UNIT-II

Inverse Laplace Transforms :

If $\mathcal{L}\{f(t)\} = \bar{f}(s)$ then $f(t) = \mathcal{L}^{-1}\{\bar{f}(s)\}$ is called

Inverse Laplace Transform.

Eg: $\mathcal{L}\{1\} = \frac{1}{s} \Rightarrow \mathcal{L}^{-1}\{\frac{1}{s}\} = 1.$

$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}.$

$\Rightarrow \mathcal{L}^{-1}\{\frac{1}{s^2+a^2}\} = \frac{1}{a}\sin at$

Formulas

S.No	Laplace Transform	Inverse Laplace Transform
1.	$\mathcal{L}\{1\} = \frac{1}{s}$	$\mathcal{L}^{-1}\{\frac{1}{s}\} = 1$
2.	$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$	$\mathcal{L}^{-1}\{\frac{1}{s-a}\} = e^{at}$
3.	$\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$	$\mathcal{L}^{-1}\{\frac{1}{s+a}\} = e^{-at}$
4.	$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$	$\mathcal{L}^{-1}\{\frac{1}{s^{n+1}}\} = \frac{t^n}{n!}$
5.	$\mathcal{L}\{t^{n-1}\} = \frac{(n-1)!}{s^n}$	$\mathcal{L}^{-1}\{\frac{1}{s^n}\} = \frac{t^{n-1}}{(n-1)!}$
6.	$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$	$\mathcal{L}^{-1}\{\frac{1}{s^2+a^2}\} = \frac{1}{a}\sin at$
7.	$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$	$\mathcal{L}^{-1}\{\frac{s}{s^2+a^2}\} = \cos at$
8.	$\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}$	$\mathcal{L}^{-1}\{\frac{1}{s^2-a^2}\} = \frac{1}{a}\sinh at$
9.	$\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$	$\mathcal{L}^{-1}\{\frac{s}{s^2-a^2}\} = \cosh at$

10.	$L\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$	$L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = \frac{1}{b}e^{at}\sin bt$
11.	$L\{e^{-at}\sin bt\} = \frac{b}{(s+a)^2+b^2}$	$L^{-1}\left\{\frac{1}{(s+a)^2+b^2}\right\} = \frac{1}{b}e^{-at}\sin bt$
12.	$L\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$	$L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos bt.$
13.	$L\{e^{-at}\cos bt\} = \frac{s-a}{(s+a)^2+b^2}$	$L^{-1}\left\{\frac{s}{(s+a)^2+b^2}\right\} = e^{-at}\cos bt.$
14.	$L\{e^{at}\sinh bt\} = \frac{b}{(s-a)^2-b^2}$	$L^{-1}\left\{\frac{1}{(s-a)^2-b^2}\right\} = \frac{1}{b}e^{at}\sinh bt$
15.	$L\{e^{-at}\sinh bt\} = \frac{b}{(s+a)^2-b^2}$	$L^{-1}\left\{\frac{1}{(s+a)^2-b^2}\right\} = \frac{1}{b}e^{-at}\sinh bt$

~~Median property~~ Linear property:-

$$L^{-1}\{c_1 f_1(s) + c_2 f_2(s)\} = c_1 L^{-1}\{f_1(s)\} + c_2 L^{-1}\{f_2(s)\}.$$

Problem

1. Find $L^{-1}\left\{\frac{2s-5}{s^2-4}\right\}$

Sol: given $L^{-1}\left\{\frac{2s-5}{s^2-4}\right\}$

$$L^{-1}\left\{\frac{2s}{s^2-4}\right\} - L^{-1}\left\{\frac{5}{s^2-4}\right\}$$

$$\Rightarrow 2L^{-1}\left\{\frac{s}{s^2-2^2}\right\} - 5L^{-1}\left\{\frac{1}{s^2-2^2}\right\}$$

$$= 2\cosh 2t - 5\cdot\frac{1}{2}\sinh 2t.$$

2. $\mathcal{L}^{-1}\left\{ \frac{s^2-3s+4}{s^3} \right\}$

3. $\mathcal{L}^{-1}\left\{ \frac{2s-5}{4s^2+25} \right\}$

4. $\mathcal{L}^{-1}\left\{ \frac{4s-18}{9-s^2} \right\}$

2sol: $\mathcal{L}^{-1}\left\{ \frac{s^2-3s+4}{s^3} \right\}$

$$\mathcal{L}^{-1}\left\{ \frac{\cancel{s^2}}{s^{\cancel{3}}} \right\} - \mathcal{L}^{-1}\left\{ \frac{3\cancel{s}}{s^{\cancel{3}}} \right\} + \mathcal{L}^{-1}\left\{ \frac{4}{s^3} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{1}{s} \right\} - 3\mathcal{L}^{-1}\left\{ \frac{1}{s^2} \right\} + 4\mathcal{L}^{-1}\left\{ \frac{1}{s^3} \right\}$$

WKT $\mathcal{L}^{-1}\left\{ \frac{1}{s} \right\} = 1$

$$\mathcal{L}^{-1}\left\{ \frac{1}{s^{n+1}} \right\} = \cancel{\frac{t\, n!}{t^n}}\ \frac{t^n}{n!}$$

$$\mathcal{L}^{-1}\left\{ \frac{1}{s^n} \right\} = \cancel{\frac{(n-1)!}{t}}\ \frac{t^{n-1}}{(n-1)!}$$

$$\Rightarrow 1 - 3 \cdot \frac{t^2}{2!} + 4 \cdot \frac{t^3}{3!}$$

$$\Rightarrow 1 - \frac{3t^2}{2} + \frac{4t^3}{6}$$

$$\Rightarrow 1 - \frac{3t^2}{2} + \frac{2t^3}{3}.$$

3 sol: $\mathcal{L}^{-1}\left\{\frac{2s-5}{4s^2+25}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{2s}{4s^2+25}\right\} - \mathcal{L}^{-1}\left\{\frac{5}{4s^2+25}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{s}{4(s^2+\frac{25}{4})}\right\} - 5\mathcal{L}^{-1}\left\{\frac{1}{4(s^2+\frac{25}{4})}\right\}$$

$$\Rightarrow 2\mathcal{L}^{-1}\frac{1}{4}\left\{\frac{s}{s^2+(\frac{5}{2})^2}\right\} - 5\mathcal{L}^{-1}\frac{1}{4}\left\{\frac{1}{s^2+(\frac{5}{2})^2}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+(5/2)^2}\right\} - \frac{5}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^2+(\frac{5}{2})^2}\right\}$$

WKT $\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$

$$= \frac{1}{2}\cos\frac{5}{2}t - \frac{5}{4}\cdot\frac{1}{5/2}\sin\frac{5}{2}t$$

$$= \frac{1}{2}\cos\frac{5}{2}t - \frac{\cancel{5}}{\cancel{4}}\cdot\frac{\cancel{2}}{\cancel{5}}\sin\frac{5}{2}t$$

$$= \frac{1}{2}\left[\cos(5/2\,t) - \sin(5/2\,t)\right].$$

4 sol: $\mathcal{L}^{-1}\left\{\frac{4s-18}{9-s^2}\right\}$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{18-4s}{s^2-9}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{18}{s^2-(3)^2}\right\} - 4\mathcal{L}^{-1}\left\{\frac{s}{s^2-(3)^2}\right\}$$

$$\Rightarrow 18\,\mathcal{L}^{-1}\left\{\frac{1}{s^2-3^2}\right\} - 4\mathcal{L}^{-1}\left\{\frac{s}{s^2-3^2}\right\}$$

WKT $\mathcal{L}^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$$

$\Rightarrow 18 \cdot \frac{1}{3}\sinh 3t - 4\cosh 3t.$

$\Rightarrow 6\sinh 3t - 4\cosh 3t$

Inverse Laplace Transforms by Partial Fraction method:-

1. Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2-5s+6}\right\}$.

Sol:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-5s+6}\right\}$$

s^2-5s+6

$s^2-2s-3s+6$

$s(s-2)-3(s-2)$

$(s-2)(s-3)$.

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-2s-3s+6}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s-2)-3(s-2)}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-2)(s-3)}\right\}.$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2-5s+6}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-2)(s-3)}\right\}.$$

$$\frac{1}{(s-3)(s-2)} = \frac{A}{s-3} + \frac{B}{s-2}.$$

$$= \frac{A(s-2)+B(s-3)}{(s-3)(s-2)}$$

$\Rightarrow A(s-2)+B(s-3) = 1$

$\Rightarrow As-2A+Bs-3B = 1.$

To find A put $s=3$.

$\Rightarrow s(A+B) - (2A+3B) = 1.$

$\Rightarrow A+B=0$ & $2A+3B=-1.$

$\Rightarrow A=-B$ $\Rightarrow -2B+3B=-1$

$B=-1.$

$A=1.$

$1 = A(3-2)+0$

$A=1.$

To find B, put $s=2$.

$B(2-3)+0=1$

$B=-1.$

$$\frac{1}{(s-3)(s-2)} = \frac{1}{s-3} + \frac{-1}{s-2}.$$

$$\Rightarrow L^{-1}\left\{\frac{1}{s^2-5s+6}\right\} = L^{-1}\left\{\frac{1}{s-3} - \frac{1}{s-2}\right\}.$$

$$= L^{-1}\left\{\frac{1}{s-3}\right\} - L^{-1}\left\{\frac{1}{s-2}\right\}$$

WKT $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}.$

$$= e^{3t} - e^{2t}$$

2. Find $L^{-1}\left\{\frac{1}{(s+1)(s+2)(s+3)}\right\}.$

Sol: $L^{-1}\left\{\frac{1}{(s+1)(s+2)(s+3)}\right\}$

$$\frac{1}{(s+1)(s+2)(s+3)} = \frac{A}{(s+1)} + \frac{B}{(s+2)} + \frac{C}{(s+3)}.$$

$$1 = A(s+2)(s+3) + B(s+1)(s+3) + C(s+1)(s+2)$$

$\Rightarrow A(s^2+5s+6) + B(s^2+4s+3) + C(s^2+3s+2) = 1.$

$\Rightarrow s^2(A+B+C) + s(5A+4B+3C) + 6A+3B+2C = 1$

$\Rightarrow A+B+C=0,\quad 5A+4B+3C=0,\quad 6A+3B+2C=1$

$\Rightarrow A=-(B+C) \Rightarrow (-5B-5C)+4B+3C=0. \quad \Rightarrow (-6B-6C)+3B+2C=1$

$\Rightarrow -B-2C=0. \quad \Rightarrow -3B-4C=1.$

$-B-2C=0 \cdot \times 2$

$-3B-4C=1$

$\Rightarrow$ $-2B-\cancel{4C}=0$

$-3B-\cancel{4C}=1$

$+ \quad + \quad -$

$\boxed{B=-1}$

Bes $-B-2C=0$

$B=-2C$

$\Rightarrow C=\frac{-B}{2}$

$C=1/2$

$A=-B-C$

$=+1-1/2$

$\boxed{A=1/2}$

$$\frac{1}{(s+1)(s+2)(s+3)}=\frac{A}{s+1}+\frac{B}{s+2}+\frac{C}{s+3}$$

$$=\frac{1}{2(s+1)}-\frac{1}{s+2}+\frac{1/2}{s+3}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+2)(s+3)}\right\}=\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{(s+2)}\right\}+\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$=\frac{1}{2}e^{-1t}+-e^{-2t}+\frac{1}{2}e^{-3t}$$

$$=\frac{e^{-t}}{2}-e^{-2t}+\frac{e^{-3t}}{2}$$

(A)

3. Find $\mathcal{L}^{-1}\left\{\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\}$

Sol

$\mathcal{L}^{-1}\left\{\dfrac{2s^2-6s+5}{s(s^2}\right.$

$s^3-6s^2+11s-6 = (s-1)(s^2-5s+6)$

$= (s-1)(s^2-2s-3s+6)$

$= (s-1)(s(s-2)-3(s-2))$

$s^3-6s^2+11s-6 = (s-1)(s-2)(s-3).$

$$\mathcal{L}^{-1}\left\{\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\} = \mathcal{L}^{-1}\left\{\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}\right\}$$

$$\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)} = \frac{A}{(s-1)} + \frac{B}{(s-2)} + \frac{C}{(s-3)}$$

$$= \frac{A[(s-2)(s-3)] + B[(s-1)(s-3)] + C[(s-1)(s-2)]}{(s-1)(s-2)(s-3)}.$$

$\Rightarrow 2s^2-6s+5 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2).$

To find A put $s=1 \Rightarrow 2(1)^2-6(1)+5 = A(-1)(-2)$

$\Rightarrow 2A = 2-6+5$

$\Rightarrow 2A = 1$

$\Rightarrow A = 1/2$

To find B put $s=2 \Rightarrow 2(2)^2-6(2)+5 = B(1)(-1)$

$\Rightarrow -B = 8-12+5$

$\Rightarrow -B = 1$

$\Rightarrow B = -1$

To find c put $s=3$.

$\Rightarrow 2(3)^2 - 6(3) + 5 = c(1)(2)$

$\Rightarrow 2c = 18 - 18 + 5$

$c = 5/2$.

$$L^{-1}\left\{\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\} = L^{-1}\left\{\frac{1/2}{(s-1)} + \frac{-1}{s-2} + \frac{5/2}{s-3}\right\}$$

$$= \frac{1}{2} L^{-1}\left\{\frac{1}{s-1}\right\} + -1\, L^{-1}\left\{\frac{1}{s-2}\right\} + \frac{5}{2} L^{-1}\left\{\frac{1}{s-3}\right\}.$$

$$= \frac{1}{2} e^{1t} - 1\, e^{2t} + \frac{5}{2} e^{3t}$$

$$= \frac{e^t}{2} - e^{2t} + \frac{5e^{3t}}{2}$$

4. Find $L^{-1}\left\{\frac{1}{(s^2+4)(s^2+25)}\right\}$

Sol: Given $L^{-1}\left\{\frac{1}{(s^2+4)(s^2+25)}\right\}$

$$\frac{1}{(s^2+4)(s^2+25)} = \frac{Ax+B}{(s^2+4)} + \frac{Cx+D}{(s^2+25)}$$

$\Rightarrow 1 = Ax+B(s^2+25) + Cx+D(s^2+4)$

$$\Rightarrow n = \frac{1}{(x^2+a^2)(x^2+b^2)} = \frac{1}{(b^2-a^2)}\left[\frac{1}{(x^2+a^2)} - \frac{1}{(x^2+b^2)}\right]$$

$$\rightarrow \left[\frac{1}{(x^2+a)(x^2+b)} = \frac{1}{(b-a)}\left(\frac{1}{x^2+a} - \frac{1}{x^2+b}\right)\right]$$

$\Rightarrow a = 2 \quad b = 5.$

$$\frac{1}{(s^2+4)(s^2+25)} = \frac{1}{25-4}\left[\frac{1}{s^2+4} - \frac{1}{s^2+25}\right].$$

$$L^{-1}\left\{\frac{1}{(s^2+4)(s^2+25)}\right\} = \frac{1}{21} L^{-1}\left\{\frac{1}{s^2+4}\right\} - \frac{1}{21} L^{-1}\left\{\frac{1}{s^2+25}\right\}.$$

WKT $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at.$

$$= \frac{1}{21}\cdot\frac{1}{2}\sin 2t - \frac{1}{21}\cdot\frac{1}{5}\sin 5t.$$

$$= \frac{1}{21}\left[\frac{\sin 2t}{2} - \frac{\sin 5t}{5}\right].$$

(A)

5. Find $L^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\right\}.$

Sol: Given $L^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\right\}.$

WKT $\frac{1}{(x^2+a)(x^2+b)} = \frac{1}{b-a}\left(\frac{1}{x^2+a} - \frac{1}{x^2+b}\right)$

$$= \frac{1}{9-1}\left[L^{-1}\left\{\frac{1}{s^2+1}\right\} - L^{-1}\left\{\frac{1}{s^2+9}\right\}\right]$$

$$= \frac{1}{8}\left[L^{-1}\left\{\frac{1}{s^2+1^2}\right\} - L^{-1}\left\{\frac{1}{s^2+3^2}\right\}\right]$$

$$= \frac{1}{8}\left[\frac{1}{1}\sin t - \frac{1}{3}\sin 3t\right].$$

$$= \frac{1}{8}\left[\sin t - \frac{1}{3}\sin 3t\right].$$

6. Find $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)(s^2+25)}\right\}$.

$$\frac{1}{(x^2+a^2)(x^2+b^2)} = \frac{1}{b-a}\left(\frac{1}{x^2+a} - \frac{1}{x^2+b}\right).$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)(s^2+25)}\right\} = \frac{1}{25-4}\left(\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+25}\right\}\right).$$

$$= \frac{1}{21}\left(\cos 2t - \cos 5t\right).$$

$\left[\text{WKT } \mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at\right]$

<u>First</u> <u>shifting</u> <u>Theorem</u>:

If $\mathcal{L}^{-1}\{\bar{f}(s)\} = f(t)$ then

$$\mathcal{L}^{-1}\{\bar{f}(s-a)\} = e^{at} f(t) \text{ and}$$

$$\mathcal{L}^{-1}\{\bar{f}(s+a)\} = e^{-at} f(t).$$

1) Find $\mathcal{L}^{-1}\left\{\frac{s+2}{(s-2)^3}\right\}$.

$$\mathcal{L}^{-1}\left\{\frac{s+2}{(s-2)^3}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s-2+2+2}{(s-2)^3}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^3} + \frac{4}{(s-2)^3}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^3}\right\}.$$

WKT $L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$ & $L^{-1}\{f(s-a)\} = e^{at} f(t)$.

$$= e^{2t}\left[\frac{t^{2-1}}{(2-1)!} + 4\,\frac{t^{3-1}}{(3-1)!}\right]$$

$$= e^{2t}\left[t + \frac{4t^2}{2}\right]$$

$$= e^{2t}t + 2e^{2t}t^2.$$

2. Find $L^{-1}\left\{\frac{s}{(s+3)^2}\right\}$.

Sol: $L^{-1}\left\{\frac{s}{(s+3)^2}\right\}$.

$$L^{-1}\left\{\frac{s+3-3}{(s+3)^2}\right\}.$$

$$\Rightarrow L^{-1}\left\{\frac{s+3}{(s+3)^2}\right\} - 3L^{-1}\left\{\frac{1}{(s+3)^2}\right\}.$$

$$\Rightarrow L^{-1}\left\{\frac{1}{(s+3)}\right\} - 3L^{-1}\left\{\frac{1}{(s+3)^2}\right\}.$$

$$\Rightarrow e^{-3t} - 3.\,e^{-3t}.\,\frac{t^{2-1}}{(2-1)!}$$

$$= e^{-3t} - 3e^{-3t}t.$$

Inverse Laplace Transform of derivatives (logarithm sums):

If $L^{-1}\{F(s)\} = f(t)$ then

$$L^{-1}\{f^n(s)\} = (-1)^n t^n f(t)$$

If $n=1$ then $L^{-1}\{f'(s)\} = (-1)t\,f(t)$.

Problems-

1) Find $\mathcal{L}^{-1}\left\{ \log\left(\frac{s+a}{s+b}\right)\right\}$

Sol: $\bar{f}(s) = \log\left(\frac{s+a}{s+b}\right)$.

$\bar{f}(s) = \log(s+a) - \log(s+b)$.

Applying derivative on both sides.

$$f'(\bar{s}) = \frac{1}{s+a} - \frac{1}{s+b}.$$

Taking L^{-1} on both sides.

$$\mathcal{L}^{-1}\{f'(\bar{s})\} = \mathcal{L}^{-1}\left\{\frac{1}{s+a} - \frac{1}{s+b}\right\}$$

$$(-1)\, t^{1} f(t) = e^{-at} - e^{-bt}$$

$$\Rightarrow f(t) = \frac{-(e^{-at} - e^{-bt})}{t}$$

$$= \frac{e^{-bt} - e^{-at}}{t}$$

2. Find $\mathcal{L}^{-1}\left\{ \log\left(\frac{s+1}{s-1}\right)\right\}$.

$\bar{f}(s) = \log\left(\frac{s+1}{s-1}\right)$.

$\bar{f}(s) = \log(s+1) - \log(s-1)$.

Differentiating on both sides.

$$f'(\bar{s}) = \frac{1}{s+1} - \frac{1}{s-1}$$

Taking L^{-1} on both sides.

$$\mathcal{L}^{-1}\{f'(\bar{s})\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1} - \frac{1}{s-1}\right\}.$$

$= (-1)t^1 f(t) = e^{-t} - e^{t}$

$\Rightarrow f(t) = \dfrac{-(e^{-t} - e^{t})}{t}$

$= \dfrac{e^{t} - e^{-t}}{t}$

$\bar{f}(s) = \cot^{-1}\left(\frac{s}{2}\right)$

$\bar{f}'(s) = \dfrac{-1}{1+\left(\frac{s}{2}\right)^2}$

$= \dfrac{-1}{1+\frac{s^2}{4}}$

$= \dfrac{-4}{2^2+s^2}$

$\mathcal{L}^{-1}\{\bar{f}'(s)\} = 4\,\mathcal{L}^{-1}\left\{\dfrac{-1}{s^2+2^2}\right\}$

$= 4 \cdot \frac{1}{2} \cdot \sin 2t$

$= 2\sin 2t$

(A)

3. $\mathcal{L}^{-1}\left\{\log \dfrac{s^2+1}{s(s+1)}\right\}$

4. $\mathcal{L}^{-1}\left\{\log\left(\dfrac{1+s}{s}\right)\right\}$

5. $\mathcal{L}^{-1}\left\{\log \dfrac{s^2+1}{(s-1)^2}\right\}$

6. $\mathcal{L}^{-1}\left\{\log \dfrac{s^2-1}{s^2}\right\}$

*** (A) 7. $\mathcal{L}^{-1}\left\{\cot^{-1}\left(\frac{s}{2}\right)\right\}$.

3. $\mathcal{L}^{-1}\left\{\log \dfrac{s^2+1}{s(s+1)}\right\}$.

$\bar{f}(s) = \log \dfrac{s^2+1}{s(s+1)}$

$= \log(s^2+1) - \log(s(s+1))$

$= \log(s^2+1) - [\log(s) + \log(s+1)]$

$\bar{f}(s) = \log(s^2+1) - \log(s) - \log(s+1)$

Applying derivative on both sides

$\bar{f}'(s) = \dfrac{1}{s^2+1} \cdot 2s - \dfrac{1}{s} - \dfrac{1}{s+1}$

$= \dfrac{2s}{s^2+1} - \dfrac{1}{s} - \dfrac{1}{s+1}$

Taking L^{-1} on bs

$$L^{-1}\{\bar{f}'(s)\} = L^{-1}\left\{\frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}\right\}$$

$$= 2.\cos t - 1 - e^{-t}$$

$$(-1)t\, f(t) = 2\cos t - e^{-t} - 1$$

$$\Rightarrow f(t) = \frac{1+e^{-t}-2\cos t}{t}$$

Inverse Laplace Transform of Integrals:

If $L^{-1}\{\bar{f}(s)\} = f(t)$ then $L^{-1}\left\{\int_s^\infty \bar{f}(s)\right\} = \frac{f(t)}{t}$.

$$t\, L^{-1}\left\{\int_s^\infty \bar{f}(s)\right\} = f(t)$$

1) Find $L^{-1}\left\{\frac{1}{(s+1)^2}\right\}$

Sol: $\bar{f}(s) = \frac{1}{(s+1)^2}$.

By ILT of Integrals,

$$t\, L^{-1}\left\{\int_s^\infty \bar{f}(s)\right\} = f(t)$$

$$t\, L^{-1}\left\{\int_s^\infty \frac{1}{(s+1)^2}\right\}$$

$$= t\, L^{-1}\left[\frac{-1}{(s+1)}\right]_s^\infty$$

$$= -t.L^{-1}\left(\frac{1}{\infty} - \frac{1}{s+1}\right). = \frac{t}{s+1}$$

$t \cdot L^{-1}\left\{\frac{1}{s+1}\right\}$

$= t \cdot e^{-t}$

$= f(t)$;

2) Find $L^{-1}\left\{\frac{2}{(s-a)^3}\right\}$.

$\bar{f}(s) = \frac{2}{(s-a)^3}$

By ILT of Integrals

$t\, L^{-1}\left\{\int_s^\infty \bar{f}(s)\right\} = f(t)$

$t \cdot L^{-1}\left\{\int_s^\infty \frac{2}{(s-a)^3}\right\}$

$= t \cdot L^{-1}\left[2\int_s^\infty \frac{1}{(s-a)^3}\right]$

$= 2t\, L^{-1}\left\{\int_s^\infty (s-a)^{-3}\right\}$

$= 2t\, L^{-1}\left\{\frac{+1}{2(s-a)^2}\right\}$

$= t\, L^{-1}\left\{\frac{+1}{(s-a)^2}\right\}$

$= t \cdot \cdot + \frac{(t-a)^{2-1}}{(2-1)!}$ $\qquad L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$

$= + t \cdot (e^{at} \cancel{*} t)$.

$= t^2 e^{at}$;

Multiplication by the powers of s :-

If $L^{-1}\{\bar{f}(s)\} = f(t)$ then

$$L^{-1}\{s\,\bar{f}(s)\} = f'(t)$$

$$L^{-1}\{s^n\,\bar{f}(s)\} = f^n(t).$$

1. Find $L^{-1}\left\{\frac{s}{s^2-a^2}\right\}$.

Sol: $L^{-1}\left\{s.\frac{1}{s^2-a^2}\right\}$

$\bar{f}(s) = \frac{1}{s^2-a^2}$

$L^{-1}\{\bar{f}(s)\} = \frac{1}{a}\sinh at$

$= f(t).$

$L^{-1}\{s.\bar{f}(s)\} = f'(t)$

$= \frac{1}{a}\cosh at\,(a)$

$= \cosh at.$

2. Find $L^{-1}\left\{\frac{s}{(s-4)^5}\right\}$.

$L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$

Sol $\bar{f}(s) = \frac{1}{(s-4)^5}$

$L^{-1}\{\bar{f}(s)\} = \frac{t^4}{4!}e^{4t}$

$= \frac{e^{4t}t^4}{4!} = f(t)$

$$\mathcal{L}^{-1}\left\{ s \cdot \bar{f}(s) \right\} = f'(t)$$

$$= \frac{1}{4!}\left[\left(e^{4t} t^4\right)' \right]$$

$$= \frac{1}{4!}\left[e^{4t} \cdot 4t^4 + e^{4t} \cdot 4t^3 \right]$$

$$= \frac{e^{4t}}{4!}\left[4t^4 + 4t^3 \right].$$

$$= \frac{4e^{4t}}{4!}\left[t^4 + t^3 \right].$$

3. Find $\mathcal{L}^{-1}\left\{ \frac{s^2}{(s+a)^3} \right\}$.

Sol: $\bar{f}(s) = \frac{1}{(s+a)^3}$.

$$\mathcal{L}^{-1}\left\{ \bar{f}(s) \right\} = \frac{t^2}{(2)!} \cdot e^{-at} \qquad \mathcal{L}^{-1}\left\{ \frac{1}{s^n} \right\} = \frac{t^{n-1}}{(n-1)!}$$

$$= \frac{e^{-at} t^2}{2!}$$

$$\mathcal{L}^{-1}\left\{ s^n \bar{f}(s) \right\} = f^n(t)$$

$$\mathcal{L}^{-1}\left\{ s^2 \bar{f}(s) \right\} = f^2(t)$$

$$= \left[\left(\frac{e^{-at} t^2}{2!} \right)'' \right]$$

$$= \frac{1}{2!}\left[\left(e^{-at} 2t + t^2 \cdot e^{-at} \cdot (-a) \right)' \right].$$

$$= \frac{1}{2!}\left[e^{-at} 2 + 2t e^{-at}(-a) + (-a) \cdot t^2 e^{-at} - a + (-a) e^{-at} 2t \right]$$

$$= \frac{e^{-at}}{2!}\left[2 + -2at + t^2 a^2 - 2at \right] = \frac{e^{-at}}{2!}\left[a^2 t^2 - 4at + 2 \right].$$

Division by s :-

If $L^{-1}\{\bar{f}(s)\} = f(t)$ then

$$L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t)\,dt$$

$$L^{-1}\left\{\frac{\bar{f}(s)}{s^2}\right\} = \int_0^t\int_0^t (f(t)\,dt)\,dt$$

1) Find $L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\}$

Sol:

$$L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+a^2}\right\}$$

$$\bar{f}(s) = \frac{1}{s^2+a^2}$$

$$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{s^2+a^2}\right\}$$

$$= \frac{1}{a}\sin at$$

$$= f(t).$$

$$L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t)\,dt$$

$$= \int_0^t \frac{1}{a}\sin at\,dt$$

$$= \frac{1}{a}\cdot\left[\frac{\cos at}{a}\right]_0^t$$

$$= \frac{1}{a^2}\left(-(\cos at - \cos 0)\right)$$

$$= -\frac{1}{a^2}(\cos at - 1)$$

$$= \frac{1-\cos at}{a^2}$$

2. Find $L^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\}$

Sol: $L^{-1}\left\{\frac{1}{s^2}\ \frac{1}{s^2+a^2}\right\}$.

$$L^{-1}\left\{\frac{1}{s}\ \frac{1}{s^2+a^2}\right\} = \frac{1-\cos at}{a^2}$$

$$L^{-1}\left\{\frac{1}{s^2}\ \frac{1}{s^2+a^2}\right\} = \int_0^t \frac{1-\cos at}{a^2}\,dt$$

$$= \frac{1}{a^2}\left[\int_0^t 1\,dt - \int_0^t \cos at\,dt\right]$$

$$= \frac{1}{a^2}\left[\left[t\right]_0^t - \left[\frac{\sin at}{a}\right]_0^t\right]$$

$$= \frac{1}{a^2}\left[(t-0) - \frac{1}{a}\left(\sin at - \sin 0\right)\right]$$

$$= \frac{1}{a^2}\left[t - \frac{1}{a}(\sin at - 0)\right]$$

$$= \frac{1}{a^2}\left[t - \frac{\sin at}{a}\right] = \frac{at - \sin at}{a^3}.$$

~~Convul~~

7M Convolution Theorem

It is used to find product of two inverse laplace transforms.

$L^{-1}\{\bar{f}(s)\} = f(t)$ and $L^{-1}\{\bar{g}(s)\} = g(t)$ then

$L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = f*g$.

$L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = \int_0^t f(u)\, g(t-u)\, du$.

1. Find

1. Using convolution theorem find $L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\}$.

By convolution theorem,

$$L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = \int_0^t f(u)\, g(t-u)\, du.$$

Given $L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\}$.

$$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at} = f(t)$$

$$L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{1}{s+b}\right\} = e^{-bt} = g(t).$$

$$L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = \int_0^t e^{-au}\, e^{-b(t-u)}\, du.$$

$$= \int_0^t e^{-au}\, e^{-bt+bu}\, du.$$

$$= e^{-bt}\int_0^t e^{u(b-a)}\, du$$

$$= e^{-bt}\left[\frac{e^{u(b-a)}}{b-a}-1\right]_0^t$$

$$= \frac{e^{-bt}}{b-a}\left(e^{t(b-a)}-1\right).$$

2. $\mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s+3)}\right\}$.

Sol Given. $\mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s+3)}\right\}$.

$$\mathcal{L}^{-1}\{\bar{f}(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t} = f(t)$$

$$\mathcal{L}^{-1}\{\bar{g}(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t} = g(t)$$

By convolution theorem

$$\mathcal{L}^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = \int_0^t f(u)\, g(t-u)\, du.$$

$$= \int_0^t e^{-2u}\cdot e^{-3(t-u)}\, du.$$

$$= \int_0^t e^{-2u}\cdot e^{-3t+3u}\, du$$

$$= e^{-3t}\int_0^t e^{-2u+3u}\, du$$

$$= e^{-3t}\int_0^t e^{u}\, du$$

$$= e^{-3t}\left[e^{u}\right]_0^t$$

$$= e^{-3t}\left[e^{t}-e^{0}\right]$$

$$= e^{-3t}\left[e^{t}-1\right].$$

(A) 3. Find $L^{-1}\left\{\frac{1}{s(s^2+4)}\right\}$

Sol: Given $L^{-1}\left\{\frac{1}{s(s^2+4)}\right\}$

$$L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+4}\right\}.$$

$$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{s}\right\} = 1 = f(t)$$

$$L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t = g(t).$$

By convolution theorem,

$$L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = \int_0^t f(u)\, g(t-u)\, du.$$

$$= \int_0^t 1\cdot\frac{1}{2}\sin 2(t-u)\, du.$$

$$= \frac{1}{2}\int_0^t \sin 2(t-u)\, du.$$

$$= \frac{1}{2}\left[-\frac{\cos 2(t-u)}{-2}\right]_0^t \qquad \left(\because \int \sin x = -\cos x\right).$$

$$= +\frac{1}{2}\left[\frac{\cos 2(t-t) - \cos 2(t-0)}{2}\right]$$

$$= +\frac{1}{4}\left[\cos 0 - \cos 2t\right]$$

$$= +\frac{1}{4}\left[1-\cos 2t\right].$$

$$= \frac{-(\cos 2t - 1)}{4}.$$

(A)

4. $\mathcal{L}^{-1}\left\{\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$.

Sol:- Given $\mathcal{L}^{-1}\left\{\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$

$$\mathcal{L}^{-1}\{\bar{f}(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)}\right\} = \cos at = f(t)$$

$$\mathcal{L}^{-1}\{\bar{g}(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\} = \cos bt = g(t)$$

By convolution theorem,

$$\mathcal{L}^{-1}\{\bar{f}(s)\,\bar{g}(s)\} = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t \cos au \,.\, \cos b(t-u)\, du$$

WKT $2\cos A\cos B = \cos(A+B) + \cos(A-B)$.

$$= \int_0^t \frac{1}{2}\left[\cos(au+bt-bu) + \cos(au-bt+bu)\right] du$$

$$= \frac{1}{2}\int_0^t \cos(bt+u(a-b))\,du + \frac{1}{2}\int_0^t \cos(u(a+b)-bt)\,du$$

$$= \frac{1}{2}\left[\frac{\sin(bt+u(a-b))}{(a-b)}\right]_0^t + \frac{1}{2}\left[\frac{\sin(u(a+b)-bt)}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin(bt+t(a-b))}{(a-b)} - \frac{\sin(bt+0)}{a-b}\right] + \frac{1}{2}\left[\frac{\sin t(a+b)-bt}{a+b} - \frac{\sin 0-bt}{a+b}\right].$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b} - \frac{\sin bt}{a-b} + \frac{\sin at}{a+b} + \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{\sin at - \sin bt}{a-b} + \frac{\sin at + \sin bt}{a+b}\right]$$

5. Find $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$.

$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)}\cdot\frac{1}{(s^2+a^2)}\right\}$.

$\mathcal{L}^{-1}\{\bar{f}(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at = f(t)$

$\mathcal{L}^{-1}\{\bar{g}(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at = g(t)$

By convolution theorem,

$$\mathcal{L}^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = \int_0^t f(u)\, g(t-u)\, du.$$

$$= \int_0^t \cos au \cdot \frac{1}{a} \cancel{\sin a} \sin a(t-u)\, du.$$

$$= \frac{1}{a}\int_0^t \cos au \cdot \sin a(t-u)\, du.$$

WKT $2\sin A\cos B = \sin(A+B) - \sin(A-B)$.

$$\Rightarrow \frac{1}{2a}\int_0\left(\sin(au+at-au) - \sin(au-at+au)\right) du$$

$$= \frac{1}{2a}\left[-\frac{\cos at}{a}u - -\frac{\cos(2au-at)}{2a}\right]_0^t$$

$$\cancel{= \frac{1}{2a}\left[\frac{\cos(2au-at)}{2a} - \frac{t\cos at}{a}\right]}$$

$$= \frac{1}{2a}\left[-\frac{t\cos at}{a} + \left(\frac{\cos(2at-at)}{2a} - \frac{\cos(0-at)}{2a}\right)\right]$$

$$= \frac{1}{2a}\left[\cancel{\frac{\cos at}{2a}} - \cancel{\frac{\cos at}{2a}} + -\frac{t\cos at}{a}\right] = \frac{-t}{2a}\cos at.$$

*****

Applications of Laplace Transforms :-

7M

$L\{y\} = \bar{y}(s)$

$L\{y'\} = s\bar{y}(s) - y(0)$

$L\{y''\} = s^2\bar{y}(s) - s(y(0)) - y'(0)$

$L\{y'''\} = s^3\bar{y}(s) - s^2 y(0) - sy'(0) - y''(0)$

$\vdots$

$L\{y^n\} = s^n\bar{y}(s) - s^{n-1}y(0) - s^{n-2}y'(0) \cdots y^{n-1}(0).$

$L\{x\} = \bar{x}(s)$

$L\{x'\} = s\bar{x}(s) - x(0)$

$L\{x''\} = s^2\bar{x}(s) - sx(0) - x'(0)$

$\vdots$

$L\{x^n\} = s^n\bar{x}(s) - s^{n-1}x(0) - s^{n-2}x'(0) \cdots x^{n-1}(0).$

Working Procedure

1. Take Laplace Transform on both sides and Apply the given initial condition.
2. Keep $\bar{y}(s)$ left side and transform all other to right hand side.
3. Take inverse Laplace transform on both sides then we get the solution.

Problems

A

1. Solve by the method of transform of the equation

$y'' + 4y' + 3y = e^{-t}, \quad y(0) = y'(0) = 1.$

Sol: Given $y'' + 4y' + 3y = e^{-t}$.

Applying Laplace transform on B.S.

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = \mathcal{L}\{e^{-t}\}$$

$$\mathcal{L}\{y''\} = s^2\bar{y}(s) - sy(0) - y'(0).$$

$$\{s^2\bar{y}(s) - sy(0) - y'(0)\} + 4\{s\bar{y}(s) - y(0)\} + 3\{\bar{y}(s)\} = \frac{1}{s+a}$$

$$\{s^2\bar{y}(s) - s(1) - 1\} + 4\{s\bar{y}(s) - 1\} + 3\bar{y}(s) = \frac{1}{s+1}$$

$$\bar{y}(s)\{s^2+4s+3\} - s - 5 = \frac{1}{s+1}$$

$$\Rightarrow \cancel{\tfrac{1}{3}}\ \bar{y}(s) = \frac{s+5}{s^2+4s+3} + \frac{1}{(s+1)(s^2+4s+3)}.$$

$$\Rightarrow \bar{y}(s) = \frac{s+5}{(s+3)(s+1)} + \frac{1}{(s+1)(s+1)(s+3)}.$$

$$\frac{s+5}{(s+3)(s+1)} = \frac{A}{s+1} + \frac{B}{s+3}$$

$$s+5 = A(s+3) + B(s+1)$$

$\Rightarrow$ put $s = -1$

$$-1+5 = A(-1+3)$$

$$4 = A(2)$$

$$A = 2.$$

$\Rightarrow$ put $s = -3$.

$$-3+5 = B(-3+1)$$

$$\Rightarrow 2 = -2B$$

$$B = -1$$

$$\frac{1}{(s+1)^2(s+3)} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{s+3}.$$

$$= A(s+1)(s+3) + B(s+3) + C(s+1)^2$$

$$\Rightarrow 1 = A(s+1)(s+3) + B(s+3) + C(s+1)^2.$$

put $s = -1$.

$$\Rightarrow 1 = B(-1+3) \Rightarrow 2B = 1 \Rightarrow B = 1/2.$$

put $s=-3$.

$1 = A(0) + B(0) + C(-3+1)^2$

$1 = C(-2)^2$

$C = 1/4$

$1 = A(s^2+4s+3) + B(s+3) + C(s^2+2s+1)$

$\Rightarrow$ put $C = \frac{1}{4}$, $B = \frac{1}{2}$.

$1 = A(s)^2 + 4As + 3A + Bs + 3B + Cs^2 + 2Cs + C$.

$\Rightarrow 1 = s^2(A+C) + s(4A+B+2C) + (3A+3B+C)$.

$\Rightarrow 3A+3B+C = 1$

$\Rightarrow 3A + 3\left(\frac{1}{2}\right) + \frac{1}{4} = 1$

$\Rightarrow 3A + \frac{3}{2} + \frac{1}{4} = 1$

$\Rightarrow 3A + \frac{6+1}{4} = 1$

$\Rightarrow 3A = 1 - \frac{7}{4}$

$= -\frac{3}{4}$

$\Rightarrow A = -1/4$.

$$\frac{1}{(s+1)^2(s+3)} = \frac{-1}{4(s+1)} + \frac{1}{2(s+1)^2} + \frac{1}{4(s+3)}.$$

$$\frac{s+5}{(s+1)(s+3)} = \frac{2}{s+1} - \frac{1}{s+3}.$$

$$\Rightarrow \bar{y}(s) = \frac{2}{s+1} - \frac{1}{s+3} + \frac{1}{2(s+1)^2} + \frac{1}{4(s+3)} - \frac{1}{4(s+1)}.$$

Applying L^{-1}

$$\Rightarrow \frac{-1}{4}e^{-t} + \frac{1}{2}\frac{t^1}{1}(e^{-t}) + \frac{1}{4}e^{-3t} + 2e^{-t} - e^{-3t}.$$

A.

2. Solve the DE $\frac{d^2x}{dt^2} + 9x = \sin t$. $x(0) = 1$, $x(\frac{\pi}{2}) = 1$.

Sol:

$$x'' + 9x = \sin t$$

Using Laplace transform given that

$$x(0) = x(\tfrac{\pi}{2}) = 1.$$

Given $\frac{d^2x}{dt^2} + 9x = \sin t$.

$$x'' + 9x = \sin t.$$

Applying Laplace on both sides.

$$L\{x''\} + 9L\{x\} = L\{\sin t\}.$$

$$\{s^2\bar{x}(s) - sx(0) - x'(0)\} + 9\bar{x}(s) = \frac{1}{s^2+1}.$$

$$\bar{x}(s)(s^2+9) - sx(0) - x'(0) = \frac{1}{s^2+1}.$$

$$\bar{x}(s)(s^2+9) - s - c = \frac{1}{s^2+1}.$$

$$\Rightarrow \bar{x}(s)(s^2+9) = s + c + \frac{1}{s^2+1}$$

$$\bar{x}(s) = \left(s + c + \frac{1}{s^2+1}\right)\frac{1}{s^2+9}.$$

$$= \frac{s}{s^2+9} + \frac{c}{s^2+9} + \frac{1}{(s^2+1)(s^2+9)}.$$

$$\Rightarrow \bar{x}(s) = \frac{s+c}{s^2+9} + \frac{1}{(s^2+1)(s^2+9)}.$$

$$\Rightarrow \left\{\frac{1}{(x^2+a^2)(x^2+b^2)} = \frac{1}{(b^2-a^2)}\left[\frac{1}{(x^2+a^2)} - \frac{1}{(x^2+b^2)}\right]\right\}.$$

$$= \frac{1}{(s^2+3^2)(s^2+1^2)} = \frac{1}{-9-1}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right].$$

$$\bar{x}(s) = \frac{s}{s^2+9} + \frac{c}{s^2+9} + \frac{1}{8}\left[\frac{1}{s^2+1}\right] - \frac{1}{8}\left[\frac{1}{s^2+9}\right].$$

• Applying L^{-1} on both sides.

$$\Rightarrow L^{-1}\{\bar{x}(s)\} = L^{-1}\left\{\frac{s}{s^2+9}\right\} + c\,L^{-1}\left\{\frac{1}{s^2+9}\right\} + \frac{1}{8}L^{-1}\left\{\frac{1}{s^2+1}\right\} -$$

$$\frac{1}{8}L^{-1}\left\{\frac{1}{s^2+9}\right\}.$$

$\bar{x}(t)$

$$x(t) = \cos 3t + c\cdot\frac{1}{3}\sin 3t + \frac{1}{8}\sin t - \frac{1}{8}\cdot\frac{1}{3}\sin 3t.$$

$$x(t) = \cos 3t + \frac{c\sin 3t}{3} + \frac{\sin t}{8} - \frac{\sin 3t}{24}.$$

Given $x\left(\frac{\pi}{2}\right) = 1$

$$x\left(\frac{\pi}{2}\right) = \cos 3\frac{3\pi}{2} + c\,\frac{1}{3}\cdot\sin 3\frac{3\pi}{2} + \frac{1}{8}\sin\frac{3\pi}{2} - \frac{1}{24}\sin 3\frac{3\pi}{2}.$$

$$\Rightarrow \cos\frac{3\pi}{2} + c\cdot\frac{1}{3}\cdot\sin\frac{3\pi}{2} + \frac{1}{8}\sin\frac{3\pi}{2} - \frac{1}{24}\cdot\sin\frac{3\pi}{2}.$$

$$1 = 0 + c\cdot\frac{1}{3}\cdot(-1) + \frac{1}{8}(1) - \frac{1}{24}(-1).$$

$$1 = -\frac{c}{3} + \frac{1}{8} + \frac{1}{24}$$

$$\frac{c}{3} = -1 + \frac{1}{8} + \frac{1}{24}$$

$$\frac{c}{\cancel{3}} = \frac{-24+3+1}{\cancel{24}\,8}$$

$$c = \frac{-20}{8}$$

$$c = \frac{-5}{2}.$$