KON 426E
INTELLIGENT CONTROL SYSTEMS

LECTURE 8
11/04/2022

Fuzzy Sets

A classical set is a set with a crisp boundary.

A classical set A of real numbers greater that 6:

$$A = \{x \mid x > 6\}$$

Now let us express the set of tall people.

Let A="tall person" and x="height"in cm. Define A:

$$A = \{x | x > 170\}$$

According to this definition a person with height 170.0001cm is tall while one with height 169.9999 is not???

So, classical (crisp) sets do not reflect the true nature of human concepts and thoughts.

- A **fuzzy set** is a set without a crisp boundary.
- The translation from "**belonging to a set**" to "**not belonging to a set**" is gradual and smooth and this is characterized by **membership functions**.
- Let X be a space of objects and x be a generic element of X.
- A (classical/ordinary/crisp/nonfuzzy) set A:

  $A \subseteq X$ is defined as a collection of elements $x \in X$

  such that each x either belongs or does not belong to set A.

$$(x, 0) \quad x \notin A$$

$$(x, 1) \quad x \in A$$

- These are called **ordered pairs** and values 0 and 1 are called **characteristic functions**. In a classical set, they are either 0 or 1.

Fuzzy Sets

- Fuzzy set expresses the degree to which an element belongs to a set.

$$A = \{(x, \mu_A(x)) \mid x \in X\}$$

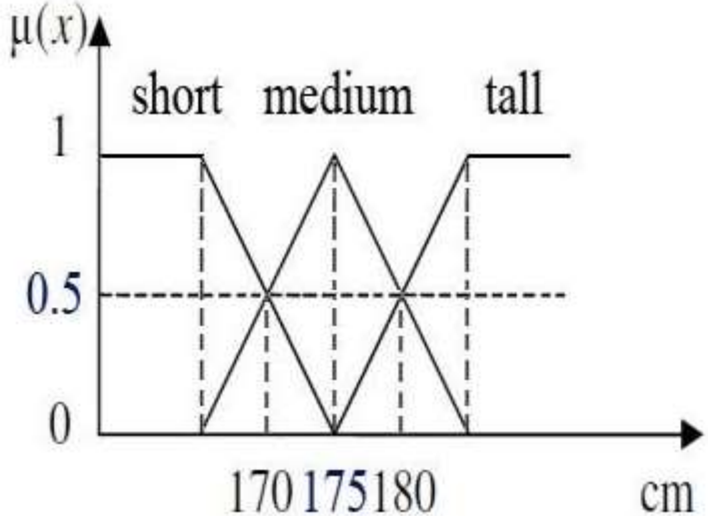
$\mu_A(x)$ is called the **membership function (MF)**

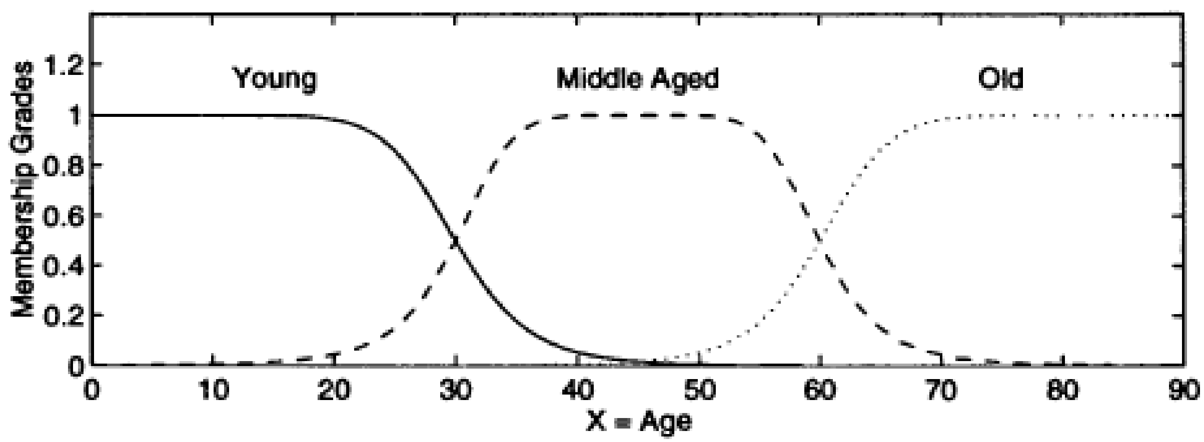
- Membership function maps each element of X to a membership value between 0 and 1.
- A classical set is a special case where the MF can take values of only 0 and 1.

X: universe of discourse

- It may consist of discrete objects or continuous space(ordered or nonordered).

Some examples of fuzzy sets and membership functions:



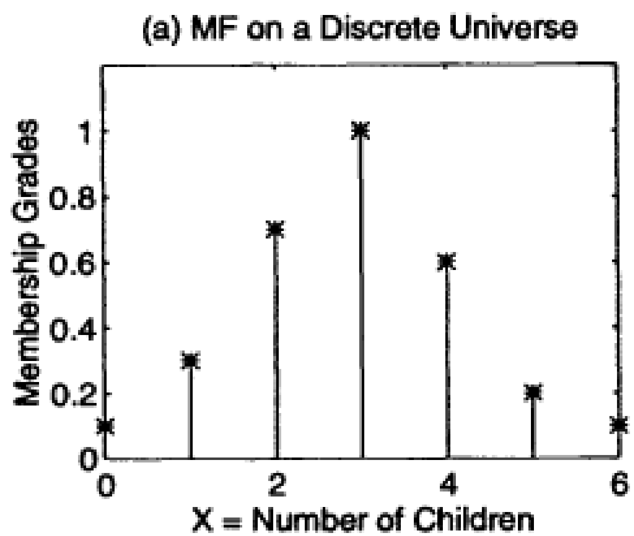


Example: Discrete ordered universe

Let $X = \{0, 1, 2, 3, 4, 5, 6\}$ be the set of numbers of children a family may choose to have. Then the fuzzy set A = "sensible number of children in a family" may be described as follows:

$$A = \{(0, 0.1), (1, 0.3), (2, 0.7), (3, 1), (4, 0.7), (5, 0.3), (6, 0.1)\}.$$

Here we have a discrete ordered universe X; the MF for the fuzzy set A is shown in Figure 2.1(a). Again, the membership grades of this fuzzy set are obviously subjective measures.



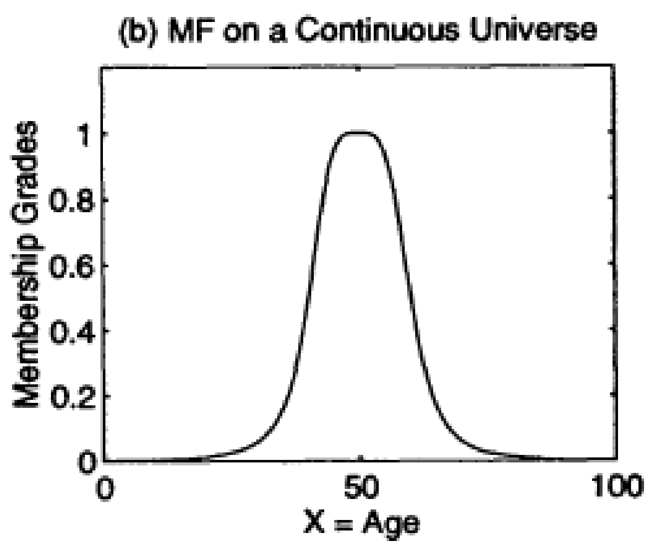
Example: Continuous ordered universe

Let $X = R^+$ be the set of possible ages for human beings. Then the fuzzy set $B =$ "about 50 years old" may be expressed as

$$B = \{(x, \mu_B(x) | x \in X\},$$

where

$$\mu_B(x) = \frac{1}{1 + \left(\frac{x-50}{10}\right)^4}.$$



Example: Discrete non-ordered universe

Let X = {San Francisco, Boston, Los Angeles} be the set of cities one may choose to live in. The fuzzy set C = "desirable city to live in" may be described as follows:

$$C = \{(\text{San Francisco},\ 0.9),\ (\text{Boston},\ 0.8),\ (\text{Los Angeles},\ 0.6)\}.$$

Notation

$$A = \begin{cases} \sum_{x_i \in X} \mu_A(x_i)/_{x_i}, & \textit{If X is a collection of discrete objects} \\ \int_X \mu_A(x)/_{x}, & \textit{If X is a continous space (e.g. the real line R)} \end{cases}$$

With this notation the previous examples can be expressed as:

$$A = 0.1/0 + 0.3/1 + 0.7/2 + 1.0/3 + 0.7/4 + 0.3/5 + 0.1/6,$$

$$B = \int_{R+} \frac{1}{1 + (\frac{x-50}{10})^4} \Big/ x,$$

$$C = 0.9/\text{San Francisco} + 0.8/\text{Boston} + 0.6/\text{Los Angeles},$$

Linguistic Variables and Linguistic Values

- Suppose the universe of discourse is X="age"

  Then **age** is a **linguistic variable**.
- We can define fuzzy sets "young", "middle-aged", and "old" with MF's $\mu_{old}(x)$, $\mu_{middleaged}(x)$, and $\mu_{old}(x)$
- In classical logic a variable can assume various values.   e.g. $x=10$
- In fuzzy logic, linguistic variables assume linguistic values.

  e.g. "age is young" or "age is old"

Some definitions regarding fuzzy sets

Support (dayanak)

The **support** of a fuzzy set A is the set of all points x in X such that $\mu_A(x) > 0$:

$$\text{support}(A) = \{x | \mu_A(x) > 0\}$$

Core (öz)

The **core** of a fuzzy set A is the set of all points x in X such that $\mu_A(x) = 1$

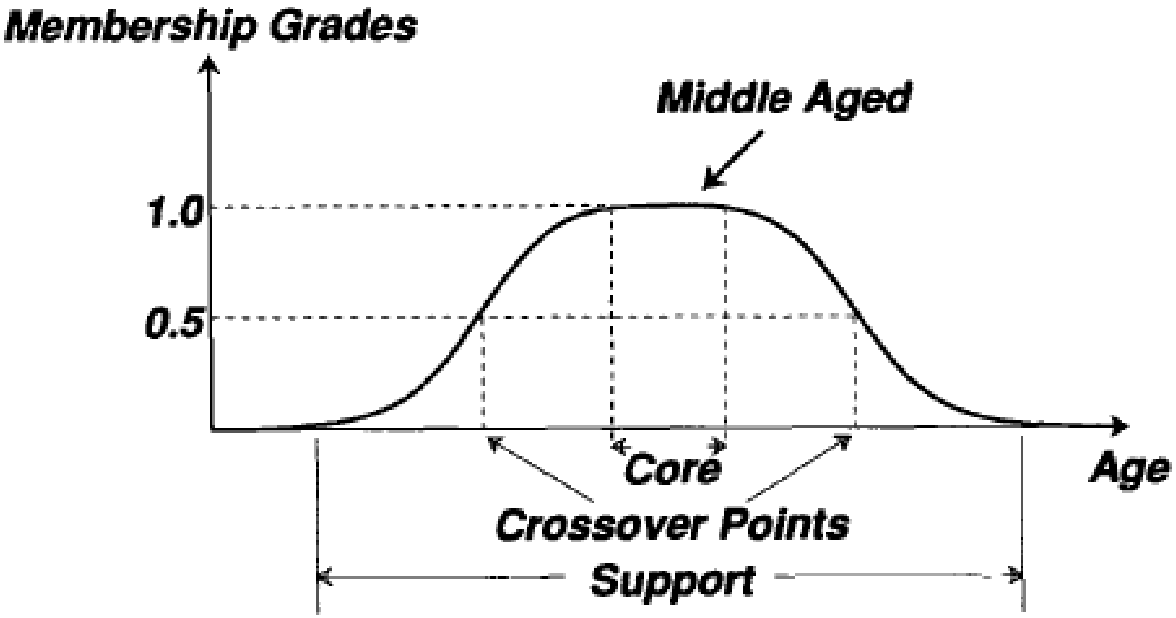
$$\text{core}(A) = \{x | \mu_A(x) = 1\}$$

$$\text{boundary}\,(A) = \{x | 0 \le \mu_A(x) < 1\}$$

Normality

A fuzzy set A is **normal** if its core is nonempty.

(We can find a point $x \in X$ such that $\mu_A(x) = 1$)



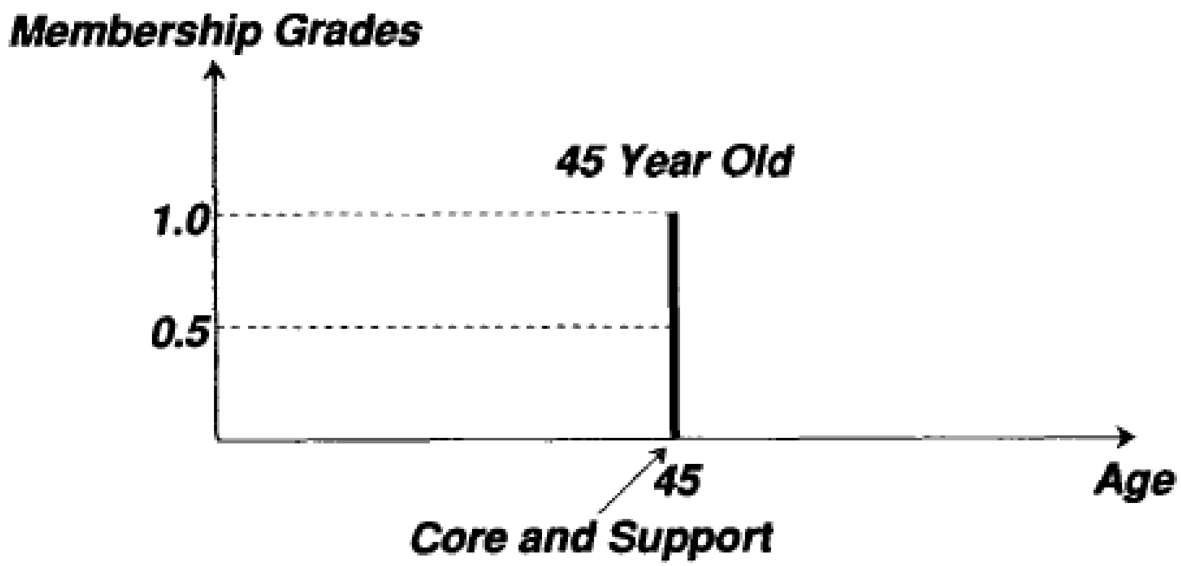
Crossover points

A **crossover point** of a fuzzy set A is a point $x \in X$ at which $\mu_A(x) = 0.5$

$$\text{crossover}(A) = \{x | \mu_A(x) = 0.5\}$$

Fuzzy Singleton

A fuzzy set whose support is a single point in X with $\mu_A(x) = 1$ is called a **fuzzy singleton**.



α–cut, stong α-cut

The α-cut (α-level set) of a fuzzy set A is a crisp set defined by:

$$A_\alpha = \{x | \mu_A(x) \geq \alpha\}$$

Strong α-cut (strong α-level set) are defined similarly:

$$A'_\alpha = \{x | \mu_A(x) > \alpha\}.$$

$$\text{support}(A) = A'_0$$

$$\text{core}(A) = A_1$$

Convexity

- For a function $f(x)$ (where $\lambda \in [0,1]$)

$$\lambda f(x_1) + (1-\lambda) f(x_2) \geq f(\lambda x_1 + (1-\lambda) x_2)$$

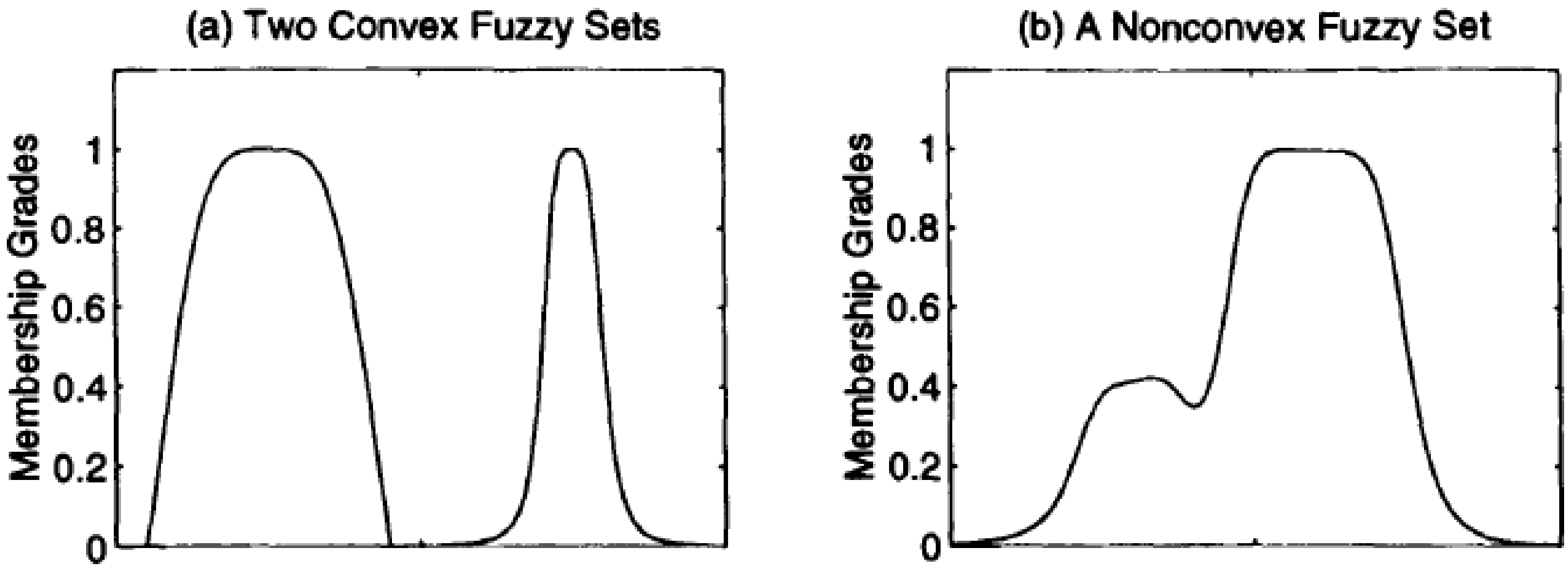
- For a crisp set

A crisp set C in R^n is convex if and only if for any two points $x_1 \in C$ and $x_2 \in C$, their convex combination $\lambda x_1 + (1-\lambda)x_2$ is still in C, where $0 \leq \lambda \leq 1$.

- For a fuzzy set

A fuzzy set A is **convex** if and only if for any $x_1,\ x_2 \in X$ and any $\lambda \in [0,1]$

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\}.$$


(a) Two Convex Fuzzy Sets
Membership Grades
1
0.8
0.6
0.4
0.2
0
(b) A Nonconvex Fuzzy Set
Membership Grades
1
0.8
0.6
0.4
0.2
0

Bandwidth for a normal and convex fuzzy set

For a normal and convex fuzzy set, the bandwidth or width is defined as the distance between two unique crossover points.

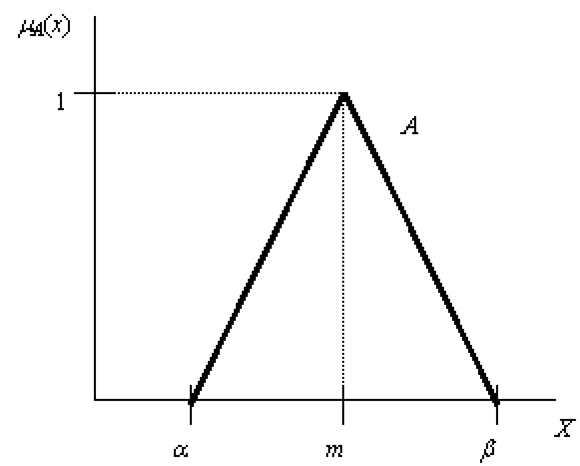
$$\mathrm{width}(A) = |x_2 - x_1|$$

where $\mu_A(x_1) = \mu_A(x_2) = 0.5$

Symmetry

A fuzzy set A is symmetric if its MF is symmetric around a certain point $x=c$.

$$\mu_A(c+x) = \mu_A(c-x) \text{ for all } x \in X$$

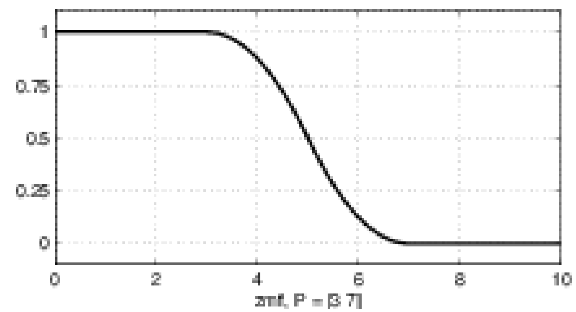


Open left, open right, closed

A fuzzy set A is **open left** if:

$$\lim_{x \to -\infty} \mu_A(x) = 1$$

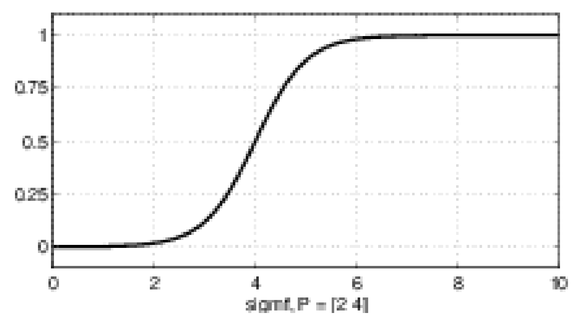
$$\lim_{x \to \infty} \mu_A(x) = 0$$



A fuzzy set A is **open right** if:

$$\lim_{x \to -\infty} \mu_A(x) = 0$$

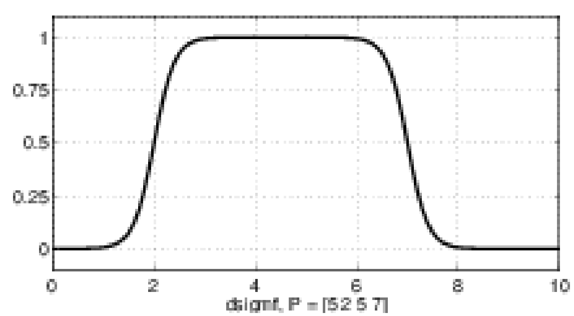
$$\lim_{x \to \infty} \mu_A(x) = 1$$



A fuzzy set A is **closed** if:

$$\lim_{x \to -\infty} \mu_A(x) = 0$$

$$\lim_{x \to \infty} \mu_A(x) = 0$$

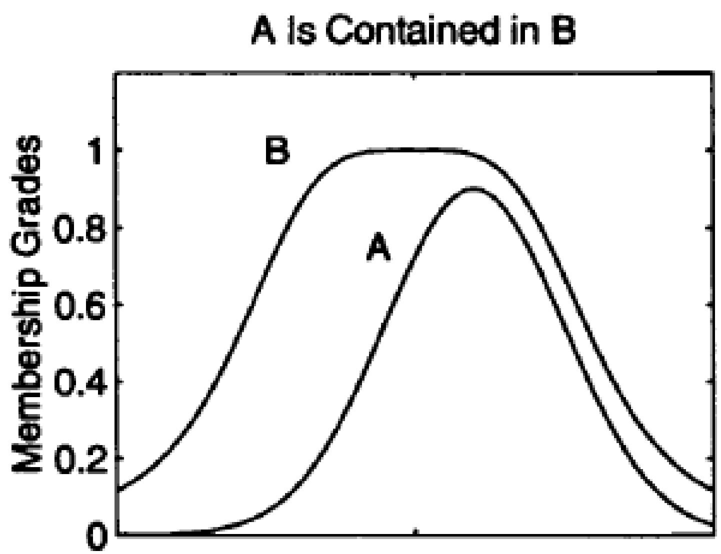


Set Theoretic Operations

Containment or subset

Fuzzy set A is contained in fuzzy set B (or A is a subset of B) if and only if $\mu_A(x) \leq \mu_B(x)$ for all x.

$$A \subseteq B \Longleftrightarrow \mu_A(x) \leq \mu_B(x)$$



The concept of $A \subseteq B$. (MATLAB file: `subset.m`)

Union (Disjunction)

The union of two fuzzy sets A and B is a fuzzy set C,

$C = A \cup B$ or $C = A \text{ OR } B$, with:

$$\mu_C(x) = \max(\mu_A(x), \mu_B(x)) = \mu_A(x) \vee \mu_B(x)$$

Union is the smallest fuzzy set containing both A and B.

Alternatively, if D is any fuzzy set that contains both A and B, then it also contains $A \cup B$

Intersection (Conjunction)

Intersection of two fuzzy sets A and B is a fuzzy set C,

$C = A \cap B$ or $C = A \text{ AND } B$, with

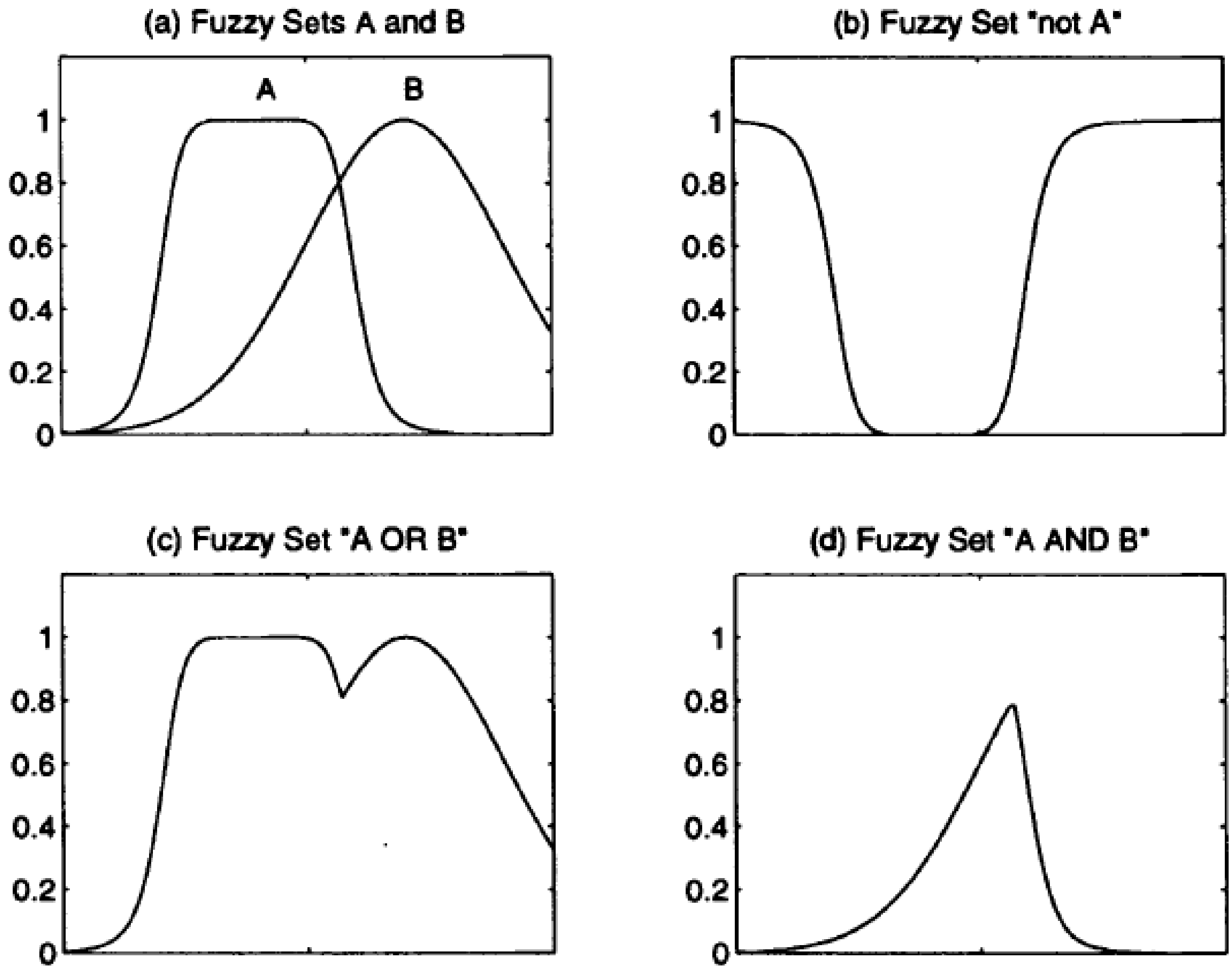
$$\mu_C(x) = \min(\mu_A(x), \mu_B(x)) = \mu_A(x) \wedge \mu_B(x)$$

Intersection of A and B is the largest fuzzy set which is contained in both A and B.

Complement (Negation)

The complement of a fuzzy set A, denoted by $\overline{A}$ ($\neg A$, NOT A) is defined as:

$$\mu_{\overline{A}}(x) = 1 - \mu_A(x)$$



(MATLAB file: `fuzsetop.m`)

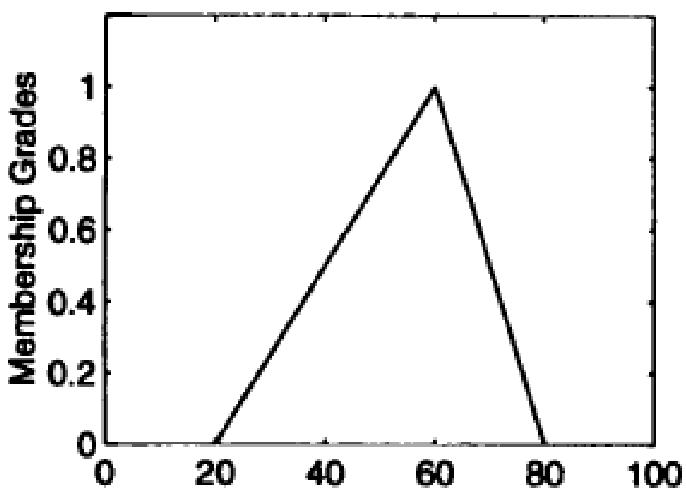
- Max, min and $\mu_{\overline{A}}(x) = 1 - \mu_A(x)$ operators are the classical or standard operators.
- But other operators are possible, as we will see later.
- If MF's are restricted to 0 and 1, these operators will boil down to corresponding operators for ordinary (crisp) sets.

Membership Functions of One-Dimension

Triangular MF

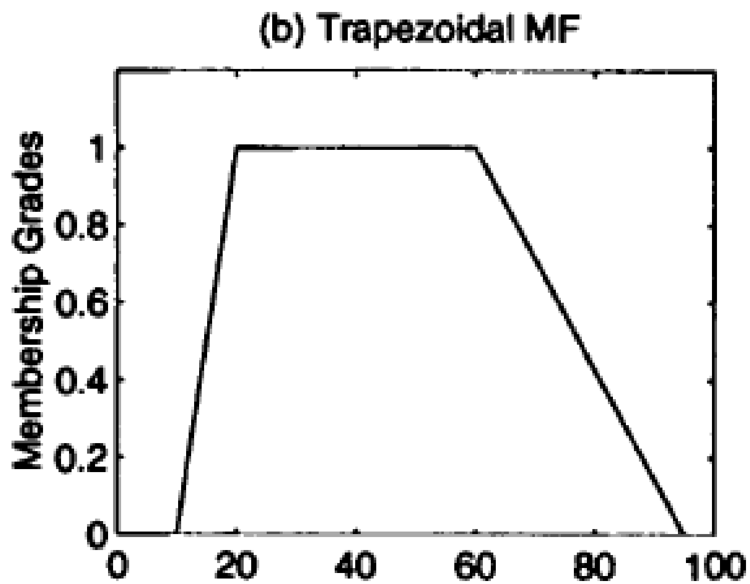
$$\text{triangle}(x; a, b, c) = \begin{cases} 0, & x \le a. \\ \dfrac{x-a}{b-a}, & a \le x \le b. \\ \dfrac{c-x}{c-b}, & b \le x \le c. \\ 0, & c \le x. \end{cases}$$

(a) Trianguler MF



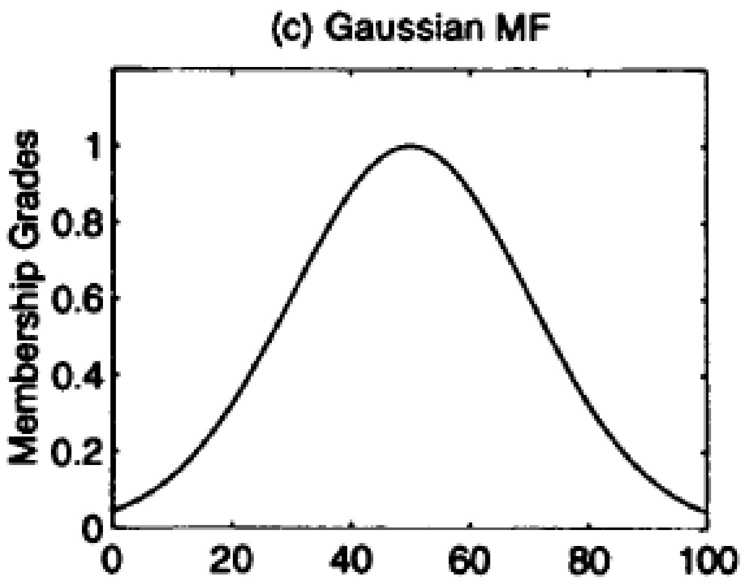
Trapezoidal MF

$$\text{trapezoid}(x;a,b,c,d) = \begin{cases} 0, & x \leq a. \\ \dfrac{x-a}{b-a}, & a \leq x \leq b. \\ 1, & b \leq x \leq c. \\ \dfrac{d-x}{d-c}, & c \leq x \leq d. \\ 0, & d \leq x. \end{cases}$$



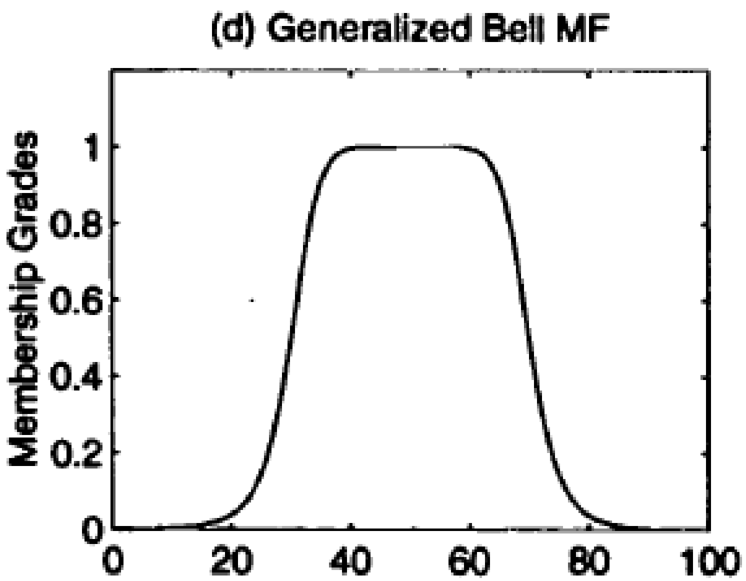
Gaussian MF

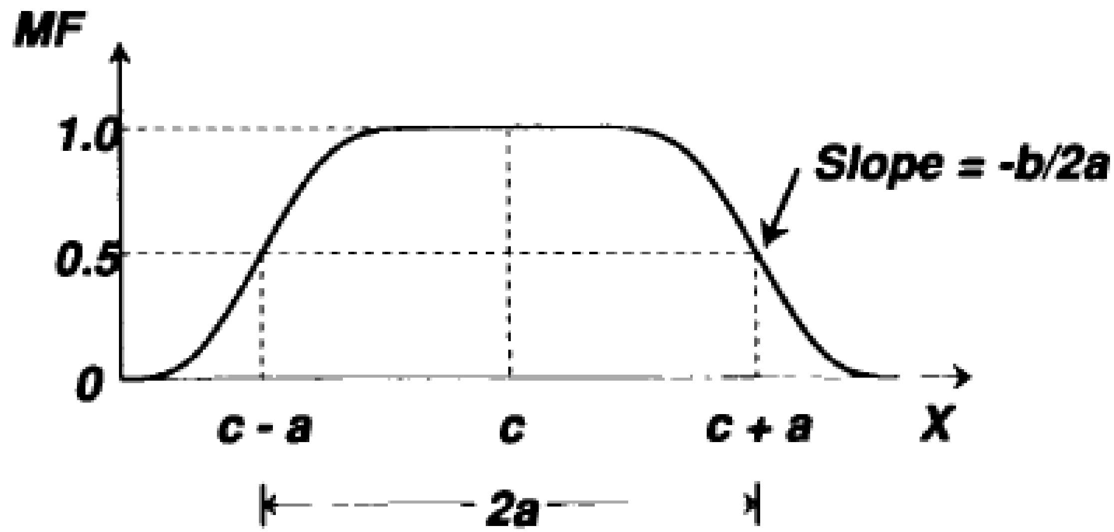
$$\text{gaussian}(x; c, \sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$$



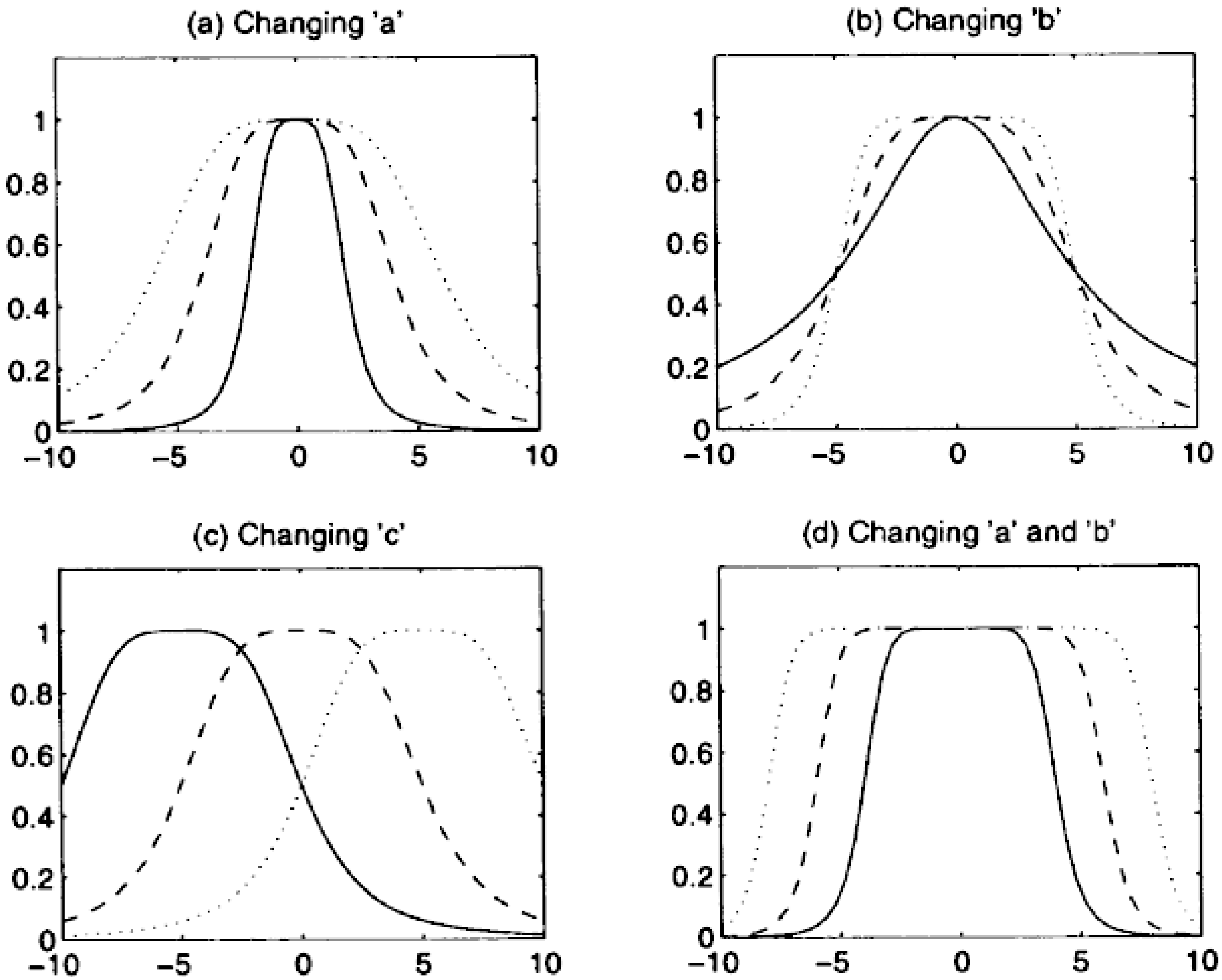
Generalized Bell MF

$$\text{bell}(x; a, b, c) = \frac{1}{1 + \left|\frac{x - c}{a}\right|^{2b}}$$





Physical meaning of parameters of the generalized bell MF



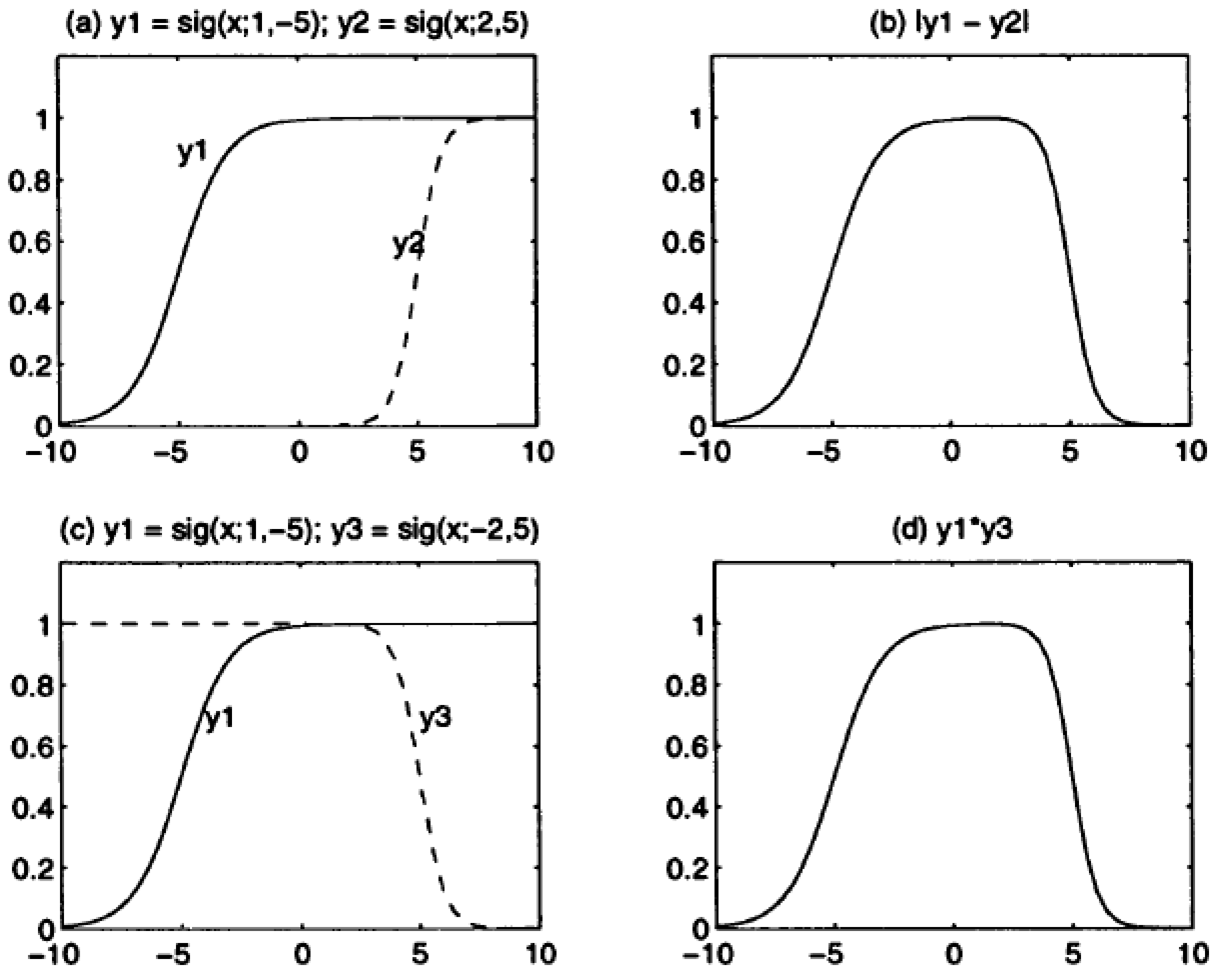
(MATLAB file: `allbells.m`)

Sigmoidal MF

$$\mathrm{sig}(x; a, c) = \frac{1}{1 + \exp[-a(x - c)]}$$

where a controls the slope at the crossover point $x = c$.

Closed and asymmetric MF's based on sigmoidal functions



(MATLAB file: disp_sig.m)

Parameterized Membership Functions

S-shaped MF

smf R2020

S-shaped membership function

collapse all in pag

Syntax

```
y = smf(x,params)
```

Description

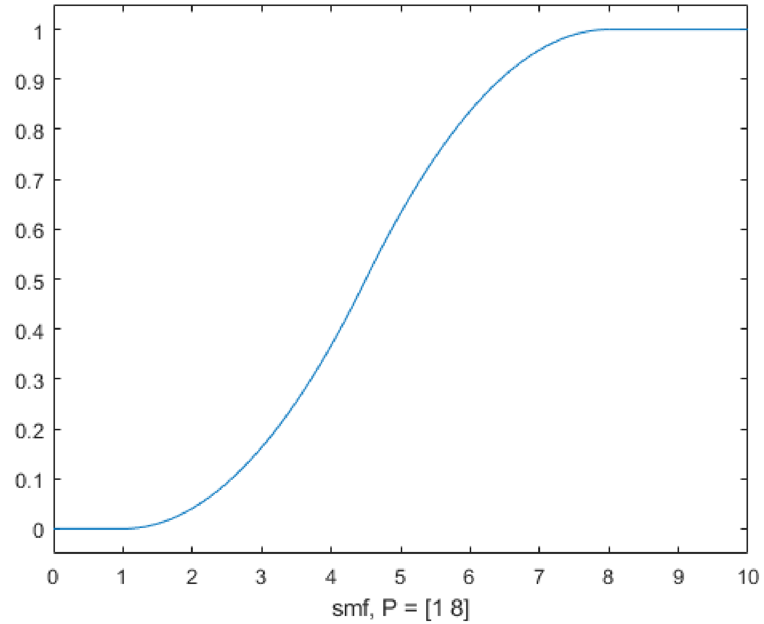
This function computes fuzzy membership values using a spline-based S-shaped membership function. You can also compute this membership function using a `fismf` object. For more information, see fismf Object.

This membership function is related to the `zmf` and `pimf` membership functions

`y = smf(x,params)` returns fuzzy membership values computed using the spli

examp

```
x = 0:0.1:10;
y = smf(x,[1 8]);
plot(x,y)
xlabel('smf, P = [1 8]')
ylim([-0.05 1.05])
```



Z-shaped MF

zmf

Z-shaped membership function

Syntax

```
y = zmf(x,params)
```

Description

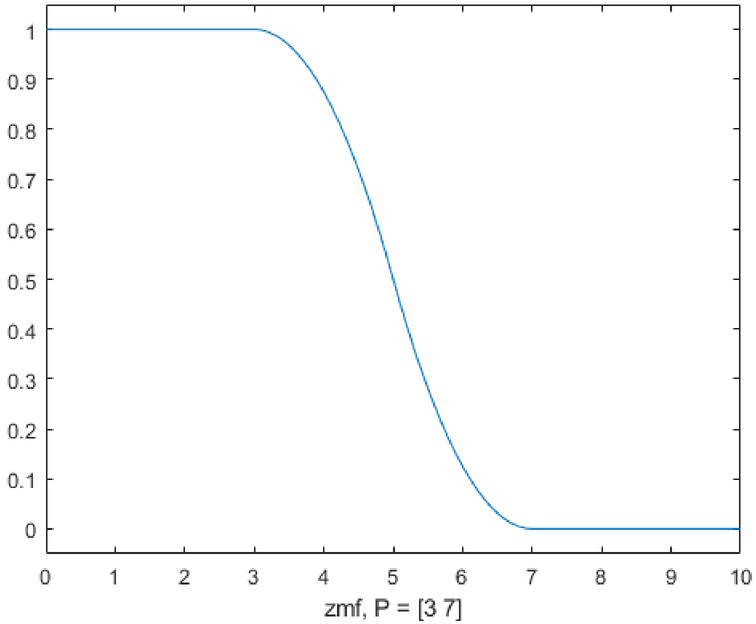
This function computes fuzzy membership values using a spline-based Z-shaped membership function. You can also compute this membership function using a `fis`
more information, see fismf Object.

This membership function is related to the `smf` and `pimf` membership functions.

`y = zmf(x,params)` returns fuzzy membership values computed using the spline-based Z-shaped membership function given by:

$$f(x;a,b)=\begin{cases}1, & x\le a\\ 1-2\left(\frac{x-a}{b-a}\right)^2, & a\le x\le \frac{a+b}{2}\\ 2\left(\frac{x-b}{b-a}\right)^2, & \frac{a+b}{2}\le x\le b\\ 0, & x\ge b\end{cases}$$

To specify the a and b parameters, use `params`.



Pi-shaped MFs

Pi-shaped membership function

Syntax

```
y = pimf(x,params)
```

Description

This function computes fuzzy membership values using a spline-based pi-shaped membership function. You can also compute this membership function usin
more information, see fismf Object.

This membership function is related to the `smf` and `zmf` membership functions.

`y = pimf(x,params)` returns fuzzy membership values computed using a spline-based pi-shaped membership function. This membership function is the pro
`smf` function and a `zmf` function, and is given by:

$$f(x;a,b,c,d)=\begin{Bmatrix} 0, & x\le a \\ 2\left(\frac{x-a}{b-a}\right)^2, & a\le x\le \frac{a+b}{2} \\ 1-2\left(\frac{x-b}{b-a}\right)^2, & \frac{a+b}{2}\le x\le b \\ 1, & b\le x\le c \\ 1-2\left(\frac{x-c}{d-c}\right)^2, & c\le x\le \frac{c+d}{2} \\ 2\left(\frac{x-d}{d-c}\right)^2, & \frac{c+d}{2}\le x\le d \\ 0, & x\ge d \end{Bmatrix}$$