

Assignment 14

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Download codes from

<https://github.com/KUSUMAPRIYAPULAVARTY/assignment14>

1 QUESTION

Let p, m, n be positive integers and F a field. Let \mathbf{V} be the space of $m \times n$ matrices over F and \mathbf{W} the space of $p \times n$ matrices over F . Let \mathbf{B} be a fixed $p \times m$ matrix and let T be the linear transformation from \mathbf{V} into \mathbf{W} defined by $T(\mathbf{A}) = \mathbf{BA}$. Prove that T is invertible if and only if $p = m$ and \mathbf{B} is an invertible $m \times m$ matrix.

2 SOLUTION

Parameter	Description
p, m, n	Positive integers
F	Field
\mathbf{V}	Space of $m \times n$ matrices over F
\mathbf{W}	Space of $p \times n$ matrices over F
\mathbf{B}	Fixed $p \times m$ matrix
Linear transformation $T : \mathbf{V} \rightarrow \mathbf{W}$	$T(\mathbf{A}) = \mathbf{BA}$

TABLE 0: Input Parameters

$$T(\mathbf{A}) = \mathbf{BA} \quad (2.0.1)$$

So, \mathbf{B} is the transformation matrix.
 \mathbf{B} is invertible if

1) T is one to one mapping, that is

$$\mathbf{BA} = \mathbf{BA}' \quad (2.0.2)$$

$$\Rightarrow \mathbf{A} = \mathbf{A}' \quad (2.0.3)$$

2) T must be onto, that is $\text{range}(\mathbf{B}) = \mathbf{W}$

2.1 Case 1

Let us assume that T is invertible with T^{-1} from \mathbf{W} to \mathbf{V}

For $\mathbf{C} \in \mathbf{W}$

$$T^{-1}(\mathbf{C}) = \mathbf{B}^{-1}\mathbf{C} = \mathbf{A} \quad (2.1.1)$$

$$\mathbf{C} = \mathbf{BA} \quad (2.1.2)$$

Consider the following

$$T^{-1}(\mathbf{C}) = \mathbf{B}^{-1}(\mathbf{BA}) = \mathbf{A} \quad (2.1.3)$$

$$\Rightarrow \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}_{m \times m} \quad (2.1.4)$$

$$T(\mathbf{A}) = \mathbf{B}(\mathbf{B}^{-1}\mathbf{C}) = \mathbf{C} \quad (2.1.5)$$

$$\Rightarrow \mathbf{BB}^{-1} = \mathbf{I}_{p \times p} \quad (2.1.6)$$

where \mathbf{I} is the identity matrix.

But

$$\mathbf{BB}^{-1} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I} \quad (2.1.7)$$

So, from (2.1.4), (2.1.6), (2.1.7)

$$p = m \quad (2.1.8)$$

So, \mathbf{B} is an invertible $m \times m$ matrix

2.2 Case 2

Consider $p = m$ and \mathbf{B} is an invertible $m \times m$ matrix.

Verifying if T is onto,

Let the set of matrices $\{\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_{mn}\}$ be the basis for \mathbf{V}

Any matrix $\mathbf{A} \in \mathbf{V}$ can be written as

$$\mathbf{A} = \sum_{i=1}^{mn} \alpha_i \mathbf{A}_i \quad (2.2.1)$$

where $\alpha_i \in F$

The set $\mathbf{M} = \{\mathbf{BA}_1, \mathbf{BA}_2, \dots, \mathbf{BA}_{mn}\}$ lie in \mathbf{W}

$$c_1(\mathbf{BA}_1) + c_2(\mathbf{BA}_2) + \dots + c_{mn}(\mathbf{BA}_{mn}) = \mathbf{0} \quad (2.2.2)$$

$$\Rightarrow \mathbf{B}(c_1\mathbf{A}_1 + c_2\mathbf{A}_2 + \dots + c_{mn}\mathbf{A}_{mn}) = \mathbf{0} \quad (2.2.3)$$

Since \mathbf{B} is non-singular,

$$(c_1\mathbf{A}_1 + c_2\mathbf{A}_2 + \dots + c_{mn}\mathbf{A}_{mn}) = \mathbf{0} \quad (2.2.4)$$

$$\implies c_1, c_2, \dots, c_{mn} = 0 \quad (2.2.5)$$

because $\{\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_{mn}\}$ are linearly independent
So, \mathbf{M} forms basis for \mathbf{W}

Any vector $\mathbf{C} \in \mathbf{W}$ can be written as

$$\mathbf{C} = \sum_{i=1}^{mn} \beta_i \mathbf{B}\mathbf{A}_i \text{ where } \beta_i \in F \quad (2.2.6)$$

$$= \mathbf{B} \left(\sum_{i=1}^{mn} \beta_i \mathbf{A}_i \right) \quad (2.2.7)$$

$$= \mathbf{B}\mathbf{A} \text{ (from (2.2.1))} \quad (2.2.8)$$

So, $\text{range}(\mathbf{B}) = \mathbf{W}$

Consider the matrix $\mathbf{A}, \mathbf{A}' \in \mathbf{V}$ such that

$$\mathbf{B}\mathbf{A} = \mathbf{B}\mathbf{A}' \quad (2.2.9)$$

$$\mathbf{B}^{-1}(\mathbf{B}\mathbf{A}) = \mathbf{B}^{-1}(\mathbf{B}\mathbf{A}') \quad (2.2.10)$$

$$(\mathbf{B}^{-1}\mathbf{B})\mathbf{A} = (\mathbf{B}^{-1}\mathbf{B})\mathbf{A}' \quad (2.2.11)$$

$$\implies \mathbf{A} = \mathbf{A}' \quad (2.2.12)$$

So, T is invertible.

2.3 Conclusion

From case 1, case 2 T is invertible if and only if $p = m$ and \mathbf{B} is an invertible $m \times m$ matrix.

2.4 Example

Let $p = m = 3, n = 4$ Let $T : \mathbf{V} \rightarrow \mathbf{W}$ adds row 2 to row 3 for a matrix $\mathbf{A} \in \mathbf{V}$

The elementary matrix that performs this is

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad (2.4.1)$$

$$\text{Let } \mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 4 & 9 & 2 & 6 \end{pmatrix} \quad (2.4.2)$$

$$T(\mathbf{A}) = \mathbf{B}\mathbf{A} \quad (2.4.3)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 4 & 9 & 2 & 6 \end{pmatrix} \quad (2.4.4)$$

$$= \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 5 & 12 & 8 & 13 \end{pmatrix} \quad (2.4.5)$$

The inverse transformation $T^{-1} : \mathbf{W} \rightarrow \mathbf{V}$ subtracts row 2 from row 3 for a matrix $\mathbf{C} \in \mathbf{W}$ and is performed by elementary matrix

$$\mathbf{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad (2.4.6)$$

$$\text{Let } \mathbf{C} = \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 5 & 12 & 8 & 13 \end{pmatrix} \quad (2.4.7)$$

$$T^{-1}(\mathbf{C}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 5 & 12 & 8 & 13 \end{pmatrix} \quad (2.4.8)$$

$$= \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 4 & 9 & 2 & 6 \end{pmatrix} \quad (2.4.9)$$

$$= \mathbf{A} \quad (2.4.10)$$

$$\text{Also, } \mathbf{U}\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad (2.4.11)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2.4.12)$$

$$\mathbf{B}\mathbf{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad (2.4.13)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2.4.14)$$

$$\implies \mathbf{B}^{-1} = \mathbf{U} \quad (2.4.15)$$

So, T is invertible and \mathbf{B} is an invertible 3×3 matrix.