A Solution for the Stresses Surrounding a Pressurized Crack in a Half-Space

Problem and solution method outlined in (Strack, 2014)

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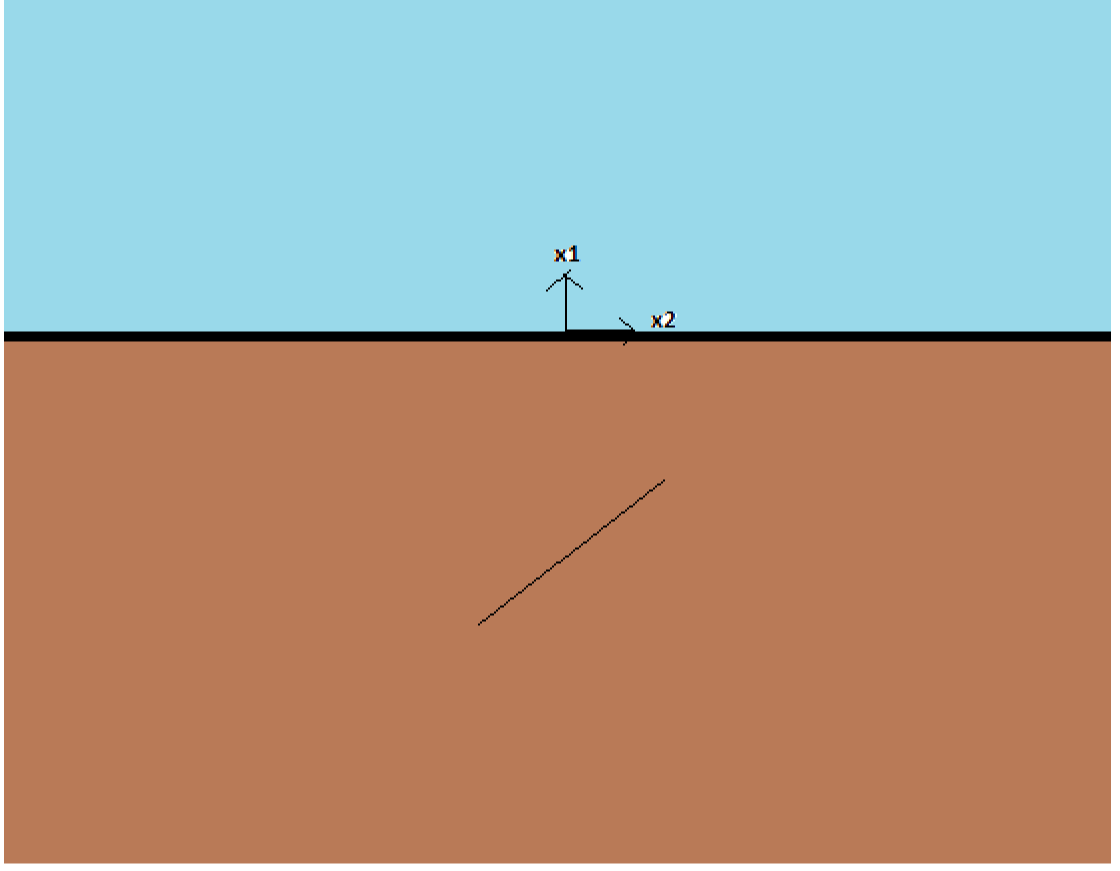
Introduction to the Problem



Three components: crack, gravity, and half-space

Muskhelishvili-Kolosov functions for displacement and stress:

$$w = \frac{1}{4G}\left\{\bar{z}\Phi'(z) + \kappa\bar{\Phi}(\bar{z}) + \Psi(z) + 2(1-2\nu)\bar{B}_2\right\}$$

$$\tau^{11} = -\bar{z}\Phi''(z) + \Psi'(z)$$

$$\tau^{12} = \Phi'(z) + \bar{\Phi}'(\bar{z}) + \left(\bar{B}_1 + B_1\right)$$

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These three equations incorporate both Hooke’s Law and equilibrium.

Start with a crack:

Boundary conditions:

- Pressure is normal to the length of the crack on the crack.
- No shear stress applied at the crack boundary.
- Components of the stress tensor are continuous across the crack.

In the direction normal to the crack:

$$\tau^{11} = p$$

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In the direction along the crack:

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On the plain of the crack:

$$\underset{\beta}{T}^{1} = \underset{\beta}{t_s} + i\,\underset{\beta}{t_n} = ip$$

Coordinate transformation:

$$Z = 2\frac{z - z_0}{L}e^{-i\beta}$$

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Displacement on a plane:

$$\underset{\beta}{w} = \frac{1}{4G} e^{i\beta} \left\{ -\bar{z}_0 \Phi'(z) - \frac{L}{2} e^{-i\beta} \bar{Z} \Phi'(z) + \kappa \bar{\Phi}(\bar{z}) + \Psi(z) \right\}$$

Choose:

$$\Psi(z) = \frac{L}{2} e^{-i\beta} Z \Phi'(z) + \bar{z}_0 \Phi'(z) + e^{-i\beta} \psi(z)$$

Introduce:

$$\phi\left(z\right)=e^{-i\beta}\Phi\left(z\right)$$

The displacement becomes:

$$\underset{\beta}{w}=\frac{1}{4G}\left\{\frac{L}{2}\left(Z-\overline{Z}\right)\phi'\left(z\right)e^{-i\beta}+\kappa\overline{\phi}\left(\overline{z}\right)+\psi\left(z\right)\right\}$$

The traction becomes:

$$\underset{\beta}{T}^1 = \frac{i}{2}\left\{\frac{L}{2}\left(Z-\overline{Z}\right)e^{2i\beta}\phi''(z)+e^{i\beta}\psi'(z)-e^{-i\beta}\overline{\phi}'(\overline{z})\right\}$$

And the jump across the crack is:

$$\left[\underset{\beta}{T}^1\right]=\frac{i}{2}\left\{e^{i\beta}\left[\psi'(z)\right]-e^{-i\beta}\left[\overline{\phi}'(\overline{z})\right]\right\}$$

So:

$$e^{-i\beta}\left[\overline{\phi}'(\overline{z})\right]=e^{i\beta}\left[\psi'(z)\right]$$

Along the crack:

$$\left[\frac{d\bar{\phi}(\bar{Z})}{dZ}\right] = -\left[\frac{d\bar{\phi}(Z)}{dZ}\right] = \left[\frac{d\psi}{dZ}\right]$$

$$\left[\bar{\phi}(Z)\right] = -\left[\psi(Z)\right] + C$$

And so:

$$\psi(Z) = \frac{a_0 + ib_0}{2\pi i} F_0(Z) \qquad \bar{\phi}(Z) = -\frac{a_0 - ib_0}{2\pi i} \bar{F}_0(Z)$$

$$F_0(Z) = \ln\frac{Z-1}{Z+1}$$

Next, the gravity:

Boundary conditions:

- Account for the weight of material
- Tractions along the boundary of the half-space are zero

Revisiting the expression for the displacement:

$$w = \frac{1}{4G}\left\{\bar{z}\Phi'(z) + \kappa\bar{\Phi}(\bar{z}) + \Psi(z) + 2(1-2\nu)\bar{B}_2\right\}$$

We want the z bar term to disappear, so we choose:

$$\Psi(z) = z\Phi'(z) + \psi(z)$$

The body force:

$$B_1 = i\rho g z \qquad B_2 = \frac{1}{2} i\rho g z^2$$

Choose:

$$\Phi(z) = -i\rho g \frac{1-2\nu}{\kappa+1}(\bar{z} - z)$$

$$\psi(z) = i\rho g \frac{1-2\nu}{\kappa+1} z^2$$

So that:

$$\tau^{11} = 2i\frac{1-2\nu}{\kappa+1}\left(\overline{z}-z\right) \qquad \tau^{12} = 2i\rho g\frac{1}{\kappa+1}\left(z-\overline{z}\right)^2$$

$$w = -1\rho g\frac{1-2\nu}{\kappa+1}\frac{1}{4G}\left(z-\overline{z}\right)^2$$

Then the half-space:

Boundary conditions:

- Traction is 0 everywhere everywhere above the half-space.

Coordinate transformation:

(Conformal mapping: Strack and Verruijt (2002))

$$Z = \frac{z}{d} \qquad \chi = \frac{Z+i}{Z-i}$$

So that:

$$\chi = e^{i\theta}$$

on the half-space boundary

And:

$$\chi\bar{\chi} \leq 1$$

underneath the half-space boundary

We can write psi' and Phi' as two Taylor series inside the circle:

$$\psi'(z) = \sum_{n=0}^{\infty} a_n \chi^n \qquad \Phi'(z) = \sum_{n=0}^{\infty} b_n \chi^n$$

So that on the boundary we can write the tractions as:

$$t_1 - it_2 = \frac{1}{2} i \left[\sum_{n=0}^{\infty} a_n e^{i\theta n} - \sum_{n=0}^{\infty} \overline{b}_n e^{-i\theta n} \right]$$

We can introduce two new functions:

$$P(z) = \Phi'(z) + \Psi'(z)$$
$$Q(z) = \Phi'(z) - \Psi'(z)$$

So that:

$$\mathrm{Im}\left(P(z)\right) = -2t_1$$
$$\mathrm{Re}\left(Q(z)\right) = 2t_2$$

Re-write P and Q:

$$P(z) = \sum_{n=0}^{\infty} \alpha_n \chi^n(z)$$

$$Q(z) = \sum_{n=0}^{\infty} \beta_n \chi^n(z)$$

Use a Couchy integral to find the constants:

$$\beta_n = \frac{1}{2\pi} \sum_{n=0}^{2\pi} 2 \underset{other}{t_2}(z) e^{in\theta} \Delta\theta$$

$$\alpha_n = \frac{i}{2\pi} \sum_{n=0}^{2\pi} -2 \underset{other}{t_1}(z) e^{-in\theta} \Delta\theta$$

Computer Code

Object Oriented Programming (OOP):

- Define objects
- Perform operations on the objects

Basic outline:

Object types:

- World
- Elements
  - Gravity
  - Half-space
  - Crack

World

Properties:

- gravity
- G
- rho
- nu
- elements = {}
- Kappa

Methods:

- Constructor
- AddElement(element)
- SolveWorld()
- Traction(z)
- w(z)
- [t11, t12] = calcTauSum(elementlist,z)

Element

Properties:

- constants
- locations
- beta

Common Methods:

- solveforconstants(tau11Other, tau12Other)
- calcPsi(z)
- calcPhiPrime(z)
- calcPhibar(z)
- calcTau11(z, alpha)
- calcTau12(z, alpha)
- calcB1 (z)
- calcB2 (z)

SolveWorld() (World class)

Take the list of elements that is in the *elements* property

while changecount > 0

for each element:

activeelement = select the active element

inactiveelements = construct an element list that contains all other elements

establish empty lists for tau11s and for tau12s. Length = length of…

…activeelement's location list

for each location in activeelement's location list:

[t11s, t12s]=call calctausum(inactiveelements, z)

save activeelement's old constants

call activeelement's solveforconstants(t11s, t12s)

check activeelement's new constants against the old constants. did they…

…change?

if so, increment changecount.

end

calcTauSum(elementList, z, alpha) (World class)

```
tau11 = 0
tau12 = 0
for each element in elementList
        tau11 = tau11 + calcTau11(z, alpha) (Element class)
        tau12 = tau11 + calcTau12(z, alpha) (Element class)
end
```

solveforconstants(inactiveelements' tau11s, inactiveelements' tau12s) (Gravity class)

```
end
```

solveforconstants(inactiveelements' tau11s, inactiveelements' tau12s) (Crack class)

z = Crack's locations property (just one location-the middle of the crack)

$$A = \begin{bmatrix} 2\,\mathrm{Re}\left(\Phi'(z)\right) & 2\,\mathrm{Im}\left(\Phi'(z)\right) \\ e^{-i\beta}\left(\phi'(z)+\psi'(z)\right) & ie^{-i\beta}\left(\phi'(z)-\psi'(z)\right) \end{bmatrix}$$

$$B = \begin{bmatrix} -tau12s \\ p - tau11s \end{bmatrix}$$

new constants = A\B

end

solveforconstants(inactiveelements' tau11s, inactiveelements' tau12s) (HalfSpace class)

(Delta theta, N, degrees will all be additional properties of this element)

for each degree: (the "n" part of beta_n)

betam = 0

alpham = 0

for each location in HalfSpace's location array (populated with N points…

…around the unit circle made by $\chi\bar{\chi} = 1$)

take the tau11 and tau12 that corresponds to the correct point

find the traction

real part of the traction is t1, the imaginary part is t2.

betam = betam + t2*e^(i m theta)Delta theta

alpham = alpham - t1 *e((-i m theta) Delta theta

alpham = alpham * i /pi

betam = betam * 1/pi

put the alpham and betam in the appropriate place in the constants list