KON 426E
INTELLIGENT CONTROL SYSTEMS

LECTURE 10
09/05/2022

How do we interpret a fuzzy IF-THEN rule?

IF x *is* A THEN y *is* B $\quad (A \rightarrow B)$

- This expression is a relation between two variables x and y. So, a fuzzy IF-THEN rule can be defined as a binary fuzzy relation R on the product space $X \times Y$

v

- There are two ways to interpret the fuzzy rule $A \rightarrow B$

  1) A is coupled with B
  2) A entails B (*entail:yol açmak, neden olmak*)

1) A is coupled with B

$$R = A \to B = A \times B = \int_{X \times Y} \mu_A(x) \tilde{*} \mu_B(y)/(x,y)$$

Examples

- Min operator (proposed by Mamdani)

$$R_m = A \times B = \int_{X \times Y} \mu_A(x) \wedge \mu_B(y)/(x,y), \text{ or } f_c(a,b) = a \wedge b$$

- Product operator (proposed by Larsen)

$$R_p = A \times B = \int_{X \times Y} \mu_A(x)\mu_B(y)/(x,y), \text{ or } f_p(a,b) = ab$$

- Bounded product operator

$$R_{bp} = A \times B = \int_{X \times Y} \mu_A(x) \odot \mu_B(y)/(x,y)$$

$$f_{bp}(a,b) = 0 \vee (a+b-1)$$

- Drastic product operator

$$R_{dp} = A \times B = \int_{X \times Y} \mu_A(x) \hat{\cdot} \mu_B(y)/(x,y) \;\; f(a,b) = a \hat{\cdot} b = \begin{cases} a & \text{if } b = 1. \\ b & \text{if } a = 1. \\ 0 & \text{otherwise.} \end{cases}$$

2) A entails B

- Material implication:

$$R = A \to B = \neg A \cup B.$$

- Propositional calculus:

$$R = A \to B = \neg A \cup (A \cap B).$$

- Extended propositional calculus:

$$R = A \to B = (\neg A \cap \neg B) \cup B.$$

- Generalization of modus ponens:

$$\mu_R(x, y) = \sup\{c \mid \mu_A(x) \tilde{*} c \le \mu_B(y) \text{ and } 0 \le c \le 1\},$$

where $R = A \to B$ and $\tilde{*}$ is a T-norm operator.

(They all reduce to the first one)

Based on these two interpretations and various T-norm and T-conorm operators, a number of qualified methods can be formulated to calculate the fuzzy relation $R = A \to B$. Note that R can be viewed as a fuzzy set with a two-dimensional MF

$$\mu_R(x, y) = f(\mu_A(x), \mu_B(y)) = f(a, b),$$

with $a = \mu_A(x)$, $b = \mu_B(y)$, where the function f, called the **fuzzy implication function**, performs the task of transforming the membership grades of x in A and y in B into those of (x, y) in $A \to B$.

Summary

- Here we will use the first interpretation.
- Fuzzy IF-THEN rule is interpreted as "It is true that A and B simultaneously hold".
- We use T-norm operators to calculate:

$$\mu_R(x, y) = f(\mu_A(x), \mu_B(y)) = f(a, b)$$

Compositional Rule of Inference

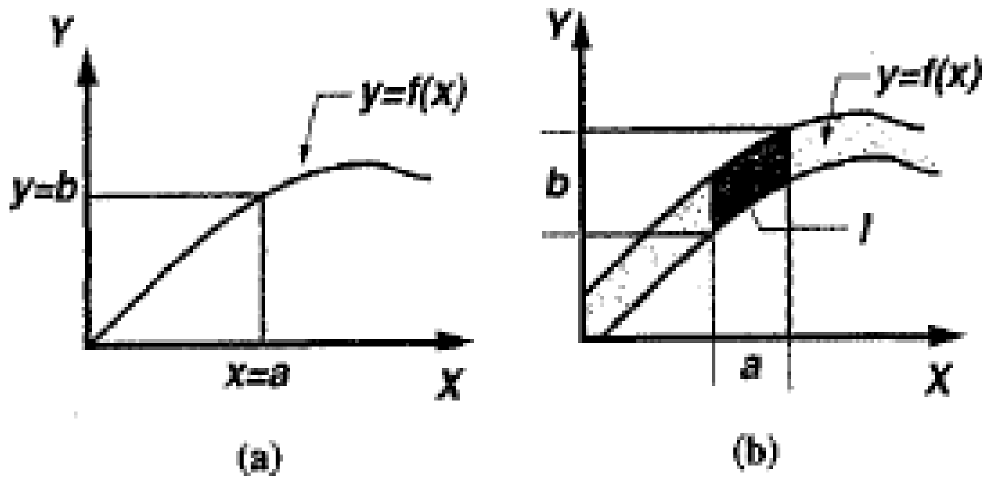


Figure 3.10. *Derivation of $y = b$ from $x = a$ and $y = f(x)$: (a) a and b are points, $y = f(x)$ is a curve; (b) a and b are intervals, $y = f(x)$ is an interval-valued function.*

- Construct a cylindrical extension of *a*.
- Find its intersection *I* with the interval-valued curve.
- Project *I* onto the *y*-axis.
- This yields the interval *y=b*.

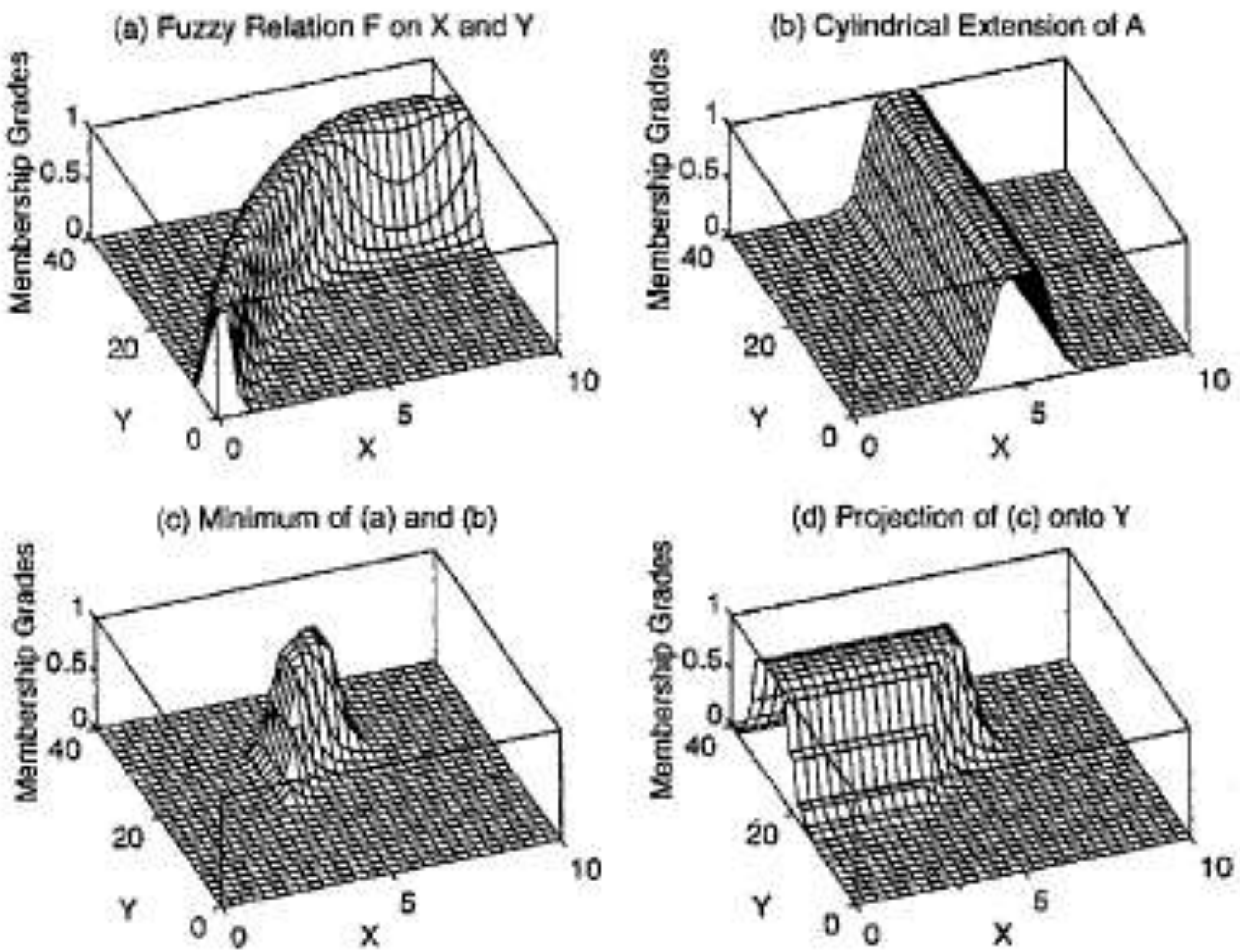


Figure 3.11. *Compositional rule of inference.* (MATLAB file: `cri.m`)

- Construct cylindrical extension c(A).
- Find $c(A) \cap F$
- Project $c(A) \cap F$ onto y-axis
- The projection gives you the fuzzy set B on y-axis.

$$\mu_{c(A)}(x,y) = \mu_A(x)$$

$$\mu_{c(A)\cap F}(x,y) = \min[\mu_A(x), \mu_F(x,y)]$$

Now, project $c(A) \cap F$ onto the y-axis:

$$\mu_B(y) = \max_x \min[\mu_A(x), \mu_F(x,y)]$$

$$B = A \circ F$$

Composition operator

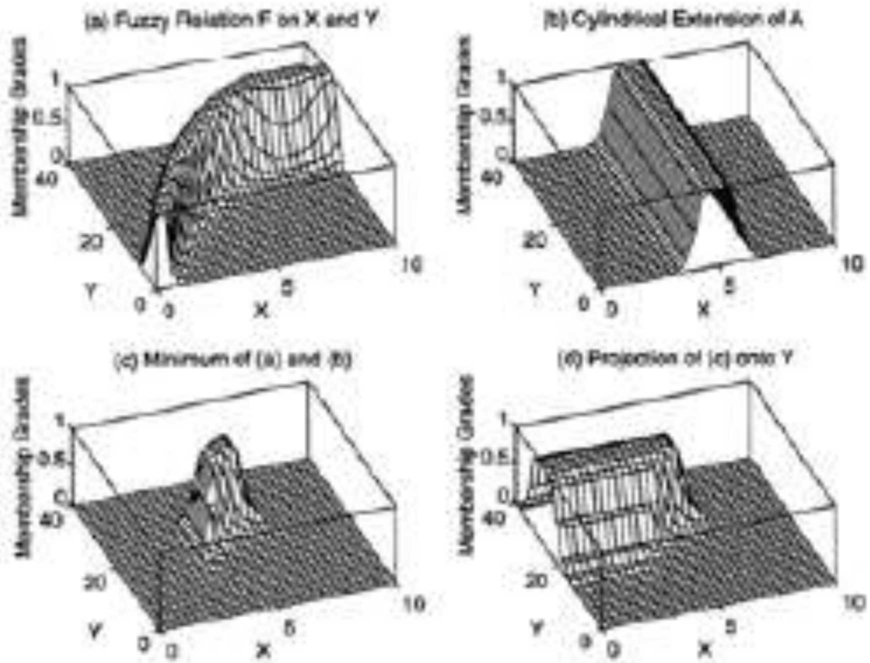


Figure 3.11. *Compositional rule of inference.* (MATLAB file: cri.m)

Fuzzy Reasoning

The basic rule of inference in traditional two-valued logic is **modus ponens**, according to which we can infer the truth of a proposition B from the truth of A and the implication $A \rightarrow B$

Modus Ponens

$$
\begin{array}{ll}
\text{premise 1 (fact):} & \text{x is } A, \\
\text{premise 2 (rule):} & \text{if x is } A \text{ then y is } B, \\
\hline
\text{consequence (conclusion):} & \text{y is } B.
\end{array}
$$

Example

Let A="the tomato is red"

B="the tomato is ripe"

Then if it is true that "the tomato is red", it is also true that "the tomato is ripe"

Approximate reasoning/Fuzzy reasoning/

Generalized modus ponens

premise 1 (fact):	x is A',
premise 2 (rule):	if x is A then y is B,
consequence (conclusion):	y is B',

(When A, B, A' and B' are of appropriate universes)

Example

A' = "the tomato is more or less red"

B' ="the tomato is more or less ripe"

Single Input with Single Antecedent

premise 1 (fact):	x is A',
premise 2 (rule):	if x is A then y is B,
consequence (conclusion):	y is B',

$$\begin{aligned}\mu_{B'}(y) &= \max_x \min[\mu_{A'}(x), \mu_R(x,y)] \\ &= \vee_x[\mu_{A'}(x) \wedge \mu_R(x,y)]\end{aligned}$$

$$B' = A' \circ R = A' \circ (A \to B)$$

$$\mu_R(x,y) = \min(\mu_A(x), \mu_B(y))$$

$$\mu_R(x,y) = \mu_A(x) \wedge \mu_B(y)$$

$$\begin{aligned}\mu_{B'}(y) &= [\vee_x(\mu_{A'}(x) \wedge \mu_A(x)] \wedge \mu_B(y) \\ &= w \wedge \mu_B(y)\end{aligned}$$

$w = \vee_x(\mu_{A'}(x) \wedge \mu_A(x))$ firing strength/degree of match/ degree of belief

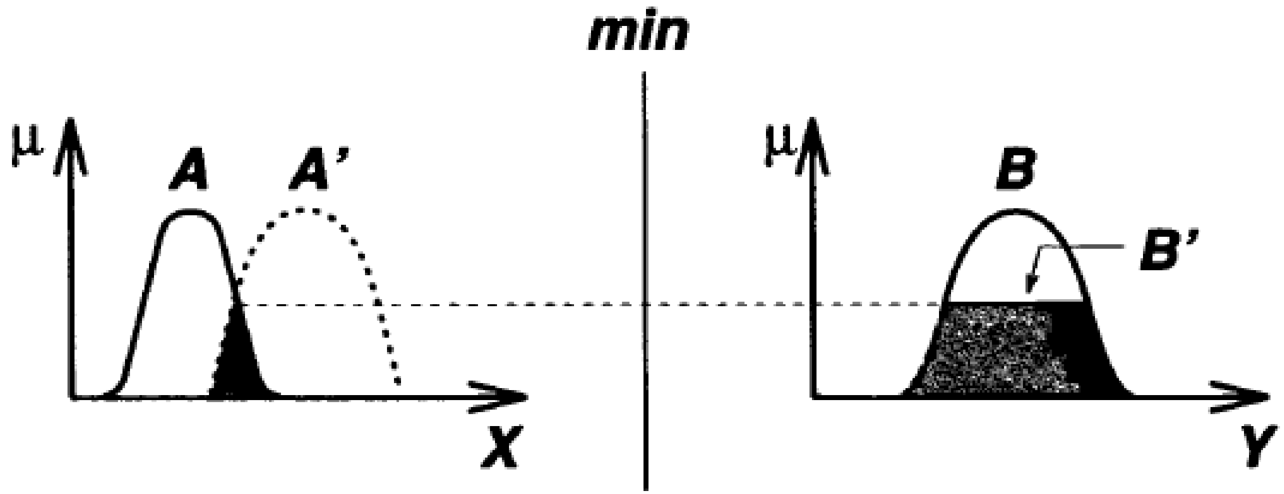


Figure 3.12. *Graphic interpretation of GMP using Mamdani's fuzzy implication and the max-min composition.*

$$\begin{aligned}\mu_{B'}(y) &= [\vee_x(\mu_{A'}(x) \wedge \mu_A(x)] \wedge \mu_B(y) \\ &= w \wedge \mu_B(y)\end{aligned}$$

Single Rule with Multiple Antecedents

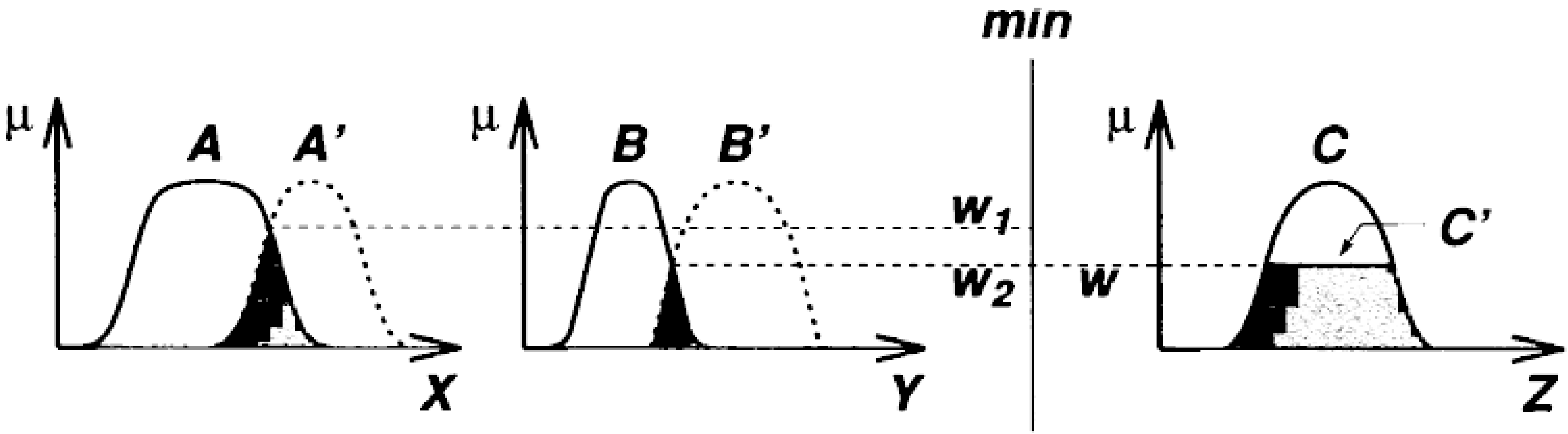
premise 1 (fact):	x is A' and y is B',
premise 2 (rule):	if x is A and y is B then z is C,
consequence (conclusion):	z is C'.

$$A \times B \to C$$

$$C' = (A' \times B') \circ (A \times B \to C)$$

$$\begin{aligned}
\mu_{C'}(z) &= \vee_{x,y}[\mu_{A'}(x) \wedge \mu_{B'}(y)] \wedge [\mu_A(x) \wedge \mu_B(y) \wedge \mu_C(z)] \\
&= \vee_{x,y}\{[\mu_{A'}(x) \wedge \mu_{B'}(y) \wedge \mu_A(x) \wedge \mu_B(y)]\} \wedge \mu_C(z) \\
&= \underbrace{\{\vee_x[\mu_{A'}(x) \wedge \mu_A(x)]\}}_{w_1} \wedge \underbrace{\{\vee_y[\mu_{B'}(y) \wedge \mu_B(y)]\}}_{w_2} \wedge \mu_C(z) \\
&= \underbrace{(w_1 \wedge w_2)}_{\text{firing strength}} \wedge \mu_C(z),
\end{aligned}$$

w_1 and w_2 are the maxima of the MFs of $A \cap A'$ and $B \cap B'$, respectively.



$$\underbrace{\{\vee_x[\mu_{A'}(x) \wedge \mu_A(x)]\}}_{w_1} \wedge \underbrace{\{\vee_y[\mu_{B'}(y) \wedge \mu_B(y)]\}}_{w_2} \wedge \mu_C(z)$$

$$\underbrace{(w_1 \wedge w_2)}_{\text{firing strength}} \wedge \mu_C(z),$$

In general, w_1 denotes the **degrees of compatibility** between A and A'; similarly for w_2. Since the antecedent part of the fuzzy rule is constructed by the connective "and," $w_1 \wedge w_2$ is called the **firing strength** or **degree of fulfillment** of the fuzzy rule, which represents the degree to which the antecedent part of the rule is satisfied.

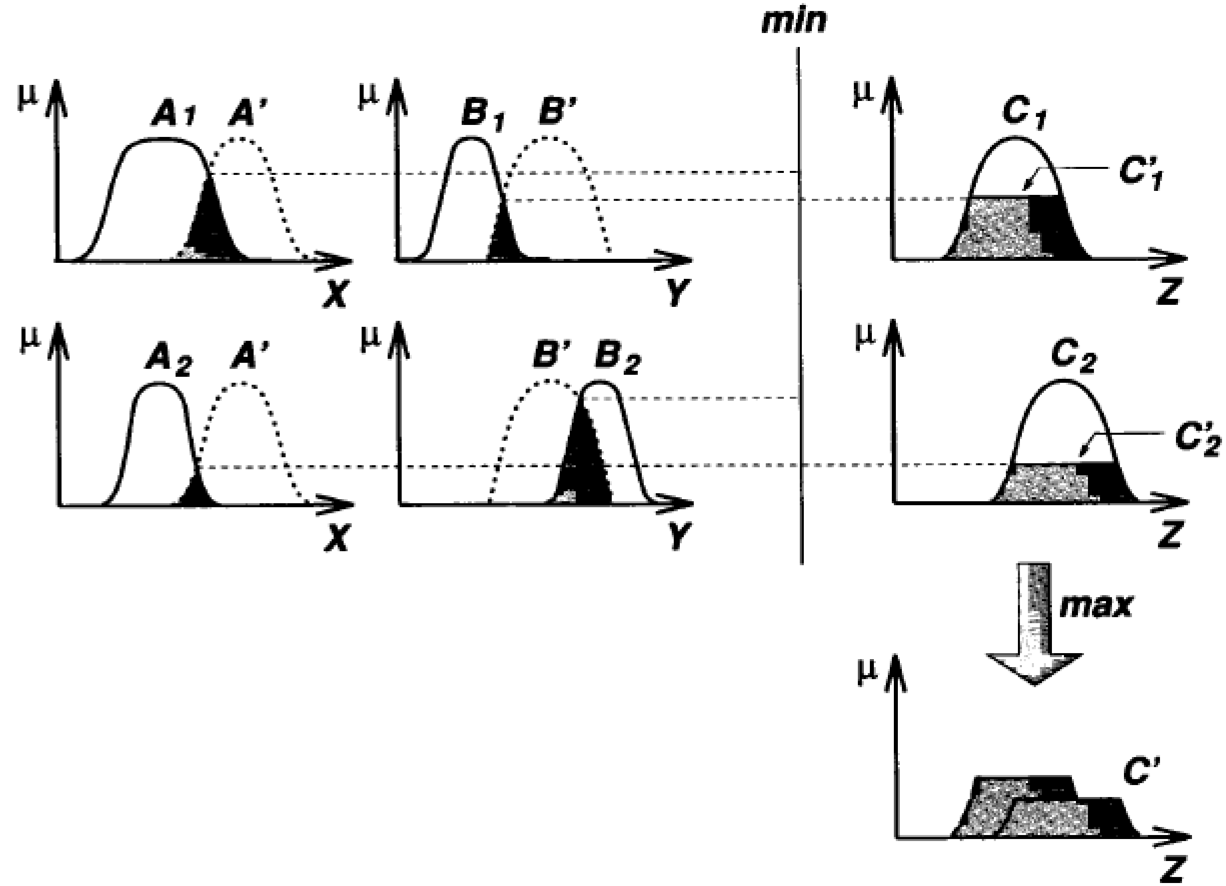
Multiple Rules with Multiple Antecedents

premise 1 (fact):	x is A' and y is B'
premise 2 (rule 1):	if x is A_1 and y is B_1 then z is C_1
premise 3 (rule 2):	if x is A_2 and y is B_2 then z is C_2
consequence (conclusion):	z is C'

$$R_1 = A_1 \times B_1 \to C_1$$

$$R_2 = A_2 \times B_2 \to C_2$$

$$\begin{aligned} C' &= (A' \times B') \circ (R_1 \cup R_2) \\ &= [(A' \times B') \circ R_1] \cup [(A' \times B') \circ R_2] \\ &= C'_1 \cup C'_2 \end{aligned}$$



$$
\begin{aligned}
C' &= (A' \times B') \circ (R_1 \cup R_2) \\
&= [(A' \times B') \circ R_1] \cup [(A' \times B') \circ R_2] \\
&= C'_1 \cup C'_2
\end{aligned}
$$

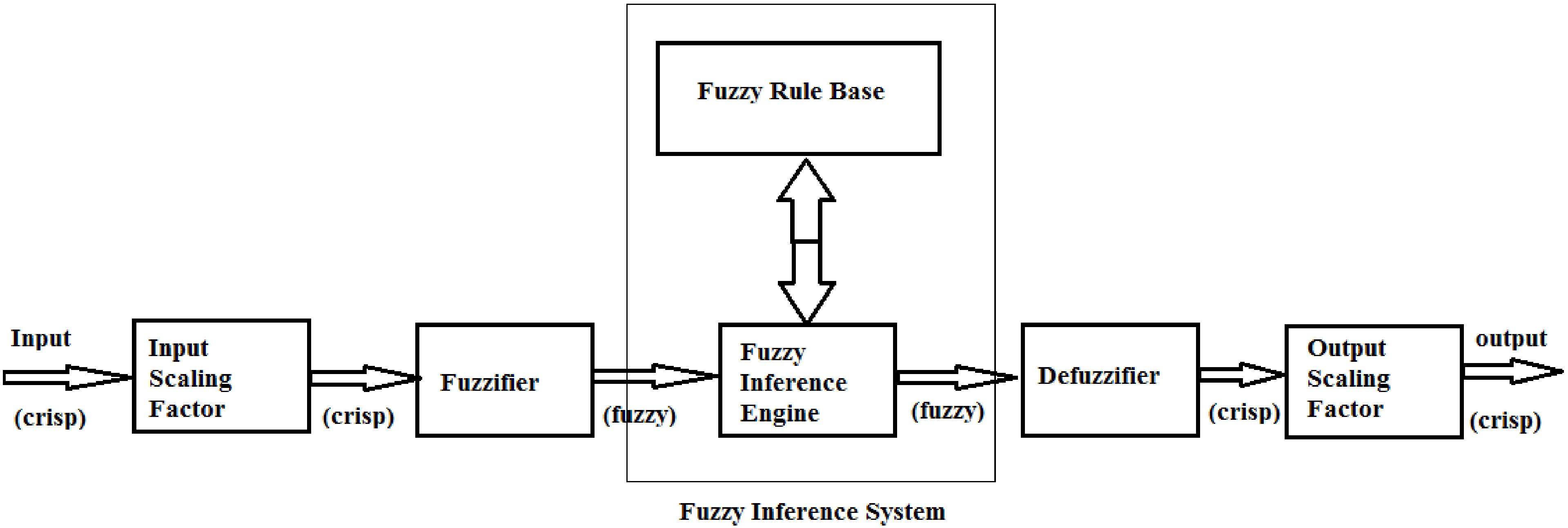
Degrees of compatibility Compare the known facts with the antecedents of fuzzy rules to find the degrees of compatibility with respect to each antecedent MF.

Firing strength Combine degrees of compatibility with respect to antecedent MFs in a rule using fuzzy AND or OR operators to form a firing strength that indicates the degree to which the antecedent part of the rule is satisfied.

Qualified (induced) consequent MFs Apply the firing strength to the consequent MF of a rule to generate a qualified consequent MF. (The qualified consequent MFs represent how the firing strength gets propagated and used in a fuzzy implication statement.)

Overall output MF Aggregate all the qualified consequent MFs to obtain an overall output MF.

Components of a Fuzzy System



- It is a nonlinear mapping from input space to output space.
- Each fuzzy IF-THEN rule describes the local behaviour of the mapping.

Fuzzification

Calculating the membership function values for the crisp inputs.

Defuzzification

Extraction of a crisp value that best represents a fuzzy set.

Defuzzification methods:

1) Centroid of area z_{COA}

2) Bisector of area z_{BOA}

3) Mean of maximum z_{MOM}

4) Smallest of maximum z_{SOM}

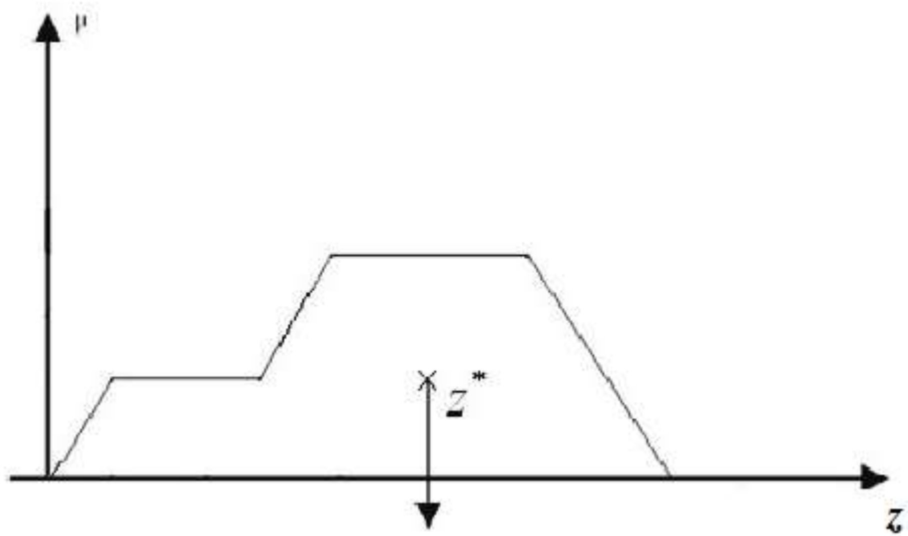
5) Largest of maximum z_{LOM}

1) Centroid of area z_{COA}

$$z_{\mathrm{COA}} = \frac{\int_Z \mu_A(z) z \, dz}{\int_Z \mu_A(z) \, dz}$$

where $\mu_A(z)$ is the aggregated output MF.

This is the most widely used defuzzification method, and it is reminiscent of the calculation of expected values of probability distributions.

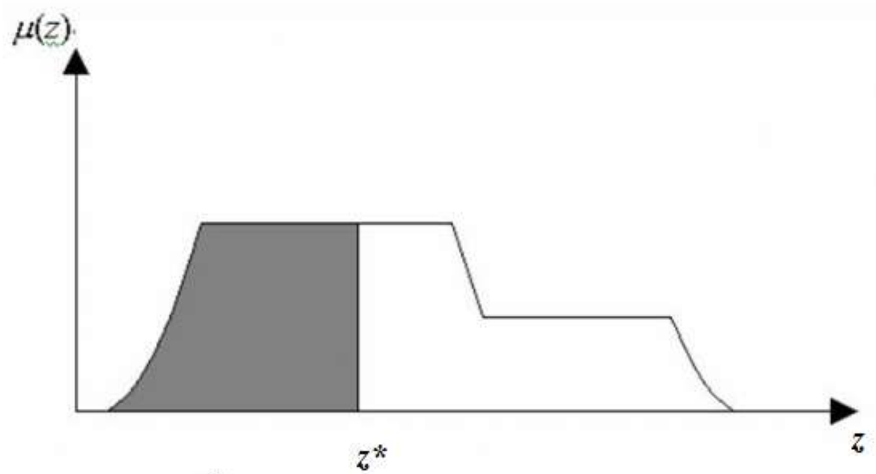


2) Bisector of area z_{BOA}

$$\int_{\alpha}^{z_{\mathrm{BOA}}} \mu_A(z)\, dz = \int_{z_{\mathrm{BOA}}}^{\beta} \mu_A(z)\, dz$$

where $\alpha = \min\{z|z \in Z\}$ and $\beta = \max\{z|z \in Z\}$

The vertical line $z = z_{\mathrm{BOA}}$ partitions the region between $z = \alpha$, $z = \beta$, $y = 0$ and $y = \mu_A(z)$ into two regions with the same area.



3) Mean of maximum z_{MOM}

It is the average of the maximizing z at which the MF reaches a maximum μ^*

$$z_{\mathrm{MOM}} = \frac{\int_{Z'} z\, dz}{\int_{Z'} dz}$$

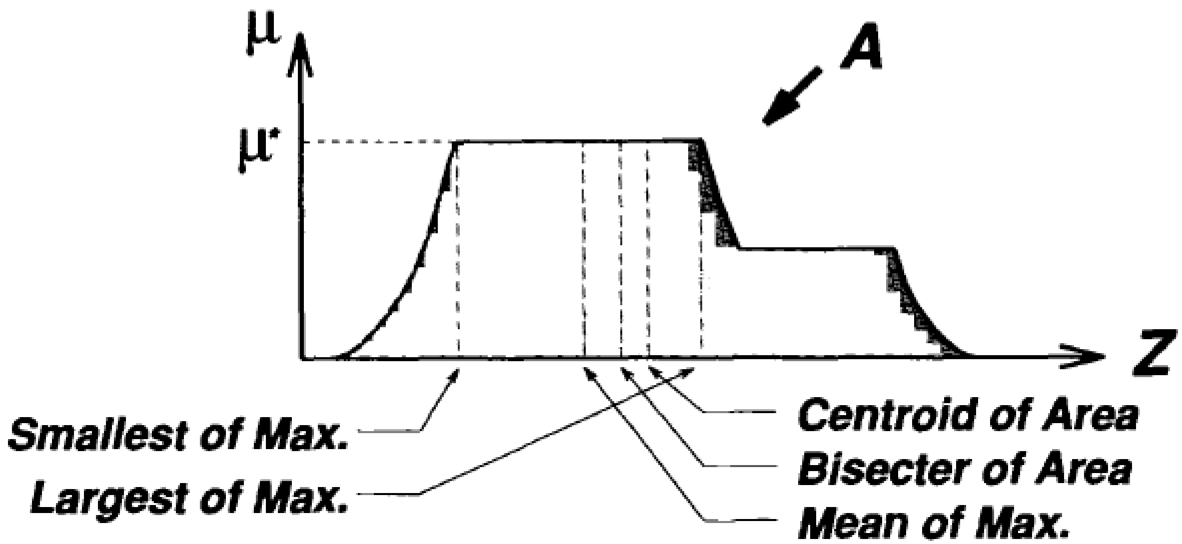
where $Z' = \{z \mid \mu_A(z) = \mu^*\}$.

4) **Smallest of maximum** z_{SOM}

It is the minimum (in terms of magnitude) of the maximing z.

5) **Largest of maximum** z_{LOM}

It is the maximum (in terms of magnitude) of the maximing z.



Various defuzzification schemes for obtaining a crisp output

- Because of their obvious bias z_{SOM} and z_{LOM} are not used as often as the other three defuzzification methods.

Mamdani Fuzzy Inference System

- proposed to control a steam engine and boiler combination by a set of linguistic rules.
- First application of FIS in control.
- "E.H. Mamdani and S. Assilian, An experiment in Linguistic Synthesis with a Fuzzy Logic Controller", International Journal of Man-Machine Studies, 7 (1), pg. 1-13, 1975.

Example: Single-input, single-output Mamdani fuzzy model

$$\begin{cases} \text{If } X \text{ is small then } Y \text{ is small.} \\ \text{If } X \text{ is medium then } Y \text{ is medium.} \\ \text{If } X \text{ is large then } Y \text{ is large.} \end{cases}$$

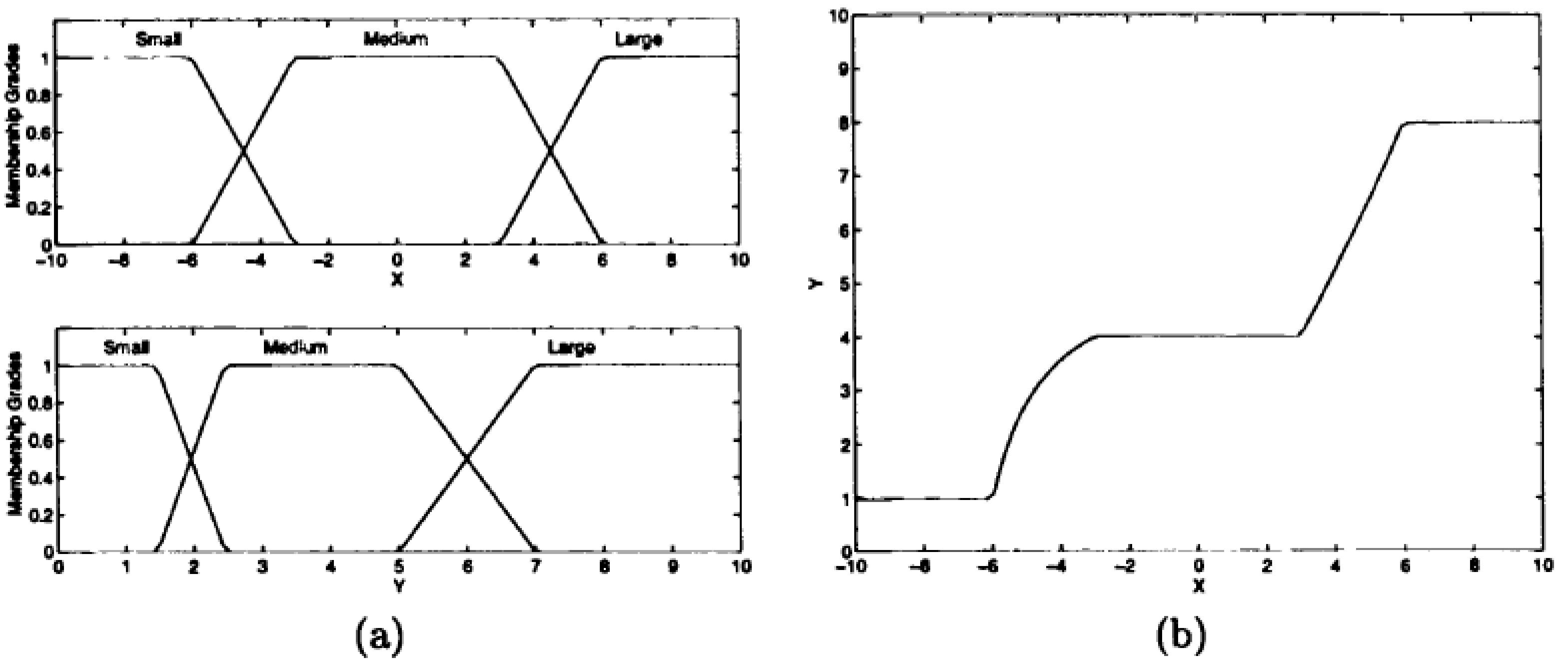


Figure 4.5. *Single-input single-output Mamdani fuzzy model in Example 4.1: (a) antecedent and consequent MFs; (b) overall input-output curve.* (MATLAB file: `mam1.m`)

Example: Two-input, single-output Mamdani fuzzy model

$$\begin{cases} \text{If } X \text{ is small and } Y \text{ is small then } Z \text{ is negative large.} \\ \text{If } X \text{ is small and } Y \text{ is large then } Z \text{ is negative small.} \\ \text{If } X \text{ is large and } Y \text{ is small then } Z \text{ is positive small.} \\ \text{If } X \text{ is large and } Y \text{ is large then } Z \text{ is positive large.} \end{cases}$$

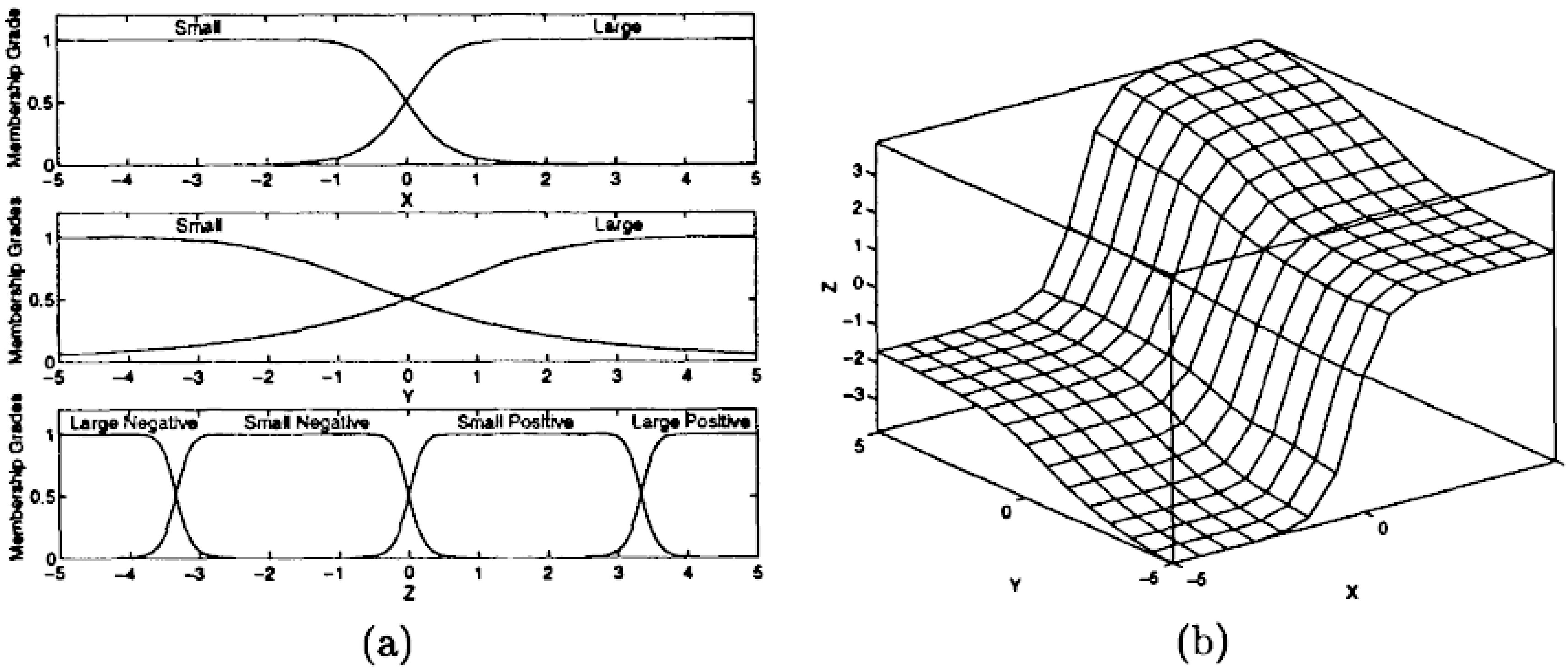
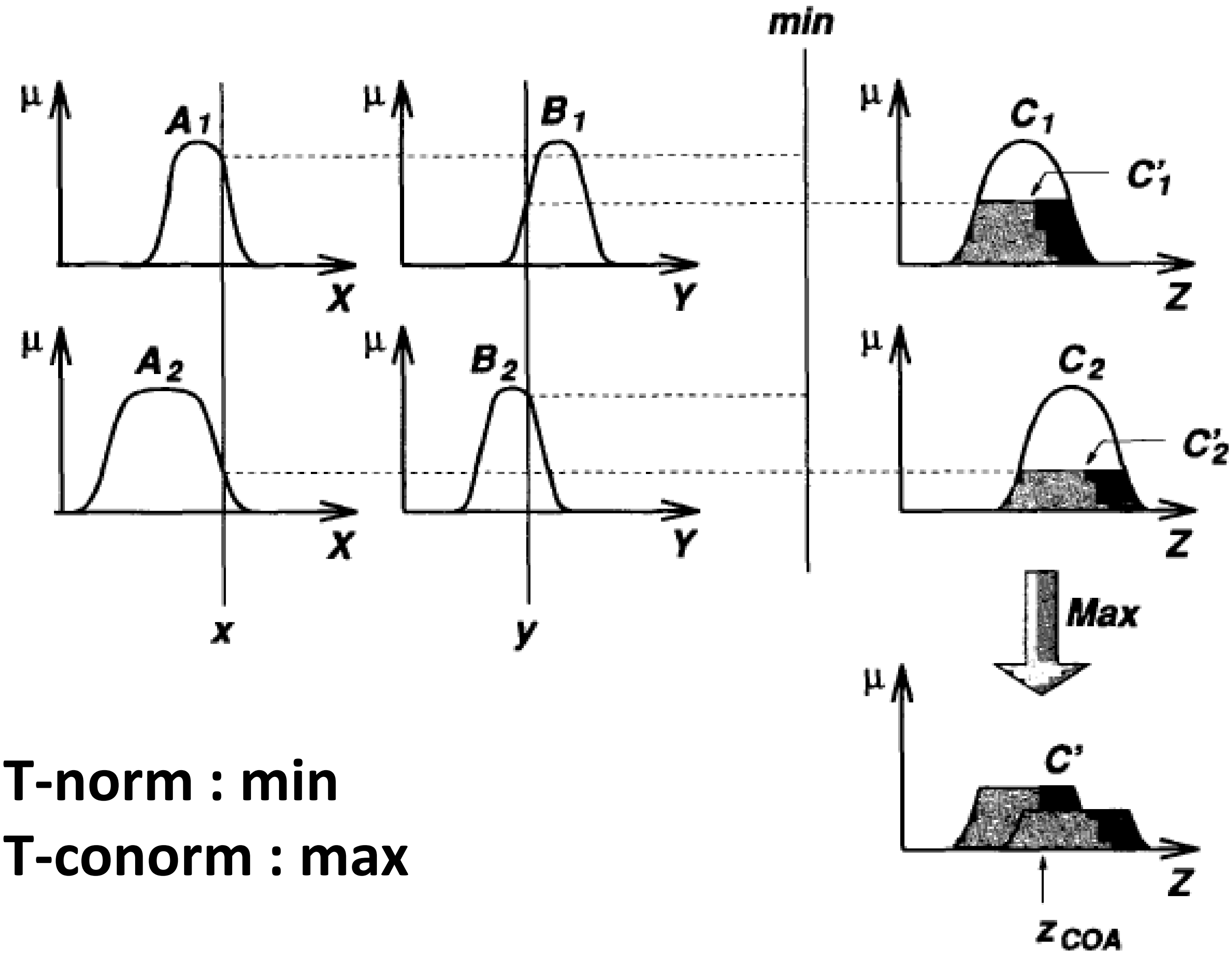
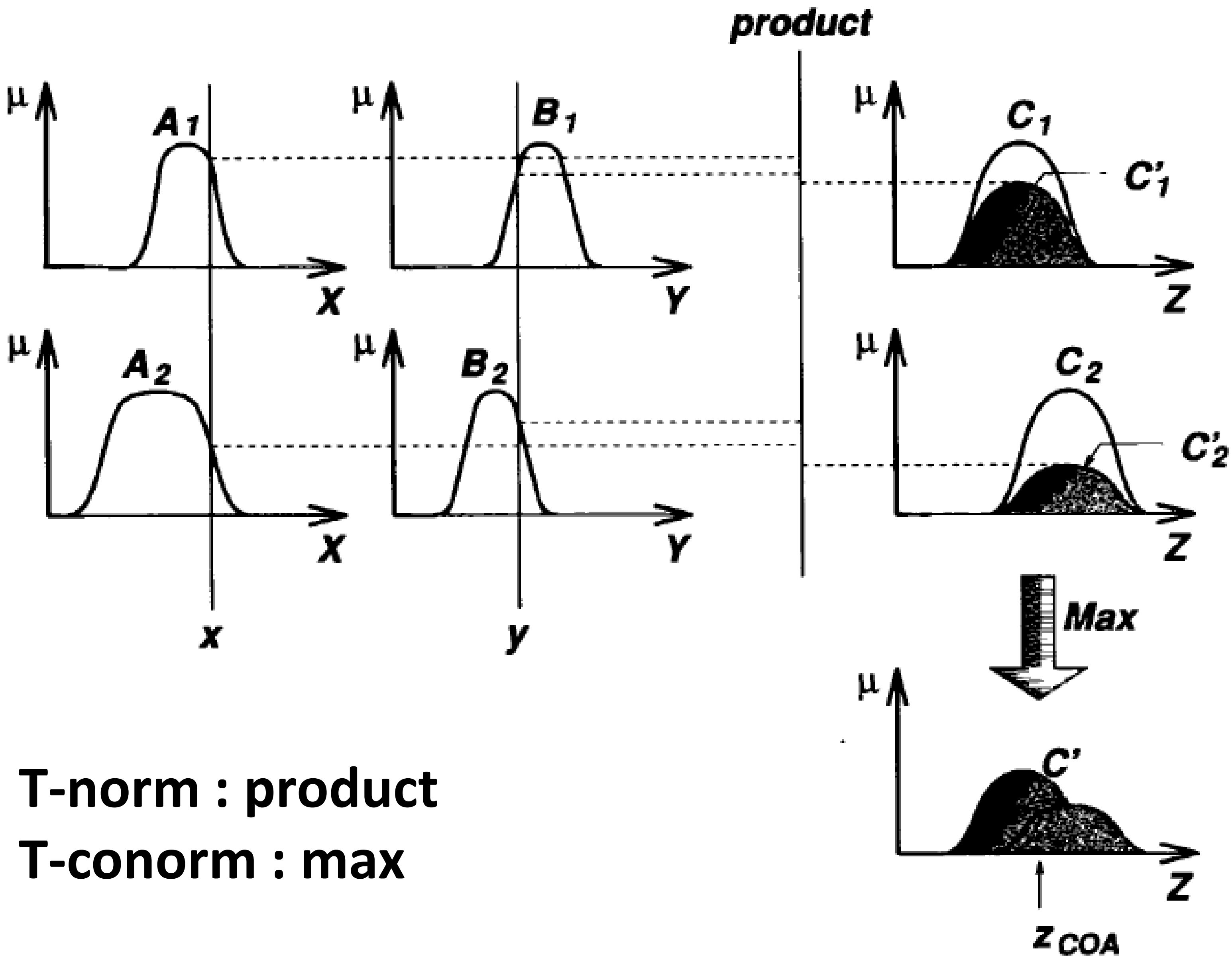


Figure 4.6. *Two-input single-output Mamdani fuzzy model in Example 4.2: (a) antecedent and consequent MFs; (b) overall input-output surface.* (MATLAB file: `mam2.m`)



T-norm : min
T-conorm : max



T-norm : product
T-conorm : max

Takagi-Sugeno-Kang (TSK) Fuzzy Models

$$\text{if } x \text{ is } A \text{ and } y \text{ is } B \text{ then } z = f(x, y)$$

where A and B are fuzzy sets in the antecedent,

$z = f(x, y)$ a crisp function in the consequent.

- Usually, $z = f(x, y)$ is a polynomial in x and y , but it can be any function.
- When $z = f(x, y)$ is a constant, we have **a zero order Sugeno fuzzy model**.
- This is a special case of **Mamdani FIS** where each rule's consequent is specified by a **fuzzy singleton**.
- When $z = f(x, y)$ is a first-order polynomial, we have a **first-order Sugeno fuzzy model**.

Example: Two-input, single-output Sugeno fuzzy model

$$
\begin{cases}
\text{If } X \text{ is small and } Y \text{ is small then } z = -x + y + 1. \\
\text{If } X \text{ is small and } Y \text{ is large then } z = -y + 3. \\
\text{If } X \text{ is large and } Y \text{ is small then } z = -x + 3. \\
\text{If } X \text{ is large and } Y \text{ is large then } z = x + y + 2.
\end{cases}
$$

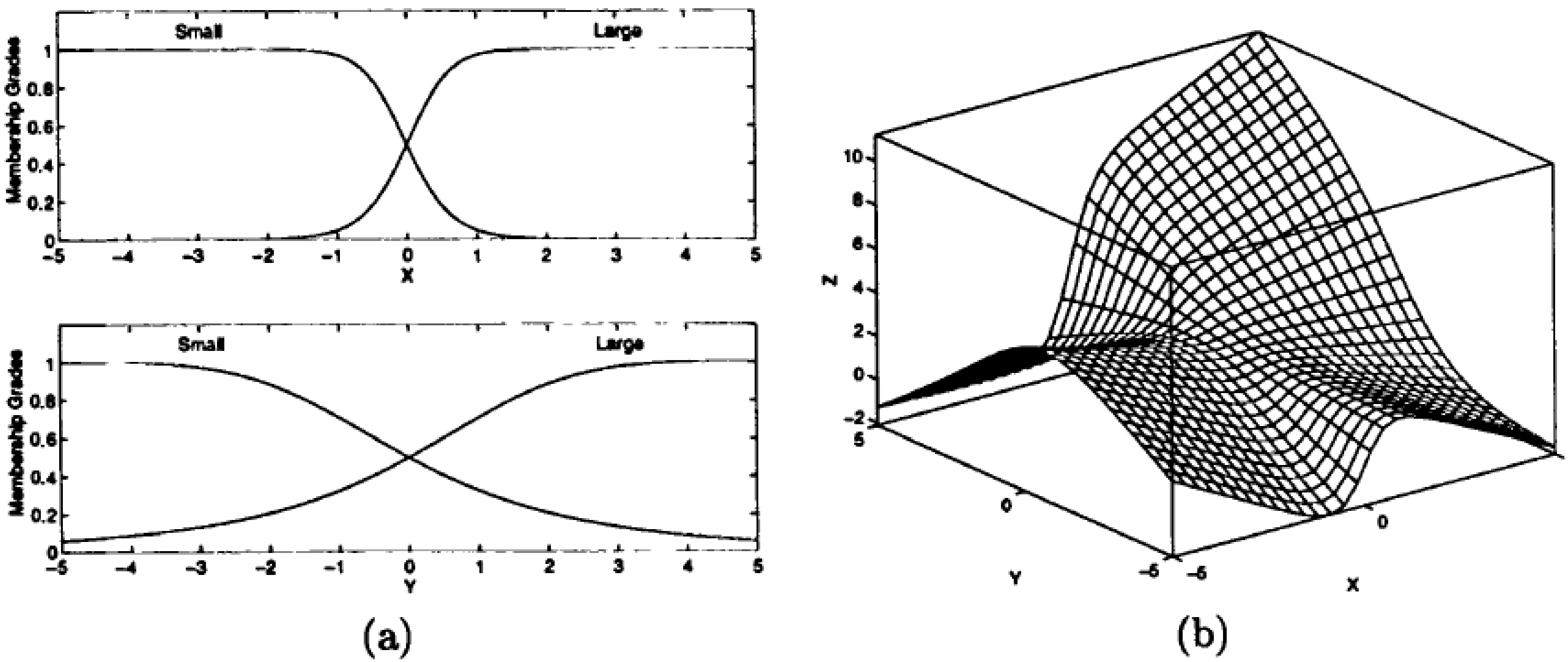
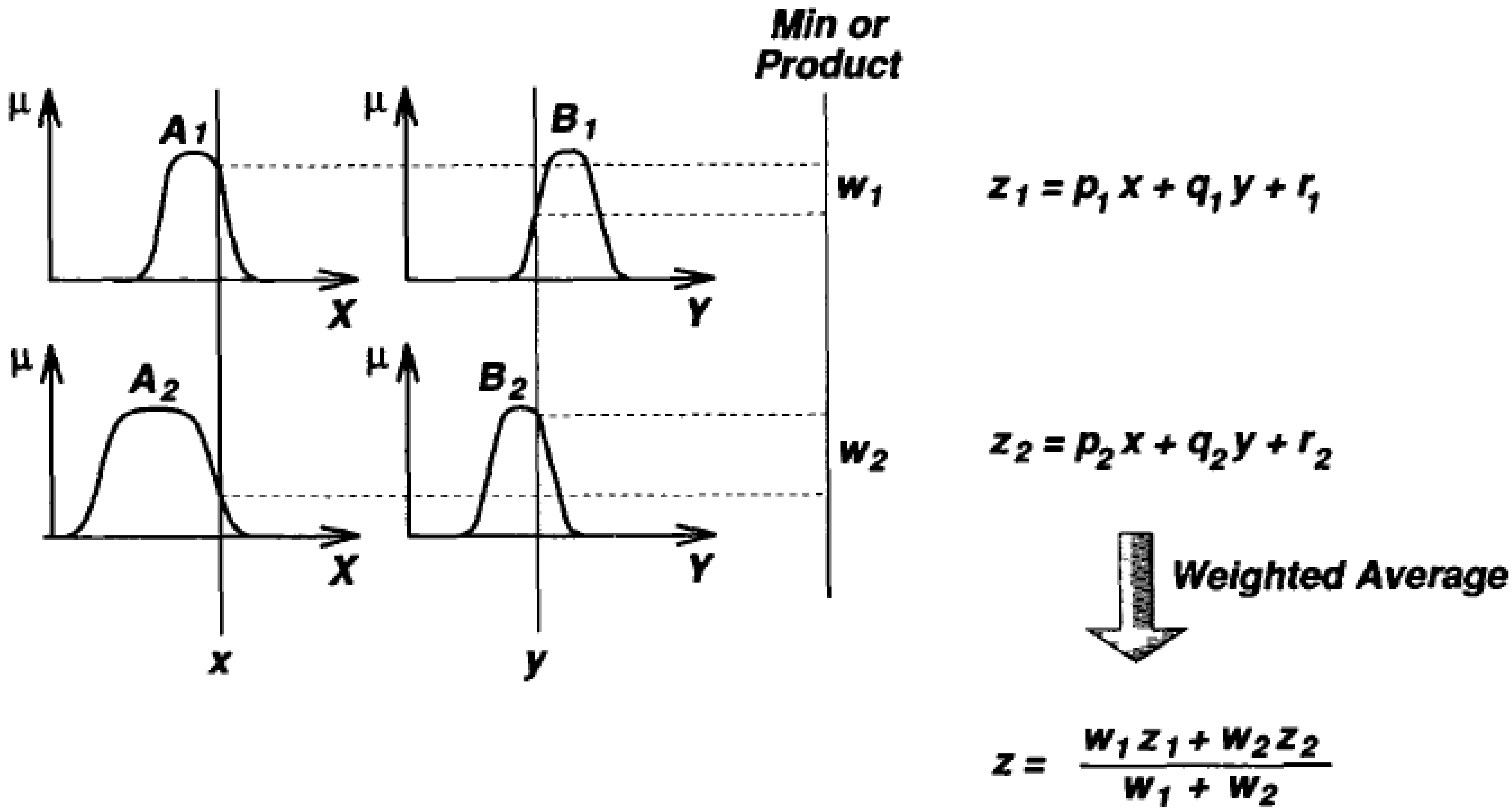


Figure 4.10. *Two-input single-output Sugeno fuzzy model in Example 4.4: (a) antecedent and consequent MFs; (b) overall input-output surface.* (MATLAB file: `sug2.m`)



The Sugeno Fuzzy Model

weighted sum operator can also be used:

$$z = w_1 z_1 + w_2 z_2$$

Tsukamoto Fuzzy Models

- The consequent part of each IF-THEN rule is represented by a fuzzy set with a monotonical (increasing or decreasing) MF. The inferred output will be a crisp value induced by the rule's firing strength.
- Each rule infers a crisp output.
- The overall output is computed as the weighted average of each rule's output.
- The Tsukamoto fuzzy model is not used frequently since it is not as transparent as either Mamdani or Sugeno fuzzy models.

Example: Single-input, single-output Tsukamoto model model

$$\begin{cases} \text{If } X \text{ is small then } Y \text{ is } C_1 \\ \text{If } X \text{ is medium then } Y \text{ is } C_2 \\ \text{If } X \text{ is large then } Y \text{ is } C_3 \end{cases}$$

$$(\sum_{i=1}^{3} w_i f_i)/(\sum_{i=1}^{3} w_i)$$

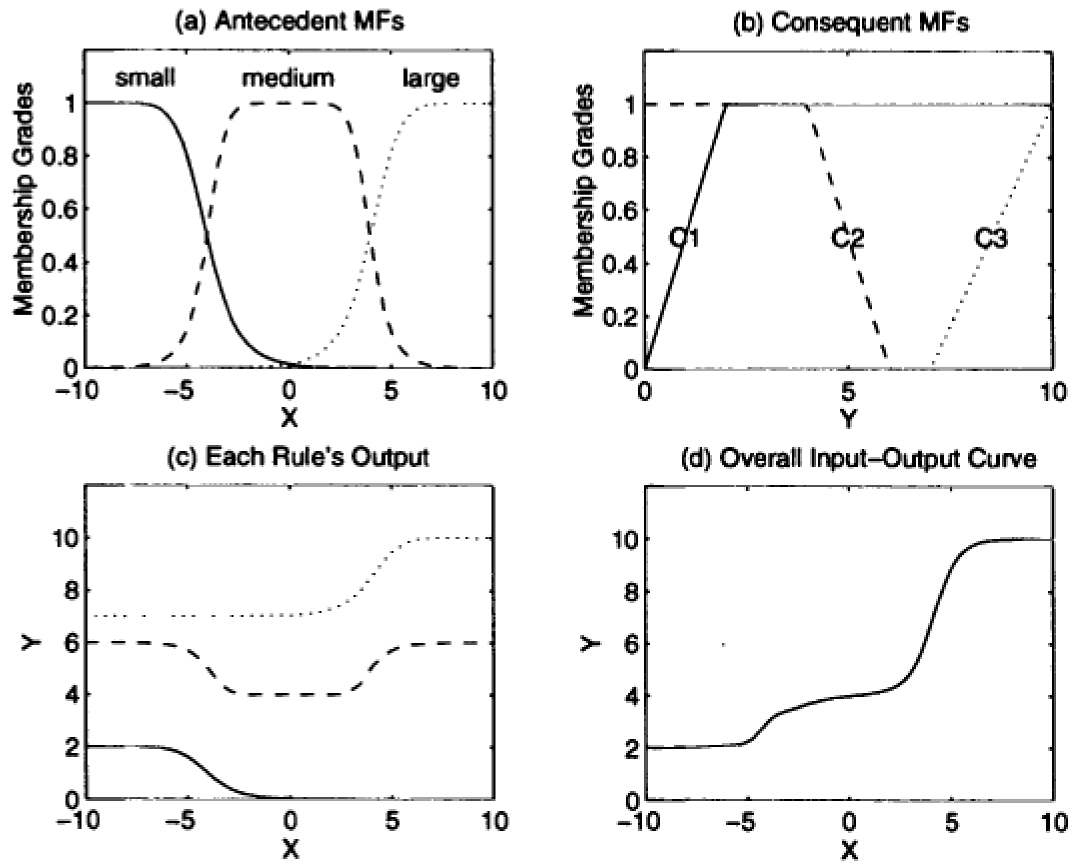
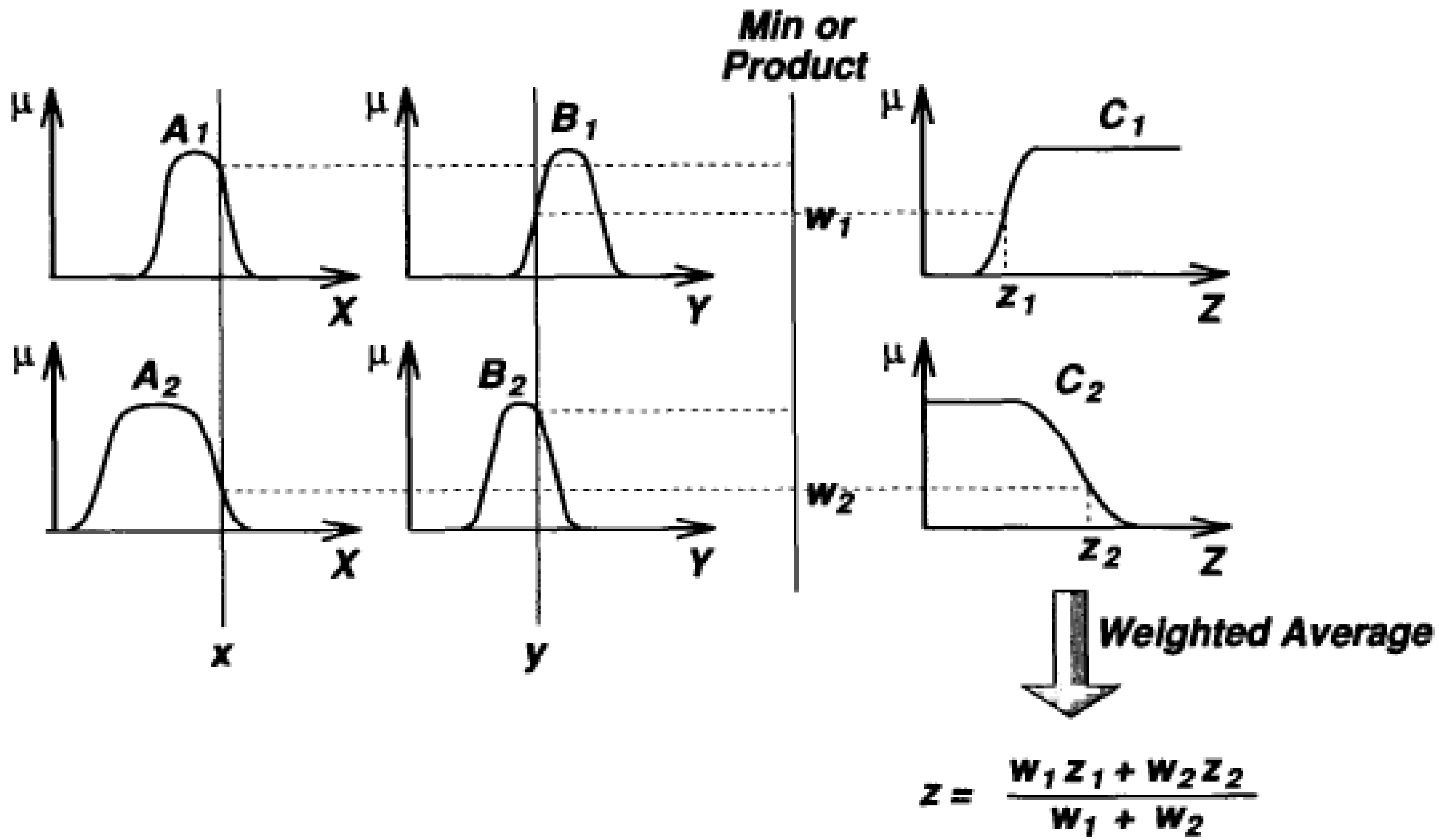


Figure 4.12. *Single-input single output Tsukamoto fuzzy model in Example 4.4: (a) antecedent MFs; (b) consequent MFs; (c) each rule's output curve; (d) overall input-output curve.* (MATLAB file: `tsu1.m`)



The Tsukamoto Fuzzy Model