Assignment 21

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https://github.com/KUSUMAPRIYAPULAVARTY/assignment21

1 QUESTION

Construct a linear operator T with minimal polynomial $x^2(x-1)^2$ and characteristic polynomial $x^3(x-1)^4$.Describe the primary decomposition of the vector space under T and find the projections on the primary components.Find a basis in which the matrix T is in Jordan form.Also find an explicit direct sum decomposition of the space into Tcyclic subspaces as in theorem 3 and give the invariant factors.

2 SOLUTION

$T : \mathbf{V} \to \mathbf{V}$ is a linear operator.
Characteristic polynomial

$$f(x) = x^3(x-1)^4 \tag{2.0.1}$$

Minimal polynomial

$$p(x) = x^2(x-1)^2 \tag{2.0.2}$$

The jordan block matrix corresponding to eigen value 0 is

$$\mathbf{J}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.3}$$

One of the possible jordan block matrix corresponding to eigen value 1 is

$$\mathbf{J}_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.0.4}$$

The jordan form of transformation matrix $\mathbf{T}$ is

$$\mathbf{J} = \begin{pmatrix} \mathbf{J}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 \end{pmatrix} \tag{2.0.5}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.0.6}$$

$\mathbf{T}, \mathbf{J}$ are similar matrices.

2.1 Primary decomposition,Finding the basis

According to primary decomposition theorem

$$\text{If } p(x) = p_1(x)^{r_1} p_2(x)^{r_2}, \tag{2.1.1}$$

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 \tag{2.1.2}$$

$$\mathbf{V}_i = \text{Null space of} (p_i(\mathbf{T}))_i^r \tag{2.1.3}$$

$$p_1(x)^{r_1} = x^2 \tag{2.1.4}$$

$$p_2(x)^{r_2} = (x-1)^2 \tag{2.1.5}$$

Consider Null space of $\mathbf{J}^2$

$$\mathbf{J}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.1.6}$$

$$\text{Nullity of } \mathbf{J}^2 = 3 \tag{2.1.7}$$

From (2.1.6),the basis for the nullspace is

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \tag{2.1.8}$$

$$\mathbf{v}_1 = \begin{pmatrix}1\\0\\0\\0\\0\\0\\0\end{pmatrix}, \mathbf{v_2} = \begin{pmatrix}0\\1\\0\\0\\0\\0\\0\end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix}0\\0\\1\\0\\0\\0\\0\end{pmatrix} \tag{2.1.9}$$

Similarly consider Nullspace of $(\mathbf{J}-\mathbf{I})^2$

$$(\mathbf{J}-\mathbf{I})^2 = \begin{pmatrix}1 & -2 & 0 & 0 & 0 & 0 & 0\\0 & 1 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix} \tag{2.1.10}$$

$$\text{Nullity of } (\mathbf{J}-\mathbf{I})^2 = 4 \tag{2.1.11}$$

From (2.1.10),the basis for the nullspace is

$$\{\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7\} \tag{2.1.12}$$

$$\mathbf{v}_4 = \begin{pmatrix}0\\0\\0\\1\\0\\0\\0\end{pmatrix}, \mathbf{v_5} = \begin{pmatrix}0\\0\\0\\0\\1\\0\\0\end{pmatrix}, \mathbf{v}_6 = \begin{pmatrix}0\\0\\0\\0\\0\\1\\0\end{pmatrix}, \mathbf{v}_7 = \begin{pmatrix}1\\0\\0\\0\\0\\0\\1\end{pmatrix} \tag{2.1.13}$$

So $\mathbf{T}$ is similar to block diagonal jordan matrix $\mathbf{J}$ in the basis

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7\} \tag{2.1.14}$$

which is the standard ordered basis. So,

$$\mathbf{T} = \mathbf{J} \tag{2.1.15}$$

2.2 Projections

The projection matrices $\mathbf{E}_1, \mathbf{E}_2$ are such that

1) for $i \in [1, 2]$

$$\mathbf{E}_i(\mathbf{v}) = \begin{cases}\mathbf{v} & \text{for } \mathbf{v} \in \mathbf{V}_i\\ 0 & \text{for } \mathbf{v} \notin \mathbf{V}_i\end{cases} \tag{2.2.1}$$

2)

$$(\mathbf{E}_i)^2 = \mathbf{E}_i \tag{2.2.2}$$

3)

$$\mathbf{E}_1 + \mathbf{E}_2 = \mathbf{I} \tag{2.2.3}$$

Therefore,

$$\mathbf{E}_1 = \begin{pmatrix}\mathbf{I}_3 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix} \tag{2.2.4}$$

$$= \begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 1 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix} \tag{2.2.5}$$

$$\mathbf{E}_2 = \begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_4\end{pmatrix} \tag{2.2.6}$$

$$= \begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 1 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 1 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix} \tag{2.2.7}$$

2.3 Cyclic Decomposition

Theorem 3(Cyclic Decomposition Theorem) Let T be a linear operator on a finite-dimensional vector space $\mathbf{V}$ and let $\mathbf{W}_0$ be a proper T-admissible subspace of $\mathbf{V}$. There exists non zero vectors $\alpha_1, \alpha_2, \ldots, \alpha_r$ in $\mathbf{V}$ with respective T-annihilators $p_1, p_2, \ldots, p_r$ such that

1)

$$\mathbf{V} = \mathbf{W}_0 \oplus \mathbf{Z}(\alpha_1; T) \oplus \mathbf{Z}(\alpha_2; T) \oplus \tag{2.3.1}$$

$$\ldots \oplus \mathbf{Z}(\alpha_r; T) \tag{2.3.2}$$

2) p_k divides p_{k-1},$k = 2, \ldots, r$

The T-cyclic subspace $\mathbf{Z}(\alpha_i; T)$ is defined as

$$\text{degree of } p_i = k \tag{2.3.3}$$

$$\implies \text{basis of } \mathbf{Z}(\alpha_i; T) = \tag{2.3.4}$$

$$\left\{\alpha_i, \mathbf{T}\alpha_i, \ldots, \mathbf{T}^{k-1}\alpha_i\right\} \tag{2.3.5}$$

Let us choose

$$\mathbf{W}_0 = \mathbf{0}, \tag{2.3.6}$$

$$p_1 = x^2(x-1)^2 \tag{2.3.7}$$

$$\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.3.8}$$

$$\mathbf{T}^2(\mathbf{T}-\mathbf{I})^2 = \mathbf{0}_{7\times 7} \tag{2.3.9}$$

$$\implies p_1(\mathbf{T})(\alpha_1) = 0 \tag{2.3.10}$$

$$\dim(\mathbf{Z}(\alpha_1; T)) = 4 \tag{2.3.11}$$

Now to chose α_2 we need to chose a vector such that

$$\alpha_2 \notin \mathbf{Z}(\alpha_1; T), p_2(\mathbf{T})\alpha_2 = 0 \tag{2.3.12}$$

$$p_2 = x(x-1)^2 \tag{2.3.13}$$

$$\frac{p_1}{p_2} = x \implies p_2 \text{ divides } p_1 \tag{2.3.14}$$

$$p_2(\mathbf{T}) = \mathbf{T}(\mathbf{T}-\mathbf{I})^2 = \tag{2.3.15}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.3.16}$$

One such vector that satisfies (2.3.12) is

$$\alpha_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.3.17}$$

$$\dim(\mathbf{Z}(\alpha_2; T)) = 3 \tag{2.3.18}$$

$$\dim \mathbf{Z}(\alpha_1; T) + \dim \mathbf{Z}(\alpha_2; T) = 7 \tag{2.3.19}$$

$$\implies \mathbf{V} = \mathbf{Z}(\alpha_1; T) \oplus \mathbf{Z}(\alpha_2; T) \tag{2.3.20}$$

is the cyclic decomposition.

2.4 *Invariant Factors*

The invariant factors are

$$p_1 = x^2(x-1)^2 \tag{2.4.1}$$

$$p_2 = x(x-1)^2 \tag{2.4.2}$$