

# Assignment 13

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Download codes from

<https://github.com/KUSUMAPRIYAPULAVARTY/assignment13>

## 1 QUESTION

Let  $F$  be a subfield of the complex numbers and let  $T$  be the function from  $F^3$  into  $F^3$  defined by

$$T(x_1, x_2, x_3) = \quad (1.0.1)$$

$$(x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3) \quad (1.0.2)$$

- (a) Verify that  $T$  is a linear transformation.  
 (b) If  $(a, b, c)$  is a vector in  $F^3$ , what are the conditions on  $a, b, c$  that the vector be in the range of  $T$ ? What is the rank of  $T$ ?  
 (c) What are the conditions on  $a, b, c$  that  $(a, b, c)$  be in the null space of  $T$ ? What is the nullity of  $T$ ?

## 2 SOLUTION

Representing the transformation in matrix form

$$T(x_1, x_2, x_3) = \mathbf{T}\mathbf{x} \quad (2.0.1)$$

$$\mathbf{T} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{pmatrix} \quad (2.0.2)$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad (2.0.3)$$

### 2.1 Part (a)

Consider the matrices  $\mathbf{x}, \mathbf{y} \in F^3$  and the scalar  $c \in F$

By the associativity of matrix multiplications, we can write

$$\mathbf{T}(c\mathbf{x} + \mathbf{y}) = \mathbf{T}(c\mathbf{x}) + \mathbf{T}\mathbf{y} \quad (2.1.1)$$

$$= c\mathbf{T}\mathbf{x} + \mathbf{T}\mathbf{y} \quad (2.1.2)$$

So,  $\mathbf{T}$  is a linear transformation.

### 2.2 Part (b)

$$\text{range}(\mathbf{T}) = \{\mathbf{y} : \mathbf{T}\mathbf{x} = \mathbf{y} \text{ where } \mathbf{x}, \mathbf{y} \in F^3\}$$

$$\mathbf{y} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (2.2.1)$$

$$\mathbf{T}\mathbf{x} = \mathbf{y} \quad (2.2.2)$$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (2.2.3)$$

Forming the augmented matrix,

$$\left( \begin{array}{ccc|c} 1 & -1 & 2 & a \\ 2 & 1 & 0 & b \\ -1 & -2 & 2 & c \end{array} \right) \quad (2.2.4)$$

$$\begin{array}{l} \xleftarrow{R_2 \leftarrow R_2 - 2R_1} \\ \xleftarrow{R_3 \leftarrow R_3 + R_1} \end{array} \left( \begin{array}{ccc|c} 1 & -1 & 2 & a \\ 0 & 3 & -4 & b - 2a \\ 0 & -3 & 4 & a + c \end{array} \right) \quad (2.2.5)$$

$$\begin{array}{l} \xleftarrow{R_3 \leftarrow R_3 + R_2} \\ \xleftarrow{R_1 \leftarrow R_1 + R_2 \times \frac{1}{3}} \end{array} \left( \begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & \frac{a+b}{3} \\ 0 & 3 & -4 & b - 2a \\ 0 & 0 & 0 & -a + b + c \end{array} \right) \quad (2.2.6)$$

$$\begin{array}{l} \xleftarrow{R_2 \leftarrow \frac{R_2}{3}} \\ \xleftarrow{R_1 \leftarrow \frac{R_1}{3}} \end{array} \left( \begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & \frac{a+b}{3} \\ 0 & 1 & -\frac{4}{3} & \frac{b-2a}{3} \\ 0 & 0 & 0 & -a + b + c \end{array} \right) \quad (2.2.7)$$

So,  $\text{rank}(\mathbf{T})=2$  and for solution to exist

$$-a + b + c = 0 \quad (2.2.8)$$

All vectors  $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in F^3$  that satisfy (2.2.8) lie in the range of  $\mathbf{T}$

### 2.3 Part (c)

$$\text{nullspace}(\mathbf{T}) = \{\mathbf{x} : \mathbf{T}\mathbf{x} = \mathbf{0} \text{ where } \mathbf{x} \in F^3\}$$

$$\mathbf{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (2.3.1)$$

$$\mathbf{T}\mathbf{x} = \mathbf{0} \quad (2.3.2)$$

Finding reduced row echelon form of  $\mathbf{T}$

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{pmatrix} \quad (2.3.3)$$

$$\begin{matrix} \xleftarrow{R_2 \leftarrow R_2 - 2R_1} \\ \xleftarrow{R_3 \leftarrow R_3 + R_1} \end{matrix} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & -3 & 4 \end{pmatrix} \quad (2.3.4)$$

$$\begin{matrix} \xleftarrow{R_3 \leftarrow R_3 + R_2} \\ \xleftarrow{R_1 \leftarrow R_1 + R_2 \times \frac{1}{3}} \end{matrix} \begin{pmatrix} 1 & 0 & \frac{2}{3} \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.3.5)$$

$$\xleftarrow{R_2 \leftarrow \frac{R_2}{3}} \begin{pmatrix} 1 & 0 & \frac{2}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 0 \end{pmatrix} \quad (2.3.6)$$

The number of free variables in the above matrix is 1 hence  $\text{nullity}(\mathbf{T}) = 1$

$$\begin{pmatrix} 1 & 0 & \frac{2}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (2.3.7)$$

$$\implies a + \frac{2}{3}c = 0 \quad (2.3.8)$$

$$b - \frac{4}{3}c = 0 \quad (2.3.9)$$

So, the null space of  $\mathbf{T}$  is set of all vectors  $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in F^3$

that satisfy (2.3.8) and (2.3.9)

**Note**

$$\text{rank}(\mathbf{T}) + \text{nullity}(\mathbf{T}) = 2 + 1 = \dim(F^3)$$