2022 SUMMER

업데이터 통계학 스터디

Chapter 7 – Point Estimation 1

Relative Efficiency

Relative Efficiency

Definition

Given two UEs $\hat{\theta}_1$ and $\hat{\theta}_2$ of θ, with variances $V(\hat{\theta}_1)$ and $V(\hat{\theta}_2)$, respectively, the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$ is defined to be the ratio.

$$eff\left(\hat{\theta}_1, \hat{\theta}_2\right) = \frac{V(\hat{\theta}_2)}{V(\hat{\theta}_1)}$$

Example 7.1

(recall) $Y_1, \cdots, Y_n$: $iid\ Unif(0, \theta)$

$$\hat{\theta}_1 = 2\overline{Y}, \quad \hat{\theta}_2 = \frac{n+1}{n}\ Y_{(n)}$$

Find the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$.

Consistency

Chebyshev Inequality

Consistency

Idea: Estimator should always get closed to the truth as number of observations increases.

Definition (Convergence in probability)

A sequence of random variables $X_1, \cdots, X_2, \cdots, X_n, \cdots$ converges in probability to a random variable X ($X_n \xrightarrow{p} X$) if for any $\exists_\varepsilon > 0$,

$$P(|X_n - X| > \varepsilon) \xrightarrow{n\to\infty} 0 \iff P(|X_n - X| \le \varepsilon) \xrightarrow{n\to\infty} 1$$

$\star$ $X_n \xrightarrow{p} c$ if $P(|X_n - c| > \varepsilon) \xrightarrow{n\to\infty} 0$ or $P(|X_n - c| \le \varepsilon) \xrightarrow{n\to\infty} 1$

Consistency

Definition (Consistency)

$\hat{\theta}_n$ based on $X_1, \cdots, X_2, \cdots, X_n$ is consistent for θ if $\hat{\theta}_n \xrightarrow{p} \theta$ as $n \to \infty$ for all values of θ.

Tools to show Consistency

Tool 1: Weak Law of Large Numbers (WLLN)

$X_1, \cdots, X_n$ are iid with mean $E(X_i) = \mu < \infty$, then

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{p} \mu$$

Proof: By Using Chebyshev Inequality.

Example 7.2

$Y_1, \cdots, Y_n$: $iid\ r.v.$ Suppose that $E(Y^2) < \infty$ and $E(\log Y) < \infty$.
Proof

1. $\frac{1}{n}\sum_1^n {Y_i}^2 \xrightarrow{p} E(Y^2)$

2. $\frac{1}{n}\sum_1^n logY_i \xrightarrow{p} E(\log Y)$

Tools to show Consistency

Tool 2: Theorems on Limiting distributions

Suppose $W_n \xrightarrow{p} a$ and $V_n \xrightarrow{p} b$. Then,

1. $c_n W_n + d_n V_n \xrightarrow{p} ca + db$. When $c_n \longrightarrow c$, $d_n \longrightarrow d$
2. $W_n V_n \xrightarrow{p} ab$
3. $W_n / V_n \xrightarrow{p} a/b$ if $b \neq 0$
4. $h(W_n) \xrightarrow{p} h(a)$ if h is continuous at a

Tools to show Consistency

- If $\bar{Y}_n \xrightarrow{p} \mu$. Then,

$$\bar{Y}_n{}^2 \xrightarrow{p} \mu^2, \quad \sqrt{\bar{Y}_n} \xrightarrow{p} \sqrt{\mu}, \quad \log(\bar{Y}_n) \xrightarrow{p} \log(\mu) \ \text{(if } \mu > 0\text{)}$$

Example 7.3

1. Show that, Sample variance(S_n^2) is always a consistent estimator of population variance (σ^2).

2. S_n^2 is unbiased and consistent for σ^2. Is S_n unbiased or consistent for σ?

Example 7.4

$Y_1, \cdots, Y_n$: $iid \;\; Exp(\theta), \theta > 0$.

Tools to show Consistency

Tool 3: Weak law of variance

If $\hat{\theta}_n$ is an UE of θ and $V(\hat{\theta}_n) \to 0$, $\hat{\theta}_n$ is consistent for θ.

Tool 3: Strong Law of variance

If $\hat{\theta}_n$ is an estimator of θ and $MSE(\hat{\theta}_n) \to 0$, $\hat{\theta}_n$ is consistent for θ.

Proof: By Using Chebyshev Inequality.

Example 7.5

$Y_1, \cdots, Y_n$: $iid \;\; N(\mu, \sigma^2)$.

Are $\bar{Y}_n$ & S_n^2 consistent estimator for μ and σ respectively ?

Example 7.6

$Y_1, \cdots, Y_n$: $iid\ \ Unif(0, \theta)$.

Are $\hat{\theta}_1 = 2\overline{Y}$ and $\hat{\theta}_2 = \frac{n+1}{n}\ Y_n$ consistent estimator for θ ?

Convergence in distribution

Convergence in distribution

Definition

Suppose X_n is a random variable with CDF $F_n(x),\ n = 1,2,\cdots$. Then $X_1, X_2, \cdots$ converges in distribution to a random variable X with CDF $F(x)$ if

$$\lim_{n\to\infty} F_n(x) = F(x)$$

Convergence in distribution

Theorem (Central Limit Theorem)

$Y_1, \cdots, Y_n$: random sample from a distribution with (μ, σ^2). Then,

$$Z_n = \frac{\sum_{i=1}^n Y_i - E(\sum_{i=1}^n Y_i)}{\sqrt{Var(\sum_{i=1}^n Y_i)}} = \frac{\bar{Y}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{D} N(0,1)$$

meaning that CDF of Z_n converges to the CDF of $N(0,1) \Rightarrow$

$$P(Z_n \leq z) \to \Phi(z) \text{ for all } z$$

$$P(a \leq Z_n \leq b) = F_{Z_n}(b) - F_{Z_n}(a) \longrightarrow \Phi(b) - \Phi(a)$$

Convergence in distribution

Theorem (Mapping Theorem)

For sequence of r.v. $X_1, \cdots, X_n$.

If $X_n \xrightarrow{D} X$, then $h(X_n) \xrightarrow{D} h(X)$ for any continuous function h.

Theorem (Limiting MGF Theorem)

X_n has CDF $F_n(x)$ and MGF $M(t;n)$ that exists for $|t| < h$. If there is a CDF $F(x)$ with MGF $M(t)$, then X_n has a limiting distribution with CDF $F(x)$.

Example 7.7

$Y_1, \cdots, Y_n$: $iid\ \ Bin(n, p)$. $\mu = np$ is a constant.

Find a limiting distribution of Y_n.

Convergence in distribution

Theorem (Slutsky's Theorem)

$$U_n \xrightarrow{D} U \text{ and } W_n \xrightarrow{p} 1 \quad \Rightarrow \quad U_n/W_n \xrightarrow{D} U.$$

Example 7.8

Prove the following proposition.

$$U_n \xrightarrow{D} N(0,1) \quad \Rightarrow \quad {W_n}^2 \xrightarrow{D} \chi^2(1)$$

Example 7.9

Prove the following proposition.

$$T_n \sim t(n) \quad \Rightarrow \quad T_n \xrightarrow{D} N(0,1)$$

Example 7.10

Prove the following proposition. (σ^2 *is known*)

$$Y_1, \cdots, Y_n \colon iid\ (\mu, \sigma^2) \quad \Rightarrow \quad \frac{\bar{Y}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{D} N(0,1)$$

Example 7.11

Prove the following proposition. $(\sigma^2 \ is \ unknown)$

$$Y_1, \cdots, Y_n: iid \ (\mu, \sigma^2) \quad \Rightarrow \quad \frac{\bar{Y}_n - \mu}{S_n/\sqrt{n}} \xrightarrow{D} N(0,1)$$