Any Questions?

$E_{23}(1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$; $\dfrac{R_3' = 1 \cdot R_2 + R_3}{E_{32}(1)}$

Linear Algebra & Its applications
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Correction: Assignment-1
for what values of "a" . . .
is missing

Subspaces

In practice, we come across subspaces in some vector space $(\mathbb{R}^n, +, \cdot)$

Four fundamental subspaces: (1) Column Space (2) Null space

They come from an $m \times n$ matrix A

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ \underbrace{a_{m1}}_{c_1} & \underbrace{a_{m2}}_{c_2} & \cdots & \underbrace{a_{mn}}_{c_n} \end{bmatrix}, \quad C(A) := L\{c_1, c_2, \ldots, c_n\}$$

$$c_k = \begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{mk} \end{bmatrix} \in \underline{(\mathbb{R}^m, +, \cdot)}$$

$$Ax = b$$

$$A = \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix}_{3 \times 2} \quad c_1 = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}, \; c_2 = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$$

$$\underline{\begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$c_1, c_2 \in \mathbb{R}^3$

$$\underline{x \underbrace{\begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}}_{c_1} + y \underbrace{\begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}}_{c_2}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Linear combination

$C(A)$

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathbb{R}^3$$

Thm/Fact: $AX = b$ has a solution if and only if

$b \in C(A)$

plane passing thru $(1,5,2)$, $(0,4,4)$, $\boxed{(0,0,0)}$

$b = \begin{bmatrix}0\\0\\1\end{bmatrix}$ for this $AX = b$ has a sol^n

$\iff \begin{bmatrix}0\\0\\1\end{bmatrix} \in C(A) = L\left\{ \begin{pmatrix}1\\5\\2\end{pmatrix}, \begin{pmatrix}0\\4\\4\end{pmatrix} \right\}$

$c_1\begin{pmatrix}1\\5\\2\end{pmatrix} + c_2\begin{pmatrix}0\\4\\4\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

solve $c_1 = 0$

$c_2 = 0 \quad \{c_2 = 1/8$

Not possible

$Ax = b = \begin{pmatrix}1\\9\\6\end{pmatrix}$, $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$ is a solution

This was importance of $C(A)$.

Null Space:

denoted by $N(A)$, A is $m \times n$ matrix.

$N(A) = \{ x \in \mathbb{R}^n : Ax = 0 \} \subseteq \mathbb{R}^n$

$[A]_{m\times n} \begin{bmatrix}\vdots \\ x \\ \vdots\end{bmatrix}_{n\times 1} \in \mathbb{R}^n$

$N(A)$ is a sol^n set of the homogenous equ^n

$Ax = 0 \subseteq \mathbb{R}^n$

Claim: $N(A)$ is subspace of $\mathbb{R}^n$

Proof: $\alpha, \beta \in N(A)$, $c \in \mathbb{R}$

(i) $\alpha + \beta \in N(A)$?? !

(ii) $c\alpha \in N(A)$??

$\alpha, \beta \in N(A)$

$A\alpha = 0$, $A\beta = 0$

$A(\alpha+\beta) = A\alpha + A\beta$

(distributive property of the matrices

$$A(\alpha+\beta) = \underline{A\alpha} + \underline{A\beta} = 0+0 = 0$$

$$\Rightarrow (\alpha+\beta) \in N(A)$$

$A(c\alpha) = 0$? Why!

$\overset{!}{=} cA\alpha = c \cdot 0 = \underline{0} \Rightarrow c\alpha$

$A(c\alpha)$

$(Ac)\alpha$

$= (cA)\alpha$

$= c(A\alpha)$

$= c\,0$

$= 0$

$N(A)$ is a subspace

$$Ax = 0 \to \boxed{I}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$c_1 + c_2 = c_3$

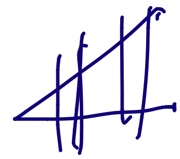
$c_1 + c_2 - c_3 = \underline{0}$

Sol^n space ? What is $N(A)$?

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad \underline{N(A) = \left\{ c\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} : c \in \mathbb{R} \right\}}$$

$L = \{ c(1,1,1) : c \in \mathbb{R} \}$ $(1,1,0) \notin L$

$(5,5,5) = 5(1,1,1) \in L$



Linearly Independent (L.I.) and Linearly dependent L.D.

$(V, +, \cdot)$

A set of vectors $S = \{v_1, \ldots, v_n\}$ is said linearly independent if $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$

$\Rightarrow c_1 = 0, c_2 = 0, \ldots c_n = 0$

$v_1, \ldots, v_n$ do not have any relation among them.

Otherwise we say S is linearly dependent

otherwise means

$c_1 v_1 + \cdots + c_n v_n = 0$ holds for at least one non-zero c_i, $i = 1, 2, \ldots, n$

$S = \{ (1,1), (1,-1) \} \subseteq \mathbb{R}^2$

Is S LI or LD ?

$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$S = \left\{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix} \right\} \quad \text{LI or LD ?}$$

$$\underset{2\times 3}{\begin{bmatrix}1 & 1 & 3\\ 1 & 2 & 4\end{bmatrix}} \underset{3\times 1}{\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}} = \begin{bmatrix}0\\0\end{bmatrix}$$

THM | $AX = 0$ $n > m$, A $m\times n$ matrix. Then it has a non-zero solution.

$A \rightsquigarrow R$ row reduced echelon form

$\Box$ ↑ Pivots

$$\begin{array}{c} \begin{bmatrix} \boxed{*} & * & 0 & 0 & * \\ 0 & 0 & \boxed{*} & 0 & * \\ 0 & 0 & 0 & \boxed{*} & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ c_1 \; c_2 \; c_3 \; c_4 \; c_5 \end{array}$$

$x_1 \leftarrow$ Pivot variable

x_2 F

x_3

x_4

x_5

free variable

$\bar{0} \in V$ | $S = \{\underline{0}\}$ is LI or L.D?

$1 \cdot \underline{0} = 0$, $c_1 = 1 \neq 0$

$\Rightarrow$ L.D.

$S = \{\underline{v}\}$ $v \neq 0$

$c\,\underline{v} = \underline{0}$

$\Rightarrow c = 0$

Pf: $c\,\underline{v} = \underline{0}$ $c \in \mathbb{R}$, $v \in \underline{V}$, $v \neq 0$.

Suppose $c \neq 0$.

$c^{-1} c\, v = c^{-1} \underline{0} = \underline{0}$

$1 v = 0$

$v = 0 \leftarrow$ a contradiction

$\Rightarrow c = 0$ $\Rightarrow \{\bar{v}\}$ is L.I.

$$\begin{bmatrix} \vdots & | & | \\ \vdots & | & | \end{bmatrix}_{c_1\, c_2\, c_3} \begin{bmatrix}x\\y\\z\end{bmatrix} = x \underset{c_1}{\begin{bmatrix} \; \end{bmatrix}} + y \underset{c_2}{\begin{bmatrix} \; \end{bmatrix}} + z \underset{c_3}{\begin{bmatrix} \; \end{bmatrix}}$$