

# Assignment 6

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Download all python codes from  
<https://github.com/KUSUMAPRIYAPULAVARTY/assignment6/tree/master/codes>

and latex-tikz codes from

<https://github.com/KUSUMAPRIYAPULAVARTY/assignment6>

## 1 QUESTION

Show that by rotating axes, equation

$$\mathbf{x}^T \begin{pmatrix} 41 & 12 \\ 12 & 34 \end{pmatrix} \mathbf{x} = 75 \quad (1.0.1)$$

can be reduced to

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 3 \quad (1.0.2)$$

## 2 EXPLANATION

The given equation is of the form

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + f = 0 \quad (2.0.1)$$

The matrix  $\mathbf{V}$  can be decomposed as

$$\mathbf{V} = \mathbf{P} \mathbf{D} \mathbf{P}^T \quad (2.0.2)$$

$$\text{where } \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (2.0.3)$$

$\lambda_1$  and  $\lambda_2$  are Eigen values of  $\mathbf{V}$ , and  $\mathbf{P}$  contains the Eigen vectors corresponding to the Eigen values  $\lambda_1$  and  $\lambda_2$

$$\mathbf{x} = \mathbf{P} \mathbf{y} + \mathbf{c} \quad (2.0.4)$$

indicates the linear transformation where  $\mathbf{P}$  indicates the rotation of axes and  $\mathbf{c}$  gives the shift of origin.

## 3 SOLUTION

$$\mathbf{V} = \begin{pmatrix} 41 & 12 \\ 12 & 34 \end{pmatrix} \quad (3.0.1)$$

$$\det(\mathbf{V}) = \begin{vmatrix} 41 & 12 \\ 12 & 34 \end{vmatrix} > 0 \quad (3.0.2)$$

So, the given equation represents an ellipse  
To find the Eigen values of  $\mathbf{V}$

$$|\lambda \mathbf{I} - \mathbf{V}| = 0 \quad (3.0.3)$$

$$\Rightarrow \begin{vmatrix} \lambda - 41 & -12 \\ -12 & \lambda - 34 \end{vmatrix} = 0 \quad (3.0.4)$$

$$\Rightarrow \lambda^2 - 75\lambda + 1250 = 0 \quad (3.0.5)$$

$$\Rightarrow \lambda_1 = 50, \lambda_2 = 25 \quad (3.0.6)$$

$$\mathbf{D} = \begin{pmatrix} 50 & 0 \\ 0 & 25 \end{pmatrix} \quad (3.0.7)$$

Finding Eigen vector  $\mathbf{p}_1$ ,

$$\lambda_1 \mathbf{I} - \mathbf{V} = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \xrightarrow[R_2 \leftarrow R_2/4]{R_1 \leftarrow R_1/3} \begin{pmatrix} 3 & -4 \\ -3 & 4 \end{pmatrix} \quad (3.0.8)$$

$$\xrightarrow{R_2 \leftarrow R_1 + R_2} \begin{pmatrix} 3 & -4 \\ 0 & 0 \end{pmatrix} \quad (3.0.9)$$

$$\Rightarrow \mathbf{p}_1 = \frac{1}{\sqrt{4^2 + 3^2}} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix} \quad (3.0.10)$$

Similarly,

$$\lambda_2 \mathbf{I} - \mathbf{V} = \begin{pmatrix} -16 & -12 \\ -12 & -9 \end{pmatrix} \xrightarrow[R_2 \leftarrow R_2/-3]{R_1 \leftarrow R_1/-4} \begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix} \quad (3.0.11)$$

$$\xrightarrow{R_2 \leftarrow R_1 - R_2} \begin{pmatrix} 4 & 3 \\ 0 & 0 \end{pmatrix} \quad (3.0.12)$$

$$\Rightarrow \mathbf{p}_2 = \frac{1}{\sqrt{4^2 + 3^2}} \begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{-3}{5} \\ \frac{4}{5} \end{pmatrix} \quad (3.0.13)$$

$$\text{Therefore, } \mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) = \begin{pmatrix} \frac{4}{5} & \frac{-3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix} \quad (3.0.14)$$

From (2.0.2)  $\mathbf{V}$  can be rewritten as

$$\mathbf{V} = \begin{pmatrix} \frac{4}{5} & \frac{-3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} 50 & 0 \\ 0 & 25 \end{pmatrix} \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix} \quad (3.0.15)$$

(1.0.1) can be now rewritten as

$$25 \left[ \mathbf{x}^T \begin{pmatrix} \frac{4}{5} & \frac{-3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix} \mathbf{x} \right] = 75 \quad (3.0.16)$$

Let us replace the variable  $\mathbf{x}$  with a variable  $\mathbf{y}$

$$\mathbf{y}^T \begin{pmatrix} \frac{4}{5} & \frac{-3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{-3}{5} & \frac{4}{5} \end{pmatrix} \mathbf{y} = 3 \quad (3.0.17)$$

$$\left[ \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{-3}{5} & \frac{4}{5} \end{pmatrix} \mathbf{y} \right]^T \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \left[ \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{-3}{5} & \frac{4}{5} \end{pmatrix} \mathbf{y} \right] = 3 \quad (3.0.18)$$

Consider the rotation transformation

$$\mathbf{x} = \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{-3}{5} & \frac{4}{5} \end{pmatrix} \mathbf{y} \quad (3.0.19)$$

Using (3.0.19) in (3.0.18), the ellipse equation can be rewritten as

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 3 \quad (3.0.20)$$

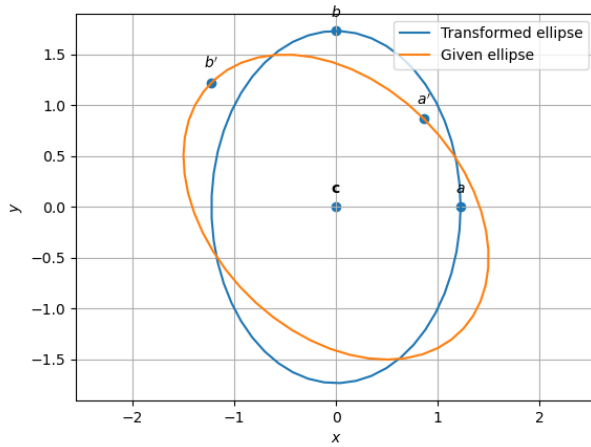


Fig. 0: plot showing the original and rotated ellipse