Assignment 14

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https://github.com/KUSUMAPRIYAPULAVARTY/assignment14

1 Question

Let p, m, n be positive integers and F a field.Let $\mathbf{V}$ be the space of $m \times n$ matrices over F and $\mathbf{W}$ the space of $p \times n$ matrices over F.Let $\mathbf{B}$ be a fixed $p \times m$ matrix and let T be the linear transformation from $\mathbf{V}$ into $\mathbf{W}$ defined by $T(\mathbf{A}) = \mathbf{BA}$.Prove that T is invertible if and only if $p = m$ and $\mathbf{B}$ is an invertible $m \times m$ matrix.

2 Solution

Parameter	**Description**
p, m, n	Positive integers
F	Field
$\mathbf{V}$	Space of $m \times n$ matrices over F
$\mathbf{W}$	Space of $p \times n$ matrices over F
$\mathbf{B}$	Fixed $p \times m$ matrix
Linear transformation $T : \mathbf{V} \to \mathbf{W}$	$T(\mathbf{A}) = \mathbf{BA}$

TABLE 0: Input Parameters

$$T(\mathbf{A}) = \mathbf{BA} \tag{2.0.1}$$

So, $\mathbf{B}$ is the transformation matrix.
$\mathbf{B}$ is invertible if

1) T is one to one mapping,that is

$$\mathbf{BA} = \mathbf{BA}' \tag{2.0.2}$$
$$\implies \mathbf{A} = \mathbf{A}' \tag{2.0.3}$$

2) T must be onto, that is range($\mathbf{B}$)=$\mathbf{W}$

2.1 Case 1

Let us assume that T is invertible with inverse transformation T_1 from $\mathbf{W}$ to $\mathbf{V}$ that satisfies

$$T(\mathbf{A}) = \mathbf{BA} \in \mathbf{W} \tag{2.1.1}$$
$$\implies T_1(\mathbf{BA}) = \mathbf{A} \in \mathbf{V} \tag{2.1.2}$$
$$\dim(\mathbf{V}) = mn, \dim(\mathbf{W}) = pn \tag{2.1.3}$$

Since T is one-one mapping, the zero vector in $\mathbf{V}, \mathbf{0}_{m\times n}$ is uniquely mapped to

$$T(\mathbf{0}_{m\times n}) = \mathbf{B0}_{m\times n} = \mathbf{0}_{p\times n} \tag{2.1.4}$$
$$\text{So, } \mathbf{BA} = \mathbf{0} \iff \mathbf{A} = \mathbf{0} \tag{2.1.5}$$

Let $\{\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_{mn}\}$ be the basis for $\mathbf{V}$

$$c_1\mathbf{V}_1 + c_2\mathbf{V}_2 + \dots + c_{mn}\mathbf{V}_{mn} = \mathbf{0} \tag{2.1.6}$$
$$\iff c_1, c_2, \dots, c_{mn} \in F = 0 \tag{2.1.7}$$

Any matrix $\mathbf{A} \in \mathbf{V}$ can be written as

$$\mathbf{A} = \sum_{i=1}^{mn} \alpha_i \mathbf{V}_i \tag{2.1.8}$$

Since T is onto, any matrix $\mathbf{C} \in \mathbf{W}$ can be expressed as

$$\mathbf{C} = \mathbf{B}\left(\sum_{i=1}^{mn} \alpha_i \mathbf{V}_i\right) \tag{2.1.9}$$
$$= \sum_{i=1}^{mn} \alpha_i (\mathbf{BV}_i) \tag{2.1.10}$$

So, the set $\mathbf{S} = \{\mathbf{BV}_1, \mathbf{BV}_2, \dots, \mathbf{BV}_{mn}\}$ forms basis of $\mathbf{W}$ if all matrices in it are linearly independent.

$$c_1(\mathbf{BV}_1) + c_2(\mathbf{BV}_2) + \dots + c_{mn}(\mathbf{BV}_{mn}) = \mathbf{0} \tag{2.1.11}$$
$$\mathbf{B}(c_1\mathbf{V}_1 + c_2\mathbf{V}_2 + \dots + c_{mn}\mathbf{V}_{mn}) = \mathbf{0} \tag{2.1.12}$$
$$(2.1.5) \implies c_1\mathbf{V}_1 + \dots + c_{mn}\mathbf{V}_{mn} = 0 \tag{2.1.13}$$
$$\iff c_1, c_2, \dots, c_{mn} = 0 (\text{from } (2.1.7)) \tag{2.1.14}$$

So,the set $\mathbf{S}$ with cardinality mn is basis for $\mathbf{W}$

$$(2.1.3) \implies pn = mn \tag{2.1.15}$$

$$p = m \tag{2.1.16}$$

(2.1.5),(2.1.16) prove that $\mathbf{B}$ is invertible $m \times m$ matrix.

2.2 Case 2

Consider $p = m$ and $\mathbf{B}$ is an invertible $m \times m$ matrix.
Verifying if T is onto,
Let the set of matrices $\{\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_{mn}\}$ be the basis for $\mathbf{V}$
Any matrix $\mathbf{A} \in \mathbf{V}$ can be written as

$$\mathbf{A} = \sum_{i=1}^{mn} \alpha_i \mathbf{A}_i \tag{2.2.1}$$

where $\alpha_i \in F$
The set $\mathbf{M} = \{\mathbf{BA}_1, \mathbf{BA}_2, \ldots, \mathbf{BA}_{mn}\}$ lie in $\mathbf{W}$

$$c_1(\mathbf{BA}_1) + c_2(\mathbf{BA}_2) + \ldots + c_{mn}(\mathbf{BA}_{mn}) = \mathbf{0} \tag{2.2.2}$$

$$\implies \mathbf{B}(c_1\mathbf{A}_1 + c_2\mathbf{A}_2 + \ldots + c_{mn}\mathbf{A}_{mn}) = \mathbf{0} \tag{2.2.3}$$

Since $\mathbf{B}$ is non-singular,

$$(c_1\mathbf{A}_1 + c_2\mathbf{A}_2 + \ldots + c_{mn}\mathbf{A}_{mn}) = \mathbf{0} \tag{2.2.4}$$

$$\iff c_1, c_2, \ldots, c_{mn} = 0 \tag{2.2.5}$$

because $\{\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_{mn}\}$ are linearly independent
So,$\mathbf{M}$ forms basis for $\mathbf{W}$
Any vector $\mathbf{C} \in \mathbf{W}$ can be written as

$$\mathbf{C} = \sum_{i=1}^{mn} \beta_i \mathbf{BA}_i \text{ where } \beta_i \in F \tag{2.2.6}$$

$$= \mathbf{B}(\sum_{i=1}^{mn} \beta_i \mathbf{A}_i) \tag{2.2.7}$$

$$= \mathbf{BA} \text{ (from (2.2.1))} \tag{2.2.8}$$

So,range($\mathbf{B}$)=$\mathbf{W}$
Consider the matrix $\mathbf{A}, \mathbf{A}' \in \mathbf{V}$ such that

$$\mathbf{BA} = \mathbf{BA}' \tag{2.2.9}$$

$$\mathbf{B}^{-1}(\mathbf{BA}) = \mathbf{B}^{-1}(\mathbf{BA}') \tag{2.2.10}$$

$$(\mathbf{B}^{-1}\mathbf{B})\mathbf{A} = (\mathbf{B}^{-1}\mathbf{B})\mathbf{A}' \tag{2.2.11}$$

$$\implies \mathbf{A} = \mathbf{A}' \tag{2.2.12}$$

So, T is invertible.

2.3 Conclusion

From case 1,case 2 T is invertible if and only if $p = m$ and $\mathbf{B}$ is an invertible $m \times m$ matrix.

2.4 Example

Let $p = m = 3, n = 4$ Let $T : \mathbf{V} \to \mathbf{W}$ adds row 2 to row 3 for a matrix $\mathbf{A} \in \mathbf{V}$
The elementary matrix that performs this is

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \tag{2.4.1}$$

$$\text{Let } \mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 4 & 9 & 2 & 6 \end{pmatrix} \tag{2.4.2}$$

$$T(\mathbf{A}) = \mathbf{BA} \tag{2.4.3}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 4 & 9 & 2 & 6 \end{pmatrix} \tag{2.4.4}$$

$$= \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 5 & 12 & 8 & 13 \end{pmatrix} \tag{2.4.5}$$

$$= \mathbf{C} \in \mathbf{W} \tag{2.4.6}$$

Let transformation $T_1 : \mathbf{W} \to \mathbf{V}$ subtracts row2 from row 3 for a matrix $\mathbf{C} \in \mathbf{W}$ and is performed by elementary matrix

$$\mathbf{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \tag{2.4.7}$$

$$\text{Let } \mathbf{C} = \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 5 & 12 & 8 & 13 \end{pmatrix} \tag{2.4.8}$$

$$T_1(\mathbf{C}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 5 & 12 & 8 & 13 \end{pmatrix} \tag{2.4.9}$$

$$= \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 4 & 9 & 2 & 6 \end{pmatrix} \tag{2.4.10}$$

$$= \mathbf{A} \tag{2.4.11}$$

$$\implies T_1(\mathbf{C}) = \mathbf{A} \tag{2.4.12}$$

$$T_1(T(\mathbf{A})) = \mathbf{A} \tag{2.4.13}$$

$$\text{and } T(\mathbf{A}) = \mathbf{C} \tag{2.4.14}$$

$$\implies T(T_1(\mathbf{C})) = \mathbf{C} \tag{2.4.15}$$

So,T_1 is the inverse transformation of T and

$$T_1 = T^{-1} \tag{2.4.16}$$

$$\mathbf{UB} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & -1 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 1 & 1\end{pmatrix} \tag{2.4.17}$$

$$= \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} \tag{2.4.18}$$

$$\mathbf{BU} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & -1 & 1\end{pmatrix} \tag{2.4.19}$$

$$= \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} \tag{2.4.20}$$

$$\implies \mathbf{B}^{-1} = \mathbf{U} \tag{2.4.21}$$

So, T is invertible and ,$\mathbf{B}$ is an invertible 3×3 matrix.