A TME226 Lecture Notes

A.1 Lecture 1

¶See Section 1.1, Eulerian, Lagrangian, material derivative

Lagrangian-Eulerian

▶Lagrangian approach

- The (fluid) particle is described by its initial position, X_i, and time, t
- In other words we "label" a particle with X_i and then follow it.
- The variation of T is expressed as dT/dt.

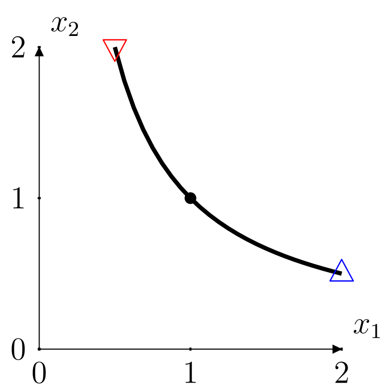
▶Eulerian approach

- We look a point, x_i, and see what happens.
- Hence T depends on both x_i and t
- The chain rule gives $\dfrac{dT}{dt} = \dfrac{\partial T}{\partial t} + \dfrac{dx_i}{dt}\dfrac{\partial T}{\partial x_i} = \dfrac{\partial T}{\partial t} + v_i \dfrac{\partial T}{\partial x_i}$

▶ $$\underbrace{\frac{dT}{dt}}_{\text{material change}} = \frac{\partial T}{\partial t} + \frac{dx_j}{dt}\frac{\partial T}{\partial x_j} = \underbrace{\frac{\partial T}{\partial t}}_{\text{local change}} + \underbrace{v_j \frac{\partial T}{\partial x_j}}_{\text{convective change}}$$

¶See Section 1.2, What is the difference between $\dfrac{dv_2}{dt}$ and $\dfrac{\partial v_2}{\partial t}$?

Difference between $\dfrac{dv_2}{dt}$ and $\dfrac{\partial v_2}{\partial t}$?



Flow path $x_2 = 1/x_1$. The filled circle shows the point $(x_1, x_2) = (1, 1)$. $\bigtriangledown$: start ($t = \ln(0.5)$); $\bigtriangleup$: end ($t = \ln(2)$).

$$x_1 = \exp(t), \quad x_2 = \exp(-t) \tag{A.1}$$

and hence $x_2 = 1/x_1$. The flow is steady (in Eulerian coordinates).

▶Equation A.1 gives the velocities

$$v_1^L = \frac{dx_1}{dt} = \exp(t), \quad v_2^L = \frac{dx_2}{dt} = -\exp(-t) \tag{A.2}$$

Eqs. A.1 and A.2 give

$$v_1^E = x_1, \quad v_2^E = -x_2$$

Note that $v_i^L = v_i^E$ but $v_i^L = v_i^L(t)$ and $v_i^E = v_i^E(x_1, x_2)$.

▶Time derivatives of the v_2 velocity:

$$\frac{dv_2^L}{dt} = \exp(-t)$$
$$\frac{dv_2^E}{dt} = \frac{\partial v_2^E}{\partial t} + v_1^E \frac{\partial v_2^E}{\partial x_1} + v_2^E \frac{\partial v_2^E}{\partial x_2} = 0 + x_1 \cdot 0 - x_2 \cdot (-1) = x_2$$

Of course $\dfrac{dv_2}{dt} = \dfrac{dv_2^E}{dt} = \dfrac{dv_2^L}{dt} = x_2 = \exp(-t)$.

▶Consider the point $(x_1, x_2) = (1, 1)$. The velocity at this point does not change in time; hence $\dfrac{\partial v_2^E}{\partial t} = 0$.

▶If we however travel with the particle then the v_2 velocity changes with time, i.e. $\dfrac{dv_2^L}{dt} = \dfrac{dv_2}{dt} = 1$ (it increases, i.e. it gets less negative with time).

¶See Section B, Introduction to tensor notation

▶a: A tensor of zeroth rank (scalar)

▶a_i: A tensor of first rank (vector) $a_i = (2, 1, 0)$

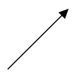
▶a_{ij}: A tensor of second rank (tensor)

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

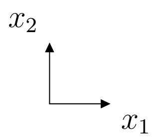
$\sigma_{ij} = \sigma_{ji}$.

What is a tensor?

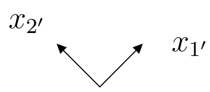
▶A tensor is a *physical* quantity. It is independent of the coordinate system. The tensor of rank one (vector) b_i below



is physically the same expressed in the coordinate system (x_1, x_2)



where $b_i = (1/\sqrt{2}, 1/\sqrt{2}, 0)^T$ and in the coordinate system $(x_{1'}, x_{2'})$



where $b_{i'} = (1, 0, 0)^T$. The tensor is the same even if its *components* are different.

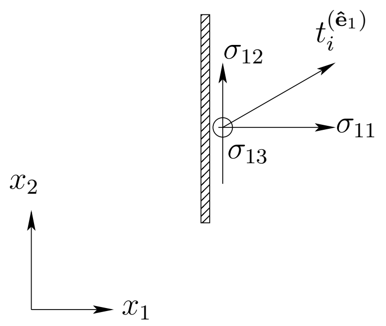
¶See Section 1.3, Viscous stress, pressure

▶The momentum balance equation derived in the continuum mechanics lectures reads

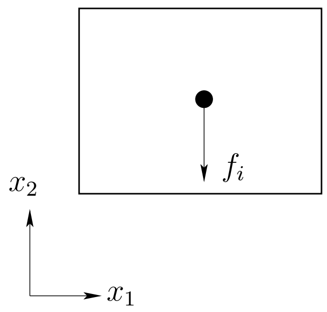
$$\boxed{\rho \dot{v}_i - \sigma_{ji,j} - \rho f_i = 0}$$

We write it as

$$\rho \frac{dv_i}{dt} = \frac{\partial \sigma_{ji}}{\partial x_j} + \rho f_i \tag{A.3}$$



Stress components and stress vector on a surface.



Volume force (per unit volume), $f_i = (0, -g, 0)$, acting in the middle of the fluid element.

Stress tensor, volume (gravitation) force and stress vector, $t_i^{(\hat{\mathbf{e}}_1)}$.

- σ_{ij} denotes the stress tensor. Stress is force per unit area. The surface stress vector is computed as

$$t_i^{(\hat{\mathbf{n}})} = \sigma_{ji} n_j$$

  where $\hat{\mathbf{n}} = n_j$ is the normal vector of the surface.

- volume forces, f_i

►The stress tensor, σ_{ij}, is split into one part which includes pressure, P, and one which includes viscous stresses (friction)

$$\sigma_{ij} = -P\delta_{ij} + \tau_{ij}$$

where $P = -\frac{1}{3}\sigma_{kk}$.
A constitutive relation can be derived for the stress tensor which reads

$$\begin{aligned}
\sigma_{ij} &= -P\delta_{ij} + 2\mu S_{ij} - \frac{2}{3}\mu S_{kk}\delta_{ij} \\
\tau_{ij} &= 2\mu S_{ij} - \frac{2}{3}\mu S_{kk}\delta_{ij} \\
S_{ij} &= \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)
\end{aligned} \tag{A.4}$$

►This expression includes the velocity gradients. Before we insert Eq. A.4 into Eq. A.3, we will look at the velocity gradient tensor, $\frac{\partial v_i}{\partial x_j}$, in some detail.

¶See Section 1.4, Strain rate tensor, vorticity

$$\frac{\partial v_i}{\partial x_j} = \frac{1}{2}\left(\underbrace{\frac{\partial v_i}{\partial x_j} + \frac{\partial v_i}{\partial x_j}}_{2\partial v_i/\partial x_j} + \underbrace{\frac{\partial v_j}{\partial x_i} - \frac{\partial v_j}{\partial x_i}}_{=0}\right)$$
$$= \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) = S_{ij} + \Omega_{ij}$$

►The vorticity reads

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v}$$
$$\omega_i = \epsilon_{ijk}\frac{\partial v_k}{\partial x_j}$$

►The vorticity represents rotation of a fluid particle. Inserting the expression for S_{ij} and Ω_{ij} gives

$$\omega_i = \epsilon_{ijk}(S_{kj} + \Omega_{kj}) = \epsilon_{ijk}\Omega_{kj} \tag{A.5}$$

where we used the fact that the product of a symmetric, S_{kj}, and an antisymmetric tensor, ϵ_{ijk}, is zero.

►Now let's invert Eq. A.5. We start by multiplying it with $\varepsilon_{i\ell m}$ so that

$$\varepsilon_{i\ell m}\omega_i = \varepsilon_{i\ell m}\epsilon_{ijk}\Omega_{kj} \tag{A.6}$$

- ϵ_{ijk} is the permutation tensor.
  - It is one if ijk is equal to 123 or any cyclic permutation, i.e. $\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1$.
  - Switch two indices and it is equal to minus one, i.e, $\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{312} = -1$.
  - If two indices are equal, then ϵ_{ijk} is zero.
- δ_{ij} is the unit or identity tensor. It is one if ijk are equal and zero otherwise, i.e.
  - $\delta_{11} = \delta_{22} = \delta_{33} = 1$
  - $\delta_{12} = \delta_{13} = \delta_{21} = \delta_{23} = \delta_{31} = \delta_{32} = 0$

Using the ε-δ-identity (see Section C) on the right side of Eq. A.6 gives

$$\varepsilon_{i\ell m}\epsilon_{ijk}\Omega_{kj} = (\delta_{\ell j}\delta_{mk} - \delta_{\ell k}\delta_{mj})\Omega_{kj} = \Omega_{m\ell} - \Omega_{\ell m} = 2\Omega_{m\ell}$$

Inserted in Eq. A.6 we get

$$\Omega_{m\ell} = \frac{1}{2}\varepsilon_{i\ell m}\omega_i = \frac{1}{2}\varepsilon_{\ell mi}\omega_i = -\frac{1}{2}\varepsilon_{m\ell i}\omega_i$$

where we first used cyclic permutation of $\varepsilon_{i\ell m}$, then used the fact that $\varepsilon_{i\ell m}$ is anti-symmetric.

▶Actually, it is easier to invert Eq. A.5 component-by-component. For ω_3 we get

$$\begin{aligned}\omega_3 = \epsilon_{3jk}\frac{\partial v_k}{\partial x_j} &= \epsilon_{321}\frac{\partial v_1}{\partial x_2} + \epsilon_{312}\frac{\partial v_2}{\partial x_1}\\ &= -\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} = \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right) = -2\Omega_{12}\end{aligned}$$

¶See Section 1.5, Product of a symmetric and antisymmetric tensor
▶The product of a symmetric, a_{ji}, and antisymmetric tensor, b_{ji}, is zero

$$a_{ij}b_{ij} = a_{ji}b_{ij} = -a_{ji}b_{ji},$$

where we used

2nd expression $a_{ij} = a_{ji}$ (symmetric)

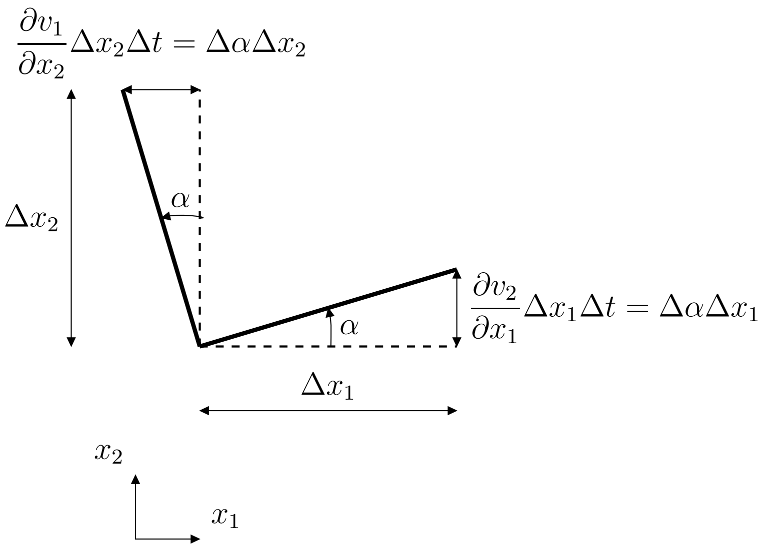
last expression $b_{ij} = -b_{ji}$ (antisymmetric)

Indices i and j are dummy indices $\Rightarrow$

$$a_{ij}b_{ij} = -a_{ij}b_{ij}$$

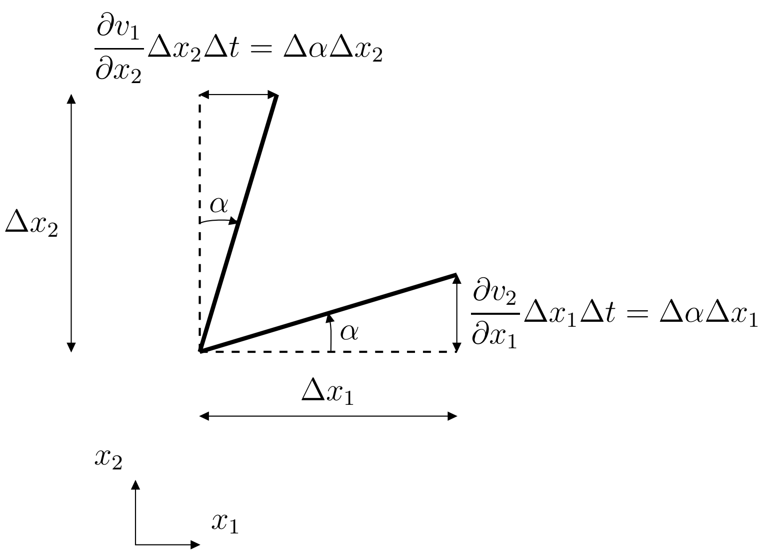
This expression says that $A = -A$ which can be only true if $A = 0$ and hence $a_{ij}b_{ij} = 0$.

¶See Section 1.6, Deformation, rotation
▶Rotation of a fluid particle ($\omega_3 > 0$, $\Omega_{12} < 0$) during time Δt

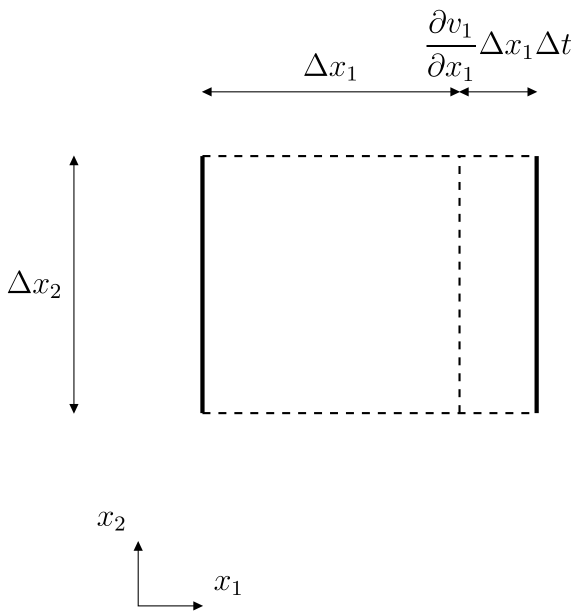


- angle rotation per unit time: $\frac{\Delta\alpha}{\Delta t} \simeq d\alpha/dt = -\partial v_1/\partial x_2 = \partial v_2/\partial x_1$
- if fluid element does not rotate as a solid body, we take the average $d\alpha/dt = (\partial v_2/\partial x_1 - \partial v_1/\partial x_2)/2$.
- Hence, the vorticity $\omega_3 = \partial v_2/\partial x_1 - \partial v_1/\partial x_2$ can be interpreted as twice the average rotation of the horizontal edge and vertical edge

▶Deformation of a fluid particle by shear during time Δt. Here $\partial v_1/\partial x_2 = \partial v_2/\partial x_1$ so that $S_{12} = (\partial v_1/\partial x_2 + \partial v_2/\partial x_1)/2 = \partial v_1/\partial x_2 > 0$.



▶Deformation of a fluid particle by elongation during time Δt. Here $S_{11} = \partial v_1/\partial x_1 > 0$



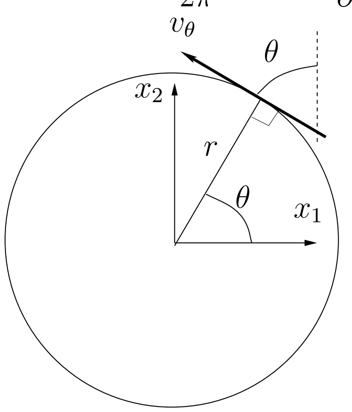
¶See Section 1.7, Irrotational and rotational flow

►Flows are often classified based on rotation: they are *rotational* ($\omega_i \neq 0$) or *irrotational* ($\omega_i = 0$)

¶See Section 1.7.1, Ideal vortex line

►Consider the ideal vortex line

$$\Phi = \frac{\Gamma\theta}{2\pi}, \quad v_k = \frac{\partial \Phi}{\partial x_k}, \quad v_\theta = \frac{\Gamma}{2\pi r}, \quad v_r = 0$$

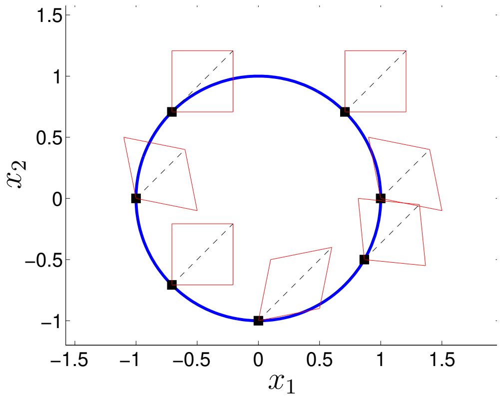


Transform v_θ into Cartesian components.

$$v_1 = -\frac{\Gamma x_2}{2\pi(x_1^2 + x_2^2)}, \quad v_2 = \frac{\Gamma x_1}{2\pi(x_1^2 + x_2^2)}.$$

$$\frac{\partial v_1}{\partial x_2} = -\frac{\Gamma}{2\pi}\frac{x_1^2 - x_2^2}{(x_1^2 + x_2^2)^2}, \quad \frac{\partial v_2}{\partial x_1} = \frac{\Gamma}{2\pi}\frac{x_2^2 - x_1^2}{(x_1^2 + x_2^2)^2}$$

$$\Rightarrow \omega_3 = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_2}{\partial x_1} = 0$$



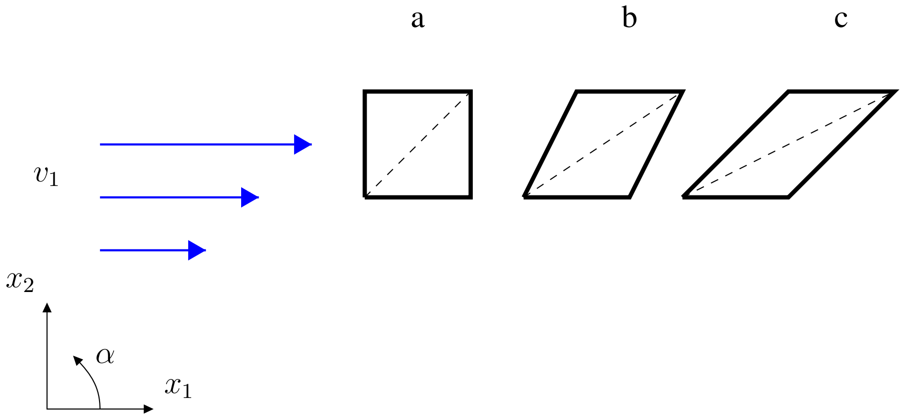
The fluid particle (i.e. its diagonal) does not rotate. The locations of the fluid particle is indicated by black, filled squares. The diagonals are shown as black dashed lines. The fluid particle is shown at $\theta = 0$, $\pi/4$, $3\pi/4$, π, $5\pi/4$, $3\pi/2$ and $-\pi/6$.

¶See Section 1.7.2, Shear flow

►Consider shear flow with $v_1 = cx_2^2$, $v_2 = 0$, see figure below. The vorticity is computed as

$$\omega_1 = \omega_2 = 0, \quad \omega_3 = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} = -2cx_2$$

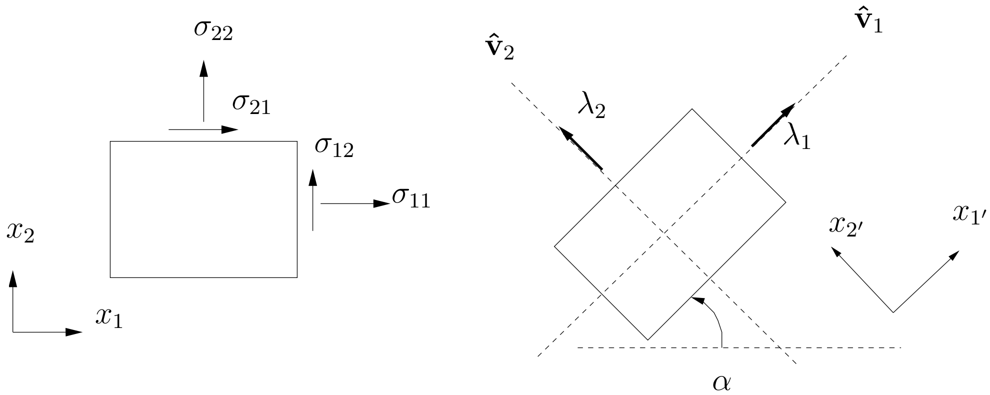
Hence the flow is rotational



The vertical edges of the fluid particles rotate according to the figure above (rotating in negative direction).

¶See Section 1.8, Eigenvalues and eigenvectors: physical interpretation

▶Eigenvalues and and eigenvectors: physical interpretation



- A two-dimensional fluid element.
  - Left: in original state;
  - right: rotated to principal coordinate directions.
  - $\lambda_1 = \sigma_{1'1'}$ and $\lambda_2 = \sigma_{2'2'}$ denote eigenvalues;
  - $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$ denote unit eigenvectors.

A.2 Lecture 2

¶See Section 2.1.1, The continuity equation

▶The continuity equation

$$\frac{d\rho}{dt} + \rho\frac{\partial v_i}{\partial x_i} = 0 \quad \text{incompressible flow gives} \quad \frac{\partial v_i}{\partial x_i} = 0$$

¶See Section 2.1.2, The momentum equation

▶The momentum equation

$$\sigma_{ij} = -P\delta_{ij} + 2\mu S_{ij} - \frac{2}{3}\mu S_{kk}\delta_{ij}$$

$$\begin{aligned}\rho\frac{dv_i}{dt} = \frac{\partial\sigma_{ji}}{\partial x_j} + \rho f_i &= -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(2\mu S_{ij} - \frac{2}{3}\mu\frac{\partial v_k}{\partial x_k}\delta_{ij}\right) + \rho f_i \\ &= -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}(2\mu S_{ij}) - \frac{2}{3}\frac{\partial}{\partial x_i}\left(\mu\frac{\partial v_k}{\partial x_k}\right) + \rho f_i\end{aligned}$$

Note that the stress tensor, σ_{ij}, depends only on S_{ij} (deformation), not on Ω_{ij} (rotation).

▶Incompressible flow gives

$$\rho\frac{dv_i}{dt} = \frac{\partial\sigma_{ji}}{\partial x_j} + \rho f_i = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}(2\mu S_{ij}) + \rho f_i$$

▶Incompressible and constant μ give

$$\rho\frac{dv_i}{dt} = -\frac{\partial P}{\partial x_i} + \mu\frac{\partial^2 v_i}{\partial x_j\partial x_j} + \rho f_i$$

¶See Section 2.2, The energy equation

▶The energy equation

$$\underbrace{\rho\frac{du}{dt}}_{\text{energy change}} = \underbrace{\sigma_{ji}\frac{\partial v_i}{\partial x_j}}_{\text{exchange of work}} \underbrace{-\frac{\partial q_i}{\partial x_i}}_{\text{exchange of heat}} \tag{A.7}$$

$$q_i = -k\frac{\partial T}{\partial x_i}$$

$$\sigma_{ij} = -P\delta_{ij} + 2\mu S_{ij} - \frac{2}{3}\mu S_{kk}\delta_{ij}$$

▶We have

$$\frac{\partial v_i}{\partial x_j} = S_{ij} + \Omega_{ij}, \quad S_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \quad \Omega_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$$

$$\sigma_{ij}\frac{\partial v_i}{\partial x_j} = \sigma_{ij}(S_{ij} + \Omega_{ij}) = \sigma_{ij}S_{ij}$$

This gives (See Eq. 2.15)

$$\sigma_{ij}\frac{\partial v_i}{\partial x_j} = \left\{-P\delta_{ij} + 2\mu S_{ij} - \frac{2}{3}\mu\frac{\partial v_k}{\partial x_k}\delta_{ij}\right\}\frac{\partial v_i}{\partial x_j}$$
$$= \left\{-P\delta_{ij} + 2\mu S_{ij} - \frac{2}{3}\mu\frac{\partial v_k}{\partial x_k}\delta_{ij}\right\}(S_{ij} + \Omega_{ij}) = -P\frac{\partial v_i}{\partial x_i} + 2\mu S_{ij}S_{ij} - \frac{2}{3}\mu S_{kk}S_{ii}$$

Insert into Eq. A.7 gives

$$\underbrace{\rho\frac{du}{dt}}_{\Delta U} = \underbrace{-P\frac{\partial v_i}{\partial x_i}}_{Rev} + \underbrace{2\mu S_{ij}S_{ij} - \frac{2}{3}\mu S_{kk}S_{ii}}_{\Phi} + \underbrace{\frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right)}_{Q}$$

▶During time, dt, the following happens:

ΔU: Change of inner energy of the fluid

Rev: Reversible work done by the fluid (compression or expansion)

Φ: Irreversible work (dissipation) done by the fluid

Q: Exchange of heat to the fluid

▶Incompressible flow (low speed, $|v_i| < \frac{1}{3}$speed of sound)

$$du = c_p dT, \quad \rho c_p\frac{dT}{dt} = \Phi + \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right)$$

Φ important for lubricant oils
▶For gases and "usual" liquids (i.e. not lubricant oils) we get (k is constant), see Eq. 2.18

$$\frac{dT}{dt} = \alpha\frac{\partial^2 T}{\partial x_i \partial x_i}, \quad \alpha = \frac{k}{\rho c_p}, \quad Pr = \frac{\nu}{\alpha}$$

¶See Section 2.3, Transformation of energy
▶k equation (multiply the momentum eq. with v_i)

$$\rho v_i\frac{dv_i}{dt} - v_i\frac{\partial \sigma_{ji}}{\partial x_j} - v_i\rho f_i = 0$$

The first term on the left side can be re-written (Trick 2, see Eq. 8.4)

$$\rho v_i\frac{dv_i}{dt} = \frac{1}{2}\rho\frac{d(v_iv_i)}{dt} = \rho\frac{dk}{dt}$$

$(v_i v_i/2 = v^2/2 = k)$ so that

$$\rho \frac{dk}{dt} = v_i \frac{\partial \sigma_{ji}}{\partial x_j} + \rho v_i f_i$$

Re-write the stress-velocity term so that (see Eq. 2.23)

$$\rho \frac{dk}{dt} = \frac{\partial v_i \sigma_{ji}}{\partial x_j} - \sigma_{ji} \frac{\partial v_i}{\partial x_j} + \rho v_i f_i$$

►Compare with the equation for inner energy

$$\rho \frac{du}{dt} = \sigma_{ji} \frac{\partial v_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i}$$

¶See Section 2.4, Left side of the transport equations

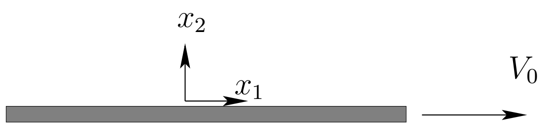
►Left-hand side $(\Psi = v_i, u, T \ldots)$

$$\begin{aligned}
\rho \frac{d\Psi}{dt} &= \rho \frac{\partial \Psi}{\partial t} + \rho v_j \frac{\partial \Psi}{\partial x_j} \\
&= \rho \frac{\partial \Psi}{\partial t} + \rho v_j \frac{\partial \Psi}{\partial x_j} + \Psi \underbrace{\left(\frac{d\rho}{dt} + \rho \frac{\partial v_j}{\partial x_j} \right)}_{=0}
\end{aligned}$$

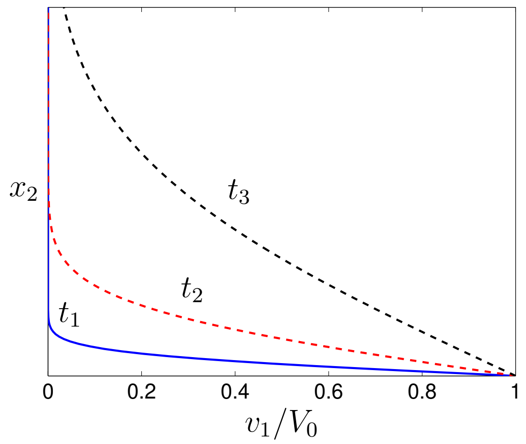
$$= \underline{\rho \frac{\partial \Psi}{\partial t}} + \underbrace{\rho v_j \frac{\partial \Psi}{\partial x_j}} + \Psi \left(\underline{\frac{\partial \rho}{\partial t}} + \underbrace{v_j \frac{\partial \rho}{\partial x_j}} + \underbrace{\rho \frac{\partial v_j}{\partial x_j}} \right) = \frac{\partial \rho \Psi}{\partial t} + \frac{\partial \rho v_j \Psi}{\partial x_j}$$

¶See Section 3.1, The Rayleigh problem

▶The Rayleigh problem



The plate moves to the right with speed V_0 for $t > 0$.



▶The v_1 velocity at three different times. $t_3 > t_2 > t_1$.

Simplificatons: $\partial v_1/\partial x_1 = \partial v_3/\partial x_3 = 0 \Rightarrow \partial v_2/\partial x_2 = 0 \Rightarrow v_2 \equiv 0$.

$$\frac{\partial v_1}{\partial t} = \nu \frac{\partial^2 v_1}{\partial x_2^2}$$

▶Similarity solution: the number of independent variables is reduced by one, from two (x_2 and t) to one (η).

$$\eta = \frac{x_2}{2\sqrt{\nu t}}, \quad \frac{\partial v_1}{\partial t} = \frac{dv_1}{d\eta}\frac{\partial \eta}{\partial t} = -\frac{x_2 t^{-3/2}}{4\sqrt{\nu}}\frac{dv_1}{d\eta} = -\frac{1}{2}\frac{\eta}{t}\frac{dv_1}{d\eta}$$

$$\frac{\partial v_1}{\partial x_2} = \frac{dv_1}{d\eta}\frac{\partial \eta}{\partial x_2} = \frac{1}{2\sqrt{\nu t}}\frac{dv_1}{d\eta}$$

$$\frac{\partial^2 v_1}{\partial x_2^2} = \frac{\partial}{\partial x_2}\left(\frac{\partial v_1}{\partial x_2}\right) = \frac{\partial}{\partial x_2}\left(\frac{1}{2\sqrt{\nu t}}\frac{dv_1}{d\eta}\right) = \frac{1}{2\sqrt{\nu t}}\frac{\partial}{\partial x_2}\left(\frac{dv_1}{d\eta}\right) = \frac{1}{4\nu t}\frac{d^2 v_1}{d\eta^2}$$

We get (see Eq. 3.6)

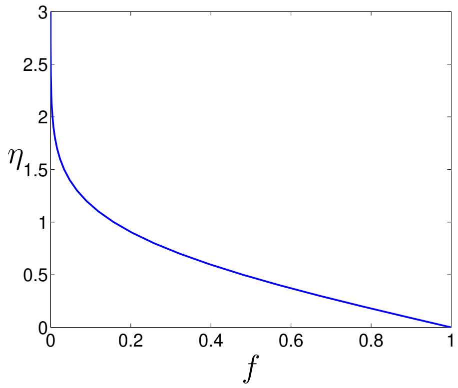
$$f = \frac{v_1}{V_0}, \quad \frac{d^2 f}{d\eta^2} + 2\eta\frac{df}{d\eta} = 0$$

▶Boundary conditions

$$\begin{aligned} v_1(x_2, t=0) = 0 &\Rightarrow f(\eta \to \infty) = 0 \\ v_1(x_2 = 0, t) = V_0 &\Rightarrow f(\eta = 0) = 1 \\ v_1(x_2 \to \infty, t) = 0 &\Rightarrow f(\eta \to \infty) = 0 \end{aligned}$$

The solution reads (see Eq. 3.11)

$$f(\eta) = 1 - \text{erf}(\eta), \quad \eta = \frac{x_2}{2\sqrt{\nu t}}, \quad f = \frac{v_1}{V_0} \tag{A.8}$$



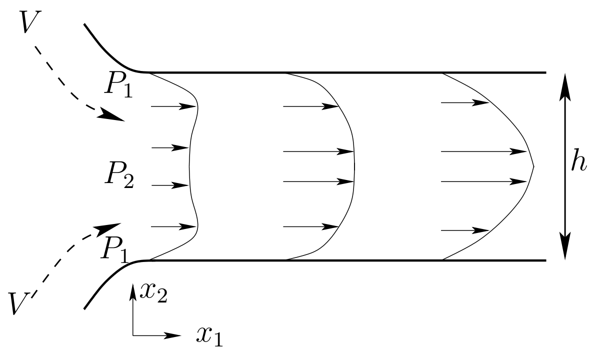
The velocity, $f = v_1/V_0$, given by Eq. A.8.

▶Boundary layer thickness defined by $v_1 = 0.01V_0$ (it would be $v_1 = 0.99V_0$ in an ordinary boundary layer). The figure above gives (for $f = 0.01$) $\eta = 1.8$ so that

$$\eta = 1.8 = \frac{\delta}{2\sqrt{\nu t}} \quad \Rightarrow \delta = 3.6\sqrt{\nu t}$$

$$\delta_{air} = 10.8cm$$
$$\delta_{water} = 2.8cm$$

¶See Section 3.2.1, Curved plates
▶The inlet part of a channel. $P_2 > P_1$



¶See Section 3.2.2, Flat plates
▶Fully developed incompressible flow in a channel. 2D and steady.
$\frac{\partial v_1}{\partial x_1} = \frac{\partial v_2}{\partial x_1} = \frac{\partial v_3}{\partial x_3} = v_3 = 0.$

$$\frac{\partial v_i}{\partial x_i} = 0 \quad \Rightarrow \frac{\partial v_2}{\partial x_2} = 0 \quad \Rightarrow v_2 = C_1(x_1) \quad \Rightarrow v_2 \equiv 0$$

►The Navier-Stokes for v_1 ($g_i = (0, -g, 0)$)

$$\cancelto{0}{\frac{\partial v_1}{\partial t}} + v_1 \cancelto{0}{\frac{\partial v_1}{\partial x_1}} + \cancel{v_2}\frac{\partial v_1}{\partial x_2} = -\frac{\partial P}{\partial x_1} + \mu\left(\cancelto{0}{\frac{\partial^2 v_1}{\partial x_1^2}} + \frac{\partial^2 v_1}{\partial x_2^2}\right) + \cancelto{0}{\rho g_1} \Rightarrow \mu\frac{\partial^2 v_1}{\partial x_2^2} = \frac{\partial P}{\partial x_1}$$

The Navier-Stokes for v_2 gives

$$\cancelto{0}{\frac{\partial v_2}{\partial t}} + v_1 \cancelto{0}{\frac{\partial v_2}{\partial x_1}} + \cancel{v_2}\frac{\partial v_2}{\partial x_2} = -\frac{\partial P}{\partial x_2} + \mu\left(\cancelto{0}{\frac{\partial^2 v_2}{\partial x_1^2}} + \cancelto{0}{\frac{\partial^2 v_2}{\partial x_2^2}}\right) - \rho g$$

$$\Rightarrow 0 = -\frac{\partial P}{\partial x_2} - \rho g \quad \Rightarrow P = -\rho g x_2 + C_1(x_1) = -\rho g x_2 + p(x_1)$$

$$\Rightarrow -\frac{\partial P}{\partial x_1} = -\frac{\partial p}{\partial x_1} \quad (p = p(x_1) \text{ is pressure at lower wall})$$

►The Navier-Stokes for v_1 (replacing $\partial P/\partial x_1$ by $\partial p/\partial x_1$, see Eq. 3.24)

$$0 = -\frac{\partial p}{\partial x_1} + \mu\frac{\partial^2 v_1}{\partial x_2^2} \quad \Rightarrow \mu\underbrace{\frac{\partial^2 v_1}{\partial x_2^2}}_{f(x_2)} = \underbrace{\frac{\partial p}{\partial x_1}}_{f(x_1)}$$

$$\Rightarrow \frac{\partial^2 v_1}{\partial x_2^2} = \frac{\partial p}{\partial x_1} = \text{const}$$

Integrate twice gives $v_1 = -\dfrac{h}{2\mu}\dfrac{dp}{dx_1}x_2\left(1 - \dfrac{x_2}{h}\right)$

¶See Section 3.3, Two-dimensional boundary layer flow over flat plate

►2D boundary layer

$$v_1\frac{\partial v_1}{\partial x_1} + v_2\frac{\partial v_1}{\partial x_2} = \nu\frac{\partial^2 v_1}{\partial x_2^2}, \quad \frac{\partial p}{\partial x_2} = 0, \quad \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0$$

(note that both terms on the left side are retained)

Streamfunction Ψ

$$v_1 = \frac{\partial \Psi}{\partial x_2}, \quad v_2 = -\frac{\partial \Psi}{\partial x_1}$$

The continuity equation is automatically satisfied

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = \frac{\partial^2 \Psi}{\partial x_1 \partial x_2} - \frac{\partial^2 \Psi}{\partial x_2 \partial x_1} = 0$$

Inserting the streamfunction into the streamwise momentum equation

$$\frac{\partial \Psi}{\partial x_2}\frac{\partial^2 \Psi}{\partial x_1 \partial x_2} - \frac{\partial \Psi}{\partial x_1}\frac{\partial^2 \Psi}{\partial x_2^2} = \nu \frac{\partial^3 \Psi}{\partial x_2^3}$$

Similarity solution: $x_1, x_2 \to \xi$; $\Psi \to g(\xi)$.

$$\xi = \left(\frac{V_{1,\infty}}{\nu x_1}\right)^{1/2} x_2, \quad \Psi = \left(\nu V_{1,\infty} x_1\right)^{1/2} g$$

First we need the derivatives $\partial\xi/\partial x_1$ and $\partial\xi/\partial x_2$

$$\frac{\partial \xi}{\partial x_1} = -\frac{1}{2}\left(\frac{V_{1,\infty}}{\nu x_1}\right)^{1/2}\frac{x_2}{x_1} = -\frac{\xi}{2x_1}, \quad \frac{\partial \xi}{\partial x_2} = \left(\frac{V_{1,\infty}}{\nu x_1}\right)^{1/2} = \frac{\xi}{x_2}$$

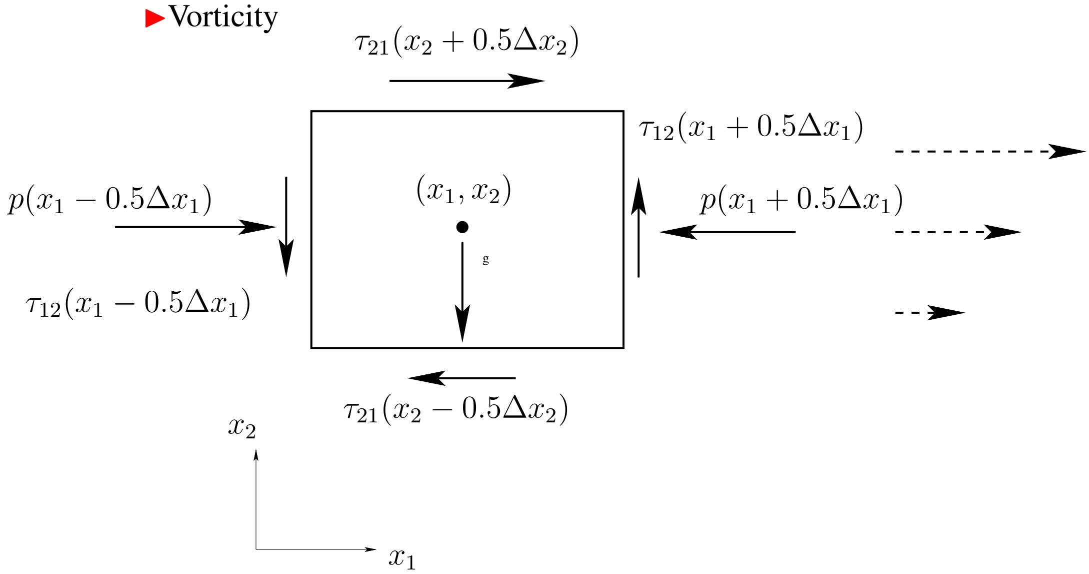
$$\begin{aligned}
\frac{\partial \Psi}{\partial x_1} &= \frac{\partial}{\partial x_1}\left(\left(\nu V_{1,\infty} x_1\right)^{1/2}\right) g + \left(\nu V_{1,\infty} x_1\right)^{1/2} g' \frac{\partial \xi}{\partial x_1} \\
&= \frac{1}{2}\left(\frac{\nu V_{1,\infty}}{x_1}\right)^{1/2} g - \left(\nu V_{1,\infty} x_1\right)^{1/2} g' \frac{\xi}{2x_1} \\
\ldots \Rightarrow &-\frac{1}{2} g g'' + g''' = 0
\end{aligned}$$

This is Blasius solution (from his PhD thesis in 1907). The numerical solution is given in Table 3.1.

A.3 Lecture 3

¶See Section 4.1, Vorticity and rotation

▶Vorticity



Surface forces. $\partial\tau_{12}/\partial x_1 = 0$, $\partial\tau_{21}/\partial x_2 > 0$.

$$\rho\frac{dv_i}{dt} = -\frac{\partial P}{\partial x_i} + \boxed{\frac{\partial \tau_{ji}}{\partial x_j}} = -\frac{\partial P}{\partial x_i}\boxed{-\mu\varepsilon_{inm}\frac{\partial\omega_m}{\partial x_n}} \tag{A.9}$$

▶change in vorticity ⇔ change in shear stresses

- irrotational flow ⇔
- potential flow ⇔
- no change in ω_i (often $\omega_i = 0$)

▶As a first step for deriving the ω_i transport eq., let's re-write the left-side of N-S:

$$v_j\frac{\partial v_i}{\partial x_j} = v_j(S_{ij} + \Omega_{ij}) = v_j\left(S_{ij} - \frac{1}{2}\varepsilon_{ijk}\omega_k\right)$$

Inserting $S_{ij} = (\partial v_i/\partial x_j + \partial v_j/\partial x_i)/2$ and multiplying by two gives

$$2v_j\frac{\partial v_i}{\partial x_j} = v_j\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - \varepsilon_{ijk}v_j\omega_k \tag{A.10}$$

The second term on the right side can be written as (Trick 2)

$$v_j\frac{\partial v_j}{\partial x_i} = \frac{1}{2}\frac{\partial(v_jv_j)}{\partial x_i}$$

With Eq. A.10 Navier-Stokes can be written (see Eq. 4.13)

$$\frac{\partial v_i}{\partial t} + \underbrace{\frac{\partial \frac{1}{2} v^2}{\partial x_i}}_{\text{no rotation}} - \underbrace{\varepsilon_{ijk} v_j \omega_k}_{\text{rotation}} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 v_i}{\partial x_j \partial x_j} + f_i \tag{A.11}$$

►Now we will derive the transport eq. for $\omega_p = \varepsilon_{pqi}\partial v_i/\partial x_q$. Multiply the Navier-Stokes equation by $\varepsilon_{pqi}\partial/\partial x_q$ so that

$$\begin{aligned}
&\varepsilon_{pqi}\frac{\partial^2 v_i}{\partial t \partial x_q} + \cancelto{0}{\varepsilon_{pqi}\frac{\partial^2 k}{\partial x_i \partial x_q}} - \varepsilon_{pqi}\varepsilon_{ijk}\frac{\partial v_j \omega_k}{\partial x_q} \\
&= \cancelto{0}{-\varepsilon_{pqi}\frac{1}{\rho}\frac{\partial^2 P}{\partial x_i \partial x_q}} + \nu \varepsilon_{pqi}\frac{\partial^3 v_i}{\partial x_j \partial x_j \partial x_q} + \cancelto{0}{\varepsilon_{pqi}\frac{\partial g_i}{\partial x_q}}
\end{aligned} \tag{A.12}$$

- Term on line 1: zero because a-sym & sym tensor
- 1st term in line 2: zero because a-sym & sym tensor
- last term: zero because g_i is constant

Re-write unsteady and viscous terms in Eq. A.12:

$$\varepsilon_{pqi}\frac{\partial^2 v_i}{\partial t \partial x_q} = \frac{\partial}{\partial t}\left(\varepsilon_{pqi}\frac{\partial v_i}{\partial x_q}\right) = \frac{\partial \omega_p}{\partial t}$$

$$\nu \varepsilon_{pqi}\frac{\partial^3 v_i}{\partial x_j \partial x_j \partial x_q} = \nu \frac{\partial^2}{\partial x_j \partial x_j}\left(\varepsilon_{pqi}\frac{\partial v_i}{\partial x_q}\right) = \nu \frac{\partial^2 \omega_p}{\partial x_j \partial x_j}$$

Inserted in Eq. A.12 gives

$$\frac{\partial \omega_p}{\partial t} - \varepsilon_{pqi}\varepsilon_{ijk}\frac{\partial v_j \omega_k}{\partial x_q} = \nu \frac{\partial^2 \omega_p}{\partial x_j \partial x_j}$$

►Transport equation for the vorticity (see Eq. 4.21)

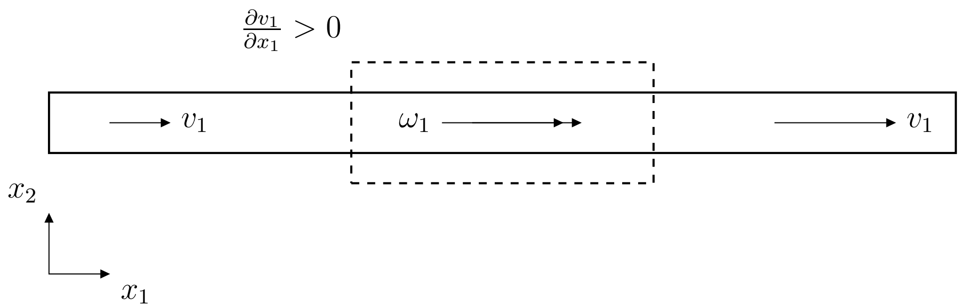
$$\frac{\partial \omega_p}{\partial t} + v_k \frac{\partial \omega_p}{\partial x_k} = \underline{\omega_k \frac{\partial v_p}{\partial x_k}} + \nu \frac{\partial^2 \omega_p}{\partial x_j \partial x_j}$$

Underlined term: vortex stretching/tilting term (a source/sink), see Eq. 4.22

$$\omega_k \frac{\partial v_p}{\partial x_k} = \begin{cases}
\omega_1 \frac{\partial v_1}{\partial x_1} & + \omega_2 \frac{\partial v_1}{\partial x_2} + \omega_3 \frac{\partial v_1}{\partial x_3}, \quad p = 1 \\
\omega_1 \frac{\partial v_2}{\partial x_1} & + \omega_2 \frac{\partial v_2}{\partial x_2} + \omega_3 \frac{\partial v_2}{\partial x_3}, \quad p = 2 \\
\omega_1 \frac{\partial v_3}{\partial x_1} & + \omega_2 \frac{\partial v_3}{\partial x_2} + \omega_3 \frac{\partial v_3}{\partial x_3}, \quad p = 3
\end{cases}$$

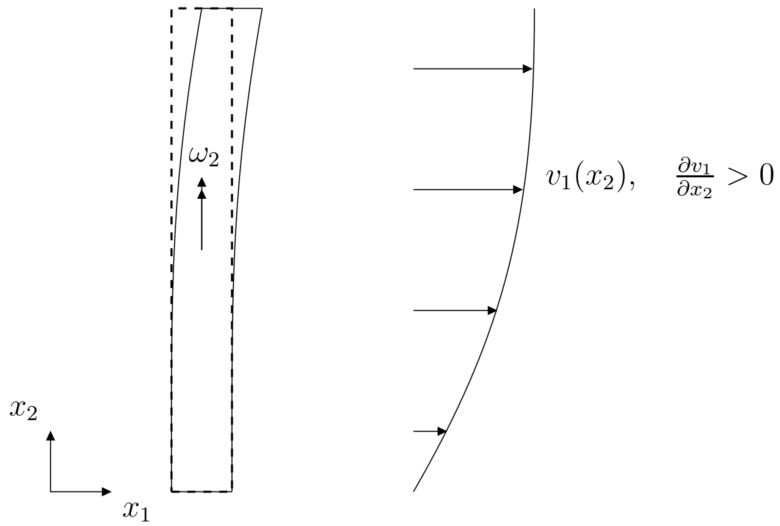
Vortex stretching and vortex tilting

▶Vortex stretching



Assume $\dfrac{\partial v_1}{\partial x_1} > 0$: the term $\omega_1 \dfrac{\partial v_1}{\partial x_1}$ will increase ω_1

▶Vortex tilting/deflection



Assume $\dfrac{\partial v_1}{\partial x_2} > 0$: the term $\omega_2 \dfrac{\partial v_1}{\partial x_2}$ will increase ω_1

¶See Section 4.3, The vorticity transport equation in two dimensions

▶2D flow: $v_i = (v_1, v_2, 0)$, $\omega_i = (0, 0, \omega_3)$ and $\dfrac{\partial}{\partial x_3} = 0$

Now the vortex stretching/tilting term $\omega_k \dfrac{\partial v_p}{\partial x_k} = \omega_3 \dfrac{\partial v_p}{\partial x_3} = 0$

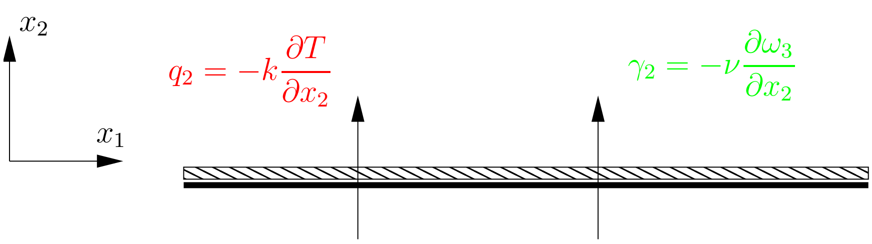
The 2D ω_3 equation reads (see Eq. 4.23)

$$\frac{\partial \omega_3}{\partial t} + v_k \frac{\partial \omega_3}{\partial x_k} = \nu \frac{\partial^2 \omega_3}{\partial x_j \partial x_j}$$

▶Consider fully developed channel flow (see Eq. 4.24)

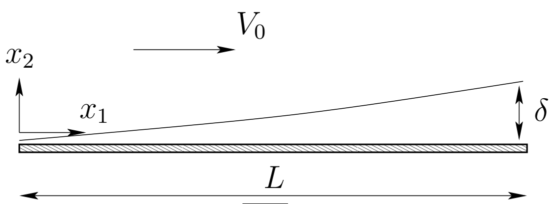
heat conduction $0 = k\dfrac{\partial^2 T}{\partial x_2^2}$

vorticity diffusion $0 = \nu\dfrac{\partial^2 \omega_3}{\partial x_2^2}$



Temperature: $q_2 = 0 \Rightarrow$ no temperature (increase)
Vorticity: $\gamma_2 = 0 \Rightarrow$ no vorticity (increase). In the self-similar region of a boundary layer this is true because $\partial p/\partial x_1 = 0$; for channel flow $\gamma_2 \neq 0$

¶See Section 4.3.1, Boundary layer thickness from the Rayleigh problem
▶Rayleigh problem: $\delta(t) = 3.6\sqrt{\nu t}$ was presented for the temperature equation. It can also be used for the vorticity equation.



Boundary layer thickness $\delta \propto \sqrt{\dfrac{L\nu}{V}} = L\sqrt{\dfrac{\nu}{VL}} \Rightarrow \dfrac{\delta}{L} \propto \sqrt{\dfrac{1}{Re_L}}$

A.4 Lecture 4

Potential flow

¶See Section 4.4, Potential flow
▶Define a potential

$$v_i = \partial\Phi/\partial x_i \tag{A.13}$$

Then the vorticity is zero

$$\omega_i = \epsilon_{ijk}\frac{\partial v_k}{\partial x_j} = \epsilon_{ijk}\frac{\partial^2\Phi}{\partial x_j \partial x_k} = 0$$

The continuity eq. reads

$$0 = \frac{\partial v_i}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\partial\Phi}{\partial x_i}\right) = \frac{\partial^2\Phi}{\partial x_i \partial x_i} \tag{A.14}$$

▶Derive the Bernoulli eq. The N-S reads (see Eqs. A.9 and A.11)

$$\frac{\partial v_i}{\partial t} + \underbrace{\frac{\partial \frac{1}{2}v^2}{\partial x_i}}_{\text{no rotation}} \underbrace{-\varepsilon_{ijk}v_j\omega_k}_{\text{rotation}} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - \nu\varepsilon_{inm}\frac{\partial\omega_m}{\partial x_n} + g_i$$

In potential flow, $\omega_i = 0$. Insert Φ (Eq. A.13) and a gravitation potential ($g_i = -\partial\mathcal{X}/\partial x_i$)

$$\frac{\partial}{\partial x_i}\left(\frac{\partial\Phi}{\partial t}\right) + \frac{\partial\frac{1}{2}v^2}{\partial x_i} + \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\partial\mathcal{X}}{\partial x_i} = 0$$

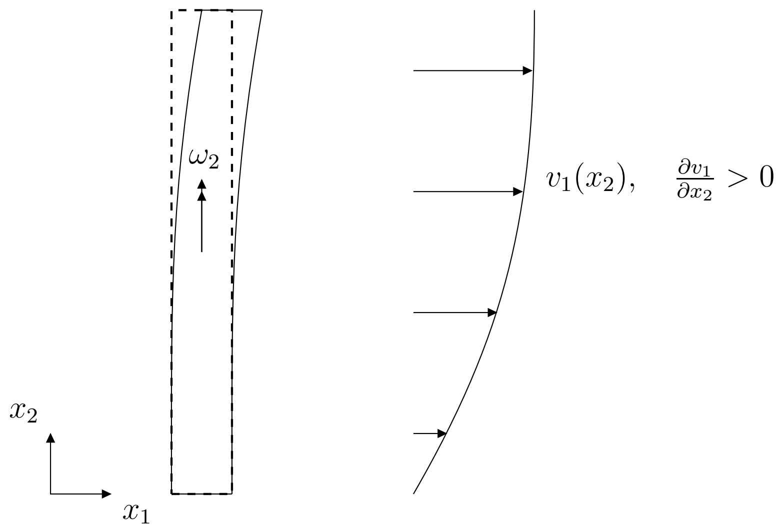
Integration gives (see Eq. 4.33)

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}v^2 + \frac{p}{\rho} + \mathcal{X} = const$$

Replacing $\mathcal{X} = -g_3 x_3 = gh$ gives the Bernoulli eq.

¶See Section 4.4.2, Complex variables for potential solutions of plane flows
▶Complex functions.
The derivative of a complex function, f, by a complex variable, z ($f = u + iv$ and $z = x + iy$) is defined only if the derivatives in the real and imaginary directions are the same, i.e.

$$\frac{df}{dz} = \lim_{\Delta z\to 0}\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$
$$= \lim_{\Delta x\to 0}\frac{f(x_0+\Delta x, iy_0) - f(x_0, iy_0)}{\Delta x} = \lim_{\Delta y\to 0}\frac{f(x_0, iy_0+i\Delta y) - f(x_0, iy_0)}{i\Delta y}. \tag{A.15}$$



which means that

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y} = -i\frac{\partial f}{\partial y} \tag{A.16}$$

Inserting $f = u + iv$ in Eq. A.16 gives

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} - i^2\frac{\partial v}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

We get (see Eq. 4.39)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

which are called the *Cauchy-Riemann* equations.

►A complex function in polar coordinates: $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$

►Fluid dynamics: define a complex potential $f = \Phi + i\Psi$ where Ψ is the streamfunction (recall the $v_1 = \frac{\partial \Psi}{\partial y}$ and $v_2 = -\frac{\partial \Psi}{\partial x}$, see Eq. 3.44). We want f to be differentiable: hence Eq. A.4 must hold (replace u and v with Φ and Ψ)

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Psi}{\partial y}, \quad \frac{\partial \Phi}{\partial y} = -\frac{\partial \Psi}{\partial x}, \tag{A.17}$$

which is satisfied (first relation $= v_1$, second $= v_2$).

Φ satisfies Laplace eq. (see Eq. A.14). Since $\omega_3 = 0$, this applies also for Ψ

$$\frac{\partial^2 \Psi}{\partial x_1^2} + \frac{\partial^2 \Psi}{\partial x_2^2} = -\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} = -\omega_3 = 0$$

Hence, f also satisfies Laplace eq.
We have now defined a complex function, $f = \Phi + i\Psi$ which satisfies Laplace eq and which has a physical meaning in fluid dynamics.

¶See Section 4.4.3, $f \propto z^n$

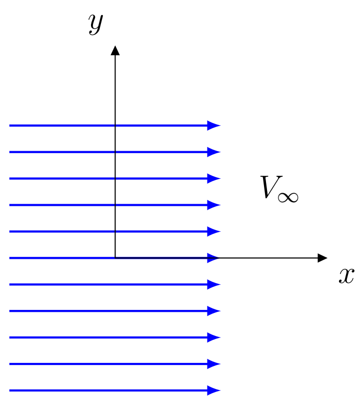
1. Now we "guess"/dream up a complex function $f = \Phi + i\Psi$
2. then we check if it satisfies the Laplace equation (i.e. the continuity equation, 4.29 and that the flow is inviscid, $\omega_3 = 0$, Eq. 4.42)
3. then we find out if f corresponds to a meaningful fluid flow situation

►We guess $f = C_1 z^n = C_1 r^n e^{in\theta} = C_1 r^n(\cos(n\theta) + i\sin(n\theta))$

Check that it satisfies Laplace eq. (it does, see Eq. 4.46) $\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$

¶See Section 4.4.3.1, *Parallel flow*
►Parallel flow, $n = 1$. $f = C_1 z = V_\infty z = V_\infty(x + iy)$
The streamfunction, Ψ, is the imaginary part of f, i.e. $\Psi = V_\infty y = V_\infty r\sin(n\theta)$ which gives

$$v_1 = \frac{\partial \Psi}{\partial y} = V_\infty, \quad v_2 = -\frac{\partial \Psi}{\partial x} = 0$$

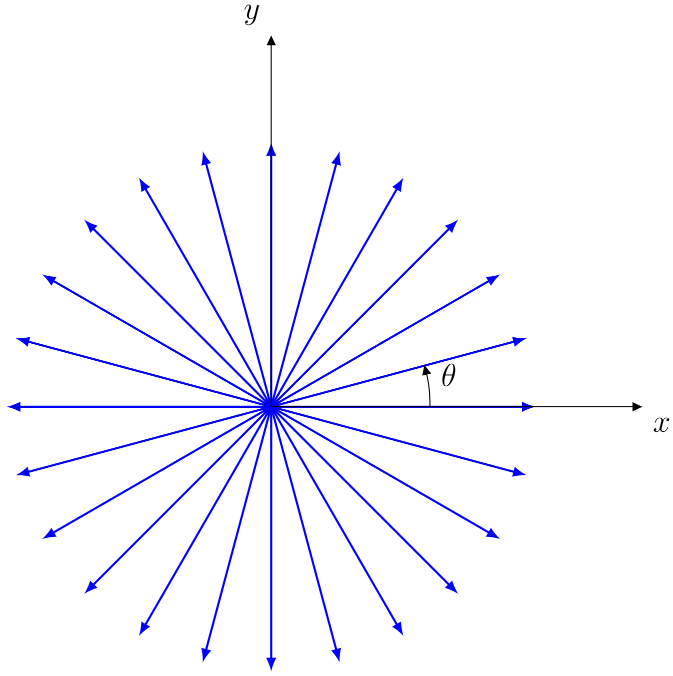


¶See Section 4.4.4, Analytical solutions for a line source
►Line source

$$f = \frac{\dot{m}}{2\pi}\ln z = \frac{\dot{m}}{2\pi}\ln\left(re^{i\theta}\right) = \frac{\dot{m}}{2\pi}\left(\ln r + \ln\left(e^{i\theta}\right)\right) = \frac{\dot{m}}{2\pi}(\ln r + i\theta) \tag{A.18}$$

Check that it satisfies Laplace eq. (it does, see 4.56)

$$v_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} = \frac{\dot{m}}{2\pi r}$$
$$v_\theta = -\frac{\partial \Psi}{\partial r} = 0$$



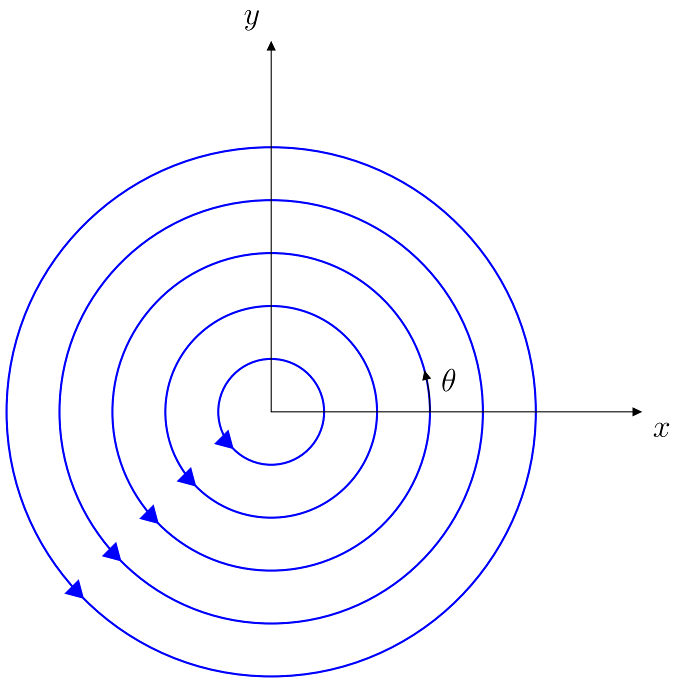
¶See Section 4.4.5, Analytical solutions for a vortex line

▶Vortex line

$$f = -i\frac{\Gamma}{2\pi}\ln z = -i\frac{\Gamma}{2\pi}\ln\left(re^{i\theta}\right) = -i\frac{\Gamma}{2\pi}\left(\ln r + \ln\left(e^{i\theta}\right)\right) = \frac{\Gamma}{2\pi}\left(-i\ln r + \theta\right)$$

Check if it satisfies Laplace eq. (it does, see 4.56)

$$v_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} = 0, \quad v_\theta = -\frac{\partial \Psi}{\partial r} = \frac{\Gamma}{2\pi r}$$



¶See Section 4.4.6, Analytical solutions for flow around a cylinder
▶Doublet: take a line source ($\dot{m} > 0$) a line sink ($\dot{m} < 0$) with a separation ε: let $\varepsilon \to 0$ which gives

$$f = \frac{\mu}{\pi z} = \frac{V_\infty r_0^2}{z}$$

where $r_0^2 = \mu/(\pi V_\infty)$. Add parallel flow ($f = V_\infty z$) gives cylinder flow

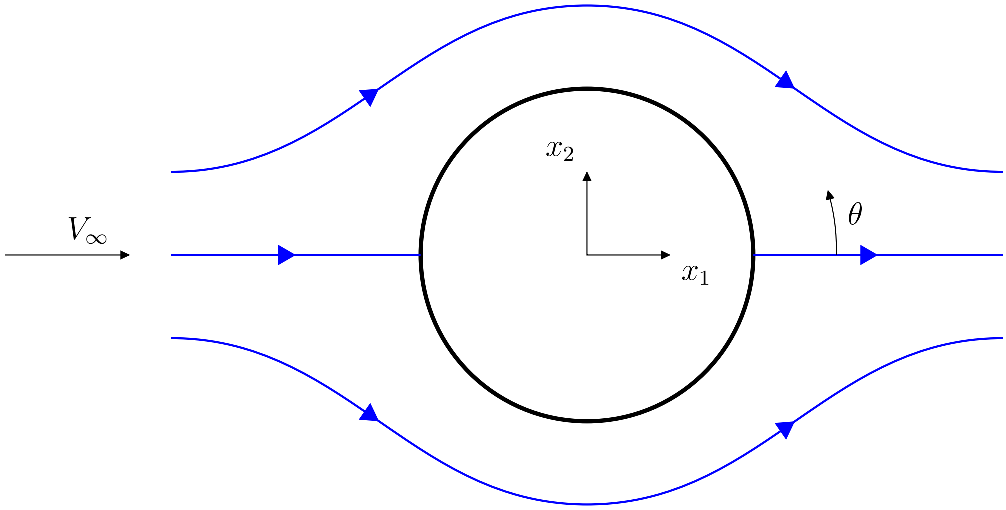
$$\begin{aligned} f = \frac{V_\infty r_0^2}{z} + V_\infty z = \frac{V_\infty r_0^2}{re^{i\theta}} + V_\infty re^{i\theta} = V_\infty \left(\frac{r_0^2}{r} e^{-i\theta} + re^{i\theta} \right) \\ = V_\infty \left(\frac{r_0^2}{r} (\cos\theta - i\sin\theta) + r(\cos\theta + i\sin\theta) \right) \end{aligned}$$

The streamfunction reads (imaginary part)

$$\Psi = V_\infty \left(r - \frac{r_0^2}{r} \right) \sin\theta$$

and we get the velocity components

$$v_r = \frac{1}{r}\frac{\partial\Psi}{\partial\theta} = V_\infty \left(1 - \frac{r_0^2}{r^2}\right)\cos\theta, \quad v_\theta = -\frac{\partial\Psi}{\partial r} = -V_\infty\left(1 + \frac{r_0^2}{r^2}\right)\sin\theta$$



¶See Section 4.4.7, Analytical solutions for flow around a cylinder with circulation
►Flow around a cylinder with circulation, Γ
We have f for a cylinder. Now we add f for a vortex line

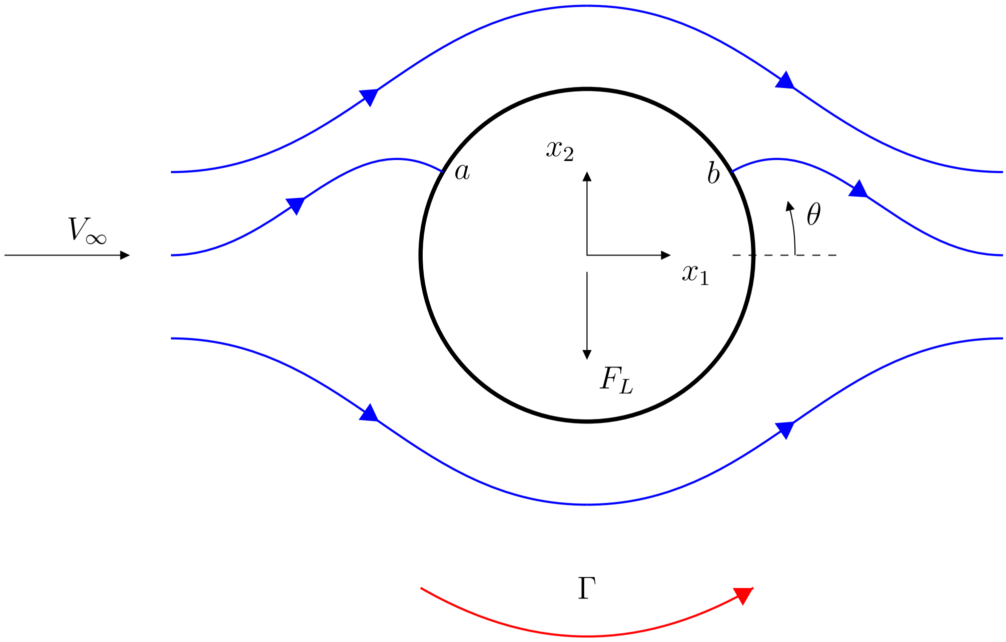
$$f = V_\infty \left(\frac{r_0^2}{r}(\cos\theta - i\sin\theta) + r(\cos\theta + i\sin\theta) \right) - i\frac{\Gamma}{2\pi}\ln z$$

The imaginary part gives the streamfunction

$$\Psi = V_\infty \left(r - \frac{r_0^2}{r} \right) \sin\theta - \frac{\Gamma}{2\pi}\ln r$$

We get the velocity components as

$$v_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} = V_\infty \left(1 - \frac{r_0^2}{r^2} \right)\cos\theta, \quad v_\theta = -\frac{\partial \Psi}{\partial r} = -V_\infty \left(1 + \frac{r_0^2}{r^2} \right)\sin\theta + \frac{\Gamma}{2\pi r}$$



The velocity at the surface, $r = r_0$, reads

$$v_{r,s} = 0$$
$$v_{\theta,s} = -2V_\infty \sin\theta + \frac{\Gamma}{2\pi r_0}$$

▶Location of the stagnation points, i.e. where $v_{\theta,s} = 0$. We get

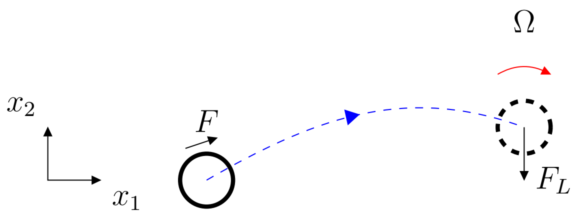
$$2V_\infty \sin\theta_{stag} = \frac{\Gamma}{2\pi r_0} \Rightarrow \theta_{stag} = \arcsin\left(\frac{\Gamma}{4\pi r_0 V_\infty}\right)$$
$$\Gamma_{max} = 4\pi V_\infty r_0.$$

▶The pressure is obtained from Bernoulli equation as (see Eq. 4.72)

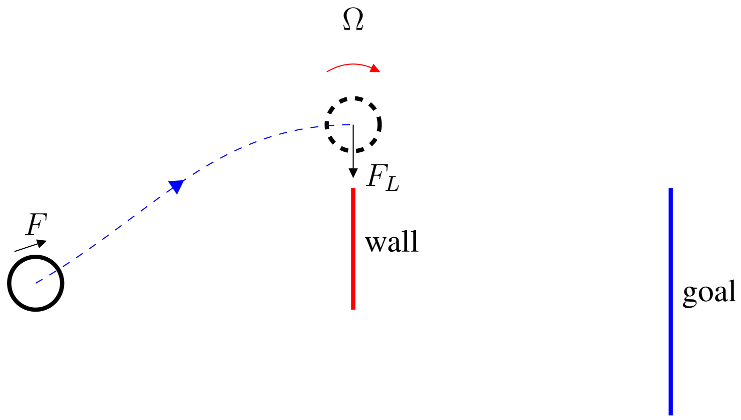
$$C_p = 1 - \frac{v_{\theta,s}^2}{V_\infty^2} = 1 - \left(-2\sin\theta + \frac{\Gamma}{2\pi r_0 V_\infty}\right)^2$$
$$= 1 - 4\sin^2\theta + \frac{4\Gamma\sin\theta}{2\pi r_0 V_\infty} - \left(\frac{\Gamma}{2\pi r_0 V_\infty}\right)^2$$

▶Integration of C_p gives drag $F_D = 0$ and lift $F_L = -\rho V_\infty \Gamma$.

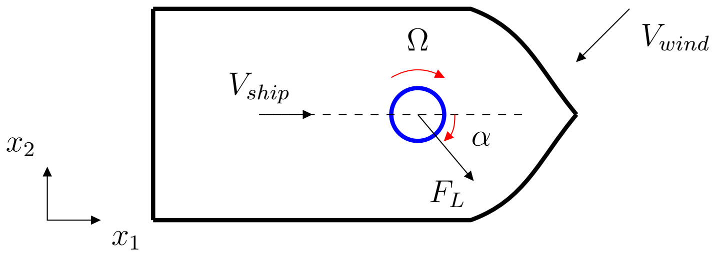
¶See Section 4.4.7.1, *The Magnus effect*
▶The Magnus' effect: three applications
▶Table tennis: loop



▶Free-kick



▶Flettner rotors: the Magus effect $\Rightarrow$ propulsion force of $F_L \cos(\alpha)$



¶See Section 4.4.8, The flow around an airfoil
▶Airfoil flow
▶The boundary layers, $\delta(x_1)$, and the wake illustrated in red.
▶Lift force $F_L = -\rho V_\infty \Gamma$

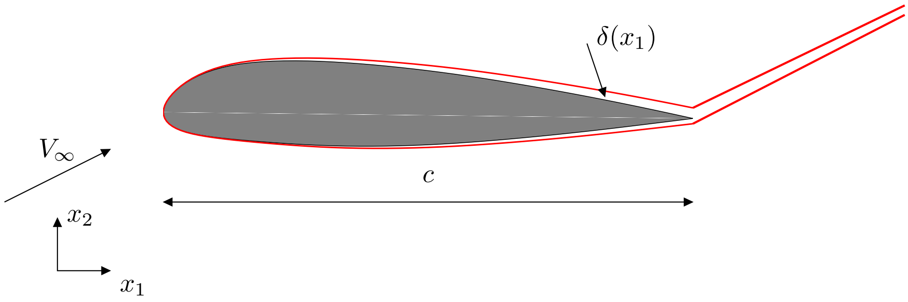


Figure A.1: Airfoil. The boundary layers, $\delta(x_1)$, and the wake illustrated in red. $x_1 = 0$ and $x_1 = c$ at leading and trailing edge, respectively.

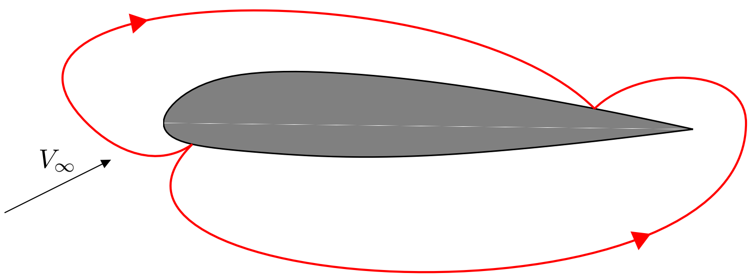


Figure A.2: Airfoil. Streamlines from potential flow. Rear stagnation point at the upper surface (suction side).

A.5 Lecture 5

¶See Section 5, Turbulence
▶Turbulence
▶$v_i = \bar{v}_i + v_i'$, is irregular and consists of eddies of different size
▶increases diffusivity

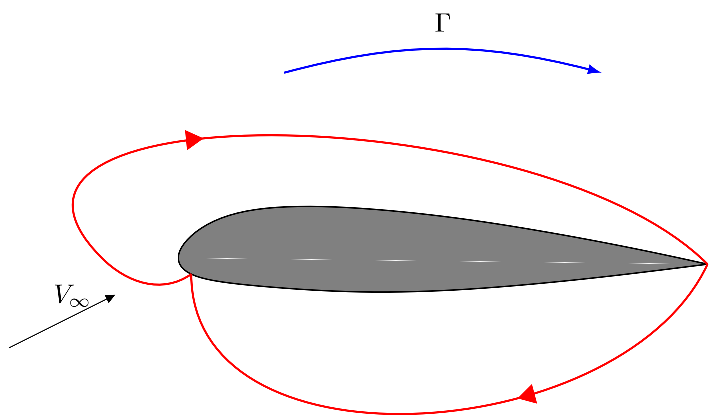
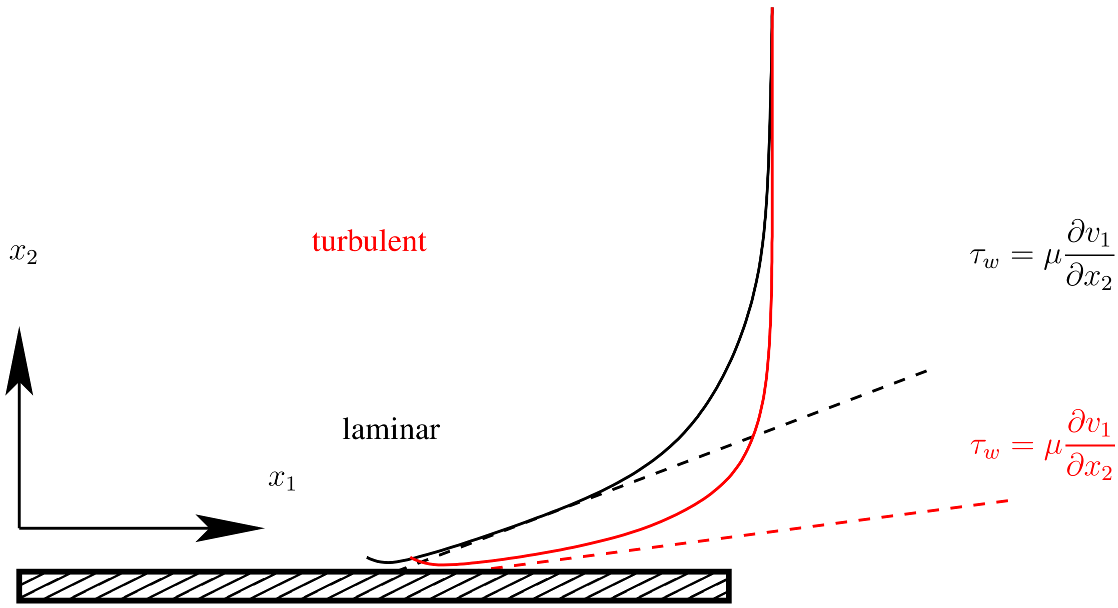


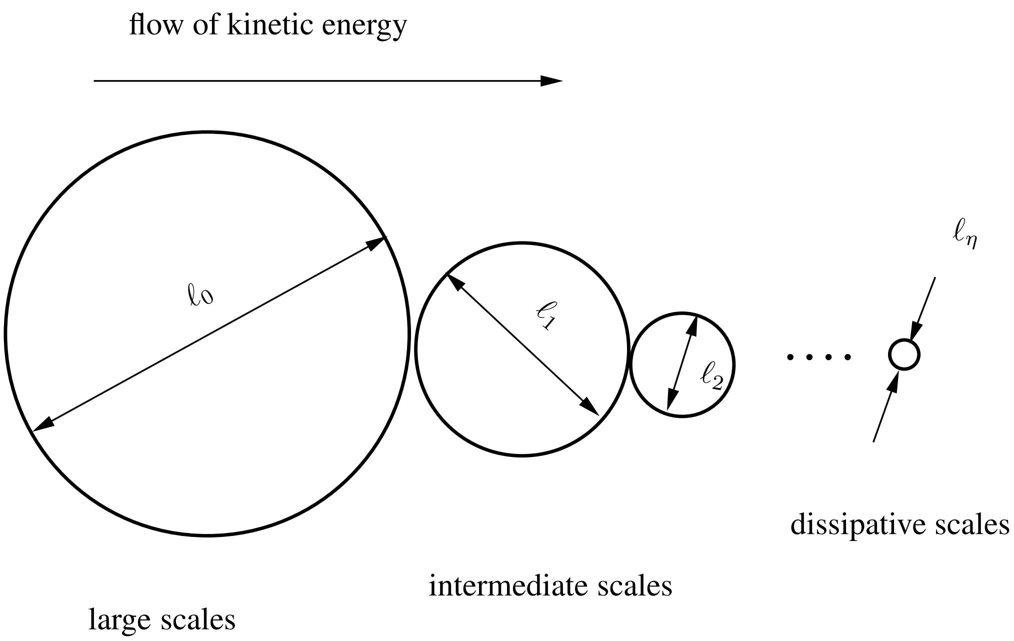
Figure A.3: Airfoil. Streamlines from potential flow with added circulation. Rear stagnation point at the trailing edge.



►occurs at large Reynolds numbers. Pipes: $Re_D = \dfrac{VD}{\nu} \simeq 2300$; boundary layers: $Re_x = \dfrac{VL}{\nu} \simeq 500\,000$.

►is three-dimensional

►is dissipative. Kinetic energy, $v_i'v_i'/2$, in the small (dissipative) eddies are transformed into thermal energy (increases temperature).



▶Dissipation $\varepsilon = \nu \overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}$
All kinetic energy (say 90%) is finally dissipated at the smallest (dissipative) scales.

▶Kinetic energy dissipative at small scales determined by ε, ν

$$\begin{array}{ccccc} v_\eta & = & \nu^a & \varepsilon^b \\ [m/s] & = & [m^2/s] & [m^2/s^3] \end{array}$$

$$\begin{aligned} 1 &= 2a + 2b, \quad [m] \\ -1 &= -a - 3b, \quad [s] \end{aligned}$$

▶This gives the Kolmogorov scales (see Eq. 5.5)

$$v_\eta = (\nu\varepsilon)^{1/4}, \; \ell_\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}, \; \tau_\eta = \left(\frac{\nu}{\varepsilon}\right)^{1/2}$$

▶Any periodic function, f, can be expressed as a Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(\kappa_n x) + b_n \sin(\kappa_n x)), \quad f = v, \quad \kappa_n = \frac{n\pi}{L}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\kappa_n x)dx$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\kappa_n x)dx$$

Parseval's formula states that (average over all eddies)

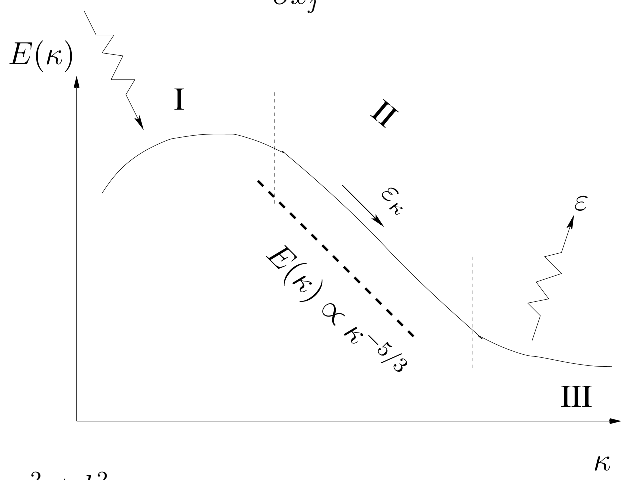
$$\int_{-L}^{L} v'^2(x)dx = \frac{L}{2}a_0^2 + L\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Time-averaging (average of over time instants):

$$\overline{v'^2} = \frac{1}{2T}\int_{-T}^{T} v'^2 dt$$

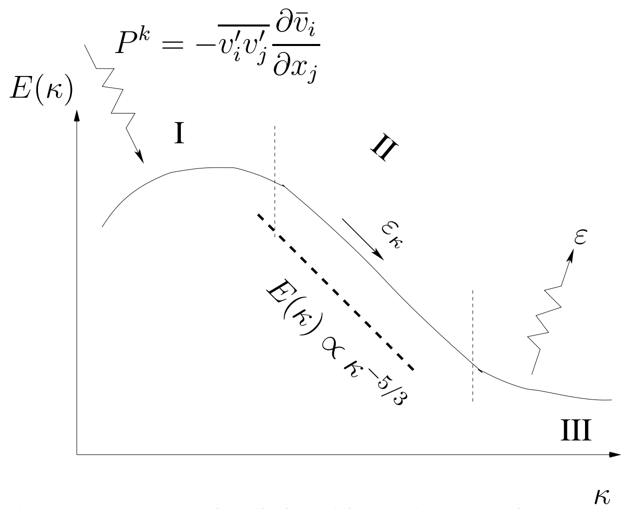
►Spectrum for turbulent kinetic energy, k

$$P^k = -\overline{v_i' v_j'}\frac{\partial \bar{v}_i}{\partial x_j}$$



$E(\kappa_n) \propto a_n^2 + b_n^2$

$$k = \int_0^\infty E(\kappa)d\kappa = \sum_0^\infty E(\kappa_n)\Delta\kappa_n$$



The turbulence spectrum is divided into three regions:

- I. Large eddies carry most of the turbulent kinetic energy. They extract energy from the mean flow.
- II. Inertial subrange. Independent of both large eddies (mean flow) and viscosity.
- III. Dissipation range. Isotropic eddies ($\overline{v_i'v_j'} = c_1\delta_{ij}$) described by the Kolmogorov scales.

►Turb. kinetic energy in Region II depends on ε and eddy size $1/\kappa$.

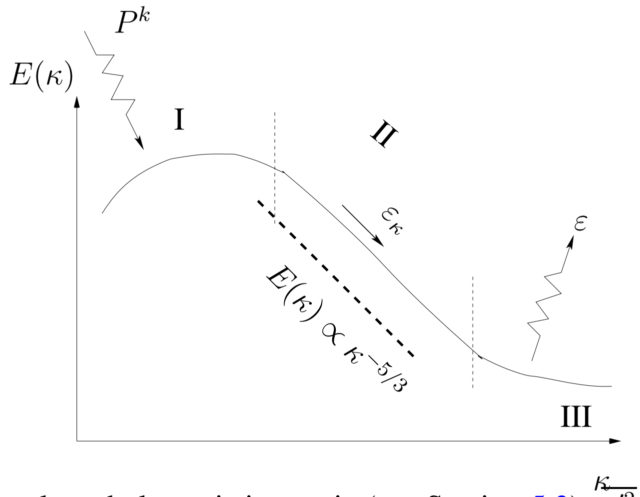
$$\begin{array}{cccc} E & = & \kappa^a & \varepsilon^b \\ [m^3/s^2] & = & [1/m] & [m^2/s^3] \end{array}$$

$$\begin{aligned} 3 &= -a + 2b, \quad [m] \\ -2 &= -3b, \quad [s] \end{aligned}$$

$b = 2/3, a = -5/3$ so that

$$E(\kappa) = \text{const.} \cdot \varepsilon^{\frac{2}{3}}\kappa^{-\frac{5}{3}}$$

▶This is called Kolmogorov spectrum or $-5/3$ law



▶Small-scale turbulence is isotropic (see Section 5.3): $\overline{v_1'^2} = \overline{v_2'^2} = \overline{v_3'^2}$. Not true instantaneously, i.e. $v_1' \neq v_2' \neq v_3'$.

Isotropy: if a coordinate direction is switched (i.e. rotated 180^o), nothing should change. $\Rightarrow \overline{v_1'v_2'}$ in both coordinate directions must be the same. $\Rightarrow \overline{v_1'v_2'} = (\overline{v_1'v_2'})_{180^o} = -\overline{v_1'v_2'} = 0$.

▶On tensor form: $\overline{v_i'v_j'} = \text{const.}\delta_{ij}$

$$\varepsilon_\kappa \sim \frac{v_\kappa^2}{\ell_\kappa/v_\kappa} \sim \frac{v_\kappa^3}{\ell_\kappa} \sim \frac{v_0^3}{\ell_0}$$

▶Relation between largest and smallest scales ($Re = v_0\ell_0/\nu$)

$$\frac{v_0}{v_\eta} = (\nu\varepsilon)^{-1/4}v_0 = \left(\nu v_0^3/\ell_0\right)^{-1/4} v_0 = (v_0\ell_0/\nu)^{1/4} = Re^{1/4}$$

$$\frac{\ell_0}{\ell_\eta} = \left(\frac{\nu^3}{\varepsilon}\right)^{-1/4} \ell_0 = \left(\frac{\nu^3\ell_0}{v_0^3}\right)^{-1/4} \ell_0 = \left(\frac{\nu^3}{v_0^3\ell_0^3}\right)^{-1/4} = Re^{3/4}$$

$$\frac{\tau_o}{\tau_\eta} = \left(\frac{\nu\ell_0}{v_0^3}\right)^{-1/2} \tau_0 = \left(\frac{v_0^3}{\nu\ell_0}\right)^{1/2} \frac{\ell_0}{v_0} = \left(\frac{v_0\ell_0}{\nu}\right)^{1/2} = Re^{1/2}$$

- This explains why DNS (Direct Numerical Simulation) is too expensive at high Re numbers: a doubling of Re number $\Rightarrow$

$$\underbrace{2^{3/4}}_{x_1\,dir} \times \underbrace{2^{3/4}}_{x_2\,dir} \times \underbrace{2^{3/4}}_{x_3\,dir} \times \underbrace{2^{1/2}}_{time} = 2^{11/4} \simeq 7 \qquad \text{(A.19)}$$

times increase in computational effort

A.6 Lecture 6

¶See Section 6, Turbulent mean flow

►The continuity and the N-S for incompr. flow with const. μ

$$\frac{\partial v_i}{\partial x_i}=0,\quad \rho\frac{\partial v_i}{\partial t}+\rho\frac{\partial v_i v_j}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\mu\frac{\partial^2 v_i}{\partial x_j\partial x_j}$$

Decompose the variables and time average

$$v_i=\bar{v}_i+v_i',\quad p=\bar{p}+p'$$

$$\frac{\partial\overline{\bar{v}_i+v_i'}}{\partial x_i}=\frac{\partial\overline{\bar{v}_i}}{\partial x_i}+\cancelto{0}{\frac{\partial\overline{v_i'}}{\partial x_i}}=\frac{\partial\bar{v}_i}{\partial x_i}$$

$$\frac{\partial\overline{v_i v_j}}{\partial x_j}=\frac{\partial}{\partial x_j}\left\{\overline{(\bar{v}_i+v_i')(\bar{v}_j+v_j')}\right\}=\frac{\partial}{\partial x_j}\left(\overline{\bar{v}_i\bar{v}_j+\bar{v}_i v_j'+\bar{v}_j v_i'+v_i'v_j'}\right)$$

$$=\frac{\partial}{\partial x_j}\left(\overline{\bar{v}_i\bar{v}_j}+\bar{v}_i\cancelto{0}{\overline{v_j'}}+\bar{v}_j\cancelto{0}{\overline{v_i'}}+\overline{v_i'v_j'}\right)=\frac{\partial\bar{v}_i\bar{v}_j}{\partial x_j}+\frac{\partial\overline{v_i'v_j'}}{\partial x_j}$$

►The steady RANS (Reynolds-Averaged Navier-Stokes) equations, see Eq. 6.9

$$\frac{\partial\bar{v}_i}{\partial x_i}=0,\quad \rho\frac{\partial\bar{v}_i\bar{v}_j}{\partial x_j}=-\frac{\partial\bar{p}}{\partial x_i}+\frac{\partial}{\partial x_j}\underbrace{\left(\mu\frac{\partial\bar{v}_i}{\partial x_j}-\rho\overline{v_i'v_j'}\right)}_{\tau_{ij,tot}}$$

¶See Section 6.1.1, Boundary-layer approximation

►RANS in developing boundary layer flow

$$\bar{v}_2\ll\bar{v}_1,\ \frac{\partial\bar{v}_1}{\partial x_1}\ll\frac{\partial\bar{v}_1}{\partial x_2},$$

$$\rho\bar{v}_1\frac{\partial\bar{v}_1}{\partial x_1}+\rho\bar{v}_2\frac{\partial\bar{v}_1}{\partial x_2}=-\frac{\partial\bar{p}}{\partial x_1}+\frac{\partial}{\partial x_2}\underbrace{\left[\mu\frac{\partial\bar{v}_1}{\partial x_2}-\rho\overline{v_1'v_2'}\right]}_{\tau_{tot}}$$

►Left side: each term include one large ($\bar{v}_1$ and $\partial/\partial x_2$) and one small ($\bar{v}_2$ and $\partial/\partial x_1$) part

¶See Section 6.2, Wall region in fully developed channel flow

►RANS in fully developed channel flow

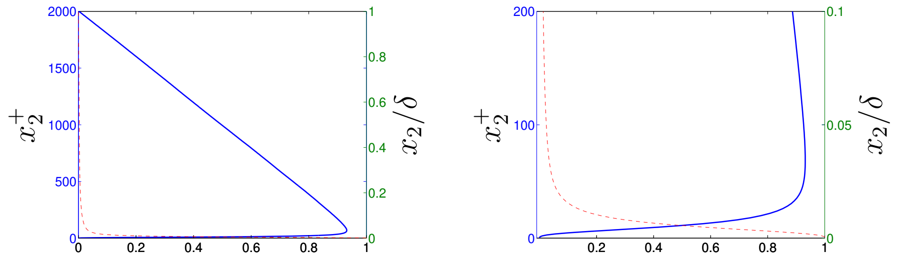
$$0=\underbrace{-\frac{\partial\bar{p}}{\partial x_1}}_{f(x_1)}+\underbrace{\frac{\partial}{\partial x_2}\left(\mu\frac{\partial\bar{v}_1}{\partial x_2}-\rho\overline{v_1'v_2'}\right)}_{g(x_2)}=-\frac{\partial\bar{p}}{\partial x_1}+\frac{\partial\tau_{tot}}{\partial x_2}$$

Integration from $x_2 = 0$ to x_2 gives

$$\tau_{tot}(x_2) - \tau_w = \frac{\partial \bar{p}}{\partial x_1} x_2 \Rightarrow \tau_{tot} = \tau_w + \frac{\partial \bar{p}}{\partial x_1} x_2 = \tau_w \left(1 - \frac{x_2}{\delta}\right)$$

Last equality: $-\dfrac{\partial \bar{p}}{\partial x_1} = \dfrac{\tau_w}{\delta}$ (force balance)

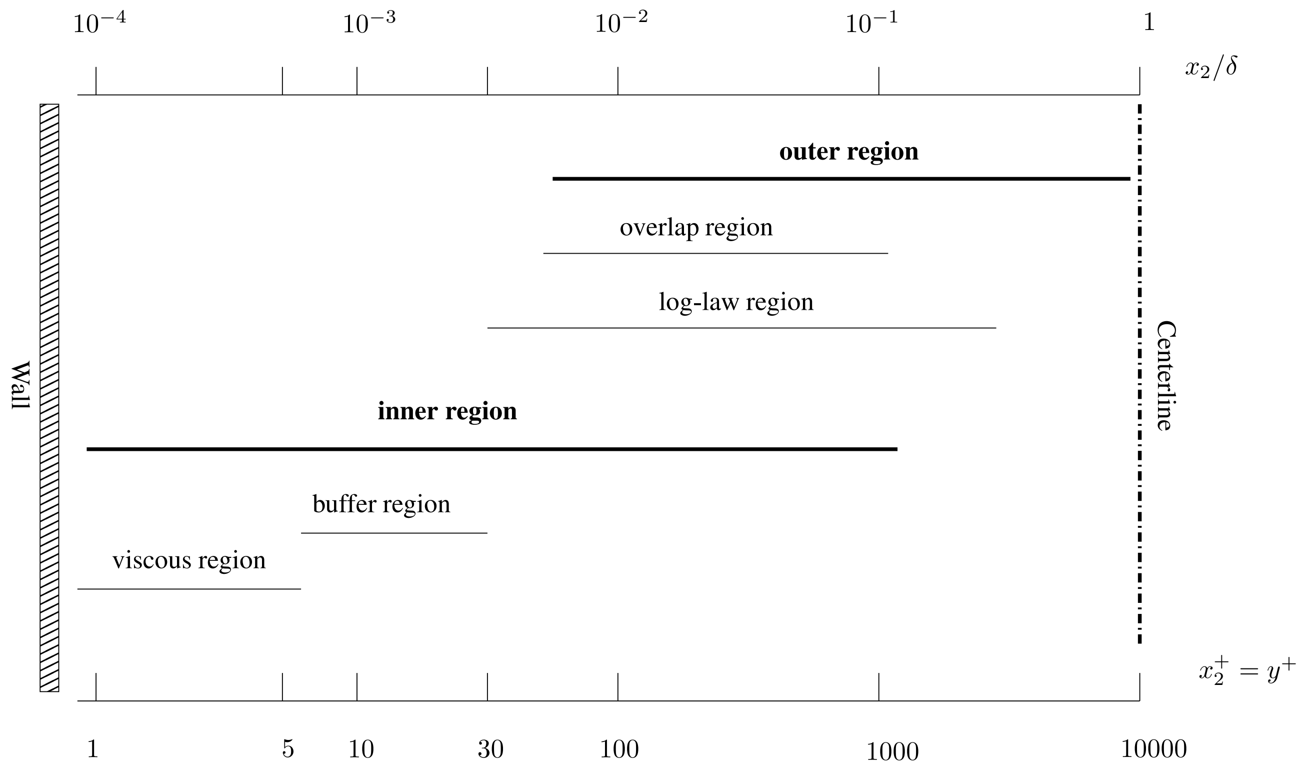
▶lower half of channel



a) full view b) zoom

—— : $-\rho\overline{v_1' v_2'}/\tau_w$; - - - : $\mu(\partial \bar{v}_1/\partial x_2)/\tau_w$.

▶The different wall regions



▶Wall shear stress (see Eq. 6.21)

$$\tau_w = \mu \frac{\partial \bar{v}_1}{\partial x_2}\bigg|_w \equiv \rho u_\tau^2 \Rightarrow u_\tau = \left(\frac{\tau_w}{\rho}\right)^{1/2}, \quad x_2^+ = \frac{x_2 u_\tau}{\nu}$$

▶The linear velocity law

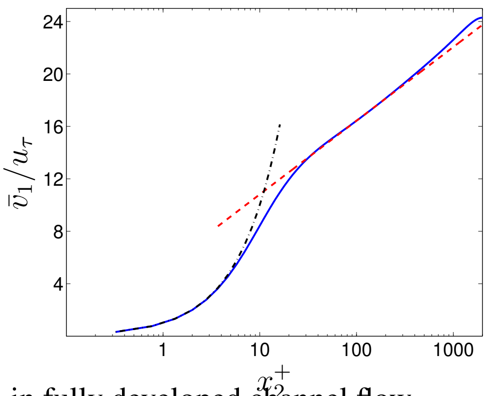
$$\frac{\partial \bar{v}_1}{\partial x_2}\bigg|_w = \frac{\tau_w}{\mu} = \frac{\rho u_\tau^2}{\mu} = \frac{u_\tau^2}{\nu}$$

Integration gives (recall that both ν and u_τ^2 are constant), see Eq. 6.24

$$\bar{v}_1 = \frac{1}{\nu} u_\tau^2 x_2 + C_1 = \frac{1}{\nu} u_\tau^2 x_2 \quad \text{or} \quad \bar{v}_1^+ = x_2^+$$

▶The log-law (turbulent region), see Eq. 6.33
Velocity scale: u_τ; length scale: $\ell \propto x_2 \Rightarrow \ell = \kappa x_2$

$$\frac{\partial \bar{v}_1}{\partial x_2} = \frac{u_\tau}{\kappa x_2} \Rightarrow \frac{\partial \bar{v}_1 / u_\tau}{\partial (x_2 u_\tau / \nu)} = \frac{1}{\kappa (x_2 u_\tau / \nu)}$$
$$\Rightarrow \frac{\partial \bar{v}_1^+}{\partial x_2^+} = \frac{1}{\kappa x_2^+} \Rightarrow \bar{v}_1^+ = \frac{1}{\kappa} \ln x_2^+ + B$$



Velocity profiles in fully developed channel flow.

▶N.B. $\bar{v}_{1,centerline}/u_\tau = 24 \quad \Rightarrow$ good estimate for u_τ ($\bar{v}_{1,centerline}/u_\tau$ increases weakly with Reynolds number)
Example: channel flow (or boundary layer), $x_2^+ = 1$ gives

water: $x_2 = \nu x_2^+ / u_\tau = 1 \cdot 10^{-6} \cdot 1/(1/24) = 2.4 \cdot 10^{-5} m = 2.4 \cdot 10^{-2} mm$

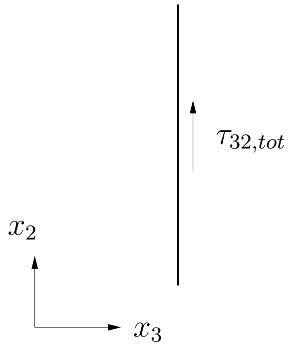
air: $x_2 = \nu x_2^+ / u_\tau = 15 \cdot 1^{-6} \cdot 1/(1/24) = 3.6 \cdot 10^{-4} m = 3.6 \cdot 10^{-1} mm$

- δ / x_2 (at $x_2^+ = 1$) is an estimate of ratio of largest to smallest turbulent length scales (δ is boundary layer thickness or channel half width)

- estimate of ε: u_τ^3/δ

¶See Section 6.3, Reynolds stresses in fully developed channel flow

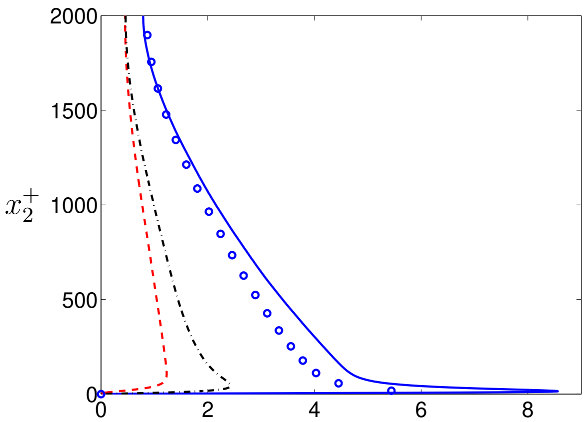
▶Symmetry plane of channel flow $\tau_{32,tot} = \tau_{32} - \rho\overline{v_3' v_2'}$



$$\tau_{32} = \mu\left(\frac{\partial \bar{v}_3}{\partial x_2} + \frac{\partial \bar{v}_2}{\partial x_3}\right) = 0, \quad \rho\overline{v_3' v_2'} = -\mu_t\left(\frac{\partial \bar{v}_3}{\partial x_2} + \frac{\partial \bar{v}_2}{\partial x_3}\right) = 0$$

because $\bar{v}_3 = \partial/\partial x_3 = 0$ (note that $\overline{v_3' v_3'} \neq 0$)

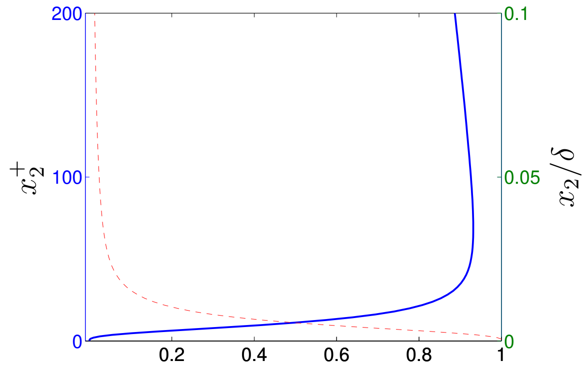
▶Normal Reynolds stresses



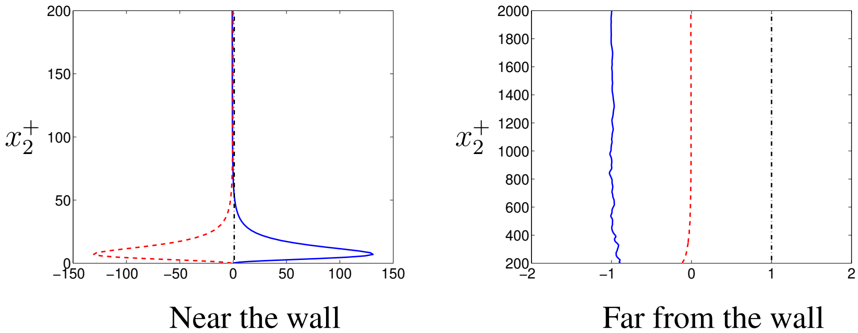
: $\rho\overline{v_1'^2}/\tau_w$; : $\rho\overline{v_2'^2}/\tau_w$; : $\rho\overline{v_3'^2}/\tau_w$

▶Forces on a fluid element

$$0 = -\frac{\partial \bar{p}}{\partial x_1} + \frac{\partial}{\partial x_2}\left(\mu\frac{\partial \bar{v}_1}{\partial x_2} - \rho\overline{v_1' v_2'}\right)$$

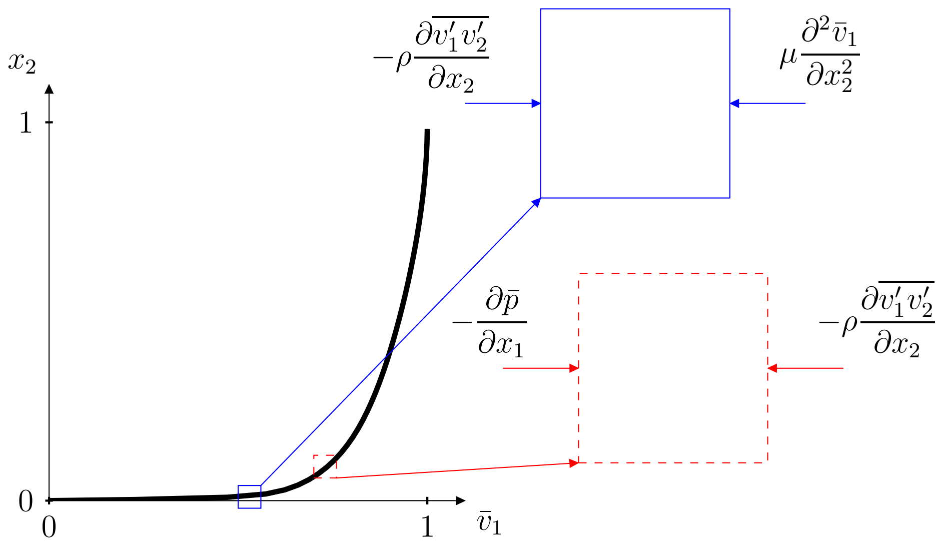


Shear stresses. —: $-\rho\overline{v_1'v_2'}/\tau_w$; - -: $\mu(\partial\bar{v}_1/\partial x_2)/\tau_w$.



Gradient of shear stresses. —: $-\rho(\partial\overline{v_1'v_2'}/\partial x_2)/\tau_w$; - -: $\mu(\partial^2\bar{v}_1/\partial x_2^2)/\tau_w$; -·-: $-(\partial\bar{p}/\partial x_1)/\tau_w$.

▶Forces in a boundary layer. The red (dashed line) and the blue (solid line) fluid particle are located at $x_2^+ \simeq 400$ and $x_2^+ \simeq 20$, respectively



$-\rho \dfrac{\partial \overline{v_1' v_2'}}{\partial x_2}$
$\mu \dfrac{\partial^2 \bar{v}_1}{\partial x_2^2}$
x_2
1
$-\dfrac{\partial \bar{p}}{\partial x_1}$
$-\rho \dfrac{\partial \overline{v_1' v_2'}}{\partial x_2}$
0
0
1
$\bar{v}_1$

A.7 Lecture 7

¶See Section 8.1, Rules for time averaging

▶Time averaging

$$\bar{v} = \frac{1}{2T}\int_{-T}^{T} v dt. \tag{A.20}$$

▶What is the difference between $\overline{v'_1 v'_2}$ and $\overline{v'_1}\ \overline{v'_2}$? Using A.20 we get

$$\overline{v'_1 v'_2} = \frac{1}{2T}\int_{-T}^{T} v'_1 v'_2 dt.$$

whereas

$$\overline{v'_1}\ \overline{v'_2} = \left(\frac{1}{2T}\int_{-T}^{T} v'_1 dt\right)\left(\frac{1}{2T}\int_{-T}^{T} v'_1 dt\right)$$

(which is zero)

▶What is the difference between $\overline{v'^2_1}$ and $\overline{v'_1}^2$? Using A.20 we get

$$\overline{v'^2_1} = \frac{1}{2T}\int_{-T}^{T} v'^2 dt.$$

whereas

$$\overline{v'_1}^2 = \left(\frac{1}{2T}\int_{-T}^{T} v' dt\right)^2.$$

(which is zero)

▶Show that $\overline{\bar{v}_1 v'^2_1} = \bar{v}_1\overline{v'^2_1}$. Using A.20 we get

$$\overline{\bar{v}_1 v'^2_1} = \frac{1}{2T}\int_{-T}^{T} \bar{v}_1 v'^2_1 dt$$

and since $\bar{v}$ does not depend on t we can take it out of the integral as

$$\bar{v}_1 \frac{1}{2T}\int_{-T}^{T} v'^2_1 dt = \bar{v}_1\overline{v'^2_1}$$

▶Show that $\bar{\bar{v}}_1 = \bar{v}_1$. Using A.20 we get

$$\bar{\bar{v}}_1 = \frac{1}{2T}\int_{-T}^{T} \bar{v}_1 dt$$

and since $\bar{v}$ does not depend on t we can take it out of the integral as

$$\bar{v}_1 \frac{1}{2T}\int_{-T}^{T} dt = \bar{v}_1 \frac{1}{2T} 2T = \bar{v}_1$$

▶RANS (see Eq. 6.9)

$$\rho\frac{\partial \bar{v}_i\bar{v}_j}{\partial x_j} = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu\frac{\partial \bar{v}_i}{\partial x_j} - \rho\overline{v_i'v_j'}\right)$$

¶See Section 11.6, The Boussinesq assumption
▶The RANS equations read (see Eq. 6.10)

$$\rho\frac{\partial \bar{v}_i\bar{v}_j}{\partial x_j} = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu\frac{\partial \bar{v}_i}{\partial x_j} - \rho\overline{v_i'v_j'}\right)$$

- The last term, the Reynolds stress, is unknown.
- It must be modeled
- This is called the *closure problem*
- We need a **turbulence model**

▶Write the diffusion term above for incompressible flow and without assuming constant viscosity (see Eq. 2.5)

$$\frac{\partial}{\partial x_j}\left\{\nu\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right) - \overline{v_i'v_j'}\right\} \tag{A.21}$$

We want to replace $\overline{v_i'v_j'}$ by a turbulent viscosity, ν_t:

$$\frac{\partial}{\partial x_j}\left\{(\nu+\nu_t)\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right)\right\} \tag{A.22}$$

Identification of Eqs. A.21 and A.22 gives (see Eq. 11.32)

$$-\overline{v_i'v_j'} = \nu_t\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right)$$

This equation is not valid upon contraction. Hence (see Eq. 11.33)

$$\overline{v_i'v_j'} = -\nu_t\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right) + \frac{1}{3}\delta_{ij}\overline{v_k'v_k'} = -2\nu_t\bar{s}_{ij} + \frac{2}{3}\delta_{ij}k \tag{A.23}$$

ν: different for different fluids (air, water, ...)

ν_t: depends on the flow, i.e. $\nu_t = \nu_t(x_i)$

¶See Section 8.2, The Exact k Equation

▶$k = \overline{v_i' v_i'}/2$ appears in the expression for the turbulence viscosity. The first step is to derive the k equation. Start by the N-S for v_i', multiply by v_i' and time average

$$\overline{v_i' \frac{\partial}{\partial x_j}\left[v_i v_j - \bar{v}_i \bar{v}_j\right]} =$$

$$\underbrace{-\frac{1}{\rho}\overline{v_i' \frac{\partial}{\partial x_i}\left[p - \bar{p}\right]}}_{\mathbf{IV}} + \underbrace{\nu \overline{v_i' \frac{\partial^2}{\partial x_j \partial x_j}\left[v_i - \bar{v}_i\right]}}_{V} + \underbrace{\overline{\frac{\partial \overline{v_i' v_j'}}{\partial x_j} v_i'}}_{\mathbf{VI}}$$

Using $v_j = \bar{v}_j + v_j'$, the left side can be rewritten as

$$\overline{v_i' \frac{\partial}{\partial x_j}\left[(\bar{v}_i + v_i')(\bar{v}_j + v_j') - \bar{v}_i \bar{v}_j\right]} = \overline{v_i' \frac{\partial}{\partial x_j}\left[\underbrace{\bar{v}_i v_j'}_{\mathbf{I}} + \underbrace{v_i' \bar{v}_j}_{\mathbf{II}} + \underbrace{v_i' v_j'}_{\mathbf{III}}\right]}.$$

▶Term I is rewritten as

$$\overline{v_i' \frac{\partial}{\partial x_j}\left(\bar{v}_i v_j'\right)} = \overline{v_i' v_j'} \frac{\partial \bar{v}_i}{\partial x_j} + \underbrace{\bar{v}_i \cancelto{0}{\overline{v_i' \frac{\partial v_j'}{\partial x_j}}}}_{\text{cont.eq.}} = \overline{v_i' v_j'} \frac{\partial \bar{v}_i}{\partial x_j}$$

▶Term II

$$\overline{v_i' \frac{\partial}{\partial x_j}\left(v_i' \bar{v}_j\right)} \underset{\text{cont. eq.}}{=} \bar{v}_j \overline{v_i' \frac{\partial v_i'}{\partial x_j}} \underset{\text{Trick 2}}{=} \bar{v}_j \frac{\partial}{\partial x_j}\left(\frac{\overline{v_i' v_i'}}{2}\right) = \bar{v}_j \frac{\partial k}{\partial x_j}$$

▶Term III

$$\overline{v_i' \frac{\partial}{\partial x_j}\left(v_i' v_j'\right)} \underset{\text{cont. eq.}}{=} \overline{v_j'\left(v_i' \frac{\partial v_i'}{\partial x_j}\right)} \underset{\text{Trick 2}}{=} \overline{v_j' \frac{\partial}{\partial x_j}\left(\frac{v_i' v_i'}{2}\right)} \underset{\text{cont. eq.}}{=} \frac{\partial}{\partial x_j}\left(\frac{\overline{v_j' v_i' v_i'}}{2}\right)$$

▶First term on the right side (Term IV)

$$-\frac{1}{\rho}\overline{v_i' \frac{\partial p'}{\partial x_i}} \underset{\text{cont. eq.}}{=} -\frac{1}{\rho}\frac{\partial \overline{p' v_i'}}{\partial x_i}$$

▶Second term on the right side (Term V)

$$\nu \overline{v_i' \frac{\partial^2 v_i'}{\partial x_j \partial x_j}} \underset{\text{Trick 1}}{=} \nu \overline{\frac{\partial}{\partial x_j}\left(\frac{\partial v_i'}{\partial x_j} v_i'\right)} - \nu \overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}} \underset{\text{Trick 2}}{=}$$

$$\nu \overline{\frac{\partial}{\partial x_j}\left(\frac{1}{2}\left(\frac{\partial v_i' v_i'}{\partial x_j}\right)\right)} - \nu \overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}} = \nu \frac{\partial^2 k}{\partial x_j \partial x_j} - \nu \overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}$$

▶Third term on the right side (Term VI)

$$\overline{\frac{\partial \overline{v'_i v'_j}}{\partial x_j} v'_i} = \frac{\partial \overline{v'_i v'_j}}{\partial x_j} \overline{v'_i} = 0$$

▶We finally get (see Eq. 8.14)

$$\underbrace{\frac{\partial \bar{v}_j k}{\partial x_j}}_{\mathrm{I}} = \underbrace{-\overline{v'_i v'_j} \frac{\partial \bar{v}_i}{\partial x_j}}_{\mathrm{II}} \underbrace{- \frac{\partial}{\partial x_j} \left[\frac{1}{\rho} \overline{v'_j p'} + \frac{1}{2} \overline{v'_j v'_i v'_i} - \nu \frac{\partial k}{\partial x_j} \right]}_{\mathrm{III}} \underbrace{- \nu \overline{\frac{\partial v'_i}{\partial x_j} \frac{\partial v'_i}{\partial x_j}}}_{\mathrm{IV}}$$

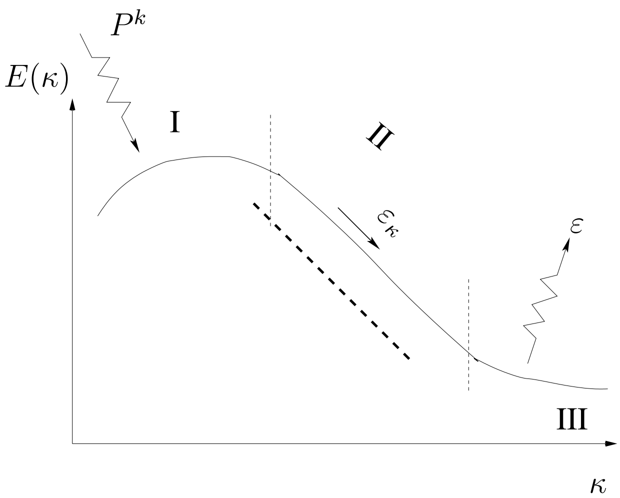
The terms have the following meaning.

I. Convection.

II. Production, P^k. The large turbulent scales extract energy from the mean flow. It is largest for small wavenumbers. It can be written as $P^k = -\overline{v'_i v'_j} \bar{S}_{ij}$, see Eq. 8.15. Hence only $\bar{S}_{ij}$ creates turbulence, not $\bar{\Omega}_{ij}$

III. The two first terms represent **turbulent diffusion** by pressure-velocity fluctuations, and velocity fluctuations, respectively. The last term is viscous diffusion.

IV. Dissipation, ε. It is largest for high wavenumbers,



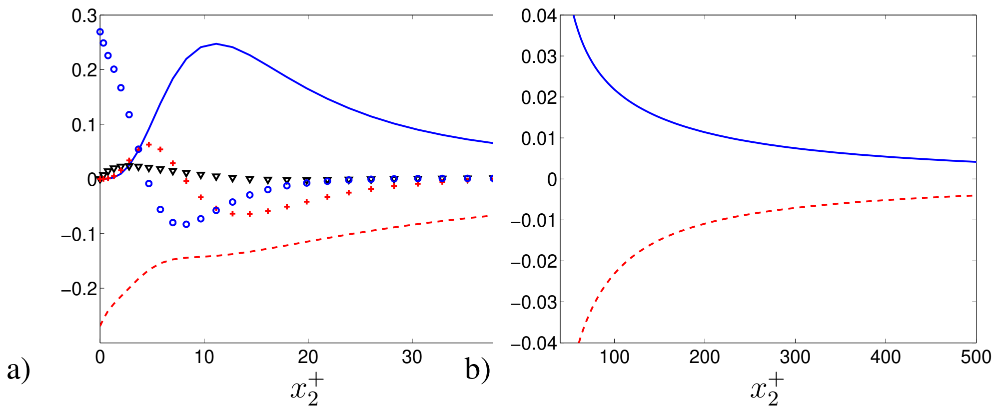
▶The transport equation for k in symbolic form reads (see Eq. 8.16)

$$C^k = P^k + D^k - \varepsilon$$

¶See Section 8.3, The Exact k Equation: 2D Boundary Layers

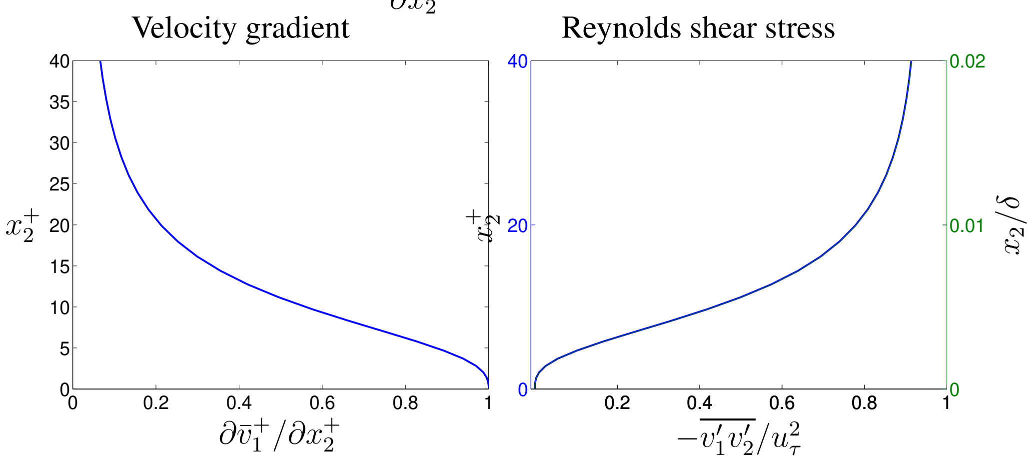
▶In 2D boundary-layer flow, $\partial/\partial x_2 \gg \partial/\partial x_1$, $\bar{v}_2 \ll \bar{v}_1$, we get

$$\frac{\partial \bar{v}_1 k}{\partial x_1}+\frac{\partial \bar{v}_2 k}{\partial x_2} = -\overline{v_1'v_2'}\frac{\partial \bar{v}_1}{\partial x_2}-\frac{\partial}{\partial x_2}\left[\frac{1}{\rho}\overline{p'v_2'}+\frac{1}{2}\overline{v_2'v_i'v_i'}-\nu\frac{\partial k}{\partial x_2}\right]-\nu\overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}$$



a) Zoom near the wall; b) Outer region. ▬: P^k; - - -: $-\varepsilon$; ∇: $-\partial\overline{v'p'}/\partial x_2$; +: $-\partial\overline{v_2'v_i'v_i'/2}/\partial x_2$; ∘: $\nu\partial^2 k/\partial x_2^2$.

▶The production term $-\overline{v_1'v_2'}\dfrac{\partial \bar{v}_1}{\partial x_2}$



A.8 Lecture 8

¶See Section 8.6, The transport equation for $\bar{v}_i\bar{v}_i/2$
►The main source term in k eq is the production term, P^k. This means that k gets energy via P^k. From where?
►Answer: from $K = \bar{v}_i\bar{v}_i/2$. Let's derive the transport eq. for K. Multiply the RANS equations by $\bar{v}_i$ so that

$$\underbrace{\bar{v}_i\left(\frac{\partial \bar{v}_i\bar{v}_j}{\partial x_j}\right)}_{\mathbf{I}} = \bar{v}_i\left(\underbrace{-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i}}_{\mathbf{II}} + \nu\underbrace{\frac{\partial^2\bar{v}_i}{\partial x_j\partial x_j}}_{\mathbf{III}} - \underbrace{\frac{\partial \overline{v_i'v_j'}}{\partial x_j}}_{\mathbf{IV}}\right)$$

Term I:

$$\bar{v}_i\frac{\partial \bar{v}_i\bar{v}_j}{\partial x_j} = \bar{v}_i\bar{v}_j\frac{\partial \bar{v}_i}{\partial x_j} + \bar{v}_i\bar{v}_i\cancelto{0}{\frac{\partial \bar{v}_j}{\partial x_j}} = \frac{1}{2}\bar{v}_j\frac{\partial \bar{v}_i\bar{v}_i}{\partial x_j} = \frac{\partial \bar{v}_j K}{\partial x_j}$$

Term II:

$$-\bar{v}_i\frac{\partial \bar{p}}{\partial x_i} \quad \text{main source term in, for example, channel flow} \quad \left(-\bar{v}_1\frac{\partial \bar{p}}{\partial x_1} > 0\right)$$

Term III:

$$\nu\bar{v}_i\frac{\partial}{\partial x_j}\left(\frac{\partial \bar{v}_i}{\partial x_j}\right) = \nu\frac{\partial}{\partial x_j}\left(\underbrace{\bar{v}_i\frac{\partial \bar{v}_i}{\partial x_j}}_{\partial/\partial_j(\bar{v}_i\bar{v}_i/2)}\right) - \nu\frac{\partial \bar{v}_i}{\partial x_j}\frac{\partial \bar{v}_i}{\partial x_j} = \nu\frac{\partial^2 K}{\partial x_j\partial x_j} - \nu\frac{\partial \bar{v}_i}{\partial x_j}\frac{\partial \bar{v}_i}{\partial x_j}.$$

Term IV:

$$-\bar{v}_i\frac{\partial \overline{v_i'v_j'}}{\partial x_j} = -\frac{\partial \bar{v}_i\overline{v_i'v_j'}}{\partial x_j} + \overline{v_i'v_j'}\frac{\partial \bar{v}_i}{\partial x_j}.$$

►$K = \frac{1}{2}\bar{v}_i\bar{v}_i$ eq. (see Eq. 8.38)

$$\frac{\partial \bar{v}_j K}{\partial x_j} = \underbrace{\overline{v_i'v_j'}\frac{\partial \bar{v}_i}{\partial x_j}}_{-P^k,\ \text{sink}} \underbrace{-\frac{\bar{v}_i}{\rho}\frac{\partial \bar{p}}{\partial x_i}}_{\text{source}} - \frac{\partial}{\partial x_j}\left(\bar{v}_i\overline{v_i'v_j'} - \nu\frac{\partial K}{\partial x_j}\right) \underbrace{-\nu\frac{\partial \bar{v}_i}{\partial x_j}\frac{\partial \bar{v}_i}{\partial x_j}}_{\varepsilon_{mean},\ \text{sink}}$$

►$k = \frac{1}{2}\overline{v_i'v_i'}$ eq. (see Eq. 8.14)

$$\frac{\partial \bar{v}_j k}{\partial x_j} = \underbrace{-\overline{v_i'v_j'}\frac{\partial \bar{v}_i}{\partial x_j}}_{P^k,\ \text{source}} - \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\overline{v_j'p'} + \frac{1}{2}\overline{v_j'v_i'v_i'} - \nu\frac{\partial k}{\partial x_j}\right] \underbrace{-\nu\overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}}_{\varepsilon,\ \text{sink}}$$

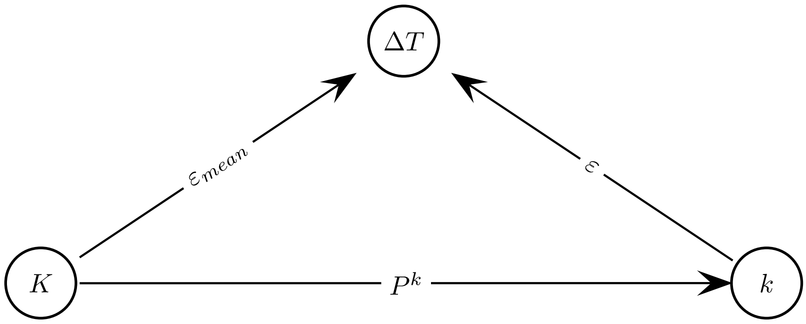


Figure A.4: Transfer of energy between mean kinetic energy (K), turbulent kinetic energy (k) and internal energy (denoted as an increase in temperature, ΔT). $K = \frac{1}{2}\bar{v}_i\bar{v}_i$ and $k = \frac{1}{2}\overline{v_i'v_i'}$.

¶See Section 11.5, The ε equation

▶Now we need to model the turbulent viscosity, see Eq. A.23. It is estimated as a turbulent velocity fluctuation times a turbulent length scale

$$\nu_t \propto \mathcal{U}\mathcal{L}$$

The velocity scale is taken as $k^{1/2}$. Recall that we have an estimate for the dissipation as $\varepsilon \propto \mathcal{U}^3/\mathcal{L}$, see Eq. 5.14. This gives $\mathcal{L} \propto k^{3/2}/\varepsilon$ which gives

$$\nu_t = C_\mu \frac{k^2}{\varepsilon}$$

where $C_\mu = 0.09$.

¶See Section 11.7.1, Production terms

▶We need modelled k and ε eqns. The exact k eq. reads (see Eq. 8.14)

$$\frac{\partial \bar{v}_j k}{\partial x_j} = -\overline{v_i'v_j'}\frac{\partial \bar{v}_i}{\partial x_j} - \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\overline{v_j'p'} + \frac{1}{2}\overline{v_j'v_i'v_i'} - \nu\frac{\partial k}{\partial x_j}\right] - \nu\overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}} \tag{A.24}$$

▶Production term needs to be modelled.

$$P^k = -\overline{v_i'v_j'}\frac{\partial \bar{v}_i}{\partial x_j} = \nu_t\left[\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}k\right]\frac{\partial \bar{v}_i}{\partial x_j} = 2\nu_t\bar{s}_{ij}\bar{s}_{ij}$$

▶Also the diffusion term needs to be modeled. Example: heat flux is modelled as (see Eq. 11.35)

$$\overline{v_i'\theta'} = -\frac{\nu_t}{\sigma_t}\frac{\partial \bar{\theta}}{\partial x_i}$$

▶The diffusion term in k eq, Eq. A.24, is modelled as

$$\frac{1}{2}\overline{v'_j v'_i v'_i} = -\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_j} \Rightarrow -\frac{1}{2}\frac{\partial \overline{v'_j v'_i v'_i}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\nu_t \frac{\partial k}{\partial x_j}\right)$$

¶See Section 11.8, The $k-\varepsilon$ model

▶Modelled k equation

$$\frac{\partial k}{\partial t} + \bar{v}_j \frac{\partial k}{\partial x_j} = 2\nu_t \bar{s}_{ij}\bar{s}_{ij} + \frac{\partial}{\partial x_j}\left\{\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right\} - \varepsilon$$

▶ε equation

$$C^\varepsilon = P^\varepsilon + D^\varepsilon - \Psi^\varepsilon$$

Use the same source terms as in k equation and add turbulent time-scale ε/k to get the right dimensions:

$$P^\varepsilon - \Psi^\varepsilon = \frac{\varepsilon}{k}(c_{\varepsilon 1}P^k - c_{\varepsilon 2}\varepsilon)$$

▶The final form of the modelled ε equation (see Eq. 11.98)

$$\frac{\partial \varepsilon}{\partial t} + \bar{v}_j \frac{\partial \varepsilon}{\partial x_j} = \frac{\varepsilon}{k}(c_{\varepsilon 1}P^k - c_{\varepsilon 2}\varepsilon) + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right]$$

▶Note that we have here omitted the buoyancy terms (they are included in Eqs. 11.97 and 11.98)

▶Summary of the $k - \varepsilon$ model.

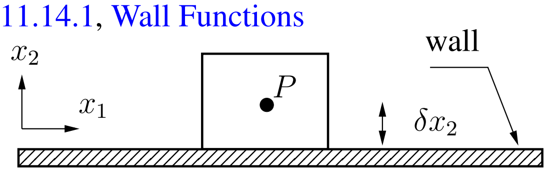
- The Reynolds stress tensor, $\overline{v_i' v_j'}$, needs to be modeled, see Eq. 6.10
- We use the Boussinsq assumption, see Eq. 11.33, to replace the unknown $\overline{v_i' v_j'}$ with the turbulent viscosity, ν_t (a new unknown).
- We make a model for ν_t , see Eq. 11.99, which includes k and ε
- We formulate modeled equations for k (Eq. 11.97) and ε (Eq. 11.98)
- Now we have closed Eq. 6.10. The equations we need to solve are
  - The time-averaged continuity equation (Eq. 6.9)
  - Three time-averaged Navier-Stokes equations (Eq. 6.10)
  - Two equations for k and ε (Eq. 11.97) and ε (Eq. 11.98)
  - The equation for turbulent viscosity, $\nu_t = C_\mu k^2/\varepsilon$ (Eq. 11.99)

A.9 Lecture 9

¶See Section 11.14, Wall boundary conditions
▶Two options for treating the wall boundary conditions.

- Coarse mesh near the walls. Assume that the logarithmic law applies. This is called *wall functions*
- Fine mesh. Modify the turbulence models to account for the viscous effects. This is called *Low-Reynolds number models*

¶See Section 11.14.1, Wall Functions



Wall-adjacent cell.

▶When using wall-functions, we don't resolve the boundary layer. The first cell center (the wall-adjacent) is placed in the log-law region ($30 < x_2^+ < 400$) and we *assume* that the velocity follows the log-law
▶The log-law reads (see Eq. 6.33)

$$\frac{\bar{v}_1}{u_\tau} = \frac{1}{\kappa} \ln\left(\frac{u_\tau x_2}{\nu}\right) + B$$

It is re-written as

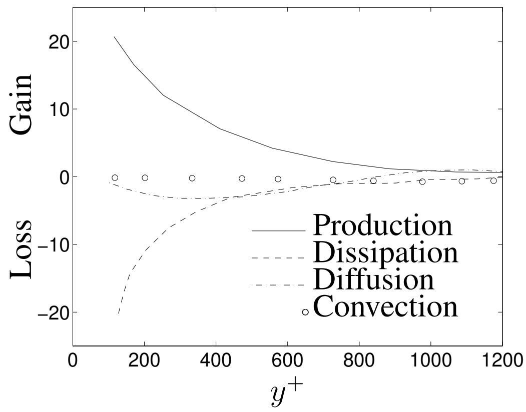
$$\frac{\bar{v}_1}{u_\tau} = \frac{1}{\kappa} \ln\left(\frac{E u_\tau x_2}{\nu}\right), \quad E = 9.0, \quad B = \frac{1}{\kappa} \ln E$$

Friction velocity is computed as (see Eq. 11.140)

$$u_\tau = \frac{\kappa \bar{v}_{1,P}}{\ln(E u_\tau \delta x_2/\nu)}$$

(subscript P denotes the wall-adjacent cell) It is obtained by iteration. Then $\tau_w = \rho u_\tau^2$ is used a force wall boundary condition.

▶B.c. for k and ε. In the log-region, the production and dissipation in the k eq. balance each other (see Fig. 8.3b) which gives (see Eq. 11.141)



Boundary along a flat plate. Energy balance in k equation.

$$0 = P^k - \rho\varepsilon = \mu_t \left(\frac{\partial \bar{v}_1}{\partial x_2}\right)^2 - \rho\varepsilon. \tag{A.25}$$

In the log-region (see Fig. 6.3b)

$$\tau_w = -\rho\overline{v_1' v_2'} = \mu_t \frac{\partial \bar{v}_1}{\partial x_2} \tag{A.26}$$

Inserting Eq. A.26 into Eq. A.25 gives

$$0 = \frac{\overline{v_1' v_2'}^2}{\nu_t} - \varepsilon = \frac{u_\tau^4}{\nu_t} - \varepsilon$$

which with $\nu_t = C_\mu k^2/\varepsilon$ gives (see Eq. 11.145)

$$k_P = C_\mu^{-1/2} u_\tau^2, \quad C_\mu = 0.09$$

►ε.

- Velocity gradient in log-region: when deriving the log-law we assumed (see Eq. 6.28): $\partial\bar{v}_1/\partial x_2 \simeq u_\tau/(\kappa x_2)$
- Shear stress in log-region $-\overline{v_1' v_2'} \simeq u_\tau^2$, see Eq. 6.26 and Fig. 6.3

Eq. A.25 gives (see Eq. 11.146)

$$\varepsilon_P = P^k = \frac{u_\tau^3}{\kappa\delta x_2}$$

¶See Section 11.14.2, Low-Re Number Turbulence Models
▶In Low-Re number models we *resolve* the boundary layer, i.e. we use a refined grid near the wall. The wall-adjacent cell at $x_2^+ < 1$. B.c. for velocity is $v_i = 0$.

However, the turbulence near the wall is not fully turbulent: the viscous effect is large. We must modify the turbulence model.

▶We start by analyzing the turbulence near the wall. Make a Taylor expansion of the fluctuating velocity, v_i', near the wall (also valid for $\bar{v}_i$)

$$\begin{aligned} v_1' &= a_0 + a_1 x_2 + a_2 x_2^2 + \ldots \\ v_2' &= b_0 + b_1 x_2 + b_2 x_2^2 + \ldots \\ v_3' &= c_0 + c_1 x_2 + c_2 x_2^2 + \ldots \end{aligned} \tag{A.27}$$

At the wall, $v_1' = v_2' = v_3' = 0$ which gives $a_0 = b_0 = c_0$.
Furthermore $\partial v_1'/\partial x_1 = \partial v_3'/\partial x_3 = 0$: continuity equation gives $\partial v_2'/\partial x_2 = 0$ so that $b_1 = 0$. Equation A.27 now reads

$$\begin{array}{lllll} v_1' & = a_1 x_2 & + & a_2 x_2^2 + \ldots \\ v_2' & = & & b_2 x_2^2 + \ldots \\ v_3' & = c_1 x_2 & + & c_2 x_2^2 + \ldots \end{array} \tag{A.28}$$

Using Eq. A.28 we can write

$$\begin{array}{lll} \overline{v_1'^2} & = \overline{a_1^2} x_2^2 + \ldots & = \mathcal{O}(x_2^2) \\ \overline{v_2'^2} & = \overline{b_2^2} x_2^4 + \ldots & = \mathcal{O}(x_2^4) \\ \overline{v_3'^2} & = \overline{c_1^2} x_2^2 + \ldots & = \mathcal{O}(x_2^2) \\ \overline{v_1' v_2'} & = \overline{a_1 b_2} x_2^3 + \ldots & = \mathcal{O}(x_2^3) \\ k & = (\overline{a_1^2 + c_1^2}) x_2^2 + \ldots & = \mathcal{O}(x_2^2) \\ \partial \bar{v}_1/\partial x_2 & = \overline{a_1} + \ldots & = \mathcal{O}(x_2^0) \\ \partial v_1'/\partial x_2 & = a_1 + \ldots & = \mathcal{O}(x_2^0) \\ \partial v_2'/\partial x_2 & = 2 b_2 x_2 + \ldots & = \mathcal{O}(x_2^1) \\ \partial v_3'/\partial x_2 & = a_1 + \ldots & = \mathcal{O}(x_2^0) \end{array} \tag{A.29}$$

¶See Section 11.14.3, Low-Re $k - \varepsilon$ Models
▶Now let's compare the exact and the modeled k eq. near the wall
The exact k eq. (see Eq. 8.14)

$$\rho\bar{v}_1\frac{\partial k}{\partial x_1}+\rho\bar{v}_2\frac{\partial k}{\partial x_2}=\underbrace{-\rho\overline{v_1'v_2'}\frac{\partial\bar{v}_1}{\partial x_2}}_{\mathcal{O}(x_2^3)}-\frac{\partial\overline{p'v_2'}}{\partial x_2}\underbrace{-\frac{\partial}{\partial x_2}\left(\frac{1}{2}\rho\overline{v_2'v_i'v_i'}\right)}_{\mathcal{O}(x_2^3)}$$
$$+\mu\frac{\partial^2 k}{\partial x_2^2}-\underbrace{\mu\overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}}_{\mathcal{O}(x_2^0)}$$

The modeled k eq. (see Eq. 11.97)

$$\rho\bar{v}_1\frac{\partial k}{\partial x_1}+\rho\bar{v}_2\frac{\partial k}{\partial x_2}=\underbrace{\mu_t\left(\frac{\partial\bar{v}_1}{\partial x_2}\right)^2}_{\mathcal{O}(x_2^4)}+\underbrace{\frac{\partial}{\partial x_2}\left(\frac{\mu_t}{\sigma_k}\frac{\partial k}{\partial x_2}\right)}_{\mathcal{O}(x_2^4)}$$
$$+\mu\frac{\partial^2 k}{\partial x_2^2}-\underbrace{\rho\varepsilon}_{\mathcal{O}(x_2^0)}$$

We used $\nu_t=C_\mu\dfrac{k^2}{\varepsilon}=\dfrac{\mathcal{O}(x_2^4)}{\mathcal{O}(x_2^0)}=\mathcal{O}(x_2^4)$

- the exact and the modeled dissipation term behave in the same way
- this is not true for the production term and the turbulent diffusion term

▶To make the modeled production term behave as $\mathcal{O}(x_2^3)$, replace C_μ with $C_\mu f_\mu$ (damping function) where $f_\mu \propto \mathcal{O}(x_2^{-1})$

This fixes also the modeled turb. diffusion term

▶Now we look at the modeled ε eq. (see Eq. 11.156)

$$\underbrace{\rho\bar{v}_1\frac{\partial\varepsilon}{\partial x_1}}_{\mathcal{O}(x_2^1)}+\underbrace{\rho\bar{v}_2\frac{\partial\varepsilon}{\partial x_2}}_{\mathcal{O}(x_2^2)}=\underbrace{C_{\varepsilon1}\frac{\varepsilon}{k}P^k}_{\mathcal{O}(x_2^1)}+\underbrace{\frac{\partial}{\partial x_2}\left(\frac{\mu_t}{\sigma_\varepsilon}\frac{\partial\varepsilon}{\partial x_2}\right)}_{\mathcal{O}(x_2^2)}$$
$$+\underbrace{\mu\frac{\partial^2\varepsilon}{\partial x_2^2}}_{\mathcal{O}(x_2^0)}-\underbrace{C_{\varepsilon2}\rho\frac{\varepsilon^2}{k}}_{\mathcal{O}(x_2^{-2})}$$

where we assumed that the production term P^k has been suitable modified so that $P^k = \mathcal{O}(x_2^3)$. The only terms that are non-zero when $x_2 \to 0$ are the viscous diffusion term and the destruction term.

▶But they can't balance each other since the first is $\propto \mathcal{O}(x_2^0)$ and the second $\propto \mathcal{O}(x_2^{-2})$.

We fix this by multiplying the destruction term by $f_2 \propto \mathcal{O}(x_2^2)$

¶See Section 11.14.5, Different ways of prescribing ε at or near the wall
▶Boundary condition for k (since $v_i' \to 0$ near the wall)

$$k = 0$$

▶Boundary condition for ε: look at the k eq. near the wall. The only non-vanishing terms are

$$0 = \mu \frac{\partial^2 k}{\partial x_2^2} - \rho\varepsilon. \tag{A.30}$$

which gives (see Eq. 11.160)

$$\varepsilon_{wall} = \nu \frac{\partial^2 k}{\partial x_2^2}.$$

▶Eq. A.30 can be used to get alternative boundary conditions of ε. Exact form of the dissipation term near the wall reads (see Eq. 8.26)

$$\varepsilon = \nu \left\{ \overline{\left(\frac{\partial v_1'}{\partial x_2}\right)^2 + \left(\frac{\partial v_3'}{\partial x_2}\right)^2} \right\}$$

where we have assumed $\partial/\partial x_2 \gg \partial/\partial x_1 \simeq \partial/\partial x_3$ and $\partial v_1'/\partial x_2 \simeq \partial v_3'/\partial x_2 \gg \partial v_2'/\partial x_2$. Taylor expansion gives (see Eq. A.28)

$$\varepsilon = \nu \left(\overline{a_1^2} + \overline{c_1^2} \right) + \ldots \tag{A.31}$$

The turbulent kinetic energy (see Eq. A.28)

$$k = \frac{1}{2} \left(\overline{a_1^2} + \overline{c_1^2} \right) x_2^2 + \ldots \tag{A.32}$$

so that

$$\left(\frac{\partial \sqrt{k}}{\partial x_2} \right)^2 = \frac{1}{2} \left(\overline{a_1^2} + \overline{c_1^2} \right) + \ldots \tag{A.33}$$

Eqs. A.31 and A.33 gives

$$\varepsilon_{wall} = 2\nu \left(\frac{\partial \sqrt{k}}{\partial x_2} \right)^2.$$

▶Often the following boundary condition is used (see Eq. 11.166)

$$\varepsilon_{wall} = \frac{2\nu k}{x_2^2} \tag{A.34}$$

where we have assumed $a_1 = c_1$ in Eqs. A.31 and A.32 so that

$$\varepsilon = 2\nu \overline{a_1^2}$$
$$k = \overline{a_1^2} x_2^2$$

which gives Eq. A.34.

- **Summary of the low-Re number model.**
  - Fine mesh near the wall. The first cell center is located at $x_2^+ \lesssim 1$.
  - This means that standard wall b.c. can be used, i.e. $\bar{v}_1 = \bar{v}_2 = \bar{v}_3 = k = 0$.
  - There are different options for the wall b.c. for ε: usually $\varepsilon_P = 2\nu k/(x_2^2)$ is prescribed for the wall-adjacent cells
- **Summary of wall-functions.**
  - Coarse mesh near the wall. The first cell center is located at $30 \lesssim x_2^+ \lesssim 400$. The point is located in the log region.
  - Friction velocity, u_τ, computed from the log-law.
  - A force/area b.c. is used for the wall-parallel velocity component: $\tau_w = \rho u_\tau^2$
  - In the log-region we know that the k equation can be simplified as $0 = P^k - \varepsilon$ which gives $k_P = C_\mu^{-1/2} u_\tau^2$ (k_P is prescribed for the wall-adjacent cells)
  - We use the simplified k equation also for ε: $0 = P^k - \varepsilon$ gives $\varepsilon_P = u_\tau^3/(\kappa x_2)$ (ε_P is prescribed for the wall-adjacent cells)