Quantum Multi-string Matching

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Presentation Overview

1. Problem Definition
2. Classical Algorithms
   - Existing Algorithms
   - Polynomial Matching
   - Polynomial Multiplication
3. Quantum Algorithms
   - Polynomial Multiplication with QFT
   - Qiskit Results

String Matching

- Text $S[0...n-1]$
- Pattern (or key) $P[0...m-1]$
- Strings from alphabet Σ
- Find set of indices i such that $S[i..i+m] = P$

Example:

- T = `"GTAT GATC TC"` (ignore spaces)
- P_1 = `"ATCT"`
- P_2 = `"TGAT"`
- P_3 = `"ACCC"`
- P_1 matches at 5, P_2 matches at 3, P_3 has no matches

- Multi-string matching: search $P_1, P_2, ...$ in S simultaneously

Classical Algorithms

- Brute Force $\longrightarrow O(mn)$
- Boyer-Moore $\longrightarrow O(n+m)$, $O(mn)$ worst case
- Knuth-Morris-Pratt $\longrightarrow O(n+m)$ worst case
- Suffix Tree/Array $\longrightarrow O(m)/O(n\log n)$ + preproc
- Karp-Rabin - rolling hash $\longrightarrow O(n+m)$ expected

<table>
<tr><td>G</td><td>T</td><td>A</td><td>T</td><td>G</td><td>A</td><td>T</td><td>C</td><td>T</td><td>C</td></tr>
<tr><td colspan="5">hash-value 84</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td colspan="5">hash-value 194</td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td colspan="5">hash-value 6</td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td colspan="5">hash-value 18</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td colspan="5">hash-value 95</td><td></td></tr>
</table>

- Polynomial Matching $\longrightarrow O(n\log n)$

Polynomial Matching

Calculate a fingerprint for P and every m character sequence in S; matching fingerprints suggest pattern match!

$$\begin{aligned} A &\mapsto -3 \\ C &\mapsto 5 \\ G &\mapsto -7 \\ T &\mapsto 11 \end{aligned}$$

$$\begin{aligned} \texttt{"ATCT"} &\longrightarrow \mathrm{T}x^3 + \mathrm{C}x^2 + \mathrm{T}x + \mathrm{A} \\ &\longrightarrow 11x^3 + 5x^2 + 11x - 3 \\ \texttt{"GTAT GATC TC"} &\longrightarrow -7x^{15} + 11x^{14} - 3x^{13} + \ldots + 11x^7 + 5x^6 \end{aligned}$$

Note that P encoding is reversed (similar to convolution)

Polynomial Matching

Given polynomials

$$P(x) = \sum_{i=0}^{m} a_i x^i, S(x) = \sum_{j=0}^{n} b_j x^j$$

then

$$R(x) = \sum_{k=0}^{m+n} c_k x^k$$

where

$$c_k = \sum_{i=0}^{k} a_i b_{k-i}, \quad \text{for } 0 \leq k \leq m+n,$$

with the convention that $a_i = 0$ for $i > m$ and $b_j = 0$ for $j > n$.

Polynomial Matching

Coefficients of $R(x)$ correspond to "dot products" between substrings.

$$
\begin{aligned}
P(x) &= 11x^3 + 5x^2 + 11x - 3 \\
S(x) &= -7x^{15} + 11x^{14} - 3x^{13} + ... + 11x^7 + 5x^6 \\
R(x) &= ... + 71x^8 + 0x^9 + 276x^{10} + 98x^{11} - 4x^{12} + ...
\end{aligned}
$$

Degrees map to index:

$$
\begin{aligned}
15 &\text{ yields index } 0 \\
14 &\text{ yields index } 1 \\
&\ldots \\
11 &\text{ yields index } 4 \\
10 &\text{ yields index } 5 \\
9 &\text{ yields index } 6
\end{aligned}
$$

Polynomial Matching

Coefficients of $R(x)$ correspond to "dot products" between substrings.

$$P(x) = 11x^3 + 5x^2 + 11x - 3$$
$$S(x) = -7x^{15} + 11x^{14} - 3x^{13} + \dots + 11x^7 + 5x^6$$
$$R(x) = \dots + 71x^8 + 0x^9 + 276x^{10} + 98x^{11} - 4x^{12} + \dots$$

Notice that $\|P\|^2 = 11^2 + 5^2 + 11^2 + (-3)^2 = 276$

We get exact fingerprint match for single patterns. Let's extend to multiple patterns...

Polynomial Matching

Add patterns together:

$$P(x) = P_1(x) + P_2(x)$$
$$S(x) = -7x^{15} + 11x^{14} - 3x^{13} + ... + 11x^7 + 5x^6$$
$$R(x) = ... + 94x^8 + 240x^9 + 272x^{10} + 64x^{11} + 296x^{12} - 60x^{13} ...$$

No more exact matches, but higher values = likelier index.

Polynomial Multiplication

Multiple algorithms:

- Naïve polynomial multiplication $\longrightarrow O(n^2)$
- Karatsuba Algorithm $\longrightarrow O(n^{\log_3 2}) = O(n^{1.59})$
- FFT $\longrightarrow O(n \log n)$

Point Value Form

Theorem (Lagrange Interpolation)

For any n points $(x_i, y_i) \in \mathbb{R}^2$ with no two x_i the same, there exists a unique polynomial $A(x)$ of degree at most $n-1$ that interpolates these points.

Given polynomials $A(x)$ and $B(x)$ of degree n, we can represent them as

$$A : \{(x_0, A(x_0)), (x_1, A(x_1)), \ldots, (x_{n-1}, A(x_{n-1}))\}$$
$$B : \{(x_0, B(x_0)), (x_1, B(x_1)), \ldots, (x_{n-1}, B(x_{n-1}))\}$$

Then their product $C(x) = A(x)B(x)$ is

$$C : \{(x_0, A(x_0)B(x_0)), (x_1, A(x_1)B(x_1)), \ldots, (x_n, A(x_{n-1})B(x_{n-1}))\}$$

Polynomial Multiplication

Algorithm idea:

1. Evaluate $A(x)$ and $B(x)$ at n points
   - $O(n)$ multiplications per point
   - $O(n)$ points
   - $O(n^2)$ overall, no speedup
2. Multiply element-wise
3. Interpolate to find $C(x)$ coefficients

But this works for any n inputs. What if we try n^{th} roots of 1?

$$\hat{a}_k = A(\omega_n^k) = \sum_{j=0}^{n} a_j e^{\frac{2\pi ijk}{n}}$$

for $0 \leq k \leq n-1$. This is DFT!

Polynomial Multiplication with FFT

1. Apply FFT to the coefficients of $A(x)$ and $B(x)$
   With $\vec{a} = [a_0 \; a_1 \ldots a_{n-1}]^T$ and $\vec{b} = [b_0 \; b_1 \ldots b_{n-1}]^T$, then

$$\vec{\alpha} = \mathrm{FFT}(\vec{a}) \qquad O(n \log n)$$

$$\vec{\beta} = \mathrm{FFT}(\vec{b}) \qquad O(n \log n)$$

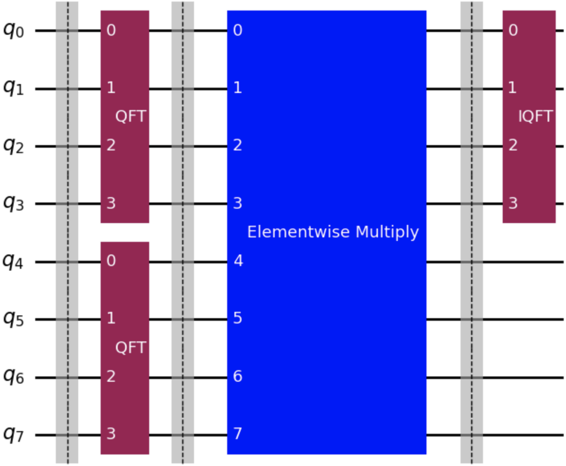
2. Perform the Hadamard (element-wise) product of coefficients

$$\begin{aligned} \vec{\gamma} &= \vec{\alpha} \odot \vec{\beta} \\ &= [\alpha_0\beta_0 \;\; \alpha_1\beta_1 \ldots \alpha_{n-1}\beta_{n-1}]^T \qquad O(n) \end{aligned}$$

3. Apply Inverse FFT

$$\vec{c} = \mathrm{IFFT}(\vec{\gamma}) \qquad O(n \log n)$$

Polynomial Multiplication with QFT



Polynomial multiplication for degree up to $n = 2^4 - 1 = 15$

Elementwise Multiply

Is there such a unitary operator? Define

$$|\alpha\rangle = \alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle + \alpha_3|11\rangle$$
$$|\beta\rangle = \beta_0|00\rangle + \beta_1|01\rangle + \beta_2|10\rangle + \beta_3|11\rangle$$
$$|\alpha \odot \beta\rangle = C\left(\alpha_0\beta_0|00\rangle + \alpha_1\beta_1|01\rangle + \alpha_2\beta_2|10\rangle + \alpha_3\beta_3|11\rangle\right)$$

for some normalization constant C

$$U|x\rangle|y\rangle = U|x\rangle|x \odot y\rangle$$

If $|y\rangle = \frac{1}{2}(|00\rangle+|01\rangle+|10\rangle+|11\rangle) \longrightarrow$ violates no-cloning theorem

Elementwise Multiply

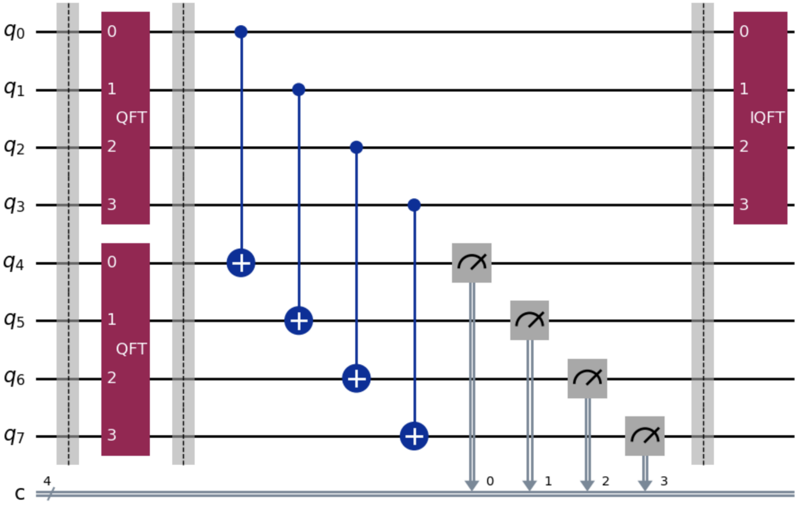
Allow probabilistic operator:

$$
\begin{aligned}
|\alpha\rangle|\beta\rangle = \alpha_0\beta_0|0000\rangle + \alpha_0\beta_1|0001\rangle + \dots \\
\alpha_1\beta_0|0100\rangle + \alpha_1\beta 1|0101\rangle + \dots \\
\dots + \alpha_2\beta_2|1010\rangle + \dots \\
\dots + \alpha_3\beta_3|1111\rangle + \dots
\end{aligned}
$$

Want when first two bits = last two bits.

Use CNOT to test for equality.

Elementwise Multiply



Elementwise Multiply

$$
\begin{aligned}
|\alpha\rangle|\beta\rangle = \alpha_0\beta_0|0000\rangle + \alpha_0\beta_1|0001\rangle + \dots \\
\alpha_1\beta_0|0100\rangle + \alpha_1\beta 1|0101\rangle + \dots \\
\dots + \alpha_2\beta_2|1010\rangle + \dots \\
\dots + \alpha_3\beta_3|1111\rangle + \dots
\end{aligned}
$$

$$
\begin{aligned}
\text{CNOT}_{0,2}\text{CNOT}_{1,3}\ |\alpha\rangle|\beta\rangle = \alpha_0\beta_0|0000\rangle + \alpha_0\beta_1|0001\rangle + \dots \\
\alpha_1\beta_0|0101\rangle + \alpha_1\beta 1|0100\rangle + \dots \\
\dots + \alpha_2\beta_2|1000\rangle + \dots \\
\dots + \alpha_3\beta_3|1100\rangle + \dots
\end{aligned}
$$

Measure the last two qubits, success when $|00\rangle \longrightarrow$ elementwise multiply in first two qubits!

Qiskit Results

Add some slides describing number of shots, probability distribution, runtime.

Current code uses statevector, not sure how many shots would be good. Need min 2^n just to get elementwise. TODO - try actual simulator

Try some larger texts

References

John Smith (2022)
Publication title
Journal Name 12(3), 45 – 678.

Annabelle Kennedy (2023)
Publication title
Journal Name 12(3), 45 – 678.