Assignment 5

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COSE 382 HW 5

Date: 2024. 10. 28 **Due:** 2024. 11.04

1. Let X and Y be i.i.d. Expo(1). Find the CDF and PDF of $Z = |X - Y|$.

2. A stick of length L (a positive constant) is broken at a uniformly random point X. Given that $X = x$, another breakpoint Y is chosen uniformly on the interval $[0, x]$.

(a) Find the joint PDF of X and Y. Be sure to specify the support.

(b) Find the marginal distribution of Y.

(c) Find the conditional PDF of X given Y = y.

3. Let X and Y have joint PDF

$$f_{X,Y}(x, y) = cxy, \text{ for } 0 < x < y < 1.$$

(a) Find c to make this a valid joint PDF.

(b) Are X and Y independent?

(c) Find the marginal PDFs of X and Y.

(d) Find the conditional PDF of Y given $X = x$.

4. Let (X, Y) be a uniformly random point in the triangle in the plane with vertices $(0, 0), (0, 1), (1, 0)$.

(a) Find the joint PDF of X and Y.

(b) Find the marginal PDF of X.

(c) Find the conditional PDF of X given Y.

(d) Find $\text{Cov}(X, Y)$

5. A chicken lays a Pois(λ) number N of eggs. Each egg hatches a chick with probability p, independently. Let X be the number which hatch, so $X|N = n \sim \text{Bin}(n, p)$. Find the correlation between N (the number of eggs) and X (the number of eggs which hatch)

6. Let $X = V + W$, $Y = V + Z$, where V, W, Z are i.i.d. Pois(λ).

(a) Find $\text{Cov}(X, Y)$.

(b) Find the conditional joint PMF of X, Y given V, $P(X = x, Y = y|V = v)$.

7. Let X and Y be i.i.d. $\mathcal{N}(0,1)$, and let S be a random sign (1 or -1, with equal probabilities) independent of (X, Y).

(a) Determine whether or not $(X, Y, SX + SY)$ is Multivariate Normal.

(b) Determine whether or not (SX, SY) is Multivariate Normal.

8. Consider a two-dimensional jointly Gaussian random vector $\mathbf{X} = [X, Y]^T$ with the mean vector $\mu = [\mu_X \ \mu_X]^T$ and the covariance matrix $\Sigma = \begin{bmatrix} \sigma_X^2 & Cov(X,Y) \\ Cov(X,Y) & \sigma_Y^2 \end{bmatrix}$. Let the correlation coefficient of X and Y be ρ. Show that the joint pdf given in the matrix form

$$f_{XY}(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^2 |\det \Sigma|}} \exp\left(-\frac{1}{2}(\mathbf{x} - \mu)\Sigma^{-1}(\mathbf{x} - \mu) \right),$$

for $\mathbf{x} = [x, y]^T$ is equivalent to the following form

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} \left[\left(\frac{x-\mu_x}{\sigma_X}\right)^2 - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 \right] \right)$$

1. Let X and Y be i.i.d. Expo(1). Find the CDF and PDF of $Z = |X - Y|$.

$X \sim \text{Expo}(1)$, $Y \sim \text{Expo}(1)$

$$\begin{cases} f(x) = e^{-x}, & x \ge 0 \\ f(y) = e^{-y}, & y \ge 0 \end{cases}$$

Convolution.

1) PDF

let $X - Y = W$, $f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(x-w)\, dx$.

i) $w \ge 0 \rightarrow x \ge 0,\ x - w \ge 0 \rightarrow x \ge w \Rightarrow x \in [w, \infty)$

$$f_W(w) = \int_w^{\infty} f_X(x) f_Y(x-w)\, dx = e^{w} \int_w^{\infty} e^{-2x}\, dx = e^{w}\left[-\tfrac{1}{2}e^{-2x}\right]_w^{\infty}$$

$$= \tfrac{1}{2}e^{-w}.$$

ii) $w < 0 \rightarrow x \ge 0,\ x - w \ge 0 \rightarrow x \ge w,\ w < 0 \Rightarrow x \in [0, \infty)$

$$f_W(w) = \int_0^{\infty} f_X(x) f_Y(x-w)\, dx = e^{w}\left[-\tfrac{1}{2}e^{-2x}\right]_0^{\infty} = \tfrac{1}{2}e^{w}$$

i)+ii) : $f_W(w) = \frac{1}{2}e^{-|w|}$, $w \in \mathbb{R}$

$Z = |W| \rightarrow$ 대칭

$$f_Z(z) = f_W(z) + f_W(-z) = e^{-z}, \quad z \ge 0.$$

$$F_Z(z) = P(Z \le z) = \int_0^z f_Z(t)\, dt = 1 - e^{-z}$$

$\boxed{\text{Ans}}$

$\boxed{\text{PDF: } e^{-z}\ (\because z \ge 0),\ \text{CDF: } 1 - e^{-z}\ (\because z \ge 0)}$

2. A stick of length L (a positive constant) is broken at a uniformly random point X. Given that $X = x$, another breakpoint Y is chosen uniformly on the interval $[0, x]$.

(a) Find the joint PDF of X and Y. Be sure to specify the support.

(b) Find the marginal distribution of Y.

(c) Find the conditional PDF of X given Y = y.

(a)

$$f(x,y) = f_X(x) f_{Y|X}(y|x) = \frac{1}{L} \cdot \frac{1}{x} = \frac{1}{Lx} \quad (\because 0 < x < L, 0 < y < x)$$

→ 길이가 L, uniform → $\frac{1}{L}$, 길이가 x, uniform → $\frac{1}{x}$.

(b)

$$f_Y(y) = \int_y^L f(x,y)\,dx = \int_y^L \frac{1}{Lx}\,dx = \frac{1}{L}\left[\ln x\right]_y^L$$

$(\because y \le x \le L)$

$$= \frac{\ln L - \ln y}{L}$$

(c)

$$f(X=x|Y=y) = \frac{f(X=x, Y=y)}{f(Y=y)} = \frac{L}{\ln L - \ln y} \cdot \frac{1}{Lx}$$

$$= \frac{1}{x(\ln L - \ln y)}$$

3. Let X and Y have joint PDF

$$f_{X,Y}(x,y) = cxy, \text{ for } 0 < x < y < 1.$$

(a) Find c to make this a valid joint PDF.

(b) Are X and Y independent?

(c) Find the marginal PDFs of X and Y.

(d) Find the conditional PDF of Y given $X = x$.

(a) $\int_0^1 \int_0^y cxy \, dx \, dy = 1$

$\rightarrow \int_0^1 \left(\int_0^y cxy \, dx \right) dy = \int_0^1 \frac{cy^3}{2} dy = \frac{c}{8}$. $\boxed{C = 8}$

(b) No, Y가 x의 range에 영향을 줌.

(c)

$f_X(x) = \int_x^1 8xy \, dy = 8x \int_x^1 y \, dy = 4x - 4x^3, \ 0 < x < 1.$

$f_Y(y) = \int_0^y 8xy \, dx = 8y \int_0^y x \, dx = 4y^3, \ 0 < y < 1.$

(d) $P(Y=y | X=x) = \frac{P(Y=y, X=x)}{P(X=x)} = \frac{8xy}{4x - 4x^3}$

$= \frac{2y}{1-x^2}, \quad x < y < 1.$

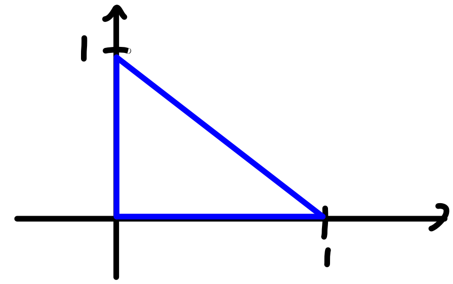
4. Let (X, Y) be a uniformly random point in the triangle in the plane with vertices $(0,0), (0,1), (1,0)$.

(a) Find the joint PDF of X and Y.

(b) Find the marginal PDF of X.

(c) Find the conditional PDF of X given Y.

(d) Find $\mathrm{Cov}(X, Y)$



(a) (X,Y) → 삼각형의 전체 면적에 비례, 삼각형 내부에서 전체 확률이 1이어야 함.

$$f_{X,Y}(x,y) = \begin{cases} 2 & (\because 0 \le x \le 1,\ 0 \le y \le 1,\ x+y \le 1) \\ 0 & \text{otherwise.} \end{cases}$$

(b)

$$f_X(x) = \int_0^{1-x} 2\,dy = 2(1-x)$$

(c)

$$f(X=x \mid Y=y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2}{2(1-y)} = \frac{1}{1-y} \quad (\because 0 \le x \le 1-y)$$

$$\Rightarrow \quad X \mid Y=y \sim \mathrm{Unif}(0, 1-y).$$

(d) $\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{12} - \frac{1}{3}\cdot\frac{1}{3} = -\frac{1}{36}.$

5. A chicken lays a Pois(λ) number N of eggs. Each egg hatches a chick with probability p, independently. Let X be the number which hatch, so $X|N = n \sim \text{Bin}(n, p)$. Find the correlation between N (the number of eggs) and X (the number of eggs which hatch)

$X \sim \text{Pois}(\lambda p)$, $Y \sim \text{Pois}(\lambda q)$ for $q = 1 - p$.

$\text{Cov}(N, X) = \text{Cov}(X+Y, X)$ (∵ X, Y는 독립)

$= \text{Cov}(X, X) + \text{Cov}(Y, X)$

$= \text{Var}(X) = \lambda p.$

$$\text{Corr}(N, X) = \frac{\lambda p}{SD(N)SD(X)} = \frac{\lambda p}{\lambda \sqrt{p}} = \frac{p}{\sqrt{p}} = \sqrt{p}.$$

6. Let $X = V + W$, $Y = V + Z$, where V, W, Z are i.i.d. Pois(λ).

(a) Find Cov(X, Y).

(b) Find the conditional joint PMF of X, Y given V, $P(X = x, Y = y | V = v)$.

(a)

$$\text{Cov}(X,Y) = \text{Cov}(V,V) + \text{Cov}(V,Z) + \text{Cov}(W,V) + \text{Cov}(W,Z)$$
$$= \text{Var}(V) = \lambda.$$

(b)
$$P(X=x, Y=y \mid V=v) = P(W = x-v, Z = y-v \mid V = v)$$
$$= P(W = x-v, Z = y-v)$$
$$= P(W = x-v)\,P(Z = y-v)$$
$$= P(X = x \mid V = v)\,P(Y = y \mid V = v)$$
$$= e^{-2\lambda} \cdot \frac{\lambda^{x-v}}{(x-v)!} \cdot \frac{\lambda^{y-v}}{(y-v)!}$$

7. Let X and Y be i.i.d. $\mathcal{N}(0,1)$, and let S be a random sign (1 or -1, with equal probabilities) independent of (X, Y).

(a) Determine whether or not $(X, Y, SX+SY)$ is Multivariate Normal.

(b) Determine whether or not (SX, SY) is Multivariate Normal.

(a)

$X+Y+(SX+SY) = (1+S)X + (1+S)Y = 0$ with probability 1/2)

$\rightarrow$ No.

(b)

$a(SX)+b(SY) = S(aX+bY)$

$aX+bY \sim N(0, a^2+b^2)$

let $Z = \dfrac{aX+bY}{\sqrt{a^2+b^2}}$,

then $S(aX+bY) = \sqrt{a^2+b^2} \cdot SZ \sim N(0, a^2+b^2)$

$\rightarrow$ Yes.

8. Consider a two-dimensional jointly Gaussian random vector $\mathbf{X} = [X, Y]^T$ with the mean vector $\mu = [\mu_X \ \mu_X]^T$ and the covariance matrix $\Sigma = \begin{bmatrix} \sigma_X^2 & Cov(X,Y) \\ Cov(X,Y) & \sigma_Y^2 \end{bmatrix}$. Let the correlation coefficient of X and Y be ρ. Show that the joint pdf given in the matrix form

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$$|\Sigma| = \sigma_x^2\sigma_y^2 - (Cov(x,y))^2$$

let $Cov(x,y) = \rho\sigma_x\sigma_y$, then $|\Sigma| = \sigma_x^2\sigma_y^2(1-\rho^2)$

$$\Sigma^{-1} = \frac{1}{1-\rho^2}\begin{bmatrix} \frac{1}{\sigma^2} & -\frac{\rho}{\sigma_x\sigma_y} \\ -\frac{\rho}{\sigma_x\sigma_y} & \frac{1}{\sigma_y^2} \end{bmatrix}$$

$$x-\mu = \begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix}$$

$$(x-\mu)^T\Sigma^{-1}(x-\mu) = \frac{1}{1-\rho^2}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} - 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2}\right]$$