Assignment 12

KUSUMA PRIYA
EE20MTECH11007

Download codes from

https://github.com/KUSUMAPRIYAPULAVARTY/assignment12

1 QUESTION

Let $\mathbf{V}$ be the set of real numbers.Regard $\mathbf{V}$ as a vector space over the field of rational numbers,with usual operations.Prove that this vector space is not finite-dimensional.

2 Solution

Given $\mathbf{V}$ is a vector space over field Q (rational numbers)
It is finite dimensional with dimensionality n if every vector $\mathbf{v}$ in $\mathbf{V}$ can be written as

$$\mathbf{v} = \sum_{i=0}^{n-1} c_i \alpha_i \tag{2.0.1}$$

$$\text{where } c_i \in Q \tag{2.0.2}$$

$$\text{and } \mathbf{B} = \{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\} \tag{2.0.3}$$

is the basis with linearly independent α_i
that is, basis is the largest set with linearly independent vectors from $\mathbf{V}$
Consider the set of vectors $\{1, x\}$, where x is irrational.
Assume there exists non zero $\beta_0, \beta_1 \in Q$ such that

$$\beta_0 + \beta_1 x = 0 \tag{2.0.4}$$

$$\implies x = -\frac{\beta_0}{\beta_1} \tag{2.0.5}$$

But x is irrational and $-\frac{\beta_0}{\beta_1}$ is rational so (2.0.5) can't be possible so $\beta_0, \beta_1 = 0$
Hence $\{1, x\}$ are independent.
Similarly for the set $\left\{1, x, x^2\right\}$
for $\beta_0, \beta_1, \beta_2 \in Q$

$$\beta_0 + \beta_1 x + \beta_2 x^2 = 0 \tag{2.0.6}$$

$\beta_1 x + \beta_2 x^2$ is irrational and β_0 is rational. Therefore

$$\beta_0 = 0 \tag{2.0.7}$$

$$\text{and } \beta_1 x + \beta_2 x^2 = 0, (x \neq 0) \tag{2.0.8}$$

$$\implies \beta_1 + \beta_2 x = 0 \tag{2.0.9}$$

$$\implies \beta_1, \beta_2 = 0 \tag{2.0.10}$$

$$\therefore \beta_0 + \beta_1 x + \beta_2 x^2 = 0 \tag{2.0.11}$$

$$\iff \beta_0, \beta_1, \beta_2 = 0 \tag{2.0.12}$$

Hence $\left\{1, x, x^2\right\}$ are independent
By induction, let us say the set $\left\{1, x, x^2, \ldots, x^n\right\}$ is independent

$$\text{for } \beta_0, \beta_1, \beta_2, \ldots, \beta_n \in Q \tag{2.0.13}$$

$$\beta_0 + \beta_1 x + \beta_2 x^2 + \ldots + \beta_n x^n = 0 \tag{2.0.14}$$

$$\iff \beta_0, \beta_1, \beta_2, \ldots, \beta_n = 0 \tag{2.0.15}$$

To prove this for the set $\mathbf{A}$=$\left\{1, x, x^2, \ldots, x^{n+1}\right\}$

$$\text{for } \beta_0, \beta_1, \beta_2, \ldots, \beta_n, \beta_{n+1} \in Q \tag{2.0.16}$$

$$\beta_0 + \beta_1 x + \ldots + \beta_n x^n + \beta_{n+1} x^{n+1} = 0 \tag{2.0.17}$$

Comparing to (2.0.7) and (2.0.8)

$$\beta_0 = 0 \tag{2.0.18}$$

$$\beta_1 + \beta_2 x + \ldots + \beta_{n+1} x^n = 0 \tag{2.0.19}$$

Comparing with (2.0.14),we have
$\beta_1, \beta_2, \ldots, \beta_{n+1} = 0$

$$\therefore \beta_0 + \beta_1 x + \ldots + \beta_n x^n + \beta_{n+1} x^{n+1} = 0 \tag{2.0.20}$$

$$\iff \beta_0, \beta_1, \beta_2, \ldots, \beta_n, \beta_{n+1} = 0 \tag{2.0.21}$$

Hence $\mathbf{A}$ has linearly independent vectors
Let the set $\mathbf{B} = \left\{1, x, x^2, \ldots, x^m\right\}$ be the largest linearly independent set in $\mathbf{V}$ and hence can form the basis leading to dimensionality $m+1$
But from induction, we have proved that $\left\{1, x, x^2, \ldots, x^m, x^{m+1}\right\}$ is also independent which is a contradiction to dimensionality being $m+1$
Hence we deduce that the vector space $\mathbf{V}$ is not finite dimensional over the field Q