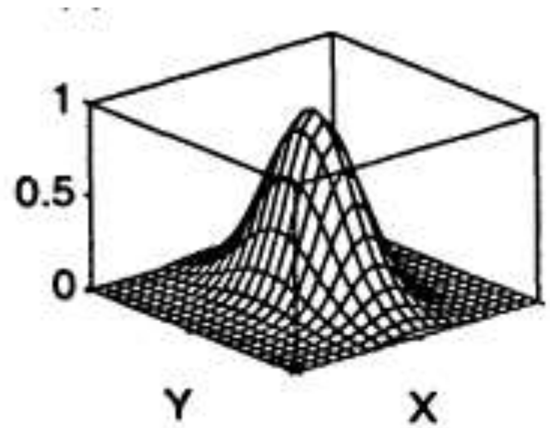
KON 426E
INTELLIGENT CONTROL SYSTEMS

LECTURE 9
25/04/2022

Membership Functions of Two Dimensions

- These are MF’s with two inputs, each in a different universe of discourse.



A Two_Dimensional MF

- There are two basic operations related with MF’s of two dimensions.

Cylindrical Extensions of One-Dimensional MF

If A is a fuzzy set in X, then its **cylindrical extension** in $X \times Y$ is a fuzzy set defined by $c(A)$:

$$c(A) = \int_{X \times Y} \mu_A(x)/(x, y)$$

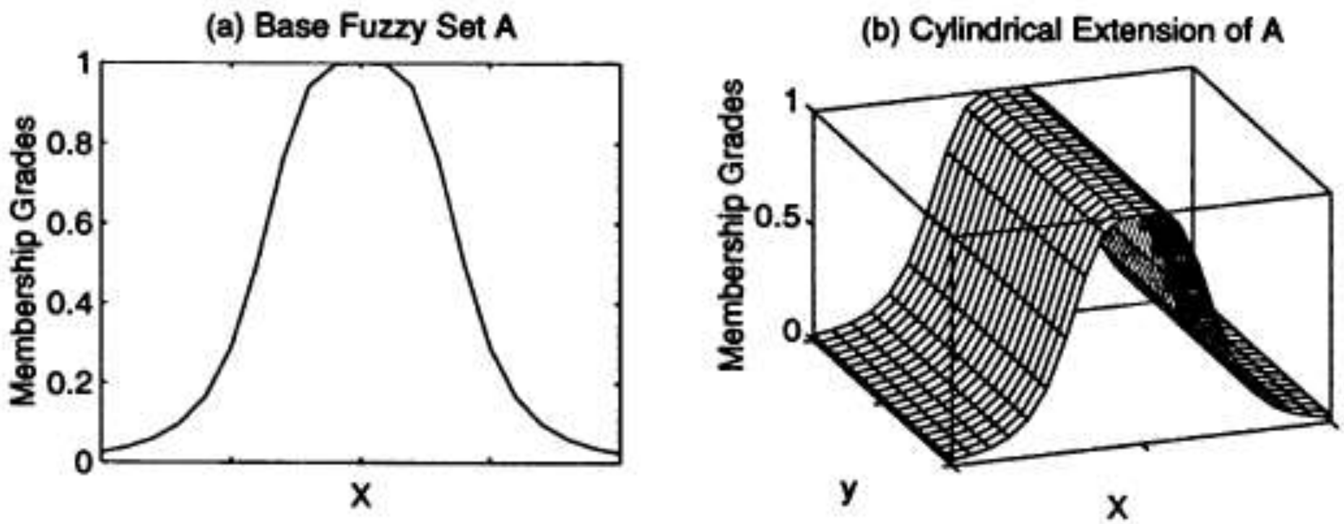


Figure 2.12. *(a) Base set A; (b) its cylindrical extension $c(A)$.* (MATLAB file: cyl_ext.m)

Projections of Fuzzy Sets

Let R be a two-dimensional fuzzy set on $X \times Y$

Then the projections of R onto X and Y are defined as:

$$R_X = \int_X [\max_y \mu_R(x,y)]/x \qquad R_Y = \int_Y [\max_x \mu_R(x,y)]/y$$

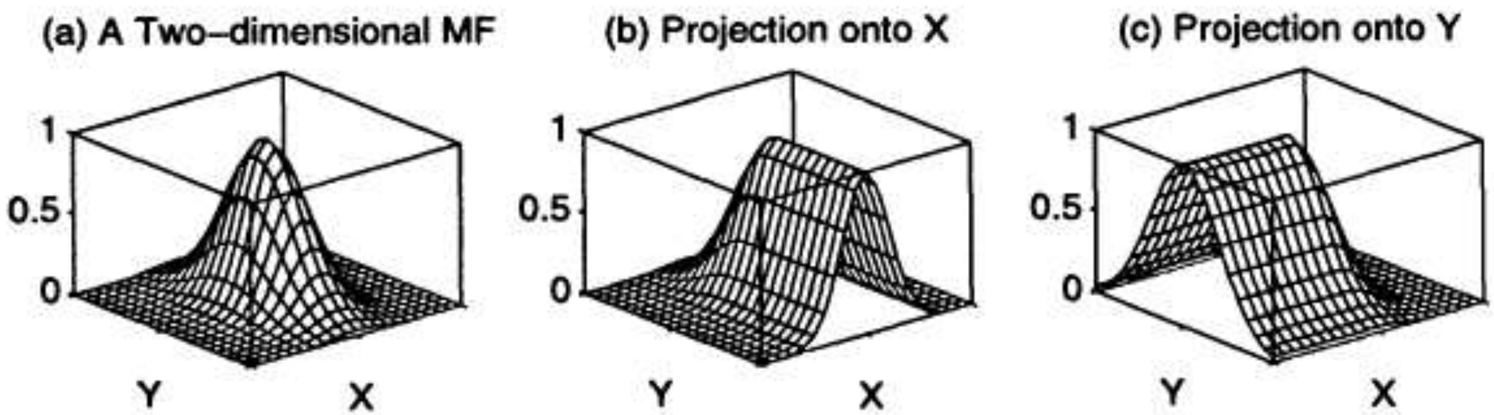


Figure 2.13. *(a) Two-dimensional fuzzy set R; (b) R_X (projection of R onto X); and (c) R_Y (projection of R onto Y).* (MATLAB file: `project.m`)

Composite and Noncomposite MF's

If an MF of two dimensions can be expressed as an analytic expression of two MF's of one-dimension, then it is **composite**. Otherwise, it is **noncomposite.**

Example:

$$\mu_A(x,y) = \exp\left[-\left(\frac{x-3}{2}\right)^2 - (y-4)^2\right]$$

$$\begin{aligned}\mu_A(x,y) &= \exp\left[-\left(\frac{x-3}{2}\right)^2\right] \exp\left[-\left(\frac{y-4}{1}\right)^2\right] \\ &= \text{gaussian}(x;3,2)\ \text{gaussian}(y;4,1)\end{aligned}$$

So, this is composite.

Example: (non-composite)

$$\mu_A(x,y) = \frac{1}{1+|x-3|\,|y-4|^{2.5}}$$

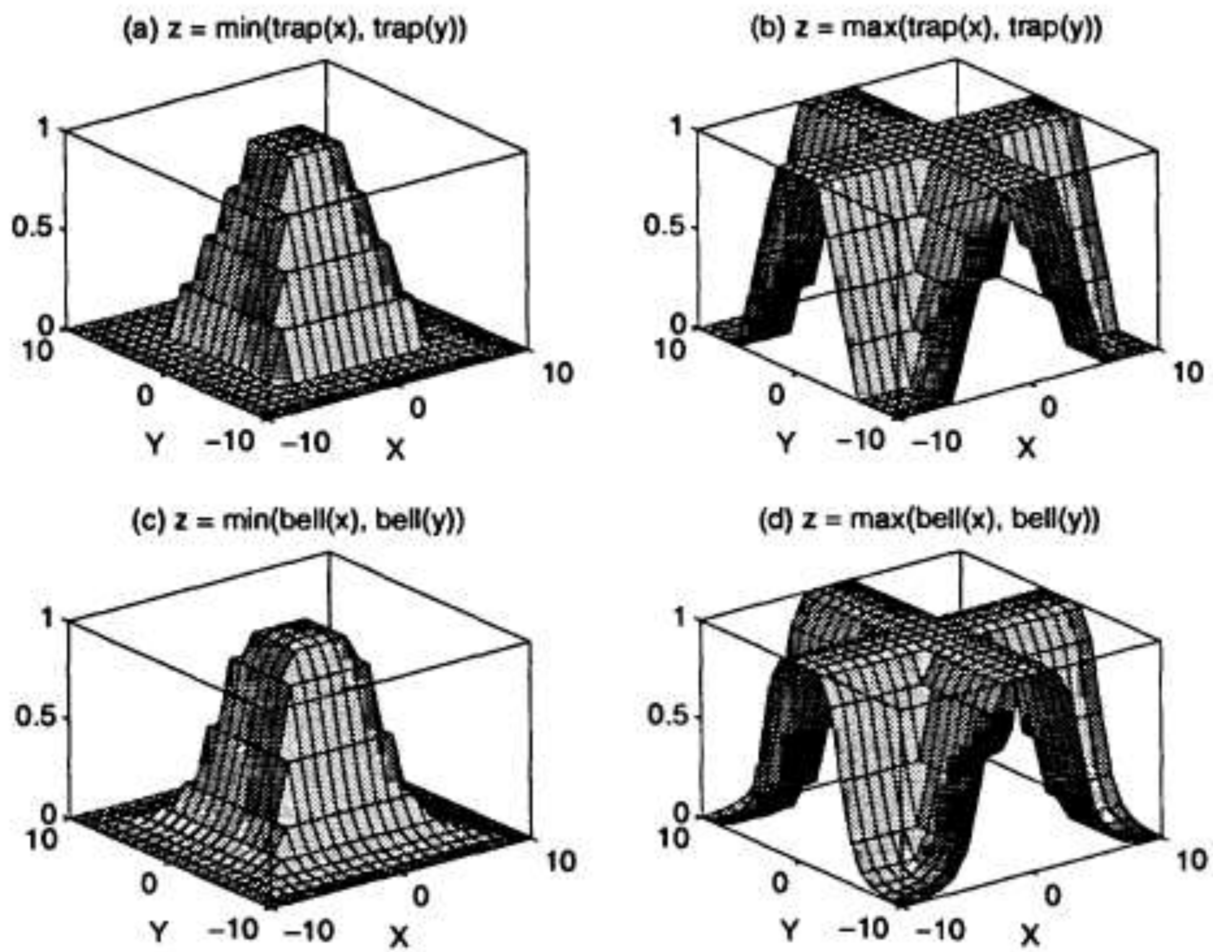


Figure 2.14. *Two-dimensional MFs defined by the min and max operators: (a)* $z = \min(\mathit{trap}(x), \mathit{trap}(y))$*; (b)* $z = \max(\mathit{trap}(x), \mathit{trap}(y))$*; (c)* $z = \min(\mathit{bell}(x), \mathit{bell}(y))$*; (d)* $z = \max(\mathit{bell}(x), \mathit{bell}(y))$. (MATLAB file: `mf2d.m`)

Fuzzy Intersection (T-norm: Triangular norm)

Intersection of two fuzzy sets A and B is specified by a $T : [0,1] \times [0,1] \to [0,1]$ which aggregates two membership grades as follows:

$$\mu_{A \cap B}(x) = T(\mu_A(x), \mu_B(x)) = \mu_A(x) \tilde{*} \mu_B(x),$$

$\tilde{*}$ is a binary operator for fuction T.

***T*-norm operator** is a two-place function $T(.,.)$ satisfying:

$T(0,0) = 0,\ T(a,1) = T(1,a) = a$	(boundary)
$T(a,b) \le T(c,d)$ if $a \le c$ and $b \le d$	(monotonicity)
$T(a,b) = T(b,a)$	(commutativity)
$T(a, T(b,c)) = T(T(a,b), c)$	(associativity).

Examples: Some frequently used T-norms

Minimum: $T_{min}(a,b) = \min(a,b) = a \wedge b.$

Algebraic product: $T_{ap}(a,b) = ab.$

Bounded product: $T_{bp}(a,b) = 0 \vee (a+b-1).$

Drastic product: $T_{dp}(a,b) = \begin{cases} a, & \text{if } b = 1. \\ b, & \text{if } a = 1. \\ 0, & \text{if } a, b < 1. \end{cases}$

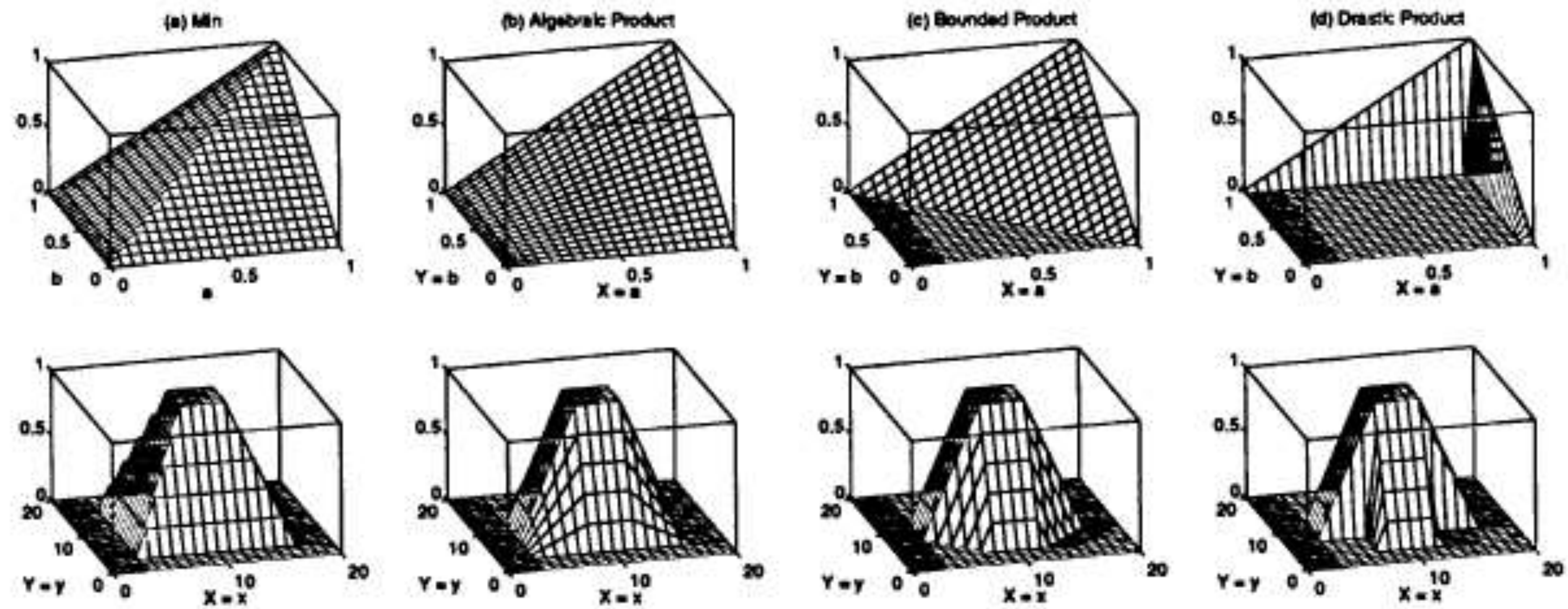


Figure 2.16. *(First row) Four T-norm operators* $T_{min}(a,b)$, $T_{ap}(a,b)$, $T_{bp}(a,b)$, *and* $T_{dp}(a,b)$; *(second row) the corresponding surfaces for* $a = trapezoid(x,3,8,12,17)$ *and* $b = trapezoid(y,3,8,12,17)$. (MATLAB file: `tnorm.m`)

Fuzzy Union (S-norm)(T-conorm)

Union of two fuzzy sets A and B is specified by a function $S : [0,1] \times [0,1] \rightarrow [0,1]$. which aggregates two membership grades as follows:

$$\mu_{A \cup B}(x) = S(\mu_A(x), \mu_B(x)) = \mu_A(x) \tilde{+} \mu_B(x)$$

S-norm operator is a two-place function $S(.,.)$ satisfying:

$$\begin{array}{ll} S(1,1) = 1,\ S(0,a) = S(a,0) = a & \text{(boundary)} \\ S(a,b) \leq S(c,d) \text{ if } a \leq c \text{ and } b \leq d & \text{(monotonicity)} \\ S(a,b) = S(b,a) & \text{(commutativity)} \\ S(a,S(b,c)) = S(S(a,b),c) & \text{(associativity).} \end{array}$$

Examples: Some frequently used S-norms

Maximum: $S(a,b) = \max(a,b) = a \vee b.$

Algebraic sum: $S(a,b) = a + b - ab.$

Bounded sum: $S(a,b) = 1 \wedge (a+b).$

Drastic sum: $S(a,b) = \begin{cases} a, & \text{if } b = 0. \\ b, & \text{if } a = 0. \\ 1, & \text{if } a, b > 0. \end{cases}$

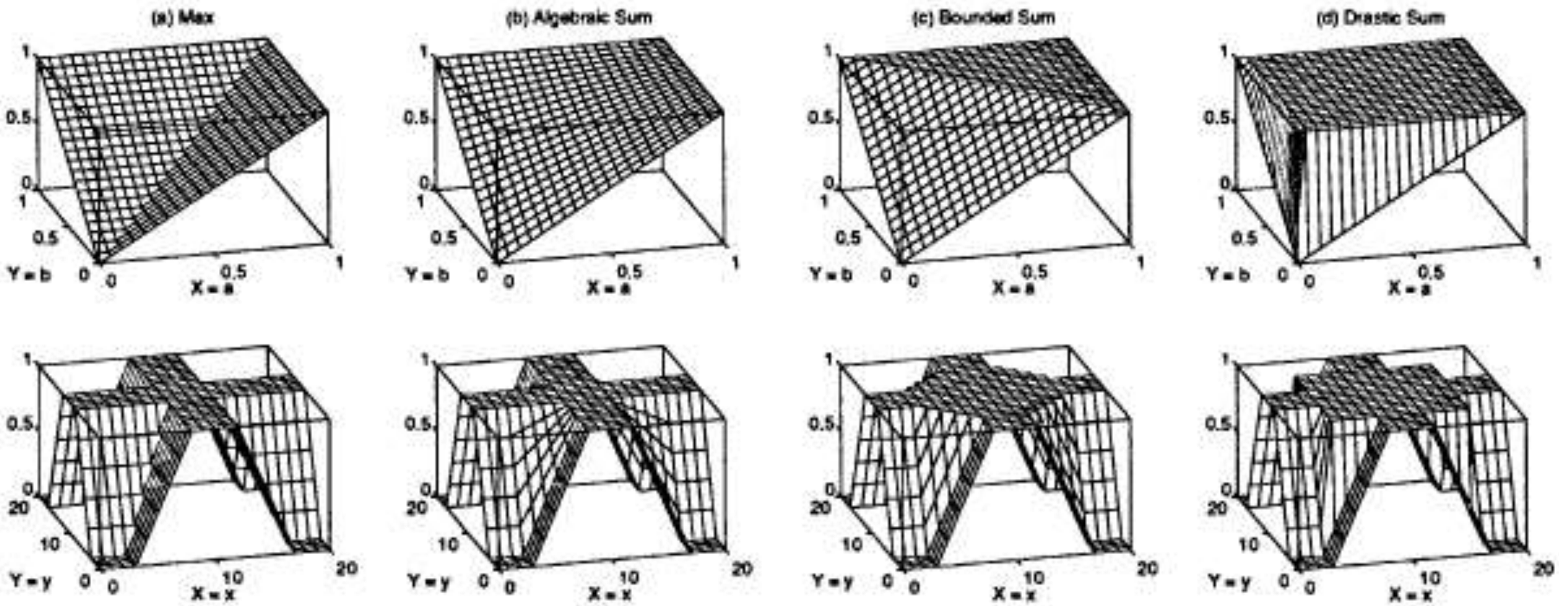


Figure 2.17. *(First row) Four T-conorm operators* $S_{min}(a,b)$, $S_{ap}(a,b)$ $S_{bp}(a,b)$ *and* $S_{dp}(a,b)$; *(second row) the corresponding surfaces for* a = *trapezoid*$(x,3,8,12,17)$ *and* b = *trapezoid*$(y,3,8,12,17)$. (MATLAB file: `tconorm.m`)

Fuzzy Complement

A fuzzy complement operator is a continuous function $N : [0,1] \rightarrow [0,1]$ which meets the following aximatic requirements:

$$
\begin{array}{ll}
N(0) = 1 \text{ and } N(1) = 0 & \text{(boundary)} \\
N(a) \geq N(b) \; if \; a \leq b & \text{(monotonicity).}
\end{array}
$$

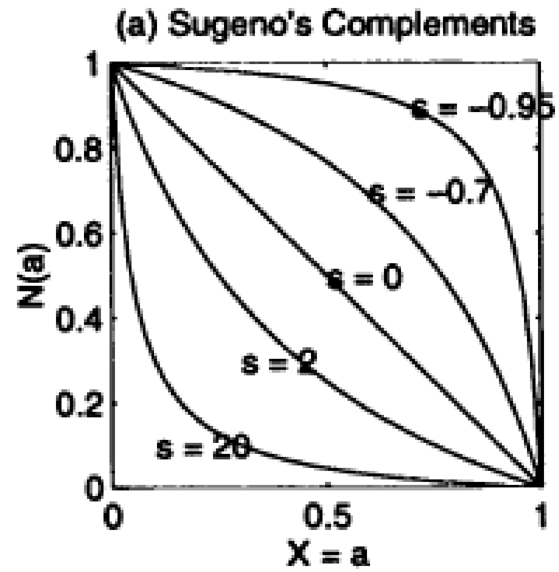
Optional requirement:

$$N(N(a)) = a \quad \text{(involution)}$$

Example: Sugeno's Complement

$$N_s(a) = \frac{1-a}{1+sa}$$

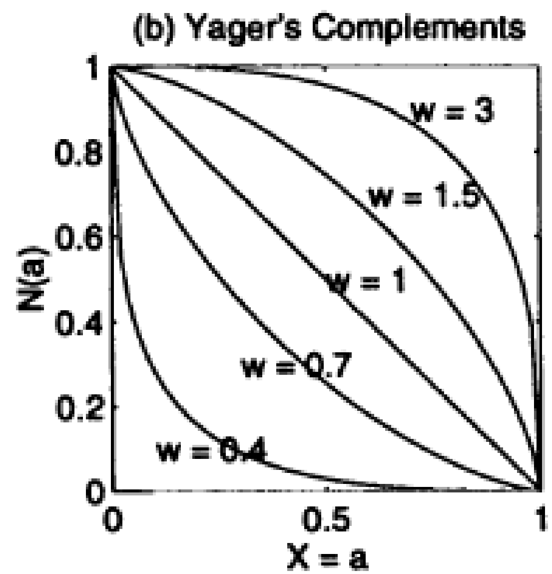
(***s*** is a parameter greater than -1)



Example: Yager's Complement

$$N_w(a) = (1 - a^w)^{1/w}$$

(***w*** is a positive parameter)



DeMorgan's Laws

For classical sets:

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

For fuzzy sets:

T-norms and S-norms are duals which support the generalization of DeMorgan's law:

$$T(a, b) = N(S(N(a), N(b)))$$
$$S(a, b) = N(T(N(a), N(b)))$$

For a given T-norm operator, we can always find a corresponding S-norm and vice versa.

Examples of Parameterized T-norms and S-norms

Yager For $q > 0$,

$$\begin{cases} T_Y(a,b,q) = 1 - \min\{1, [(1-a)^q + (1-b)^q]^{1/q}\} \\ S_Y(a,b,q) = \min\{1, (a^q + b^q)^{1/q}\} \end{cases}$$

Dubois and Prade For $\alpha \in [0,1]$,

$$\begin{cases} T_{DP}(a,b,\alpha) = ab/\max\{a,b,\alpha\}, \\ S_{DP}(a,b,\alpha) = [a+b-ab-\min\{a,b,(1-\alpha)\}/\max\{1-a,1-b,\alpha\} \end{cases}$$

Hamacher For $\gamma > 0$,

$$\begin{cases} T_H(a,b,\gamma) = ab/[\gamma + (1-\gamma)(a+b-ab)], \\ S_H(a,b,\gamma) = [a+b+(\gamma-2)ab]/[1+(\gamma-1)ab] \end{cases}$$

Frank For $s > 0$,

$$\begin{cases} T_F(a,b,s) = \log_s[1 + (s^a - 1)(s^b - 1)/(s-1)] \\ S_F(a,b,s) = 1 - \log_s[1 + (s^{1-a} - 1)(s^{1-b} - 1)/(s-1)] \end{cases}$$

Sugeno For $\lambda \geq -1$,

$$\begin{cases} T_S(a,b,\lambda) = \max\{0, (\lambda+1)(a+b-1) - \lambda ab\} \\ S_S(a,b,\lambda) = \min\{1, a+b-\lambda ab\} \end{cases}$$

Dombi For $\lambda > 0$,

$$\begin{cases} T_D(a,b,\lambda) = \dfrac{1}{1 + [(a^{-1}-1)^{\lambda} + (b^{-1}-1)^{\lambda}]^{1/\lambda}} \\ S_D(a,b,\lambda) = \dfrac{1}{1 + [(a^{-1}-1)^{-\lambda} + (b^{-1}-1)^{-\lambda}]^{-1/\lambda}} \end{cases}$$

Binary Relations (for crisp sets)

A binary relation between two sets X_1 and X_2 is a subset of $X_1 \times X_2$ (It is a collection of ordered pairs)

Let X_1 and X_2 be two sets.

The Cartesian product (the set of all ordered pairs) is:

$$X_1 \times X_2 = \{(x, y): x \in X_1, y \in X_2\}$$

Characteristic function is:

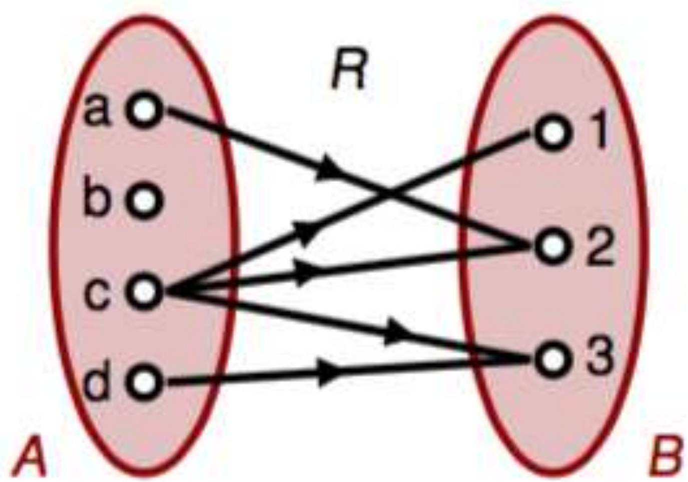
$$\varphi_R(x, y) = \begin{cases} 1, & (x, y) \in R \\ 0, & otherwise \end{cases}$$

$$R(X_1, X_2) \subseteq X_1 \times X_2$$

$$R(X_1, X_2, \ldots\ldots X_n) \subseteq X_1 \times X_2 \times X_3 \times \ldots\ldots\ldots \times X_n$$

There are three ways to show relations.

- Venn diagrams
- Listing
- Matrix method



Matrix representation:

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$R = \{\langle a,2\rangle, \langle c,1\rangle, \langle c,2\rangle, \langle c,3\rangle, \langle d,3\rangle\} \subseteq \{a,b,c,d\} \times \{1,2,3\}$$

Binary Fuzzy Relation

Let X and Y be two universes of discourse. Then:

$$\mathcal{R} = \{(\ (x,y),\ \mu_{\mathcal{R}}(x,y)\ )\ |\ (x,y) \in X \times Y\}$$

is a binary fuzzy relation in $X \times Y$

Note that $\mu_{\mathcal{R}}(x,y)$ is a two-dimensional MF.

Example

Let $X = Y = R^+$ (the positive real line) and $\mathcal{R}$ = "y is much greater than x." The MF of the fuzzy relation $\mathcal{R}$ can be subjectively defined as

$$\mu_{\mathcal{R}}(x, y) = \begin{cases} \dfrac{y - x}{x + y + 2}, & \text{if } y > x. \\ 0, & \text{if } y \le x. \end{cases}$$

If $X = \{3, 4, 5\}$ and $Y = \{3, 4, 5, 6, 7\}$, then it is convenient to express the fuzzy relation $\mathcal{R}$ as a **relation matrix**:

$$\mathcal{R} = \begin{bmatrix} 0 & 0.111 & 0.200 & 0.273 & 0.333 \\ 0 & 0 & 0.091 & 0.167 & 0.231 \\ 0 & 0 & 0 & 0.077 & 0.143 \end{bmatrix},$$

where the element at row i and column j is equal to the membership grade between the ith element of X and jth element of Y.

Example

X={New York, Paris}

Y={Pekin, New York, London}

R="city x is distant from city y"

$$R = \begin{bmatrix} 1 & 0 & 0.6 \\ 0.9 & 0.7 & 0.3 \end{bmatrix}$$

Some common examples of binary fuzzy relations

- x is close to y (x and y are numbers)
- x depends on y (x and y are events)
- x and y look alike (x and y are persons, objects, and so on)
- If x is large, then y is small (x is an observed reading and y is a corresponding action).

How are binary fuzzy relations combined?

Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two fuzzy relations defined on $X \times Y$ and $Y \times Z$ respectively.

They can be combined in **two ways**:

Max-min composition (Zadeh)

$$\mathcal{R}_1 \circ \mathcal{R}_2 = \{[(x,z), \max_y \min(\mu_{\mathcal{R}_1}(x,y), \mu_{\mathcal{R}_2}(y,z))] | x \in X, y \in Y, z \in Z\}$$

Or equivalently:

$$\begin{aligned} \mu_{\mathcal{R}_1 \circ \mathcal{R}_2}(x,z) &= \max_y \min[\mu_{\mathcal{R}_1}(x,y), \mu_{\mathcal{R}_2}(y,z)] \\ &= \vee_y \left[\mu_{\mathcal{R}_1}(x,y) \wedge \mu_{\mathcal{R}_2}(y,z)\right] \end{aligned}$$

Max-product composition

$$\mu_{\mathcal{R}_1 \circ \mathcal{R}_2}(x, z) = \max_y \left[\mu_{\mathcal{R}_1}(x, y)\mu_{\mathcal{R}_2}(y, z)\right]$$

Example

Let

$\mathcal{R}_1$ = "x is relevant to y"

$\mathcal{R}_2$ = "y is relevant to z"

be two fuzzy relations defined on $X \times Y$ and $Y \times Z$, respectively, where $X = \{1, 2, 3\}$, $Y = \{\alpha, \beta, \gamma, \delta\}$, and $Z = \{a, b\}$. Assume that $\mathcal{R}_1$ and $\mathcal{R}_2$ can be expressed as the following relation matrices:

$$\mathcal{R}_1 = \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.7 \\ 0.4 & 0.2 & 0.8 & 0.9 \\ 0.6 & 0.8 & 0.3 & 0.2 \end{bmatrix}$$

$$\mathcal{R}_2 = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.3 \\ 0.5 & 0.6 \\ 0.7 & 0.2 \end{bmatrix}.$$

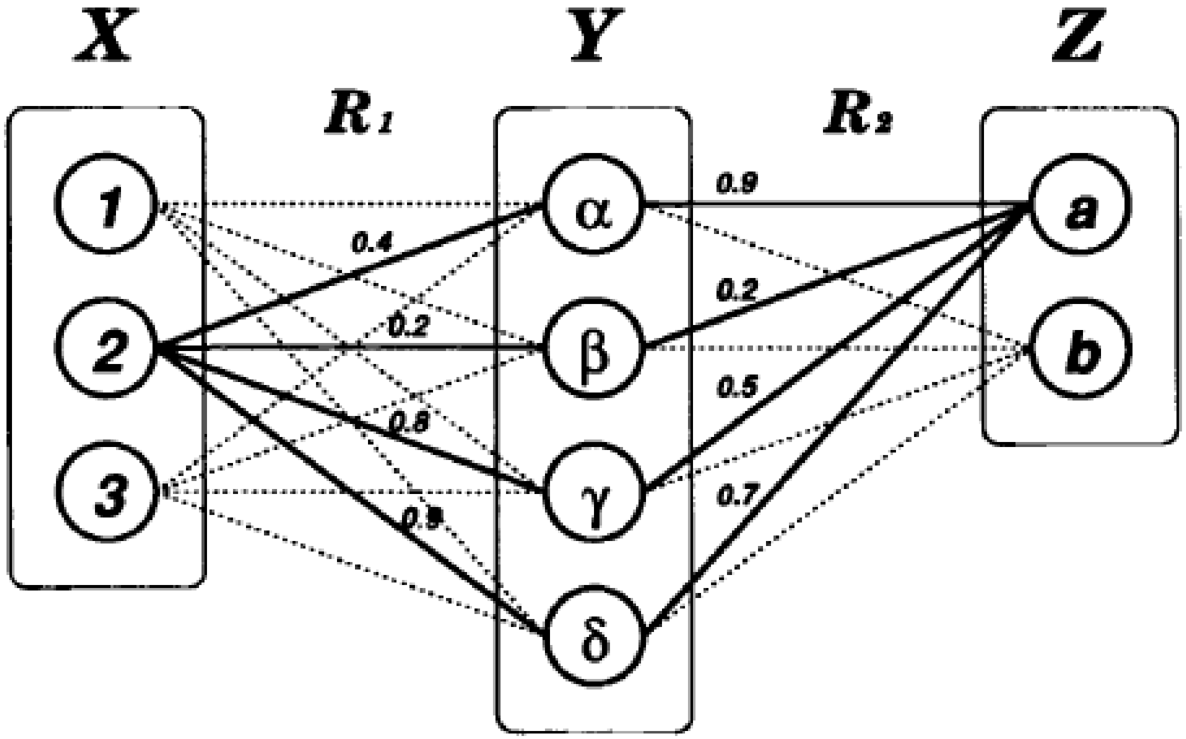
Now we want to find $\mathcal{R}_1 \circ \mathcal{R}_2$, which can be interpreted as a derived fuzzy relation "x is relevant to z" based on $\mathcal{R}_1$ and $\mathcal{R}_2$. For simplicity, suppose that we are only interested in the degree of relevance between 2 ($\in X$) and a ($\in Z$). If we adopt max-min composition, then

Max-min composition:

$$
\begin{aligned}
\mu_{\mathcal{R}_1 \circ \mathcal{R}_2}(2, a) &= \max(0.4 \wedge 0.9,\ 0.2 \wedge 0.2,\ 0.8 \wedge 0.5,\ 0.9 \wedge 0.7) \\
&= \max(0.4,\ 0.2,\ 0.5,\ 0.7) \\
&= 0.7 \text{ (by max-min composition).}
\end{aligned}
$$

Max-product composition:

$$
\begin{aligned}
\mu_{\mathcal{R}_1 \circ \mathcal{R}_2}(2, a) &= \max(0.4 \times 0.9,\ 0.2 \times 0.2,\ 0.8 \times 0.5,\ 0.9 \times 0.7) \\
&= \max(0.36,\ 0.04,\ 0.40,\ 0.63) \\
&= 0.63 \text{ (by max-product composition).}
\end{aligned}
$$



Composition of fuzzy relations

Extension Principle

(Generalizes point-to-point mapping f to a mapping between fuzzy sets)

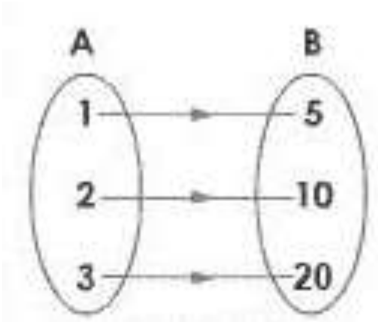
Suppose f is a function from X to Y and A is a fuzzy set on X defined by:

$$A = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \cdots + \mu_A(x_n)/x_n$$

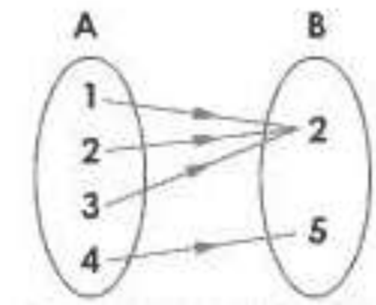
The extension principle states that the image of fuzzy set A under the mapping $f(\cdot)$ can be expressed as a fuzzy set B:

$$B = f(A) = \mu_A(x_1)/y_1 + \mu_A(x_2)/y_2 + \cdots + \mu_A(x_n)/y_n$$

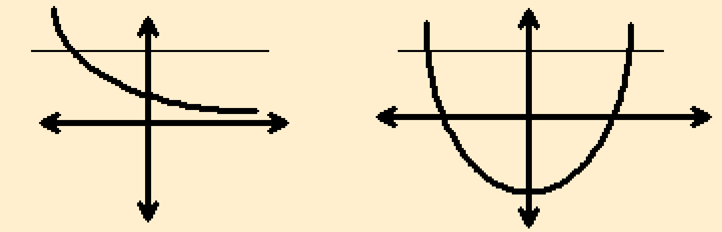
where $y_i = f(x_i),\ i = 1, \ldots, n$



One-to-one



many-to-one



one-to-one function

many-to-one function

If $f(\cdot)$ is a many-to one mapping, there exist $x_1, x_2 \in X, x_1 \neq x_2$ such that $f(x_1) = f(x_2) = y^*, y^* \in Y$

In this case, the membership grade of B at $y = y^*$ is the maximum of the membership grades of A at $x = x_1$ and $x = x_2$ since $f(x) = y^*$ may result from either $x = x_1$ or $x = x_2$.

We have:

$$\mu_B(y) = \max_{x = f^{-1}(y)} \mu_A(x)$$

Example:

Let

$$A = 0.1/-2 + 0.4/-1 + 0.8/0 + 0.9/1 + 0.3/2$$

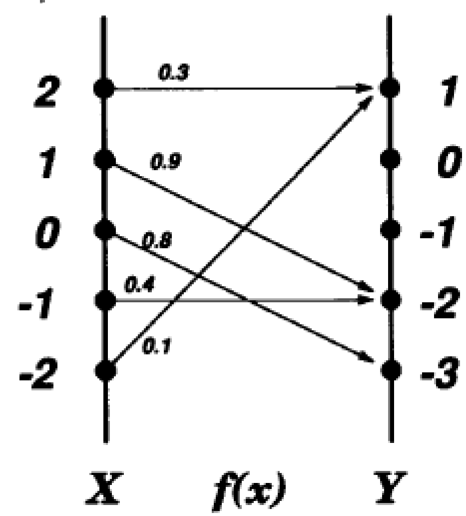
and

$$f(x) = x^2 - 3.$$

Upon applying the extension principle, we have

$$\begin{aligned} B &= 0.1/1 + 0.4/-2 + 0.8/-3 + 0.9/-2 + 0.3/1 \\ &= 0.8/-3 + (0.4 \vee 0.9)/-2 + (0.1 \vee 0.3)/1 \\ &= 0.8/-3 + 0.9/-2 + 0.3/1, \end{aligned}$$

where $\vee$ represents max.



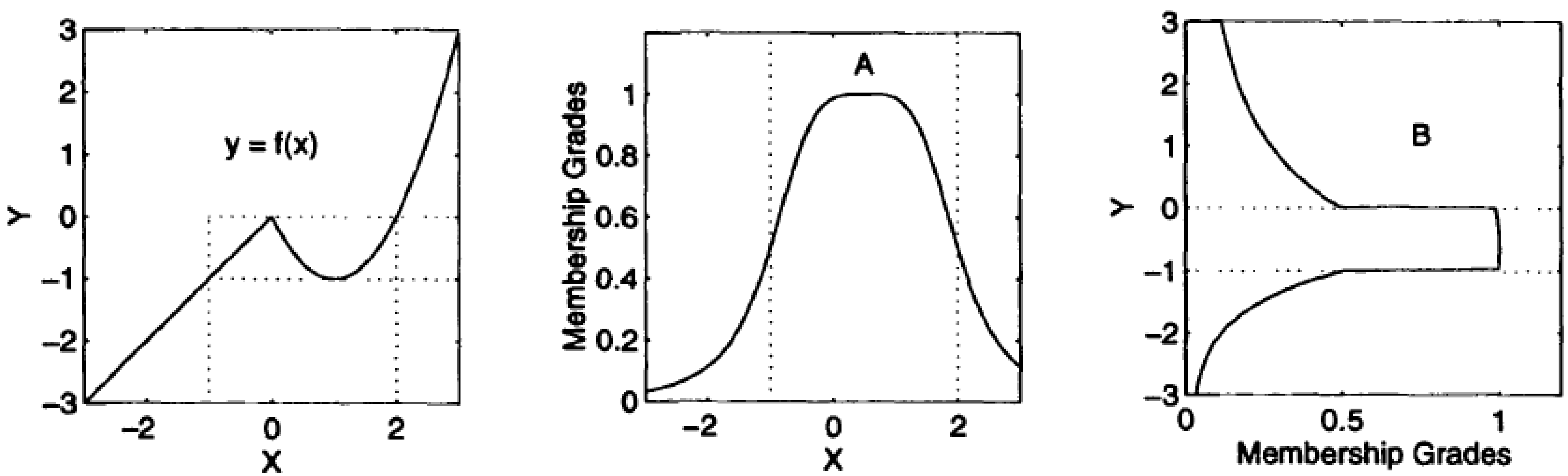
Example (continuous universe of discourse)

Let

$$\mu_A(x) = \text{bell}(x; 1.5, 2, 0.5)$$

and

$$f(x) = \begin{cases} (x-1)^2 - 1, & \text{if } x \geq 0. \\ x, & \text{if } x \leq 0. \end{cases}$$



- Upto now, $y = f(x)$ is a single-variable function.
- Now, suppose f is a mapping from an n-dimensional product space $X_1 \times X_2 \times \ldots\ldots\ldots \times X_n \to Y$ such that:

  $f(x_1, \ldots, x_n) = y$
- There is a fuzzy set A_i in each X_i, $i = 1, \ldots, n.$
- Since each element in an input vector $(x_1, \ldots, x_n)$

  occurs **simultaneously**, this implies an **AND** operation.
- Membership grade $\mu_B(y)$ of fuzzy set B induced by the mapping f should be the minimum of the membership grades of the constituent fuzzy sets A_i, $i = 1, \ldots, n.$

Formal Definition of the Extension Principle

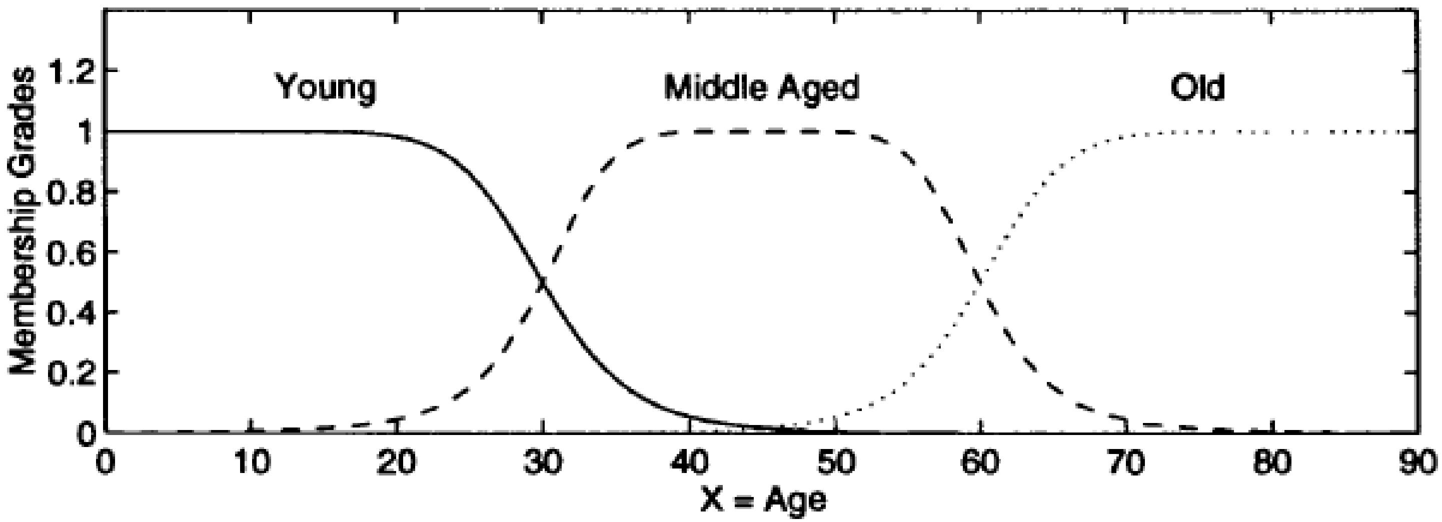
Suppose f is a mapping from an n-dimensional Cartesian product space $X_1 \times X_2 \times \cdots X_n$ to a one-dimensional universe Y such that $y = f(x_1, \ldots, x_n)$ and suppose that $A_1, \ldots, A_n$ are n fuzzy sets in $X_1, \ldots, X_n$ respectively.

Then the extension principle asserts that the fuzzy set B induced by mapping f is defined by:

$$\mu_B(y) = \begin{cases} \max\limits_{(x_1,\ldots,x_n),\,(x_1,\ldots,x_n)=f^{-1}(y)} [\min_i \ \mu_{A_i}(x_i)], & \text{if } f^{-1}(y) \neq \emptyset. \\ 0, & \text{if } f^{-1}(y) = \emptyset. \end{cases}$$

FUZZY IF-THEN RULES

Linguistic Variables



- Here, the **linguistic variable** is the "age".
- It is defined on the **universe of discourse** $X=[0,90]$.
- The **term set** of age is the set of linguistic values or linguistic terms: $T(age)=\{young, middle\text{-}aged, old\}$
- **"age is young":** assignment of a linguistic value to a linguistic variable.

Fuzzy IF-THEN rules

(Fuzzy Rule/Fuzzy Implication/Fuzzy Conditional Statement)

IF x *is* A THEN y *is* B

antecedent *consequence*

premise *conclusion*

A and B are linguistic values defined by fuzzy sets on universes of discourses X and Y, respectively.

Examples: **IF** *pressure is high*, **THEN** *volume is small*

IF *road is slippery*, **THEN** *driving is dangerous*

IF *a tomato is red*, **THEN** *it is ripe*

IF *speed is high*, **THEN** *apply the brake a little*

Example:

For a basic control application:

Linguistic values:

PS: Positive small

PM:Positive medium

PB: Positive big

NS:Negative small

NM:Negative medium

NB: Negative big

e: tracking error

$\dot{e}$: derivative of the tracking error

u: control input

IF e *is PS and* $\dot{e}$ *is PM* **THEN** u *is PS*

IF e *is NM and* $\dot{e}$ *is NB* **THEN** u *is NM*

Orthogonality

A term set $T = t_1, \ldots, t_n$ of a linguistic variable x on the universe X is orthogonal if it fulfills the following property:

$$\sum_{i=1}^{n} \mu_{t_i}(x) = 1, \ \forall x \in X$$

where the t_i 's are convex and normal fuzzy sets defined on X and these fuzzy sets make up the term set T.

Linguistic Modifiers

Concentration and Dilation of Linguistic Values

If A is a linguistic value, characterized by a fuzzy set with membership function $\mu_A(\cdot)$, then you can modify A as A^k :

$$A^k = \int_X [\mu_A(x)]^k / x$$

Concentration (Applying **VERY** or **TOO**)

$$\mathrm{CON}(A) = A^2$$

Dilation (Applying **MORE OR LESS**)

$$\mathrm{DIL}(A) = A^{0.5}$$

Example

Let
$$\mu_{\text{young}}(x) = \text{bell}(x, 20, 2, 0) = \frac{1}{1 + (\frac{x}{20})^4},$$

$$\mu_{\text{old}}(x) = \text{bell}(x, 30, 3, 100) = \frac{1}{1 + (\frac{x-100}{30})^6},$$

where x is the age of a given person, with the interval $[0, 100]$ as the universe of discourse.

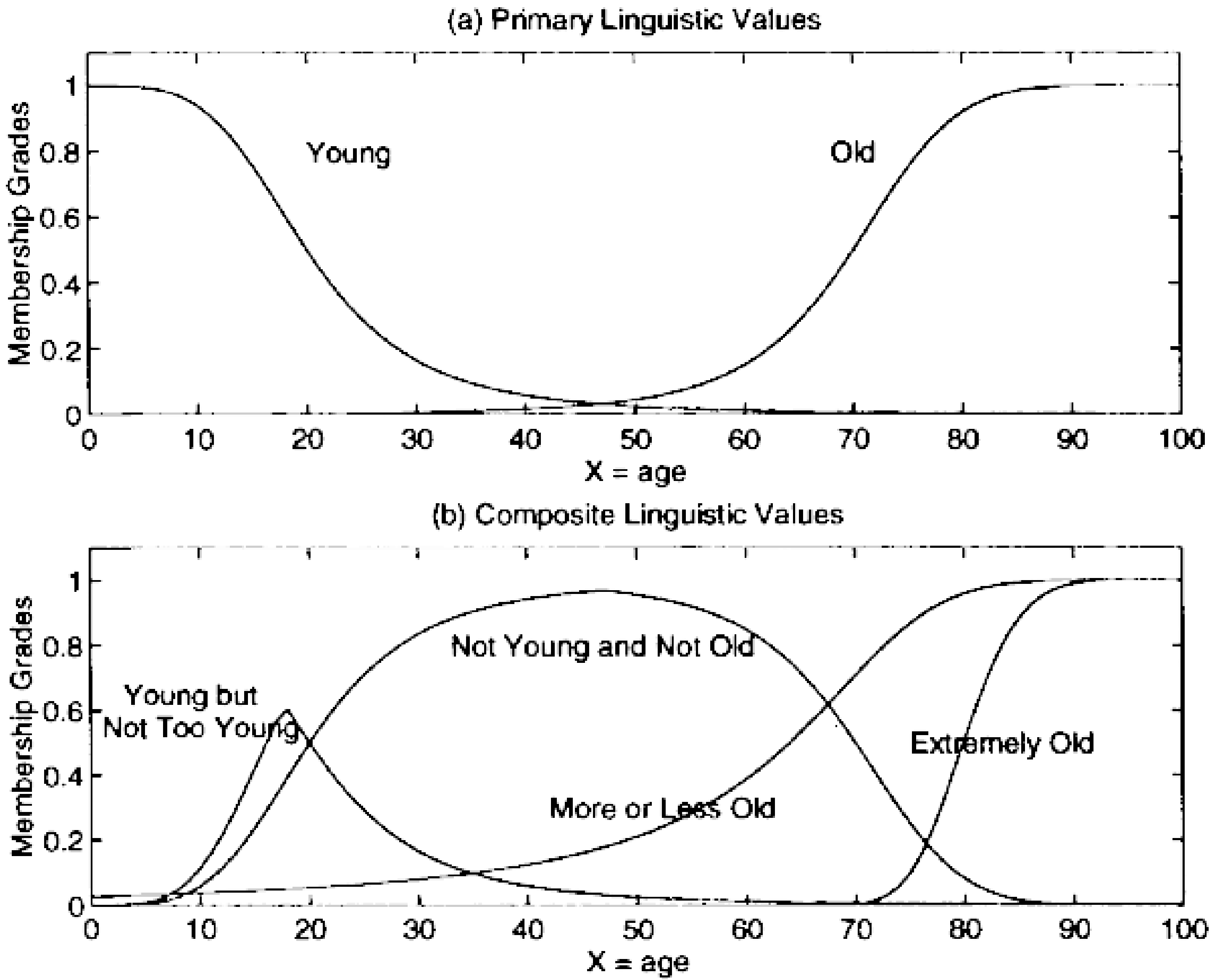
- *more or less old* = DIL(*old*) = *old*$^{0.5}$
$$= \int_X \sqrt{\frac{1}{1 + (\frac{x-100}{30})^6}} \Big/ x.$$
- *not young and not old* = $\neg$*young* $\cap$ $\neg$*old*
$$= \int_X \left[1 - \frac{1}{1 + (\frac{x}{20})^4}\right] \wedge \left[1 - \frac{1}{1 + (\frac{x-100}{30})^6}\right] \Big/ x.$$

- *young but not too young* = $young \cap \neg young^2$

$$= \int_X \left[\frac{1}{1+(\frac{x}{20})^4}\right] \wedge \left[1 - \left(\frac{1}{1+(\frac{x}{20})^4}\right)^2\right] \Big/ x.$$

- *extremely old*

$$= \mathrm{CON}(\mathrm{CON}(\mathrm{CON}(old))) = ((old^2)^2)^2 = \int_X \left[\frac{1}{1+(\frac{x-100}{30})^6}\right]^8 \Big/ x.$$



(a) Primary Linguistic Values
Membership Grades
1
0.8
0.6
0.4
0.2
0
Young
Old
0
10
20
30
40
50
60
70
80
90
100
X = age
(b) Composite Linguistic Values
Membership Grades
1
0.8
0.6
0.4
0.2
0
Not Young and Not Old
Young but
Not Too Young
Extremely Old
More or Less Old
0
10
20
30
40
50
60
70
80
90
100
X = age