Assignment 8

2023320060

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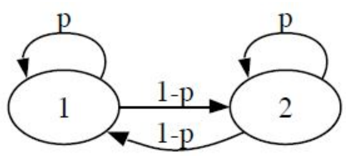
COSE 382 HW 8

Date: 2024. 12. 02 **Due:** 2024. 12. 09

1. Consider the Markov chain shown below, where $0 < p < 1$ and the labels on the arrows indicate transition probabilities

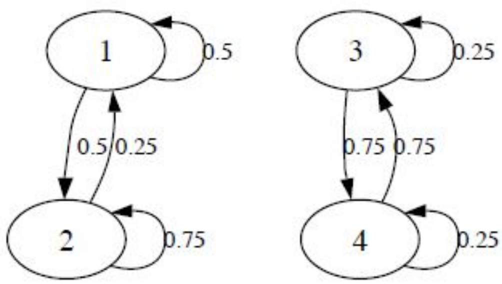


(a) Find the transition matrix Q

(b) Find the stationary distribution

(c) What happens to Q^n as $n \to \infty$

2. Consider the Markov chain shown below, with state space $\{1, 2, 3, 4\}$ and the labels on the arrows indicate transition probabilities

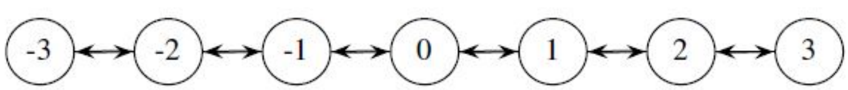


(a) Find the transition matrix Q

(b) Which states (if any) are recurrent? Which states (if any) are transient?

(c) Find two different stationary distributions for the chain

3. A Markov chain $X_0, X_1, \cdots$ with state space $\{-3, -2, -1, 0, 1, 2, 3\}$ proceeds as follows. The chain starts at $X_0 = 0$. If X_n is not an endpoint (-3 or 3), then X_{n+1} is X_{n-1} or X_{n+1}, each with probability $1/2$. Otherwise, the chain gets reflected off the endpoint, i.e., from 3 it always goes to 2 and from -3 it always goes to -2. A diagram of the chain is shown below.

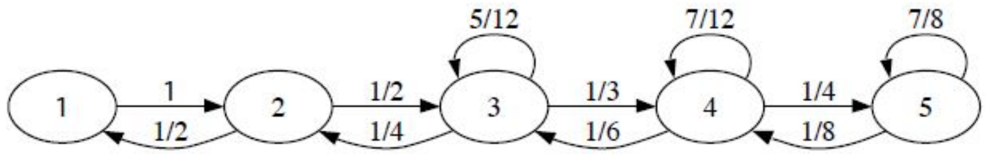


(a) Is $|X_0|, |X_1|, |X_2|, \cdots$ also a Markov chain?

(b) Let sgn be the sign function: $\text{sgn}(x) = 1$ if $x > 0$, $\text{sgn}(x) = -1$ if $x < 0$, and $\text{sgn}(0) = 0$. Is $\text{sgn}(X_0), \text{sgn}(X_1), \text{sgn}(X_2), \cdots$ a Markov chain?

(c) Find the stationary distribution of the chain $X_0, X_1, X_2, \cdots$

4. Find the stationary distribution of the Markov chain shown below, without using matrices. The number above each arrow is the corresponding transition probability



5. Let $\{X_n\}$ be a Markov chain on states $\{0, 1, 2\}$ with transition matrix

$$\begin{pmatrix} 0.8 & 0.2 & 0 \\ 0 & 0.8 & 0.2 \\ 0 & 0 & 1 \end{pmatrix}$$

The chain starts at $X_0 = 0$. Let T be the time it takes to reach state 2 :

$$T = \min\{n : X_n = 2\}.$$

Find $E(T)$ and $\text{Var}(T)$.

6. Let us consider random walk on a weighted undirected network. Suppose that an undirected network is given, where each edge (i, j) has a nonnegative weight w_{ij} assigned to it (we allow $i = j$ as a possibility). We assume that $w_{ij} = w_{ji}$ since the edge from i to j is considered the same as the edge from j to i. When (i, j) is not an edge, we set $w_{ij} = 0$

When at node i, the next step is determined by choosing an edge attached to i with probabilities proportional to the weights.

(a) Let $v_i = \sum_j w_{ij}$ for all nodes i. Show that the stationary distribution of node i is proportional to v_i.

(b) Show that every reversible Markov chain can be represented as a random walk on a weighted undirected network.

7. There are two urns with a total of $2N$ distinguishable balls. Initially, the first urn has N white balls and the second urn has N black balls. At each stage, we pick a ball at random from each urn and interchange them. Let X_n be the number of black balls in the first urn at time n. This is a Markov chain on the state space $\{0, 1, \ldots, N\}$.

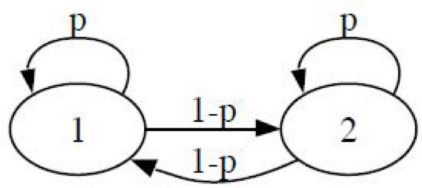
(a) Give the transition probabilities of the chain.

(b) Show that $(s_0, s_1, \ldots, s_N)$ where

$$s_i = \frac{\binom{N}{i}\binom{N}{N-i}}{\binom{2N}{N}}$$

is the stationary distribution, by verifying the reversibility condition.

1. Consider the Markov chain shown below, where $0 < p < 1$ and the labels on the arrows indicate transition probabilities



(a) Find the transition matrix Q

(b) Find the stationary distribution

(c) What happens to Q^n as $n \to \infty$

(a)
$$Q = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

(b)
$Q \rightarrow$ symmetric,

Stationary distribution $\sim$ unif$(\frac{1}{2}, \frac{1}{2})$.

$\pi Q = \pi$ ($\because \pi = (\pi_1, \pi_2)$)

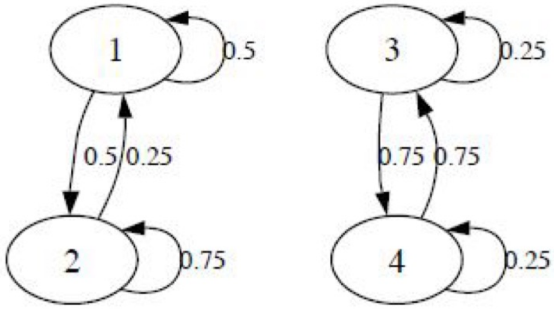
$$[\pi_1, \pi_2] \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix} \Rightarrow \frac{1}{2} = \pi_1 = \pi_2 \quad (\because \pi_1 + \pi_2 = 1)$$

(c) Q의 고윳값

$\lambda_1 = 1, \ \lambda_2 = 2p-1$

$$Q^n \rightarrow \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

2. Consider the Markov chain shown below, with state space $\{1, 2, 3, 4\}$ and the labels on the arrows indicate transition probabilities



(a) Find the transition matrix Q

(b) Which states (if any) are recurrent? Which states (if any) are transient?

(c) Find two different stationary distributions for the chain

(a)
$$Q = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.25 & 0.75 & 0 & 0 \\ 0 & 0 & 0.25 & 0.75 \\ 0 & 0 & 0.75 & 0.25 \end{pmatrix}$$

(b) All of the states are reccurent.
None of the states are transient.

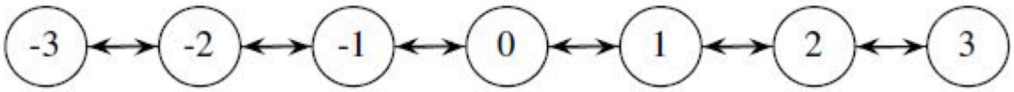
(c) $Q_1 = \begin{pmatrix} 0.5 & 0.5 \\ 0.25 & 0.75 \end{pmatrix}$, $[\pi_1(1), \pi_1(2)]\, Q_1 = [\pi_1(1), \pi_1(2)]$

$\sum \pi_1(n) = 1 \Rightarrow \pi_1 = \left[\frac{1}{3}, \frac{2}{3}\right]$

$Q_2 = \begin{pmatrix} 0.25 & 0.75 \\ 0.75 & 0.25 \end{pmatrix}$, $[\pi_2(1), \pi_2(2)]\, Q_2 = [\pi_2(1), \pi_2(2)]$

$\sum \pi_2(m) = 1 \Rightarrow \pi_2 = \left[\frac{1}{2}, \frac{1}{2}\right]$

3. A Markov chain $X_0, X_1, \cdots$ with state space $\{-3, -2, -1, 0, 1, 2, 3\}$ proceeds as follows. The chain starts at $X_0 = 0$. If X_n is not an endpoint (-3 or 3), then X_{n+1} is X_{n-1} or X_{n+1}, each with probability $1/2$. Otherwise, the chain gets reflected off the endpoint, i.e., from 3 it always goes to 2 and from -3 it always goes to -2. A diagram of the chain is shown below.



(a) Is $|X_0|, |X_1|, |X_2|, \cdots$ also a Markov chain?

(b) Let sgn be the sign function: $\text{sgn}(x) = 1$ if $x > 0$, $\text{sgn}(x) = -1$ if $x < 0$, and $\text{sgn}(0) = 0$. Is $\text{sgn}(X_0), \text{sgn}(X_1), \text{sgn}(X_2), \cdots$ a Markov chain?

(c) Find the stationary distribution of the chain $X_0, X_1, X_2, \cdots$

(a) True.

→ It can be viewed as the chain on state space 0, 1, 2, 3 that moves left or right with equal probability, except that at 0 it bounces back to 1 and at 3 it bounces back to 2.

(b) False

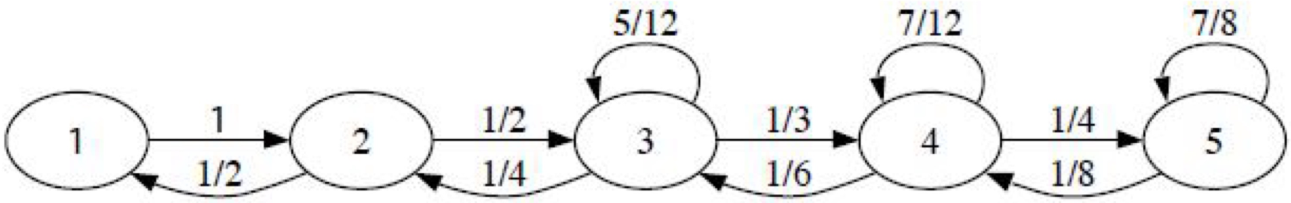
→ knowing that the chain was at 0 recently affects how far the chain can be from the origin.

Ex) $p(\text{sgn}(X_2)=1 \mid \text{sgn}(X_1)=1) > p(\text{sgn}(X_2)=1 \mid \text{sgn}(X_1)=1, \text{sgn}(X_0)=0)$.

(c) $(1, 2, 2, 2, 2, 2, 1)$

$\rightarrow \frac{1}{12}(1, 2, 2, 2, 2, 2, 1)$

4. Find the stationary distribution of the Markov chain shown below, without using matrices. The number above each arrow is the corresponding transition probability



let q_{ij} be the transition prob from i to j.

Solve for s in terms of s_1.

And $q_{ij} = 2q_{ji}$, for $j = i+1$.

Then,

$$s_1 q_{12} = s_2 q_{21},\ s_2 = 2s_1$$

$$s_2 q_{23} = s_3 q_{32},\ s_3 = 2s_2 = 4s_1$$

$$s_3 q_{34} = s_4 q_{43},\ s_4 = 2s_3 = 8s_1$$

$$s_4 q_{45} = s_5 q_{54},\ s_5 = 2s_4 = 16s_1.$$

$$\Rightarrow \left(\frac{1}{31}, \frac{2}{31}, \frac{4}{31}, \frac{8}{31}, \frac{16}{31}\right)$$

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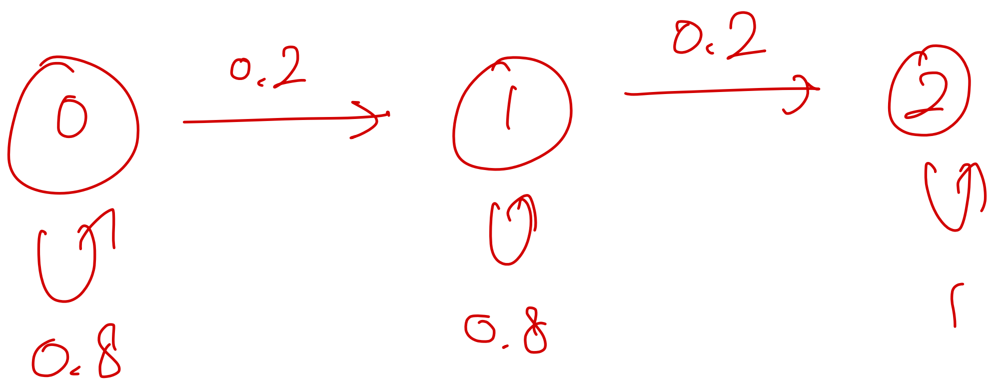

$T_1 \sim FS(0.2)$, $T_2 \sim FS(0.2)$

$E(T) = 5 + 5 = 10$

$\mathrm{Var}(T) = 20 + 20 = 40$

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(a)

$$P_{ij} = \frac{W_{ij}}{\Sigma_K W_{ik}} = \frac{W_{ij}}{V_i}, \quad \text{조건} : \pi_i P_{ij} = \pi_j P_{ji} \quad \forall i,j$$

정규화 상수 C,

$$\pi_i P_{ij} = \frac{W_{ij}}{C}, \quad \pi_j P_{ji} = \frac{W_{ji}}{C}$$

$\Rightarrow$ 비례. $\pi_i \propto V_i$ 는 정상 분포 조건 만족

(b) let $W_{ij} = S_i q_{ij}$

i) 대칭성 : $W_{ij} = S_i q_{ij} = S_j q_{ji} = W_{ji}$

ii) $P_{ij} = \dfrac{W_{ij}}{\Sigma_k W_{ik}} = \dfrac{S_i q_{ij}}{\Sigma_k S_i q_{ik}}$

이때, $\Sigma_k S_i q_{ik} = S_i \cdot \Sigma_k q_{ik} = S_i$ 이므로: $P_{ij} = q_{ij}$.

따라서 statement는 True.

7. There are two urns with a total of $2N$ distinguishable balls. Initially, the first urn has N white balls and the second urn has N black balls. At each stage, we pick a ball at random from each urn and interchange them. Let X_n be the number of black balls in the first urn at time n. This is a Markov chain on the state space $\{0, 1, \ldots, N\}$.

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is the stationary distribution, by verifying the reversibility condition.

(a)

(∵ p_{ij} → transition probability from i to j.)

first urns → i개의 black, N-i개의 white

second urns → N-i개의 black과 i개의 white

i) $i \to i+1$

second에서 검은 공, first에서 흰공

$$P_{i,i+1} = \left(\frac{N-i}{N}\right)^2$$

ii) $i \to i-1$

first에서 검은 공, second에서 흰공

$$P_{i,i-1} = \left(\frac{i}{N}\right)^2$$

iii) $i \to i$

두 항아리가 같은 색

$$P_{i,i} = 2 \cdot \frac{i}{N} \cdot \frac{N-i}{N}$$

(b)

$$s_i = \frac{\binom{N}{i}\binom{N}{N-i}}{\binom{2N}{N}}, \quad \underbrace{s_i P_{i,i+1} = s_{i+1} P_{i+1,i}}$$

⇒ 가역성의 조건이다.

i) $$s_i P_{i,i+1} = \frac{\binom{N}{i}\binom{N}{N-i}}{\binom{2N}{N}}\left(\frac{N-i}{N}\right)^2$$

ii) $$s_{i+1} P_{i+1,i} = \frac{\binom{N}{i+1}\binom{N}{N-(i+1)}}{\binom{2N}{N}}\left(\frac{i+1}{N}\right)^2$$

$$\binom{N}{i}\binom{N}{N-i}(N-i)^2 \underset{?}{=} \binom{N}{i+1}\binom{N}{N-(i+1)}\left(\frac{i+1}{N}\right)^2$$

확인해야 함

⇒ 같다.

⇒ 가역성 (o)