2022 SUMMER

업데이터 통계학 스터디

Chapter 7 – Point Estimation 1

Relative Efficiency

Relative Efficiency

$V + B^2 = MSE$

Definition

$B=0$

Given two UEs $\hat{\theta}_1$ and $\hat{\theta}_2$ of θ, with variances $V(\hat{\theta}_1)$ and $V(\hat{\theta}_2)$, respectively, the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$ is defined to be the ratio.

$$eff(\hat{\theta}_1, \hat{\theta}_2) = \frac{V(\hat{\theta}_2)}{V(\hat{\theta}_1)} > 1 \rightsquigarrow V(\hat{\theta}_2) > V(\hat{\theta}_1)$$

$$= \frac{MSE(\hat{\theta}_2)}{MSE(\hat{\theta}_1)}$$

$\hat{\theta}_1$ is useful.

Example 7.1

(recall) $Y_1, \cdots, Y_n$: $iid\ Unif(0, \theta)$

max Y_i

$$\hat{\theta}_1 = 2\overline{Y}, \quad \hat{\theta}_2 = \frac{n+1}{n} Y_{(n)}$$

order statistic

$Y_{(1)} \le Y_{(2)} \le \cdots \le Y_{(n)}$

Find the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$.

$V(\hat{\theta}_1) = MSE = \frac{\theta^2}{3n}$

$V(\hat{\theta}_2) = MSE = \frac{\theta^2}{n(n+2)}$

$eff(\hat{\theta}_1, \hat{\theta}_2) = \frac{V(\hat{\theta}_2)}{V(\hat{\theta}_1)} = \frac{3}{n+2} < 1 \quad n > 1$

$\hat{\theta}_2$ Good

Consistency

Chebyshev Inequality

X: r.v. $\quad E(X) = \mu < \infty \quad , \quad Var(X) = \sigma^2 < \infty$

$$\rightarrow \left(\begin{array}{l} P(|X-\mu| \ge k\sigma) \le \dfrac{1}{k^2} \\ P(|X-\mu| \le k\sigma) \ge 1 - \dfrac{1}{k^2} \end{array} \right)$$

X: c.r.v.

$$P(|X-\mu| \ge k\sigma) = \int_{\{x:\, |x-\mu| \ge k\sigma\}} f(x)\, dx.$$

$$\frac{|x-\mu|}{k\sigma} \ge 1 \qquad \frac{(x-\mu)^2}{k^2\sigma^2} \ge 1.$$

$$\le \int_{\{x:\, |x-\mu| \ge k\sigma\}} \frac{(x-\mu)^2}{k^2\sigma^2} f(x)\, dx. \quad //$$

$$\frac{|x-\mu|}{k\sigma} \ge 1.$$

$$\leq \int_{\{x \in \mathbb{R}\}} \frac{(x-\mu)^2}{k^2\sigma^2} f(x)\, dx.$$

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

$$\frac{1}{k^2\sigma^2} \underbrace{\int_{\mathbb{R}} (x-\mu)^2 f(x)\, dx}_{Var(X) = \sigma^2} = \frac{1}{k^2}.$$

Consistency

<u>Idea</u>: Estimator should always get closed to the truth as number of observations increases.

<u>Definition (Convergence in probability)</u>

A sequence of random variables $X_1, \cdots, X_2, \cdots, X_n, \cdots$ converges in probability to a random variable X ($X_n \xrightarrow{p} X$) if for any $\exists_\varepsilon > 0$,

$X_n \approx X$

$$P(|X_n - X| > \varepsilon) \xrightarrow{n\to\infty} 0 \Leftrightarrow P(|X_n - X| \leq \varepsilon) \xrightarrow{n\to\infty} 1$$

$$\star\ X_n \xrightarrow{p} c \quad \text{if} \quad P(|X_n - c| > \varepsilon) \xrightarrow{n\to\infty} 0 \quad \text{or} \quad P(|X_n - c| \leq \varepsilon) \xrightarrow{n\to\infty} 1$$

Consistency

Definition (Consistency)

$\hat{\theta}_n$ based on $X_1, \cdots, X_2, \cdots, X_n$ is consistent for θ if $\hat{\theta}_n \xrightarrow{p} \theta$ as $n \to \infty$ for all values of θ.

Tools to show Consistency

Tool 1: Weak Law of Large Numbers (WLLN)

$X_1, \cdots, X_n$ are iid with mean $E(X_i) = \mu < \infty$, then

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{p} \mu$$

Proof: By Using Chebyshev Inequality.

$\sigma^2 < \infty$

$$P(|X-\mu| \ge k\sigma) \le \frac{1}{k^2} \qquad k = \frac{\varepsilon}{\sigma} \quad (\varepsilon > 0)$$

$$P(|X-\mu| \ge \varepsilon) \le \frac{\sigma^2}{\varepsilon^2}$$

일치성

$\forall \varepsilon > 0.$ $$P(|\bar{Y}_n - \mu| \ge \varepsilon) \le \frac{\sigma^2/n}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2} \xrightarrow{n\to\infty} 0.$$

$X = \bar{Y}_n$ n sample 수.

$$\bar{Y}_n \xrightarrow{p} \mu.$$

Example 7.2

$Y_1, \cdots, Y_n$: *iid r.v.* Suppose that $E(Y^2) < \infty$ and $E(\log Y) < \infty$.

Proof

$\overline{Y_n^2} \to E(Y^2)$

By WLLN, $\overline{Y_n^2} \xrightarrow{p} E(Y^2)$.

1. $\frac{1}{n}\sum_1^n Y_i^2 \xrightarrow{p} E(Y^2)$

2. $\frac{1}{n}\sum_1^n \log Y_i \xrightarrow{p} E(\log Y)$

$\overline{(\log Y)}_n \xrightarrow{p} E(\log Y)$

Tools to show Consistency

Tool 2: Theorems on Limiting distributions

Suppose $W_n \xrightarrow{p} a$ and $V_n \xrightarrow{p} b$. Then,

r.v.

수열

1. $c_n W_n + d_n V_n \xrightarrow{p} ca + db$. When $c_n \longrightarrow c$, $d_n \longrightarrow d$ $(n \to \infty)$
2. $W_n V_n \xrightarrow{p} ab$
3. $W_n / V_n \xrightarrow{p} a/b$ if $b \neq 0$
4. $h(W_n) \xrightarrow{p} h(a)$ if h is continuous at a

Tools to show Consistency

- If $\bar{Y}_n \xrightarrow{p} \mu$. Then,

$$\bar{Y}_n^{\,2} \xrightarrow{p} \mu^2, \quad \sqrt{\bar{Y}_n} \xrightarrow{p} \sqrt{\mu}, \quad \log(\bar{Y}_n) \xrightarrow{p} \log(\mu) \ (\text{if } \mu > 0)$$

② or ④   ④   ④

$h = u^2$

Example 7.3

$Y_1, \dots, Y_n$ iid (μ, σ^2) $\Rightarrow$ $\bar{Y}_n$: UE of μ. & consistent for μ

by WLLN

1. Show that, Sample variance(S_n^2) is always a consistent estimator of population variance (σ^2).

S_n^2 : UE of σ^2.

2. S_n^2 is unbiased and consistent for σ^2. Is S_n unbiased or consistent for σ?

Under estimator of σ

$$\boxed{S_n^2 \xrightarrow{p} \sigma^2.}$$

$$S_n \xrightarrow{p} \sigma$$

by Tool 2

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

$E(Y)$ by tool 1.

$$= \frac{1}{n-1}\left(\sum_{i=1}^{n} Y_i^2 - n(\bar{Y}_n)^2\right)$$

$E(Y^2)$ by tool 1.

~~tool~~

$$= \frac{n}{n-1}\left\{E(Y^2) - \{E(Y)\}^2\right\} = \frac{n}{n-1} Var(Y) \xrightarrow{p} var(Y)$$

Example 7.4

$Y_1, \cdots, Y_n$: iid $Exp(\theta), \theta > 0.$

$Y_i \overset{iid}{\sim} Exp(\theta) := Gamma(1, \theta)$

$E(Y_i) = \theta$ $\quad$ $Var(Y_i) = \theta^2.$

$E(Y_i^2) = 2\theta^2$

$\bar{Y}_n \xrightarrow{p} \theta$

(By tool.1)

$\frac{1}{\bar{Y}_n} \xrightarrow{p} \frac{1}{\theta}$

by Tool.2

$\log \bar{Y}_n \xrightarrow{p} \log \theta.$

$(\bar{Y}_n)^2 \xrightarrow{p} \theta^2$

Tool.2

$\frac{1}{n} \sum Y_i^2 \xrightarrow{p} E(Y^2)$ by Tool.1

$= 2\theta^2$

Tools to show Consistency

√ Tool 3: Weak law of variance   WMS : 약한 것.

If $\hat{\theta}_n$ is an UE of θ and $\boxed{V(\hat{\theta}_n) \to 0}$, $\hat{\theta}_n$ is consistent for θ.

√ Tool 3: Strong Law of variance   "SJ" : 빡센 것. , 강한 것

If $\hat{\theta}_n$ is an estimator of θ and $MSE(\hat{\theta}_n) \to 0$, $\hat{\theta}_n$ is consistent for θ.

$V(\hat{\theta}_n) + \{B(\hat{\theta}_n)\}^2 \to 0$

Proof: By Using Chebyshev Inequality.

$$P(|\hat{X} - X| \geq k\sigma) \leq \frac{1}{k^2}$$

let, $\underline{\varepsilon = k\sigma}_{\,/\!/}$ $\rightarrow$ $\frac{1}{k} = \frac{\sigma}{\varepsilon}$

$$P(\underline{|\hat{\theta}_n - \theta|}_{\,/\!/} \geq \varepsilon) \leq \frac{Var(\hat{\theta}_n)}{\varepsilon^2} \longrightarrow 0.$$

$$\underline{\hat{\theta}_n \xrightarrow{P} \theta.}_{\,/\!/}$$

Example 7.5

$Y_1, \cdots, Y_n$: $iid \;\; N(\mu, \sigma^2)$.

Are $\bar{Y}_n$ & S_n^2 consistent estimator for μ and σ^2 respectively ?

(i) $\overline{Y_n}$

① UE of μ.

② $var(\overline{Y_n}) = \frac{\sigma^2}{n} \to 0$.

$\therefore$ By Tool 3, $\overline{Y_n}$ is a consistent estimator for μ.

(ii) S_n^2

① UE of σ^2.

② $Var\left(\underbrace{\boxed{\frac{(n-1)S_n^2}{\sigma^2}}}_{}\right) = \frac{(n-1)^2}{\sigma^4} var(S_n^2) = 2(n-1)$

$\chi^2(n-1)$

$var(S_n^2) = \frac{2\sigma^4}{n-1} \to 0$.

$\therefore$ By Tool 3, $S_n^2 \xrightarrow{p} \sigma^2$.

Example 7.6

$E(\bar{Y}) =$ 모평균

$Var(\bar{Y}) = \dfrac{\text{모분산}}{n}$

$Y_1, \cdots, Y_n$: *iid* $Unif(0, \theta)$. $\frac{\theta}{2} < \infty$ $\qquad \frac{\theta^2}{12}$

$Var(\hat{\theta}_1) = Var(2\bar{Y})$

$= 4Var(\bar{Y})$

Are $\hat{\theta}_1 = 2\bar{Y}$ and $\hat{\theta}_2 = \frac{n+1}{n} Y_{(n)}$ consistent estimator for θ ?

(i) $\hat{\theta}_1 = 2\bar{Y}$ $\qquad$ $\hat{\theta}_1 =$ consistent estimator

$\bar{Y} \to \frac{\theta}{2}$ by Tool. 1

$2\bar{Y} \to \theta$

$\left(\begin{array}{l} \hat{\theta}_1 \text{ is an UE of } \theta. \\ Var(\hat{\theta}_1) = \frac{\theta^2}{3n} \to 0. \end{array} \right)$ Tool 3.

(ii) $\hat{\theta}_2 = \frac{n+1}{n} Y_{(n)}$

$\left(\begin{array}{l} \hat{\theta}_2 \text{ is an UE of } \theta. \\ Var(\hat{\theta}_2) = \frac{(n+1)^2}{n^2} \cdot \boxed{\frac{\theta^2}{n(n+2)}} \to 0. \end{array} \right)$ Tool 3.

앞에서 구함

Convergence in distribution

Convergence in distribution

Definition

Suppose X_n is a random variable with CDF $F_n(x),\ n = 1,2,\cdots$. Then $X_1, X_2, \cdots$ converges in distribution to a random variable X with CDF $F(x)$ if

$$\lim_{n\to\infty} F_n(x) = F(x)$$

Convergence in distribution

Theorem (Central Limit Theorem) CLT

개별

분포인지 모름

$Y_1, \cdots, Y_n$: random sample from a distribution with (μ, σ^2). Then,

$$Z_n = \frac{\sum_{i=1}^{n} Y_i - E(\sum_{i=1}^{n} Y_i)}{\sqrt{Var(\sum_{i=1}^{n} Y_i)}} = \frac{\bar{Y}_n - \mu}{\sigma / \sqrt{n}} \xrightarrow{D} N(0,1)$$

meaning that CDF of Z_n converges to the CDF of $N(0,1) \Rightarrow$

$$P(Z_n \le z) \to \Phi(z) \text{ for all } z$$

$$P(a \le Z_n \le b) = F_{Z_n}(b) - F_{Z_n}(a) \longrightarrow \Phi(b) - \Phi(a)$$

Convergence in distribution

$\chi^2(1) = Z^2$

Theorem (Mapping Theorem)

For sequence of r.v. $X_1, \cdots, X_n$.

If $X_n \xrightarrow{D} X$, then $h(X_n) \xrightarrow{D} h(X)$ for any continuous function h.

Theorem (Limiting MGF Theorem)

X_n has CDF $F_n(x)$ and MGF $M(t;n)$ that exists for $|t| < h$. If there is a CDF $F(x)$ with MGF $M(t)$, then X_n has a limiting distribution with CDF $F(x)$.

Example 7.7

고정값

$Y_1, \cdots, Y_n$: *iid* $Bin(n,p)$. $\mu = np$ is a constant.

Find a limiting distribution of Y_n.

$Y_n \xrightarrow{D} \text{poisson}(\mu)$

$Y_n \sim Bin(n,p)$

$Y_n = V_1 + \cdots + V_n$, V_i : iid Bernoulli(p)

$E(e^{tV}) = M_V(t) = p \cdot e^t + (1-p)$

$E(e^{tY_n}) = \{E(e^{tV})\}^n$

$= (pe^t + (1-p))^n$

$p = \frac{\mu}{n}$.

$M(t) = \left(1 - \frac{\mu}{n} + \frac{\mu}{n} e^t\right)^n$

$= \left(\left(1 + \frac{\mu}{n}(e^t - 1)\right)^{\frac{n}{\mu(e^t-1)}}\right)^{\mu(e^t-1)}$

$e^{1 \times \mu(e^t-1)}$

$n \to \infty$: $e^{\mu(e^t-1)}$

Poisson(μ)의 MGF.

Bin(n, p)

$Y_n \xrightarrow{D} X$. poisson(np)

By MGF Thm

Convergence in distribution

Theorem (Slutsky's Theorem) 증명X

$$U_n \xrightarrow{D} U \text{ and } W_n \xrightarrow{p} 1 \quad \Rightarrow \quad U_n/W_n \xrightarrow{D} U.$$

$$\frac{U_n}{W_n} \xrightarrow{D} U$$

Example 7.8

Prove the following proposition.

mapping theorem

$$\left(U_n \xrightarrow{D} N(0,1)\right) \Rightarrow U_n^2 \xrightarrow{D} \chi^2(1)$$

$W_n = 1$

$$\frac{U_n^2}{W_n} \Rightarrow \frac{U_n^2}{1}$$

Example 7.9

Prove the following proposition.

$$T_n \sim t(n) \;\Rightarrow\; T_n \xrightarrow{D} N(0,1)$$

∴ by Slutsky theorem:

$T_n = \dfrac{Z}{\sqrt{W_n/n}}$, $Z \sim N(0,1)$, $\sqrt{W_n/n} \xrightarrow{P} 1$, $\sim N(0,1)$

$$T_n = \frac{Z}{\sqrt{W_n/n}} \qquad Z \sim N(0,1) \qquad W_n \sim \chi^2(n)$$

$\chi^2(\nu)$

평균 $= \nu$

분산 $= 2\nu$

$$\cdot\ \frac{W_n}{n} = \frac{\chi^2(n)}{n} = \frac{\overbrace{\chi^2(1) + \cdots + \chi^2(1)}^{n개}}{n} = E(\chi^2(1)) \xrightarrow{P} 1 \quad (\because \text{WLLN})$$

Tool 2

$$\frac{W_n}{n} \xrightarrow{P} 1. \qquad \sqrt{\frac{W_n}{n}} \xrightarrow{P} 1$$

Example 7.10

μ

Prove the following proposition. (σ^2 *is known*)

$$Y_1, \cdots, Y_n : iid\ (\mu, \sigma^2) \quad \Rightarrow \quad \frac{\bar{Y}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{D} N(0,1)$$

By CLT.

CLT

Example 7.11

Prove the following proposition. (σ^2 *is unknown*)

$$Y_1, \cdots, Y_n : iid\ (\mu, \sigma^2) \quad \Rightarrow \quad \frac{\bar{Y}_n - \mu}{S_n/\sqrt{n}} \xrightarrow{D} N(0,1)$$

$S_n^2 \xrightarrow{P} \sigma^2$

$S_n \xrightarrow{P} \sigma$ (By Tool 2)

$\frac{S_n}{\sigma} \xrightarrow{P} 1$ (By Tool 2)

$$= \frac{\dfrac{\bar{Y}_n - \mu}{\sigma/\sqrt{n}}}{\dfrac{S_n}{\sigma}}$$

$\frac{\bar{Y}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{\text{By CLT}} N(0,1)$

$\frac{S_n}{\sigma} \xrightarrow{P} 1$

$\sim N(0,1)$

By Slutsky's Theorem.