

# Assignment 14

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Download codes from

<https://github.com/KUSUMAPRIYAPULAVARTY/assignment14>

## 2.1 Case 1

Let us assume that  $T$  is invertible with inverse transformation  $T_1$  from  $\mathbf{W}$  to  $\mathbf{V}$  that satisfies

$$T(\mathbf{A}) = \mathbf{BA} \in \mathbf{W} \quad (2.1.1)$$

$$\implies T_1(\mathbf{BA}) = \mathbf{A} \in \mathbf{V} \quad (2.1.2)$$

$$\dim(\mathbf{V}) = mn, \dim(\mathbf{W}) = pn \quad (2.1.3)$$

**1 QUESTION**  
Let  $p, m, n$  be positive integers and  $F$  a field. Let  $\mathbf{V}$  be the space of  $m \times n$  matrices over  $F$  and  $\mathbf{W}$  the space of  $p \times n$  matrices over  $F$ . Let  $\mathbf{B}$  be a fixed  $p \times m$  matrix and let  $T$  be the linear transformation from  $\mathbf{V}$  into  $\mathbf{W}$  defined by  $T(\mathbf{A}) = \mathbf{BA}$ . Prove that  $T$  is invertible if and only if  $p = m$  and  $\mathbf{B}$  is an invertible  $m \times m$  matrix.

Let  $\{\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_{mn}\}$  be the basis for  $\mathbf{V}$ . Any matrix  $\mathbf{A} \in \mathbf{V}$  can be written as

$$\mathbf{A} = \sum_{i=1}^{mn} \alpha_i \mathbf{V}_i \quad (2.1.4)$$

Since  $T$  is onto, any matrix  $\mathbf{C} \in \mathbf{W}$  can be expressed as

$$\mathbf{C} = \mathbf{B} \left( \sum_{i=1}^{mn} \alpha_i \mathbf{V}_i \right) \quad (2.1.5)$$

$$= \sum_{i=1}^{mn} \alpha_i (\mathbf{B}\mathbf{V}_i) \quad (2.1.6)$$

So,  $\{\mathbf{B}\mathbf{V}_1, \mathbf{B}\mathbf{V}_2, \dots, \mathbf{B}\mathbf{V}_{mn}\}$  forms basis of  $\mathbf{W}$

$$(2.1.3) \implies pn = mn \quad (2.1.7)$$

$$p = m \quad (2.1.8)$$

Since  $T$  is one-one mapping, the zero vector in  $\mathbf{V}$ ,  $\mathbf{0}_{m \times n}$  is uniquely mapped to

$$T(\mathbf{0}_{m \times n}) = \mathbf{B}\mathbf{0}_{m \times n} = \mathbf{0}_{m \times n} \quad (2.1.9)$$

$$\text{So, } \mathbf{BA} = \mathbf{0} \iff \mathbf{A} = \mathbf{0} \quad (2.1.10)$$

This proves that  $\mathbf{B}$  is invertible  $m \times m$  matrix.

$$T(\mathbf{A}) = \mathbf{BA} \quad (2.0.1)$$

So,  $\mathbf{B}$  is the transformation matrix.  
 $\mathbf{B}$  is invertible if

1)  $T$  is one to one mapping, that is

$$\mathbf{BA} = \mathbf{BA}' \quad (2.0.2)$$

$$\implies \mathbf{A} = \mathbf{A}' \quad (2.0.3)$$

2)  $T$  must be onto, that is  $\text{range}(\mathbf{B}) = \mathbf{W}$

## 2.2 Case 2

Consider  $p = m$  and  $\mathbf{B}$  is an invertible  $m \times m$  matrix.

Verifying if  $T$  is onto,

Let the set of matrices  $\{\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_{mn}\}$  be the basis for  $\mathbf{V}$

Any matrix  $\mathbf{A} \in \mathbf{V}$  can be written as

$$\mathbf{A} = \sum_{i=1}^{mn} \alpha_i \mathbf{A}_i \quad (2.2.1)$$

where  $\alpha_i \in F$

The set  $\mathbf{M} = \{\mathbf{BA}_1, \mathbf{BA}_2, \dots, \mathbf{BA}_{mn}\}$  lie in  $\mathbf{W}$

$$c_1(\mathbf{BA}_1) + c_2(\mathbf{BA}_2) + \dots + c_{mn}(\mathbf{BA}_{mn}) = \mathbf{0} \quad (2.2.2)$$

$$\implies \mathbf{B}(c_1\mathbf{A}_1 + c_2\mathbf{A}_2 + \dots + c_{mn}\mathbf{A}_{mn}) = \mathbf{0} \quad (2.2.3)$$

Since  $\mathbf{B}$  is non-singular,

$$(c_1\mathbf{A}_1 + c_2\mathbf{A}_2 + \dots + c_{mn}\mathbf{A}_{mn}) = \mathbf{0} \quad (2.2.4)$$

$$\implies c_1, c_2, \dots, c_{mn} = 0 \quad (2.2.5)$$

because  $\{\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_{mn}\}$  are linearly independent

So,  $\mathbf{M}$  forms basis for  $\mathbf{W}$

Any vector  $\mathbf{C} \in \mathbf{W}$  can be written as

$$\mathbf{C} = \sum_{i=1}^{mn} \beta_i \mathbf{BA}_i \text{ where } \beta_i \in F \quad (2.2.6)$$

$$= \mathbf{B} \left( \sum_{i=1}^{mn} \beta_i \mathbf{A}_i \right) \quad (2.2.7)$$

$$= \mathbf{BA} \text{ (from (2.2.1) )} \quad (2.2.8)$$

So,  $\text{range}(\mathbf{B}) = \mathbf{W}$

Consider the matrix  $\mathbf{A}, \mathbf{A}' \in \mathbf{V}$  such that

$$\mathbf{BA} = \mathbf{BA}' \quad (2.2.9)$$

$$\mathbf{B}^{-1}(\mathbf{BA}) = \mathbf{B}^{-1}(\mathbf{BA}') \quad (2.2.10)$$

$$(\mathbf{B}^{-1}\mathbf{B})\mathbf{A} = (\mathbf{B}^{-1}\mathbf{B})\mathbf{A}' \quad (2.2.11)$$

$$\implies \mathbf{A} = \mathbf{A}' \quad (2.2.12)$$

So,  $T$  is invertible.

### 2.3 Conclusion

From case 1, case 2  $T$  is invertible if and only if  $p = m$  and  $\mathbf{B}$  is an invertible  $m \times m$  matrix.

### 2.4 Example

Let  $p = m = 3, n = 4$  Let  $T : \mathbf{V} \rightarrow \mathbf{W}$  adds row 2 to row 3 for a matrix  $\mathbf{A} \in \mathbf{V}$

The elementary matrix that performs this is

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad (2.4.1)$$

$$\text{Let } \mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 4 & 9 & 2 & 6 \end{pmatrix} \quad (2.4.2)$$

$$T(\mathbf{A}) = \mathbf{BA} \quad (2.4.3)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 4 & 9 & 2 & 6 \end{pmatrix} \quad (2.4.4)$$

$$= \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 5 & 12 & 8 & 13 \end{pmatrix} \quad (2.4.5)$$

$$= \mathbf{C} \in \mathbf{W} \quad (2.4.6)$$

Let transformation  $T_1 : \mathbf{W} \rightarrow \mathbf{V}$  subtracts row 2 from row 3 for a matrix  $\mathbf{C} \in \mathbf{W}$  and is performed by elementary matrix

$$\mathbf{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad (2.4.7)$$

$$\text{Let } \mathbf{C} = \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 5 & 12 & 8 & 13 \end{pmatrix} \quad (2.4.8)$$

$$T_1(\mathbf{C}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 5 & 12 & 8 & 13 \end{pmatrix} \quad (2.4.9)$$

$$= \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 4 & 9 & 2 & 6 \end{pmatrix} \quad (2.4.10)$$

$$= \mathbf{A} \quad (2.4.11)$$

$$\implies T_1(\mathbf{C}) = \mathbf{A} \quad (2.4.12)$$

$$T_1(T(\mathbf{A})) = \mathbf{A} \quad (2.4.13)$$

$$\text{and } T(\mathbf{A}) = \mathbf{C} \quad (2.4.14)$$

$$\implies T(T_1(\mathbf{C})) = \mathbf{C} \quad (2.4.15)$$

So,  $T_1$  is the inverse transformation of  $T$  and

$$T_1 = T^{-1} \quad (2.4.16)$$

$$\mathbf{UB} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad (2.4.17)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2.4.18)$$

$$\mathbf{BU} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad (2.4.19)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2.4.20)$$

$$\implies \mathbf{B}^{-1} = \mathbf{U} \quad (2.4.21)$$

So,  $T$  is invertible and  $\mathbf{B}$  is an invertible  $3 \times 3$  matrix.