Quantum Multi-string Matching

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Presentation Overview

String Matching

- Text $S[0...n-1]$
- Pattern (or key) $P[0...m-1]$
- Strings from alphabet Σ
- Find set of indices i such that $S[i..i+m] = P$

Example:

- T = "GTAT GATC TC" (ignore spaces)
- P_1 = "ATCT"
- P_2 = "TGAT"
- P_3 = "ACCC"
- P_1 matches at 5, P_2 matches at 3, P_3 has no matches

- Multi-string matching: search $P_1, P_2, ...$ in S simultaneously

Classical Algorithms

- Brute Force $\longrightarrow O(mn)$
- Boyer-Moore $\longrightarrow O(n+m)$, $O(mn)$ worst case
- Knuth-Morris-Pratt $\longrightarrow O(n+m)$ worst case
- Suffix Tree/Array $\longrightarrow O(m)/O(n \log n)$ + preproc
- Karp-Rabin - rolling hash $\longrightarrow O(n+m)$ expected

G	T	A	T	G	A	T	C	T	C
hash-value 84									
	hash-value 194								
		hash-value 6							
			hash-value 18						
				hash-value 95					

- Polynomial Matching $\longrightarrow O(n \log n)$

Polynomial Matching

Calculate a fingerprint for P and every m character sequence in S; matching fingerprints suggest pattern match!

$$A \mapsto -3$$
$$C \mapsto 5$$
$$G \mapsto -7$$
$$T \mapsto 11$$

$$\text{"ATCT"} \longrightarrow Tx^3 + Cx^2 + Tx + A$$
$$\longrightarrow 11x^3 + 5x^2 + 11x - 3$$
$$\text{"GTAT GATC TC"} \longrightarrow -7x^{15} + 11x^{14} - 3x^{13} + \ldots + 11x^7 + 5x^6$$

Note that P encoding is reversed (similar to convolution)

Polynomial Matching

Given polynomials

$$P(x) = \sum_{i=0}^{m} a_i x^i, S(x) = \sum_{j=0}^{n} b_j x^j$$

then

$$R(x) = \sum_{k=0}^{m+n} c_k x^k$$

where

$$c_k = \sum_{i=0}^{k} a_i b_{k-i}, \quad \text{for } 0 \leq k \leq m + n,$$

with the convention that $a_i = 0$ for $i > m$ and $b_j = 0$ for $j > n$.

Polynomial Matching

Coefficients of $R(x)$ correspond to "dot products" between substrings.

$$P(x) = 11x^3 + 5x^2 + 11x - 3$$
$$S(x) = -7x^{15} + 11x^{14} - 3x^{13} + ... + 11x^7 + 5x^6$$
$$R(x) = ... + 71x^8 + 0x^9 + 276x^{10} + 98x^{11} - 4x^{12} + ...$$

Degrees map to index:

15 yields index 0

14 yields index 1

...

11 yields index 4

10 yields index 5

9 yields index 6

Polynomial Matching

Coefficients of $R(x)$ correspond to "dot products" between substrings.

$$P(x) = 11x^3 + 5x^2 + 11x - 3$$
$$S(x) = -7x^{15} + 11x^{14} - 3x^{13} + ... + 11x^7 + 5x^6$$
$$R(x) = ... + 71x^8 + 0x^9 + 276x^{10} + 98x^{11} - 4x^{12} + ...$$

Notice that $\|P\|^2 = 11^2 + 5^2 + 11^2 + (-3)^2 = 276$

We get exact fingerprint match for single patterns. Let's extend to multiple patterns...

Polynomial Matching

Add patterns together:

$$P(x) = P_1(x) + P_2(x)$$
$$S(x) = -7x^{15} + 11x^{14} - 3x^{13} + ... + 11x^7 + 5x^6$$
$$R(x) = ... + 94x^8 + 240x^9 + 272x^{10} + 64x^{11} + 296x^{12} - 60x^{13}...$$

No more exact matches, but higher values = likelier index.

Polynomial Multiplication

Multiple algorithms:

- Naïve polynomial multiplication $\longrightarrow O(n^2)$
- Karatsuba Algorithm $\longrightarrow O(n^{\log_3 2}) = O(n^{1.59})$
- FFT $\longrightarrow O(n \log n)$

Point Value Form

Theorem (Lagrange Interpolation)

For any n points $(x_i, y_i) \in \mathbb{R}^2$ with no two x_i the same, there exists a unique polynomial $A(x)$ of degree at most $n-1$ that interpolates these points.

Given polynomials $A(x)$ and $B(x)$ of degree n, we can represent them as

$$A : \{(x_0, A(x_0)), (x_1, A(x_1)), \ldots, (x_{n-1}, A(x_{n-1}))\}$$
$$B : \{(x_0, B(x_0)), (x_1, B(x_1)), \ldots, (x_{n-1}, B(x_{n-1}))\}$$

Then their product $C(x) = A(x)B(x)$ is

$$C : \{(x_0, A(x_0)B(x_0)), (x_1, A(x_1)B(x_1)), \ldots, (x_n, A(x_{n-1})B(x_{n-1}))\}$$

Polynomial Multiplication

Algorithm idea:

1. Evaluate $A(x)$ and $B(x)$ at n points
   - $O(n)$ multiplications per point
   - $O(n)$ points
   - $O(n^2)$ overall, no speedup
2. Multiply element-wise
3. Interpolate to find $C(x)$ coefficients

But this works for any n inputs. What if we try the nth roots of unity?

$$\hat{a}_k = A(\omega_n^k) = \sum_{j=0}^{n} a_j e^{\frac{2\pi ijk}{n}}$$

for $0 \leq k \leq n - 1$. This is DFT!

Polynomial Multiplication with FFT

1. Apply FFT to the coefficients of $A(x)$ and $B(x)$
   With $\vec{a} = [a_0 \ a_1 \ldots a_{n-1}]^T$ and $\vec{b} = [b_0 \ b_1 \ldots b_{n-1}]^T$, then

$$\vec{\alpha} = \text{FFT}(\vec{a}) \qquad O(n \log n)$$
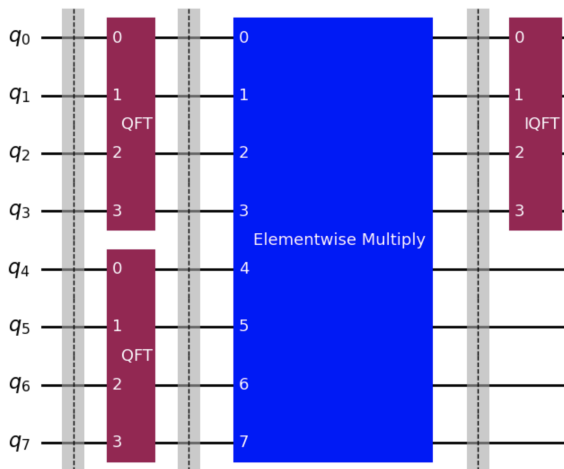$$\vec{\beta} = \text{FFT}(\vec{b}) \qquad O(n \log n)$$

2. Perform the Hadamard (element-wise) product of coefficients

$$\vec{\gamma} = \vec{\alpha} \odot \vec{\beta}$$
$$= [\alpha_0 \beta_0 \quad \alpha_1 \beta_1 \ldots \alpha_{n-1} \beta_{n-1}]^T \qquad O(n)$$

3. Apply Inverse FFT

$$\vec{c} = \text{IFFT}(\vec{\gamma}) \qquad O(n \log n)$$

Polynomial multiplication for degree up to $n = 2^4 - 1 = 15$

Is there such a unitary operator? Define

$$|\alpha\rangle = \alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle + \alpha_3|11\rangle$$
$$|\beta\rangle = \beta_0|00\rangle + \beta_1|01\rangle + \beta_2|10\rangle + \beta_3|11\rangle$$
$$|\alpha \odot \beta\rangle = C\left(\alpha_0\beta_0|00\rangle + \alpha_1\beta_1|01\rangle + \alpha_2\beta_2|10\rangle + \alpha_3\beta_3|11\rangle\right)$$

for some normalization constant C

$$U|x\rangle|y\rangle = U|x\rangle|x \odot y\rangle$$

If $|y\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right) \longrightarrow$ violates no-cloning theorem
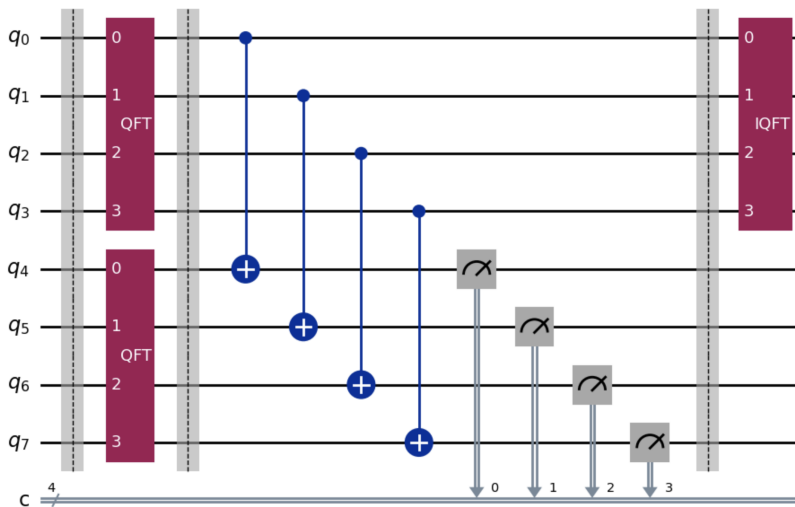
Allow probabilistic operator:

$$|\alpha\rangle|\beta\rangle = \alpha_0\beta_0|0000\rangle + \alpha_0\beta_1|0001\rangle + \dots$$
$$\alpha_1\beta_0|0100\rangle + \alpha_1\beta 1|0101\rangle + \dots$$
$$\dots + \alpha_2\beta_2|1010\rangle + \dots$$
$$\dots + \alpha_3\beta_3|1111\rangle + \dots$$

Want when first two bits = last two bits.

Use CNOT to test for equality.

Elementwise Multiply

Elementwise Multiply

$$|\alpha\rangle|\beta\rangle = \alpha_0\beta_0|0000\rangle + \alpha_0\beta_1|0001\rangle + \ldots$$
$$\alpha_1\beta_0|0100\rangle + \alpha_1\beta1|0101\rangle + \ldots$$
$$\cdots + \alpha_2\beta_2|1010\rangle + \ldots$$
$$\cdots + \alpha_3\beta_3|1111\rangle + \ldots$$

$$\text{CNOT}_{0,2}\text{CNOT}_{1,3}\,|\alpha\rangle|\beta\rangle = \alpha_0\beta_0|0000\rangle + \alpha_0\beta_1|0001\rangle + \ldots$$
$$\alpha_1\beta_0|0101\rangle + \alpha_1\beta1|0100\rangle + \ldots$$
$$\cdots + \alpha_2\beta_2|1000\rangle + \ldots$$
$$\cdots + \alpha_3\beta_3|1100\rangle + \ldots$$

Measure the last two qubits, success when $|00\rangle \longrightarrow$ elementwise multiply in first two qubits!

Algorithmic Analysis

Let n be the degree of the polynomial produced by multiplying $A(x)$ and $B(x)$.
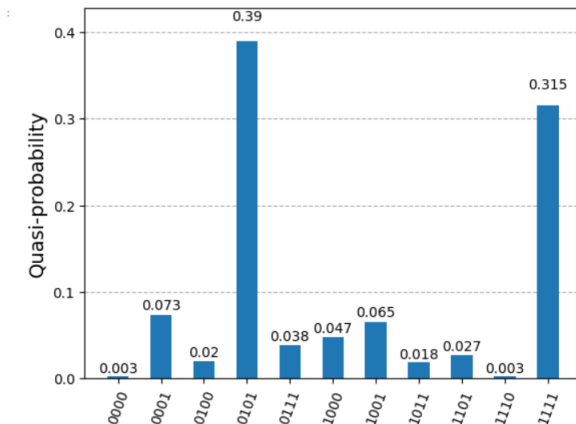
Initial QFT: $O(n^2)$ gates.

The element-wise multiplication: $O(n)$ CNOT gates, but requries an expected $O(2^n)$ measurements for a result where the last n qubits are all zero.

Final IQFT: $O(n^2)$ gates.

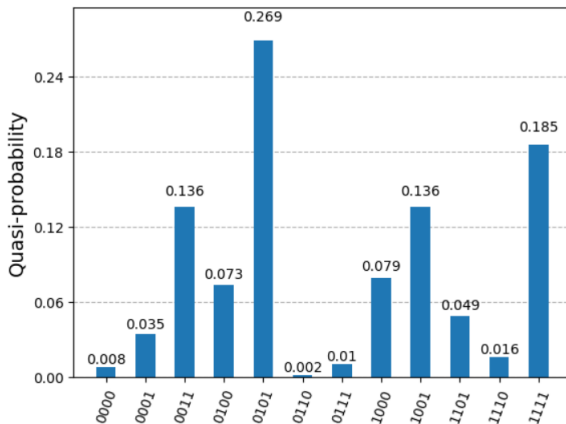So the total cost is $O(n^2)$ gates, but expected $O(2^n)$ shots.

Text: "`GTAT GATC TC`" Key: "`ATCT`"

Notice that the highest quasi-probability is for state $0101_2 = 5$!
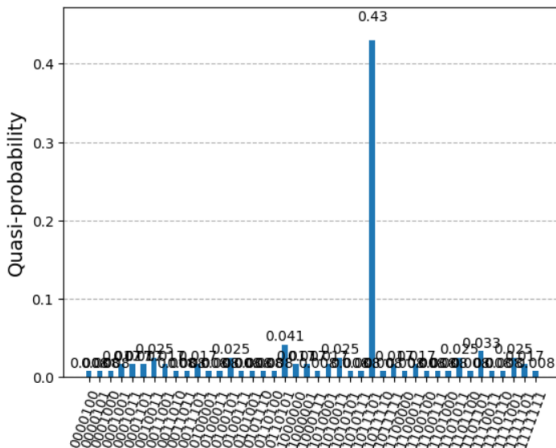
Multi-String Matching

- Text: "GTAT GATC TC"
- Key 1: "ATCT"
- Key 2: "TGAT"
- Key 3: "ACCC"

More Qiskit Results

- Text: 90 randomly generated characters $\in A, T, C, G$
- Key: 52 characters

Qiskit Results

Add some slides describing number of shots, probability distribution, runtime.

Current code uses statevector, not sure how many shots would be good. Need min 2^n just to get elementwise. TODO - try actual simulator

Try some larger texts

References

📄 John Smith (2022)
Publication title
Journal Name 12(3), 45 – 678.

📄 Annabelle Kennedy (2023)
Publication title
Journal Name 12(3), 45 – 678.