Assignment 21

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Download codes from

https://github.com/KUSUMAPRIYAPULAVARTY/assignment21

1 QUESTION

Construct a linear operator T with minimal polynomial $x^2(x-1)^2$ and characteristic polynomial $x^3(x-1)^4$.Describe the primary decomposition of the vector space under T and find the projections on the primary components.Find a basis in which the matrix T is in Jordan form.Also find an explicit direct sum decomposition of the space into Tcyclic subspaces as in theorem 3 and give the invariant factors.

2 Solution

Statement	Solution
Jordan Form	
Given	Linear operator $$T : \mathbf{V} \to \mathbf{V} \quad (2.0.1)$$ $$\text{Characteristic polynomial} f(x) = x^3(x-1)^4 \quad (2.0.2)$$ $$\text{Minimal polynomial} p(x) = x^2(x-1)^2 \quad (2.0.3)$$
The jordan block corresponding to eigen value 0	$$\mathbf{J}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.4)$$
One of the possible jordan blocks corresponding to eigen value 1	$$\mathbf{J}_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.0.5)$$
The jordan form of transformation matrix $\mathbf{T}$	$$\mathbf{J} = \begin{pmatrix} \mathbf{J}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 \end{pmatrix} \quad (2.0.6)$$ $$= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.0.7)$$
Primary Decomposition,Finding the basis	

According to primary decomposition theorem	$$\text{If } p(x) = p_1(x)^{r_1} p_2(x)^{r_2}, \tag{2.0.8}$$ $$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 \tag{2.0.9}$$ $$\mathbf{V}_i = \text{Null space of}(p_i(\mathbf{T}))_i^r \tag{2.0.10}$$ $$p_1(x)^{r_1} = x^2 \tag{2.0.11}$$ $$p_2(x)^{r_2} = (x-1)^2 \tag{2.0.12}$$
Null space of $\mathbf{J}^2$	$$\mathbf{J}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.0.13}$$ $$\text{Nullity of } \mathbf{J}^2 = 3 \tag{2.0.14}$$ From (2.0.13),the basis for the nullspace is $$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \tag{2.0.15}$$ $$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.0.16}$$
Nullspace of $(\mathbf{J}-\mathbf{I})^2$	$$(\mathbf{J}-\mathbf{I})^2 = \begin{pmatrix} 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.0.17}$$ $$\text{Nullity of } (\mathbf{J}-\mathbf{I})^2 = 4 \tag{2.0.18}$$ From (2.0.17),the basis for the nullspace is $$\{\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7\} \tag{2.0.19}$$ $$\mathbf{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v_5} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_7 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{2.0.20}$$

<table>
<tr><td>$$\mathbf{T} = \mathbf{J} \tag{2.0.21}$$</td><td>$\mathbf{T}$ is similar to block diagonal jordan matrix $\mathbf{J}$ in the basis
$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7\} \tag{2.0.22}$$
which is the standard ordered basis.</td></tr>
<tr><td colspan="2">Finding the projections</td></tr>
<tr><td>The projection matrices $\mathbf{E}_1, \mathbf{E}_2$ are such that</td><td>1) for $i \in [1, 2]$
$$\mathbf{E}_i(\mathbf{v}) = \begin{cases} \mathbf{v} & \text{for } \mathbf{v} \in \mathbf{V}_i \\ 0 & \text{for } \mathbf{v} \notin \mathbf{V}_i \end{cases} \tag{2.0.23}$$
2)
$$(\mathbf{E}_i)^2 = \mathbf{E}_i \tag{2.0.24}$$
3)
$$\mathbf{E}_1 + \mathbf{E}_2 = \mathbf{I} \tag{2.0.25}$$</td></tr>
<tr><td>The projection matrices are</td><td>
$$\mathbf{E}_1 = \begin{pmatrix} \mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{2.0.26}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.0.27}$$
$$\mathbf{E}_2 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_4 \end{pmatrix} \tag{2.0.28}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.0.29}$$
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<tr><td colspan="2">Cyclic Decomposition</td></tr>
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Cyclic decomposition theorem(Theorem 3)	Let T be a linear operator on a finite-dimensional vector space $\mathbf{V}$ and let $\mathbf{W}_0$ be a proper T-admissible subspace of $\mathbf{V}$. There exists non zero vectors $\alpha_1, \alpha_2, \ldots, \alpha_r$ in $\mathbf{V}$ with respective T-annihilators $p_1, p_2, \ldots, p_r$ such that 1) $$\mathbf{V} = \mathbf{W}_0 \oplus \mathbf{Z}(\alpha_1; T) \oplus \mathbf{Z}(\alpha_2; T) \oplus \quad (2.0.30)$$ $$\ldots \oplus \mathbf{Z}(\alpha_r; T) \quad (2.0.31)$$ 2) p_k divides p_{k-1},$k = 2, \ldots, r$
The T-cyclic subspace $\mathbf{Z}(\alpha_i; T)$ is defined as	$$\text{degree of } p_i = k \quad (2.0.32)$$ $$\implies \text{basis of } \mathbf{Z}(\alpha_i; T) = \quad (2.0.33)$$ $$\left\{\alpha_i, \mathbf{T}\alpha_i, \ldots, \mathbf{T}^{k-1}\alpha_i\right\} \quad (2.0.34)$$
Finding the cyclic subspaces $$p_1 = x^2(x-1)^2 \quad (2.0.37)$$	Let us choose $$\mathbf{W}_0 = \mathbf{0} \quad (2.0.35)$$ $$(2.0.36)$$ $$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad (2.0.38)$$ $$\mathbf{T}^2(\mathbf{T}-\mathbf{I})^2 = \mathbf{0}_{7\times 7} \quad (2.0.39)$$ $$\implies p_1(\mathbf{T})(\alpha_1) = 0 \quad (2.0.40)$$ $$\text{Basis } \mathbf{B_1} = \left\{\alpha_1, \mathbf{T}\alpha_1, \mathbf{T}^2\alpha_1, \mathbf{T}^3\alpha_1\right\} \quad (2.0.41)$$ $$\text{rank of } \begin{pmatrix} \alpha_1 \\ \mathbf{T}\alpha_1 \\ \mathbf{T}^2\alpha_1 \\ \mathbf{T}^3\alpha_1 \end{pmatrix} = \quad (2.0.42)$$ $$\text{rank of } \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 3 & 1 & 3 & 1 \\ 0 & 0 & 0 & 4 & 1 & 4 & 1 \end{pmatrix} = 4 \quad (2.0.43)$$ $$\dim(\mathbf{Z}(\alpha_1; T)) = 4 \quad (2.0.44)$$
$$p_2 = x(x-1)^2 \quad (2.0.45)$$	Now to chose α_2 we need to chose a vector such that

	$$\alpha_2 \notin \mathbf{Z}(\alpha_1; T), p_2(\mathbf{T})\alpha_2 = 0 \quad (2.0.46)$$ $$p_2 = x(x-1)^2 \quad (2.0.47)$$ $$\frac{p_1}{p_2} = x \implies p_2 \text{ divides } p_1 \quad (2.0.48)$$ $$p_2(\mathbf{T}) = \mathbf{T}(\mathbf{T}-\mathbf{I})^2 = \quad (2.0.49)$$ $$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.0.50)$$ One such vector that satisfies (2.0.46) is $$\alpha_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad (2.0.51)$$ $$\text{Basis } \mathbf{B_2} = \left\{\alpha_2, \mathbf{T}\alpha_2, \mathbf{T}^2\alpha_2\right\} \quad (2.0.52)$$ $$\text{rank of } \begin{pmatrix} \alpha_2 \\ \mathbf{T}\alpha_2 \\ \mathbf{T}^2\alpha_2 \end{pmatrix} = \quad (2.0.53)$$ $$\text{rank of } \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 & 0 \end{pmatrix} = 3 \quad (2.0.54)$$ $$\dim(\mathbf{Z}(\alpha_2; T)) = 3 \quad (2.0.55)$$ $$\dim\mathbf{Z}(\alpha_1; T) + \dim\mathbf{Z}(\alpha_2; T) = 7 \quad (2.0.56)$$ $$\implies \mathbf{V} = \mathbf{Z}(\alpha_1; T) \oplus \mathbf{Z}(\alpha_2; T) \quad (2.0.57)$$ is the cyclic decomposition. Basis for $\mathbf{V}$ is $\{\mathbf{B}_1, \mathbf{B}_2\}$
Invariant Factors	
Invariant factors are	$$p_1 = x^2(x-1)^2 \quad (2.0.58)$$ $$p_2 = x(x-1)^2 \quad (2.0.59)$$

Table1:Solution