Assignment 15

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1 QUESTION

Let $\mathbf{V}$ be the set of complex numbers regarded as a vector space over the field of real numbers. We define a function T from $\mathbf{V}$ into the space of 2×2 real matrices, as follows. If $z = x + iy$ with x and y real numbers, then

$$T(z) = \begin{pmatrix} x + 7y & 5y \\ -10y & x - 7y \end{pmatrix} \qquad (1.0.1)$$

How would you describe the range of T?

2 SOLUTION

$$T : \mathbf{V} \to R^{2 \times 2} \qquad (2.0.1)$$

where $R^{2 \times 2}$, is the space of all 2×2 real matrices

$$T(z) = \begin{pmatrix} x + 7y & 5y \\ -10y & x - 7y \end{pmatrix} \qquad (2.0.2)$$

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad (2.0.3)$$

$$T(\mathbf{x}) = \begin{pmatrix} \begin{pmatrix} 1 & 7 \end{pmatrix}\mathbf{x} & \begin{pmatrix} 0 & 5 \end{pmatrix}\mathbf{x} \\ \begin{pmatrix} 0 & -10 \end{pmatrix}\mathbf{x} & \begin{pmatrix} 1 & -7 \end{pmatrix}\mathbf{x} \end{pmatrix} \qquad (2.0.4)$$

$$= \begin{pmatrix} \begin{pmatrix} 1 & 7 \\ 0 & -10 \end{pmatrix}\mathbf{x} & \begin{pmatrix} 0 & 5 \\ 1 & -7 \end{pmatrix}\mathbf{x} \end{pmatrix} \qquad (2.0.5)$$

$$\text{Let } \mathbf{A} = \begin{pmatrix} 1 & 7 \\ 0 & -10 \end{pmatrix} \qquad (2.0.6)$$

$$\mathbf{B} = \begin{pmatrix} 0 & 5 \\ 1 & -7 \end{pmatrix} \qquad (2.0.7)$$

$$\implies T(\mathbf{x}) = \begin{pmatrix} \mathbf{Ax} & \mathbf{Bx} \end{pmatrix} \qquad (2.0.8)$$

$$T(\mathbf{x}) = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix}\begin{pmatrix} \mathbf{x} & \mathbf{0}_{2 \times 1} \\ \mathbf{0}_{2 \times 1} & \mathbf{x} \end{pmatrix} \qquad (2.0.9)$$

The kronecker product of $\mathbf{I}, \mathbf{x}$ gives the block matrix

$$\mathbf{I}_{2 \times 2} \otimes \mathbf{x}_{2 \times 1} = \begin{pmatrix} \mathbf{x} & \mathbf{0} \\ \mathbf{0} & \mathbf{x} \end{pmatrix}_{4 \times 2} \qquad (2.0.10)$$

$$(2.0.9) \implies T(\mathbf{x}) = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix}\mathbf{I} \otimes \mathbf{x} \qquad (2.0.11)$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix}\mathbf{I} \otimes \left[\begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \end{pmatrix} \right] \qquad (2.0.12)$$

$$= x\begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix}\mathbf{I} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix}\mathbf{I} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (2.0.13)$$

$$\mathbf{I} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad (2.0.14)$$

$$\mathbf{I} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad (2.0.15)$$

Kronecker product in the first term of (2.0.13) picks out 1st columns of $\mathbf{A}, \mathbf{B}$ and in the second term picks out 2nd columns of $\mathbf{A}, \mathbf{B}$ so

$$\text{range}(T) = \text{span of } \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 5 \\ -10 & -7 \end{pmatrix} \right\} \qquad (2.0.16)$$