Assignment 12

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Download codes from

https://github.com/KUSUMAPRIYAPULAVARTY/assignment12

1 QUESTION

Let $\mathbf{V}$ be the set of real numbers.Regard $\mathbf{V}$ as a vector space over the field of rational numbers,with usual operations.Prove that this vector space is not finite-dimensional.

2 SOLUTION

Given $\mathbf{V}$ is a vector space over field Q (rational numbers)
It is finite dimensional with dimensionality n if every vector $\mathbf{v}$ in $\mathbf{V}$ can be written as

$$\mathbf{v} = \sum_{i=0}^{n-1} c_i \alpha_i \tag{2.0.1}$$

$$\text{where } c_i \in Q \tag{2.0.2}$$

$$\text{and } \mathbf{B} = \{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\} \tag{2.0.3}$$

is the basis with linearly independent α_i
Let us consider the set of polynomials in k of degree less than equal to n

$$f(k) = a_0 + a_1 k + a_2 k^2 + \ldots + a_{n-1} k^{n-1} \tag{2.0.4}$$

Checking if it a subspace of $\mathbf{V}$
Any function of the form $f(ck_1 + k_2)$ (where $c \in Q$) becomes

$$f(ck_1 + k_2) = a_0 + a_1(ck_1 + k_2) + \tag{2.0.5}$$

$$a_2(ck_1 + k_2)^2 + \ldots + a_{n-1}(ck_1 + k_2)^{n-1} \tag{2.0.6}$$

which is also a polynomial of degree less than equal to n implying it is closed under scalar multiplication and vector addition.
Hence it forms a subspace with basis

$$\mathbf{B} = \left\{1, k, k^2, \ldots, k^{n-1}\right\} \tag{2.0.7}$$

Clearly the set of vectors $\left\{1, k, k^2, k^3, \ldots\ k^{n-1}\right\}$ forms a linearly independent set since
for $\beta_0, \beta_1, \beta_2, \ldots \in Q$

$$\beta_0(1) + \beta_1(k) + \beta_2(k^2) + \ldots = 0 \tag{2.0.8}$$

$$\implies \beta_0 = 0, k(\beta_1 + \beta_2(k) + \ldots) = 0 \tag{2.0.9}$$

$$\implies \beta_1 = 0, k(\beta_2 + \beta_3(k) + \ldots) = 0 \tag{2.0.10}$$

$$\text{Therefore } \beta_0(1) + \beta_1(k) + \beta_2(k^2) + \ldots = 0 \tag{2.0.11}$$

$$\iff \beta_0, \beta_1, \beta_2, \ldots = 0 \tag{2.0.12}$$

and they also span the subspace as

$$f(k) = \sum_{i=0}^{n-1} a_i k^i \tag{2.0.13}$$

So, $\mathbf{B}$ forms a basis for the subspace
We know that e^k is a vector that lies in $\mathbf{V}$

$$e^k = \sum_{i=0}^{\infty} \frac{k^i}{i!} \tag{2.0.14}$$

$$= 1 + \frac{k}{1!} + \frac{k^2}{2!} + \frac{k^3}{3!} + \ldots \tag{2.0.15}$$

Representing e^k as a polynomial function,

$$f(k) = e^k = \sum_{i=0}^{\infty} c_i k^i \tag{2.0.16}$$

$$\text{where } c_i = \frac{1}{i!} \in Q \tag{2.0.17}$$

Since, (2.0.16) refers to an infinite summation, n is ∞
So, a of subspace $\mathbf{V}$ is not finite dimensional over Q
So, the vector space $\mathbf{V}$ of real numbers is not finite dimensional over the field of rational numbers Q