2022 SUMMER

업데이터 통계학 스터디

Chapter ~~7~~ 8 – Point Estimation 2

Sufficiency

$Y_1 \cdots Y_n \xrightarrow[\text{Reduction}]{} S = \sum Y_i$

Sufficiency

Ex. $Y_1, \cdots, Y_n$: $iid\ Bernoulli(p)$

Inference about p uses count of successes $S = \sum_1^n Y_i \sim Bin(n, p)$

Do we lose information about p in going from n observations to 1 sum?

Sufficiency

Ex. $Y_1, \cdots, Y_n$: $iid\ Unif(0, \theta)$, Good estimator based on W $= Y_{(n)}$

Does W contain all information about θ available from data?

Data reduction: $Y_1, \cdots, Y_n \to S = \sum_1^n Y_i$ or W $= Y_{(n)}$

Idea: A sufficient statistic compresses data without losing information about the parameter.

Sufficient Statistics

Definition (SS)

$Y_1, \cdots, Y_n$: random sample from a distribution with unknown θ.

A statistic $U = U(Y_1, \cdots, Y_n)$ is a sufficient statistic if conditional distribution of $Y_1, \cdots, Y_n$ given U does not depend on θ.

$Y_1 \cdots Y_n | U$

SS

Meaning: if you already know $\boxed{U = U(Y_1, \cdots, Y_n)}$, any other statistic does not have any extra information about θ.

Example 8.1

$Y_1, \cdots, Y_n$: $iid\ Bernoulli(p)$

Let $S = \sum_1^n Y_i \sim Bin(n, p)$. Find a distribution of $Y_1, \cdots, Y_n | S$?

ss:

$$P(Y_1 = y_1, \cdots, Y_n = y_n \mid S = s)$$

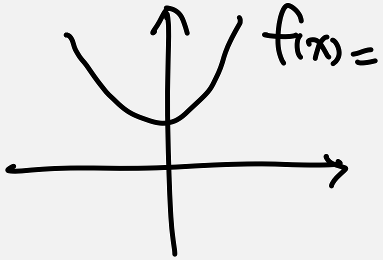
$$= \frac{P(Y_1 = y_1, \cdots Y_n = y_n \cap S = s)}{P(S=s)} = \begin{cases} \dfrac{P(Y_1 = y_1, \cdots, Y_n = y_n)}{P(S=s)} & (\Sigma y_i = s) \\ 0 & \Sigma y_i \neq s \text{ 존재 X} \quad \text{o.w} \end{cases}$$

$$= \boxed{\frac{1}{{}_nC_s}}$$

p에 의존 X.

$(\Sigma y_i = s)$

Likelihood Function



$f(x) = x^2 + a = g(a)$

<u>Definition</u>

$y_1, \cdots, y_n$: sample observations taken on corresponding random variables $Y_1, \cdots, Y_n$ whose distributions depend on θ. The likelihood function of the sample is iid

$$L(\theta | y_1, \cdots, y_n) = f(y_1|\theta) \times \cdots \times f(y_n|\theta) \overset{iid}{=} (f(y|\theta))^n$$

Likelihood = joint probability/density function of $(Y_1, \cdots, Y_n)$

[illegible]

Example 8.2

$Y_1, \cdots, Y_n :$ $Exponential(\theta)$

$Y_i \sim Exp(\theta)$ $\qquad f_{Y_i} : \frac{1}{\theta} e^{-\frac{y_i}{\theta}}$

$L(\theta | y_1, \cdots, y_n) = ?$

$= f(y_1, \cdots, y_n | \theta)$

$= f(y_1|\theta) \cdots f(y_n|\theta)$

$= \prod_{i=1}^{n} f(y_i|\theta)$

$= \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{y_i}{\theta}} I_{(0,\infty)}(y_i) = \underline{\frac{1}{\theta^n} e^{-\frac{\sum y_i}{\theta}}} \Big/ \underline{\prod_{i=1}^{n} I_{(0,\infty)}(y_i)}$

Factorization Theorem

Theorem

$U = U(Y_1, \cdots, Y_n)$ is a SS for θ iff we can write the likelihood in the form as

SS

$$L(\theta) = g(\boxed{u(y_1, \cdots, y_n)}, \theta)h(y_1, \cdots, y_n) \quad (\theta, y)$$

θ o $\qquad$ θ x

구간 제한 X.

Example 8.3

$Y_1, \cdots, Y_n$: *iid Exponential*(θ)

Find a SS for θ

(iid x) $L(\theta) = \underbrace{\frac{1}{\theta^n} e^{-\frac{\sum y_i}{\theta}}}_{g(y_1 \cdots y_n \mid \theta)} \underbrace{\prod_{i=1}^{n} I_{(0,\infty)}(y_i)}_{h(y_1 \cdots y_n)}$ (실수 전체)

By Factorization ✓

$\sum y_i$ is a SS for θ.

Example 8.4

$Y_1, \cdots, Y_n$: Geometric(p)

Find a SS for p

$f(y_i|p) = p(1-p)^{y_i-1} \quad (y_i \in \mathbb{N})$

$$L(p) = f(y_1|p) \cdots f(y_n|p)$$

$$= p^n (1-p)^{\sum y_i - n} \prod_{i=1}^{n} I_{\mathbb{N}}(y_i)$$

$$= \underbrace{\left(\frac{p}{1-p}\right)^n \cdot (1-p)^{\sum y_i}}_{g} \underbrace{\prod_{i=1}^{n} I_{\mathbb{N}}(y_i)}_{h}$$

$g(\sum y_i, p) \qquad h(y_1, \cdots, y_n)$

$\sum y_i$ is a SS for p.

By F-T.

Example 8.5

$Y_1, \cdots, Y_n$: $Bernoulli(p)$

Find a SS for p

$$f(y_i|p) = p^{y_i}(1-p)^{1-y_i} I_{\{0,1\}}(y_i)$$

$$L(p) = \underbrace{p^{\sum y_i}(1-p)^{n-\sum y_i}}_{g(\sum y_i,\, p)} \underbrace{\prod_{i=1}^{n} I_{\{0,1\}}(y_i)}_{h(y_1 \cdots y_n)}$$

$$\therefore \sum y_i \quad SS$$

Example 8.6

$Y_1, \cdots, Y_n$: $\underline{Uniform(0,\theta)}$

Find a SS for θ

$$f(y|\theta) = \begin{cases} \frac{1}{\theta} & (0<y<\theta) \\ 0 & \text{o.w.} \end{cases}$$

$$= \frac{1}{\theta} I_{(0,\theta)}(y)$$

$$L(\theta) = \underline{\frac{1}{\theta^n} \prod_{i=1}^{n} I_{(0,\theta)}(y_i)}$$

θ 존재.

$$\prod_{i=1}^{n} I_{(0,\theta)}(y_i)$$

$$= I_{(0,\theta)}(\min(y_i)) \, I_{(0,\theta)}(\max(y_i))$$

$$= I_{(0,\infty)}(\min(y_i)) \cdot I_{(0,\theta)}(\max(y_i))$$

$$= \underbrace{\frac{1}{\theta^n} I_{(0,\theta)}(\max(y_i))}_{g(\max y_i,\, \theta)} \; \underbrace{I_{(0,\infty)}(\min y_i)}_{h}$$

$\max Y_i$ SS

Example 8.7

$$f(y|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-\frac{y}{\beta}} I_{(0,\infty)}(y)$$

$Y_1, \cdots, Y_n$: *iid Gamma*(α,β)

$$L(\alpha,\beta) = \left\{\frac{1}{\beta^\alpha \Gamma(\alpha)}\right\}^n \left\{\prod y_i\right\}^{\alpha-1} \cdot e^{-\frac{\sum y_i}{\beta}} \prod_{i=1}^{n} I_{(0,\infty)}(y_i)$$

1) α is known, Find a SS for β

2) β is known, Find a SS for α

3) α and β are both unknown, SS for (α,β)

① $$L(\beta) = \underbrace{\frac{1}{\beta^{\alpha n}} e^{-\sum y_i/\beta}}_{g(\sum y_i,\beta)} \times \underbrace{\left\{\frac{1}{\Gamma(\alpha)}\right\}^n \left(\prod y_i\right)^{\alpha-1} \prod I_{(0,\infty)}(y_i)}_{h}$$

② $$L(\alpha) = \underbrace{\left\{\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\right\}^{n}\left(\prod y_i\right)^{\alpha-1}}_{g(\prod y_i,\ \alpha)} \cdot \underbrace{e^{-\sum y_i/\beta}\prod_{i=1}^{n} I_{(0,\infty)}(y_i)}_{h}$$

③ $$L(\alpha,\beta) = \underbrace{\left\{\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\right\}^{n}\left(\prod y_i\right)^{\alpha-1} e^{-\sum y_i/\beta}}_{g(\prod y_i,\ \sum y_i,\ \alpha,\ \beta)} \cdot \underbrace{\prod_{i=1}^{n} I_{(0,\infty)}(y_i)}_{h}$$

$(\prod y_i, \sum y_i)$ 가 (α,β) 의 Jointly SS.

Example 8.8

$$f(y|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

$Y_1, \cdots, Y_n$: $N(\mu, \sigma^2)$

$$\boxed{L(\mu,\sigma^2) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \cdot e^{-\frac{\Sigma(y_i-\mu)^2}{2\sigma^2}}}$$

1) σ^2 is known, Find a SS for μ

2) μ is known, Find a SS for σ^2

3) μ and σ^2 are both unknown, SS for (μ, σ^2)

① $L(\mu) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\Sigma(y_i-\bar{y}+\bar{y}-\mu)^2}$

$\Sigma(y_i-\bar{y})^2 + n(\bar{y}-\mu)^2 + \cancel{2(\bar{y}-\mu)\Sigma(y_i-\bar{y})} \to 0$

$$= e^{-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2} \times \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\Sigma(y_i-\bar{y})^2}$$

$\bar{y}$ g / h

② $$L(\sigma^2) = \underbrace{\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\sum(y_i-\mu)^2}}_{g} \times \underbrace{I_{\mathbb{R}}(y_i)}_{h}$$

$$\underline{\sum(y_i-\mu)^2} \;//$$

③ $$L(\mu,\sigma^2) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \frac{1}{\sigma^n} e^{-\frac{1}{2\sigma^2}\sum(y_i-\bar{y})^2 - \frac{n}{2\sigma^2}(\bar{y}-\mu)^2}$$

$$= \frac{1}{\sigma^n} e^{-\frac{1}{2\sigma^2}\sum y_i^2 + \frac{\mu}{\sigma^2}\sum y_i - \frac{n\mu^2}{2\sigma^2}} \times \left(\frac{1}{\sqrt{2\pi}}\right)^n I_{\mathbb{R}}(y_i)$$

$(\sum y_i^2, \sum y_i)$ Jointly SS for (μ,σ^2)

$$= \frac{1}{\sigma^n} e^{\underbrace{-\frac{1}{2\sigma^2}\sum(y_i-\bar{y})^2}_{\downarrow} - \underbrace{\frac{n}{2\sigma^2}(\bar{y}-\mu)^2}_{\downarrow}} \quad \text{"}$$

$(\sum(y_i-\bar{y}), \bar{y})$ Jointly SS for "

Sufficient Statistics

$V \xrightarrow[\text{reduction}]{\text{Data}} U$ SS

∴ SS

복원 해주는 data 잃지 않음

$L(\theta) = \frac{1}{\theta^n} e^{-\frac{\frac{1}{n}\sum y_i}{\theta/n}} \cancel{\prod_{i=1}^{n}} I_{(0,\infty)}(y_i)$

Notes:

- Any 1-1 function of a sufficient statistic is a sufficient statistic.

  Ex. $Exp(\theta)$: $U = \sum Y_i$, $V = \frac{1}{n}\sum Y_i$

  $Y_1 \cdots Y_n \longrightarrow \sum y_i$

  SS SS

- Any statistic from which a sufficient statistic is calculated is also a sufficient statistic. Ex) Random sample itself

- Many possible SS's ⇒ MSS(Minimal Sufficient Statistics)

  MSS