Assignment 14

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Download codes from

https://github.com/KUSUMAPRIYAPULAVARTY/assignment14

1 QUESTION

Let p, m, n be positive integers and F a field.Let $\mathbf{V}$ be the space of $m \times n$ matrices over F and $\mathbf{W}$ the space of $p \times n$ matrices over F.Let $\mathbf{B}$ be a fixed $p \times m$ matrix and let T be the linear transformation from $\mathbf{V}$ into $\mathbf{W}$ defined by $T(\mathbf{A}) = \mathbf{BA}$.Prove that T is invertible if and only if $p = m$ and $\mathbf{B}$ is an invertible $m \times m$ matrix.

2 Solution

$$T(\mathbf{A}) = \mathbf{BA} \tag{2.0.1}$$

So, $\mathbf{B}$ is the transformation matrix.
$\mathbf{B}$ is invertible if

1) T is one to one mapping,that is

$$\mathbf{BA} = \mathbf{BA'} \tag{2.0.2}$$
$$\implies \mathbf{A} = \mathbf{A'} \tag{2.0.3}$$

2) T must be onto, that is range($\mathbf{B}$)=$\mathbf{W}$

2.1 Case 1

Let us assume that T is invertible with T^{-1} from $\mathbf{W}$ to $\mathbf{V}$
For $\mathbf{C} \in \mathbf{W}$

$$T^{-1}(\mathbf{C}) = \mathbf{B}^{-1}\mathbf{C} = \mathbf{A} \tag{2.1.1}$$
$$\mathbf{C} = \mathbf{BA} \tag{2.1.2}$$

Consider the following

$$T^{-1}(\mathbf{C}) = \mathbf{B}^{-1}(\mathbf{BA}) = \mathbf{A} \tag{2.1.3}$$
$$\implies \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}_{m\times m} \tag{2.1.4}$$
$$T(\mathbf{A}) = \mathbf{B}(\mathbf{B}^{-1}\mathbf{C}) = \mathbf{C} \tag{2.1.5}$$
$$\implies \mathbf{BB}^{-1} = \mathbf{I}_{p\times p} \tag{2.1.6}$$

where $\mathbf{I}$ is the identity matrix.
But

$$\mathbf{BB}^{-1} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I} \tag{2.1.7}$$

So,from (2.1.4),(2.1.6),(2.1.7)

$$p = m \tag{2.1.8}$$

So,$\mathbf{B}$ is an invertible $m \times m$ matrix

2.2 Case 2

Consider $p = m$ and $\mathbf{B}$ is an invertible $m \times m$ matrix.
Since $\mathbf{B}$ is invertible,

$$\text{rank}(\mathbf{B}) = m \tag{2.2.1}$$
$$\implies \text{range}(\mathbf{B}) = \mathbf{W} \tag{2.2.2}$$

Consider the matrix $\mathbf{A}, \mathbf{A'} \in \mathbf{V}$ such that

$$\mathbf{BA} = \mathbf{BA'} \tag{2.2.3}$$
$$\mathbf{B}^{-1}(\mathbf{BA}) = \mathbf{B}^{-1}(\mathbf{BA'}) \tag{2.2.4}$$
$$(\mathbf{B}^{-1}\mathbf{B})\mathbf{A} = (\mathbf{B}^{-1}\mathbf{B})\mathbf{A'} \tag{2.2.5}$$
$$\implies \mathbf{A} = \mathbf{A}^{1} \tag{2.2.6}$$

So, T is invertible.

2.3 Conclusion

From case 1,case 2 T is invertible if and only if $p = m$ and $\mathbf{B}$ is an invertible $m \times m$ matrix.