

LINEAR ALGEBRA (MA20105)

Problems Sheet-1

Notation: In the following $\mathbb{F} = \mathbb{R}$.

1. Prove that $(\mathbb{F}^n, +)$ is a vector space over \mathbb{F} , where $+$ is coordinate wise addition and $c(a_1, \dots, a_n) := (ca_1, \dots, ca_n)$, $(a_1, \dots, a_n) \in \mathbb{F}^n$, $c \in \mathbb{F}$.

Hints: This is done in the class for $n = 3$.

2. Let $M_{m \times n}(\mathbb{F})$ be the set of all $m \times n$ matrices over \mathbb{F} . Prove that $M_{m \times n}(\mathbb{F})$ forms a vector space under matrix addition and scalar multiplication.

Hints: Proof is similar to 1.

3. On \mathbb{R}^n , define two operations

$$\alpha \oplus \beta = \alpha - \beta$$

$$c * \alpha = -c\alpha.$$

The operations on the right are the usual ones. Which of the axioms for a vector space are satisfied by $(\mathbb{R}^n, \oplus, *)$?

4. On \mathbb{R}^2 , define

$$(x, y) + (x_1, y_1) = (x + x_1, 0)$$

$$c(x, y) = (cx, 0).$$

Is \mathbb{R}^2 , with these operations, a vector space over \mathbb{R} ? Provide reasons.

Hints: No. For a reason, try to find out what is the zero element in this '+' operation.

5. Let $P[x]_{\leq n} := \{a_0 + a_1x + \dots + a_nx^n : a_i \in \mathbb{F}, 1 \leq i \leq n\}$ (set of all polynomials of degree $\leq n$). Let $p(x) = a_0 + a_1x + \dots + a_nx^n$, $q(x) = b_0 + b_1x + \dots + b_nx^n \in P[x]_{\leq n}$. Define

$$p(x) + q(x) := (a_0 + b_0) + (a_1 + b_1)x + \dots + (a_n + b_n)x^n$$

$$cp(x) := ca_0 + ca_1x + \dots + ca_nx^n.$$

Prove that $P[x]_{\leq n}$ is a vector space over \mathbb{F} with the above operations.

Subspace related problems

1. Which of the following sets of vectors $\alpha = (a_1, \dots, a_n)$ in \mathbb{R}^n are subspaces of \mathbb{R}^n ($n \geq 3$)?
- (i) all α such that $a_1 \geq 0$; **Hints:** Let S be such a set. $(1, 0, \dots, 0) \in S$ but $-1(1, 0, \dots, 0) = (-1, 0, \dots, 0) \notin S$. So S is not a subspace.
 - (ii) all α such that $a_1 + 3a_2 = a_3$; **Hints:** Yes, it is a subspace.
 - (iii) all α such that $a_2 = a_1^2$; **Hints:** No. Closure property with respect to addition does not satisfy.

- (iv) all α such that $a_1 a_2 = 0$; **Hints:** No. Let S be such a set. $a = (3, 0, \dots, 0), b = (0, 5, \dots, 0) \in S$ but $(a + b) = (3, 5, \dots, 0) \notin S$.
- (v) all α such that $a_1 \in \mathbb{Q}$?; **Hints:** No. Let S be such a set. $a = (1, 0, \dots, 0) \in S$ but $\sqrt{2}(a) = (\sqrt{2}, 0, \dots, 0) \notin S$ as $\sqrt{2} \notin \mathbb{Q}$.
2. Let V be the real vector space of all functions from $\mathbb{R} \rightarrow \mathbb{R}$. Which of the following sets of functions are subspaces of V ?
- (i) all f such that $f(x^2) = f(x)^2$; **Hints:** No. Find a counter example.
- (ii) all f such that $f(0) = f(1)$; **Hints:** yes. Check the criterion to be a subspace.
- (iii) all f such that $f(3) = 1 + f(-5)$; **Hints:** No. Done in the class.
- (iv) all f such that $f(-1) = 0$; **Hints:** yes. Check the criterion to be a subspace.
- (v) all f which are continuous. **Hints:** yes. Check the criterion to be a subspace.
3. If S_1, S_2, \dots, S_k are subsets of a vector space V , the set of all sums $\alpha_1 + \alpha_2 + \dots + \alpha_k$ of vectors $\alpha_i \in S_i$ is called the sum of the subsets S_1, \dots, S_k and is denoted by $S_1 + \dots + S_k$ or by $\sum_{i=1}^k S_i$. If W_1, \dots, W_k are subspaces of V , then check that the sum $\sum_{i=1}^k W_i$ is a subspace of V which contains each of the subspaces $W_i, 1 \leq i \leq k$. **Hints:** Done in the class for $k = 2$.
4. Prove that the union of two subspaces W_1, W_2 of a vector space V is not a subspace in general but the intersection is. If $W_1 \cup W_2$ is a subspace of V , then prove that $W_1 \subset W_2$ or $W_2 \subset W_1$. **Hints:** Discussed in the class.
5. Let W_1 and W_2 be two subspaces of a vector space V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$. Prove that for each vector $\alpha \in V$ there are unique vectors $\alpha_1 \in W_1, \alpha_2 \in W_2$ such that $\alpha = \alpha_1 + \alpha_2$. **Hints:** First do it for $V = \mathbb{R}^2, W_1 = x\text{-axis}$ and $W_2 = y\text{-axis}$. Then generalize.