FOURIER SERILERI

ELM207 Analog Elektronik

Giriş

Bir Fourier serisi periyodik bir $f(t)$ fonksiyonunun, kosinüs ve sinüslerin sonsuz toplamı biçiminde bir açılımdır.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

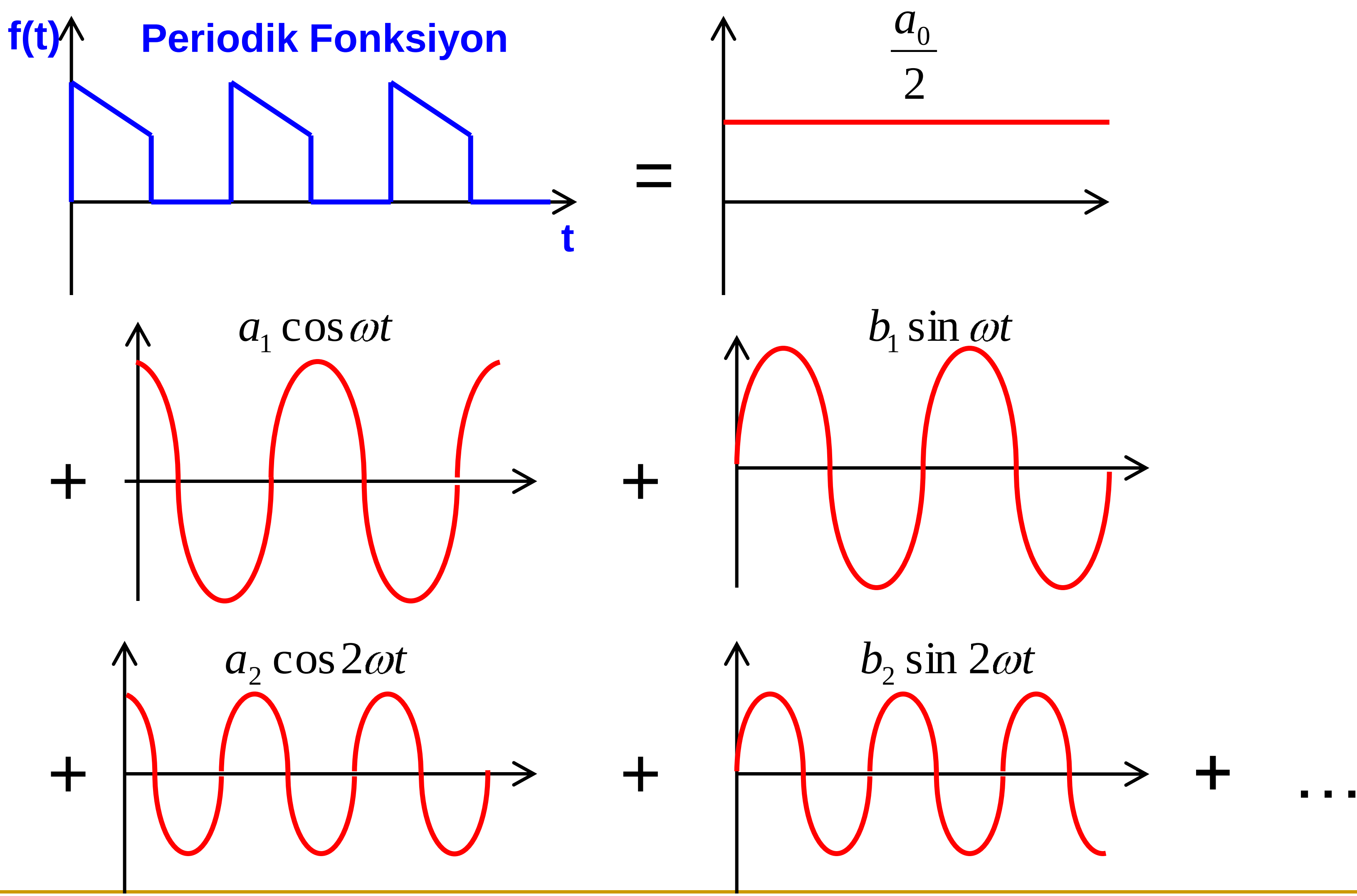
$$\omega = \frac{2\pi}{T}$$

Başka deyişle, herhangi bir periyodik fonksiyon sabit bir değer, kosinüs ve sinüs fonksiyonlarının toplamı olarak ifade edilebilir:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

$$\begin{aligned} &= \frac{a_0}{2} \quad + (a_1 \cos \omega t + b_1 \sin \omega t) \\ &\qquad + (a_2 \cos 2\omega t + b_2 \sin 2\omega t) \\ &\qquad + (a_3 \cos 3\omega t + b_3 \sin 3\omega t) \quad + \ldots \end{aligned}$$

Fourier serisi hesaplamaları *harmonik analiz* olarak bilinir ve keyfi bir fonksiyonun bir dizi basit terimlere ayrılarak, ayrık terimler olarak çözülmesi ve yeniden birleştirilip orjinal problemin çözümü için oldukça kullanışlı bir yoldur. Böylelikle problem istenilen ya da pratik olan bir yaklaşıklıkta çözülebilir.


f(t)
Periodik Fonksiyon
t
=
$\frac{a_0}{2}$
+
$a_1 \cos \omega t$
+
$b_1 \sin \omega t$
+
$a_2 \cos 2\omega t$
+
$b_2 \sin 2\omega t$
+
...

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

burada $\omega = \frac{2\pi}{T} = \text{Temel frekans}$

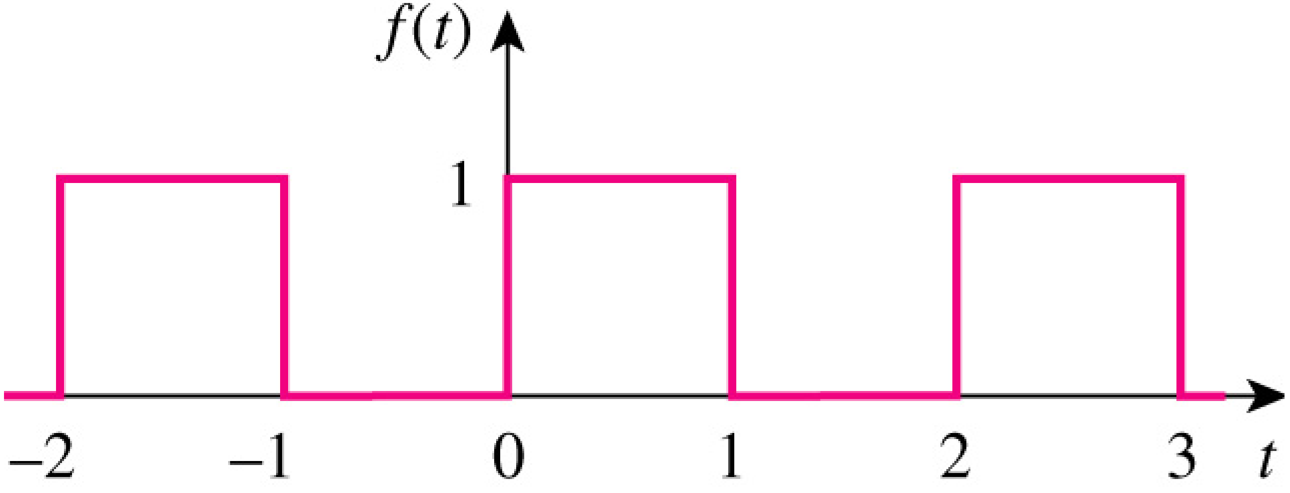
$$\boxed{a_0 = \frac{2}{T}\int_0^T f(t)dt}$$

$$\boxed{a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t dt} \qquad \boxed{b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t dt}$$

*integral limiti olarak $\int_{-T/2}^{T/2}$ kullanabiliriz

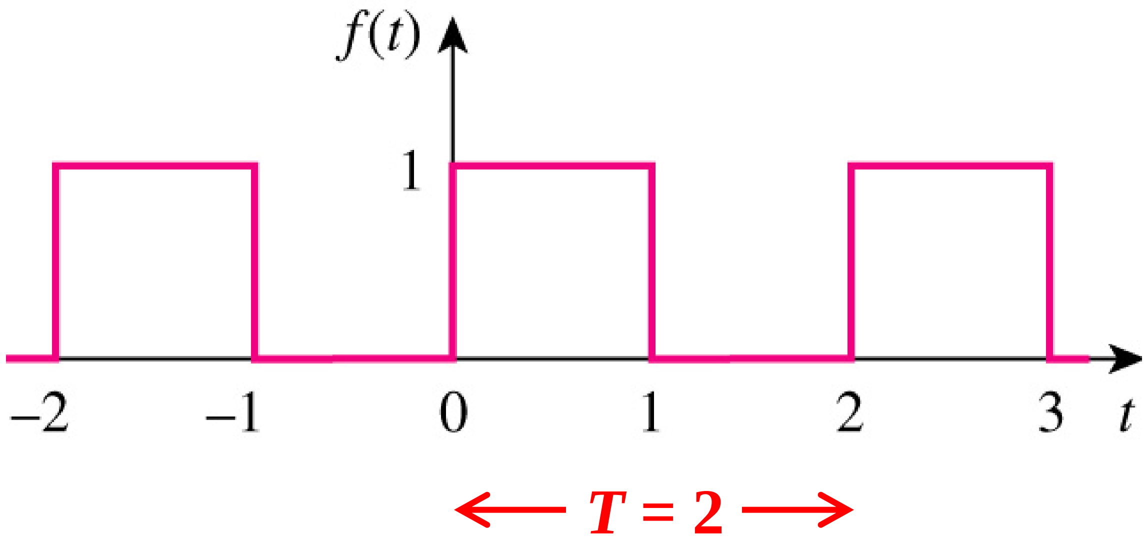
Örnek 1

Aşağıdaki dalga biçiminin Fourier serisi gösterimini bulunuz.



Çözüm

İlk önce, fonksiyonun periyodu ve tanımı belirlenir:



$$f(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases} \qquad f(t+2) = f(t)$$

Sonra, a_0, a_n ve b_n katsayıları bulunur :

$$a_0 = \frac{2}{T}\int_0^T f(t)dt = \frac{2}{2}\int_0^2 f(t)dt = \int_0^1 1dt + \int_1^2 0dt = 1 - 0 = 1$$

Ya da, $\int_a^b f(t)dt$ [*a*,*b*] aralığı boyunca grafiğin altındaki toplam alan olduğundan

$$a_0 = \frac{2}{T}\int_0^T f(t)dt = \frac{2}{T}\times\begin{pmatrix} [0,T]boyunca \\ alan \end{pmatrix} = \frac{2}{2}\times(1\times1) = 1$$

$$a_n = \frac{2}{T}\int_0^2 f(t)\cos n\omega t dt$$

$$= \int_0^1 1\cos n\pi t dt + \int_1^2 0 dt = \left[\frac{\sin n\pi t}{n\pi}\right]_0^1 = \frac{\sin n\pi}{n\pi}$$

n tamsayıdır ve, $\sin n\pi = 0$

olduğundan $\sin\pi = \sin 2\pi = \sin 3\pi = \ldots = 0$

Dolayısıyla, $a_n = 0$.

$$b_n = \frac{2}{T}\int_0^2 f(t)\sin n\omega t dt$$

$$= \int_0^1 1\sin n\pi t dt + \int_1^2 0 dt = \left[-\frac{\cos n\pi t}{n\pi}\right]_0^1 = \frac{1-\cos n\pi}{n\pi}$$

$$\cos\pi = \cos 3\pi = \cos 5\pi = \ldots = -1$$
$$\cos 2\pi = \cos 4\pi = \cos 6\pi = \ldots = 1$$

Ya da $\cos n\pi = (-1)^n$

Dolayısıyla, $b_n = \dfrac{1-(-1)^n}{n\pi} = \begin{cases} 2/n\pi & , \quad n \text{ tek} \\ 0 & , \quad n \text{ çift} \end{cases}$

Sonuçta,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \left[\frac{1-(-1)^n}{n\pi} \right] \sin n\pi t$$

$$= \frac{1}{2} + \frac{2}{\pi} \sin \pi t + \frac{2}{3\pi} \sin 3\pi t + \frac{2}{5\pi} \sin 5\pi t + \dots$$

Bazı faydalı tanımlar

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x$$

n tamsayı olduğundan,

$$\sin n\pi = 0 \qquad \cos n\pi = (-1)^n$$

$$\sin 2n\pi = 0 \qquad \cos 2n\pi = 1$$

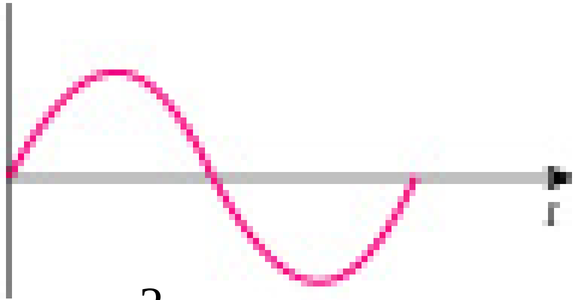
- Fourier serisi terimlerinin toplamı orjinal dalga biçimini verir

- Örnek 1'den,

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sin \pi t + \frac{2}{3\pi}\sin 3\pi t + \frac{2}{5\pi}\sin 5\pi t + \ldots$$

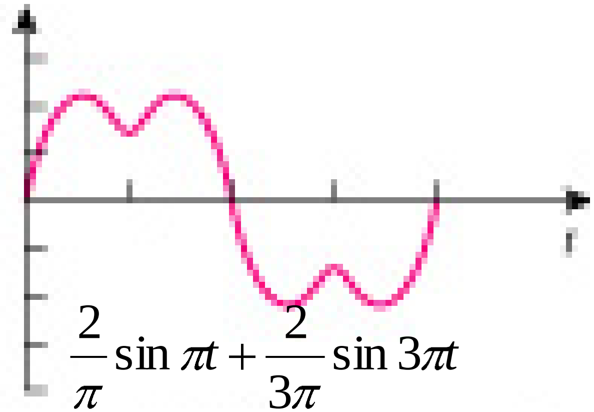
- Toplamın kare dalga vereceği gösterilebilir:

(a)



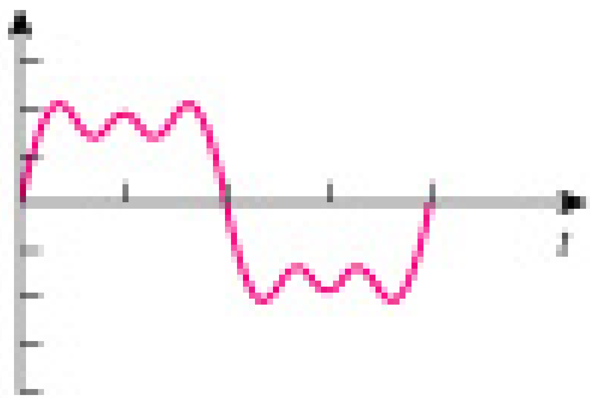
$$\frac{2}{\pi}\sin \pi t$$

(b)



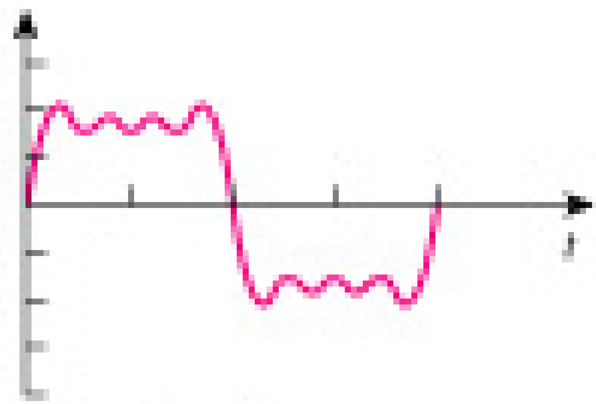
$$\frac{2}{\pi}\sin \pi t + \frac{2}{3\pi}\sin 3\pi t$$

(c)



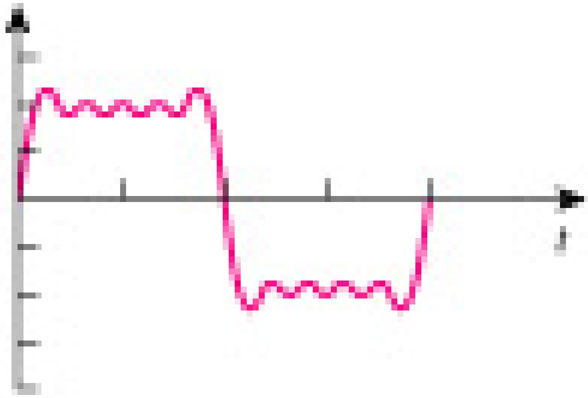
$$\frac{2}{\pi}\sin \pi t + \frac{2}{3\pi}\sin 3\pi t + \frac{2}{5\pi}\sin 5\pi t$$

(d)



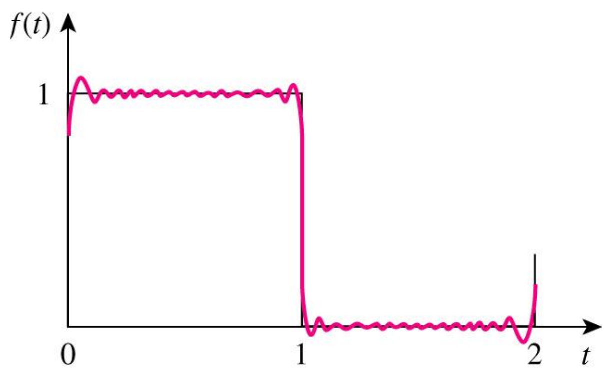
$$\frac{2}{\pi}\sin \pi t + \frac{2}{3\pi}\sin 3\pi t + \frac{2}{5\pi}\sin 5\pi t + \frac{2}{7\pi}\sin 7\pi t$$

(e)



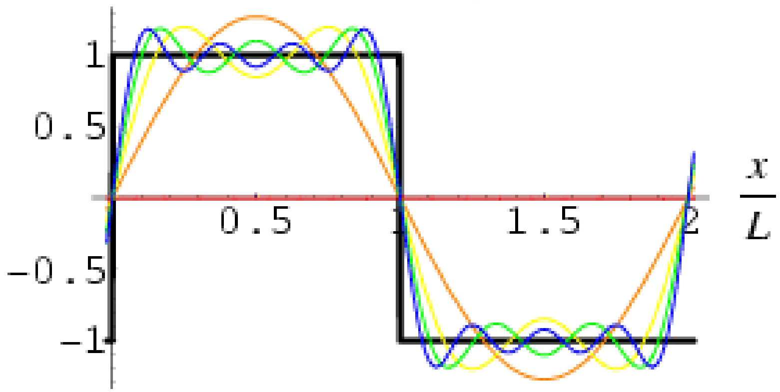
$$\frac{2}{\pi}\sin \pi t+\frac{2}{3\pi}\sin 3\pi t+\frac{2}{5\pi}\sin 5\pi t+\frac{2}{7\pi}\sin 7\pi t+\frac{2}{9\pi}\sin 9\pi t$$

(f)

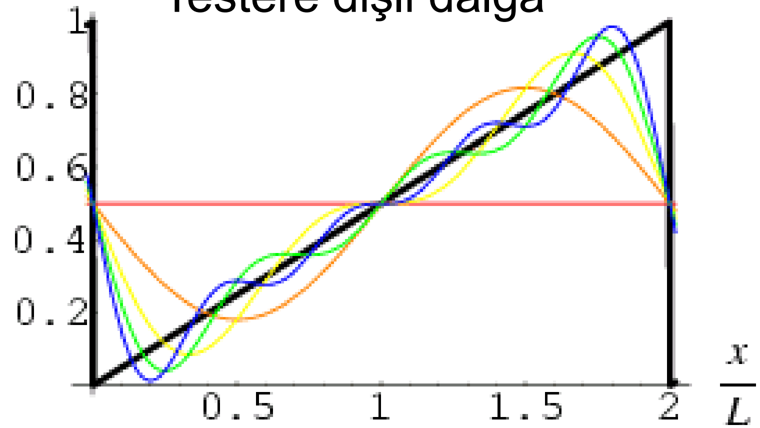


$$\frac{1}{2}+\frac{2}{\pi}\sin \pi t+\frac{2}{3\pi}\sin 3\pi t+\ldots+\frac{2}{23\pi}\sin 23\pi t$$

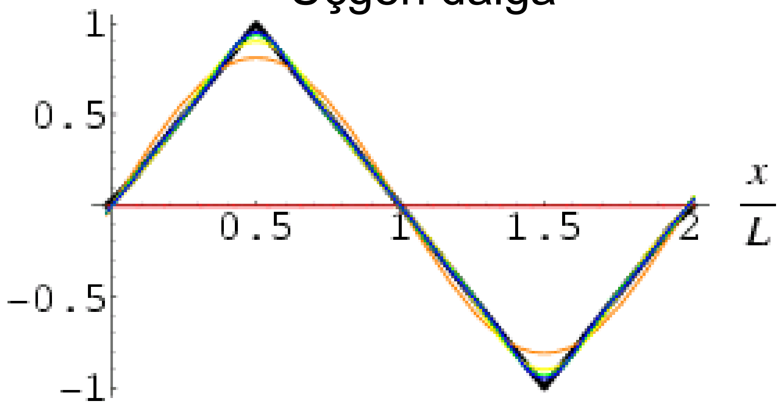
Kare dalga



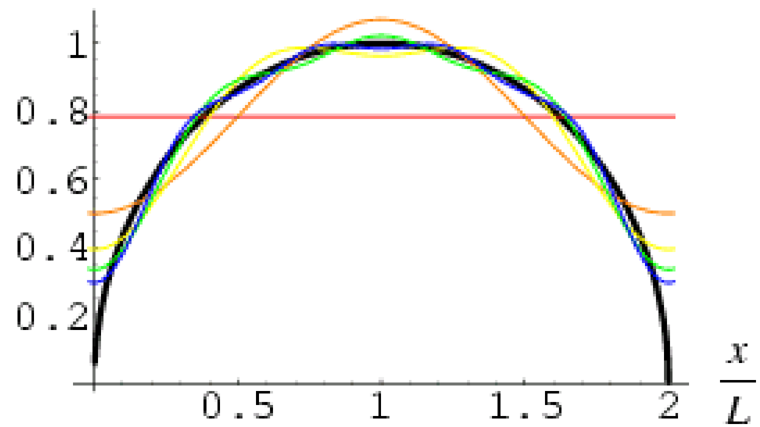
Testere dişli dalga



Üçgen dalga



Yarı çember



Örnek 2

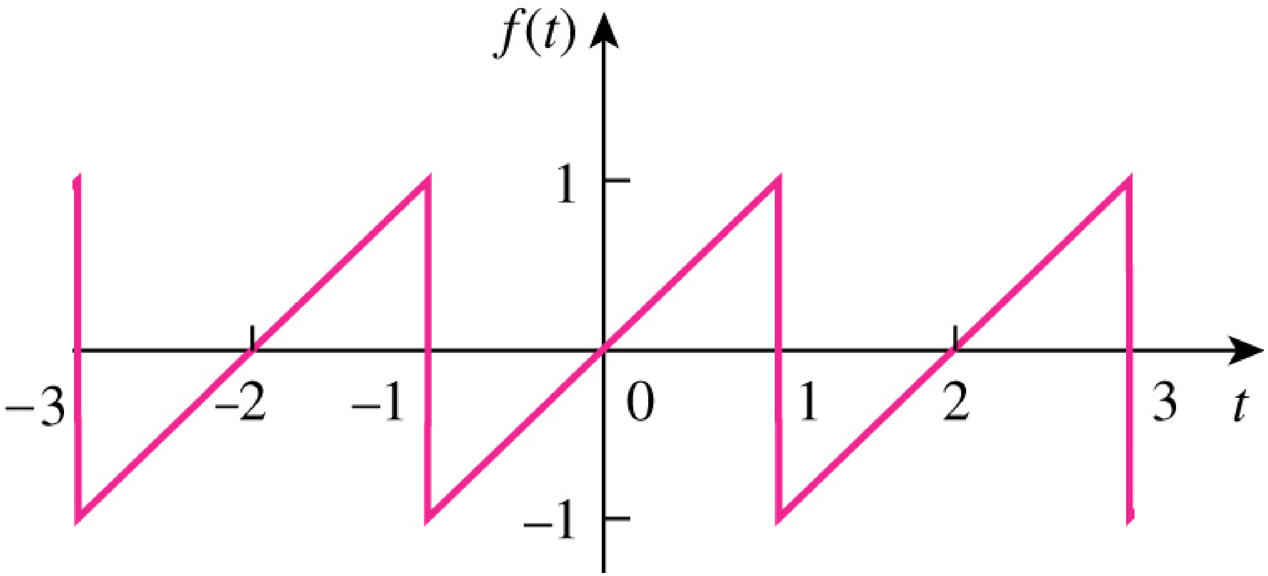
$$f(t) = t, \quad -1 \leq t \leq 1$$

$$f(t+2) = f(t)$$

$f(t)$'nin grafiğini çiziniz, $\quad -3 \leq t \leq 3$.

$f(t)$'nin Fourier serisini hasaplayınız.

Çözüm



$T = 2$

$$\omega = \frac{2\pi}{T} = \pi$$

Katsayıları hesaplayalım:

$$a_0 = \frac{2}{T}\int_{-1}^{1} f(t)dt$$

$$= \frac{2}{2}\int_{-1}^{1} t dt = \left[\frac{t^2}{2}\right]_{-1}^{1} = \frac{1-1}{2} = 0$$

$$a_n = \frac{2}{T}\int_{-1}^{1} f(t)\cos n\omega t dt = \int_{-1}^{1} t\cos n\pi t dt$$

$$= \left[\frac{t\sin n\pi t}{n\pi}\right]_{-1}^{1} - \int_{-1}^{1}\frac{\sin n\pi t}{n\pi}dt$$

$$= \frac{\sin n\pi - [-\sin(-n\pi)]}{n\pi} + \left[\frac{\cos n\pi t}{n^2\pi^2}\right]_{-1}^{1}$$

$$= 0 + \frac{\cos n\pi - \cos(-n\pi)}{n^2\pi^2}$$

$$= \frac{\cos n\pi - \cos n\pi}{n^2\pi^2} = 0$$

$$\cos(-x) = \cos x$$

$$b_n = \frac{2}{T}\int_{-1}^{1} f(t)\sin n\omega t dt = \int_{-1}^{1} t\sin n\pi t dt$$

$$= \left[-\frac{t\cos n\pi t}{n\pi}\right]_{-1}^{1} + \int_{-1}^{1}\frac{\cos n\pi t}{n\pi}dt$$

$$= \frac{-\cos n\pi + [-\cos(-n\pi)]}{n\pi} + \left[\frac{\sin n\pi t}{n^2\pi^2}\right]_{-1}^{1}$$

$$= -\frac{2\cos n\pi}{n\pi} + \frac{\sin n\pi - \sin(-n\pi)}{n^2\pi^2}$$

$$= -\frac{2\cos n\pi}{n\pi} = -\frac{2(-1)^n}{n\pi} = \frac{2(-1)^{n+1}}{n\pi}$$

Sonuçta,

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\omega t+b_n\sin n\omega t)$$

$$=\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n\pi}\sin n\pi t$$

$$=\frac{2}{\pi}\sin \pi t-\frac{2}{2\pi}\sin 2\pi t+\frac{2}{3\pi}\sin 3\pi t-\ldots$$

Örnek 3

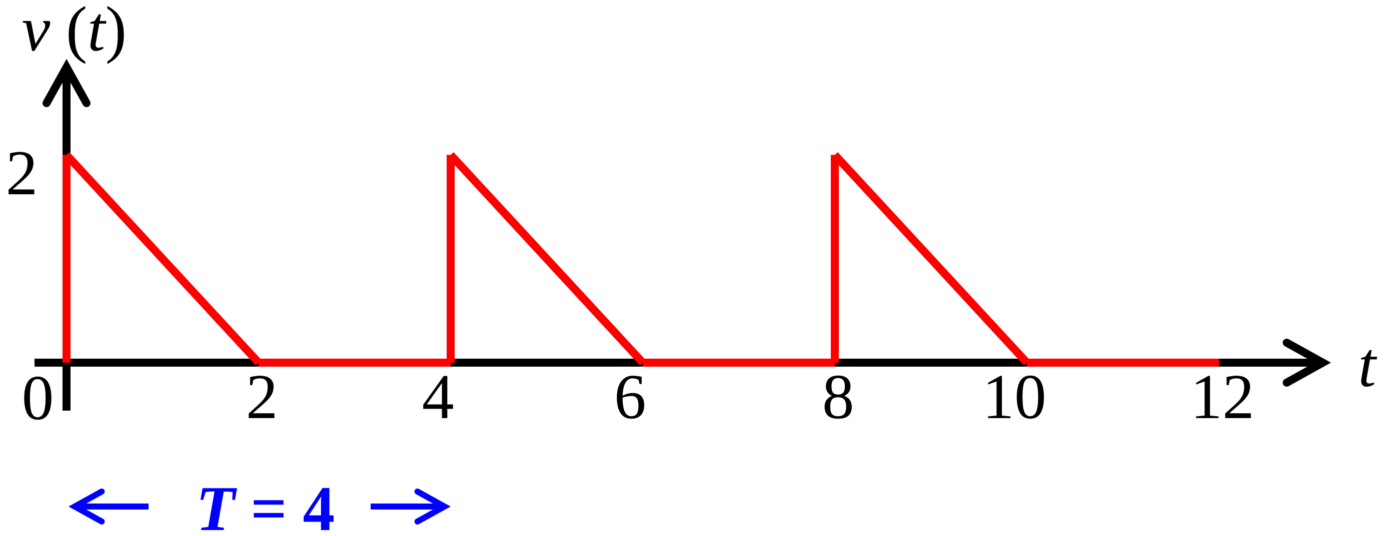
$$v(t) = \begin{cases} 2-t & , \quad 0 < t < 2 \\ 0 & , \quad 2 < t < 4 \end{cases}$$

$$v(t+4) = v(t)$$

$v\ (t)$ grafiğini çiziniz, $0 \leq t \leq 12.$

$v\ (t)$'nin Fourier serisi açılımını hesaplayınız.

Çözüm



$$\omega = \frac{2\pi}{T} = \frac{\pi}{2}$$

Katsayılar:

$$a_0 = \frac{2}{T}\int_0^4 v(t)dt$$

$$= \frac{2}{4}\left\{\int_0^2 (2-t)dt + \int_2^4 0dt\right\}$$

$$= \frac{1}{2}\int_0^2 (2-t)dt = \frac{1}{2}\left[2t - \frac{t^2}{2}\right]_0^2 = 1$$

$$a_n = \frac{2}{T}\int_0^4 v(t)\cos n\omega t dt = \frac{1}{2}\int_0^2 (2-t)\cos n\omega t dt + \int_2^4 0$$

$$= \frac{1}{2}\left[\frac{(2-t)\sin n\omega t}{n\omega}\right]_0^2 + \frac{1}{2}\int_0^2 \frac{\sin n\omega t}{n\omega} dt$$

$$= 0 + \frac{1}{2}\left[-\frac{\cos n\omega t}{n^2\omega^2}\right]_0^2$$

$$= \frac{1-\cos 2n\omega}{2n^2\omega^2} = \frac{2(1-\cos n\pi)}{n^2\pi^2} = \frac{2[1-(-1)^n]}{n^2\pi^2}$$

$$b_n = \frac{2}{T}\int_0^4 v(t)\sin n\omega t dt = \frac{1}{2}\int_0^2 (2-t)\sin n\omega t dt + \int_2^4 0$$

$$= \frac{1}{2}\left[\frac{-(2-t)\cos n\omega t}{n\omega}\right]_0^2 - \frac{1}{2}\int_0^2 \frac{\cos n\omega t}{n\omega} dt$$

$$= \frac{1}{n\omega} - \frac{1}{2}\left[\frac{\sin n\omega t}{n^2\omega^2}\right]_0^2$$

$$= \frac{1}{n\omega} - \frac{\sin 2n\omega}{2n^2\omega^2} = \frac{1}{n\omega} = \frac{2}{n\pi}$$

$$\sin 2n\omega = \sin n\pi = 0$$

Sonuçta,

$$v(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \left\{ \frac{2[1-(-1)^n]}{n^2\pi^2} \cos\left(\frac{n\pi t}{2}\right) + \frac{2}{n\pi} \sin\left(\frac{n\pi t}{2}\right) \right\}$$

Simetri

- Simetri fonksiyonları:

  (i) **çift** simetri

  (ii) **tek** simetri

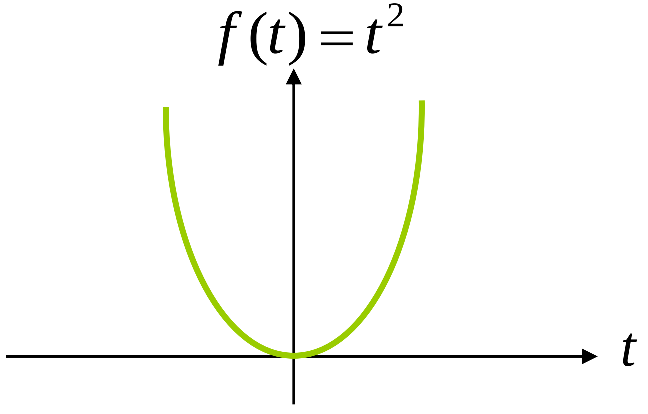
Çift simetri

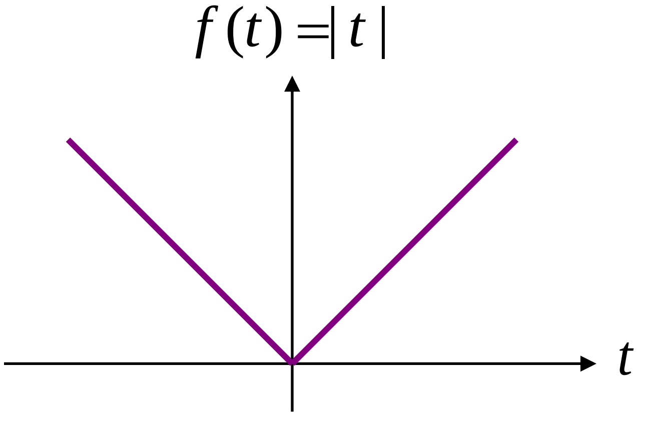
- Herhangi $f(t)$ fonksiyonu grafiğin düşey eksenine göre simetrik ise **çifttir,** yani

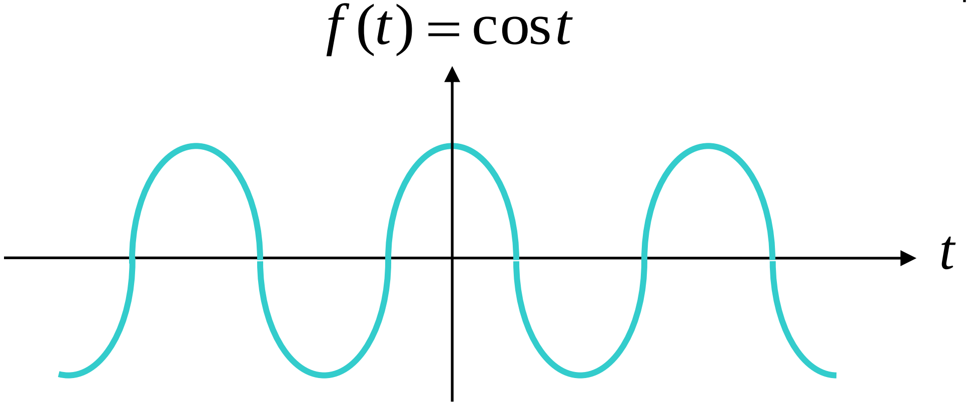
$$f(-t) = f(t)$$

Çift simetri (devam)

- **çift** fonksiyonlara örnek:

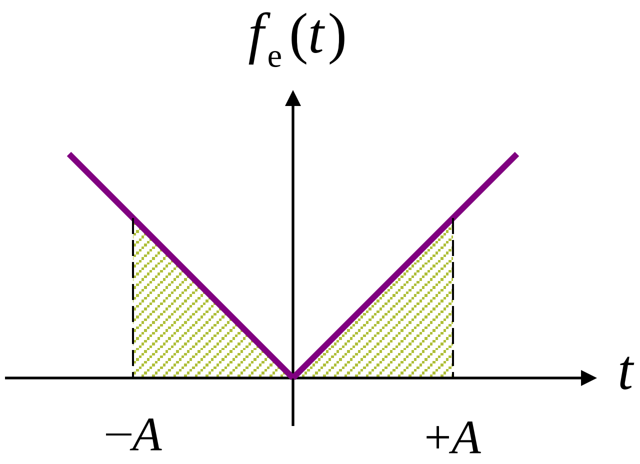






Çift simetri (devam)

- $-A$ dan $+A$ ya **çift** bir fonksiyonun integrali 0 dan $+A$ ya integralinin iki katıdır



$$\int_{-A}^{+A} f_e(t)dt = 2\int_{0}^{+A} f_e(t)dt$$

Tek simetri

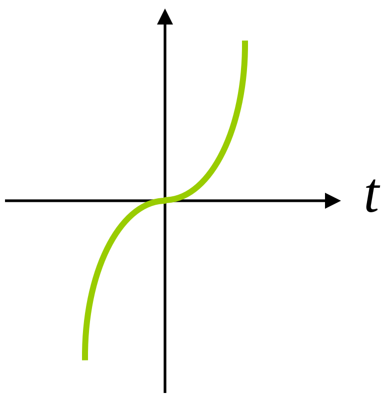
- Herhangi $f(t)$ fonksiyonu grafiğin düşey eksenine göre asimetrik ise **tektir,** yani

$$f(-t) = -f(t)$$

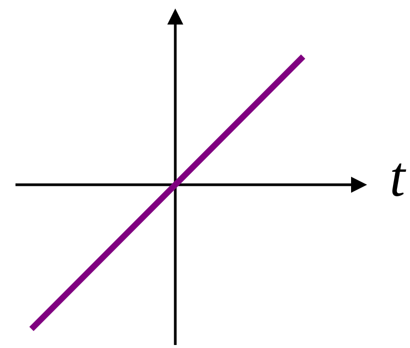
Tek simetri (devam)

- **Tek** fonksiyonlara örnek:

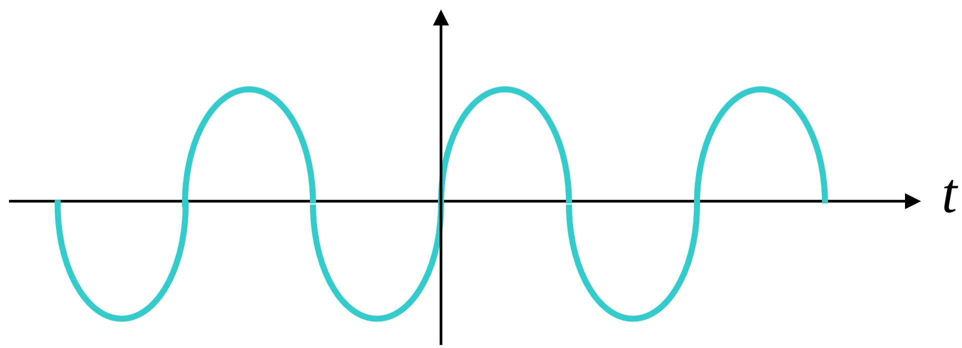
$$f(t) = t^3$$



$$f(t) = t$$

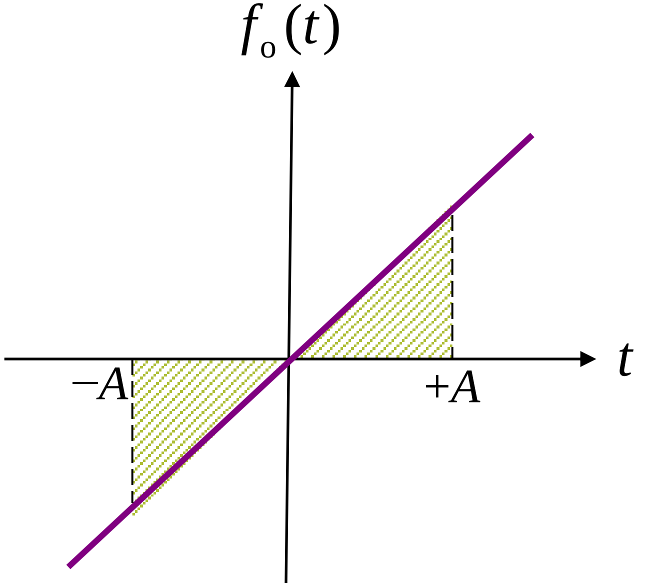


$$f(t) = \sin t$$



Tek simetri (devam)

- $-A$ dan $+A$ ya **tek** bir fonksiyonun integrali sıfırdır



$$\int_{-A}^{+A} f_o(t)\,dt = 0$$

Çift ve tek fonksiyonlar

Çift ve tek fonksiyonların çarpım özellikleri:

- (çift) (çift) = (çift)
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- (çift) (tek) = (tek)
- (tek) (çift) = (tek)

Simetri

çift ve tek fonksiyonların özelliklerinden:

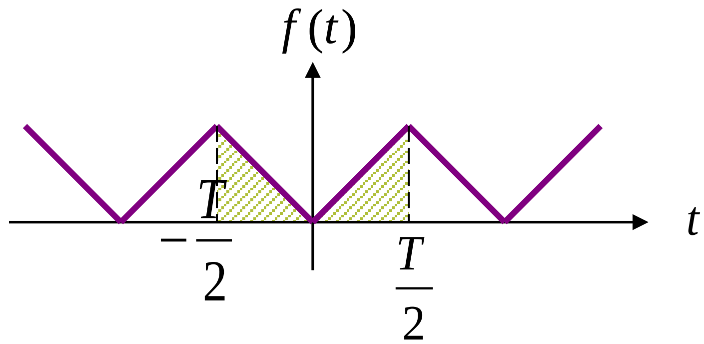
- çift periyodik bir fonksiyon için;

$$a_n = \frac{4}{T} \int_0^{T/2} f(t) \cos n\omega t dt \qquad b_n = 0$$

- tek periyodik bir fonksiyon için;

$$a_0 = a_n = 0 \qquad b_n = \frac{4}{T} \int_0^{T/2} f(t) \sin n\omega t dt$$

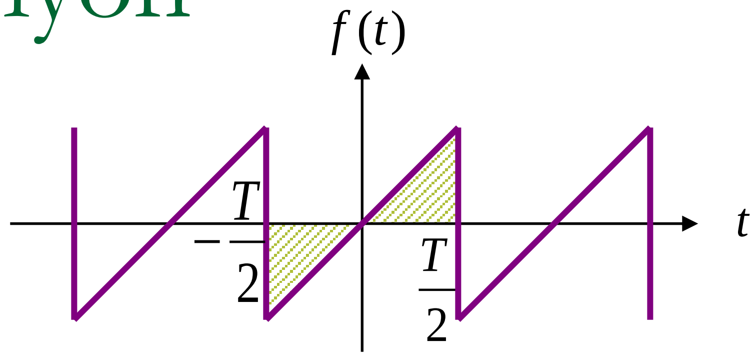
Çift fonksiyon



$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} \underbrace{\underbrace{f(t)}_{(\text{çift})}\,\underbrace{\cos n\omega t}_{(\text{çift})}}_{=\,(\text{çift})}\,dt = \frac{4}{T}\int_{0}^{T/2} f(t)\cos n\omega t\,dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} \underbrace{\underbrace{f(t)}_{(\text{çift})}\,\underbrace{\sin n\omega t}_{(\text{tek})}}_{=\,(\text{tek})}\,dt = 0$$

Tek fonksiyon



$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} \underbrace{f(t)}_{\text{(tek)}}\,dt = 0$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} \underbrace{\underbrace{f(t)}_{\text{(tek)}}\ \underbrace{\cos n\omega t}_{\text{(çift)}}}_{\substack{\parallel \\ \text{(tek)}}}\,dt = 0$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} \underbrace{\underbrace{f(t)}_{\text{(tek)}}\ \underbrace{\sin n\omega t}_{\text{(tek)}}}_{\substack{\parallel \\ \text{(çift)}}}\,dt = \frac{4}{T}\int_{0}^{T/2} f(t)\sin n\omega t\,dt$$

Örnek 4

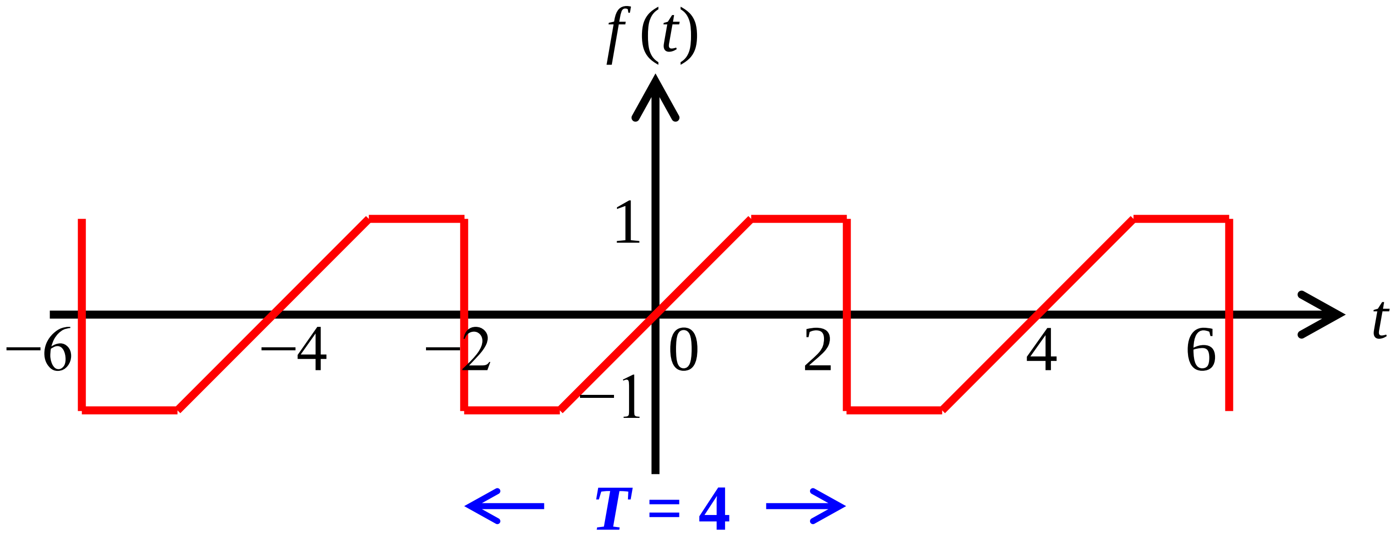
$$f(t) = \begin{cases} -1 & , \quad -2 < t < -1 \\ t & , \quad -1 < t < 1 \\ 1 & , \quad 1 < t < 2 \end{cases}$$

$$f(t+4) = f(t)$$

$f(t)$'nin grafiğini çiziniz, $-6 \leq t \leq 6.$

$f(t)$'nin Fourier serisi açılımını hesaplayınız

Çözüm



$$\omega = \frac{2\pi}{T} = \frac{\pi}{2}$$

Katsayıları hesaplayalım. $f(t)$ tek fonksiyon olduğundan,

$$a_0 = \frac{2}{T}\int_{-2}^{2} f(t)dt = 0$$

ve

$$a_n = \frac{2}{T}\int_{-2}^{2} f(t)\cos n\omega t dt = 0$$

$$b_n = \frac{2}{T}\int_{-2}^{2} f(t)\sin n\omega t dt = \frac{4}{T}\int_{0}^{2} f(t)\sin n\omega t dt$$

$$= \frac{4}{4}\left[\int_{0}^{1} t\sin n\omega t dt + \int_{1}^{2} 1\sin n\omega t dt\right]$$

$$= \left[-\frac{t\cos n\omega t}{n\omega}\right]_0^1 + \int_0^1 \frac{\cos n\omega t}{n\omega} dt + \left[-\frac{\cos n\omega t}{n\omega}\right]_1^2$$

$$= -\frac{\cos n\omega}{n\omega} + \left[\frac{\sin n\omega t}{n^2\omega^2}\right]_0^1 - \frac{\cos 2n\omega - \cos n\omega}{n\omega}$$

$$= -\frac{\cos 2n\omega}{n\omega} + \frac{\sin n\omega}{n^2\omega^2} = -\frac{2\cos n\pi}{n\pi}$$

$$\sin 2n\omega = \sin n\pi = 0$$

Sonuçta,

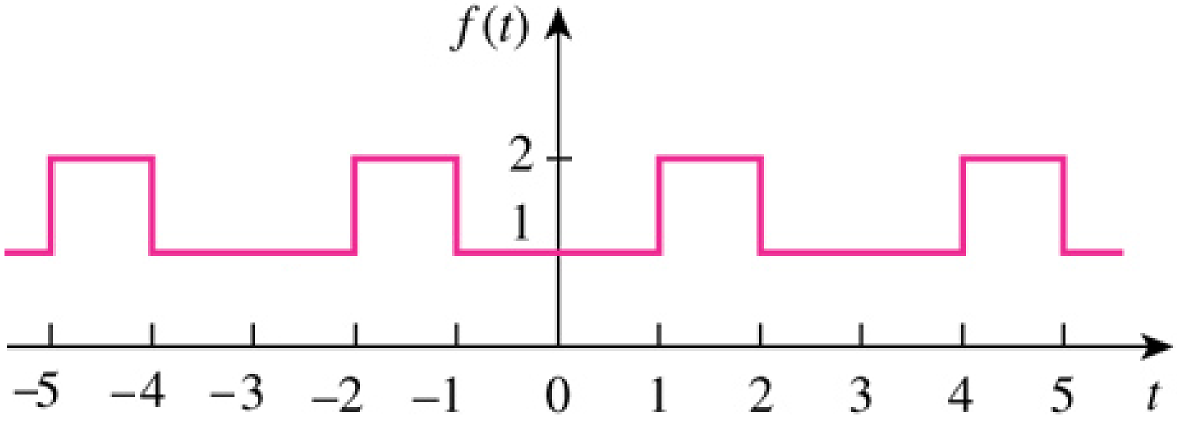
$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\omega t+b_n\sin n\omega t)$$

$$=\sum_{n=1}^{\infty}\left(-\frac{2\cos n\pi}{n\pi}\right)\sin\frac{n\pi t}{2}$$

$$=2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\pi}\sin\frac{n\pi t}{2}$$

Örnek 5

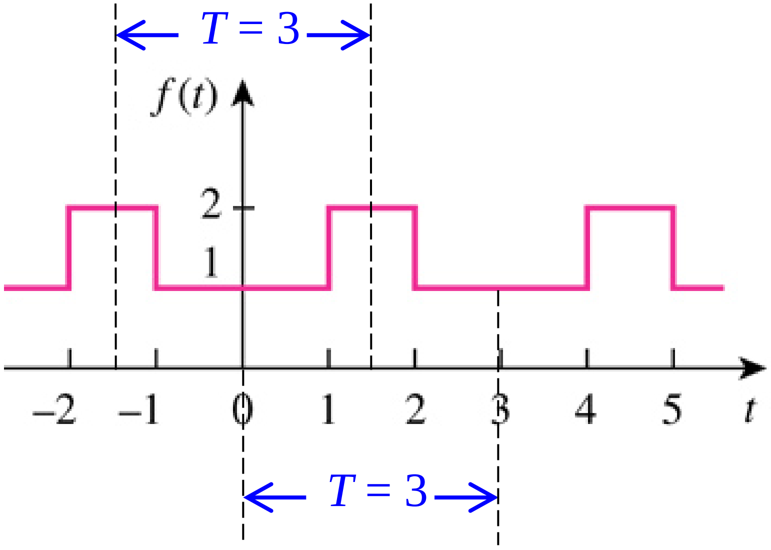
$f(t)$'nin Fourier serisi açlımını hesaplayınız.



Çözüm

Fonksiyonu tarif edelim;

$$f(t)=\begin{cases}1 & , \quad 0<t<1 \\ 2 & , \quad 1<t<2 \\ 1 & , \quad 2<t<3\end{cases}$$



$$f(t+3)=f(t)$$

ve $\qquad \omega=\frac{2\pi}{T}=\frac{2\pi}{3}$

Katsayıları hesaplayalım.

$$a_0 = \frac{2}{T}\int_0^3 f(t)dt = \frac{2}{3}\left[\int_0^1 1dt + \int_1^2 2dt + \int_2^3 1dt\right] = \frac{2}{3}\left[(1-0)+2(2-1)+(3-2)\right] = \frac{8}{3}$$

Ya da, $f(t)$ çift bir fonksiyon olduğundan,

$$a_0 = \frac{2}{T}\int_0^3 f(t)dt = \frac{4}{T}\int_0^{3/2} f(t)dt = \frac{4}{3}\left[\int_0^1 1dt + \int_1^{3/2} 2dt\right] = \frac{4}{3}\left[(1-0)+2\left(\frac{3}{2}-1\right)\right] = \frac{8}{3}$$

Veya, basitçe

$$a_0 = \frac{2}{T}\int_0^3 f(t)dt = \frac{2}{T}\times\begin{pmatrix}\text{Bir periyod boyunca}\\ \text{toplam alan}\end{pmatrix} = \frac{2}{3}\times 4 = \frac{8}{3}$$

$$a_n = \frac{2}{T}\int_0^3 f(t)\cos n\omega t dt = \frac{4}{T}\int_0^{3/2} f(t)\cos n\omega t dt$$

$$= \frac{4}{3}\left[\int_0^1 1\cos n\omega t dt + \int_1^{3/2} 2\cos n\omega t dt\right]$$

$$= \frac{4}{3}\left[\frac{\sin n\omega t}{n\omega}\right]_0^1 + \frac{4}{3}\left[\frac{2\sin n\omega t}{n\omega}\right]_1^{3/2}$$

$$= \frac{4}{3n\omega}\left[\sin n\omega + 2\left(\sin\frac{3n\omega}{2} - \sin n\omega\right)\right]$$

$$= \frac{4}{3n\omega}\left(2\sin\frac{3n\omega}{2} - \sin n\omega\right) \quad ; \quad \omega = \frac{2\pi}{3}$$

$$= \frac{2}{n\pi}\left(2\sin n\pi - \sin\frac{2n\pi}{3}\right) = -\frac{2}{n\pi}\sin\frac{2n\pi}{3}$$

ve $b_n = 0$ $f(t)$ çift bir fonksiyon olduğundan.

Sonuçta,

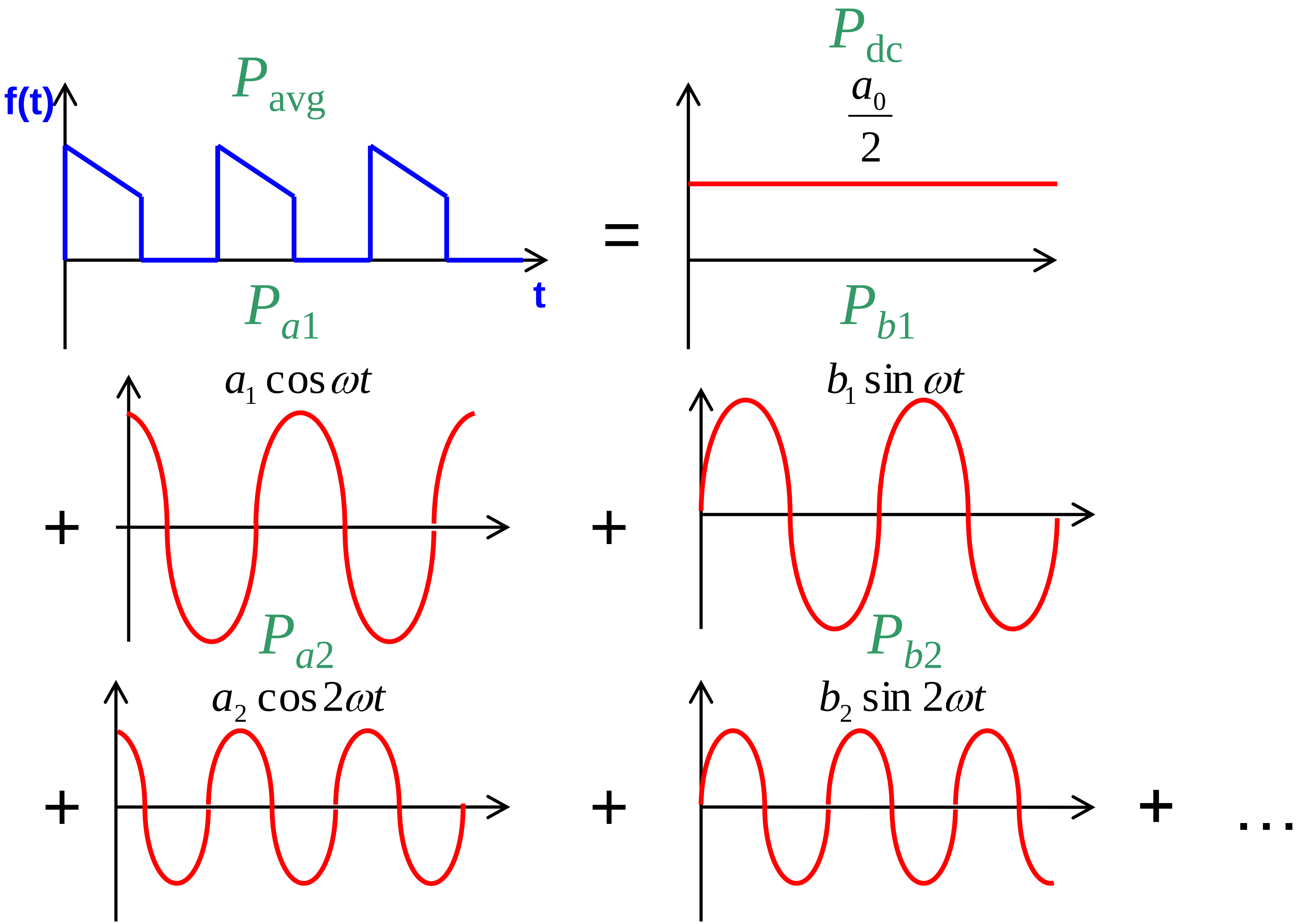
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

$$= \frac{4}{3} + \sum_{n=1}^{\infty} \left(-\frac{2}{n\pi} \sin \frac{2n\pi}{3} \right) \cos \frac{2n\pi t}{3}$$

$$= \frac{4}{3} - \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\frac{1}{n} \sin \frac{2n\pi}{3} \right) \cos \frac{2n\pi t}{3}$$

Parseval Teoremi

- Parserval teoremi periyodik bir sinyaldeki ortalama gücün, sinyalin DC bileşenindeki ortalama güç ve harmoniklerindeki ortalama güçlerin toplamına eşit olduğunu ifade eder.



P_{avg}
f(t)
t
=
P_{dc}
$\frac{a_0}{2}$
P_{a1}
P_{b1}
$a_1 \cos \omega t$
$b_1 \sin \omega t$
+
+
P_{a2}
P_{b2}
$a_2 \cos 2\omega t$
$b_2 \sin 2\omega t$
+
+
+
$\cdots$

- Sinüzoidal sinyal için (kosinüs ve sinüs),

$$P = \frac{V_{\mathrm{rms}}^2}{R} = \frac{\left(V_{\mathrm{peak}} \big/ \sqrt{2} \right)^2}{R} = \frac{1}{2}\frac{V_{\mathrm{peak}}^2}{R}$$

- Sadelik açısından sıklıkla, $R = 1\Omega$, olarak alırız,

$$P = \frac{1}{2} V_{\mathrm{peak}}^2$$

- Sinüzoidal sinyal için (kosinüs ve sinüs),

$$P_{\text{avg}} = P_{\text{dc}} + P_{a_1} + P_{b_1} + P_{a_2} + P_{b_2} + \dots$$

$$= \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}a_1^2 + \frac{1}{2}b_1^2 + \frac{1}{2}a_2^2 + \frac{1}{2}b_2^2 + \dots$$

$$\therefore P_{\text{avg}} = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Üstel Fourier serileri

- Euler eşitliğinden,

$$e^{\pm jx} = \cos x \pm j \sin x$$

dolayısıyla

$$\cos x = \frac{e^{jx} + e^{-jx}}{2} \qquad \text{ve} \qquad \sin x = \frac{e^{jx} - e^{-jx}}{j2}$$

Fourier serisi gösterimi aşağıdaki gibi olur;

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\omega t+b_n\sin n\omega t)$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\left(\frac{e^{jn\omega t}+e^{-jn\omega t}}{2}\right)+b_n\left(\frac{e^{jn\omega t}-e^{-jn\omega t}}{j2}\right)\right]$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\left(\frac{e^{jn\omega t}+e^{-jn\omega t}}{2}\right)-jb_n\left(\frac{e^{jn\omega t}-e^{-jn\omega t}}{2}\right)\right]$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[\left(\frac{a_n-jb_n}{2}\right)e^{jn\omega t}+\left(\frac{a_n+jb_n}{2}\right)e^{-jn\omega t}\right]$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(\frac{a_n-jb_n}{2}\right)e^{jn\omega t}+\sum_{n=1}^{\infty}\left(\frac{a_n+jb_n}{2}\right)e^{-jn\omega t}$$

Burada, $c_n = \dfrac{a_n - jb_n}{2}$, $c_{-n} = \dfrac{a_n + jb_n}{2}$

Diyelim ve $c_0 = \dfrac{a_0}{2}$ Dolayısıyla,

$$f(t) = \underbrace{\frac{a_0}{2}}_{c_0} + \sum_{n=1}^{\infty} \underbrace{\left(\frac{a_n - jb_n}{2}\right)}_{c_n} e^{jn\omega t} + \sum_{n=1}^{\infty} \underbrace{\left(\frac{a_n + jb_n}{2}\right)}_{c_{-n}} e^{-jn\omega t}$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega t} + \sum_{n=1}^{\infty} c_{-n} e^{-jn\omega t}$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega t} + \sum_{n=-1}^{-\infty} c_n e^{jn\omega t}$$

$$= \sum_{n=-\infty}^{-1} c_n e^{jn\omega t} + c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega t} = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t}$$

Sonra, c_n katsayısı,

$$c_n = \frac{a_n - jb_n}{2}$$

$$= \frac{1}{2}\frac{2}{T}\int_0^T f(t)\cos n\omega t dt - \frac{j}{2}\frac{2}{T}\int_0^T f(t)\sin n\omega t dt$$

$$= \frac{1}{T}\left[\int_0^T f(t)\cos n\omega t dt - j\int_0^T f(t)\sin n\omega t dt\right]$$

$$= \frac{1}{T}\int_0^T f(t)[\cos n\omega t - j\sin n\omega t]dt$$

$$= \frac{1}{T}\int_0^T f(t)e^{-jn\omega t}dt$$

- Çoğu durumda kompleks Fourier serileri trigonometrik Fourier serilerinden daha kolay elde edilir.
- Özetle, kompleks ve trigonometrik Fourier serileri arasındaki ilişki:

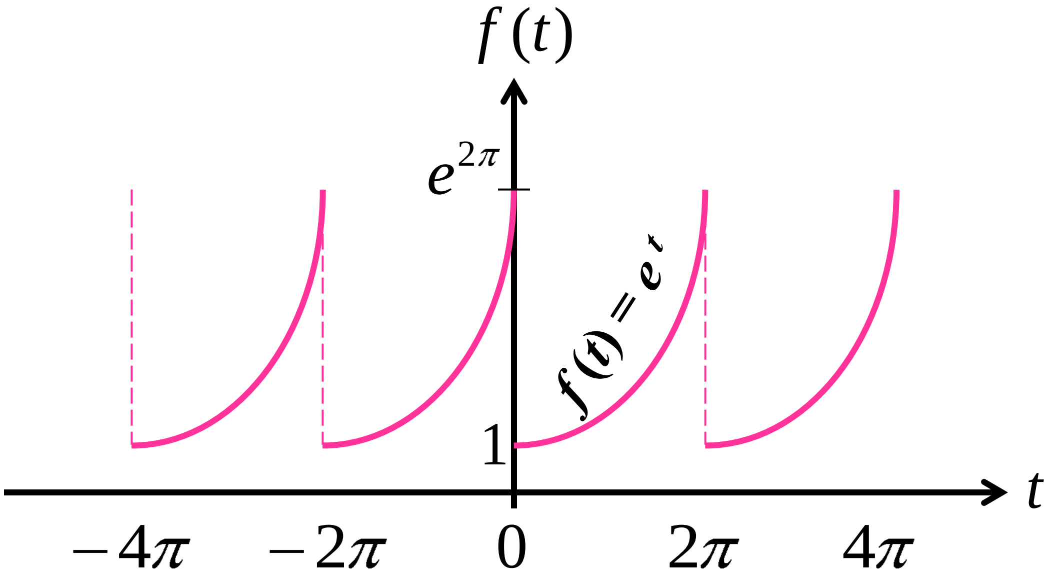
$$c_0 = \frac{a_0}{2} = \frac{1}{T}\int_0^T f(t)dt \qquad c_n = \frac{1}{T}\int_0^T f(t)e^{-jn\omega t}dt$$

$$c_n = \frac{a_n - jb_n}{2}$$

$$c_{-n} = \frac{a_n + jb_n}{2}$$ Ya da $$c_{-n} = \bar{c}_n$$

Örnek 6

Aşağıdaki fonksiyonun kompleks Fourier serisini bulunuz



Çözüm

$T = 2\pi$ Dolayısıyla $\omega = 1$

$$c_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^t\,dt$$

$$= \frac{1}{2\pi}\left[e^t\right]_0^{2\pi} = \frac{e^{2\pi}-1}{2\pi}$$

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-jn\omega t}dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^t e^{-jnt}dt = \frac{1}{2\pi}\int_0^{2\pi} e^{(1-jn)t}dt$$

$$= \frac{1}{2\pi}\left[\frac{e^{(1-jn)t}}{1-jn}\right]_0^{2\pi}$$

$$= \frac{e^{2\pi(1-jn)}-1}{2\pi(1-jn)} = \frac{e^{2\pi}e^{-j2n\pi}-1}{2\pi(1-jn)} = \frac{e^{2\pi}-1}{2\pi(1-jn)}$$

dolayısıyla

$$e^{-j2n\pi} = \cos 2n\pi - j\sin 2n\pi = 1-0 = 1$$

$$c_n\Big|_{n=0} = \left.\frac{e^{2\pi}-1}{2\pi(1-jn)}\right|_{n=0} = \frac{e^{2\pi}-1}{2\pi} = c_0$$

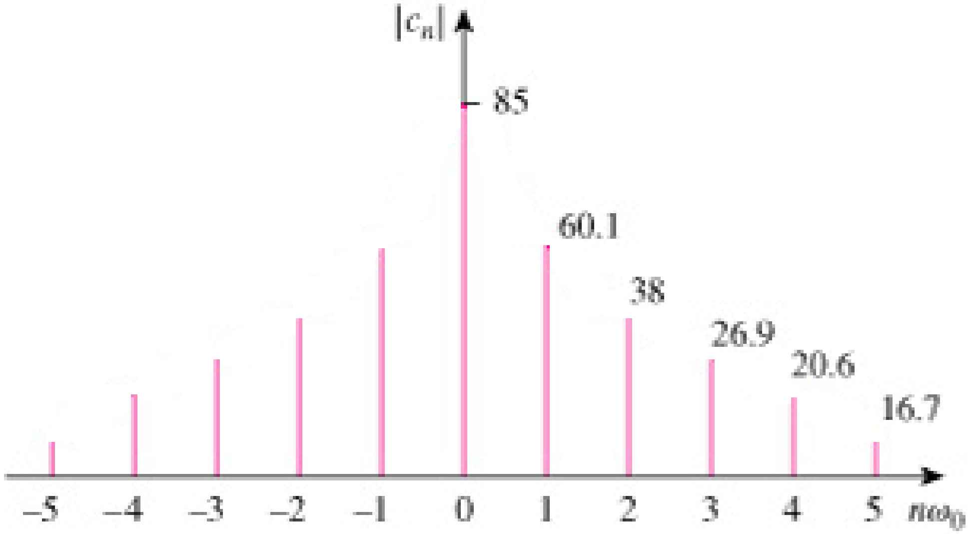
Sonuçta,

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t} = \sum_{n=-\infty}^{\infty} \frac{e^{2\pi}-1}{2\pi(1-jn)} e^{jnt}$$

*Not: c_0 , c_n de $n = 0$ konularak hesaplanabilirse de, bazen bu mümkün olmayabilir. Dolayısıyla, c_0'ı tek başına hesaplamak daha iyi olabilir.

c_n kompleks bir terimdir, ve $n\omega$'ye bağlıdır. Dolayısıyla, $n\omega$ 'ye karşılık $|c_n|$ grafiğini çizebiliriz.

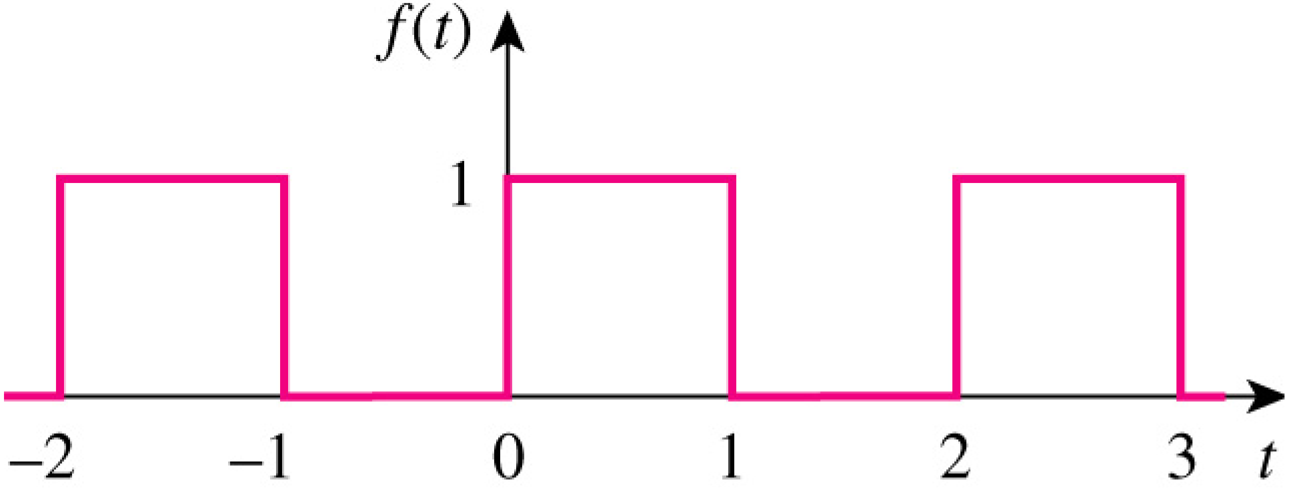
$$|c_n| = \frac{e^{2\pi} - 1}{2\pi\sqrt{1+n^2}}$$



Başka deyişle, (t) zaman bölgesindeki $f\,(t)$ fonksiyonunu, $(n\omega)$ frekans bölgesindeki c_n fonksiyonuna dönüştürdük.

Örnek 7

Örnek 1'deki fonksiyonun kompleks Fourier serisini hesaplayınız.



Çözüm

$$\omega = \pi$$

$$c_0 = \frac{1}{T}\int_0^T f(t)dt = \frac{1}{2}\int_0^1 1dt = \frac{1}{2}$$

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-jn\omega t}dt = \frac{1}{2}\int_0^1 1e^{-jn\pi t}dt + \int_1^2 0$$

$$= \frac{1}{2}\left[\frac{e^{-jn\pi t}}{-jn\pi}\right]_0^1 = \frac{j}{2n\pi}(e^{-jn\pi} - 1)$$

Fakat $e^{-jn\pi} = \cos n\pi - j\sin n\pi = \cos n\pi = (-1)^n$

Böylece, $c_n = \dfrac{j}{2n\pi}(e^{-jn\pi} - 1)$

$$= \frac{j}{2n\pi}[(-1)^n - 1] = \begin{cases} -j/n\pi & , \quad n \text{ tek} \\ 0 & , \quad n \text{ çift} \end{cases}$$

*Burada $c_n\big|_{n=0} \neq c_0$.

Dolayısıyla, $f(t) = \displaystyle\sum_{n=-\infty}^{\infty} c_n e^{jn\omega t} = \frac{1}{2} - \sum_{\substack{n=-\infty \\ n\neq 0 \\ n \text{ tek}}}^{\infty} \frac{j}{n\pi} e^{jn\pi t}$

Grafik çizimi aşağıdadır,

$$c_0 = \frac{1}{2} \qquad\qquad |c_n| = \begin{cases} \dfrac{1}{n\pi}, & n \text{ tek} \\ 0, & n \text{ çift} \end{cases}$$