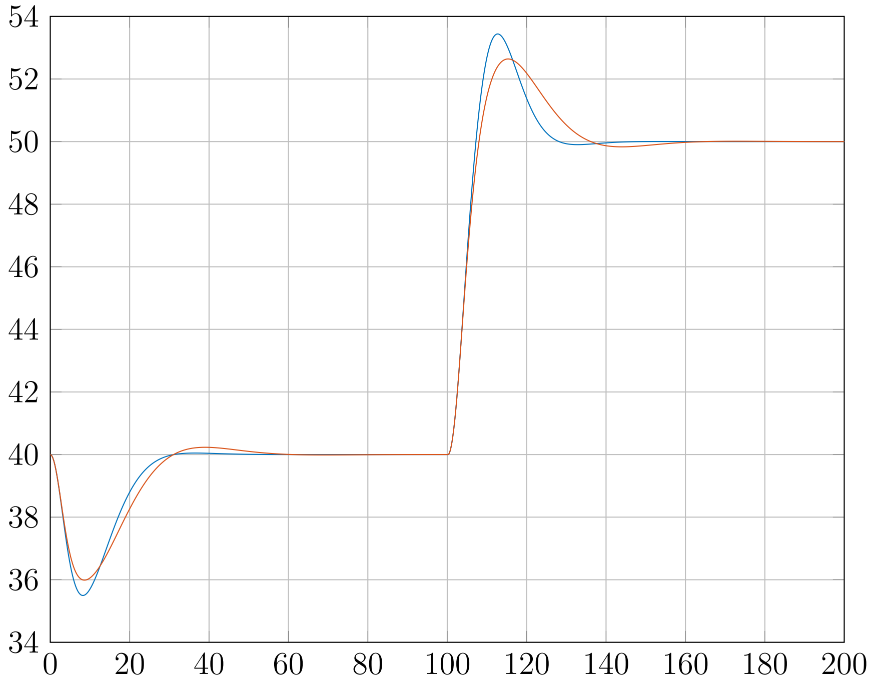

Figure 18: Step response using a discrete controller (red) with sampling time $T_s = 0.6$ sec and a continuous one with a zero-order hold (blue) with sampling time $h = 0.6$.

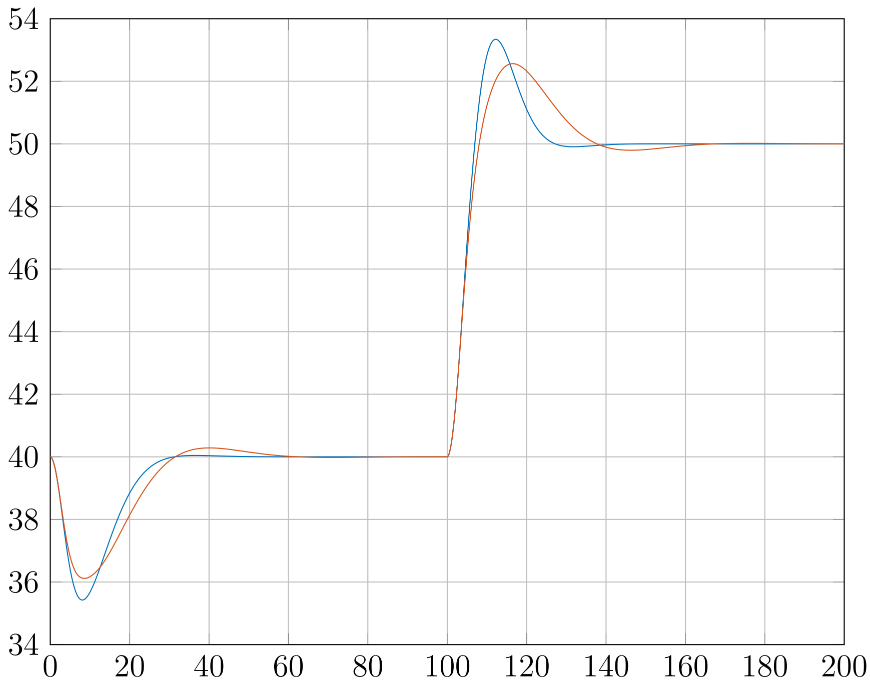

Figure 19: Step response using a discrete controller (red) with sampling time $T_s = 0.8$ sec and a continuous one with a zero-order hold (blue) with sampling time $h = 0.8$.

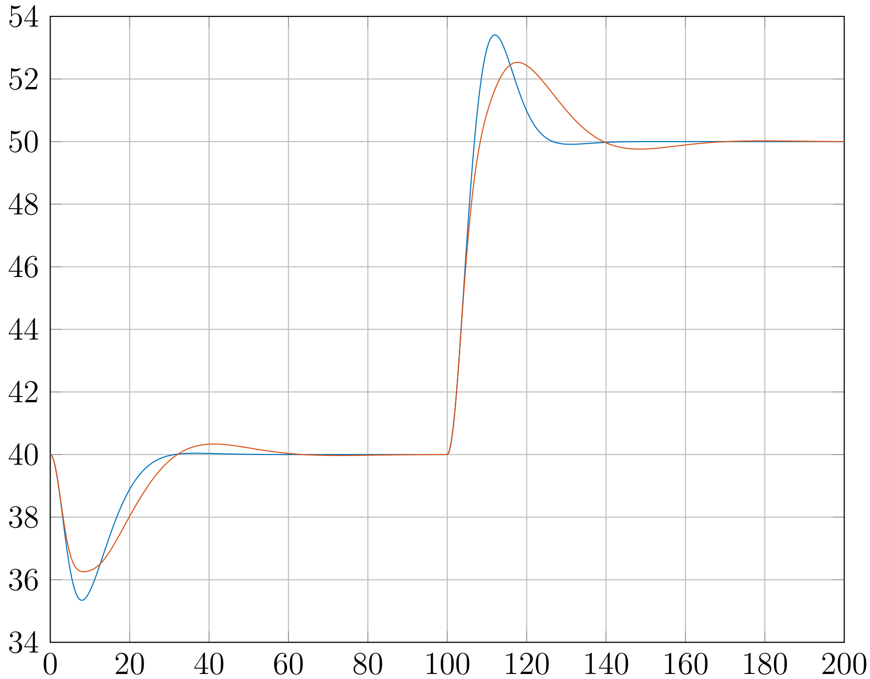

Figure 20: Step response using a discrete controller (red) with sampling time $T_s = 1$ sec and a continuous one with a zero-order hold (blue) with sampling time $h = 1$.

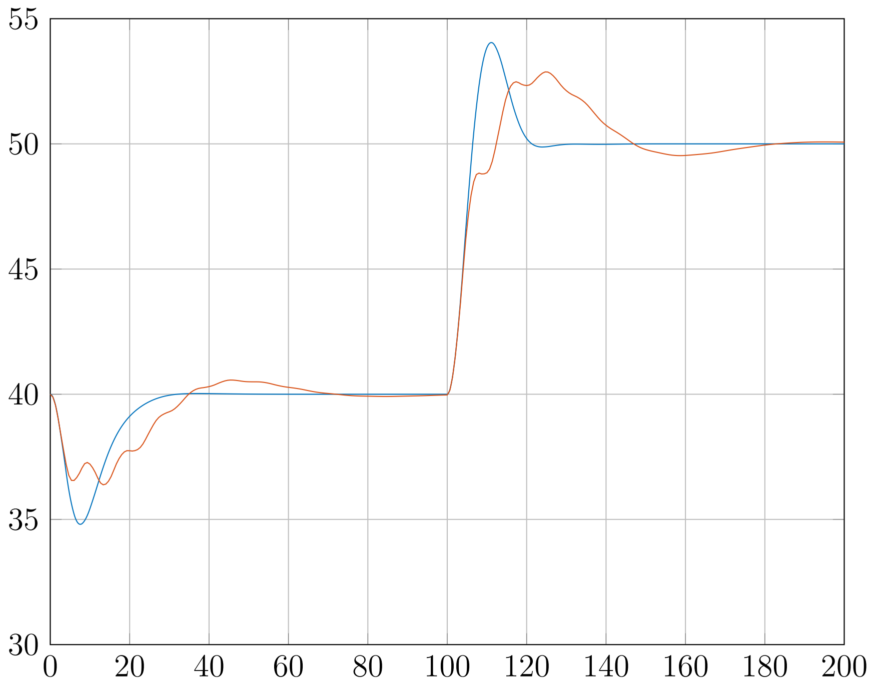

Figure 21: Step response using a discrete controller (red) with sampling time $T_s = 2$ sec and a continuous one with a zero-order hold (blue) with sampling time $h = 2$.

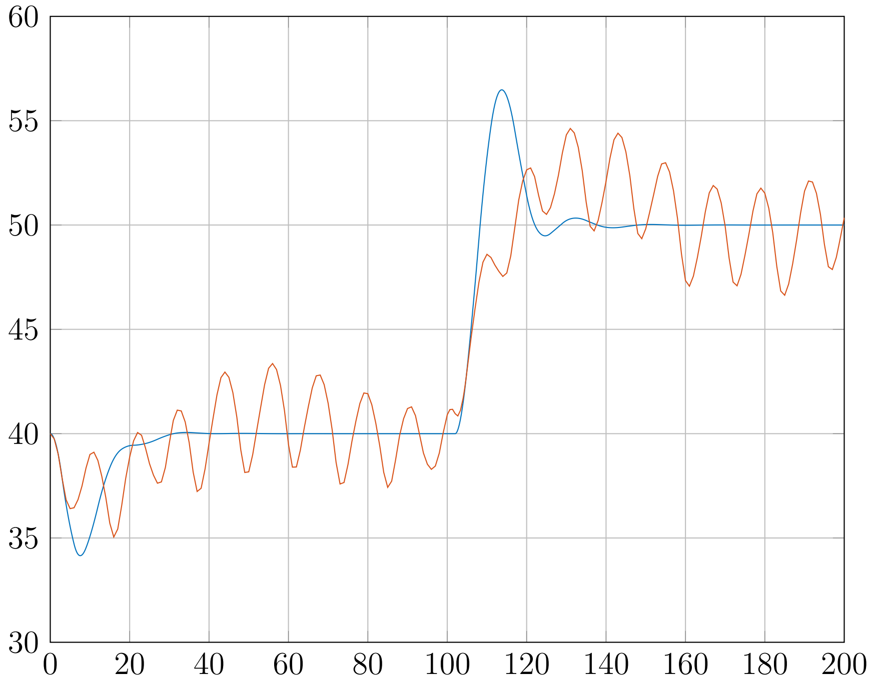

Figure 22: Step response using a discrete controller (red) with sampling time $T_s = 3$ sec and a continuous one with a zero-order hold (blue) with sampling time $h = 3$.

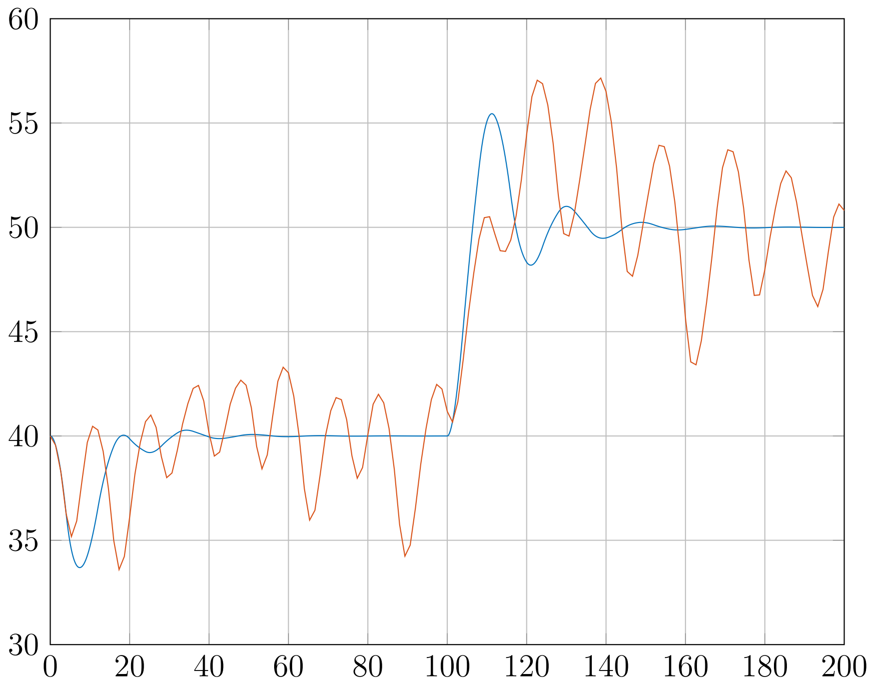

Figure 23: Step response using a discrete controller (red) with sampling time $T_s = 4$ sec and a continuous one with a zero-order hold (blue) with sampling time $h = 4$.

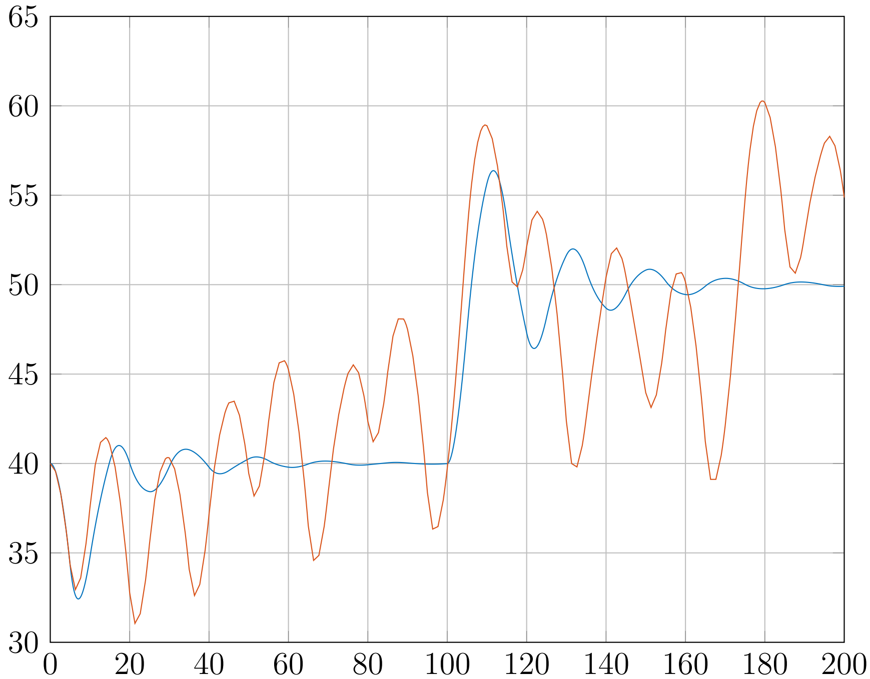

Figure 24: Step response using a discrete controller (red) with sampling time $T_s = 5$ sec and a continuous one with a zero-order hold (blue) with sampling time $h = 5$.

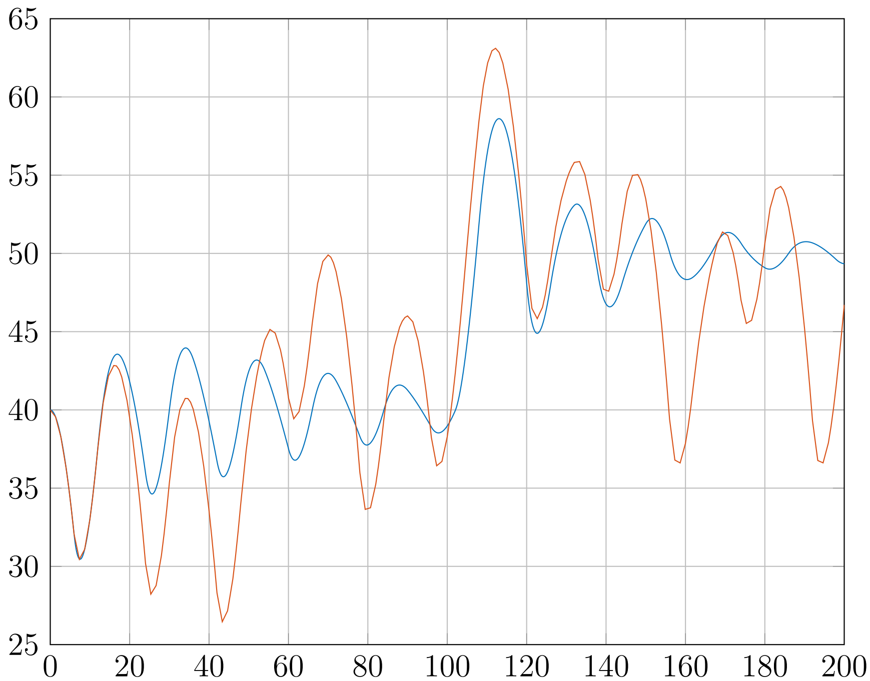

Figure 25: Step response using a discrete controller (red) with sampling time $T_s = 6$ sec and a continuous one with a zero-order hold (blue) with sampling time $h = 6$.

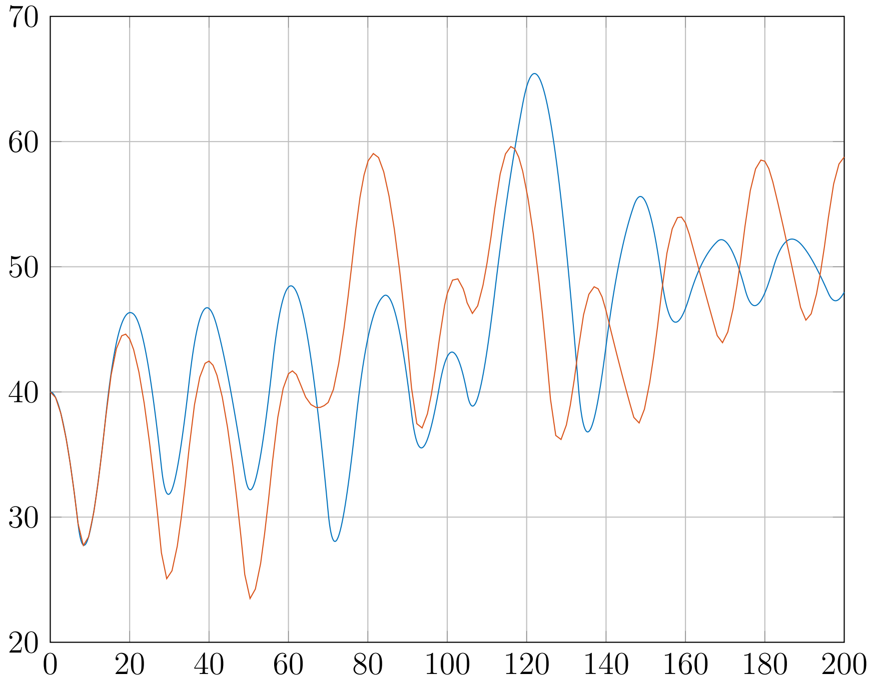

Figure 26: Step response using a discrete controller (red) with sampling time $T_s = 7$ sec and a continuous one with a zero-order hold (blue) with sampling time $h = 7$.

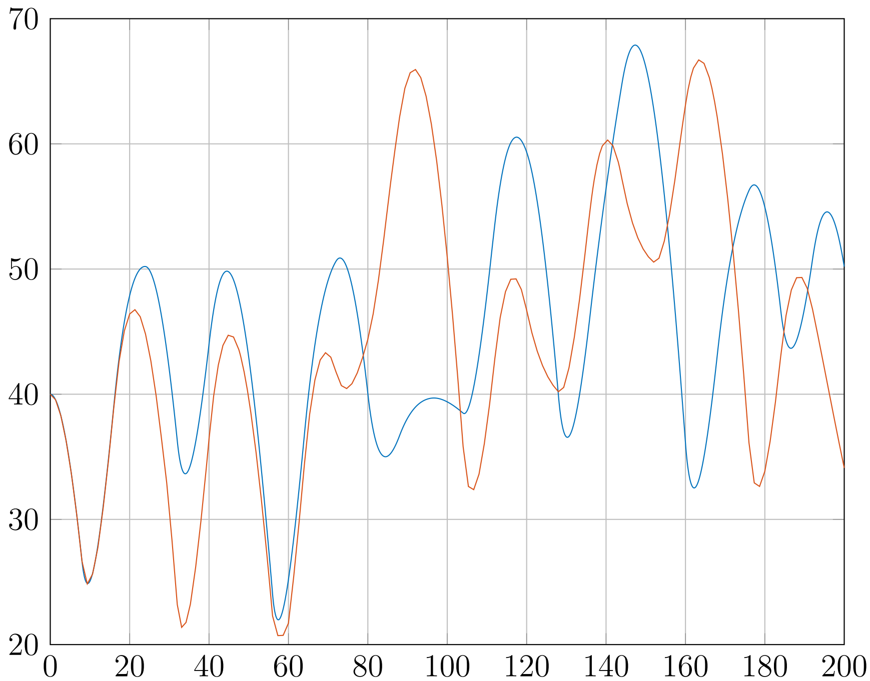

Figure 27: Step response using a discrete controller (red) with sampling time $T_s = 8$ sec and a continuous one with a zero-order hold (blue) with sampling time $h = 8$.

7 Question 9

In theory[1], as a general rule of thumb, the sampling period should be selected so that

$$0.08 < T_s\omega_c < 0.3$$

where ω_c is the crossover frequency of the open-loop system, hence with $\omega_c = 0.3619$ rad/s

$$0.220 < T_s < 0.829 \; sec$$

A second way one could look at this is to choose a sampling time such that there are 4 to 10 samples per rise time T_r. In this case where $T_r = 4.95$ sec, the resulting range of acceptable values for the sampling time is

$$0.495 < T_s < 1.2375 \; sec$$

Another way one could look at this is to select a sampling frequency that is much higher than twice the bandwidth of the closed-loop system. Given our setting, the bandwidth $\omega_0 = 0.5819$ rad/s and

$$10\omega_0 < \omega_s < 30\omega_0$$

$$0.36 < T_s < 1.08 \; sec$$

8 Question 10

The highest sampling rate that results in the step response satisfying the control requirements was found to be $T_s = 0.5$ sec. This value lies within all intervals proposed by the empirical rules considered in the previous question.

The difference in performance can be accounted for by the discretization of the PID parameters, since they are approximated.

Figure 28 illustrates the step response for $T_s = 0.5$ and table 4 shows the resulting control performance.

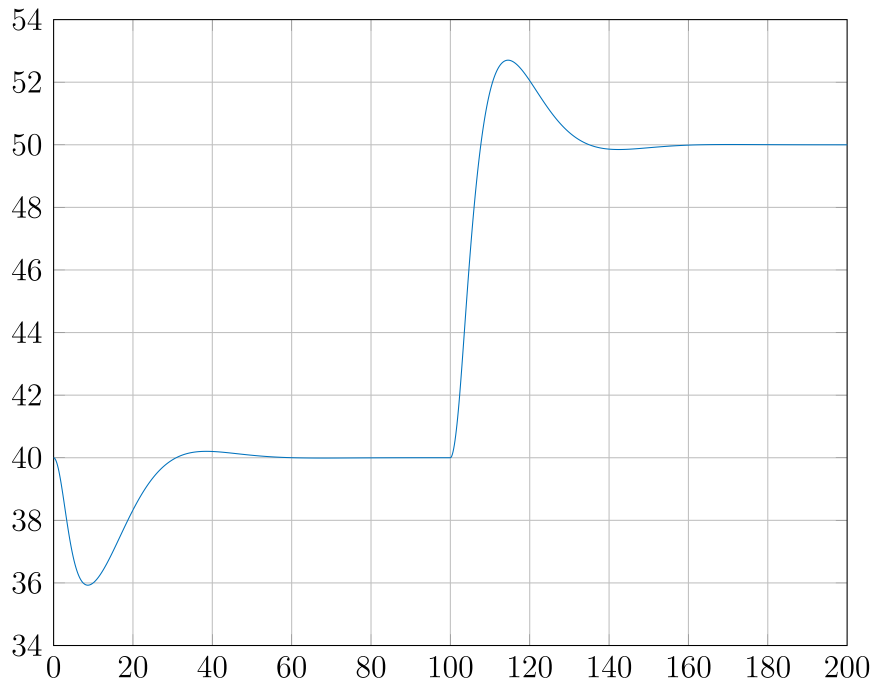

Figure 28: Step response of the system using a discrete PID controller with a sampling time of $T_s = 0.5$ sec.

Rise time	5.15
Overshoot(%)	27.0
Settling time	29.2

Table 4: The rise time, overshoot and settling time of the step response featured in figure 28.

9 Question 11

Figure 29 shows the step response for the discrete controller with a sampling time of $T_s = 4$ sec. Its performance is stable but oscillatory.

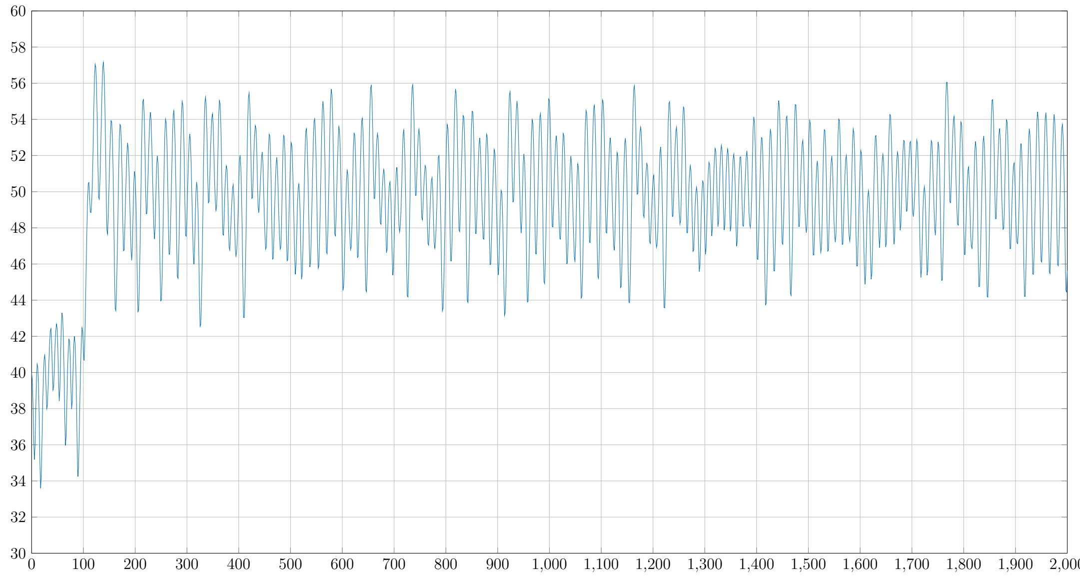

Figure 29: Step response using a discrete controller with sampling time $T_s = 4$.

10 Question 12

The sampled process' transfer function is

$$G_d = \frac{0.09162z + 0.07393}{z^2 - 1.45z + 0.5254} \tag{1}$$

The coefficients a_i and b_i are featured in table 5.

a_1	0.09162
a_2	0.07393
b_1	-1.45
b_2	0.5254

Table 5: Coefficients of the tranfer function of the sampled process.

11 Question 13

Inside the unit circle.

12 Question 14

The poles of the continuous time closed-loop system are featured in table 6, while the poles of the discrete time closed-loop system are featured in table 7.

The pole polynomial of the discrete time closed-loop system is given by equation 2:

$$P_d(z) = z^4 - 1.2061z^3 + 0.5495z^2 - 0.0924z + 0.0051 \tag{2}$$

$p_{0,1}$	-0.5
p_2	$-0.16 + 0.12i$
p_3	$-0.16 - 0.12i$

Table 6

$p_{0,1}$	0.1353
p_2	$0.4677 + 0.2435i$
p_3	$0.4677 - 0.2435i$

Table 7

13 Question 15

The transfer function of the discrete time closed-loop system is

$$\begin{aligned}
H_{cl}(z) &= \frac{F(z)\cdot G(z)}{1+F(z)\cdot G(z)} \\
&= \frac{\frac{(c_0z^2+c_1z+c_2)(a_1z+a_2)}{(z-1)(z+r)(z^2+b_1z+b_2)}}{1+\frac{(c_0z^2+c_1z+c_2)(a_1z+a_2)}{(z-1)(z+r)(z^2+b_1z+b_2)}} \\
&= \frac{\frac{(c_0z^2+c_1z+c_2)(a_1z+a_2)}{(z-1)(z+r)(z^2+b_1z+b_2)}}{\frac{(z-1)(z+r)(z^2+b_1z+b_2)+(c_0z^2+c_1z+c_2)(a_1z+a_2)}{(z-1)(z+r)(z^2+b_1z+b_2)}} \\
&= \frac{(c_0z^2+c_1z+c_2)(a_1z+a_2)}{(z-1)(z+r)(z^2+b_1z+b_2)+(c_0z^2+c_1z+c_2)(a_1z+a_2)}
\end{aligned}$$

The pole polynomial $P_d(z)$ is defined as

$$P_d(z) = z^4 + d_0z^3 + d_1z^2 + d_2z + d_3$$

and is equal to that of equation 2.
The denominator of $H_{cl}(z)$ and $P_d(z)$ should be equal:

$$(z-1)(z+r)(z^2+b_1z+b_2)+(c_0z^2+c_1z+c_2)(a_1z+a_2) = z^4 + d_0z^3 + d_1z^2 + d_2z + d_3$$

$$\begin{aligned}
&z^4 \\
&+(b_1-1+r+c_0a_1)z^3 \\
&+(b_2-b_1-r+rb_1+c_0a_2+c_1a_1)z^2 \\
&+(-b_2-b_1r+rb_2+c_1a_2+c_2a_1)z \\
&-rb_2+c_2a_2 \\
&= z^4 + d_0z^3 + d_1z^2 + d_2z + d_3
\end{aligned}$$

Hence

$$\begin{aligned}
d_0 &= b_1 - 1 + r + c_0 a_1 \\
d_1 &= b_2 - b_1 - r + r b_1 + c_0 a_2 + c_1 a_1 \\
d_2 &= -b_2 - b_1 r + r b_2 + c_1 a_2 + c_2 a_1 \\
d_3 &= -r b_2 + c_2 a_2
\end{aligned}$$

or

$$\begin{aligned}
d_0 - b_1 + 1 &= r + c_0 a_1 \\
d_1 - b_2 + b_1 &= -r + r b_1 + c_0 a_2 + c_1 a_1 \\
d_2 + b_2 &= -b_1 r + r b_2 + c_1 a_2 + c_2 a_1 \\
d_3 &= -r b_2 + c_2 a_2
\end{aligned}$$

or, in matrix form

$$\begin{bmatrix} 1 & a_1 & 0 & 0 \\ b_1 - 1 & a_2 & a_1 & 0 \\ b_2 - b_1 & 0 & a_2 & a_1 \\ -b_2 & 0 & 0 & a_2 \end{bmatrix} \begin{bmatrix} r \\ c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} d_0 - b_1 + 1 \\ d_1 - b_2 + b_1 \\ d_2 + b_2 \\ d_3 \end{bmatrix} \tag{3}$$

14 Question 16

Solving equation 3 for the calculated values of a_i, b_j and d_k gives

r	0.4543
c_0	8.6181
c_1	−10.3688
c_2	3.2976

Table 8: The unknown coefficients r, c_0, c_1, c_2 resolved from solving equation 3.

hence

$$F_d(z) = \frac{8.6181 z^2 - 10.3688 z + 3.2976}{(z-1)(z+0.4543)} \tag{4}$$

The desired and actual poles of the discrete closed-loop tranfer function are featured in tables 9 and 10 respectively. What we found is that, while the two complex poles reside where they should, the double pole at $z = 0.1353$ cannot be set, and is represented by two distinct poles at either side of it, at a distance of approximately e^{-T_s}.

$p^d_{0,1}$	0.1353
p^d_2	$0.4677 + 0.2435i$
p^d_3	$0.4677 - 0.2435i$

Table 9: The desired poles of the closed-loop transfer function.

p_0	0.1528
p_1	0.1197
p_3	$0.4667 + 0.2430i$
p_4	$0.4667 - 0.2430i$

Table 10: The actual poles of the closed-loop transfer function.

15 Question 17

Figure 30 illustrates the step responses for the discrete system and the system with a discretized controller and a continuous process for a sampling time of $T_s = 4$ sec.

The performance of the discrete controller - discrete plant is far better than that of the discretized controller - continuous plant. This is due to the fact that the designed controller is not a PID controller and that it is designed in a way such that the discrete closed-loop system has exactly the same poles as the continuous closed-loop system.

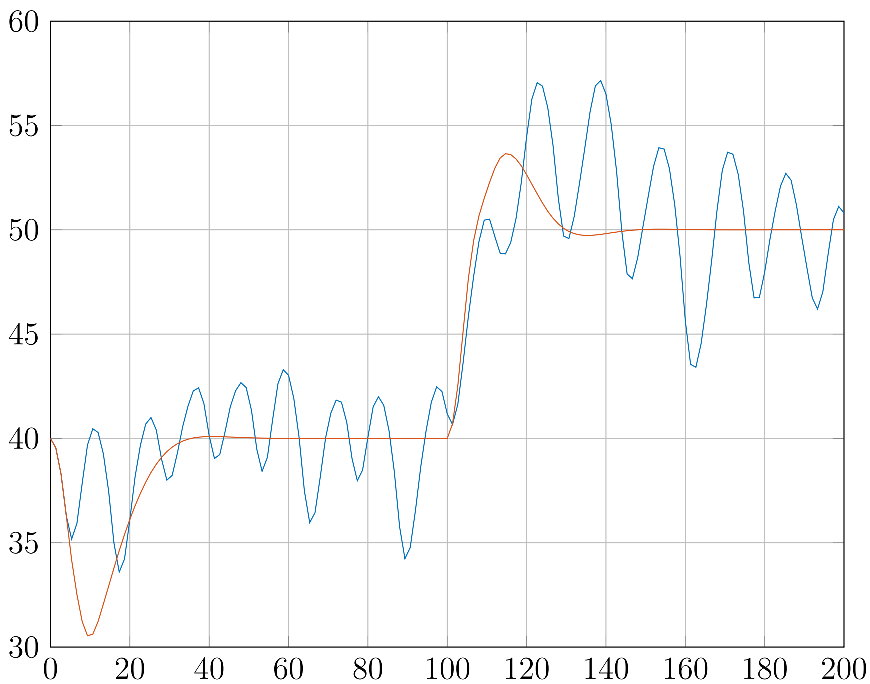

Figure 30: Step response of the discrete system (red) and the system with a disretized controller and continuous process (blue). Sampling time $T_s = 4$ sec.

16 Question 18

Since the values of the signal are non-negative, we can use the full extent of the 10 bits in the conversion process. The resolution obtained from using 10 bits to represent values inside the $[0, 100]$ interval is $2^{-10}(100 - 0) = 0.09765625$.

17 Question 20

Figure 31 shows the step response of the system, with a quantization interval of $q = \frac{100}{2^6}$. It is apparent that the steady state output oscillates, with the steady state output bias error being more than 1% of the desired steady state value. However, if a quantization interval of $\frac{100}{2^7}$ is used (fig. 32), then the output is still oscillating but within 1% of the steady state output value.

Furthermore, due to the fact that we consider values inside the interval $[-100, 100]$, at least $7 + 1 = 8$ bits will be needed to quantize the system and not be heavily degraded. If the steady state output bias error is desired to be less than 1%, then, more than 7 bits should be used.

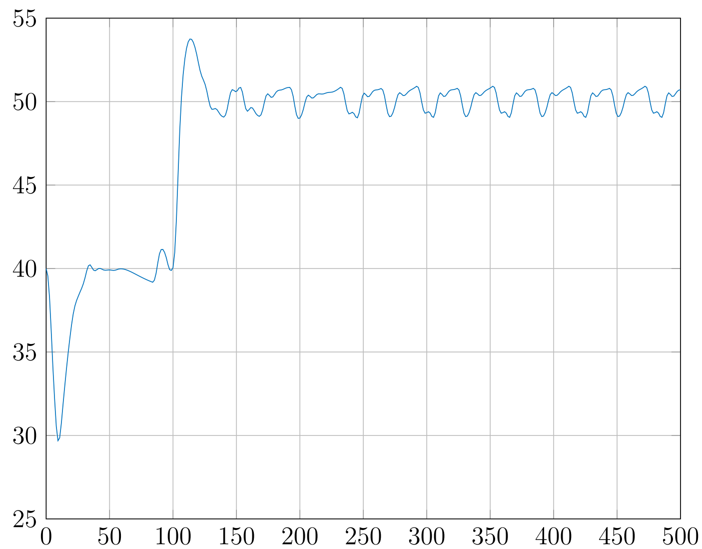

Figure 31

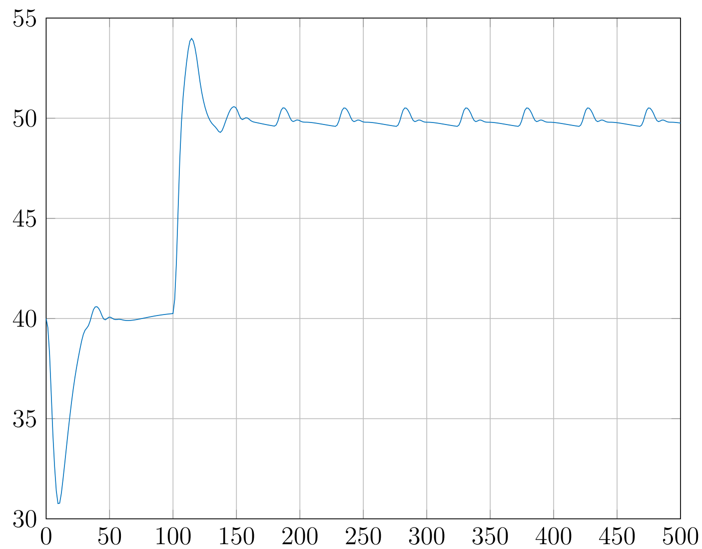

Figure 32

References

[1] William Levine et al. The Control Handbook. CRC Press, 1996.