Taylor's Theorem, for Rogues and Pirates *- Eric Mjolsness UCI 9/17*

I. Taylor's theorem for the exponential function - no remainder, no real proof, just calculate.

A. Euler's formula for the exponential function

$$e^x = exp(x) \equiv lim_{n\to\infty}(1+\frac{x}{n})^n$$

B. Use the binomial theorem

$$(1+\frac{x}{n})^n = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{n}\right)^k = \sum_{k=0}^{n}\frac{x^k}{k!}\frac{n(n-1)\ldots(n-k+1)}{n^k}$$

C. Take limits with wanton carelessness

$$\exp(x) = lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{n}\right)^k = \sum_{k=0}^{\infty}\frac{x^k}{k!}\left(lim_{n\to\infty}\frac{n(n-1)\ldots(n-k+1)}{n^k}\right)$$, so

$\exp(x) = \sum_{k=0}^{\infty}\frac{x^k}{k!}$. PBH (Pirate Booty in the Hold). 

II. Taylor's theorem for everything else: other functions of one variable

A. Use linear interpolation to guess $f(x \equiv x_0 + \epsilon)$ where ϵ is very small:

$f(x_0+\epsilon) \simeq f(x_0) + \epsilon f'(x_0) + \ldots$ // even smaller corrections

B. Express this approximation as an *operator* on the function $f(x)$, involving the Derivative operator $D_x \equiv \frac{d}{dx}$ and the do-nothing Identity operator $I = 1$:

$$f(x_0+\epsilon) \simeq \left(I + \epsilon\frac{d}{dx}\right)\circ f(x_0) + \ldots$$

C. To estimate $f(x_0 + a)$ where a is not small, let $\epsilon = a/n$ and apply this operator *n* times:

$$f(x_0+a) = f(x_0 + a/n + \ldots a/n) = f(x_0 + \epsilon + \ldots\epsilon) \simeq \left(1+\epsilon\frac{d}{dx}\right)^n \circ f(x_0)\ .$$

Take epsilon as small as you need, by taking the $n \to \infty$ limit with Euler's formula:

$$f(x_0+a) = lim_{n\to\infty}\left(1+\frac{a}{n}\frac{d}{dx}\right)^n \circ f(x_0) = \exp\left(a\frac{d}{dx}\right)\circ f(x_0)$$

D. Now apply Part I, Taylor's theorem for just the exponential, to get it for everything else:

$f(x_0+a) = \exp\left(a\frac{d}{dx}\right)\circ f(x_0) = \sum_{k=0}^{\infty}\frac{a^k}{k!}\frac{d^k}{dx^k}\circ f(x_0) = \sum_{k=0}^{\infty}\frac{a^k}{k!}\frac{d^k f(x_0)}{dx^k}$; PBH. 

III. Multivariable Taylor's theorem is now easy, using product notation:

$$f(\mathbf{x}_0+\mathbf{a}) = f(\mathbf{x}_0+\sum_{i=1}^{d} a_i\hat{\mathbf{e}}_i) = \left[\prod_{i=1}^{d}\exp\Big(a_i\frac{d}{dx_i}\Big)\right]\circ f(\mathbf{x}_0) = \exp\Big(\sum_{i=1}^{d} a_i\frac{d}{dx_i}\Big)\circ f(\mathbf{x}_0)$$

$$f(\mathbf{x}_0+\mathbf{a}) = \exp\Big(\sum_{i=1}^{d} a_i\frac{d}{dx_i}\Big)\circ f(\mathbf{x}_0) = \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{i_1+\ldots i_d=k, i_l\geq 0}\binom{k}{i_1\ldots i_d}\left[\Big(\prod_{l=1}^{d} a_l^{i_l}\frac{d^{i_l}}{dx_i^{i_l}}\Big)f(\mathbf{x}_0)\right]$$

$$f(\mathbf{x}_0+\mathbf{a}) = \sum_{k=0}^{\infty}\sum_{i_1+\ldots i_d=k, i_l\geq 0}\frac{1}{i_1!\ldots i_d!}\Big(\prod_{l=1}^{d} a_l^{i_l}\Big)\left[\Big(\prod_{l=1}^{d}\frac{d^{i_l}}{dx_i^{i_l}}\Big)f(\mathbf{x}_0)\right];$$ PBH. 

For example d=2:

$$f(\mathbf{x}_0+\mathbf{a}) = \sum_{k=0}^{\infty}\sum_{i+j=k, i,j\geq 0}\frac{a_1^i a_2^j}{i!j!}\frac{d^k f(\mathbf{x}_0)}{dx_1^i dx_2^j}$$

Or for example d=3:

$$f(\mathbf{x}_0+\mathbf{a}) = \sum_{n=0}^{\infty}\sum_{i+j+k=n, i,j,k\geq 0}\frac{a_1^i a_2^j a_3^k}{i!j!k!}\frac{d^n f(\mathbf{x}_0)}{dx_1^i dx_2^j x_3^k}$$

IV. Loose ends

A. In I, if the limit defining "$exp(x)$" exists, *calculate* that $exp(kx) = (exp(x))^k$.
B. Write out the sum of the k=0, 1, 2 terms.
C. Look up the remainder term