Assignment 14

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Download codes from

https://github.com/KUSUMAPRIYAPULAVARTY/assignment14

1 Question

Let p, m, n be positive integers and F a field.Let $\mathbf{V}$ be the space of $m \times n$ matrices over F and $\mathbf{W}$ the space of $p \times n$ matrices over F.Let $\mathbf{B}$ be a fixed $p \times m$ matrix and let T be the linear transformation from $\mathbf{V}$ into $\mathbf{W}$ defined by $T(\mathbf{A}) = \mathbf{BA}$.Prove that T is invertible if and only if $p = m$ and $\mathbf{B}$ is an invertible $m \times m$ matrix.

2 Solution

$$T(\mathbf{A}) = \mathbf{BA} \tag{2.0.1}$$

So, $\mathbf{B}$ is the transformation matrix.
Transformation matrix is invertible if

1) T is one to one mapping,that is for a matrix $\mathbf{A} \in \mathbf{V}$ there exists a unique $\mathbf{C} \in \mathbf{W}$ such that
$$T(\mathbf{A}) = \mathbf{BA} = \mathbf{C} \tag{2.0.2}$$
2) T must be onto, that is range($\mathbf{B}$)=$\mathbf{W}$

2.1 Case 1

Let us assume that T is invertible.
Let inverse of T is U from $\mathbf{W}$ to $\mathbf{V}$

$$U(\mathbf{C}) = \mathbf{UC} = \mathbf{A} \tag{2.1.1}$$

where $\mathbf{U}_{m\times p}$ is the inverse transformation matrix.

$$U(\mathbf{C}) = \mathbf{U}(\mathbf{BA}) = \mathbf{A} \tag{2.1.2}$$
$$\implies \mathbf{UB} = \mathbf{I}_{m\times m} \tag{2.1.3}$$
$$T(\mathbf{A}) = \mathbf{B}(\mathbf{UC}) = \mathbf{C} \tag{2.1.4}$$
$$\implies \mathbf{BU} = \mathbf{I}_{p\times p} \tag{2.1.5}$$

where $\mathbf{I}$ is the identity matrix.
Matrix $\mathbf{B}$ is invertible only if

$$\mathbf{BU} = \mathbf{UB} = \mathbf{I} \tag{2.1.6}$$

So,from (2.1.3),(2.1.5),(2.1.6)

$$p = m \tag{2.1.7}$$

So, if T is invertible linear transformation, then matrix $\mathbf{B}_{m\times m}$ is invertible and $p = m$

2.2 Case 2

Consider $p = m$ and $\mathbf{B}$ is an invertible $m \times m$ matrix.
Since $\mathbf{B}$ is invertible,

$$\text{rank}(\mathbf{B}) = m \tag{2.2.1}$$
$$\implies \text{range}(\mathbf{T}) = \mathbf{W} \tag{2.2.2}$$

Consider the matrix $\mathbf{A} \in \mathbf{V}$.Since $\mathbf{B}^{-1}$ exists, we can write

$$\mathbf{B}^{-1}(T(\mathbf{A})) = \mathbf{B}^{-1}(\mathbf{BA}) \tag{2.2.3}$$
$$= (\mathbf{B}^{-1}\mathbf{B})\mathbf{A} \tag{2.2.4}$$
$$= \mathbf{A} \tag{2.2.5}$$

So, every $T(\mathbf{A}) \in \mathbf{W}$ is uniquely transformed to $\mathbf{A} \in \mathbf{V}$ by $\mathbf{B}^{-1}$
So, T is invertible.

2.3 Conclusion

From case 1,case 2 T is invertible if and only if $p = m$ and $\mathbf{B}$ is an invertible $m \times m$ matrix.