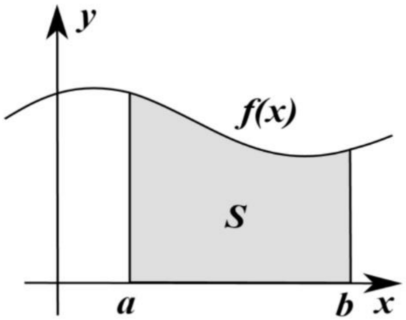
Numerical Analysis

Newton-Cotes Integration Formula

목 차

Lecture Note : Numerical Analysis - Newton-Cotes Integration Formula

1. Introduction to the Numerical Integration and Newton-Cotes Integration



○ Numerical Integration

$$S = \int_a^b f(x)dx \approx I_n = \int_a^b f_n(x)dx \quad \text{where} \quad f_n(x) \text{ is the approximating function}$$

○ Exact Integration of Polynomials and Newton-Cotes Integration

$$\int_a^b x^k dx = \left.\frac{x^{k+1}}{k+1}\right|_a^b = \begin{cases} \ln b - \ln a & \text{if } k = -1 \\ \dfrac{b^{k+1}}{k+1} - \dfrac{a^{k+1}}{k+1} & \text{therwise} \end{cases}$$

For $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \cdots + a_{m-1}x^{m-1} + a_m x^m = \sum a_j x^j$

$$I = \int_{x_1}^{x_2} f(x)dx = \int_{x_1}^{x_2} \sum a_j x^j dx = \sum a_j \int_{x_1}^{x_2} x^j dx = \sum \frac{a_j}{j+1}\left(x_2^{\,j+1} - x_1^{\,j+1}\right)$$

The Newton-Cotes integration method approximates $f(x)$ **with interpolating polynomials to numerically calculate the integration of** $I = \int_a^b f(x)dx$.

Divided-Difference Interpolation Formula

2. Divided-Difference Interpolation Formula revisited and Integration

- n^{th} order polynomial interpolation function can be defined using the divided-difference formula as

$$\boxed{\begin{aligned} f_n(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1) + \cdots + b_{n-1}(x-x_0)(x-x_1)\cdots(x-x_{n-1}) \\ +b_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})(x-x_n) \end{aligned}}$$

$$\boxed{\begin{aligned} b_0 &= f[x_0] \\ b_1 &= f[x_1, x_0] \\ b_2 &= f[x_2, x_1, x_0] \end{aligned}} \qquad \boxed{\begin{aligned} &\vdots \\ b_{n-2} &= f[x_{n-2}\cdots, x_2, x_1, x_0] \\ b_{n-1} &= f[x_{n-1}, x_{n-2}\cdots, x_2, x_1, x_0] \\ b_n &= f[x_n, x_{n-1}, x_{n-2}\cdots, x_2, x_1, x_0] \end{aligned}}$$

In general, if we define the following divided difference formula,

0^{th} order: $\boxed{f[x_i] = f(x_i)}$

1^{st} order: $\boxed{f[x_i, x_j] = \dfrac{f[x_i] - f[x_j]}{(x_i - x_j)}}$

2^{nd} order: $\boxed{f[x_i, x_j, x_k] = \dfrac{f[x_i, x_j] - f[x_j, x_k]}{(x_i - x_k)}}$

……..

n^{th} order: $\boxed{f[x_n, x_{n-1}, x_{n-1}, \cdots, x_1, x_0] = \dfrac{f[x_n, x_{n-1}, x_{n-2}, \cdots, x_1] - f[x_{n-1}, x_{n-2}, \cdots, x_1, x_0]}{(x_n - x_0)}}$

Trapezoidal Integration Formula

- In case of equal spacing such as $h = |x_{j+1} - x_j|, \quad j = 0, 1, \cdots, n$

  - 1st order with two points x_0, x_1

$$f_n(x) = b_0 + b_1(x - x_0)$$

$$b_0 = f[x_0] = f(x_0)$$

$$b_1 = f[x_1, x_0] = \frac{f[x_1] - f[x_0]}{(x_1 - x_0)} = \frac{f(x_1) - f(x_0)}{h}$$

$$I = \int_{x_0}^{x_1} f(x)dx \approx \int_{x_0}^{x_1} f_n(x)dx = \int_{x_0}^{x_1} \{b_0 + b_1(x - x_0)\}dx$$

$$= b_0(x - x_0) + \frac{1}{2}b_1(x - x_0)^2 \Big|_{x_0}^{x_1} = b_0(x_1 - x_0) + \frac{1}{2}b_1(x_1 - x_0)^2 = b_0 h + \frac{1}{2}b_1 h^2$$

$$= f(x_0)h + \frac{1}{2}\frac{f(x_1) - f(x_0)}{h}h^2$$

$$= f(x_0)h + \frac{1}{2}\{f(x_1) - f(x_0)\}h$$

$$= \frac{h}{2}\{f(x_0) + f(x_1)\}$$

Trapezoidal Formula $\boxed{I \approx \frac{h}{2}\{f(x_0) + f(x_1)\}}$

Simpson's Integration Formula

- 2nd order with three points x_0, x_1, x_2

$$f_n(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$
$$= b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_0 + x_0 - x_1)$$
$$= b_0 + b_1(x - x_0) + b_2(x - x_0)^2 - b_2(x - x_0)(x_1 - x_0)$$

$$b_0 = f[x_0] = f(x_0)$$
$$b_1 = f[x_1, x_0] = \frac{f[x_1] - f[x_0]}{(x_1 - x_0)} = \frac{f(x_1) - f(x_0)}{h}$$
$$b_2 = f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{(x_2 - x_0)} = \frac{f(x_2) - 2f(x_1) + f(x_0)}{2h^2}$$

$$I = \int_{x_0}^{x_2} f(x)dx \approx \int_{x_0}^{x_2} f_n(x)dx$$
$$= b_0(x - x_0) + \frac{1}{2}b_1(x - x_0)^2 + \frac{1}{3}b_3(x - x_0)^3 - \frac{1}{2}b_2(x - x_0)^2(x_1 - x_0)\Big|_{x_0}^{x_2}$$
$$= b_0(x_2 - x_0) + \frac{1}{2}b_1(x_2 - x_0)^2 + \frac{1}{3}b_2(x_2 - x_0)^3 - \frac{1}{2}b_2(x_2 - x_0)^2(x_1 - x_0) \leftarrow x_2 - x_0 = 2h$$
$$= 2b_0h + 2b_1h^2 + \frac{8}{3}b_2h^3 - 2b_2h^3$$

Simpson's Integration Formula

$$I = \int_{x_0}^{x_2} f(x)dx \approx \int_{x_0}^{x_2} f_n(x)dx$$

$$= 2f(x_0)h + 2\{f(x_1) - f(x_0)\}h + \frac{1}{3}\{f(x_2) - 2f(x_1) + f(x_0)\}h$$

$$= \frac{1}{3}\{f(x_2) + 4f(x_1) + f(x_0)\}h$$

Simpson's 1/3 Rule $\boxed{I \approx \frac{h}{3}\{f(x_2) + 4f(x_1) + f(x_0)\}}$

Simpson's Integration Formula

- 3rd order with four points x_0, x_1, x_2, x_3

$$
\begin{aligned}
f_n(x) &= b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1) + b_3(x-x_0)(x-x_1)(x-x_2) \\
&= b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1) + b_3(x-x_0)(x-x_1)(x-x_2) \\
&= b_0 + b_1(x-x_0) + b_2(x-x_0)^2 - b_2(x-x_0)(x_1-x_0) \\
&\quad + b_3(x-x_0)(x-x_0+x_0-x_1)(x-x_0+x_0-x_2) \\
&= b_0 + b_1(x-x_0) + b_2(x-x_0)^2 - b_2(x-x_0)(x_1-x_0) \\
&\quad + b_3(x-x_0)(x-x_0-h)(x-x_0-2h) \\
&= b_0 + b_1(x-x_0) + b_2(x-x_0)^2 - b_2(x-x_0)(x_1-x_0) + b_3(x-x_0)^3 - 3b_3(x-x_0)^2 h + 2(x-x_0)h^2
\end{aligned}
$$

$$b_0 = f[x_0] = f(x_0)$$

$$b_1 = f[x_1, x_0] = \frac{f[x_1] - f[x_0]}{(x_1 - x_0)} = \frac{f(x_1) - f(x_0)}{h}$$

$$b_2 = f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{(x_2 - x_0)} = \frac{f(x_2) - 2f(x_1) + f(x_0)}{2h^2}$$

$$b_3 = f[x_3, x_2, x_1, x_0] = \frac{f[x_3, x_2, x_1] - f[x_2, x_1, x_0]}{(x_3 - x_0)} = \frac{f(x_3) - 3f(x_2) + 3f(x_1) - f(x_0)}{6h^3}$$

Simpson's Integration Formula

$$I=\int_{x_0}^{x_3} f(x)dx \approx \int_{x_0}^{x_3} f_n(x)dx$$

$$=\left.\begin{matrix} b_0(x-x_0)+\frac{1}{2}b_1(x-x_0)^2+b_2(x-x_0)(x-x_1)+\frac{1}{3}b_1(x-x_0)^3-\frac{1}{2}b_2(x-x_0)^2(x_1-x_0) \\ +\frac{1}{4}b_3(x-x_0)^4-b_3(x-x_0)^3h+b_3(x-x_0)^2h^2 \end{matrix}\right|_{x_0}^{x_3}$$

$$=\begin{pmatrix} b_0(x_3-x_0)+\frac{1}{2}b_1(x_3-x_0)^2+\frac{1}{3}b_2(x_3-x_0)^3-\frac{1}{2}b_2(x_3-x_0)^2(x_1-x_0) \\ +\frac{1}{4}b_3(x_3-x_0)^4-b_3(x_3-x_0)^3h+b_3(x_3-x_0)^2h^2 \end{pmatrix} \leftarrow x_3-x_0=3h$$

$$=3b_0h+\frac{9}{2}b_1h^2+9b_2h^3-\frac{9}{2}b_2h^3+\frac{81}{4}b_3h^4-27b_3h^4+9b_3h^4$$

$$=3b_0h+\frac{9}{2}b_1h^2+\frac{9}{2}b_2h^3+\frac{9}{4}b_3h^4$$

$$=3f(x_0)h+\frac{9}{2}\left\{\frac{f(x_1)-f(x_0)}{h}\right\}h^2+\frac{9}{2}\left\{\frac{f(x_2)-2f(x_1)+f(x_0)}{2h^2}\right\}h^3+\frac{9}{4}\left(\frac{f(x_3)-3f(x_2)+3f(x_1)-f(x_0)}{6h^3}\right)h^4$$

$$=3f(x_0)h+\frac{9}{2}\{f(x_1)-f(x_0)\}h+\frac{9}{4}\{f(x_2)-2f(x_1)+f(x_0)\}h+\frac{3}{8}\big(f(x_3)-3f(x_2)+3f(x_1)-f(x_0)\big)h$$

$$=\frac{3}{8}\{f(x_3)+3f(x_2)+3f(x_1)+f(x_0)\}h$$

Simpson's 3/8 Rule $\boxed{I \approx \frac{3h}{8}\{f(x_3)+3f(x_2)+3f(x_1)+f(x_0)\}}$

Application of Integration Formula over Multiple Nodes

3. Multiple Application of Newton-Cotes Integration Formula

Trapezoidal Formula $$I_k \approx \frac{h}{2}\{f(x_k)+f(x_{k+1})\}, \quad x\in[x_k, x_{k+1}]$$

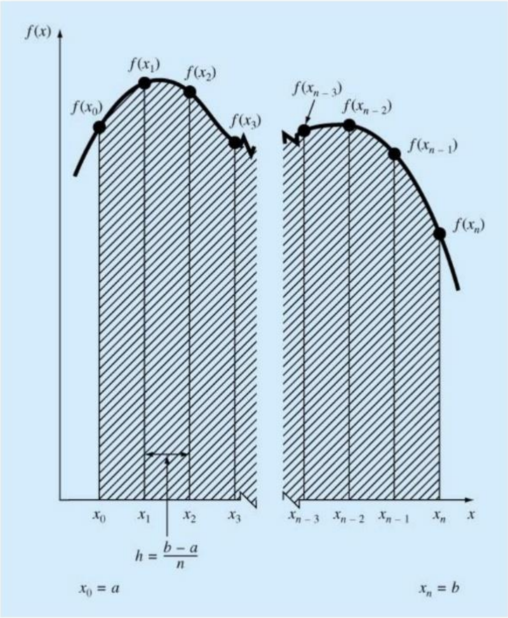
Simpson's 1/3 Rule $$I_k \approx \frac{h}{3}\{f(x_k)+4f(x_{k+1})+f(x_{k+2})\}, \quad x\in[x_k, x_{k+2}]$$

Simpson's 3/8 Rule $$I \approx \frac{3h}{8}\{f(x_k)+3f(x_{k+1})+3f(x_{k+2})+f(x_{k+3})\}, \quad x\in[x_k, x_{k+3}]$$

(3-1) Trapezoidal

$$\boxed{I_k \approx \frac{h}{2}\{f(x_k)+f(x_{k+1})\}, \quad x\in[x_k, x_{k+1}]}$$

$$\begin{aligned} I &= \int_a^b f(x)dx \\ &\approx \sum_{k=0}^{n-1} I_k \\ &= \sum_{k=0}^{n-1}\left[\frac{h}{2}\{f(x_k)+f(x_{k+1})\}\right] \\ &= \frac{h}{2}\{f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\} \end{aligned}$$



Application of Integration Formula over Multiple Nodes

(3-2) Simpson's 1/3 Rule

$$\boxed{I_{2k} \approx \frac{h}{3}\{f(x_{2k}) + 4f(x_{2k+1}) + f(x_{2k+2})\}, \quad x \in [x_k, x_{k+2}], \quad 2k = 0,2,4,6,\dots,n-2}$$

$$\begin{aligned}
I &= \int_a^b f(x)dx \approx \sum_{k=0}^{n/2-1} I_{2k} \\
&= \sum_{k=0}^{n/2-1}\left[\frac{h}{3}\{f(x_{2k}) + 4f(x_{2k+1}) + f(x_{2k+2})\}\right] = \frac{h}{3}\left\{f(x_0) + 4\sum_{k=0}^{n/2-1} f(x_{2k+1}) + 2\sum_{k=0}^{n/2-1} f(x_{2k+2}) + f(x_n)\right\} \\
&= \frac{h}{3}\left\{f(x_0) + 4\sum_{k=1,3,5,\cdots}^{n-1} f(x_n) + 2\sum_{k=2,4,6,\cdots}^{n-2} f(x_k) + f(x_n)\right\}
\end{aligned}$$

(3-3) Simpson's 8/3 Rule

$$\boxed{I_{3k} \approx \frac{3h}{8}\{f(x_{3k}) + 3f(x_{3k+1}) + 3f(x_{3k+2}) + f(x_{3k+3})\}, \quad x \in [x_k, x_{k+3}], \quad 3k = 0,3,6,9,\dots,n-3}$$

$$\begin{aligned}
I &= \int_a^b f(x)dx \approx \sum_{k=0}^{n/3-1} I_{3k} \\
&= \sum_{k=0}^{n/2-1}\left[\frac{3h}{8}\{f(x_{3k}) + 3f(x_{3k+1}) + 3f(x_{3k+2}) + f(x_{3k+3})\}\right] \\
&= \frac{3h}{8}\left\{f(x_0) + 3\sum_{k=0}^{n/3-1}\big(f(x_{3k+1}) + f(x_{3k+2})\big) + 2\sum_{k=0}^{n/2-1} f(x_{3k+3}) + f(x_n)\right\} \\
&= \frac{h}{3}\left\{f(x_0) + 3\sum_{k=1,4,7,\cdots}^{n-2}\big(f(x_k) + f(x_{k+1})\big) + 2\sum_{k=3,6,9,\cdots}^{n-3} f(x_k) + f(x_n)\right\}
\end{aligned}$$

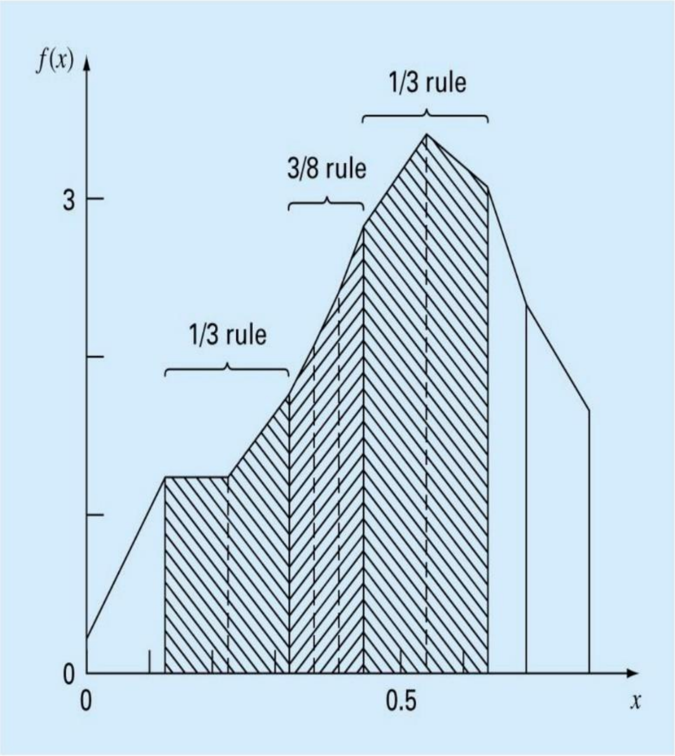
Handling Unequally-Spaced Nodes

4. Integration With unequal segments

○ **By using the divided-difference formula, we can drive the related formula**

$$
\begin{aligned}
f_n(x) = b_0 &+ b_1(x-x_0) + b_2(x-x_0)(x-x_1) \\
&+ \cdots + b_{n-1}(x-x_0)(x-x_1)\cdots(x-x_{n-1}) \\
&+ b_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})(x-x_n)
\end{aligned}
$$

$$
\begin{aligned}
b_0 &= f[x_0] \\
b_1 &= f[x_1, x_0] \\
b_2 &= f[x_2, x_1, x_0] \\
&\vdots \\
b_{n-2} &= f[x_{n-2} \cdots, x_2, x_1, x_0] \\
b_{n-1} &= f[x_{n-1}, x_{n-2} \cdots, x_2, x_1, x_0] \\
b_n &= f[x_n, x_{n-1}, x_{n-2} \cdots, x_2, x_1, x_0]
\end{aligned}
$$



End of Lecture