Unit - V

Vector Integration

Line Integral

Any integral which evaluated along A and B in curve C, is called Line integral, is denoted by

$$\oint_C \bar{F}.d\bar{r}$$

where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

$$\Rightarrow d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$$

$$\bar{F}.d\bar{r} = f_1\,dx + f_2\,dy + f_3\,dz$$

Work done by the force ~~is~~ in a moving particle from A to B is

$$W = \oint_A^B \bar{F}.d\bar{r}$$

(A)

1. If $\bar{F} = 3xy\bar{i} - y^2\bar{j}$, Evaluate $\int_C \bar{F}.d\bar{r}$ along the curve C, $y = 2x^2$ in the XY plane from $(0,0)$ to $(1,2)$.

Sol Given $\bar{F} = 3xy\bar{i} - y^2\bar{j}$

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

In XY plane, $z = 0$

$$\therefore \bar{r} = x\bar{i} + y\bar{j}$$

$$d\bar{r} = dx\,\bar{i} + dy\,\bar{j}$$

Given curve $y = 2x^2$

$$\Rightarrow dy = 4x.dx$$

Now $\bar{F}.d\bar{r} = (3xy\bar{i} - y^2\bar{j})(dx\bar{i} + dy\bar{j})$

$$= 3xy\,dx - y^2 dy.$$

$$= 3x(2x^2)dx - (4x^4)(4x\,dx)$$

$$= 6x^3 dx - 16x^5 dx.$$

$$\therefore \oint_C \bar{F}.d\bar{r} = \int_0^1 6x^3 dx - 16x^5 dx.$$

$$= 6\left[\frac{x^4}{4}\right]_0^1 - 16\left[\frac{x^6}{6}\right]_0^1$$

$$= \frac{6}{4}(1-0) - \frac{16}{6}(1-0)$$

$$= \frac{6}{4} - \frac{16}{6}$$

$$= \frac{-28}{24}$$

$$= \frac{-14}{12} = \frac{-7}{6}.$$

(A) $\therefore \oint_C \bar{F}.d\bar{r} = \frac{-7}{6}.$

2. Estimate the work done when a force $\bar{F} = (x^2 - y^2 + x)\bar{i} - (2xy + y)\bar{j}$ moves a particle from origin to (1,1) along a parabola $y^2 = x$.

Given force $\bar{F} = (x^2-y^2+x)\bar{i} - (2xy+y)\bar{j}$.

Let $\bar{r} = x\bar{i} + y\bar{j}$.

$d\bar{r} = dx\,\bar{i} + dy\,\bar{j}$

Given curve $y^2 = x$.

$\Rightarrow dx = 2y\,dy$.

Now work done $= \oint_A^B \bar{F}.d\bar{r}$

$$= \oint_0^1 [(x^2-y^2+x)\bar{i} - (2xy+y)\bar{j}].[dx\,\bar{i} + dy\,\bar{j}]$$

$$= \int_0^1 [((y^2)^2 - y^2 + y^2)\bar{i} - (2y^2y + y)\bar{j}].[2y\,dy\,\bar{i} + dy\,\bar{j}]$$

$$= \int_0^1 ([y^4 - \cancel{y^2} + \cancel{y^2}]\bar{i} - (2y^3+y)\bar{j}).(2y\,dy\,\bar{i} + dy\,\bar{j})$$

$$= \int_0^1 (y^4\bar{i} - 2y^3\bar{j} + y\bar{j})(2y\,dy\,\bar{i} + dy\,\bar{j})$$

$$= \int_0^1 2y^5dy - 2y^3dy - y\,dy.$$

$$= \left[\frac{\cancel{2}y^6}{\cancel{6}3} - \cancel{2}\frac{y^4}{\cancel{4}2} - \frac{y^2}{2}\right]_0^1$$

$$= \left(\frac{1}{3} - \frac{1}{2} - \frac{1}{2}\right) = \frac{1}{3} - \frac{1}{2} - \frac{1}{2}$$

$$= \frac{1}{3} - 1 = \frac{-2}{3}$$

7M

Surface Integral

To evaluate surface integral we have to take the projection of the surface on any one of the coordinate planes XY, YZ, ZX, is

$$\iint_R \bar{F}\cdot\bar{n}\, ds$$

In XY plane, $\displaystyle\iint_R \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$

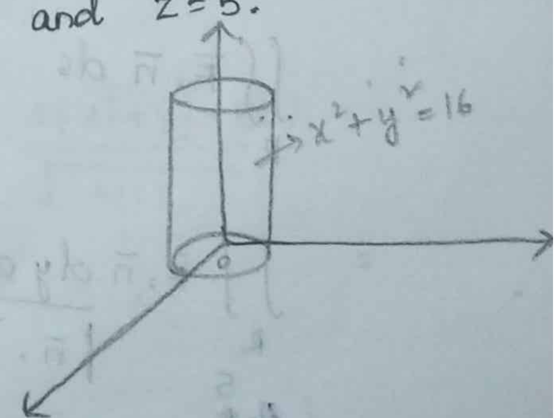
In YZ plane, $\displaystyle\iint_R \bar{F}\cdot\bar{n}\,\frac{dy\,dz}{|\bar{n}\cdot\bar{i}|}$

In ZX plane, $\displaystyle\iint_R \bar{F}\cdot\bar{n}\,\frac{dz\,dx}{|\bar{n}\cdot\bar{j}|}$

where R is the region of integration.

(A)

1. Evaluate $\iint \bar{F}\cdot\bar{n}\, ds$ where $\bar{F} = z\bar{i} + x\bar{j} - 3y^2z\bar{k}$ where S is in surface of a cylinder, $S = x^2+y^2 = 16$ included in the first octant between $z=0$ and $z=5$.



Sol: Given $\bar{F} = z\bar{i} + x\bar{j} - 3y^2z\bar{k}$

$$S = x^2+y^2 = 16$$

WKT $\bar{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

$$= \frac{2x\bar{i} + 2y\bar{j}}{\sqrt{(2x)^2 + (2y)^2}}$$

$$= \frac{2x\bar{i} + 2y\bar{j}}{2(4)}$$

$$\bar{n} = \frac{x\bar{i} + y\bar{j}}{4}$$

$$\bar{F}.\bar{n} = (z\bar{i} + x\bar{j} - 3y^2 z\bar{k})\left(\frac{x\bar{i} + y\bar{j}}{4}\right)$$

$$= \frac{zx + xy}{4}$$

$$= \frac{x(z+y)}{4}$$

Let R be the region in surface yz plane,

$z : 0$ to 5

$x^2 + y^2 = 16$

$0 + y^2 = 16$

$y = 4$

$y : 0$ to 4.

$$\bar{n} = \frac{x\bar{i} + y\bar{j}}{4}$$

$$\bar{n}.\bar{i} = \frac{(x\bar{i} + y\bar{j})}{4}\bar{i} = \frac{x}{4}$$

$$\therefore \iint_R \bar{F}.\bar{n}\, ds$$

$$= \iint_R \bar{F}.\bar{n} \frac{dy\,dz}{|\bar{n}.\bar{i}|}$$

$$= \left[\int_0^4 \int_0^5 \frac{x(z+y)}{\cancel{4}} \cdot \frac{\cancel{x}}{\left|\frac{x}{4}\right|} \cdot dydz\right]$$

$$= \int_0^4 \int_0^5 \frac{\cancel{x}(z+y)}{\cancel{4}} \frac{dy\,dz}{\cancel{x}/\cancel{4}} = \int_0^4 \int_0^5 (z+y)\,dy\,dz.$$

$$= \int_0^4 \left(\frac{z^2}{2} + yz\right)_0^5 dy.$$

$$= \int_0^4 \frac{25}{2} + 5y\,dy.$$

$$= \left[\frac{25}{2} y + \frac{5y^2}{2}\right]_0^4$$

$$= \frac{25}{\cancel{2}} (\overset{2}{\cancel{4}}) + \frac{5}{\cancel{2}} (\overset{8}{\cancel{16}})$$

$$= 50 + 40$$

$$= 90.$$

2. If $\bar{F} = yz\bar{i} + zx\bar{j} + xy\bar{k}$. Evaluate $\iint \bar{F}.\bar{n}\,ds$ over the surface $x^2+y^2+z^2=1$ in the first octant.

Sol: Given $\bar{F} = yz\bar{i} + zx\bar{j} + xy\bar{k}$

$S = x^2+y^2+z^2=1$

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\bar{i} + 2y\bar{j} + 2z\bar{k}}{\sqrt{(2x)^2+(2y)^2+(2z)^2}}$$

$$= \frac{2(x\bar{i}+y\bar{j}+z\bar{k})}{\sqrt{4x^2+4y^2+4z^2}} = \frac{2(x\bar{i}+y\bar{j}+z\bar{k})}{\sqrt{4(1)}}.$$

$$= x\bar{i}+y\bar{j}+z\bar{k}$$

$\bar{F}.\bar{n} = (yz\bar{i} + zx\bar{j} + xy\bar{k})(x\bar{i} + y\bar{j} + z\bar{k})$

$= xyz + xyz + xyz$

$= 3xyz.$

In XY plane, $z = 0$.

from S, $x^2+y^2+z^2=1$

$\Rightarrow x^2+y^2+0=1$

$\Rightarrow x^2+y^2=1.$

put $x=0 \Rightarrow y=1$

put $y=0 \Rightarrow x=1.$

$\bar{n}.\bar{k} = z$

$\therefore \iint \bar{F}.\bar{n}\, ds = \int_0^1\int_0^1 \bar{F}.\bar{n}\, \frac{dx\,dy}{|\bar{n}.\bar{k}|}$

$= \int_0^1\int_0^1 3xy\cancel{z}.\ \frac{dx\,dy}{\cancel{z}}$

$= \int_0^1\int_0^1 3xy\, dx\, dy.$

$= \int_0^1 \left[3x\frac{y^2}{2}\right]_0^1 \cancel{dy}\, dx$

$= \int_0^1 \frac{3x}{2}\, dx = \left[\frac{3}{2}\frac{x^2}{2}\right]_0^1$

$= \frac{3}{4}.$

Volume integrals

$\iiint_V dv$ is called Volume integral.

$\bar{F} = f_1 \bar{i} + f_2 \bar{j} + f_3 \bar{k}$ then

volume integral is given by

$$\iiint_V \bar{F}\, dv = \bar{i} \iiint f_1\, dx dy dz + \bar{j} \iiint f_2\, dx dy dz + \bar{k} \iiint f_3\, dx dy dz.$$

1. If $\bar{F} = 2xz\,\bar{i} - x\bar{j} + y^2 \bar{k}$ Evaluate $\int_v \bar{F}.dv$, where v is the region bounded by the surface $x=0, x=2$; $y=0, y=6$ $z=0$, $z=x^2$, $z=4$.

sol: $\bar{F} = 2xz\,\bar{i} - x\bar{j} + y^2\bar{k}$

$f_1 = 2xz$ $f_2 = -x$ $f_3 = y^2$

$$\iiint \bar{F}.dv = \bar{i} \int_{x=0}^{2} \int_{y=0}^{6} \int_{z=x^2}^{4} 2xz\, dx dy dz + \bar{j} \int_0^2 \int_0^6 \int_{x^2}^{4} -x\, dx dy dz + \bar{k} \int_0^2 \int_0^6 \int_{x^2}^{4} y^2 dx dy dz.$$

$$= \bar{i} \int_0^2 \int_0^6 \left[2x \frac{z^2}{2} \right]_{x^2}^{4} dx dy - \bar{j} \int_0^2 \int_0^6 x(z)_{x^2}^{4} dx dy + \bar{k} \int_0^2 \int_0^6 y^2 (z)_{x^2}^{4} dx dy.$$

$$= \bar{i} \int_0^2 \int_0^6 x(16 - x^4)\, dx dy - \bar{j} \int_0^2 \int_0^6 x(4 - x^2)\, dx dy + \bar{k} \int_0^2 \int_0^6 y^2 (4 - x^2)\, dx dy.$$

$$= \bar{i} \int_0^2 (16x - x^5) (y)_0^6\, dx - \bar{j} \int_0^2 (4x - x^3)(y)_0^6\, dx + \bar{k} \int_0^2 (4 - x^2) \left(\frac{y^3}{3}\right)_0^6 dx$$

$$= \bar{i} \int_0^2 (16x - x^5)\, 6\, dx - \bar{j} \int_0^2 6(4x - x^3)\, dx + \bar{k} \int_0^2 (4 - x^2)\, 72\, dx.$$

$$= \bar{i} \int_0^2 (16xy - x^5 y)_0^6$$

$$= \bar{i} \int_0^2 (16x - x^5)\, 6\, dx - \bar{j} \int_0^2 6(4x - x^2)\, dx + \bar{k} \int_0^2 (4 - x^2)\, 72\, dx.$$

$$= \bar{i}\,.\,6\left[16\frac{x^2}{2} - \frac{x^6}{6}\right]_0^2 - \bar{j}\,.\,6\left(4\frac{x^2}{2} - x^2\right)dx + \bar{k}\int_0^2 (4-x^2)\,72\,dx$$

$$= \bar{i}\,.\,128 - \bar{j}\,24 + \bar{k}\,.\,-384.$$

$$= 128\bar{i} - 24\bar{j} - 384\bar{k}.$$

7M/14M **Vector Integral Theorems:-**

1. Green's Theorem
2. Stokes Theorem
3. Gauss Divergence Theorem

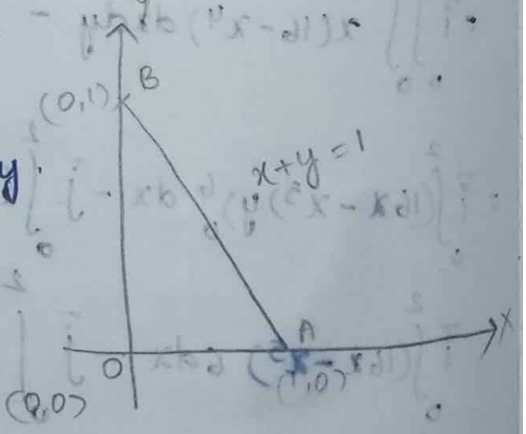
Green's theorem:-

Statement: If M and N are continuous functions of x and y. having first order partial derivative. Then

$$\oint_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

(A)

1. Verify Green's Theorem for $\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ where c is the boundary of region bounded by $x=0$, $y=0$ and $x+y=1$.



Sol: Given $\int (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$

$M = 3x^2 - 8y^2$

$N = 4y - 6xy.$

$$\oint_C M\,dx + N\,dy = \int_{OA} + \int_{AB} + \int_{BO}$$

Along $\overline{OA}$:

$O(0,0)$, $A(1,0)$. Along x-axis $y=0$, $dy=0$.

$x : 0 \text{ to } 1.$

$$\int (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy.$$

$$= \int_0^1 (3x^2 - 0)\,dx + 0 \quad = \left[\frac{3x^3}{3}\right]_0^1$$

$$= 1.$$

Along $\overline{AB}$:

$A(1,0), \quad B(0,1)$

$x + y = 1$

$x = 1 - y$

$dx = -dy.$

$$\int_{AB} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$= \int_1^0 3x^2 - 8(1 - x^2)\,dx + 4(1 - x) - 6x(1 - x)(-dx)).$$

$$= \int_1^0 (3x^2 - 8 - 8x^2 + 16x)\,dx + (-4 + 10x - 6x^2)\,dx.$$

$$= \int_1^0 (-11x^2 + 26x - 12)\,dx$$

$$= \frac{8}{3}$$

Along the line $\overline{BO}$.

$x = 0, \quad dx = 0.$

$B(0,1), \quad O(0,0)$

$$\int_{BO} (0 - 8y^2)0 + (4y - 6xy)\,dy.$$

$$= \int_1^0 0 + 4y\,dy.$$

3. $= 4\left[\frac{y^2}{2}\right]_1^0$

$= 2.$

$$\oint_C M dx + N dy = 1 + \frac{8}{3} - 2$$

$$= \frac{8}{3} - 1$$

$$= \frac{5}{3} = LHS.$$

$\Rightarrow$ RHS $= \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy$ $N = 4y - 6xy.$ $M = 3x^2 - 8y^2.$

$$\frac{\partial N}{\partial x} = 0 - 6(1)y.$$
$$= -6y.$$

$$\frac{\partial M}{\partial y} = 0 - 16y.$$
$$= -16y.$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -6y + 16y$$
$$= 10y.$$

$$\iint\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy = \int_{x=0}^{1}\int_{y=0}^{1} 10y \, dx dy.$$

$$= \int_0^1 \left(10\frac{y^2}{2}\right)_0^{1-x} dx.$$

$$= \int_0^1 5(1-x)^2 dx$$

$$= \int_0^1 5 + 5x^2 - 10x \, .dx = 5 + \frac{5}{3} - \frac{10}{2} = \frac{5}{3}$$

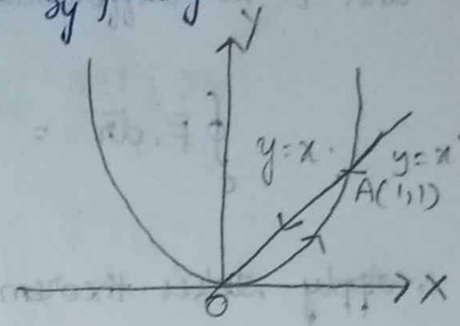
= RHS.

Hence proved.

A
2- Verify Green's theorem for $\int (xy+y^2)dx + x^2 dy$ where c is bounded by $y = x$ and $y = x^2$.

Given curves $y = x$ and $y = x^2$.

By Green's theorem, $\oint_C M dx + N dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy$.



Along OA:- $y = x^2$

$O(0,0)$, $A(1,1)$. $dy = 2x dx$.

$$\int_{C_1} M dx + N dy = \int xy + y^2 dx + x^2 dy.$$

$$= \int_0^1 (x(x^2) + x^4) dx + x^2 2x dx = \int_0^1 (x^3 + x^4) dx + 2x^3 dx.$$

$$= \frac{x^4}{4} + \frac{x^5}{5} + 2 \cdot \frac{x^4}{4}\Big]_0^1 = \frac{1}{4} + \frac{1}{5} + \frac{2}{4} = \frac{19}{20}.$$

Along AO:- $y = x \Rightarrow dy = dx$. $A(1,1)$, $O(0,0)$.

$$\int_{C_2} M dx + N dy = \int_1^0 xy + y^2 dx + x^2 dy = \int_1^0 x^2 + x^2 dx + x^2 dx.$$

$$= \int_1^0 3x^2 dx = \left[\frac{3x^3}{3}\right]_1^0 = -1.$$

$$\oint_C M dx + N dy = \frac{19}{20} - 1 = \frac{-1}{20} = LHS.$$

$\Rightarrow RHS = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy. \Rightarrow \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - x - 2y = x - 2y.$

$\frac{\partial M}{\partial y} = x + 2y$ $\quad \frac{\partial N}{\partial x} = 2x$ $\quad$ Limits: $x: 0$ to 1 $\quad y = x^2$ & $y = x$ $\Rightarrow x^2 = x \Rightarrow x(x-1) = 0$

$y: x^2$ to x $\Rightarrow x = 0, 1$

Now, $\iint_S \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} dx dy = \int_0^1 \int_{x^2}^{x} (x - 2y) dx dy = \int_0^1 \left[xy - \frac{2y^2}{2}\right]_{x^2}^{x} dx$

$$= \int_0^1 \left[xy - \frac{2y^2}{2}\right]_{x^2}^{x} dx = \int_0^1 (x(x) - (x^2)) - (x(x^2) - (x^2)^2) dx.$$

$$= \int_0^1 0 - x^3 + x^4 dx = \left(\frac{-x^4}{4} + \frac{x^5}{5}\right)_0^1 = -\frac{1}{4} + \frac{1}{5} - 0 = \frac{-5+4}{20} = \frac{-1}{20}$$

$= RHS$

Hence proved.

Stoke's Theorem (Relation between line and surface Integral)

Let S be an open surface bounded by a closed curve C and $\bar{F}$ is differentiable vector function then

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds$$

1. Apply stoke's theorem to evaluate $\int y\,dx + z\,dy + x\,dz$, where C is the curve of intersection of sphere $x^2+y^2+z^2=a^2$, $x+z=a$.

Sol: Given $\oint y\,dx + z\,dy + x\,dz$

$x^2+y^2+z^2=a^2$ and $x+z=a$.

By stokes theorem

$$\oint_C \bar{F}\cdot dr = \iint_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds.$$

Given eq of plane $x+z=a$.

$$\frac{x}{a}+\frac{z}{a}=1.$$

and $x^2+y^2+z^2=a^2$

$OA = OB = a$

$O(0,0,0)$ $A(a,0,0)$, $B(0,0,a)$.

$\therefore$ Length of the diameter $AB = \sqrt{a^2+a^2+0} = \sqrt{2}\,a$.

$\therefore$ Radius of the circle $r = \dfrac{a}{\sqrt{2}}$.

$\bar{F} = y\,dx + z\,dy + x\,dz$

$$\text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial(x)}{\partial y} - \frac{\partial(z)}{\partial z}\right) - \bar{j}\ \frac{\partial(x)}{\partial x} - \frac{\partial(y)}{\partial z} + \bar{k}\ \frac{\partial(z)}{\partial x} - \frac{\partial(y)}{\partial y}.$$

$$= \bar{i}(0-1) - \bar{j}(1-0) + \bar{k}(0-1).$$

$$= -\bar{i} - \bar{j} - \bar{k}.$$

$$= -(\bar{i}+\bar{j}+\bar{k}).$$

$\bar{n} = \frac{\nabla s}{|\nabla s|}$, where $s = x+z = a$.

$$\nabla s = \bar{i}\frac{\partial}{\partial x}(x+z-a) + \bar{k}\frac{\partial}{\partial z}(x+z-a).$$

$$= \bar{i}(1) + \bar{k}(1).$$

$$\bar{n} = \frac{(1)\bar{i} + \bar{k}(1)}{\sqrt{1^2+1^2}}$$

$$\bar{n} = \frac{\bar{i}+\bar{k}}{\sqrt{2}}$$

$$\therefore \text{curl}\,\bar{F}\cdot\bar{n} = -(\bar{i}+\bar{j}+\bar{k})\cdot\frac{\bar{i}+\bar{k}}{\sqrt{2}}$$

$$= \frac{-(1+1)}{\sqrt{2}}$$

$$= \frac{-2}{\sqrt{2}} = -\sqrt{2}.$$

Hence by stoke's theorem. $\oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds.$

$$= \iint_S -\sqrt{2}\,ds$$

$$= -\sqrt{2} \int_S 1\, ds$$

$$= -\sqrt{2}\,(S)$$

$$= -\sqrt{2}\left(\pi r^2\right)$$

$$= -\sqrt{2}\left(\pi \cdot \left(\frac{a}{\sqrt{2}}\right)^2\right)$$

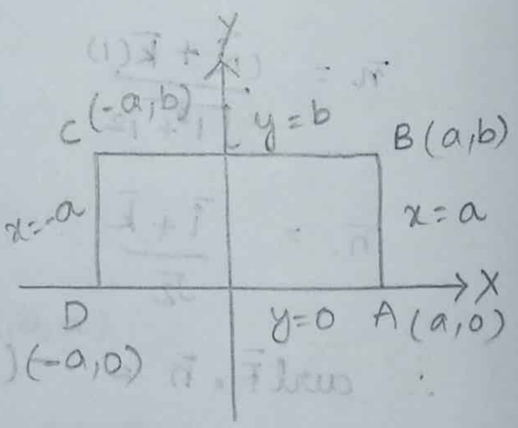
$$= -\sqrt{2}\,\pi \frac{a^2}{(\sqrt{2})^2}$$

$$= \frac{-\pi a^2}{\sqrt{2}}.$$

2. Verify stoke's theorem for $\bar{F} = (x^2+y^2)\bar{i} - 2xy\bar{j}$ taken around rectangle bounded by the lines ; $x = \pm a, y = 0, y = b$.

Sol: Given $\bar{F} = (x^2+y^2)\bar{i} - 2xy\bar{j}$

Let ABCD be the rectangle with vertices $A(a,0)$, $B(a,b)$, $C(-a,b)$, $D(-a,0)$.



By stoke's theorem.

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot\bar{n}\, ds.$$

$$\bar{F}\cdot d\bar{r} = \left[(x^2+y^2)\bar{i} - 2xy\bar{j}\right]\left[dx\,\bar{i} + dy\,\bar{j}\right]$$

$$= (x^2+y^2)\,dx - 2xy\,dy.$$

LHS : $\oint_C \bar{F}\cdot d\bar{r}$

$= \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$

Along AB :

$x = a$ A $(a,0)$ B (a,b).

$\Rightarrow dx = 0.$

$$\int_{AB} \bar{F}\cdot d\bar{r} = \int_0^b (a^2+y^2)\,dx - 2ay\,dy.$$

$$= \int_0^b (a^2+y^2)\,0 - 2ay\,dy.$$

$$= \int_0^b -2ay\,dy = -2a\left.\frac{y^2}{2}\right)_0^b = -ab^2.$$

Along BC :

$y = b$ B (a,b) C $(-a,b)$.

$\Rightarrow dy = 0$

$$\int_{BC} \bar{F}\cdot d\bar{r} = \int_a^{-a} (x^2+b^2)\,dx - 0$$

$$= \left[\frac{x^3}{3} + b^2x\right]_a^{-a}$$

$$= \frac{-a^3}{3} - ab^2 - \left(+\frac{a^3}{3} + ab^2\right)$$

$$= \frac{-2a^3}{3} - 2ab^2 = \frac{-2a(a^2+3b^2)}{3}$$

Along CD :

$x = -a$ C $(-a,b)$ D $(-a,0)$.

$\Rightarrow dx = 0$

$$\int_{CD} \bar{F}\cdot d\bar{r} = \int_b^0 (a^2+y^2)\,dx - (-2ay)\,dy$$

$$= \int_b^0 2ay\,dy.$$

$$= \cancel{2}a\left[\frac{y^2}{\cancel{2}}\right]_b^0$$

$$= -ab^2$$

Along DA:

$y = 0$ $\quad D(-a, 0) \quad A(a, 0)$

$\Rightarrow dy = 0$

$$\int_{DA} \bar{F}.d\bar{r} = \int_{-a}^{a} (x^2+0)\,dx - 2x(0)\,dy.$$

$$= \int_{-a}^{a} x^2\,dx + 0$$

$$= \left[\frac{x^3}{3}\right]_{-a}^{a} = \frac{a^3}{3} + \frac{a^3}{3} = \frac{2a^3}{3}.$$

$$\oint_C \bar{F}.d\bar{r} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$

$$= -ab^2 + \frac{-2a(a^2+6b^2)}{3} + -ab^2 + \frac{2a^3}{3}.$$

$$= -ab^2 + \frac{-2a^3}{\cancel{3}} - 2ab^2 - ab^2 + \frac{2a^3}{\cancel{3}}$$

$$= -4ab^2.$$

RHS: $\iint_S \text{curl}\,\bar{F}.\bar{n}\,ds.$

vector perpendicular to XY plane is $\bar{n} = \bar{k}$.

$$\operatorname{curl}\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2+y^2) & -2xy & 0 \end{vmatrix}$$

$$= \bar{i}(0) - \bar{j}(0-0) + \bar{k}(-2(1)(y) - (0+2y))$$

$$= \bar{k}(-4y)$$

$$= -4y\bar{k}.$$

$$\operatorname{curl}\bar{F}\cdot\bar{n} = (-4y\bar{k})(\bar{k})$$

$$= -4y.$$

$$\iint_S \operatorname{curl}\bar{F}\cdot\bar{n}\, ds = \int_{x=-a}^{a}\int_{y=0}^{b} -4y\, dx\, dy$$

$$= \int_{-a}^{a}\int_{0}^{b} -4y\, dx\, dy.$$

$$= -4\int_{-a}^{a}\left[\frac{y^2}{2}\right]_0^b dx$$

$$= -4\int_{-a}^{a}\frac{b^2}{2}\, dx.$$

$$= -2b^2\left[x\right]_{-a}^{a}$$

$$= -2b^2\left[a+a\right]$$

$$= -4ab^2.$$

LHS = RHS.

Gauss Divergence theorem :- (Relation b/w Volume & surface integral).

Let S be closed surface enclosing a volume V.

If $\bar{F}$ is differential vector point function then

$$\int_V \operatorname{div} \bar{F}\, dv = \int_S \bar{F}\cdot\bar{n}\, ds \quad \iint_S (f_1\, dy\, dz + f_2\, dz\, dx + f_3\, dx\, dy)$$

or

$$\iiint \operatorname{div} \bar{F}\, dv = \iint \bar{F}\cdot\bar{n}\, ds. \quad = \iiint_V \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx\,dy\,dz$$

(A)

1. By transforming into triple integral evaluate $\iint x^3\, dy\, dz + x^2 y\, dz\, dx + x^2 z\, dx\, dy$. where S is closed surface consisting of the cylinder $x^2+y^2=a^2$ and the circular disk $z=0$ and $z=b$.

Sol: Let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

$F_1 = x^3 \quad F_2 = x^2 y \quad F_3 = x^2 z$

$\frac{\partial F_1}{\partial x} = 3x^2, \quad \frac{\partial F_2}{\partial y} = x^2(1) = x^2 \quad \frac{\partial F_3}{\partial z} = x^2(1) = x^2$

div F =

$\Rightarrow 3x^2 + x^2 + x^2$

$= 5x^2$.

By Gauss divergence theorem

$$\iint x^3\, dy\, dz + x^2 y\, dz\, dx + x^2 z\, dx\, dy = \iiint \operatorname{div} F\, dv$$

$$= \iiint 5x^2\, dx\, dy\, dz$$

$z = 0$ to b.

$x^2 + y^2 = a^2 \Rightarrow y^2 = a^2 - x^2$

$y = \sqrt{a^2 - x^2}$

$y = \pm\sqrt{a^2 - x^2}$

$x^2 = a^2 \Rightarrow x = \pm a$

$$\int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{0}^{b} 5x^2\,dx\,dy\,dz$$

$$20\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}}\int_{0}^{b} x^2\,dx\,dy\,dz.$$

$$20\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}} x^2[z]_0^b\,dx\,dy.$$

$$20\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}} x^2 b\,dx\,dy.$$

$$20b\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}} x^2(y)\,dx\,dy$$

$$20b\int_{0}^{a} x^2(y)_0^{\sqrt{a^2-x^2}}\,dx.$$

$$= 20b\int_{0}^{a} x^2\sqrt{a^2-x^2}\,dx.$$

put $x = a\sin\theta$

$dx = a\cos\theta$

$\theta = 0$ to $\pi/2$

$$20b\int_{0}^{\pi/2} a^2\sin^2\theta\left(\sqrt{a^2-a^2\sin^2\theta}\right)a\cos\theta\,d\theta.$$

$$= 20b\int_{0}^{\pi/2} a^2\sin^2\theta\,(a)\cos\theta\;a\cos\theta\,d\theta.$$

$$= 20b\int_{0}^{\pi/2} a^4\sin^2\theta\cos^2\theta\,d\theta.$$

$$= 5a^4b\int_{0}^{\pi/2}(2\sin\theta\cos\theta)^2\,d\theta$$

$$= 5a^4b \int_0^{\pi/2} (\sin 2\theta)^2 \, d\theta$$

$$= 5a^4b \int_0^{\pi/2} \sin^2 4\theta \, d\theta$$

$$= 5a^4b \int_0^{\pi/2} \frac{1-\cos 4\theta}{2} \, d\theta$$

$$= 5a^4b \left[\theta - \frac{\sin 4\theta}{2} \right]_0^{\pi/2}$$

$$= 5a^4b \left[\frac{\pi}{4} - 0 \right]$$

$$= \frac{5\pi a^4 b}{4}$$

2. Verify gauss divergence theorem for $\bar{F}$ taken over the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$ where $\bar{F} = x^2\bar{i} + z\bar{j} + yz\bar{k}$.