Assignment-1 is uploaded
Due on 21^{st} Sept

$(V, +, \cdot)$ over $\mathbb{R}$

$+ : V \times V \to V \quad a, b \mapsto a + b$ four axioms

$\cdot : \mathbb{R} \times V \to V \quad r, a \mapsto ra$ " "

$v_1, \ldots, v_n \in V, \; c_i \in \mathbb{R}$

$\alpha = c_1 v_1 + \cdots + c_n v_n \in V$?

$c_i v_i \in V \qquad n = 1, \; c_1 v_1 \in V$

$n = 2$
$c_1 v_1 \in V$
$c_2 v_2 \in V$
$c_1 v_1 + c_2 v_2 \in V$

$c_1 v_1 + \cdots + c_{n-1} v_{n-1} \in V$
$c_n v_n \in V$

$I \Rightarrow \alpha \in V.$

$S = \{v_1, \ldots, v_m\}$

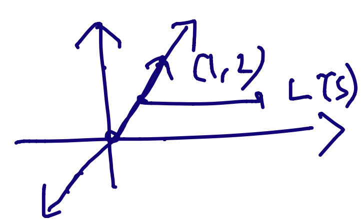
$L(S) := \{c_1 v_1 + \cdots + c_m v_m : c_i \in \mathbb{R}, 1 \le i \le m\}$

$v_1 \in L(S)$? $\quad v_1 \in L(S) \quad c_1 = 1, \; c_2 = \cdots = c_m = 0$

In this way, $v_i \in L(S)$, $1 \le i \le m$ $\qquad v_1 + v_2 \in L(S)$

$\mathbb{R}^2, \quad S = \{(1,2)\}$

$L(S) := \{c(1,2) : c \in \mathbb{R}\}$



$S = \{(1,0), (0,1)\}$

$L(S) = ? \{c_1(1,0) + c_2(0,1) : c_1, c_2 \in \mathbb{R}\}$

$= \{(c_1, c_2) : c_1 \in \mathbb{R}, c_2 \in \mathbb{R}\} = \mathbb{R}^2$

linear span of the set S.

Subspace: A non-empty subset S of V is called subspace

S is called subspace of $\underline{V}$ if (i) $\underline{v}, \underline{w} \in S$
$\Rightarrow v + w \in S$

(ii) c a scalar, $v \in S$
$\Rightarrow cv \in S$.

$(V, +, \cdot)$ vector space is given

$S \subseteq (V, +, \cdot)$

Check that $(S, +, \cdot)$ is vector space.

Is $L(S)$ subspace of V?

Answer: Yes. Proof: $L(S) \subseteq V$

$S = \{v_1, \cdots, v_m\} \subseteq V$

$v, w \in L(S)$, $v = c_1 v_1 + \cdots + c_m v_m$
$w = d_1 v_1 + \cdots + d_m v_m$

$$v + w = c_1 v_1 + \cdots + c_m v_m + d_1 v_1 + \cdots + d_m v_m$$
$$= \underline{(c_1 + d_1) v_1 + \cdots + (c_m + d_m) v_m}$$
$$\underline{e_i = c_i + d_i} \in \mathbb{R}$$
$$v + w = e_1 v_1 + \cdots + e_m v_m \in L(S)$$

(ii) $c \in \mathbb{R}$, $v \in L(S)$

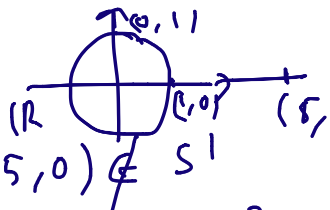
$$cv = c(c_1 v_1 + \cdots + c_m v_m) = cc_1 v_1 + \cdots + cc_m v_m$$
$$f_i = cc_i \in \mathbb{R}, \ i = 1, \cdots, m$$
$$cv = f_1 v_1 + \cdots + f_m v_m \in L(S)$$

So $v + w \in L(S)$ and $cv \in L(S)$

<u>Hence $L(S)$ is a subspace of V.</u>

$S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subseteq (\mathbb{R}^2, +, \cdot)$

Is S^1 subspace of $\mathbb{R}^2$?



$(1,0) \in S^1$, $5 \in \mathbb{R}$

$5(1,0) = (5,0) \notin S^1$

S^1 is <u>not</u> <u>a</u> subspace of $\mathbb{R}^2$.

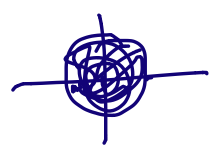
Every subspace S of V has the zero vector.

$v \in S$, $-1 \in \mathbb{R}$, by (ii) $(-1)v \in S$, $-v \in S$

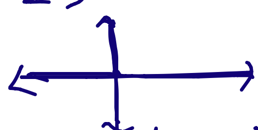
by (i) $v + (-v) \in S$

$0 \in S$

$D^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$



Is D^1 a subspace of $\mathbb{R}^2$?

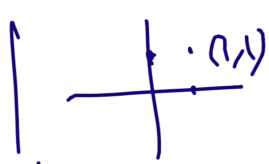
 $\subseteq \mathbb{R}^2$

$S_2 = \{(x,0) : x \in \mathbb{R}\} \cup \{(0,y) : y \in \mathbb{R}\}$

Is S_2 a subspace of $\mathbb{R}^2$?

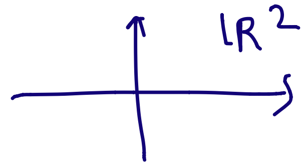
No: $(1,0), (0,1) \in S_2$

$(1,0) + (0,1) = (1,1) \notin S_2$



<u>Q1.</u> What are all subspaces of $(\mathbb{R}^2, +, \cdot)$?

$(V, +, \cdot)$



① $\{\underline{0}\} \leq V$ ← subspace

↑ Trival subspace

② $(V, +, \cdot) \subseteq (V, +, \cdot)$ ← Improper Subspace

subspace

Answer to Q_1: (i) $\{(0,0)\}$ (ii) Any line passing through origin.

(iii) $\mathbb{R}^2$

Q2: What are all subspaces of $\mathbb{R}^n$, $n \geq 1$?

$W = \{ax + by + cz = 0\} \subseteq \mathbb{R}^3$

Is W a subspace of $\mathbb{R}^3$? (Exercise)

$\boxed{y = mx}$, $\boxed{m \in \mathbb{R}}$

$y = 2x$ $\qquad \mathbb{R}^n$

$y = 3x$

Four Fundamental Subspaces.

A is $m \times n$ real matrix.

① Column space ② Null Space

$\quad\; \| \qquad\qquad\qquad\qquad \|$

$\quad C(A) \qquad\qquad\qquad N(A)$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$\quad c_1 \qquad c_2 \qquad\qquad c_n$

$$c_k = \begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{mk} \end{bmatrix} \in \mathbb{R}^m \quad 1 \le k \le n$$

$$C(A) :\equiv L\{c_1, \cdots, c_n\} \le \mathbb{R}^m$$

$S \subseteq V$ $\quad$ V

$L(S)$: $\quad S = \{v_1, \cdots, v_n\}$

$$= \{c_1 v_1 + \cdots + c_n v_n : c_i \in \mathbb{R}\}$$

$$c_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, c_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \cdots, c_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

$$c_1, c_2, \cdots, c_n \in (\mathbb{R}^m, +, \cdot)$$

$$L\{c_1, \cdots, c_n\} = \{d_1 c_1 + \cdots + d_n c_n : d_i \in \mathbb{R}\}$$

is a subspace $\mathbb{R}^m$

A $\quad m \times n$ matrix

$$\begin{bmatrix} a_{11} & \cdot & a_{1n} \\ a_{m1} & & a_{mn} \end{bmatrix}$$

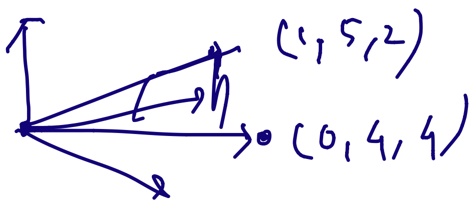
$$\begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix}_{3 \times 2}$$

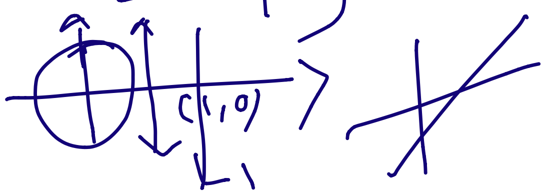
$$c_1 = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}, c_2 = \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix}$$

$$L\{c_1, c_2\} = L\left\{\begin{pmatrix}1\\5\\2\end{pmatrix}, \begin{pmatrix}0\\4\\4\end{pmatrix}\right\} \subseteq \mathbb{R}^3$$

$$= c_1\begin{pmatrix}1\\5\\2\end{pmatrix} + c_2\begin{pmatrix}0\\4\\4\end{pmatrix} : c_1, c_2 \in \mathbb{R}$$



$$\begin{bmatrix}1 & 2\\5 & 10\\2 & 4\end{bmatrix}$$



$\tilde{A}$ $\qquad \left[\, A \;\middle|\; b \,\right]$

P U $\tilde{b}$

$A = LU \qquad LUx = b$

$Lc = b \qquad Ux = c$

A $n \times n$ invertible matrix

$R_j' = cR_j$

$R_j' = cR_i + R_j \;\Big|\; R_i \leftrightarrow R_j$