Assignment 8

KUSUMA PRIYA
EE20MTECH11007

Download all python codes from

https://github.com/KUSUMAPRIYAPULAVARTY/assignment8/tree/master/codes

and latex-tikz codes from

https://github.com/KUSUMAPRIYAPULAVARTY/assignment8

1 Question

Check if the lines L_1, L_2 are skew.If so, find the closest points on those lines using Singular Value Decomposition(SVD)

$$L_1 : \mathbf{x} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \tag{1.0.1}$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{1.0.2}$$

2 Explanation

The matrix M of dimensions $(m \times n)$ can be decomposed using SVD as

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{2.0.1}$$

where,columns of $\mathbf{U}_{(m\times m)}$ are eigen vectors of $\mathbf{M}\mathbf{M}^T$
columns of $\mathbf{V}_{(n\times n)}$ are eigen vectors of $\mathbf{M}^T\mathbf{M}$
$\mathbf{S}$ is a diagonal matrix containing singular values of $\mathbf{M}$. Also, $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrices

$$\mathbf{U}\mathbf{U}^T = \mathbf{U}^T\mathbf{U} = \mathbf{I} \tag{2.0.2}$$

$$\mathbf{V}\mathbf{V}^T = \mathbf{V}^T\mathbf{V} = \mathbf{I} \tag{2.0.3}$$

3 Solution

Given line equations intersect if

$$\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{3.0.1}$$

This can be written as

$$\begin{pmatrix} 3 & 1 \\ 2 & 2 \\ 6 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} \tag{3.0.2}$$

$$\mathbf{M}\mathbf{x} = \mathbf{b} \tag{3.0.3}$$

The augmented matrix is

$$\begin{pmatrix} 3 & 1 & 5 \\ 2 & 2 & -1 \\ 6 & 2 & -1 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 - R_1 \times \frac{2}{3}]{R_3 \leftarrow R_3 - 2\times R_1} \begin{pmatrix} 3 & 1 & 5 \\ 0 & \frac{5}{3} & -\frac{13}{3} \\ 0 & 0 & -11 \end{pmatrix} \tag{3.0.4}$$

So, the given pair of lines do not intersect and also their direction vectors are not parallel. Hence they are skew lines.
To find U,

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 3 & 1 \\ 2 & 2 \\ 6 & 2 \end{pmatrix} \begin{pmatrix} 3 & 2 & 6 \\ 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 10 & 8 & 20 \\ 8 & 8 & 16 \\ 20 & 16 & 40 \end{pmatrix} \tag{3.0.5}$$

To calculate its Eigen values,

$$\begin{vmatrix} 10-\lambda & 8 & 20 \\ 8 & 8-\lambda & 16 \\ 20 & 16 & 40-\lambda \end{vmatrix} = 0 \tag{3.0.6}$$

$$\implies \lambda^3 + 58\lambda^2 + 80\lambda = 0 \tag{3.0.7}$$

$$\lambda_1 = 29 - \sqrt{761}, \lambda_2 = 29 + \sqrt{761}, \lambda_3 = 0 \tag{3.0.8}$$

with corresponding Eigen vectors as

$$\mathbf{u_1} = \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 + 1 + \left(\frac{21+\sqrt{761}}{16}\right)^2}} \begin{pmatrix} \frac{1}{2} \\ \frac{-21-\sqrt{761}}{16} \\ 1 \end{pmatrix} \tag{3.0.9}$$

$$\mathbf{u_2} = \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 + 1 + \left(\frac{-21+\sqrt{761}}{16}\right)^2}} \begin{pmatrix} \frac{1}{2} \\ \frac{-21+\sqrt{761}}{16} \\ 1 \end{pmatrix} \tag{3.0.10}$$

$$\mathbf{u_3} = \frac{1}{\sqrt{(-2)^2 + 1}} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \tag{3.0.11}$$

Solving, the $\mathbf{U}$ matrix becomes

$$\mathbf{U} = \begin{pmatrix} \frac{8}{\sqrt{1522+42\sqrt{761}}} & \frac{8}{\sqrt{1522-42\sqrt{761}}} & -\frac{2}{\sqrt{5}} \\ \frac{-21-\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} & \frac{-21+\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} & 0 \\ \frac{16}{\sqrt{1522+42\sqrt{761}}} & \frac{16}{\sqrt{1522-42\sqrt{761}}} & \frac{1}{\sqrt{5}} \end{pmatrix} \tag{3.0.12}$$

Also, from the obtained Eigen values,the $\mathbf{S}$ matrix becomes

$$\mathbf{S} = \begin{pmatrix} \sqrt{29-\sqrt{761}} & 0 \\ 0 & \sqrt{29+\sqrt{761}} \\ 0 & 0 \end{pmatrix} \tag{3.0.13}$$

The Moore-Penrose pseudo inverse of $\mathbf{S}$ is given by

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{29-\sqrt{761}}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{29+\sqrt{761}}} & 0 \end{pmatrix} \tag{3.0.14}$$

Now to find $\mathbf{V}$,
Rewriting (2.0.1)

$$\mathbf{V} = (\mathbf{M}^T\mathbf{U})\mathbf{S}_+{}^T \tag{3.0.15}$$

$\mathbf{M}^T\mathbf{U}$ becomes

$$\begin{pmatrix} 3 & 2 & 6 \\ 1 & 2 & 2 \end{pmatrix} \tag{3.0.16}$$

$$\begin{pmatrix} \frac{8}{\sqrt{1522+42\sqrt{761}}} & \frac{8}{\sqrt{1522-42\sqrt{761}}} & -\frac{2}{\sqrt{5}} \\ \frac{-21-\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} & \frac{-21+\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} & 0 \\ \frac{16}{\sqrt{1522+42\sqrt{761}}} & \frac{16}{\sqrt{1522-42\sqrt{761}}} & \frac{1}{\sqrt{5}} \end{pmatrix} \tag{3.0.17}$$

$$= \begin{pmatrix} \frac{78-2\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} & \frac{78+2\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} & 0 \\ \frac{-2-2\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} & \frac{-2+2\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} & 0 \end{pmatrix} \tag{3.0.18}$$

Therefore from (3.0.14),(3.0.15),(3.0.18),

$$\mathbf{V} = \begin{pmatrix} \frac{78-2\sqrt{761}}{\sqrt{1522+42\sqrt{761}}\sqrt{29-\sqrt{761}}} & \frac{78+2\sqrt{761}}{\sqrt{1522-42\sqrt{761}}\sqrt{29+\sqrt{761}}} \\ \frac{-2-2\sqrt{761}}{\sqrt{1522+42\sqrt{761}}\sqrt{29-\sqrt{761}}} & \frac{-2+2\sqrt{761}}{\sqrt{1522-42\sqrt{761}}\sqrt{29+\sqrt{761}}} \end{pmatrix} \tag{3.0.19}$$

Now, to calculate x

$$\mathbf{Mx} = \mathbf{b} \tag{3.0.20}$$

$$\implies \mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{3.0.21}$$

$$\implies \mathbf{SV}^T\mathbf{x} = \mathbf{U}^T\mathbf{b} \tag{3.0.22}$$

$$\implies \mathbf{x} = \mathbf{V}(\mathbf{S}_+(\mathbf{U}^T\mathbf{b})) \tag{3.0.23}$$

Calculating $\mathbf{U}^T\mathbf{b}$, we have

$$\begin{pmatrix} \frac{45+\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} \\ \frac{45-\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} \end{pmatrix} \tag{3.0.24}$$

$$\mathbf{S}_+(\mathbf{U}^T\mathbf{b}) = \begin{pmatrix} \frac{45+\sqrt{761}}{\sqrt{1522+42\sqrt{761}}\sqrt{29-\sqrt{761}}} \\ \frac{45-\sqrt{761}}{\sqrt{1522-42\sqrt{761}}\sqrt{29+\sqrt{761}}} \end{pmatrix} \tag{3.0.25}$$

$\mathbf{V}(\mathbf{S}_+(\mathbf{U}^T\mathbf{b}))$

$$= \begin{pmatrix} \frac{78-2\sqrt{761}}{\sqrt{1522+42\sqrt{761}}\sqrt{29-\sqrt{761}}} & \frac{78+2\sqrt{761}}{\sqrt{1522-42\sqrt{761}}\sqrt{29+\sqrt{761}}} \\ \frac{-2-2\sqrt{761}}{\sqrt{1522+42\sqrt{761}}\sqrt{29-\sqrt{761}}} & \frac{-2+2\sqrt{761}}{\sqrt{1522-42\sqrt{761}}\sqrt{29+\sqrt{761}}} \end{pmatrix} \tag{3.0.26}$$

$$\begin{pmatrix} \frac{45+\sqrt{761}}{\sqrt{1522+42\sqrt{761}}\sqrt{29-\sqrt{761}}} \\ \frac{45-\sqrt{761}}{\sqrt{1522-42\sqrt{761}}\sqrt{29+\sqrt{761}}} \end{pmatrix} \tag{3.0.27}$$

Solving,

$$\mathbf{x} = \begin{pmatrix} \frac{8371}{15220} \\ \frac{-15981}{15220} \end{pmatrix} = \begin{pmatrix} \frac{11}{20} \\ \frac{-21}{20} \end{pmatrix} \tag{3.0.28}$$

Verifying the solution,

$$\mathbf{Mx} = \mathbf{b} \tag{3.0.29}$$

$$\implies \mathbf{M}^T\mathbf{Mx} = \mathbf{M}^T\mathbf{b} \tag{3.0.30}$$

$$\mathbf{M}^T\mathbf{b} = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} \tag{3.0.31}$$

$$= \begin{pmatrix} 7 \\ 1 \end{pmatrix} \tag{3.0.32}$$

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 2 \\ 6 & 2 \end{pmatrix} \tag{3.0.33}$$

$$= \begin{pmatrix} 49 & 19 \\ 19 & 9 \end{pmatrix} \tag{3.0.34}$$

$$\text{From, (3.0.30)} \begin{pmatrix} 49 & 19 \\ 19 & 9 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 7 \\ 1 \end{pmatrix} \tag{3.0.35}$$

Solving for $\mathbf{x}$

$$\begin{pmatrix} 49 & 19 & 7 \\ 19 & 9 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1 \times \frac{19}{49}} \begin{pmatrix} 49 & 19 & 7 \\ 0 & \frac{80}{49} & \frac{-84}{49} \end{pmatrix} \tag{3.0.36}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 \times \frac{1}{49}} \begin{pmatrix} 1 & \frac{19}{49} & \frac{7}{49} \\ 0 & \frac{80}{49} & \frac{-84}{49} \end{pmatrix} \tag{3.0.37}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2 \times \frac{19}{80}} \begin{pmatrix} 1 & 0 & \frac{11}{20} \\ 0 & \frac{80}{49} & \frac{-84}{49} \end{pmatrix} \tag{3.0.38}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 \times \frac{49}{80}} \begin{pmatrix} 1 & 0 & \frac{11}{20} \\ 0 & 1 & \frac{-21}{20} \end{pmatrix} \tag{3.0.39}$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{11}{20} \\ \frac{-21}{20} \end{pmatrix} \tag{3.0.40}$$