LINEAR ALGEBRA (MA20105)

Problems Sheet-1

Notation: In the following $\mathbb{F} = \mathbb{R}$.

1. Prove that $(\mathbb{F}^n, +)$ is a vector space over $\mathbb{F}$, where $+$ is coordinate wise addition and $c(a_1, \ldots, a_n) := (ca_1, \ldots, ca_n)$, $(a_1, \ldots, a_n) \in \mathbb{F}^n$, $c \in \mathbb{F}$.

2. Let $M_{m\times n}(\mathbb{F})$ be the set of all $m \times n$ matrices over $\mathbb{F}$. Prove that $M_{m\times n}(\mathbb{F})$ forms a vector space under matrix addition and scalar multiplication.

3. On $\mathbb{R}^n$, define two operations
$$\alpha \oplus \beta = \alpha - \beta$$
$$c * \alpha = -c\alpha.$$
The operations on the right are the usual ones. Which of the axioms for a vector space are satisfied by $(\mathbb{R}^n, \oplus, *)$?

4. On $\mathbb{R}^2$, define
$$(x, y) + (x_1, y_1) = (x + x_1, 0)$$
$$c(x, y) = (cx, 0).$$
Is $\mathbb{R}^2$, with these operations, a vector space over $\mathbb{R}$? Provide reasons.

5. Let $P[x]_{\le n} := \{a_0 + a_1x + \cdots + a_nx^n : a_i \in \mathbb{F}, 1 \le i \le n\}$ (set of all polynomials of degree $\le n$). Let $p(x) = a_0 + a_1x + \cdots + a_nx^n$, $q(x) = b_0 + b_1x + \cdots + b_nx^n \in P[x]_{\le n}$. Define
$$p(x) + q(x) := (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$$
$$cp(x) := ca_0 + ca_1x + \cdots + ca_nx^n.$$
Prove that $P[x]_{\le n}$ is a vector space over $\mathbb{F}$ with the above operations.

Subspace related problems

1. Which of the follwing sets of vectors $\alpha = (a_1, \cdots, a_n)$ in $\mathbb{R}^n$ are subspaces of $\mathbb{R}^n (n \ge 3)$?
(i) all α such that $a_1 \ge 0$;
(ii) all α such that $a_1 + 3a_2 = a_3$;
(iii) all α such that $a_2 = a_1^2$;
(iv) all α such that $a_1a_2 = 0$;
(v) all α such that $a_1 \in \mathbb{Q}$?

2. Let V be the real vector space of all functions from $\mathbb{R} \to \mathbb{R}$. Which of the following sets of functions are subspaces of V?
   (i) all f such that $f(x^2) = f(x)^2$;
   (ii) all f such that $f(0) = f(1)$;
   (iii) all f such that $f(3) = 1 + f(-5)$;
   (iv) all f such that $f(-1) = 0$;
   (v) all f which are continuous.

3. If $S_1, S_2, \dots S_k$ are subsets of a vector space V, the set of all sums $\alpha_1 + \alpha_2 + \cdots + \alpha_k$ of vectors $\alpha_i \in S_i$ is called the sum of the subsets $S_1, \dots S_k$ and is denoted by $S_1 + \cdots S_k$ or by $\sum_{i=1}^k S_i$. If $W_1, \dots W_k$ are subspaces of V, then check that the sum $\sum_{i=1}^k W_i$ is a subspace of V which contains each of the subspaces W_i, $1 \leq i \leq k$.

4. Prove that the union of two subspaces W_1, W_2 of a vector space V is not a subspace in general but the intersection is. If $W_1 \cup W_2$ is a subspace of V, then prove that $W_1 \subset W_2$ or $W_2 \subset W_1$.

5. Let W_1 and W_2 be two subspaces of a vector space V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$. Prove that for each vector $\alpha \in V$ there are unique vectors $\alpha_1 \in W_1, \alpha_2 \in W_2$ such that $\alpha = \alpha_1 + \alpha_2$.