VT16 – EP2200 – Home assignment I

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1 Problem 1

a)

$$\begin{aligned}P(A \cup B) &= P(A) + P(B) - P(A, B) \Leftrightarrow \\ P(A, B) &= P(A) + P(B) - P(A \cup B) \\ &= 0.4 + 0.7 - 0.9 = 0.2\end{aligned}$$

b)

$$P(\bar{A}, B) = P(B - A) = P(B) - P(A, B) = 0.7 - 0.2 = 0.5$$

c)

$$\begin{aligned}P(A - B) &= P(A) - P(A, B) \\ &= P(A \cup B) - P(B) \\ &= 0.9 - 0.7 = 0.2\end{aligned}$$

d)

$$\begin{aligned}P(\bar{A} - B) &= P(S - A - B) \\ &= 1 - P(A) - P(B) + P(A, B) \\ &= 1 - 0.4 - 0.7 + 0.2 = 0.1\end{aligned}$$

with S denoting the sample space
e)

$$\begin{aligned}P(\bar{A} \cup B) &= P((S - A) \cup B) \\ &= P(S - A) + P(B) - P((S - A), B) \\ &= 0.6 + 0.7 - 0.5 = 0.8\end{aligned}$$

with S denoting the sample space

f)

$$\begin{aligned}P(A,(\bar{A}\cup B)) &= P(A)+P(\bar{A}\cup B)-P(A\cup(\bar{A}\cup B))\\ &= P(A)+P(\bar{A}\cup B)-P(S)\\ &= 0.4+0.8-1=0.2\end{aligned}$$

with S denoting the sample space

2 Problem 2

$$P(\texttt{coffee}/\texttt{cake})=\frac{P(\texttt{coffee},\texttt{cake})}{P(\texttt{cake})}=\frac{0.2}{0.4}=0.5$$

3 Problem 3

$$P(\texttt{spam}/\texttt{refinance})=\frac{P(\texttt{spam},\texttt{refinance})}{P(\texttt{refinance})}=\frac{10^{-2}}{P(\texttt{refinance})}$$

We can calculate $P(\texttt{refinance})$ from the law of total probability:

$$\begin{aligned}P(\texttt{refinance}) &= \sum_{i=1}^{2}P(\texttt{class}_i,\texttt{refinance})\\ &= P(\texttt{spam},\texttt{refinance})+P(\overline{\texttt{spam}},\texttt{refinance})\\ &= 10^{-2}+10^{-5}\end{aligned}$$

Hence,

$$P(\texttt{spam}/\texttt{refinance})=\frac{10^{-2}}{10^{-2}+10^{-5}}$$

4 Problem 4

a) Since X and Y are independent:

$$\begin{aligned}P(X\leq 2\ \texttt{and}\ Y\leq 2) &= P(X\leq 2)P(Y\leq 2)\\ &= (P_X(1)+P_X(2))(P_Y(1)+P_Y(2))\\ &= (\frac{1}{4}+\frac{1}{8})(\frac{1}{6}+\frac{1}{6})=\frac{1}{4}\end{aligned}$$

b) From Demorgan's law:

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

Hence

$$\overline{(X > 2 \cup Y > 2)} = \overline{(X > 2)} \cap \overline{(Y > 2)} = (X \leq 2) \cap (Y \leq 2)$$

and using the identity

$$P(A) = 1 - P(\bar{A})$$

we get that

$$P(X > 2 \cup Y > 2) = 1 - P(X \leq 2, Y \leq 2) = 1 - \frac{1}{4} = \frac{3}{4}$$

c) Since X and Y are independent, knowing Y does not give us information about X:

$$P(X > 2/Y > 2) = P(X > 2) = P_X(3) + P_X(4) = \frac{1}{8} + \frac{1}{2} = \frac{5}{8}$$

d)

$$\begin{aligned} X < Y = &\{(X = 1, Y = 2), (X = 1, Y = 3), (X = 1, Y = 4), \\ &(X = 2, Y = 3), (X = 2, Y = 4), (X = 3, Y = 4)\} \end{aligned}$$

And X, Y are independent, hence

$$\begin{aligned} P(X < Y) &= P_X(1)P_Y(2) + P_X(1)P_Y(3) + P_X(1)P_Y(4) \\ &+ P_X(2)P_Y(3) + P_X(2)P_Y(4) + P_X(3)P_Y(4) \\ &= \frac{7}{24} \end{aligned}$$

5 Problem 5

$$\begin{aligned} E[Y] &= E[-2X + 3] = -2E[X] + 3 \Leftrightarrow \\ 1 &= -2E[X] + 3 \Leftrightarrow \\ E[X] &= 1 \end{aligned}$$

and

$$
\begin{aligned}
E[Y^2] = E[(-2X+3)^2] &= E[4X^2 - 12X + 9] \Leftrightarrow \\
9 &= 4E[X^2] - 12E[X] + 9 \Leftrightarrow \\
E[X^2] &= 3E[X] = 3
\end{aligned}
$$

Hence

$$Var(X) = E[X^2] - E^2[X] = 3 - 1 = 2$$

6 Problem 6

a) The total area under the probability density function should be equal to one:

$$
\begin{aligned}
\int_{-\infty}^{+\infty} f_X(x)dx = \int_0^{+\infty} ce^{-4x}dx = -\frac{c}{4}e^{-4x}\Big|_0^{+\infty} = 1 \Leftrightarrow \\
c = 4
\end{aligned}
$$

b)

$$F_X(x) = \int_{-\infty}^{x} f_X(x)dx = \int_0^{x} 4e^{-4x}dx = -4e^{-4x}\Big|_0^{x} + ct = 1 - e^{-4x} + ct$$

For the CDF, the following must hold:

$$F_X(0) = 0$$

and

$$F_X(+\infty) = 1$$

Hence, $ct = 0$, and

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-4x}, & x \geq 0 \end{cases}$$

c)

$$P(2 < X < 5) = F_X(5) - F_X(2) = e^{-8} - e^{-20}$$

d)

$$E[X] = \int_{-\infty}^{+\infty} x f_X(x)dx = \int_0^{+\infty} 4xe^{-4x} = \frac{1}{4}$$

7 Problem 7

I suppose continuous random variables X, Y for this exercise. For discrete ones, only the integral switches places with a finite sum.

a)

$$E[cX] = \int_{-\infty}^{+\infty} cxf_X(x) = c\int_{-\infty}^{+\infty} xf_X(x) = cE[X]$$

b)

$$\begin{aligned}
E[X+Y] &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(x+y)f_{XY}(x,y)dxdy \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}xf_{XY}(x,y)dxdy + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}yf_{XY}(x,y)dxdy \\
&= E[X] + E[Y]
\end{aligned}$$

where $f_XY(x,y)$ is the joint probability distribution of X and Y.

c)

$$\begin{aligned}
Var[cX] &= E[(cX)^2] - E^2(cX) = E[c^2X^2] - (cE[X])^2 \\
&= c^2E[X^2] - c^2E^2[X] = c^2(E[X^2] - E^2[X]) \\
&= c^2Var[X]
\end{aligned}$$

d)

$$\begin{aligned}
Var[X+Y] &= E[(X+Y)^2] - E^2[X+Y] = E[X^2+2XY+Y^2] - (E[X]+E[Y])^2 \\
&= E[X^2+2XY+Y^2] - E^2[X] - 2E[X]E[Y] - E^2[Y] \\
&= E[X^2] + 2E[XY] + E[Y^2] - E^2[X] - 2E[X]E[Y] - E^2[Y] \\
&= E[X^2] - E^2[X] + E[Y^2] - E^2[Y] \\
&= Var[X] + Var[Y]
\end{aligned}$$

where $E[XY] = E[X]E[Y]$, since X, Y are independent:

$$\begin{aligned}
E[XY] &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}xyf_{XY}(x,y)dxdy \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}xyf_X(x)f_Y(y)dxdy \\
&= \int_{-\infty}^{+\infty}xf_X(x)dx\int_{-\infty}^{+\infty}yf_Y(y)dy \\
&= E[X]E[Y]
\end{aligned}$$

e)

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{+\infty} xp(x)dx = \int_{-\infty}^{+\infty} x(\alpha_1 p_1(x) + \alpha_2 p_2(x))dx \\
&= \alpha_1 \int_{-\infty}^{+\infty} xp_1(x)dx + \alpha_2 \int_{-\infty}^{+\infty} xp_2(x)dx \\
&= \alpha_1 E_{p1}[X] + \alpha_2 E_{p2}[X]
\end{aligned}
$$

$$
\begin{aligned}
E[X^2] &= \int_{-\infty}^{+\infty} x^2 p(x)dx = \int_{-\infty}^{+\infty} x^2(\alpha_1 p_1(x) + \alpha_2 p_2(x))dx \\
&= \alpha_1 \int_{-\infty}^{+\infty} x^2 p_1(x)dx + \alpha_2 \int_{-\infty}^{+\infty} x^2 p_2(x)dx \\
&= \alpha_1 E_{p1}[X^2] + \alpha_2 E_{p2}[X^2]
\end{aligned}
$$