Assignment 4

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Download all python codes from

https://github.com/KUSUMAPRIYAPULAVARTY/assignment4/tree/master/codes

and latex-tikz codes from

https://github.com/KUSUMAPRIYAPULAVARTY/assignment4

1 QUESTION

Prove that the following equations represent two straight lines.Also find their point of intersection and the angle between them

$$3y^2 - 8xy - 3x^2 - 29x + 3y - 18 = 0 \tag{1.0.1}$$

2 Explanation

The general form of equation representing a pair of straight lines is

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.1}$$

This can be represented as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.2}$$

$$\text{where, } \mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} \tag{2.0.4}$$

This represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \tag{2.0.5}$$

3 Solution

$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix}$ of (1.0.1) becomes

$$\begin{vmatrix} -3 & -4 & -\frac{29}{2} \\ -4 & 3 & \frac{3}{2} \\ -\frac{29}{2} & \frac{3}{2} & -18 \end{vmatrix} \tag{3.0.1}$$

Expanding equation (3.0.1), we get zero.
Hence given equation represents a pair of straight lines.

4 Finding the individual lines

Slopes of the individual lines are roots of equation

$$cm^2 + 2bm + a = 0 \tag{4.0.1}$$

$$\implies 3m^2 - 8m - 3 = 0 \tag{4.0.2}$$

$$\text{Solving, } m = 3, -\frac{1}{3} \tag{4.0.3}$$

The normal vectors of the lines then become

$$\mathbf{n_1} = \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \tag{4.0.4}$$

$$\mathbf{n_2} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \tag{4.0.5}$$

Equations of the lines can therefore be written as

$$\begin{pmatrix} \frac{1}{3} & 1 \end{pmatrix}\mathbf{x} = c \tag{4.0.6}$$

$$\implies \begin{pmatrix} 1 & 3 \end{pmatrix}\mathbf{x} = c_1, \tag{4.0.7}$$

$$\begin{pmatrix} -3 & 1 \end{pmatrix}\mathbf{x} = c_2 \tag{4.0.8}$$

$$\implies \left[\begin{pmatrix} 1 & 3 \end{pmatrix}\mathbf{x} - c_1\right]\left[\begin{pmatrix} -3 & 1 \end{pmatrix}\mathbf{x} - c_2\right] \tag{4.0.9}$$

represents the equation specified in (1.0.1)
Comparing the equations, we have

$$\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 29 \\ -3 \end{pmatrix} \tag{4.0.10}$$

$$\tag{4.0.11}$$

Row reducing the augmented matrix

$$\begin{pmatrix} 1 & -3 & 29 \\ 3 & 1 & -3 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 3\times R_1} \begin{pmatrix} 1 & -3 & 29 \\ 0 & 10 & -90 \end{pmatrix} \tag{4.0.12}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 \times \frac{1}{10}} \begin{pmatrix} 1 & -3 & 29 \\ 0 & 1 & -9 \end{pmatrix} \tag{4.0.13}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + 3\times R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -9 \end{pmatrix} \tag{4.0.14}$$

$$\implies c_2 = 2 \text{ and } c_1 = -9 \tag{4.0.15}$$

The individual line equations therefore become

$$\begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = -9, \tag{4.0.16}$$

$$\begin{pmatrix} -3 & 1 \end{pmatrix} \mathbf{x} = 2 \tag{4.0.17}$$

Note that the convolution of the normal vectors, should satisfy the below condition

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} * \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{4.0.18}$$

The LHS part of (4.0.18) can be rewritten using toeplitz matrix as

$$\begin{pmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -8 \\ 3 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{4.0.19}$$

5 Intersection Point

The augmented matrix for the set of equations represented in (4.0.16), (4.0.17) is

$$\begin{pmatrix} 1 & 3 & -9 \\ -3 & 1 & 2 \end{pmatrix} \tag{5.0.1}$$

Row reducing the matrix

$$\begin{pmatrix} 1 & 3 & -9 \\ -3 & 1 & 2 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + 3 \times R_1} \begin{pmatrix} 1 & 3 & -9 \\ 0 & 10 & -25 \end{pmatrix} \tag{5.0.2}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{3}{10} \times R_2} \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 10 & -25 \end{pmatrix} \tag{5.0.3}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{10}} \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & -\frac{5}{2} \end{pmatrix} \tag{5.0.4}$$

$$\text{Hence, the intersection point is} \begin{pmatrix} -\frac{3}{2} \\ -\frac{5}{2} \end{pmatrix} \tag{5.0.5}$$

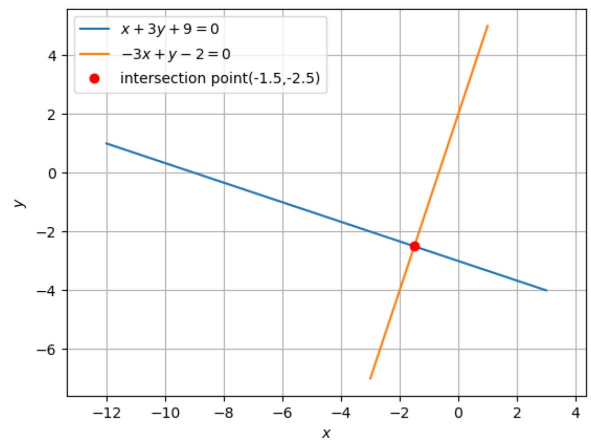

Fig. 0: plot showing intersection of lines

6 Angle Between The Lines

Angle between two lines θ can be given by

$$\cos\theta = \frac{\mathbf{n_1}^T \mathbf{n_2}}{\|\mathbf{n_1}\| \, \|\mathbf{n_2}\|} \tag{6.0.1}$$

From (4.0.16), (4.0.17),

$$\cos\theta = \frac{\begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \end{pmatrix}}{\sqrt{(3)^2 + 1} \times \sqrt{(-3)^2 + 1}} = 0 \tag{6.0.2}$$

$$\implies \theta = 90^\circ \tag{6.0.3}$$