Numerical Analysis

Gauss-Quadrature Integration Formula

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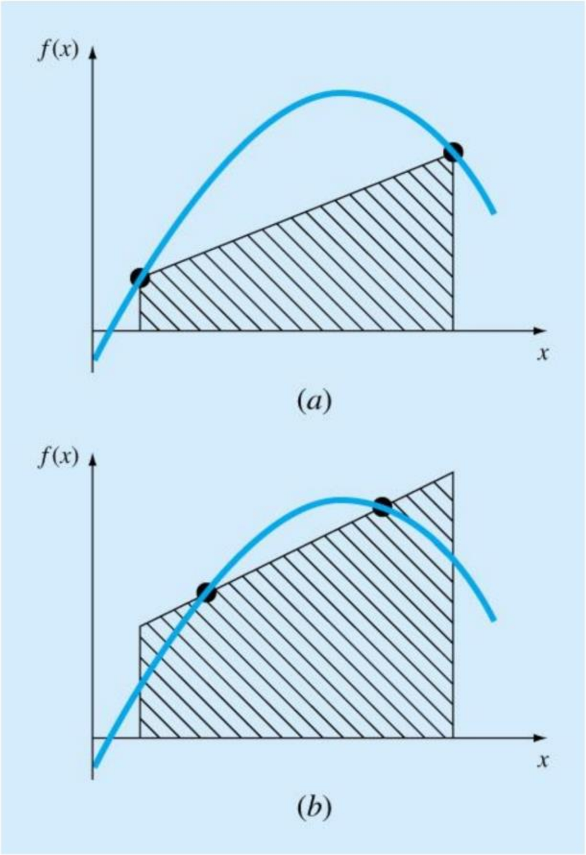
Lecture Note for Numerical Analysis- Gauss Quadrature Formula

1. Concept of Gauss Quadrature

- **Gauss quadrature** implements a strategy of positioning any two points on a curve to define a straight line that would balance the positive and negative errors.

- Hence the area evaluated under this straight line provides an improved estimate of the integral.



Standard Form for Gauss-Quadrature Integration Formula

2. Standard Form of Integration for Gauss Quadrature Application

- **Gauss quadrature** can be standardized by using the integration interval of

$$\boxed{x \in [-1, 1]}$$

to calculate

$$\boxed{I = \int_{x_0}^{x_f} f(x)dx, \quad x \in [x_0, x_f]}$$

- **Affine Transformation**

$$\boxed{t = \frac{2}{x_f - x_0}x - \frac{x_f + x_0}{x_f - x_0}, \quad t \in [-1, 1]} \rightarrow \boxed{x = \frac{x_f - x_0}{2}t + \frac{x_f + x_0}{2}}$$

Using Affine transformation we can redefine above integration as

$$dt = \frac{2}{x_f - x_0}dx \rightarrow dx = \frac{x_f - x_0}{2}dt$$

$$I = \int_{x_0}^{x_f} f(x)dx = \frac{x_f - x_0}{2}\int_{-1}^{1} f\left(\frac{x_f - x_0}{2}t + \frac{x_f + x_0}{2}\right)dt = \frac{x_f - x_0}{2}\int_{-1}^{1} g(t)dt$$

Derivation of Gauss-Quadrature Integration Formula

3. Derivation of Gauss Quadrature Formula

○ **If we use the standard form, the integration can be estimated by integrating the following form**

$$\boxed{I = \int_{-1}^{1} f(x)dx}$$

○ **General form of n-point Gauss Quadrature Formula**

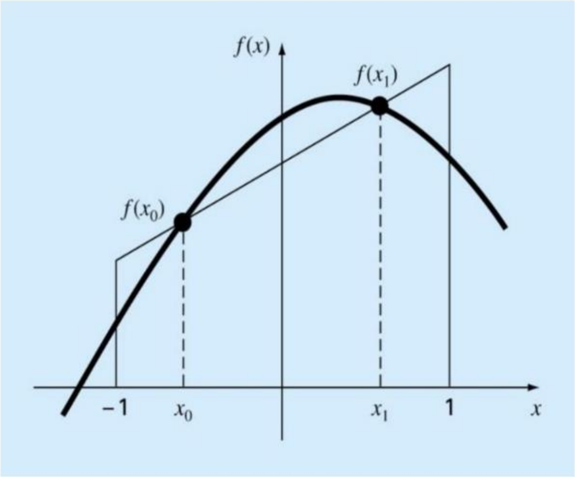
$$\boxed{\begin{aligned} I &= \int_{-1}^{1} f(x)dx \\ &\approx w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3) + w_4 f(x_4) + \cdots + w_{n-1} f(x_{n-1}) \end{aligned}}$$

where

$x_j,\quad j = 0,1,2,\cdots,n-1$: Gauss Quadrature points

$w_j,\quad j = 0,1,2,\cdots,n-1$: Gauss Quadrature weights

$x_j's, w_j's$ should be determined to balance the positive and negative errors



Derivation of Gauss-Quadrature Integration Formula

(3-1) Two-point Gauss quadrature formula to exactly integrate $I=\int_{-1}^{1} f(x)dx$ with $f(x)=1,x,x^2,x^3$

$$I = w_0 f(x_0) + w_1 f(x_1)$$

$$f(x)=1 \rightarrow w_0 f(x_0)+w_1 f(x_1) = w_0 + w_1 = \int_{-1}^{1} 1dx = 2$$

$$f(x)=x \rightarrow w_0 f(x_0)+w_1 f(x_1) = w_0 x_0 + w_1 x_1 = \int_{-1}^{1} xdx = 0$$

$$f(x)=x^2 \rightarrow w_0 f(x_0)+w_1 f(x_1) = w_0 x_0^{\,2} + w_1 x_1^{\,2} = \int_{-1}^{1} x^2dx = \frac{2}{3}$$

$$f(x)=x^3 \rightarrow w_0 f(x_0)+w_1 f(x_1) = w_0 x_0^{\,3} + w_1 x_1^{\,3} = \int_{-1}^{1} x^3dx = 0$$

$$\rightarrow \boxed{\begin{aligned} &w_0 + w_1 = 2 \\ &w_0 x_0 = -w_1 x_1 \\ &w_0 x_0^{\,2} + w_1 x_1^{\,2} = \frac{2}{3} \\ &w_0 x_0^{\,3} = -w_1 x_1^{\,3} \end{aligned}}$$

From the 2nd and 4th equations: $x_0^{\,2} = x_1^{\,2} \rightarrow \boxed{x_0 = -x_1}$

From the 2nd equation: $w_0 x_0 + w_1 x_1 = w_0 x_0 - w_1 x_0 = x_0(w_0 - w_1) = 0 \rightarrow \boxed{w_0 = w_1}$ since $x_0 \neq 0$

From the 1st equation: $2w_0 = 2 \rightarrow \boxed{w_0 = w_1 = 1}$

From the 3rd equation: $w_0 x_0^{\,2} + w_1 x_1^{\,2} = \frac{2}{3} \rightarrow 2x_0^{\,2} = \frac{2}{3} \rightarrow \boxed{\begin{aligned} &x_0 = -\frac{1}{\sqrt{3}} = -0.5773503\cdots \\ &x_1 = \frac{1}{\sqrt{3}} = 0.5773503\cdots \end{aligned}}$

Therefore the integration formula becomes: $I = w_0 f(x_0) + w_1 f(x_1) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

Derivation of Gauss-Quadrature Integration Formula

(3-2) Performance of two-point Gauss quadrature formula

(a) 2nd order polynomial function: $f(x) = a + bx + cx^2$

- Exact integration of $I = \int_{-1}^{1} f(x)dx \rightarrow I = \int_{-1}^{1} (a + bx + cx^2)dx = 2a + \frac{2}{3}c$

- Gauss Quadrature formula results in the same value

$$I = w_0 f(x_0) + w_1 f(x_1) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = \left(a - b\frac{1}{\sqrt{3}} + c\frac{1}{3}\right) + \left(a + b\frac{1}{\sqrt{3}} + c\frac{1}{3}\right) = 2a + \frac{2}{3}c$$

(b) 3rd order polynomial function: $f(x) = a + bx + cx^2 + dx^3$

- Exact integration: $I = \int_{-1}^{1} (a + bx + cx^2 + dx^3)dx = 2a + \frac{2}{3}c$

- Gauss Quadrature formula results in the same value

$$I = w_0 f(x_0) + w_1 f(x_1)$$

$$= f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = \left(a - b\frac{1}{\sqrt{3}} + c\frac{1}{3} - d\frac{1}{3\sqrt{3}}\right) + \left(a + b\frac{1}{\sqrt{3}} + c\frac{1}{3} + d\frac{1}{3\sqrt{3}}\right) = 2a + \frac{2}{3}c$$

(c) 4th order polynomial function: $f(x) = a + bx + cx^2 + dx^3 + ex^4$

- Exact integration: $I = \int_{-1}^{1} (a + bx + cx^2 + dx^3 + ex^4)dx = 2a + \frac{2}{3}c + \frac{2}{5}e$

- Gauss Quadrature formula generates some error

$$I = w_0 f(x_0) + w_1 f(x_1) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$= \left(a - b\frac{1}{\sqrt{3}} + c\frac{1}{3} - d\frac{1}{3\sqrt{3}} + e\frac{1}{9}\right) + \left(a + b\frac{1}{\sqrt{3}} + c\frac{1}{3} + d\frac{1}{3\sqrt{3}} + e\frac{1}{9}\right) = 2a + \frac{2}{3}c + \frac{2}{9}e$$

Example Applications of Gauss-Quadrature Integration Formula

4. Application of Gauss Quadrature Formula

○ **Given**

- Number of Gauss Quadrature Point: n
- Data for Gauss Quadrature Point
  - (a) Arguments : $\tau_j, \quad j = 1,2,\cdots,n \quad (\tau_j \in (-1,1))$
  - (b) Weights : $w_j, \quad j = 1,2,\cdots,n$
- Function and the integration interval: $\boxed{I = \int_{x_0}^{x_f} f(x)dx, \quad x \in [x_0, x_f]}$

○ **Transform using the affine transformation to get the standard form**

$$d\tau = \frac{2}{x_f - x_0}dx \rightarrow dx = \frac{x_f - x_0}{2}d\tau$$

$$I = \int_{x_0}^{x_f} f(x)dx = \frac{x_f - x_0}{2}\int_{-1}^{1} f\left(\frac{x_f - x_0}{2}\tau + \frac{x_f + x_0}{2}\right)d\tau$$

$$= \frac{x_f - x_0}{2}\int_{-1}^{1} g(\tau)d\tau = \frac{x_f - x_0}{2}\tilde{I} \leftarrow \tilde{I} = \int_{-1}^{1} g(\tau)d\tau$$

○ **Integration Formula**

$$\tilde{I} = \int_{-1}^{1} g(\tau)d\tau = \sum_{j=1}^{n} w_j g(\tau_j)$$

$$I = \left(\frac{x_f - x_0}{2}\right)\tilde{I} = \left(\frac{x_f - x_0}{2}\right)\sum_{j=1}^{n} w_j g(\tau_j) = \left(\frac{x_f - x_0}{2}\right)\sum_{j=1}^{n} w_j f\left(\frac{x_f - x_0}{2}\tau_j + \frac{x_f + x_0}{2}\right)$$

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Example Applications of Gauss-Quadrature Integration Formula

○ **Weights and function arguments x in Gauss Quadrature for the different number of quadrature points**

Points n	Weighting Factors w_j	Function Arguments τ_j
2	$w_1 = 1.000000000$ $w_2 = 1.000000000$	$\tau_1 = -0.577350269$ $\tau_2 = 0.577350269$
3	$w_1 = 0.555555556$ $w_2 = 0.888888889$ $w_3 = 0.555555556$	$\tau_1 = -0.774596669$ $\tau_2 = 0.000000000$ $\tau_3 = 0.774596669$
4	$w_1 = 0.347854845$ $w_2 = 0.652145155$ $w_3 = 0.652145155$ $w_4 = 0.347854845$	$\tau_1 = -0.861136312$ $\tau_2 = -0.339981044$ $\tau_3 = 0.339981044$ $\tau_4 = 0.861136312$
5	$w_1 = 0.236926885$ $w_2 = 0.478628670$ $w_3 = 0.568888889$ $w_4 = 0.478628670$ $w_5 = 0.236926885$	$\tau_1 = -0.906179846$ $\tau_2 = -0.538469310$ $\tau_3 = 0.000000000$ $\tau_4 = 0.538469310$ $\tau_5 = 0.906179846$

Example Applications of Gauss-Quadrature Integration Formula

Example 1]

○ **Given for n=3**

$$I = \int_0^5 f(x)dx, \quad x \in [0, 5]$$
$$f(x) = 3x^2 + 2x$$

Points n	Weighting Factors w_j	Function Arguments τ_j
3	$w_1 = 0.555555556$ $w_2 = 0.888888889$ $w_3 = 0.555555556$	$\tau_1 = -0.774596669$ $\tau_2 = \ 0.000000000$ $\tau_3 = \ 0.774596669$

→ Exact Integration $I = \int_0^5 (3x^2 + 2x)dx = (x^3 + x^2)\Big|_0^5 = 5^3 + 5^2 = 150.0$

○ **Solution using Gauss-Quadrature Formula**

$$I = 2.5\sum_{j=1}^{n} w_j f(2.5\tau_j + 2.5) \leftarrow \frac{x_f - x_0}{2} = 2.5, \quad \frac{x_f + x_0}{2} = 2.5$$
$$= 2.5[w_1 f(2.5\tau_1 + 2.5) + w_2 f(2.5\tau_2 + 2.5) + w_3 f(2.5\tau_3 + 2.5)]$$

$$x_1 = 2.5\tau_1 + 2.5 = -2.5 \times 0.774596669 + 2.5 = 0.56350833$$
$$x_2 = 2.5\tau_2 + 2.5 = 2.5 \times 0.0 + 2.5 \qquad = 2.5$$
$$x_3 = 2.5\tau_3 + 2.5 = 2.5 \times 0.774596669 + 2.5 = 4.436491672$$
$$f_1 = f(0.56350833) = (3x^2 + 2x)\Big|_{x=0.56350833} = 2.079641560$$
$$f_2 = f(2.5) \qquad = (3x^2 + 2x)\Big|_{x=2.5} \qquad = 23.75$$
$$f_3 = f(4.436491672) = (3x^2 + 2x)\Big|_{x=4.436491672} = 67.920358425$$
$$I = 2.5(0.555555556 f_1 + 0.888888889 f_2 + 0.555555556 f_3) = 150.0$$

Example Applications of Gauss-Quadrature Integration Formula

5. Pseudo code for the Gauss Quadrature

```
Program main
  % Input
       n          =3;
       xmin       = 0.0;
       xmax       = 5.0;
       dx_plus    = 0.5*(xmax+xmin);
       dx_minus= 0.5*(xmax-xmin);
  % Guass quadrature nodes and weights
       call gauss_node(n,tau,w);
  % Integration using Guass quadrature formula
       gauss_integral       = 0.0;
       do j=1, n
            x = dx_minus*tau(j)+dx_plus;
            call f(x,y);
            gauss_integral = gauss_integral + w(j)*y;
       end do;
       gauss_integral = dx_minus*guass_integral;
end program main
```

Example Applications of Gauss-Quadrature Integration Formula

function gauss_node(n,tau,w)

if n=2, then

$w_1 = 1.000000000, w_2 = 1.000000000$

$\tau_1 = -0.577350269, \quad \tau_2 = \ 0.577350269$

else if n=3, then

$w_1 = 0.555555556, w_2 = 0.888888889, w_3 = 0.555555556$

$\tau_1 = -0.774596669, \tau_2 = \ 0.000000000, \tau_3 = \ 0.774596669$

else if n=4, then

$w_1 = 0.347854845, w_2 = 0.652145155, w_3 = 0.652145155, w_4 = 0.347854845$

$\tau_1 = -0.861136312, \tau_2 = -0.339981044, \tau_3 = \ 0.339981044, \tau_4 = \ 0.861136312$

else

print*,‘node number n exceeds the maximum allowed node number= 4.’

print*, ‘Please add the node information (Gauss quadrature points and weights’

stop

end if

end function gauss_node

```
function f(x,y);% Function to be integrated, which should be specified by the user
        y=3.0*x*x*x+2.0*x;
end function f
```

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Appendix: Table for Gauss-Quadrature Integration Formula

Appendix: Table for Gauss Quadrature Nodes and Weights

N	No	Node	weight		N	No	Node	weight
1	1	0.000000000000E+00	.200000000000E+01		2	1	-.577350269190E+00	.100000000000E+01
						2	0.577350269190E+00	.100000000000E+01
3	1	-.774596669241E+00	0.555555555556E+00		4	1	-.861136311594E+00	.347854845137E+00
	2	0.000000000000E+00	0.888888888889E+00			2	-.339981043585E+00	.652145154863E+00
	3	0.774596669241E+00	0.555555555556E+00			3	0.339981043585E+00	.652145154863E+00
						4	0.861136311594E+00	.347854845137E+00
5	1	-.906179845939E+00	0.236926885056E+00		6	1	-.932469514203E+00	0.171324492379E+00
	2	-.538469310106E+00	0.478628670499E+00			2	-.661209386466E+00	0.360761573048E+00
	3	0.000000000000E+00	0.568888888889E+00			3	-.238619186083E+00	0.467913934573E+00
	4	0.538469310106E+00	0.478628670499E+00			4	0.238619186083E+00	0.467913934573E+00
	5	0.906179845939E+00	0.236926885056E+00			5	0.661209386466E+00	0.360761573048E+00
						6	0.932469514203E+00	0.171324492379E+00
7	1	-.949107912343E+00	0.129484966169E+00		8	1	-.960289856498E+00	0.101228536290E+00
	2	-.741531185599E+00	0.279705391489E+00			2	-.796666477414E+00	0.222381034453E+00
	3	-.405845151377E+00	0.381830050505E+00			3	-.525532409916E+00	0.313706645878E+00
	4	0.000000000000E+00	0.417959183673E+00			4	-.183434642496E+00	0.362683783378E+00
	5	0.405845151377E+00	0.381830050505E+00			5	0.183434642496E+00	0.362683783378E+00
	6	0.741531185599E+00	0.279705391489E+00			6	0.525532409916E+00	0.313706645878E+00
	7	0.949107912343E+00	0.129484966169E+00			7	0.796666477414E+00	0.222381034453E+00
						8	0.960289856498E+00	0.101228536290E+00
9	1	-.968160239508E+00	0.812743883616E-01		10	1	-.973906528517E+00	0.666713443087E-01
	2	-.836031107327E+00	0.180648160695E+00			2	-.865063366689E+00	0.149451349151E+00
	3	-.613371432701E+00	0.260610696403E+00			3	-.679409568299E+00	0.219086362516E+00
	4	-.324253423404E+00	0.312347077040E+00			4	-.433395394129E+00	0.269266719310E+00
	5	0.000000000000E+00	0.330239355001E+00			5	-.148874338982E+00	0.295524224715E+00
	6	0.324253423404E+00	0.312347077040E+00			6	0.148874338982E+00	0.295524224715E+00
	7	0.613371432701E+00	0.260610696403E+00			7	0.433395394129E+00	0.269266719310E+00
	8	0.836031107327E+00	0.180648160695E+00			8	0.679409568299E+00	0.219086362516E+00
	9	0.968160239508E+00	0.812743883616E-01			9	0.865063366689E+00	0.149451349151E+00
						10	0.973906528517E+00	0.666713443087E-01

Appendix: Table for Gauss-Quadrature Integration Formula

11	1	-.978228658146E+00	0.556685671162E-01	12	1	-.981560634247E+00	0.471753363865E-01
	2	-.887062599768E+00	0.125580369465E+00		2	-.904117256370E+00	0.106939325995E+00
	3	-.730152005574E+00	0.186290210928E+00		3	-.769902674194E+00	0.160078328543E+00
	4	-.519096129207E+00	0.233193764592E+00		4	-.587317954287E+00	0.203167426723E+00
	5	-.269543155952E+00	0.262804544510E+00		5	-.367831498998E+00	0.233492536538E+00
	6	0.000000000000E+00	0.272925086778E+00		6	-.125233408511E+00	0.249147045813E+00
	7	0.269543155952E+00	0.262804544510E+00		7	0.125233408511E+00	0.249147045813E+00
	8	0.519096129207E+00	0.233193764592E+00		8	0.367831498998E+00	0.233492536538E+00
	9	0.730152005574E+00	0.186290210928E+00		9	0.587317954287E+00	0.203167426723E+00
	10	0.887062599768E+00	0.125580369465E+00		10	0.769902674194E+00	0.160078328543E+00
	11	0.978228658146E+00	0.556685671162E-01		11	0.904117256370E+00	0.106939325995E+00
					12	0.981560634247E+00	0.471753363865E-01

End of Lecture