Assignment 20

KUSUMA PRIYA
EE20MTECH11007

Download codes from

https://github.com/KUSUMAPRIYAPULAVARTY/assignment20

1 Question

For every 4×4 real symmetric non-singular matrix $\mathbf{A}$ there exists a positive integer p such that

1) $p\mathbf{I}+\mathbf{A}$ is positive definite
2) $\mathbf{A}^p$ is positive definite
3) $\mathbf{A}^{-p}$ is positive definite
4) $\exp(p\mathbf{A})-\mathbf{I}$ is positive definite

2 Theory

A matrix is real symmetric implies its eigen values are real and eigen vectors are orthogonal,that is its eigen value decomposition is

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{2.0.1}$$

$\mathbf{D}$ is the diagonal matrix containing the real eigen values of $\mathbf{A}$
$\mathbf{P}$ has the corresponding eigen vectors

$$\mathbf{P}\mathbf{P}^T = \mathbf{P}^T\mathbf{P} = \mathbf{I} \tag{2.0.2}$$

A real matrix is positive definite if

$$\mathbf{x}^T\mathbf{A}\mathbf{x} > 0 \tag{2.0.3}$$
$$\implies \mathbf{x}^T\lambda\mathbf{x} > 0 \tag{2.0.4}$$
$$\implies \lambda\mathbf{x}^T\mathbf{x} > 0 \tag{2.0.5}$$
$$\implies \lambda > 0 \tag{2.0.6}$$

In other words, all the eigen values of A are positive

3 Solution

Let $\mathbf{A}$ be

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{3.0.1}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} \tag{3.0.2}$$

3.1 Choice 1

$$p\mathbf{I}+\mathbf{A} = \mathbf{P}(p\mathbf{I})\mathbf{P}^T + \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{3.1.1}$$
$$= \mathbf{P}\mathbf{D}_1\mathbf{P}^T \tag{3.1.2}$$

$$\mathbf{D}_1 = \begin{pmatrix} \lambda_1+p & 0 & 0 & 0 \\ 0 & \lambda_2+p & 0 & 0 \\ 0 & 0 & \lambda_3+p & 0 \\ 0 & 0 & 0 & \lambda_4+p \end{pmatrix} \tag{3.1.3}$$

Some of the eigen values of $\mathbf{A}$ may be negative. All the eigen values in $\mathbf{D}_1$ are positive only if

$$p > |\lambda_i| \ \forall i \in [1,4] \tag{3.1.4}$$

3.2 Choice 2

$$\mathbf{A}^2 = \mathbf{A}\mathbf{A} \tag{3.2.1}$$
$$= (\mathbf{P}\mathbf{D}\mathbf{P}^T)(\mathbf{P}\mathbf{D}\mathbf{P}^T) \tag{3.2.2}$$
$$= \mathbf{P}\mathbf{D}^2\mathbf{P}^T \tag{3.2.3}$$
$$\text{Similarly, } \mathbf{A}^p = \mathbf{P}\mathbf{D}^p\mathbf{P}^T \tag{3.2.4}$$

$$\mathbf{D}^p = \begin{pmatrix} \lambda_1^p & 0 & 0 & 0 \\ 0 & \lambda_2^p & 0 & 0 \\ 0 & 0 & \lambda_3^p & 0 \\ 0 & 0 & 0 & \lambda_4^p \end{pmatrix} \tag{3.2.5}$$

Some of the eigen values of $\mathbf{A}$ may be negative and $\mathbf{A}^p$ is positive definite only if p is even.

3.3 Choice 3

$$\mathbf{A}^{-p} = \mathbf{P}\mathbf{D}^{-p}\mathbf{P}^T \tag{3.3.1}$$

$$\mathbf{D}^{-p} = \begin{pmatrix} \lambda_1^{-p} & 0 & 0 & 0 \\ 0 & \lambda_2^{-p} & 0 & 0 \\ 0 & 0 & \lambda_3^{-p} & 0 \\ 0 & 0 & 0 & \lambda_4^{-p} \end{pmatrix} \tag{3.3.2}$$

Some of the eigen values of $\mathbf{A}$ may be negative and $\mathbf{A}^{-p}$ is positive definite only if p is even.

3.4 Choice 4

$$\exp(p\mathbf{A}) = \sum_{k=0}^{\infty} \frac{(p\mathbf{A})^k}{k!} \tag{3.4.1}$$

$$\implies \exp(p\mathbf{A}) = \mathbf{P}\exp(p\mathbf{D})\mathbf{P}^T \tag{3.4.2}$$

$$\implies \exp(p\mathbf{A}) - \mathbf{I} = \tag{3.4.3}$$

$$\mathbf{P}\exp(p\mathbf{D})\mathbf{P}^T - \mathbf{PIP}^T \tag{3.4.4}$$

$$= \mathbf{P}(\exp(p\mathbf{D}) - \mathbf{I})\mathbf{P}^T \tag{3.4.5}$$

$$\exp(p\mathbf{D}) - \mathbf{I} = \tag{3.4.6}$$

$$\begin{pmatrix} e^{\lambda_1} - 1 & 0 & 0 & 0 \\ 0 & e^{\lambda_2} - 1 & 0 & 0 \\ 0 & 0 & e^{\lambda_3} - 1 & 0 \\ 0 & 0 & 0 & e^{\lambda_4} - 1 \end{pmatrix} \tag{3.4.7}$$

$\mathbf{A}$ is non-singular

$$\implies \forall i \in [1,4], \lambda_i \neq 0 \tag{3.4.8}$$

$$e^{\lambda_i} < 1 \tag{3.4.9}$$

So, $\exp(p\mathbf{A}) - \mathbf{I}$ is not positive definite.

3.5 Answer

The choices would be option 1,2,3