Quantum Multi-string Matching

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Presentation Overview

1. Problem Definition
2. Classical Algorithms
   - Existing Algorithms
   - Polynomial Matching
   - Polynomial Multiplication
3. Quantum Algorithms
   - Polynomial Multiplication with QFT

String Matching

- Text $S[0...n-1]$
- Pattern (or key) $P[0...m-1]$
- Strings from alphabet Σ
- Find set of indices i such that $S[i..i+m] = P$

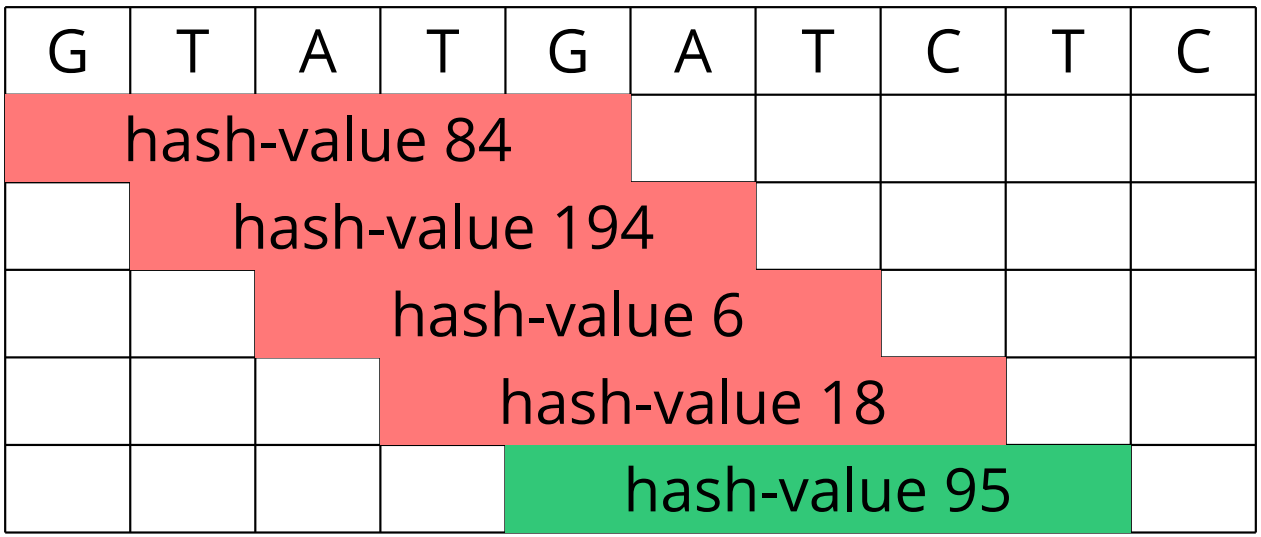
Example:

- T = "`GTAT GATC TC`" (ignore spaces)
- P_1 = "`ATCT`"
- P_2 = "`TGAT`"
- P_3 = "`ACCC`"
- P_1 matches at 5, P_2 matches at 3, P_3 has no matches

- Multi-string matching: search $P_1, P_2, ...$ in S simultaneously

Classical Algorithms

- Brute Force $\longrightarrow O(mn)$
- Boyer-Moore $\longrightarrow O(n+m)$, $O(mn)$ worst case
- Knuth-Morris-Pratt $\longrightarrow O(n+m)$ worst case
- Suffix Tree/Array $\longrightarrow O(m)/O(n \log n)$ + preproc
- Karp-Rabin - rolling hash $\longrightarrow O(n+m)$ expected



- Polynomial Matching $\longrightarrow O(n \log n)$

Polynomial Matching

Calculate a fingerprint for P and every m character sequence in S; matching fingerprints suggest pattern match!

$$A \mapsto -3$$
$$C \mapsto 5$$
$$G \mapsto -7$$
$$T \mapsto 11$$

$$\texttt{"ATCT"} \longrightarrow 11x^3 + 5x^2 + 11x - 3$$
$$\texttt{"GTAT GATC TC"} \longrightarrow -7x^{15} + 11x^{14} - 3x^{13} + \ldots + 11x^7 + 5x^6$$

Note that P encoding is reversed (similar to convolution)

Polynomial Matching

Given polynomials

$$P(x) = \sum_{i=0}^{m} a_i x^i, S(x) = \sum_{j=0}^{n} b_j x^j$$

then

$$R(x) = \sum_{k=0}^{m+n} c_k x^k$$

where

$$c_k = \sum_{i=0}^{k} a_i b_{k-i}, \quad \text{for } 0 \leq k \leq m+n,$$

with the convention that $a_i = 0$ for $i > m$ and $b_j = 0$ for $j > n$.

Polynomial Matching

Coefficients of $R(x)$ correspond to "dot products" between substrings.

$$P(x) = 11x^3 + 5x^2 + 11x - 3$$
$$S(x) = -7x^{15} + 11x^{14} - 3x^{13} + \ldots + 11x^7 + 5x^6$$
$$R(x) = \ldots + 71x^8 + 0x^9 + 276x^{10} + 98x^{11} - 4x^{12} + \ldots$$

Degrees map to index:

15 yields index 0

14 yields index 1

...

11 yields index 4

10 yields index 5

9 yields index 6

Polynomial Matching

Coefficients of $R(x)$ correspond to "dot products" between substrings.

$$P(x) = 11x^3 + 5x^2 + 11x - 3$$
$$S(x) = -7x^{15} + 11x^{14} - 3x^{13} + ... + 11x^7 + 5x^6$$
$$R(x) = ... + 71x^8 + 0x^9 + 276x^{10} + 98x^{11} - 4x^{12} + ...$$

Notice that $\|P\|^2 = 11^2 + 5^2 + 11^2 + (-3)^2 = 276$

We get exact fingerprint match for single patterns. Let's extend to multiple patterns...

Polynomial Matching

Add patterns together:

$$P(x) = P_1(x) + P_2(x)$$
$$S(x) = -7x^{15} + 11x^{14} - 3x^{13} + ... + 11x^7 + 5x^6$$
$$R(x) = ... + 94x^8 + 240x^9 + 272x^{10} + 64x^{11} + 296x^{12} - 60x^{13}...$$

No more exact matches, but higher values = likelier index.

Polynomial Multiplication

For two polynomials of degree n,

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_(n-1)x^{n-1}$$
$$B(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_(n-1)x^{n-1}$$

The goal is to simply to find their product:

$$C(x) = \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} a_k b_k x^{j+k}$$

- Naïve polynomial multiplication $\longrightarrow O(n^2)$
- Karatsuba Algorithm $\longrightarrow O(n^{\log_3 2}) = O(n^{1.59})$
- FFT $\longrightarrow O(n \log n)$

Polynomial Multiplication

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$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_(n-1) x^{n-1}$$
$$B(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_(n-1) x^{n-1}$$

The goal is to simply to find their product:

$$C(x) = \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} a_k b_k x^{j+k}$$

The naïve approach is to directly evaluate the summation (i.e. long multiplication/grade-school multiplication).

Efficient Polynomial Multiplication

Multiple algorithms for multiplication:

- Naïve polynomial multiplication $\longrightarrow O(n^2)$
- Karatsuba Algorithm $\longrightarrow O(n^{\log_3 2}) = O(n^{1.59})$
- FFT $\longrightarrow O(n \log n)$

Polynomial Multiplication with FFT

1. Apply FFT to the coefficients of $A(x)$ and $B(x)$
   With $\vec{a} = [a_0\ a_1 \dots a_{n-1}]^T$ and $\vec{b} = [b_0\ b_1 \dots b_{n-1}]^T$, then

$$\vec{\alpha} = \text{FFT}(\vec{a}) \qquad O(n \log n)$$
$$\vec{\beta} = \text{FFT}(\vec{b}) \qquad O(n \log n)$$

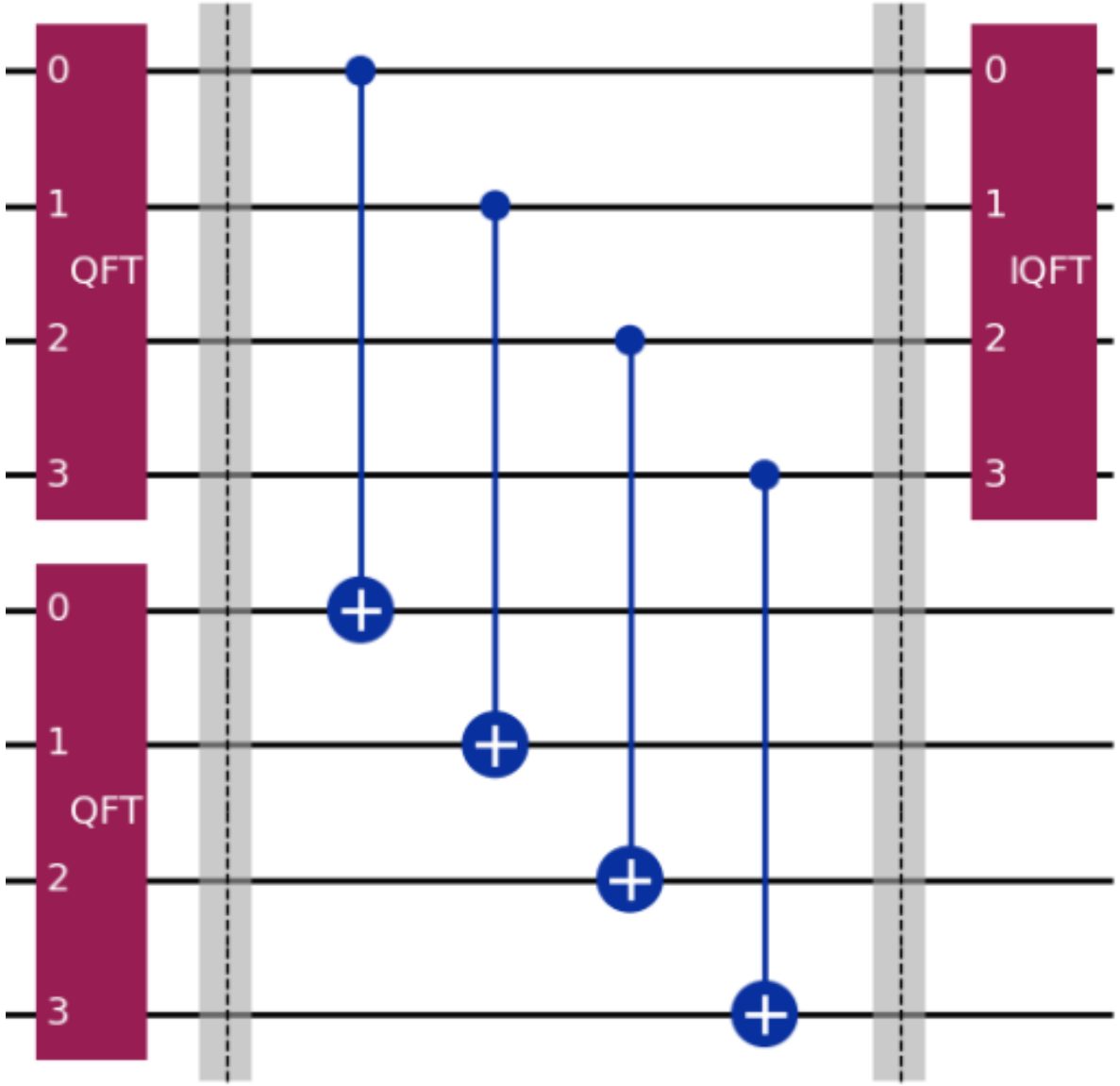
2. Perform the Hadamard (element-wise) product of coefficients

$$\begin{aligned} \vec{\gamma} &= \vec{\alpha} \odot \vec{\beta} \\ &= [\alpha_0\beta_0 \quad \alpha_1\beta_1 \dots \alpha_{n-1}\beta_{n-1}]^T \end{aligned} \qquad O(n)$$

3. Apply Inverse FFT

$$\vec{c} = \text{IFFT}(\vec{\gamma}) \qquad O(n \log n)$$

Polynomial Multiplication with QFT



Using Measurement for Validity

To ensure the validity of quantum element-wise multiplication, we require the last qubits to be measured as all zero.

Consider the simplest case: the element-wise product of two qubits' states

$$\begin{aligned} &CNOT_{1\to 2}((a_0|0\rangle + a_1|1\rangle) \otimes (b_0|0\rangle + b_1|1\rangle)) \\ &= CNOT_{1\to 2}(a_0b_0|00\rangle + a_0b_1|01\rangle + a_1b_0|10\rangle + a_1b_1|11\rangle) \\ &= a_0b_0|00\rangle + a_0b_1|01\rangle + a_1b_0|11\rangle + a_1b_1|10\rangle \end{aligned}$$

Note how the a_0b_0 and a_1b_1 products are only associated with terms where the second qubit is $|0\rangle$, and a coefficient product can be "chosen" using the state of the first qubit.

Lists

Bullet Points and Numbered Lists

- Lorem ipsum dolor sit amet, consectetur adipiscing elit
- Aliquam blandit faucibus nisi, sit amet dapibus enim tempus
  - Lorem ipsum dolor sit amet, consectetur adipiscing elit
  - Nam cursus est eget velit posuere pellentesque
- Nulla commodo, erat quis gravida posuere, elit lacus lobortis est, quis porttitor odio mauris at libero

1. Nam cursus est eget velit posuere pellentesque
2. Vestibulum faucibus velit a augue condimentum quis convallis nulla gravida

Blocks of Highlighted Text

Block Title

Lorem ipsum dolor sit amet, consectetur adipiscing elit. Integer lectus nisl, ultricies in feugiat rutrum, porttitor sit amet augue.

Example Block Title

Aliquam ut tortor mauris. Sed volutpat ante purus, quis accumsan.

Alert Block Title

Pellentesque sed tellus purus. Class aptent taciti sociosqu ad litora torquent per conubia nostra, per inceptos himenaeos.

Suspendisse tincidunt sagittis gravida. Curabitur condimentum, enim sed venenatis rutrum, ipsum neque consectetur orci.

Multiple Columns

Subtitle

Heading

1. Statement
2. Explanation
3. Example

Lorem ipsum dolor sit amet, consectetur adipiscing elit. Integer lectus nisl, ultricies in feugiat rutrum, porttitor sit amet augue. Aliquam ut tortor mauris. Sed volutpat ante purus, quis accumsan dolor.

Table

Subtitle

Treatments	Response 1	Response 2
Treatment 1	0.0003262	0.562
Treatment 2	0.0015681	0.910
Treatment 3	0.0009271	0.296

Table: Table caption

Definitions & Examples

Definition

A prime number is a number that has exactly two divisors.

Example

- 2 is prime (two divisors: 1 and 2).
- 3 is prime (two divisors: 1 and 3).
- 4 is not prime (three divisors: 1, 2, and 4).

You can also use the `theorem`, `lemma`, `proof` and `corollary` environments.

Theorem, Corollary & Proof

Theorem (Mass–energy equivalence)

$E = mc^2$

Corollary

$x + y = y + x$

Proof.

$\omega + \phi = \epsilon$ □

Equation

$$\cos^3\theta = \frac{1}{4}\cos\theta + \frac{3}{4}\cos 3\theta \tag{1}$$

Verbatim

Example (Theorem Slide Code)

```
\begin{frame}
\frametitle{Theorem}
\begin{theorem}[Mass--energy equivalence]
$E = mc^2$
\end{theorem}
\end{frame}
```

Slide without title.

Citing References

An example of the `\cite` command to cite within the presentation:

This statement requires citation [Smith, 2022, Kennedy, 2023].

References


John Smith (2022)
Publication title
Journal Name 12(3), 45 – 678.


Annabelle Kennedy (2023)
Publication title
Journal Name 12(3), 45 – 678.

Acknowledgements

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The End

Questions? Comments?