Assignment 13

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Download codes from

https://github.com/KUSUMAPRIYAPULAVARTY/assignment13

1 QUESTION

Let F be a subfield of the complex numbers and let T be the function from F^3 into F^3 defined by

$$T(x_1, x_2, x_3) = \tag{1.0.1}$$

$$(x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3) \tag{1.0.2}$$

(a) Verify that T is a linear transformation.
(b) If (a, b, c) is a vector in F^3, what are the conditions on a, b, c that the vector be in the range of T ?What is the rank of T?
(c) What are the conditions on a, b, c that (a, b, c) be in the null space of T?What is the nullity of T?

2 Solution

Representing the transformation in matrix form

$$T(x_1, x_2, x_3) = \mathbf{T}\mathbf{x} \tag{2.0.1}$$

$$\mathbf{T} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{2.0.3}$$

2.1 Part (a)

Consider the matrices $\mathbf{x}, \mathbf{y} \in F^3$ and the scalar $c \in F$
By the associativity of matrix multiplications, we can write

$$\mathbf{T}(c\mathbf{x} + \mathbf{y}) = \mathbf{T}(c\mathbf{x}) + \mathbf{T}\mathbf{y} \tag{2.1.1}$$

$$= c\mathbf{T}\mathbf{x} + \mathbf{T}\mathbf{y} \tag{2.1.2}$$

So, $\mathbf{T}$ is a linear transformation.

2.2 Part (b)

range($\mathbf{T}$)=$\left\{\mathbf{y} : \mathbf{T}\mathbf{x} = \mathbf{y} \text{ where } \mathbf{x}, \mathbf{y} \in F^3\right\}$

$$\mathbf{y} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{2.2.1}$$

$$\mathbf{T}\mathbf{x} = \mathbf{y} \tag{2.2.2}$$

$$\implies \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{2.2.3}$$

Forming the augmented matrix,

$$\begin{pmatrix} 1 & -1 & 2 & a \\ 2 & 1 & 0 & b \\ -1 & -2 & 2 & c \end{pmatrix} \tag{2.2.4}$$

$$\xleftrightarrow[R_3 \leftarrow R_3 + R_1]{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 1 & -1 & 2 & a \\ 0 & 3 & -4 & b - 2a \\ 0 & -3 & 4 & a + c \end{pmatrix} \tag{2.2.5}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 + R_2 \times \frac{1}{3}]{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & 0 & \frac{2}{3} & \frac{a+b}{3} \\ 0 & 3 & -4 & b - 2a \\ 0 & 0 & 0 & -a + b + c \end{pmatrix} \tag{2.2.6}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{3}} \begin{pmatrix} 1 & 0 & \frac{2}{3} & \frac{a+b}{3} \\ 0 & 1 & \frac{-4}{3} & \frac{b-2a}{3} \\ 0 & 0 & 0 & -a + b + c \end{pmatrix} \tag{2.2.7}$$

So, rank($\mathbf{T}$)=2 and for solution to exist

$$-a + b + c = 0 \tag{2.2.8}$$

All vectors $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in F^3$ that satisfy (2.2.8) lie in the range of $\mathbf{T}$

2.3 Part (c)

nullspace(T)=$\left\{\mathbf{x} : \mathbf{T}\mathbf{x} = \mathbf{0} \text{ where } \mathbf{x} \in F^3\right\}$

$$\mathbf{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{2.3.1}$$

$$\mathbf{T}\mathbf{x} = \mathbf{0} \tag{2.3.2}$$

Finding reduced row echelon form of **T**

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{pmatrix} \tag{2.3.3}$$

$$\xleftrightarrow[R_3 \leftarrow R_3 + R_1]{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & -3 & 4 \end{pmatrix} \tag{2.3.4}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 + R_2 \times \frac{1}{3}]{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & 0 & \frac{2}{3} \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.3.5}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{3}} \begin{pmatrix} 1 & 0 & \frac{2}{3} \\ 0 & 1 & \frac{-4}{3} \\ 0 & 0 & 0 \end{pmatrix} \tag{2.3.6}$$

The number of free variables in the above matrix is 1 hence nullity(**T**) =1

$$\begin{pmatrix} 1 & 0 & \frac{2}{3} \\ 0 & 1 & \frac{-4}{3} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.3.7}$$

$$\implies a + \frac{2}{3}c = 0 \tag{2.3.8}$$

$$b - \frac{4}{3}c = 0 \tag{2.3.9}$$

So, the null space of **T** is set of all vectors $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in F^3$ that satisfy (2.3.8) and (2.3.8)

<u>**Note**</u>

rank(**T**)+nullity((**T**)=2+1=dim(F^3)