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Numerical Solutions for Stiff Ordinary Differential Equation Systems

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Abstract

The initial value problems with stiff ordinary differential equation systems (SODEs) occur in many fields of engineering science, particularly in the studies of electrical circuits, vibrations, chemical reactions and so on. In this paper we introduce a method based on the modification of the power series method proposed by Guzel [1] for numerical solution of stiff (or non-stiff) ordinary differential equation systems of the first-order with initial condition. Using this modification, the SODEs were successfully solved resulting in good solutions. Some numerical examples have been presented to show the capability of the approach method.

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1. Introduction

Consider the system of ordinary differential equations with the initial conditions as follows:

$$y' = f(x,y) \quad , \quad x \in [0,T] \tag{1.1}$$

$$y(0) = y_0 \tag{1.2}$$

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with theoretical solution of $y(x)$. Here

$y = (y_1, y_2, \ldots, y_n)^t$, $f = (f_1, f_2, \ldots, f_n)^t$, $f_i = f_i(x, y_1, y_2, \ldots, y_n)$, $i = 1, 2, \ldots, n$

where the i*th* equation of (1,1) ,$y'_i = f_i(x, y_1, y_2, \ldots, y_n)$, a mapping depending on the independent variable x and n unknown functions $y_1, y_2, \ldots, y_n$, with $y_0 = (y_{0,1}, y_{0,2}, \ldots, y_{0,n})^t$ as initial conditions. We assume that (1.1) with the initial conditions (1.2) has a unique solution.

Let the solution of (1.1) with the initial conditions (1.2) be of the form

$$y = y_0 + ex \qquad (1.3)$$

where e is a unknown vector which has same size as y_0. Substitute (1.3) into (1.1) and neglecting higher order terms, we have the linear equation of e of the form

$$Ae = B \qquad (1.4)$$

where A and B are constant matrices.

Having solved the linear equation (1.4), the coefficient of x in (1.3) can be determined. By repeating the above procedure for higher order terms, we can obtain the arbitrary order of power series of the solution (1.1) at neighbourhood of $x=0$.

2. Modified power series method

The interval [0,T] in (1.1) is divided into n individual sub-intervals as:

$0 = x_0 \prec x_1 \prec x_2 \prec \ldots \prec x_i \prec x_{i+1} \prec \ldots \prec x_n = T$

Also let , $h = \dfrac{T}{n}$, $x_i = x_0 + ih, i = 1, \ldots, n$.

Suppose the solution of (1.1), with initial conditions (1.2), within $[0, x_1]$ to be as:

$$y^{(1)}(x) = y_0 + ex \qquad (2.1)$$

where $e = (e_1, e_2, \ldots, e_n)^t$ is a unknown vector .

If we substitute (2.1) into (1.1) system will derive as:

$$Ae - B + Q(x) = 0 \quad , \quad Q(x) = (Q_1(x), Q_2(x), \ldots, Q_n(x))^t \qquad (2.2)$$

Where $A_{n\times n}$ and $B_{n\times 1}$ are matrices with known constant values and $Q_i(x)$, $i = 1, 2, \ldots, n$ are polynomials with the order greater than zero. By neglecting $Q(x)$ in (2.2) and solving the system of $Ae = B$, the vector e and therefore the coefficient of x in (2.1) is obtained. We set $y_1^{(1)} = e$.

In the next step, we assume that the solution of the equation (1.1) with the initial conditions (1.2) to be:

$$y^{(1)}(x) = y_0 + y_1^{(1)}x + ex^2 \qquad (2.3)$$

By substituting (2.3) in (1.1), we have the following system:

$$(Ae - B)x + Q(x) = 0 \tag{2.4}$$

By neglecting and solving the system of $Ae = B$, the unknown vector e and therefore the coefficient of x^2 in (2.3) is obtained. We set $y_2^{(1)} = e$, then by repeating the above procedure for m iteration, a power series of the following form is derived:

$$y^{(1)}(x) = y_0 + y_1^{(1)}x + y_2^{(1)}x^2 + ... + y_m^{(1)}x^m \tag{2.5}$$

Each element in (2.5) has the following form:

$$y_i^{(1)}(x) = y_{0,i} + y_{1,i}^{(1)}x + y_{2,i}^{(1)}x^2 + ... + y_{m,i}^{(1)}x^m \quad , \; i = 1, 2, ..., n$$

Equation (2.5) is an approximation for the exact solution, *y(x)*, of the system (1.1) in the interval $[0, x_1]$. Using (2.5), the approximate solution of the system (1.1) at $x = x_1$ that $y^{(1)}(x_1)$ is obtained. Therefore we set:

$$y^{(1)}(x_1) = y_1 \tag{2.6}$$

Consider the interval $[x_1, x_2]$, By changing variable $x = \chi + h$, in the system (1.1) and equation (2.6), an ordinary differential equation systems with the initial conditions is formed:

$$y'(\chi + h) = f(\chi + h, y(\chi + h)) \tag{2.7}$$

$$y'(0) = y_1$$

By solving the ordinary differential equation systems with the initial conditions in (2.7), the procedure in previous step gives:

$$y^{(2)}(\chi + h) = y_1 + y_1^{(2)}\chi + y_2^{(2)}\chi^2 + ... + y_m^{(2)}\chi^m \tag{2.8}$$

Then, after substituting $\chi = x - h$ in (2.8) we have:

$$y^{(2)}(x) = y_1 + y_1^{(2)}(x - h) + y_2^{(2)}(x - h)^2 + ... + y_m^{(2)}(x - h)^m \tag{2.9}$$

The solution (2.9) is an approximation for exact solution of the system (1.1) in the neighborhood of $x = x_1$.

Finally, if $y^{(n-1)}(x)$ is an approximate solution which is obtained the above procedure in the interval $[x_{n-2}, x_{n-1}]$, we have:

$$y^{(n-1)}(x_{n-1}) = y_{n-1} \tag{2.10}$$

By changing variable $x = \chi + nh$, in both (1.1) and (2.10), an ordinary differential equation systems with the initial conditions is obtained:

$$y'(\chi + nh) = f(\chi + nh, y(\chi + nh)) \tag{2.11}$$

$$y'(0) = y_{n-1}$$

By solving (2.11) with the mentioned method and by applying $\chi = x - nh$ the following solution is derived:

$$y^{(n)}(x) = y_{n-1} + y_1^{(n)}(x - nh) + y_2^{(n)}(x - nh)^2 + ... + y_m^{(n)}(x - nh)^m \tag{2.12}$$

The solution (2.12) is an approximation for the exact solution, *y(x)*, of the system (1.1) in the interval $\left[x_{n-1}, x_n\right]$.

3. Numerical example

a. *Example 1.* Consider a stiff system of differential equations taken from [1-3]. We are looking for approximate values of $y_1(x)$ and $y_2(x)$ at $x=1$. We have:

$$y_1' = -1002 y_1 + 1000 y^2{}_2 \quad x \in [0,1] \tag{3.1}$$

$$y_2' = y_1 - y_2(1+y_2)$$

where the initial conditions is:

$y_1(0) = 1 \quad and \quad y_2(0) = 1$

The theoretical solution is:

$y_1(x) = \exp(-2t)$

$y_2(x) = \exp(-t)$

Suppose h=0.02 and consider the interval [0,0.02]. According to the initial conditions, assume the solution of the system (3.1) to be:

$$y_1^{(1)} = y_{0,1} + e_1 x \Rightarrow y_1{}^{(1)}(x) = 1 + e_1 x \tag{3.2}$$

$$y_2^{(1)} = y_{0,2} + e_2 x \Rightarrow y_2{}^{(1)}(x) = 1 + e_2 x$$

By substituting (3.2) into (3.1) we have:

$$e_1 + 2 + Q_1(x) = 0 \tag{3.3}$$

$$1 + e_2 + Q_2(x) = 0$$

where

$$Q_1(x) = (1002 e_1 - 2000 e_2)x - 1000 e_2^2 x^2$$

$$Q_2(x) = (-e_1 + 3e_2)x + e_2^2 x^2$$

Neglecting $Q_1(x)$ and $Q_2(x)$ in (3.3) gives the following system:

$$Ae = B \tag{3.4}$$

where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad B = \begin{bmatrix} -2 \\ -1 \end{bmatrix} \quad , \quad e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

By solving the linear system of (3.4), we have:

$$e = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

Therefore,

$$y_1^{(1)}(x) = 1 - 2x \qquad (3.5)$$
$$y_2^{(1)}(x) = 1 - x$$

From (3.5), the solution of (3.1) can be supposed as:

$$y_1^{(1)}(x) = 1 - 2x + e_1 x^2 \qquad (3.6)$$
$$y_2^{(1)}(x) = 1 - x + e_2 x^2$$

Substituting (3.6) into (3.1), we have:

$$(2e_1 - 4)x + Q_1(x) = 0 \qquad (3.7)$$
$$(2e_2 - 1)x + Q_2(x) = 0$$

where $Q_1(x)$ and $Q_2(x)$ are polynomials of order higher than 1.
By neglecting the higher order terms in (3.7), we have the linear system Ae = B where

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \ , \ B = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \ , \ e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

Then the solution of the above system will result $e = (2 \ , \ \frac{1}{2})^t$. Therefore,

$$y_1^{(1)}(x) = 1 - 2x + 2x^2$$
$$y_2^{(1)}(x) = 1 - x + \frac{1}{2}x^2$$

By repeating the above procedure we have:

$$y_1^{(1)}(x) = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 + ... \qquad (3.8)$$
$$y_2^{(1)}(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + ...$$

Using (3.8), the approximate solution of the system at point x=0.02 is calculated as it follows:

$$y_1(0.2) = 0.96078943915232 \qquad (3.9)$$
$$y_2(0.2) = 0.98019867330676$$

By using the variable change $x = \chi + 0.02$ in (3.1) and (3.9), a ordinary differential equation system with the initial condition is constructed:

$$y_1'(\chi + 0.02) = -1002 y_1(\chi + 0.02) + 1000 y_2(\chi + 0.02) \qquad (3.10)$$
$$y_2'(\chi + 0.02) = y_1(\chi + 0.02) - y_2(\chi + 0.02)(1 + y_2(\chi + 0.02))$$
$$y_1(0) = 0.96078943915232$$
$$y_2(0) = 0.98019867330676$$

By solving the system of (3.10), similar to previous steps, the power series of the system within the interval [0.02,0.04] is given:

$$y_1(x) = 0.96078943915232 - 1.9215788783067\chi + 1.92157887831763\chi^2 \tag{3.11}$$
$$-1.28105258988239\chi^3 + 0.64052738356588\chi^4 + ...$$
$$y_2(x) = 0.98019867330676 - 0.98019867330676\chi + 0.49009933665336\chi^2$$
$$-0.16336644554678\chi^3 + 0.04084161029807\chi^4 + ...$$

By changing variable χ =*x-0.02* in (3.11), an approximate solution for (3.1) is derived. We have:

$$y_1(0.04) = 0.92311634638664 \tag{3.12}$$
$$y_2(0.04) = 0.96078943915232$$

In the next step, using *x=* χ *+0.04* in the interval [0.04,0.06] and applying the above procedure, we can calculate the power series of approximate solution of (3.1) at neighborhood *x*=0.04. Eventually, by repeating the procedure over the interval [0.98,1], we have:

$$y_1(1) = 0.135533528323661$$
$$y_2(1) = 0.36787944117144$$

All computations above, were undertaken by using power series of order *m=60*.

b. *Example 2.* Consider a second-order ordinary differential equation such as:

$$z''(x) + z(x) = 0.001(\cos x + i\sin x)$$

Defined on a long interval, $0 \le x \le 10\pi$, with the following initial conditions:
$z(0) = 1$ and $z'(0) = 0.9995i$
The analytical solution is $z(x) = \cos x + 0.0005x\sin x + i(\sin x - 0.0005x\cos x)$.
The above system was discussed by Liu [4]. here, we convert this problem in the following form:

$$\begin{aligned}
&u_1'(x) - u_2(x) = 0\\
&u_2'(x) + u_1(x) - u_5(x) = 0\\
&u_3'(x) - u_4(x) = 0\\
&u_4'(x) + u_3(x) - u_6(x) = 0 \qquad 0 \le x \le 10\pi\\
&u_5'(x) + u_6(x) = 0\\
&u_6'(x) - u_5(x) = 0
\end{aligned}$$

The corresponding initial conditions as:
$u_1(0) = 1$, $u_2(0) = 0, u_3(0) = 0$, $u_4(0) = 0.995, u_5(0) = 0.001$ and $u_6(0) = 0$
The analytical solution of this system is:

$u_1(x) = \cos x + 0.0005x\sin x$, $u_2(x) = -0.9995\sin x + 0.0005x\cos x$

$u_3(x) = \sin x - 0.0005x\cos x$, $u_4(x) = 0.9995\cos x + 0.0005x\sin x$

$u_5(x) = 0.001\cos x$, $u_6(x) = 0.001\sin x$

We apply the modified power series method with *h=0.1*, and the power series degree *m=20*, to solve the above equation.
Also we solve the above equation by power series method discussed in [1] with *m=20*.
In Table 1,we report the error $|z(x) - z_m(x)|$ at $x = 0.1, x = 0.2$, $x = \pi, x = 5\pi$, , $x = 10\pi$,with $z(x) = u_1(x) + iu_3(x)$.

Table 1: Comparison of modified power series method with original power series method.

x	Z(x)	Theoretical solution	Modified power series method Degree m=20	Error	Power series method Degree m=20	Error
0.1	$u_1(x)$	0.9950091569488	0.9950091569488	0	0.9950091569488	0
	$u_3(x)$	0.9978366643856	0.9978366643856	$1.38\times10^{-}$	0.9978366643856	1.38×10^{-17}
0.2	$u_1(x)$	0.9800864447743	0.9800864447743	$1.11\times10^{-}$	0.9800864447743	0
	$u_3(x)$	0.1985713241372	0.1985713241372	2.77×10	0.1985713241372	0
π	$u_1(x)$	-1	-1	$1.11\times10^{-}$	-0.9999999999518	7.48×10^{-11}
	$u_3(x)$	0.0015707963268	0.0015707963268	3.64×10	0.0015707958034	5.23×10^{-10}
5π	$u_1(x)$	-1	-1	2.22×10	1.2348241×10^{5}	1.23×10^{5}
	$u_3(x)$	0.0078539816339	0.0078539816339	7.45×10	-1.6780115×10^{5}	1.67×10^{5}
10π	$u_1(x)$	1	1	2.22×10	2.5330718×10^{11}	2.53×10^{11}
	$u_3(x)$	-.01570796326795	-0.0157079632679	2.02×10	-1.6623477×10^{11}	1.66×10^{11}

Table 1 proves that, as we approach the end points of the interval, the error of the power series method is increased, whilst the modified method shows excellent results at those points.

c. *Example 2.* A two-dimensional SODEs is considered [5].

$$y_1' = -500000.5y_1 + 499999.5y_2 \qquad y_1(0) = 0$$
$$y_2' = 499999.5y_1 - 500000.5y_2 \qquad y_2(0) = 2$$

The exact solutions are:

$$y_1(t) = -e^{\lambda_1 t} + e^{\lambda_2 t}$$
$$y_2(t) = e^{\lambda_1 t} + e^{\lambda_2 t}$$
$$\lambda_1 = -10^6 \ , \ \lambda_2 = -1$$

This problem, by itself, is strongly stiff (the corresponding stiff ratio is 10^6), it will become weakly stiff for a larger time.

We solve the above equation with modification power series method, with *h=0.00001* and *m=40*.

The numerical results are given in Table 2.

Table 2: Comparison of modified power series method with exact solutions.

t	Y	Theoretical solution	Modified power series method Degree m=40	Error
0.2	y_1	0.81873075307793	0.81873075307787	6.20×10^{-14}
	y_2	0.81873075307793	0.81873075307787	6.20×10^{-14}
0.4	y_1	0.67032004603547	0.67032004603536	1.02×10^{-13}
	y_2	0.67032004603547	0.67032004603536	1.02×10^{-13}
0.6	y_1	0.54880614800516	0.54880614800510	6.05×10^{-14}
	y_2	0.54880614800516	0.54880614800510	6.05×10^{-14}
0.8	y_1	0.44932447085050	0.44932447085045	4.48×10^{-14}
	y_2	0.44932447085050	0.44932447085045	4.48×10^{-14}
1	y_1	0.36787576239613	0.36787576239610	4.41×10^{-14}
	y_2	0.36787576239613	0.36787576239610	4.41×10^{-14}

4. Conclusions

The modified power series method is a powerful method for solving stiff ordinary differential equation systems. The result of applying this method on some SODEs show high capability of method respect to other method and even the original

which is presented in [1]. The simplicity and easy-to-apply in programming are two special features of this method.
Besides, the modified power series method is the most suitable method for those systems which have a wide range of solution intervals (example 2). This method helps the approximate solution not to diverge from exact ones.

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