VT16 – EP2200 – Home assignment II

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1 Problem 1

The first packet will wait, on average, the sum of the interarrival times between it and the 50^{th} packet. Since the packets arrive in a Poisson fashion, the interarrival times are exponentially distributed. Due to the memoryless property of the exponential distribution, the mean interarrival time between packets is

$$E[T] = \frac{1}{\lambda} = 1 \text{ ms}$$

Hence, the mean waiting time for the first packet is

$$\overline{T}_w = \sum_{i=1}^{50-1} E[T] = \frac{49}{\lambda} = 49 \text{ ms}$$

The probability that k packets are transmitted in a time interval of $\Delta t = 10$ ms is

$$P_k = \frac{(\lambda \Delta t)^k}{k!} e^{-\lambda \Delta t} = \frac{10^k}{k!} e^{-10}$$

The average number of packets in a block is

$$E[P] = \lambda \Delta t = 10 \text{ packets/block}$$

2 Problem 2

For the missing pieces of the diagonal of the state transition intensity matrix, the following equation holds:

$$q_{i,i} = -q_i = -\sum_{i \neq j} q_{i,j}$$

Hence matrix Q is formed as

$$Q = \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -4 & 3 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 3 & 0 & -3 \end{bmatrix}$$

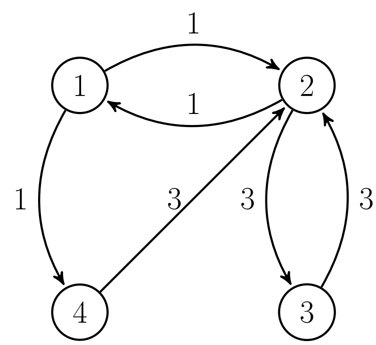

Figure 1: The Markovian chain corresponding to the above Q matrix

Given a state s that can transition to states $\{s_i\}$, the time the system spends in state s is an exponentially distributed random variable $T = min(\{s_i\})$, which means that its parameter will be $\lambda = \sum \lambda_{s \to s_i}$, and the mean time the system spends in s will be $\overline{T} = \frac{1}{\lambda}$.

Ergo, $\overline{T}_1 = \frac{1}{1+1} = 0.5$ sec, $\overline{T}_2 = \frac{1}{1+3} = 0.25$ sec, $\overline{T}_3 = \frac{1}{3} = 0.33$ sec, and $\overline{T}_4 = \frac{1}{3} = 0.33$ sec.

In steady state $[\pi_1\ \pi_2\ \pi_3\ \pi_4] \cdot \mathrm{Q} = 0$, where π_i is the probability of the radio being in state i. Solving the system of equations gives

$$[\pi_1\ \pi_2\ \pi_3\ \pi_4] = [\frac{3}{16}\ \frac{3}{8}\ \frac{3}{8}\ \frac{1}{16}]$$

Hence there is a probability of $\frac{3}{16} = 18.75\%$ that the radio is in stand-by, $\frac{3}{8} = 37.5\%$ that the radio is listening to the radio channel for incoming packets, or receiving a packet, and $\frac{1}{16} = 6.25\%$ that the radio is transmitting a packet.

3 Problem 3

a) With Little's result, using N_{min} as the minimum number of simultaneous listeners, λ_{min} as the minimum request intensity and T the length of a song, then

$$N_{min} = \lambda_{min} T \Leftrightarrow \lambda_{min} = \frac{N_{min}}{T} = 5/3 \text{ requests per minute, or}$$

$$\lambda_{min} = \frac{5 \cdot 24 \cdot 60}{3} = 2400 \text{ requests per day}$$

b) With Little's result, using N as the number of people in the exhibition, T the average duration of their visit and λ the rate at which the tickets to exhibition are made available

$$N = \lambda T \Leftrightarrow \lambda = \frac{N}{T} = \frac{250}{40} \text{ available tickets per minute, or}$$

$$\lambda = \frac{250 \cdot 60}{40} = 375 \text{ available tickets per hour}$$

4 Problem 4

a)

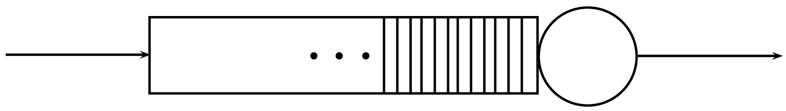

Figure 2: The block diagram of the queue. There are infinite places in the queue and only one server.

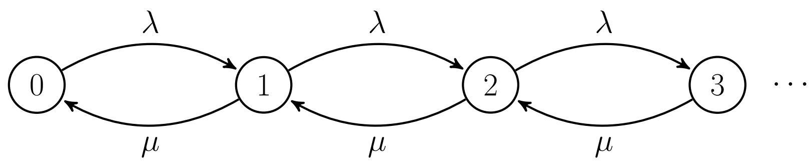

Figure 3: The Markovian chain describing the system

b) A state is defined by the number of packets in the system. Considering local balance equations we get

$$
\begin{aligned}
&\lambda\pi_0 = \mu\pi_1 \Leftrightarrow \pi_1 = \frac{\lambda}{\mu}\pi_0 \\
&\lambda\pi_1 = \mu\pi_2 \Leftrightarrow \pi_2 = \frac{\lambda}{\mu}\pi_1 = \left(\frac{\lambda}{\mu}\right)^2\pi_0 \\
&\lambda\pi_2 = \mu\pi_3 \Leftrightarrow \pi_3 = \left(\frac{\lambda}{\mu}\right)^3\pi_0 \\
&\dots
\end{aligned}
$$

In other words, in general

$$\pi_i = \left(\frac{\lambda}{\mu}\right)^i\pi_0$$

In order to identify all π_i we resort to the general law of

$$\sum_{i=0}^{\infty}\pi_i = 1$$

from where we calculate that

$$\pi_0 = 1 - \frac{\lambda}{\mu}$$

and

$$\pi_i = \left(\frac{\lambda}{\mu}\right)^i(1 - \frac{\lambda}{\mu})$$

The values for the intensities λ and μ can be derived from

$$\lambda = \frac{1}{2 \cdot 10^{-3}} = 500$$
$$\mu = \frac{1}{10^{-3}} = 1000$$

Hence the probability that the system is empty is given by

$$\pi_0 = 1 - \frac{\lambda}{\mu} = 0.5$$

and, in general, the probability that there are i packets in the system (in the queue and in the server) is

$$\pi_i = \left(\frac{\lambda}{\mu}\right)^i (1 - \frac{\lambda}{\mu}) = 0.5^{i+1}$$

The average number of packets waiting for transmission N_q is the average number of packets in the queue, which is

$$N_q = N - N_s \tag{1}$$

where N is the average number of packets in the system, and N_s is the average number of packets being transmitted. Hence, finding N_q means finding N and N_s.

The average number of packets in the system is defined as

$$\begin{aligned}
N &= \sum_{i=0}^{\infty} i \cdot \pi_i = \sum_{i=0}^{\infty} i \left(\frac{\lambda}{\mu}\right)^i \pi_0 = \pi_0 \sum_{i=0}^{\infty} i \left(\frac{\lambda}{\mu}\right)^i \\
&= \pi_0 \left(\frac{\lambda}{\mu}\right) \sum_{i=0}^{\infty} i \left(\frac{\lambda}{\mu}\right)^{i-1} = \pi_0 \rho \sum_{i=0}^{\infty} i \rho^{i-1} \\
&= \pi_0 \rho \sum_{i=0}^{\infty} \frac{d(\rho^i)}{d\rho} = \pi_0 \rho \frac{d}{d\rho} \left(\sum_{i=0}^{\infty} \rho^i \right) \\
&= \pi_0 \rho \frac{d}{d\rho} \left(\frac{1}{1-\rho} \right) = (1-\rho)\rho \frac{1}{(1-\rho)^2} = \frac{\rho}{1-\rho} \\
&= \frac{\lambda}{\mu - \lambda} = \frac{500}{1000 - 500} = 1 \text{ packet}
\end{aligned}$$

The average number of packets under transmission is

$$N_s = \frac{\lambda}{\mu} = \frac{500}{1000} = 0.5 \text{ packets}$$

Hence, from equation 1:

$$N_q = 1 - 0.5 = 0.5 \text{ packets}$$

c) The probability of at least n packets existing in the system is equal to that of 1 minus the probability of $n-1$ existing in the system at most:

$$P(k \geq n) = 1 - P(k < n) = 1 - P(k \leq n-1)$$

But

$$P(k \leq n-1) = \sum_{k=0}^{n-1} \pi_i = \sum_{k=0}^{n-1} (1 - \frac{\lambda}{\mu}) \left(\frac{\lambda}{\mu}\right)^k$$

$$= (1 - \frac{\lambda}{\mu}) \sum_{k=0}^{n-1} \left(\frac{\lambda}{\mu}\right)^k = (1 - \frac{\lambda}{\mu}) \frac{1 - \left(\frac{\lambda}{\mu}\right)^n}{1 - \frac{\lambda}{\mu}}$$

$$= 1 - \left(\frac{\lambda}{\mu}\right)^n$$

Hence

$$P(k \geq n) = 1 - P(k \leq n-1) = \left(\frac{\lambda}{\mu}\right)^n = 0.5^n$$

Intuitively, this makes sense, since the probability of the system having at least 0 packets is 1, and as the number of packets in the condition increases, the probability decreases, since the system is stable.

d) The arriving packet will have to wait for transmission for the duration of k transmission times, supposing that k packets are in the system, 1 being transmitted and $k-1$ waiting for transmission in the queue. These times are distributed exponentially. Assuming $X_i, 1 \leq i \leq k$ is a random variable, independent of all $X_j, i \neq j$ describing the service time of the i^{th} packet, then the waiting time for packet $k+1$ will be the sum of X_i:

$$W = X_1 + X_2 + \cdots + X_k = \sum_{i=1}^{k} X_i$$

The addition of X_1, although the first packet is assumed to be in transmission is made because of the memoryless property of the exponential distribution.

This summation of random variables denotes their convolution in time, or multiplication in the (continuous) Laplace domain. The probability density function of X_i is

$$f_{X_i}(t) = \mu e^{-\mu t}$$

and its Laplace transform is

$$f^*_{T_i}(s) = \mu \frac{1}{s + \mu}$$

Hence the probability density function of the waiting time W of packet $k+1$ in the Laplace domain, given that there are k packets in the system is

$$f_W(s|k) = \prod_{i=1}^{k} f^*_{T_i}(s) = \prod_{i=1}^{k} \frac{\mu}{s+\mu} = \Big(\frac{\mu}{s+\mu}\Big)^k$$

The probability density function of the waiting time W will thus be the marginalization over all possible values of k, starting from $k = 1$ because in order for waiting time to exist, there should be one packet under transmission in the server

$$\begin{aligned} f_W(s) &= \sum_{k=1}^{\infty} \Big(\frac{\mu}{s+\mu}\Big)^k \alpha_k = \sum_{k=1}^{\infty} \Big(\frac{\mu}{s+\mu}\Big)^k \pi_k \\ &= \sum_{k=1}^{\infty} \Big(\frac{\mu}{s+\mu}\Big)^k (1-\rho)\rho^k = (1-\rho) \sum_{k=1}^{\infty} \Big(\frac{\lambda}{s+\mu}\Big)^k \\ &= (1-\rho)\Big(-1 + \sum_{k=0}^{\infty} \Big(\frac{\lambda}{s+\mu}\Big)^k\Big) = (1-\rho)\Big(-1 + \frac{1}{1 - \dfrac{\lambda}{s+\mu}}\Big) \\ &= (1-\rho)\Big(-1 + \frac{s+\mu}{s+\mu-\lambda}\Big) = (1-\rho)\frac{\lambda}{s+\mu-\lambda} \end{aligned}$$

Where the arriving packet $k+1$ observes the system being in state k with probability α_k is equal to the probability that the system is indeed in state k, π_k, due to the PASTA property.

Turning $f_W(s)$ in the time domain, we get

$$\begin{aligned} f_W(t) &= (1-\rho)\lambda e^{-(\mu-\lambda)t} = (1 - \frac{\lambda}{\mu})\lambda e^{-(\mu-\lambda)t} \\ &= \rho(\mu-\lambda)e^{-(\mu-\lambda)t} \end{aligned}$$

The CDF of the waiting time W for packet $k+1$ is thus

$$F_W(t) = P(W \le t) = 1 - \rho e^{-(\mu-\lambda)t}$$

since $F_W(\infty) = 1$ and $F_W(0) = 1 - \rho = \pi_0$

The probability that an arriving packet waits in the queue more than $T = 5$ms before being transmitted is

$$\begin{aligned} P(W > T) &= 1 - P(W \le T) = 1 - (1 - \rho e^{-(\mu-\lambda)T}) = \frac{\lambda}{\mu} e^{-(\mu-\lambda)T} \\ &= \frac{500}{1000} e^{-(1000-500)5\cdot 10^{-3}} = 0.5e^{-2.5} = 0.041 \end{aligned}$$

5 Problem 5