Any Questions about Basis. Basis $B \subseteq V$

$V$ is a vector space.

(i) $L(B) = V$ (ii) $B$ is L.I. linear span of $B$

$L(B) = \{c_1 v_1 + \cdots + c_n v_n ; c_i \in \mathbb{R}\}$ $B = \{v_1, \cdots, v_n\}$

$(\mathbb{R}^3, +) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$ $\boxed{L(B) = \mathbb{R}^3}$

$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ $L(B) \subseteq \mathbb{R}^3$

$L(B) \subseteq \mathbb{R}^3$ ✓

and $\underline{\mathbb{R}^3 \subseteq L(B)}$ $\Rightarrow L(B) = \mathbb{R}^3$

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$, want to prove that $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ can be written as a linear combination of elts of $B$

Here $B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in L(B)$$

$\mathbb{R}^3 \subseteq L(B) \therefore L(B) = \mathbb{R}^3$

$$S = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ -3 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ 4 \end{pmatrix} \right\} \subseteq \mathbb{R}^3$$

Want to check whether $S$ is L.I.

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$c_1 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 6 \\ -3 \end{pmatrix} + c_3 \begin{pmatrix} 3 \\ 9 \\ 3 \end{pmatrix} + c_4 \begin{pmatrix} 2 \\ 7 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$AC = 0, \quad C = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

saying $S$ is L.I. $\Longleftrightarrow$ $AC = 0$ has no non-trivial solution.

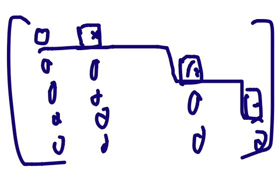
$A \rightsquigarrow$ echelon form (e.f)

$\rightarrow$ row reduced echelon form

e.f will look like

1. Pivots are 1st non-zero entry in their rows.
2. Below each pivot in a column of zeros, obtained by elimination
3. Each pivot lie to the right of the pivot in the row above.

zero rows come Last.



$$\begin{bmatrix} \boxed{1} & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \underset{R_3' = R_1 + R_3}{\overset{R_2' = -2R_1 + R_2}{\sim}} \begin{bmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{3} & 3 \\ 0 & 0 & 6 & 6 \end{bmatrix} \sim R_3' = -2R_2 + R_3$$

$$\begin{bmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{3} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ echelon form}$$

To make row reduced.

$$R_2' = \frac{1}{3}R_2 \begin{bmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_1' = -3R_2 + R_1 \quad R = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

row reduced echelon form

To make row reduced:

1. Each pivot should one
2. All other elts in the column containing pivots should be zero.

Sol$^n$ of $AX = 0$ and $RX = 0$ are same.

$$R = \begin{bmatrix} \boxed{1} & 3 & 0 & -1 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Pivot variable

free variable

Goal is to read off the solutions.

Pivot variable and free variable

Pivot variables are those that correspond to columns with pivots.

Free variables are those that correspond to columns without pivots.

$c_2 = r$, $c_4 = s$ free variable

$$c_1 + 3c_2 - c_4 = 0$$
$$c_3 + c_4 = 0$$

Method to find a special solutions:
Assign 1 to one free variable and assign 0

to other free variables
Two free variables in this case, so we get two

special solution
① $c_2 = 1$ and $c_4 = 0$
② $c_4 = 1$ and $c_2 = 0$

For ①: $c_1 = -3$
$c_4 = 0$ $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

For ②: $c_4 = 1$, $c_2 = 0$
we get $c_1 = 1$
$c_3 = -1$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Complete solution
$$c \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad c, d \in \mathbb{R}$$