Assignment 6

2023320060

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COSE 382 HW 6

Date: 2024. 11. 11 **Due:** 2024. 11. 18

1. For $Z \sim \mathcal{N}(0,1)$
   (a) Find the PDF of Z^3.
   (b) Find the PDF of Z^4.

2. Let $U \sim \text{Unif}(0, \frac{\pi}{2})$. Find the PDF of $\sin(U)$.

3. Let X and Y have joint PDF $f_{X,Y}(x,y)$, and transform $(X,Y) \mapsto (T,W)$ linearly by letting
$$T = aX + bY \text{ and } W = cX + dY,$$
where a, b, c, d are constants such that $ad - bc \neq 0$.
   (a) Find the joint PDF $f_{T,W}(t,w)$ (in terms of $f_{X,Y}$ as a function of t and w).
   (b) For a special case where $T = X + Y,\ W = X - Y$, write down $f_{T,W}(t,w)$.

4. Let X and Y be independent positive r.v.s, with PDFs f_X and f_Y, respectively. Let T be the ratio X/Y and $W = X$.
   (a) Find the joint PDF of T and W, using a Jacobian.
   (b) Find the marginal PDF of T, as a single integral

5. Let X and Y be i.i.d. Expo(λ), and transform them to $T = X + Y, W = X/Y$.
   (a) Find the joint PDF of T and W. Are they independent?
   (b) Find the marginal PDFs of T and W.

6. Let X and Y be i.i.d. Unif(0,1). Find the joint distribution of $U = X + Y$ and $V = X - Y$.

7. Let X and Y be i.i.d. Gaussian Normal $\mathcal{N}(0,1)$. Let
$$R = \sqrt{X^2 + Y^2} \text{ and}$$
$$U = \begin{cases} \tan^{-1}(Y/X) & x > 0 \\ \tan^{-1}(Y/X) + \pi & x < 0, y \geq 0 \\ \tan^{-1}(Y/X) - \pi & x < 0, y < 0 \end{cases}$$
Find the pdf of R.

8. Let T and V be random variables with the joint pdf
$$f_{T,V}(t,v) = \frac{1}{\sqrt{\pi}\Gamma(n/2)} \frac{1}{2^{(n+1)/2}} \frac{1}{\sqrt{n}} v^{(n+1)/2-1} e^{-(v/2)\left(1+t^2/n\right)} \quad \text{(for } v > 0\text{)}.$$
Compute the marginal pdf of T.

1. For $Z \sim \mathcal{N}(0,1)$

   (a) Find the PDF of Z^3.

   (b) Find the PDF of Z^4.

$Z \sim N(0,1)$, $f_Z(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)$

$f_Y(y) = f_Z(z) \left|\frac{dz}{dy}\right|$ where $Y = g(z)$, $z = g^{-1}(y)$

(a) $Y = Z^3$, $Z = Y^{1/3}$, $\frac{dz}{dy} = \frac{d}{dy}(y^{1/3}) = \frac{1}{3} y^{-2/3}$

$$f_Y(y) = f_Z(y^{1/3}) \left|\frac{dz}{dy}\right| = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y^{1/3})^2}{2}\right) \cdot \frac{1}{3} y^{-2/3}$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^{2/3}}{2}\right) \cdot \frac{1}{3} |y|^{-2/3}.$$

(b) $Y = Z^4$, $Z = \pm Y^{1/4}$, $\frac{dz}{dy} = \frac{d}{dy}(y^{1/4}) = \frac{1}{4} y^{-3/4}$

$$f_Y(y) = f_Z(y^{1/4}) \left|\frac{dz}{dy}\right| = f_Z(y^{1/4}) \cdot \left|\frac{1}{4} y^{-3/4}\right| + f_Z(-y^{1/4}) \cdot \left|\frac{1}{4} y^{-3/4}\right|$$

$$= 2 \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y^{1/4})^2}{2}\right) \cdot \frac{1}{4} y^{-3/4}$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^{1/2}}{2}\right) \cdot \frac{1}{2} y^{-3/4}, \quad y \geq 0.$$

2. Let $U \sim \text{Unif}(0, \frac{\pi}{2})$. Find the PDF of $\sin(U)$.

$f_U(u) = \frac{2}{\pi}, \quad 0 \le u \le \frac{\pi}{2}$

$f_Y(y) = f_U(u) \left| \frac{du}{dy} \right|$ where $Y = \sin(U)$, $U = \sin^{-1}(Y)$

Let $Y = \sin(U)$.

$\frac{du}{dy} = \frac{d}{dy}(\sin^{-1}(y)) = \frac{1}{\sqrt{1-y^2}}$.

$f_Y(y) = f_U(\sin^{-1}(y)) \left| \frac{dy}{dy} \right|$

$= \frac{2}{\pi} \cdot \frac{1}{\sqrt{1-y^2}}, \quad 0 \le y \le 1.$

3. Let X and Y have joint PDF $f_{X,Y}(x,y)$, and transform $(X,Y) \mapsto (T,W)$ linearly by letting

$$T = aX + bY \text{ and } W = cX + dY,$$

where a, b, c, d are constants such that $ad - bc \neq 0$.

(a) Find the joint PDF $f_{T,W}(t,w)$ (in terms of $f_{X,Y}$ as a function of t and w).

(b) For a special case where $T = X + Y,\ W = X - Y$, write down $f_{T,W}(t,w)$.

(a) $J = \begin{bmatrix} \frac{\partial T}{\partial X} & \frac{\partial T}{\partial Y} \\ \frac{\partial W}{\partial X} & \frac{\partial W}{\partial Y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$

$|\det(J)| = |ad - bc|.$

$f_{T,W}(t,w) = f_{X,Y}(x,y) \cdot \frac{1}{|\det(J)|}$

$= f_{X,Y}(x,y) \cdot \frac{1}{|ad-bc|}$

(b) $J = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$

$|\det(J)| = 2.$

$f_{T,W}(t,w) = f_{X,Y}\left(\frac{t+w}{2}, \frac{t-w}{2}\right) \cdot \frac{1}{2}.$

4. Let X and Y be independent positive r.v.s, with PDFs f_X and f_Y, respectively. Let T be the ratio X/Y and $W = X$.

(a) Find the joint PDF of T and W, using a Jacobian.

(b) Find the marginal PDF of T, as a single integral

(a)

$$J = \frac{\partial(x,y)}{\partial(t,w)} = \begin{pmatrix} \frac{\partial X}{\partial T} & \frac{\partial X}{\partial W} \\ \frac{\partial Y}{\partial T} & \frac{\partial Y}{\partial W} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{W}{T^2} & \frac{1}{T} \end{pmatrix}.$$

$$|\det(J)| = \frac{W}{T^2}.$$

$$f_{T,W}(t,w) = f_X(x) f_Y(y) \cdot \frac{w}{t^2} = f_X(w) f_Y\left(\frac{w}{t}\right) \cdot \frac{w}{t^2}, \quad t>0, w>0.$$

(b)

$$f_T(t) = \int_0^\infty f_X(x) f_Y\left(\frac{x}{t}\right) \cdot \frac{x}{t^2}\, dx$$

5. Let X and Y be i.i.d. Expo(λ), and transform them to $T = X + Y, W = X/Y$.

(a) Find the joint PDF of T and W. Are they independent?

(b) Find the marginal PDFs of T and W.

(a) Let $U = X/(X+Y)$, T and W are independent with $T \sim \text{Gamma}(2, \lambda)$ and $U \sim \text{Unif}(0,1)$.

$$P(W \le w) = P(U \le w/(w+1)) = w/(w+1), \quad w > 0.$$

$$f_W(w) = \frac{(w+1) - w}{(w+1)^2} = \frac{1}{(w+1)^2}, \quad w > 0.$$

$$f_{T,W}(t,w) = (\lambda t)^2 e^{-\lambda t} \cdot \frac{1}{t} \cdot \frac{1}{(w+1)^2}, \quad t > 0, w > 0.$$

(b)

$$f_T(t) = (\lambda t)^2 e^{-\lambda t} \cdot \frac{1}{t}, \quad t > 0.$$

$$f_W(w) = \frac{1}{(w+1)^2}, \quad w > 0.$$

6. Let X and Y be i.i.d. Unif(0,1). Find the joint distribution of $U = X + Y$ and $V = X - Y$.

$$X = \frac{U+V}{2}, \quad Y = \frac{U-V}{2}.$$

$$J = \begin{bmatrix} \frac{\partial X}{\partial U} & \frac{\partial X}{\partial V} \\ \frac{\partial Y}{\partial U} & \frac{\partial Y}{\partial V} \end{bmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

$$|\det(J)| = \frac{1}{2}.$$

$$0 \le X = \frac{U+V}{2} \le 1, \quad 0 \le Y = \frac{U-V}{2} \le 1.$$

$$\rightsquigarrow \quad 0 \le U \le 2, \quad |V| \le U$$

$$f_{U,V}(u,v) = f_{X,Y}\left(\frac{u+v}{2}, \frac{u-v}{2}\right) \cdot |\det(J)| = 1 \cdot \frac{1}{2} = \frac{1}{2},$$

$$0 \le u \le 2, \quad |v| \le u.$$

7. Let X and Y be i.i.d. Gaussian Normal $\mathcal{N}(0,1)$. Let

$$R = \sqrt{X^2+Y^2} \text{ and}$$

$$U = \begin{cases} \tan^{-1}(Y/X) & x > 0 \\ \tan^{-1}(Y/X) + \pi & x < 0, y \geq 0 \\ \tan^{-1}(Y/X) - \pi & x < 0, y < 0 \end{cases}$$

Find the pdf of R.

$R = \sqrt{X^2+Y^2}$ → 직교 좌표계 ~ 극좌표계 변환에서 보았음.

$$\underbrace{f_R(r) = re^{-r^2/2}}_{\text{레일리 분포}}, \quad r \geq 0.$$

8. Let T and V be random variables with the joint pdf

$$f_{T,V}(t,v) = \frac{1}{\sqrt{\pi}\Gamma(n/2)} \frac{1}{2^{(n+1)/2}} \frac{1}{\sqrt{n}} v^{(n+1)/2-1} e^{-(v/2)\left(1+t^2/n\right)} \quad (\text{for } v > 0).$$

Compute the marginal pdf of T.

$$f_T(t) = \int_0^\infty f_{T,V}(t,v)\,dv.$$

$$= \int_0^\infty \frac{1}{\sqrt{\pi}\Gamma(n/2)} \cdot \frac{1}{2^{(n+1)/2}} \cdot \frac{1}{\sqrt{n}} \cdot v^{(n+1)/2-1} e^{-(v/2)(1+t^2/n)}\,dv.$$

이때,

$$\int_0^\infty v^{a-1} e^{-bv}\,dv = \frac{\Gamma(a)}{b^a}, \quad a>0,\ b>0.$$

$$\left(\because\ a = \frac{n+1}{2},\ b = \frac{1+t^2/n}{2}\right)$$: α 꼴에 이용...

$$f_T(t) = \frac{1}{\sqrt{\pi}\Gamma(n/2)} \cdot \frac{1}{2^{(n+1)/2}} \cdot \frac{1}{\sqrt{n}} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right)}{\left(\frac{1+t^2/n}{2}\right)^{(n+1)/2}}$$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n}\,\Gamma\left(\frac{n}{2}\right)} \left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}}$$