MODULE - 4

# Moments

## Moments

The $r^{th}$ moment of a variable $x$ about any point, say $A$, usually denoted by $\mu_r'$ is given by

For individual series $\mu_r' = \frac{\sum_{i=1}^{n} (x_i - A)^r}{n}$

For discrete and continuous series

$$\mu_r' = \frac{\sum_i f_i (x_i - A)^r}{N}, \quad N = \sum_i f_i.$$

## Raw Moments

When the deviations are taken from origin $(0,0)$, we get the $r^{th}$ order raw moment as -

For individual series $\mu_r' = \frac{\sum_{i=1}^{n} x_i^r}{n}$

For discrete and continuous series.

$$\mu_r' = \frac{\sum_i f_i x_i^r}{N}$$

## Central Moments

When the deviations are taken from mean then we get the r th order central moment as

For individual series $\mu_r = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^r}{n}$

For discrete and continuous series.

$$\mu_r = \frac{\sum f_i(x_i - \bar{x})^r}{N}$$

Q. Find the 1st 4 moments about the value 8 for the following data 7, 8, 10, 12, 6.

$\Rightarrow$ $\mu_r' = \frac{\sum (x-8)^r}{n}$

$\mu_1' = \frac{\sum (x-8)}{n} = 3/5$

$\mu_2' = \frac{\sum (x-8)^2}{n} = \frac{25}{5} = 5$

$\mu_3' = \frac{\sum (x-8)^3}{n} = 63/5$

$\mu_4' = \frac{\sum (x-8)^4}{n} = 289/5$

| $x$ | $d = x-8$ | $d^2$ | $d^3$ | $d^4$ |
|---|---|---|---|---|
| 7 | -1 | 1 | -1 | 1 |
| 8 | 0 | 0 | 0 | 0 |
| 10 | 2 | 4 | 8 | 16 |
| 12 | 4 | 16 | 64 | 256 |
| 6 | -2 | 4 | -8 | 16 |
| | 3 | 25 | 63 | 289 |

$$\begin{array}{r} 64 \\ \times\ 4 \\ \hline 256 \end{array}$$

## Recurrence Relation

Central Moments and Raw moments can connected by the following Recurrence Relation.

$$\mu_r = \mu_r' - rC_1 \mu_{r-1}' \mu_1' + rC_2 \mu_{r-2}' \mu_1'^2 + \ldots + (-1)^r \mu_1'^r$$

$$\left( nC_r = \frac{n!}{r!(n-r)!} \right)$$

When $r = 1$, $\mu_1 = \mu_1' - 1C_1 \mu_0' \mu_1'$

$$= \mu_1' - 1 \mu_1' = 0.$$

$$1C_1 = \frac{1!}{1!(1-1)!} = \frac{1!}{1!0!} = \frac{1}{1 \times 1} = 1$$

$$\left[ \mu_0' = \frac{\Sigma (x)^0}{n} = \frac{\sum_{i=1}^{n}}{} = \frac{n}{n} = 1 \right].$$

When $r=2$, $\mu_2 = \mu_2' - (\mu_1')^2$

When $r=3$, $\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$

When $r=4$, $\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$

Q- Find the 1st 4 central moments for the values 10, 7, 5, 6, 12.

$\Rightarrow$ $\bar{x} = \dfrac{\sum_{i=1}^{n} x}{n} = \dfrac{40}{5} = 8$

$$\mu_r = \frac{\sum (x-\bar{x})^r}{n}$$

$$\mu_1 = \frac{\sum (x-8)}{n} = 0$$

$$\mu_2 = \frac{\sum (x-8)^2}{n} = \frac{34}{5}$$

$$\mu_3 = \frac{\sum (x-8)^3}{n} = \frac{36}{5}$$

$$\mu_4 = \frac{\sum (x-8)^4}{n} = \frac{370}{5} = 74.$$

| $x$ | $d = (x-8)$ | $d^2$ | $d^3$ | $d^4$ |
|---|---|---|---|---|
| 10 | 2 | 4 | 8 | 16 |
| 7 | -1 | 1 | -1 | 1 |
| 5 | -3 | 9 | -27 | 81 |
| 6 | -2 | 4 | -8 | 16 |
| 12 | 4 | 16 | 64 | 256 |
| | 0 | 34 | 36 | 370 |

Q. For a distribution $\mu_1' = -2$, $\mu_2' = 35$, $\mu_3' = -105$, $\mu_4' = 817$. Find the 1st 4 central moments.

$\Rightarrow$ $\mu_1 = 0$.

$\mu_2 = \mu_2' - (\mu_1')^2$

$= 35 - (-2)^2 =$

$= 35 - 4 = 31$ //

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$

$= (-105) - 3 \times 35 \times (-2) + 2 \times (-8)$

$= 89$ //

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$

$= 817 - 4 \times (-105)(-2) + 6 \times (35) \times (-2)^2 - 3 \times (-2)^4$

$= 769$

Q- Calculate the 1st 4 moments about mean for the following data

x : 1, 2, 3, 4, 5, 6, 7, 8, 9.

f : 1, 6, 13, 25, 30, 22, 9, 5, 2.

| $x$ | $f$ | $fx$ | $d = x - \bar{x}$ | $fd^2$ | $fd^3$ | $fd^4$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | $1 - 4.9 = -3.9$ | 15.21 | -59.319 | 231.3 |
| 2 | 6 | 12 | $2 - 4.9 = -2.9$ | 50.46 | [illegible] | 424.3 |
| 3 | 13 | 39 | -1.9 | 46.93 | -89.167 | 169.4 |
| 4 | 25 | 100 | -0.9 | 20.25 | -18.22 | 16.402 |
| 5 | 30 | 150 | 0.1 | 0.3 | 0.03 | 0.003 |
| 6 | 22 | 132 | 1.1 | 26.62 | 29.2 | 32.2 |
| 7 | 9 | 63 | 2.1 | 39.69 | 83.3 | 175 |
| 8 | 5 | 40 | 3.1 | 48 | 148.9 | 461.7 |
| 9 | 2 | 18 | 4.1 | 33.62 | 137.8 | 565.1 |
| | 113 | 555 | | 281.08 | ~~12269~~ 86.453 | 2075.403 |

$(M_1')^4$

$$\text{Mean} = \frac{\Sigma fx}{\Sigma f} = \frac{555}{113} = 4.9$$

$$\mu_1 = 0$$

$$\mu_2 = \frac{\Sigma f(x-\bar{x})^2}{\Sigma f} = \frac{\Sigma fd^2}{\Sigma f} = \frac{281.08}{113} = 2.48$$

$$\mu_3 = \frac{\Sigma f(x-\bar{x})^3}{\Sigma f} = \frac{\Sigma fd^3}{\Sigma f} = \frac{86.4}{113} = 0.765$$

$$\mu_4 = \frac{\Sigma f(x-\bar{x})^4}{\Sigma f} = \frac{\Sigma fd^4}{\Sigma f} = \frac{2075.403}{113} = 18.36$$

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Q Find 1st 4 raw moments and 4 central moments from the following data.

class - 0-10, 10-20, 20-30, 30-40, 40-50.

f : 2, 4, 6, 5, 3.

| class | f | $x$ | $fx$ | $fx^2$ | $fx^3$ | $fx^4$ |
|---|---|---|---|---|---|---|
| 0-10 | 2 | 5 | 10 | 50 | 250 | 1250 |
| 10-20 | 4 | 15 | 60 | 900 | 13,500 | 202,500 |
| 20-30 | 6 | 25 | 150 | 3,750 | 93,750 | 2343750. |
| 30-40 | 5 | 35 | 175 | 6125 | 214375 | 7503125 |
| 40-50 | 3 | 45 | 135 | 6075 | 273375 | 12301875 |
| | 20 | | 530 | 16900 | 595250 | 22352500 |

and $Q_3$ are not

## Skewness and Kurtosis

Symmetric Distribution:- A frequency distribution is said to be symmetric if the frequencies are distributed symmetrically on either side of an average. In a symmetrical frequency distribution, the no. of items above the mean and below the mean would be the same and the items are symmetrically arranged about the mean. In a symmetrical distribution mean, median and mode coincide and they lie at the centre of the distribution. ~~As the distribution~~ Here, $Q_1$ and $Q_3$ are equidistant from median.

Skewness:- Skewness means lack of symmetry. The word skewness literally denotes asymmetry. For a skewed distribution mean, median and mode are not equal.

Positive skewness:-

Skewness is said to be positive when,

mean > median > mode.

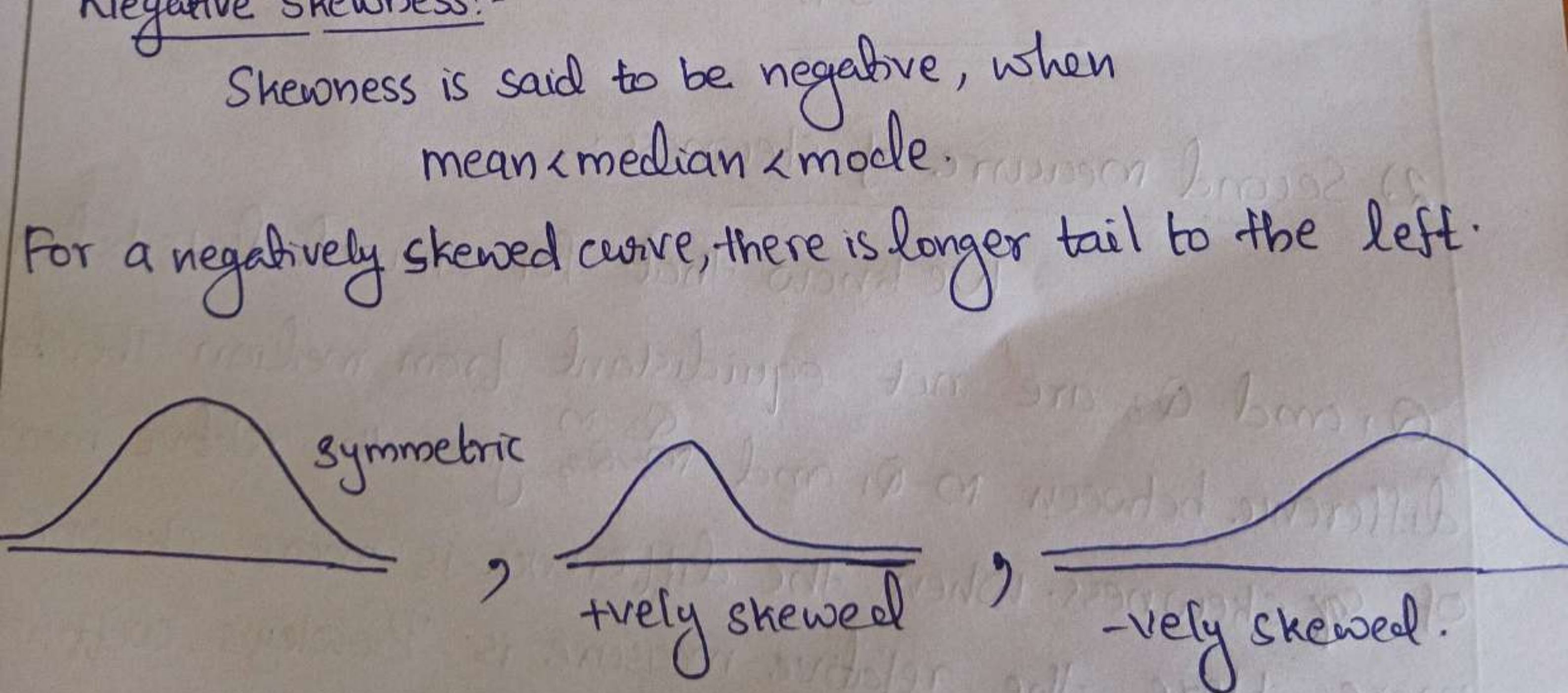
For a positively skewed curve, there is longer tail to the right

Negative Skewness:-

Skewness is said to be negative, when

mean < median < mode.

For a negatively skewed curve, there is longer tail to the left.



## Measure of Skewness

Measure of skewness gives an idea about the direction and extent of asymmetry in a series. We can also compare two or more series and say which ~~is~~ series has more skewness. Measure of skewness may be absolute or relative. Relative measure of skewness are also known as coefficient of skewness.

1) First measure of skewness: ~~For a skewed~~

We know that for a skewed distribution mean, median and mode are not equal. ~~Therefore~~ Therefore the distance between the mean and mode can be used to measure skewness.

When mean-mode $>0$, then skewness is +ve.

When mean-mode $<0$, then skewness is -ve.

Here the relative measure is, Karl Pearson coefficient of skewness, and is defined as,

$$J = \frac{\text{mean}-\text{mode}}{S.D}, \quad -3 < J < 3.$$

2) Second measure of skewness :-

We know that for skewed distribution $Q_1$ and $Q_3$ are not equidistant from median. Therefore the difference between $M-Q_1$ and $Q_3-M$ gives the relative measure of ~~sq~~ skewness. When the difference is more, skewness is more. Here the relative measure is Bowley's coefficient of skewness.

$$S_B = \frac{(Q_3-M)-(M-Q_1)}{Q_3-Q_1} = \frac{Q_3+Q_1-2M}{Q_3-Q_1}$$

3) Third measure of skewness:

We know that for a skewed distribution 1st decile and 9th decile are not equidistant from median. Then, the difference between $D_9$-median and median-$D_1$ give the absolute measure of skewness.

Here the relative measure is Kelley's coefficient of skewness, $S_K = \frac{D_9 + D_1 - 2\,median}{D_9 - D_1}$

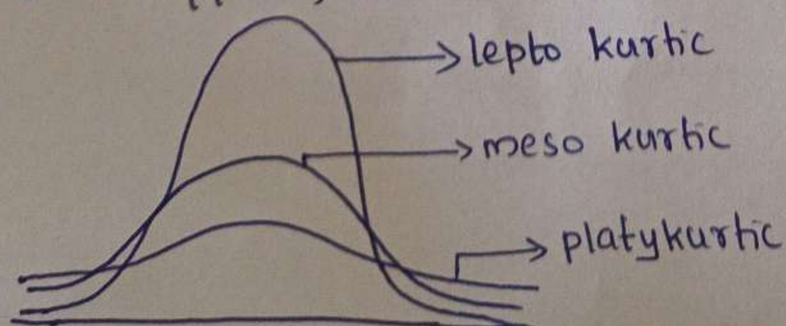
4) Fourth measure of skewness:-

This is based on moments.

$$\text{Coef. of skewness} = \frac{\mu_3}{\sqrt{\mu_2^3}}$$

II Kurtosis:

Kurtosis indicates whether a distribution is flat topped or peaked. A measure of Kurtosis is therefore a measure of peakedness. When a frequency curve is more peaked than the normal curve it is called leptokurtic. And when it is more flat than the normal curve then it is called platy kurtic. When a curve is neither peaked not flat topped, it is mesokurtic.



## Measure of Kurtosis

The measure of Kurtosis is derived from moments

Kurtosis can be defined as,

$\beta_2 = \mu_4 / \mu_2^2$ , when $\beta_2 = 3$, the curve is mesokurtic

when $\beta_2 < 3$, then it is platy kurtic

when $\beta_2 > 3$, then it is lepto kurtic

Given coefficient of skewness = $-0.23$, Mean = 47.2 and S.D = 12. Find Mode and Median of the ~~dif~~ distribution.

We have Karl pearson coefficient of skewness.

$$S = \frac{\text{mean} - \text{mode}}{SD}$$

$$-0.23 = \frac{47.2 - \text{mode}}{12}$$

$$(-0.23) \times 12 = 47.2 - \text{mode}$$

$$\text{mode} = 47.2 + 2.76$$

$$\text{mode} = 49.96$$

Imp Mode = 3 median − 2 mean.

$$49.96 = 3 \times \text{median} - 2(47.2)$$

$$3\,\text{Median} = 49.96 + 2(47.2)$$

$$= \underline{\underline{48.12}}$$

Q. You are given mean = 50. Coefficient of Variance = 40, skewness = -0.4. Find SD, ~~and~~ Mode and median.

$\Rightarrow$ C.V = 40.

$CV = \frac{S.D}{\bar{x}} \times 100.$

$40 = \frac{SD}{50} \times 100.$

$SD = \frac{40 \times 50}{100} = 20\%.$

$\text{Skewness} = \frac{\text{mean} - \text{mode}}{S.D.}$

$-0.4 \cancel{\text{Mode}} = \frac{50 - \text{mode}}{20} =$

~~mode =~~

$\cancel{-0.4 \times \text{mode} = \frac{50}{20}}$

$\cancel{\text{mode} = \frac{50 \times -0.4}{20}} =$

$-0.4 \times 20 = 50 - \text{mode}$

$-8 \quad 50 - \text{mode}$

$\text{mode} = \cancel{-0.4 \times 20}$

$= 58$

median = ,

Mode = 3 ~~Med~~ 2 mean

The median, mode & C. skewness for a certain distribution are respectively 17.4, 15.3 and 0.35. Calculate C. of variation.

$$\text{Mode} = 3M - 2\bar{x}$$

$$\Rightarrow 15.3 = 3 \times 17.4 - 2\bar{x}$$

$$\rightarrow 2\bar{x} = 3(17.4) - 15.3$$

$$\bar{x} =$$

$$S = \frac{\text{mean} - \text{mode}}{S.D.}$$

$$0.35 = \frac{18.45 - 15.3}{S.D.}$$

$$S.D = \frac{18.45 - 15.3}{0.35} = \ldots$$

$$CV = \frac{SD}{\bar{x}} \times 100$$