Assignment 14

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Download codes from

https://github.com/KUSUMAPRIYAPULAVARTY/assignment14

1 Question

Let p, m, n be positive integers and F a field.Let $\mathbf{V}$ be the space of $m \times n$ matrices over F and $\mathbf{W}$ the space of $p \times n$ matrices over F.Let $\mathbf{B}$ be a fixed $p \times m$ matrix and let T be the linear transformation from $\mathbf{V}$ into $\mathbf{W}$ defined by $T(\mathbf{A}) = \mathbf{BA}$.Prove that T is invertible if and only if $p = m$ and $\mathbf{B}$ is an invertible $m \times m$ matrix.

2 Solution

Parameter	Description
p, m, n	Positive integers
F	Field
$\mathbf{V}$	Space of $m \times n$ matrices over F
$\mathbf{W}$	Space of $p \times n$ matrices over F
$\mathbf{B}$	Fixed $p \times m$ matrix
Linear transformation $T : \mathbf{V} \to \mathbf{W}$	$T(\mathbf{A}) = \mathbf{BA}$

TABLE 0: Input Parameters

$$T(\mathbf{A}) = \mathbf{BA} \tag{2.0.1}$$

So, $\mathbf{B}$ is the transformation matrix.
$\mathbf{B}$ is invertible if

1) T is one to one mapping,that is

$$\mathbf{BA} = \mathbf{BA}' \tag{2.0.2}$$
$$\implies \mathbf{A} = \mathbf{A}' \tag{2.0.3}$$

2) T must be onto, that is range($\mathbf{B}$)=$\mathbf{W}$

2.1 Case 1

Let us assume that T is invertible with T^{-1} from $\mathbf{W}$ to $\mathbf{V}$
For $\mathbf{C} \in \mathbf{W}$

$$T^{-1}(\mathbf{C}) = \mathbf{B}^{-1}\mathbf{C} = \mathbf{A} \tag{2.1.1}$$
$$\mathbf{C} = \mathbf{BA} \tag{2.1.2}$$

Consider the following

$$T^{-1}(\mathbf{C}) = \mathbf{B}^{-1}(\mathbf{BA}) = \mathbf{A} \tag{2.1.3}$$
$$\implies \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}_{m\times m} \tag{2.1.4}$$
$$T(\mathbf{A}) = \mathbf{B}(\mathbf{B}^{-1}\mathbf{C}) = \mathbf{C} \tag{2.1.5}$$
$$\implies \mathbf{BB}^{-1} = \mathbf{I}_{p\times p} \tag{2.1.6}$$

where $\mathbf{I}$ is the identity matrix.
But

$$\mathbf{BB}^{-1} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I} \tag{2.1.7}$$

So,from (2.1.4),(2.1.6),(2.1.7)

$$p = m \tag{2.1.8}$$

So,$\mathbf{B}$ is an invertible $m \times m$ matrix

2.2 Case 2

Consider $p = m$ and $\mathbf{B}$ is an invertible $m \times m$ matrix.
Verifying if T is onto,
Let the set of matrices $\{\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_{mn}\}$ be the basis for $\mathbf{V}$
Any matrix $\mathbf{A} \in \mathbf{V}$ can be written as

$$\mathbf{A} = \sum_{i=1}^{mn} \alpha_i \mathbf{A}_i \tag{2.2.1}$$

where $\alpha_i \in F$
The set $\mathbf{M} = \{\mathbf{BA}_1, \mathbf{BA}_2, \dots, \mathbf{BA}_{mn}\}$ lie in $\mathbf{W}$

$$c_1(\mathbf{BA}_1) + c_2(\mathbf{BA}_2) + \dots + c_{mn}(\mathbf{BA}_{mn}) = \mathbf{0} \tag{2.2.2}$$
$$\implies \mathbf{B}(c_1\mathbf{A}_1 + c_2\mathbf{A}_2 + \dots + c_{mn}\mathbf{A}_{mn}) = \mathbf{0} \tag{2.2.3}$$

Since $\mathbf{B}$ is non-singular,

$$(c_1\mathbf{A}_1 + c_2\mathbf{A}_2 + \ldots + c_{mn}\mathbf{A}_{mn}) = \mathbf{0} \tag{2.2.4}$$
$$\implies c_1, c_2, \ldots, c_{mn} = 0 \tag{2.2.5}$$

because $\{\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_{mn}\}$ are linearly independent
So,$\mathbf{M}$ forms basis for $\mathbf{W}$
Any vector $\mathbf{C} \in \mathbf{W}$ can be written as

$$\mathbf{C} = \sum_{i=1}^{mn} \beta_i \mathbf{B}\mathbf{A}_i \text{ where } \beta_i \in F \tag{2.2.6}$$
$$= \mathbf{B}(\sum_{i=1}^{mn} \beta_i \mathbf{A}_i) \tag{2.2.7}$$
$$= \mathbf{B}\mathbf{A} \text{ (from (2.2.1))} \tag{2.2.8}$$

So,range($\mathbf{B}$)=$\mathbf{W}$
Consider the matrix $\mathbf{A}, \mathbf{A}' \in \mathbf{V}$ such that

$$\mathbf{B}\mathbf{A} = \mathbf{B}\mathbf{A}' \tag{2.2.9}$$
$$\mathbf{B}^{-1}(\mathbf{B}\mathbf{A}) = \mathbf{B}^{-1}(\mathbf{B}\mathbf{A}') \tag{2.2.10}$$
$$(\mathbf{B}^{-1}\mathbf{B})\mathbf{A} = (\mathbf{B}^{-1}\mathbf{B})\mathbf{A}' \tag{2.2.11}$$
$$\implies \mathbf{A} = \mathbf{A}' \tag{2.2.12}$$

So, T is invertible.

2.3 Conclusion

From case 1,case 2 T is invertible if and only if $p = m$ and $\mathbf{B}$ is an invertible $m \times m$ matrix.

2.4 Example

Let $p = m = 3, n = 4$ Let $T : \mathbf{V} \to \mathbf{W}$ adds row 2 to row 3 for a matrix $\mathbf{A} \in \mathbf{V}$
The elementary matrix that performs this is

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \tag{2.4.1}$$

$$\text{Let } \mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 4 & 9 & 2 & 6 \end{pmatrix} \tag{2.4.2}$$
$$T(\mathbf{A}) = \mathbf{B}\mathbf{A} \tag{2.4.3}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 4 & 9 & 2 & 6 \end{pmatrix} \tag{2.4.4}$$
$$= \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 5 & 12 & 8 & 13 \end{pmatrix} \tag{2.4.5}$$
$$= \mathbf{C} \in \mathbf{W} \tag{2.4.6}$$

Let transformation $T_1 : \mathbf{W} \to \mathbf{V}$ subtracts row2 from row 3 for a matrix $\mathbf{C} \in \mathbf{W}$ and is performed by elementary matrix

$$\mathbf{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \tag{2.4.7}$$
$$\text{Let } \mathbf{C} = \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 5 & 12 & 8 & 13 \end{pmatrix} \tag{2.4.8}$$
$$T_1(\mathbf{C}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 5 & 12 & 8 & 13 \end{pmatrix} \tag{2.4.9}$$
$$= \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 4 & 9 & 2 & 6 \end{pmatrix} \tag{2.4.10}$$
$$= \mathbf{A} \tag{2.4.11}$$
$$\implies T_1(\mathbf{C}) = \mathbf{A} \tag{2.4.12}$$
$$T_1(T(\mathbf{A})) = \mathbf{A} \tag{2.4.13}$$
$$\text{and } T(\mathbf{A}) = \mathbf{C} \tag{2.4.14}$$
$$\implies T(T_1(\mathbf{C})) = \mathbf{C} \tag{2.4.15}$$

So,T_1 is the inverse transformation of T and

$$T_1 = T^{-1} \tag{2.4.16}$$
$$\text{Also, } \mathbf{U}\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \tag{2.4.17}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.4.18}$$
$$\mathbf{B}\mathbf{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \tag{2.4.19}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.4.20}$$
$$\implies \mathbf{B}^{-1} = \mathbf{U} \tag{2.4.21}$$

So, T is invertible and ,$\mathbf{B}$ is an invertible 3×3 matrix.