Assignment 6

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Download all python codes from

https://github.com/KUSUMAPRIYAPULAVARTY/assignment6/tree/master/codes

and latex-tikz codes from

https://github.com/KUSUMAPRIYAPULAVARTY/assignment6

1 Question

Show that by rotating axes,equation

$$\mathbf{x}^T \begin{pmatrix} 41 & 12 \\ 12 & 34 \end{pmatrix} \mathbf{x} = 75 \tag{1.0.1}$$

can be reduced to

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 3 \tag{1.0.2}$$

2 Explanation

The given equation is of the form

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + f = 0 \tag{2.0.1}$$

The matrix V can be decomposed as

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{2.0.2}$$

$$\text{where } \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.0.3}$$

λ_1 and $_2$ are Eigen values of $\mathbf{V}$, and $\mathbf{P}$ contains the Eigen vectors corresponding to the Eigen values λ_1 and λ_2

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{2.0.4}$$

indicates the linear transformation where $\mathbf{P}$ indicates the rotation of axes and $\mathbf{c}$ gives the shift of origin.

3 Solution

$$\mathbf{V} = \begin{pmatrix} 41 & 12 \\ 12 & 34 \end{pmatrix} \tag{3.0.1}$$

$$det(\mathbf{V}) = \begin{vmatrix} 41 & 12 \\ 12 & 34 \end{vmatrix} > 0 \tag{3.0.2}$$

So,the given equation represents an ellipse
To find the Eigen values of $\mathbf{V}$

$$\left|\lambda \mathbf{I} - \mathbf{v}\right| = 0 \tag{3.0.3}$$

$$\implies \begin{vmatrix} \lambda - 41 & -12 \\ -12 & \lambda - 34 \end{vmatrix} = 0 \tag{3.0.4}$$

$$\implies \lambda^2 - 75\lambda + 1250 = 0 \tag{3.0.5}$$

$$\implies \lambda_1 = 50, \lambda_2 = 25 \tag{3.0.6}$$

$$\mathbf{D} = \begin{pmatrix} 50 & 0 \\ 0 & 25 \end{pmatrix} \tag{3.0.7}$$

Finding Eigen vector $\mathbf{p}_1$,

$$\lambda_1 \mathbf{I} - \mathbf{V} = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2/4]{R_1 \leftarrow R_1/3} \begin{pmatrix} 3 & -4 \\ -3 & 4 \end{pmatrix} \tag{3.0.8}$$

$$\xleftrightarrow{R_2 \leftarrow R_1 + R_2} \begin{pmatrix} 3 & -4 \\ 0 & 0 \end{pmatrix} \tag{3.0.9}$$

$$\implies \mathbf{p_1} = \frac{1}{\sqrt{4^2 + 3^2}} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix} \tag{3.0.10}$$

Similarly,

$$\lambda_2 \mathbf{I} - \mathbf{V} = \begin{pmatrix} -16 & -12 \\ -12 & -9 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2/-3]{R_1 \leftarrow R_1/-4} \begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix} \tag{3.0.11}$$

$$\xleftrightarrow{R_2 \leftarrow R_1 - R_2} \begin{pmatrix} 4 & 3 \\ 0 & 0 \end{pmatrix} \tag{3.0.12}$$

$$\implies \mathbf{p_2} = \frac{1}{\sqrt{4^2 + 3^2}} \begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{-3}{5} \\ \frac{4}{5} \end{pmatrix} \tag{3.0.13}$$

$$\text{Therefore, } \mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \begin{pmatrix} \frac{4}{5} & \frac{-3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix} \tag{3.0.14}$$

From (2.0.2) $\mathbf{V}$ can be rewritten as

$$\mathbf{V} = \begin{pmatrix} \frac{4}{5} & \frac{-3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} 50 & 0 \\ 0 & 25 \end{pmatrix} \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{-3}{5} & \frac{4}{5} \end{pmatrix} \tag{3.0.15}$$

(1.0.1) can be now rewritten as

$$25 \left[\mathbf{x}^T \begin{pmatrix} \frac{4}{5} & \frac{-3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{-3}{5} & \frac{4}{5} \end{pmatrix} \mathbf{x} \right] = 75 \tag{3.0.16}$$

$$\left[\begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{-3}{5} & \frac{4}{5} \end{pmatrix} \mathbf{x} \right]^T \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \left[\begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{-3}{5} & \frac{4}{5} \end{pmatrix} \mathbf{x} \right] = 3 \tag{3.0.17}$$

Consider the rotation transformation

$$\mathbf{x} = \mathbf{P}\mathbf{y} \tag{3.0.18}$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{4}{5} & \frac{-3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix} \mathbf{y} \tag{3.0.19}$$

$$\mathbf{P}^{-1}\mathbf{x} = \mathbf{P}^{-1}\mathbf{P}\mathbf{y} \tag{3.0.20}$$

$$\implies \mathbf{y} = \mathbf{P}^{-1}\mathbf{x} \tag{3.0.21}$$

$$\text{But, } \mathbf{P}^{-1} = \mathbf{P}^{T} \tag{3.0.22}$$

$$\implies \mathbf{y} = \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{-3}{5} & \frac{4}{5} \end{pmatrix} \mathbf{x} \tag{3.0.23}$$

Using (3.0.19) in (3.0.17), the ellipse equation can be rewritten as

$$\mathbf{y}^{T} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{y} = 3 \tag{3.0.24}$$

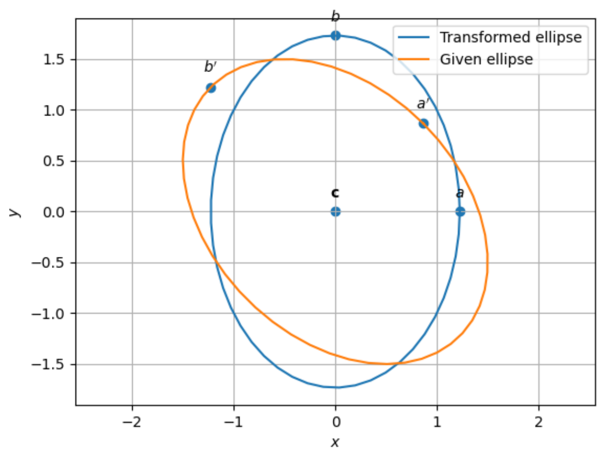

Fig. 0: plot showing the original and rotated ellipse