Expressing $\frac{dN}{d\Omega}$ in terms of the m-parameters that appear in $\rho(\mathrm{X})$ of lepton pair production

As written in Eq. 6 from [1], the angular distribution of dilepton pairs is given by

$$\frac{dN}{d\Omega} = \frac{1}{\sigma}\frac{d\sigma}{d\Omega} = \frac{3}{4\pi} tr(\rho(\ell)\rho(\mathrm{X})). \tag{1}$$

From this expression and following the steps described in [1], we obtained Eq. 13 from [1]

$$\begin{aligned}\frac{dN}{d\Omega} &= \frac{1}{4\pi} + \frac{3}{4\pi} S_x \sin\theta\cos\phi \\ &+ \frac{3}{4\pi} S_y \sin\theta\sin\phi + \frac{3}{4\pi} S_z \cos\theta \\ &+ c\rho_0(\frac{1}{\sqrt{3}} - \sqrt{3}\cos^2\theta) - c\rho_1 \sin(2\theta)\cos\phi \\ &+ c\rho_2 \sin^2\theta\cos(2\phi) \\ &+ c\rho_3 \sin^2\theta \sin(2\phi) - c\rho_4 \sin(2\theta)\sin\phi\end{aligned} \tag{2}$$

where S_x, S_y and S_z are the components of the spin vector, and $c = 3/(8\sqrt{2}\pi)$, and the expression has been normalized.

In order to by-pass the use of matrix calculations and complex notation when coding the fit of $\frac{dN}{d\Omega}$ to the data discussed in the Quantum Tomography procedure [1], we have now express it in terms of the m parameters that appear in $\rho(\mathrm{X})(\mathrm{m}) = \mathrm{M}(\mathrm{m}) \cdot \mathrm{M}^{\dagger}(\mathrm{m})$. This expression, in the same form as presented in Eq. 2, is given by

$$\begin{aligned}\frac{dN}{d\Omega} &= \frac{1}{4}(1 + m_3^2) \\ &+ 0.22 m_3 m_9 \sin\theta\cos\phi \\ &- 0.22 m_3 m_7 \sin\theta\sin\phi \\ &+ 0.22(m_2 m_5 + m_7 m_8 - m_6 m_9)\cos\theta \\ &+ \frac{1}{4}(1 - 3m_3^2)\cos^2\theta \\ &- \frac{1}{2} m_3 m_6 \sin(2\theta)\cos\phi \\ &+ \frac{1}{2}(m_2^2 + m_8^2 + m_9^2 + \frac{1}{2}m_3^2 - \frac{1}{2})\sin^2\theta\cos(2\phi) \\ &- \frac{1}{2}(m_2 m_4 + m_6 m_8 + m_7 m_8)\sin^2\theta\sin(2\phi) \\ &- \frac{1}{2} m_3 m_8 \sin(2\theta)\sin\phi.\end{aligned} \tag{3}$$

Remember that there are 8 independent m parameters, one of them is given by requiring that $\rho(\mathrm{X})$ is normalized, *i.e.* $\text{Træ}(\mathrm{X}) = 1$. In particular, we note that m_1 is not present in Eq. 3 since it is fixed by $\text{Træ}(\mathrm{X}) = 1$, which requires that $m_1^2 + m_2^2 + \ldots + m_9^2 = 1$.

As discussed in [1], remember that to perform the fit to the data $-1 \leq m_\alpha \leq 1$, for all α that appear in Eq. 3.

By comparing Eq. 2 and Eq. 3, we can also express the spin vector in terms of the m parameters directly, resulting in

$$S_x \longrightarrow 0.22 m_3 m_9 \tag{4}$$

$$S_y \longrightarrow -0.22 m_3 m_7 \tag{5}$$

$$S_z \longrightarrow 0.22(m_2 m_5 + m_7 m_8 - m_6 m_9) \tag{6}$$

Here we note that m_4 does not appear in any of the spin vector components.

Finally, we integrate $\frac{dN}{d\Omega}$ using Eq. 3 over ϕ in the $0 \leq \phi \leq 2\pi$ interval, and over $\sin\theta$ in $0 \leq \theta \leq \pi$, resulting in

$$\begin{aligned}\frac{dN}{d\cos\theta} &= 1.5708(1 + m_3^2) + 1.3823(m_2 m_5 + m_7 m_8 - m6 m_9)\cos\theta \\ &+ (1.5708 - 4.71239 m_3^2)\cos^2\theta,\end{aligned} \tag{7}$$

and

$$\begin{aligned}\frac{dN}{d\phi} &= \frac{2}{3} + 0.345575 m_3 m_9 \cos\phi \\ &+ \frac{1}{3}(2m_2^2 + m_3^2 + 2m_8^2 + 2m_9^2 - 1)\cos 2\phi \\ &- 0.345575 m_3 m_7 \sin\phi \\ &- \frac{2}{3}(m_2 m_4 + m_6 m_8 + m_7 m_8)\sin 2\phi\end{aligned} \tag{8}$$

Note that in Eq. 7 the m_4 parameter does not appear in the expression, while m_5 does not appear in Eq. 9. Also note that S_z clearly appears in $\frac{dN}{d\cos\theta}$, while S_x and S_y appear in $\frac{dN}{d\phi}$.

These observations could be useful for measurements that have already been published where only one of the two angle dependence have been reported. In that case, $\rho(\mathrm{X})$ cannot be reconstructed since two m parameters will be missing, m_1 and either m_4 or m_5. Fortunately, the spin vector component(s) can be obtained. To do so equations 7 and 9 can be transformed in terms of the traditional conventions found in the literature, as discussed in [1].

Roofit fits

We have written a code in Roofit to peform the unbinned log likelihood fit to the data. By using Roofit we have built a PDF using expressions of Eq. 7 and Eq. 9, and generated toy data based on these PDFs, separately. We then fitted these PDFs to the generated toy data as shown in Fig. 1. This plot is the basis of a "closure test" of the fitting procedure. For the "real data" analysis, we prefer to do a simultaneous fit to both $\frac{dN}{d\cos\theta}$ and $\frac{dN}{d\phi}$, and do projections in either of them.

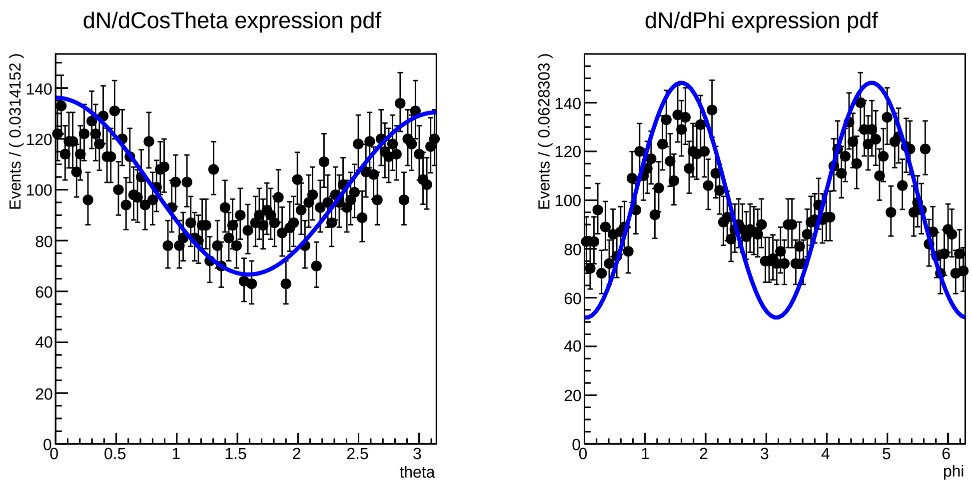

Figure 1: Fit of the "model" given by expressions of Eq. 7 and Eq. 9 to the toy data generated for the corresponding PDFs.

Example 1: Generated J/ψ sample with transverse polarization

We have generated coherent J/ψ photoproduced events in ultra-peripheral PbPb collisions with transverse polarization, using the STARLIGHT generator [2] which only considers transversely polarized photons. Figure 2 shows the $\frac{dN}{d\cos\theta}$ and $\frac{dN}{d\phi}$ distributions obtained from the generated sample.

We have then performed an unbinned log-likelihood fit using a PDF built from Eq. 3, so simultaneously fitting $\frac{dN}{d\cos\theta}$ and $\frac{dN}{d\phi}$, to the generated data. Figure 3 shows a projection of the results of the fit in $\frac{dN}{d\cos\theta}$.

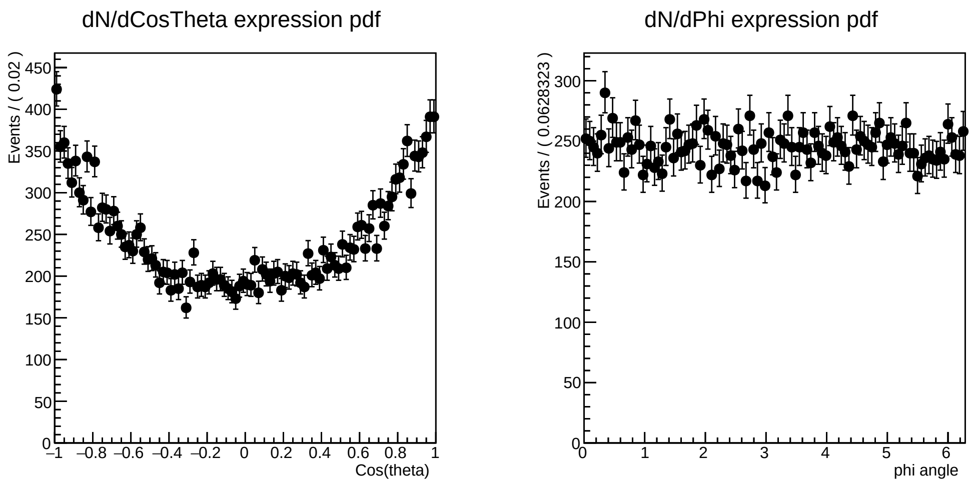

Figure 2: $\frac{dN}{d\cos\theta}$ and $\frac{dN}{d\phi}$ obtained from the generated coherent J/ψ photoproduction sample with transverse polarization.

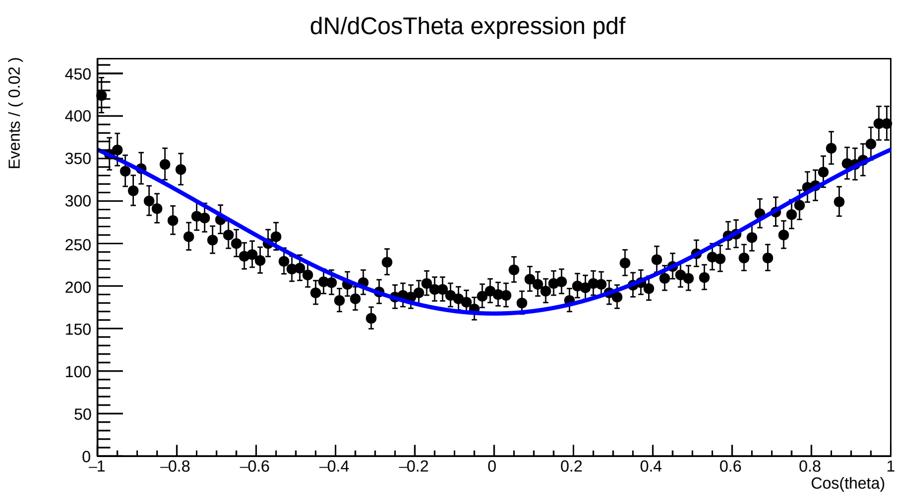

Figure 3: Projection in $\frac{dN}{d\cos\theta}$ of the fitted data using an unbinned log-likelihood fit using a PDF built from Eq. 3. Results are for coherent J/ψ photoproduced events using a transverse polarization.

Next steps

Next step will be to carry out the fitting results for the case of longitudinal polarization and no polarization, as well as for the two-photon process. We would like to explore what terms of the density matrix are relevant for the different processes, and whether the general behavior of the (X,Y,Z) coordinates or the eigenvalues of the probe density matrix can effectively distinguish between "signal" and "background" sources.

Another question to be answered is what is the minimum number of events needed to ensure the fit stability.

Impact of parity conservation

We can now express the angular distribution considering parity conservation, which eliminates 5-additional m-parameters, resulting in only 3-parameters: m_2, m_3 and m_6, while only two paramaters are independent:

$$\begin{aligned}\frac{dN}{d\Omega} &= \frac{1}{4}(1+m_3^2)\\ &+ \frac{1}{4}(1-3m_3^2)\cos^2\theta\\ &- \frac{1}{2}m_3 m_6 \sin(2\theta)\cos\phi\\ &+ \frac{1}{4}(2m_2^2+m_3^2-1)\sin^2\theta\cos(2\phi).\end{aligned} \tag{9}$$

The corresponding density matrix is

$$\rho_X = \begin{pmatrix} \mathrm{m_6}^2 & 0 & \mathrm{m}_3 m_6 \\ 0 & \mathrm{m}_2^2 & 0 \\ \mathrm{m}_3 m_6 & 0 & \mathrm{m}_3^2 \end{pmatrix} \tag{10}$$

which means that we can re-write the angular distribution in terms of the matrix elements of ρ_X, with only two independent parameters

$$\begin{aligned}\frac{dN}{d\Omega} &= \frac{1}{4}(1+\rho_X^{33})\\ &+ \frac{1}{4}(1-3\rho_X^{33})\cos^2\theta\\ &- \frac{1}{2}\rho_X^{13}\sin(2\theta)\cos\phi\\ &+ \frac{1}{4}(2\rho_X^{22}+\rho_X^{33}-1)\sin^2\theta\cos(2\phi).\end{aligned} \tag{11}$$

Note that in Eq. 10 the normalization of the trace is not required, while it has been required for obtaining the expression in Eq. 9. Thus, only two parameters are needed to describe the angular distribution, say, $m2$ and $m3$. This leads to

$$\rho_X^{13} = \sqrt{\rho_X^{33}(1-\rho_X^{22}-\rho_X^{33})} \equiv \sqrt{\rho_X^{33}\rho_X^{11}} \tag{12}$$

and

$$\rho_X^{22}+\rho_X^{33} < 1 \tag{13}$$

When fitting the data using Eq. 11, one will need to write explicitly ρ_X^{13} using Eq. 12 and demanding the relationship in Eq. 13. Alternatively, one can use the Eq. 11 as written, but requiring that $\rho_X^{13} < 1-\rho_X^{22}+\rho_X^{33}$ in the fitting to the data.

Note that Eq. 11 is equivalent to the angular distribution in terms of the λ_j parameters

$$\frac{dN}{d\Omega} = \frac{1}{\lambda_\theta + 3}(1 + \lambda_\theta \cos^2\theta + \lambda_{\theta\phi} \sin(2\theta)\cos\phi + \lambda_\phi \sin^2\theta \cos(2\phi)) \tag{14}$$

At the same time, our expression of Eq. 11 has three major advantages:

- All the terms in our expression are linear;
- All the terms in our expression are linked to the density matrix, and
- Our expression provides a clear way to impose positivity constrains using the requirement that the density matrix must have non-negative eigenvalues. In practice this means that the m-paramaters are defined between -1 and 1.

It is also interesting to observe that when the term $\sin(2\theta)\cos\phi$ vanishes (for whatever reason), linked to the $\lambda_{\theta\phi}$ paramater, the description of the normalized angular distribution only requires one paramater:

$$\begin{aligned}\frac{dN}{d\Omega} &= \frac{1}{4}(1 + \rho_X^{33}) \\ &+ \frac{1}{4}(1 - 3\rho_X^{33})\cos^2\theta \\ &+ \frac{1}{4}(1 - \rho_X^{33})\sin^2\theta\cos(2\phi).\end{aligned} \tag{15}$$

When both $\lambda_{\theta\phi} = 0$ and $\lambda_\phi = 0$, the normalized distribution does not require any paramater since it is $\frac{dN}{d\Omega} = \frac{1}{2}(1 - \cos\theta^2)$.

Angular distribution in terms of the Lambda parameters

Just for completeness, the matrix elements of ρ_X are related to the λ_j as follows

$$\rho_X^{33} = \frac{1 - \lambda_\theta}{\lambda_\theta + 3} \tag{16}$$

$$\rho_X^{22} = \frac{2\lambda_\phi + \lambda_\theta + 1}{\lambda_\theta + 3} \tag{17}$$

$$\rho_X^{11} = \frac{\lambda_\theta - 2\lambda_\phi + 1}{\lambda_\theta + 3} \tag{18}$$

$$\rho_X^{13} \equiv= \rho_X^{31} = \frac{-2\lambda_{\theta\phi}}{\lambda_\theta + 3} \tag{19}$$

As mentioned above, only two-independent paramaters are needed, thus,

$$\lambda_{\theta\phi} = -\frac{1}{2}\sqrt{(1 - \lambda_\theta)(\lambda_\theta - 2\lambda_\phi + 1)} \tag{20}$$

References

[1] J. C. Martens, J. P. Ralston and J. D. Tapia Takaki, "Quantum tomography for collider physics: Illustrations with lepton pair production," Eur. Phys. J. C **78** (2018) no.1, 5 doi:10.1140/epjc/s10052-017-5455-8 [arXiv:1707.01638 [hep-ph]].

[2] S. R. Klein, J. Nystrand, J. Seger, Y. Gorbunov and J. Butterworth, Comput. Phys. Commun. **212** (2017), 258-268 doi:10.1016/j.cpc.2016.10.016 [arXiv:1607.03838 [hep-ph]].