Assignment 9

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Download codes from

https://github.com/KUSUMAPRIYAPULAVARTY/assignment9

1 Question

Prove that if two homogenous systems of linear equations in two unknowns have the same solutions, then they are equivalent.

2 Solution

Let the two systems of homogenous equations be

$$\mathbf{A}\mathbf{x} = \mathbf{0} \tag{2.0.1}$$

$$\mathbf{B}\mathbf{x} = \mathbf{0} \tag{2.0.2}$$

$$\implies \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ \vdots & \vdots \\ A_{n1} & A_{n2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{2.0.3}$$

$$\text{and} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ \vdots & \vdots \\ B_{n1} & B_{n2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{2.0.4}$$

2.1 Case 1

Let us assume that the solution is unique. Since they have the same solution, both $\mathbf{A}, \mathbf{B}$ must have their rank as 2.
Let the matrix $\mathbf{A}$ upon rowreducing becomes $\mathbf{R_1}$ with rank 2

$$\mathbf{R_1} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \tag{2.1.1}$$

and matrix $\mathbf{B}$ becomes $\mathbf{R_2}$ with rank 2

$$\mathbf{R_2} = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \tag{2.1.2}$$

Consider $\mathbf{R_1}$
Performing elementary row operation $R_1 \leftarrow R_1 \times \frac{1}{a_1}$ and $R_2 \leftarrow R_2 \times \frac{1}{a_2}$ using elementary matrices,

$$\mathbf{E_1} = \begin{pmatrix} \frac{1}{a_1} & 0 & 0 & \dots & 0 \\ 0 & 1 & 0\dots & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \tag{2.1.3}$$

$$\mathbf{E_2} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{a_2} & 0\dots & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \tag{2.1.4}$$

$$\mathbf{E_2}(\mathbf{E_1}\mathbf{R_1}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \tag{2.1.5}$$

Performing $R_1 \leftarrow R_1 \times b_1$ using elementary matrix

$$\mathbf{E_3} = \begin{pmatrix} b_1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0\dots & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \tag{2.1.6}$$

$$\mathbf{E_3}(\mathbf{E_2}(\mathbf{E_1}\mathbf{R_1})) = \begin{pmatrix} b_1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \tag{2.1.7}$$

Performing $R_2 \leftarrow R_2 \times b_2$ using elementary matrix

$$\mathbf{E_4} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & b_2 & 0\dots & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \tag{2.1.8}$$

$$\mathbf{E_4}(\mathbf{E_3}(\mathbf{E_2}(\mathbf{E_1}\mathbf{R_1}))) = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} = \mathbf{R_2} \tag{2.1.9}$$

$$\implies \mathbf{R_2} = \mathbf{E}\mathbf{R_1} \tag{2.1.10}$$

This proves that $\mathbf{R_2}$ can be obtained by linear combinations of $\mathbf{R_1}$
But $\mathbf{R_1}$ is a linear combination of $\mathbf{A}$ and $\mathbf{R_2}$ is a linear combination of $\mathbf{B}$
Hence, $\mathbf{B}$ can be obtained by linear combinations of $\mathbf{A}$

2.2 Case 2

Let us assume that (2.0.3),(2.0.4) have infinitely many solutions
So,

$$\text{either rank}(\mathbf{A}) = \text{rank}(\mathbf{B}) = 1 \tag{2.2.1}$$

$$\text{or rank}(\mathbf{A}) = \text{rank}(\mathbf{B}) = 0 \tag{2.2.2}$$

Rank zero indicates both $\mathbf{A}$ and $\mathbf{B}$ are null matrices and are equivalent.
If, rank of $\mathbf{A}$ = rank of $\mathbf{B}$ = 1

$$\mathbf{R_1} = \begin{pmatrix} a_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \tag{2.2.3}$$

$$\mathbf{R_2} = \begin{pmatrix} b_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \tag{2.2.4}$$

Clearly $\mathbf{R_1}$ and $\mathbf{R_2}$ are related through an elementary row operation as

$$\mathbf{R_2} = \mathbf{E}\mathbf{R_2} \tag{2.2.5}$$

$$\text{where } \mathbf{E} = \begin{pmatrix} \frac{b_1}{a_1} & 0 & 0 & \dots & 0 \\ 0 & 1 & 0\dots & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \tag{2.2.6}$$

So, (2.0.4) can be expressed as linear combinations of (2.0.3) indicating that the two systems of equations are equivalent.