

ODE Review

1. Solve 1st ODE (No $S(x-c)$)

(i) Separation variable

$$\text{e.g. } y' + y^3 \cos(t^2) = 0 \Rightarrow \frac{dy}{dt} = -y^3 \cos(t^2)$$

$$\Rightarrow \int \frac{dy}{y^3} = \int -\cos(t^2) dt$$

Use I.C. $y(0) = y_0$ to determine C

(ii) Integrating factor

$$\text{e.g. } t y' - 4y = 2t^5 e^{-t} \Rightarrow y' - \frac{4}{t} y = 2t^4 e^{-t}$$

$$\text{Solve } \mu \text{ s.t. } \frac{\mu'}{\mu} = -\frac{4}{t} \Rightarrow \int \frac{d\mu}{\mu} = -\frac{4}{t} dt$$

$$\Rightarrow \mu = t^{-4}$$

$$\mu \times \boxed{} \Rightarrow (\mu y)' = 2t^4 e^{-t} \cdot \mu$$

$$\stackrel{\int \bullet}{\Rightarrow} \mu y = \int 2t^4 e^{-t} \mu dt \dots$$

2. Solve 2nd ODE (No $\delta(x-c)$)

Sol $y(t) = \text{General sol. } (y_g) + \text{Particular sol. } (y_p)$

- Find Fundamental/General sol. (by roots of char. polyn.)

(i) different real roots (r_1, r_2)

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

(ii) same real roots (r_1, r_1)

$$y(t) = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$$

(iii) complex roots ($r = \alpha \pm \beta i$)

$$y(t) = c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t)$$

- Find particular sol (by the type of source term)

$$y'' + a y' + b y = f(t)$$

source $f(t)$	Guess $y_p(t)$
$K e^{at}$	$k e^{at}, k t e^{at}, k t^2 e^{at}, \dots$
$K_m t^m + \dots + K_1 t + K_0$	$k_m t^m + \dots + k_1 t + k_0$
$K_1 \cos(bt) + K_2 \sin(bt)$	$k_1 \cos(bt) + k_2 \sin(bt)$

\Rightarrow Plug $y_p(t)$ back into ODE to solve coefficient k_i

$$\Rightarrow y(t) = y_g(t) + y_p(t)$$

use I.C. to solve c_1, c_2 .

3. Solve 2nd ODE (w/ $\delta(x-c)$) \Rightarrow Laplace transform \mathcal{L}

$$y'' + ay' + by = f(t) \xrightarrow{\mathcal{L}} (s^2 + as + b)Y - y(0)s - y'(0) - ay(0) = \mathcal{L}[f]$$

$$\text{Find } y(t) \text{ s.t. } \mathcal{L}[y(t)] = \frac{\mathcal{L}[f] + y(0)s + y'(0) + ay(0)}{s^2 + as + b}$$

E.g.: (i) $\mathcal{L}^{-1}\left[e^{-4s} \frac{2}{s^2 + 6s + 16} \right] = \mathcal{L}^{-1}\left[\frac{2}{\sqrt{7}} e^{-4s} \frac{\sqrt{7}}{(s+3)^2 + (\sqrt{7})^2} \right]$

translation & $u(t-c)$ $\leftarrow \mathcal{L}[e^{-3t} \sin(\sqrt{7}t)]$

$$= \frac{2}{\sqrt{7}} u(t-4) e^{-3(t-4)} \sin(\sqrt{7}(t-4))$$

(ii) $\mathcal{L}^{-1}\left[e^{-2s} \frac{1}{(s^2 - 2s - 15)} \right] = \mathcal{L}^{-1}\left[e^{-2s} \frac{1}{(s-5)^2 - 16} \right]$

$$= \mathcal{L}^{-1}\left[e^{-2s} \frac{1}{(s-5)(s+3)} \right] \quad \frac{A}{s-5} + \frac{B}{s+3} \text{ w/ } A = \frac{1}{8}, B = -\frac{1}{8}$$

$$= \mathcal{L}^{-1}\left[e^{-2s} \left(\frac{1}{8} \frac{1}{s-5} - \frac{1}{8} \frac{1}{s+3} \right) \right] \quad \mathcal{L}[e^{-3t}]$$

$$= u(t-2) \left(\frac{1}{8} e^{5(t-2)} - \frac{1}{8} e^{-3(t-2)} \right)$$

4. Solve matrix form ODE

- Change $y'' + ay' + by = 0 \rightarrow X' = AX$ w/ $X = \begin{pmatrix} y \\ y' \end{pmatrix}$

$$\rightarrow A = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}$$

- Solve eigenvalue / eigenvector of $A \Rightarrow$ Fundamental / General sol

(i) different real e-value λ_1, λ_2 w/ e-vector v_1, v_2

$$\vec{X}_1(t) = e^{\lambda_1 t} \vec{v}_1, \quad \vec{X}_2(t) = e^{\lambda_2 t} \vec{v}_2$$

(ii) same real roots λ w/ e-vector $v = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\vec{X}_1(t) = e^{\lambda t} \begin{pmatrix} a \\ b \end{pmatrix}, \quad \vec{X}_2(t) = e^{\lambda t} \begin{pmatrix} at \\ a + b\lambda t \end{pmatrix}$$

(iii) complex e-value $\lambda = \alpha \pm \beta i$, $v = \vec{a} \pm \vec{b}$

$$\vec{X}_1(t) = e^{\alpha t} \cos(\beta t) \vec{a} - e^{\alpha t} \sin(\beta t) \vec{b}$$

$$\vec{X}_2(t) = e^{\alpha t} \sin(\beta t) \vec{a} + e^{\alpha t} \cos(\beta t) \vec{b}$$

\Rightarrow General sol $\vec{X}(t) = c_1 \vec{X}_1(t) + c_2 \vec{X}_2(t)$

Use I.C. to solve c_1, c_2

5. Equilibrium

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}, \quad \text{solve } \begin{cases} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{cases}$$

E.g. $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_2 \\ \sin(x_1) \end{pmatrix}, \Rightarrow x_2 = 0 \Rightarrow \sin(x_1) = 0 \Rightarrow (n\pi, 0) \quad n \in \mathbb{Z}$

6. Linearization (Jacobian matrix)

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = DF \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad DF = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}$$

7. Type of Equilibrium.

Solve e-value / e-vector of DF

