Assignment 14

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Download codes from

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https://github.com/KUSUMAPRIYAPULAVARTY/
    assignment14
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1 Question

Let p, m, n be positive integers and F a field.Let $\mathbf{V}$ be the space of $m \times n$ matrices over F and $\mathbf{W}$ the space of $p \times n$ matrices over F.Let $\mathbf{B}$ be a fixed $p \times m$ matrix and let T be the linear transformation from $\mathbf{V}$ into $\mathbf{W}$ defined by $T(\mathbf{A}) = \mathbf{BA}$.Prove that T is invertible if and only if $p = m$ and $\mathbf{B}$ is an invertible $m \times m$ matrix.

2 Solution

$$T(\mathbf{A}) = \mathbf{BA} \tag{2.0.1}$$

So, $\mathbf{B}$ is the transformation matrix.
Transformation matrix is invertible if

1) T is one to one mapping,that is for a matrix $\mathbf{A} \in \mathbf{V}$ there exists a unique $\mathbf{C} \in \mathbf{W}$ such that
$$T(\mathbf{A}) = \mathbf{BA} = \mathbf{C} \tag{2.0.2}$$
2) T must be onto, that is range($\mathbf{B}$)=$\mathbf{W}$ that is,
$$\text{rank}(\mathbf{B}) = p \tag{2.0.3}$$

Let us assume that T is invertible. Therefore,there exists an operator U from $\mathbf{W}$ to $\mathbf{V}$ such that

$$U(\mathbf{C}) = \mathbf{UC} = \mathbf{A} \tag{2.0.4}$$

where $\mathbf{U}_{m \times p}$ is the inverse transformation matrix.
If $\mathbf{B}$ is one to one, onto then from (2.0.2) and (2.0.4) $\mathbf{U}$ is also one to one which implies range($\mathbf{U}$) is $\mathbf{V}$

$$\text{rank}(\mathbf{U}) = m \tag{2.0.5}$$

But the relation between transformation matrix and its inverse is

$$\mathbf{U} = \mathbf{B}^{-1} \tag{2.0.6}$$

which means that the rank of $\mathbf{B}, \mathbf{U}$ must be same.
So,from (2.0.3),(2.0.5) we can write

$$p = m \tag{2.0.7}$$

So, if T is invertible linear transformation, then matrix $\mathbf{B}$ is invertible and $p = m$

Now,consider it the other way.We consider $p = m$ and $\mathbf{B}$ is an invertible $m \times m$ matrix and wish to prove that T is an invertible transformation. Since $\mathbf{B}$ is invertible, rank of $\mathbf{B}$ is m so T is onto. To prove T is one to one, consider the matrix $\mathbf{A} \in \mathbf{V}$.Since $\mathbf{B}^{-1}$ exists, we can write

$$\mathbf{B}^{-1}(T(\mathbf{A})) = \mathbf{B}^{-1}(\mathbf{BA}) \tag{2.0.8}$$
$$= (\mathbf{B}^{-1}\mathbf{B})\mathbf{A} \tag{2.0.9}$$
$$= \mathbf{A} \tag{2.0.10}$$

So, $\mathbf{B}^{-1}$ is the transformation matrix that maps every $T(\mathbf{A}) \in \mathbf{W}$ to $\mathbf{A} \in \mathbf{V}$
So, T is invertible.
Hence,from both the cases of consideration, T is invertible if and only if $p = m$ and $\mathbf{B}$ is an invertible $m \times m$ matrix.