VT16 – EP2200 – Home assignment II

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1 Problem 1

The first packet will wait, on average, the sum of the interarrival times between it and the 50^{th} packet. Since the packets arrive in a Poisson fashion, the interarrival times are exponentially distributed. Due to the memoryless property of the exponential distribution, the mean interarrival time between packets is

$$E[T] = \frac{1}{\lambda} = 1 \text{ ms}$$

Hence, the mean waiting time for the first packet is

$$\overline{T}_w = \sum_{i=1}^{50-1} E[T] = \frac{49}{\lambda} = 49 \text{ ms}$$

The probability that k packets are transmitted in a time interval of $\Delta t = 10$ ms is

$$P_k = \frac{(\lambda \Delta t)^k}{k!} e^{-\lambda \Delta t} = \frac{10^k}{k!} e^{-10}$$

The average number of packets in a block is

$$E[P] = \lambda \Delta t = 10 \text{ packets/block}$$

2 Problem 2

For the missing pieces of the diagonal of the state transition intensity matrix, the following equation holds:

$$q_{i,i} = -\sum_{i \neq j} q_{i,j}$$

Hence matrix Q is formed as

$$Q = \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -4 & 3 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 3 & 0 & -3 \end{bmatrix}$$

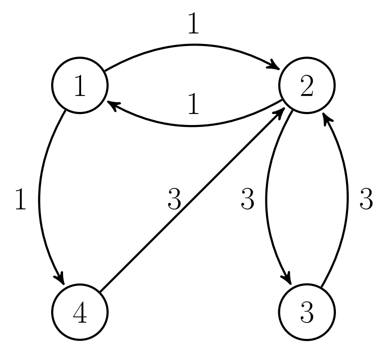

Figure 1: The Markovian chain corresponding to the above Q matrix

In steady state $[\pi_1\ \pi_2\ \pi_3\ \pi_4] \cdot \mathrm{Q} = 0$, where π_i is the probability of the radio being in state i. Solving the system of equations gives

$$[\pi_1\ \pi_2\ \pi_3\ \pi_4] = [\frac{3}{16}\ \frac{3}{8}\ \frac{3}{8}\ \frac{1}{16}]$$

Hence there is a probability of $\frac{3}{16} = 18.75\%$ that the radio is in stand-by, $\frac{3}{8} = 37.5\%$ that the radio is listening to the radio channel for incoming packets, or receiving a packet, and $\frac{1}{16} = 6.25\%$ that the radio is transmitting a packet.

This means that if we observe the state distribution of the radio over a period of time T, then as T approaches infinity, we will observe that the radio is in state 1 for $0.1875T$ time units, in state 2 for $0.375T$ time units, as is also the case for the time spent in state 3, and in state 4 for $0.0625T$ time units.

3 Problem 3

a) With Little's result, using N_{min} as the minimum number of simultaneous listeners, λ_{min} as the minimum request intensity and T the length of a song, then

$$N_{min} = \lambda_{min} T \Leftrightarrow \lambda_{min} = \frac{N_{min}}{T} = 5/3 \text{ requests per minute, or}$$

$$\lambda_{min} = \frac{5 \cdot 24 \cdot 60}{3} = 2400 \text{ requests per day}$$

b) With Little's result, using N as the number of people in the exhibition, T the average duration of their visit and λ the rate at which the tickets to exhibition are made available

$$N = \lambda T \Leftrightarrow \lambda = \frac{N}{T} = \frac{250}{40} \text{ available tickets per minute, or}$$

$$\lambda = \frac{250 \cdot 60}{40} = 375 \text{ available tickets per hour}$$

4 Problem 4

a)

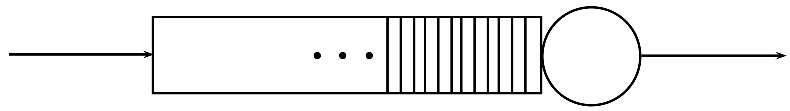

Figure 2: The block diagram of the queue. There are infinite places in the queue and only one server.

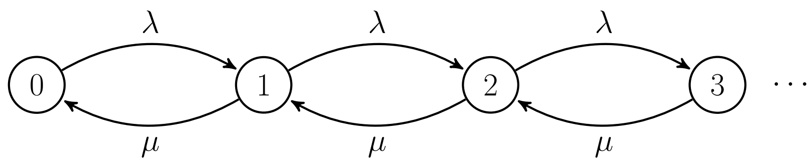

Figure 3: The Markovian chain describing the system

b) A state is defined by the number of packets in the system. Considering local balance equations we get

$$\begin{aligned}
&\lambda\pi_0 = \mu\pi_1 \Leftrightarrow \pi_1 = \frac{\lambda}{\mu}\pi_0 \\
&\lambda\pi_1 = \mu\pi_2 \Leftrightarrow \pi_2 = \frac{\lambda}{\mu}\pi_1 = \left(\frac{\lambda}{\mu}\right)^2\pi_0 \\
&\lambda\pi_2 = \mu\pi_3 \Leftrightarrow \pi_3 = \left(\frac{\lambda}{\mu}\right)^3\pi_0 \\
&\ldots
\end{aligned}$$

In other words, in general

$$\pi_i = \left(\frac{\lambda}{\mu}\right)^i\pi_0$$

In order to identify all π_i we resort to the general law of

$$\sum_{i=0}^{\infty}\pi_i = 1$$

from where we calculate that

$$\pi_0 = 1 - \frac{\lambda}{\mu}$$

and

$$\pi_i = \left(\frac{\lambda}{\mu}\right)^i(1 - \frac{\lambda}{\mu})$$

The values for the intensities λ and μ can be derived from

$$\lambda = \frac{1}{2 \cdot 10^{-3}} = 500$$
$$\mu = \frac{1}{10^{-3}} = 1000$$

Hence the probability that the system is empty is given by

$$\pi_0 = 1 - \frac{\lambda}{\mu} = 0.5$$

and, in general, the probability that there are i packets in the system (in the queue and in the server) is

$$\pi_i = \left(\frac{\lambda}{\mu}\right)^i (1 - \frac{\lambda}{\mu}) = 0.5^{i+1}$$

The average number of packets waiting for transmission N_q is the average number of packets in the queue, which is

$$N_q = N - N_s \tag{1}$$

where N is the average number of packets in the system, and N_s is the average number of packets being transmitted. Hence, finding N_q means finding N and N_s.

The average number of packets in the system is defined as

$$\begin{aligned}
N &= \sum_{i=0}^{\infty} i \cdot \pi_i = \sum_{i=0}^{\infty} i \left(\frac{\lambda}{\mu}\right)^i \pi_0 = \pi_0 \sum_{i=0}^{\infty} i \left(\frac{\lambda}{\mu}\right)^i \\
&= \pi_0 \left(\frac{\lambda}{\mu}\right) \sum_{i=0}^{\infty} i \left(\frac{\lambda}{\mu}\right)^{i-1} = \pi_0 \rho \sum_{i=0}^{\infty} i \rho^{i-1} \\
&= \pi_0 \rho \sum_{i=0}^{\infty} \frac{d(\rho^i)}{d\rho} = \pi_0 \rho \frac{d}{d\rho} \left(\sum_{i=0}^{\infty} \rho^i \right) \\
&= \pi_0 \rho \frac{d}{d\rho} \left(\frac{1}{1-\rho} \right) = (1-\rho)\rho \frac{1}{(1-\rho)^2} = \frac{\rho}{1-\rho} \\
&= \frac{\lambda}{\mu - \lambda} = \frac{500}{1000 - 500} = 1 \text{ packet}
\end{aligned}$$

The average number of packets under transmission is

$$N_s = \frac{\lambda}{\mu} = \frac{500}{1000} = 0.5 \text{ packets}$$

Hence, from equation 1:

$$N_q = 1 - 0.5 = 0.5 \text{ packets}$$

c) The probability of at least n packets existing in the system is equivalent to that of 1 minus the probability of $n-1$ existing in the system at most:

$$P(k \geq n) = 1 - P(k < n) = 1 - P(k \leq n-1)$$

But

$$\begin{aligned}
P(k \leq n-1) &= \sum_{k=0}^{n-1} \pi_i = \sum_{k=0}^{n-1} (1 - \frac{\lambda}{\mu}) \left(\frac{\lambda}{\mu}\right)^k \\
&= (1 - \frac{\lambda}{\mu}) \sum_{k=0}^{n-1} \left(\frac{\lambda}{\mu}\right)^k = (1 - \frac{\lambda}{\mu}) \frac{1 - \left(\frac{\lambda}{\mu}\right)^n}{1 - \frac{\lambda}{\mu}} \\
&= 1 - \left(\frac{\lambda}{\mu}\right)^n
\end{aligned}$$

Hence

$$P(k \geq n) = 1 - P(k \leq n-1) = \left(\frac{\lambda}{\mu}\right)^n = 0.5^n$$

5 Problem 5