Assignment 17

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Download codes from

https://github.com/KUSUMAPRIYAPULAVARTY/assignment17

1 Question

Let $\mathbb{F}$ be a field of characteristic zero and let $\mathbf{V}$ be a finite dimensional vector space over $\mathbb{F}$.If $\alpha_1, \alpha_2, \dots, \alpha_m$ are finitely many vectors in $\mathbf{V}$, each different from the zero vector, prove that there is a linear functional f on $\mathbf{V}$ such that

$$f(\alpha_i) \neq 0, i = 1, 2, \dots, m \tag{1.0.1}$$

2 Theorem

We will make use of the following theorem
Let $\mathbf{V}$ be a finite dimensional vector space over the field $\mathbb{F}$ and let $\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}\}$ be a basis for $\mathbf{V}$.Then there exists a unique dual basis $\{f_1, f_2, \dots, f_n\}$ for $\mathbf{V}^*$ such that

$$f_i(\mathbf{v}_j) = \delta_{ij} \tag{2.0.1}$$

and $\mathbf{V}^*$ is the space of all linear functionals on $\mathbf{V}$

3 Solution

Let

$$\mathbf{B} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{pmatrix} \tag{3.0.1}$$

$$\mathbf{B}^* = \begin{pmatrix} \mathbf{f}_1 & \mathbf{f}_2 & \dots & \mathbf{f}_n \end{pmatrix} \tag{3.0.2}$$

Rewriting (2.0.1)

$$(\mathbf{f}_i)^T \mathbf{v}_j = \delta_{ij} \tag{3.0.3}$$

$$\therefore \text{ from (2.0.1) } (\mathbf{B}^*)^T \mathbf{B} = \mathbf{I} \tag{3.0.4}$$

$$\alpha_i = \sum_{k=1}^{n} a_k \mathbf{v}_k \tag{3.0.5}$$

$$= \mathbf{Ba} \tag{3.0.6}$$

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \tag{3.0.7}$$

Since α_i vectors are non-zeros, there exists atleast one non zero $a_l \in \mathbb{F}$ where $l \in [1, n]$

$$\implies \mathbf{a} \neq \mathbf{0} \tag{3.0.8}$$

Any linear functional $f : \mathbf{V} \to \mathbb{F}$ can be expressed as

$$\mathbf{f} = \mathbf{B}^* \mathbf{c} \tag{3.0.9}$$

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, c_i \in \mathbb{F} \text{ where } i \in [1, n] \tag{3.0.10}$$

Consider

$$f(\alpha_i) = \mathbf{f}^T \alpha_i \tag{3.0.11}$$

$$= (\mathbf{B}^* \mathbf{c})^T \alpha_i \tag{3.0.12}$$

$$= \mathbf{c}^T (\mathbf{B}^*)^T \mathbf{Ba} \tag{3.0.13}$$

$$= \mathbf{c}^T \mathbf{a} \tag{3.0.14}$$

A linear functional with $c_i \neq 0 \forall i \in [1, n]$ always lies in $\mathbf{V}^*$. And for $\mathbf{a} \neq \mathbf{0}$,

$$f(\alpha_i) = \mathbf{c}^T \mathbf{a} \neq 0, i = 1, 2, \dots, m \tag{3.0.15}$$