Assignment 9

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Download codes from

https://github.com/KUSUMAPRIYAPULAVARTY/assignment9

1 Question

Prove that if two homogenous systems of linear equations in two unknowns have the same solutions, then they are equivalent.

2 Solution

Let the two systems of homogenous equations be

$$\mathbf{Ax} = \mathbf{0} \tag{2.0.1}$$

$$\mathbf{Bx} = \mathbf{0} \tag{2.0.2}$$

$$\implies \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ \vdots & \vdots \\ A_{n1} & A_{n2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{2.0.3}$$

$$\text{and } \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ \vdots & \vdots \\ B_{n1} & B_{n2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{2.0.4}$$

2.1 Case 1

Let us assume that the solution is unique. Since they have the same solution, both $\mathbf{A}, \mathbf{B}$ must have their rank as 2.
Let the reduced row echelon form of $\mathbf{A}$ be $\mathbf{R_1}$ and $\mathbf{B}$ be $\mathbf{R_2}$
Finding the reduced row echelon form of $\mathbf{A}$,

$$\mathbf{R_1} = \mathbf{CA} \tag{2.1.1}$$

where $\mathbf{C}$ is a product of elementary matrices

$$\mathbf{C} = \mathbf{E_5 E_4 E_3 E_2 E_1} \tag{2.1.2}$$

$$\mathbf{E_1} = \begin{pmatrix} \frac{1}{A_{11}} & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \tag{2.1.3}$$

$$\mathbf{E_2} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -A_{21} & 1 & 0 & \dots & 0 \\ -A_{31} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ -A_{n1} & 0 & 0 & \dots & 1 \end{pmatrix} \tag{2.1.4}$$

$$\mathbf{E_3} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{A_{22}-\frac{A_{21}A_{12}}{A_{11}}} & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \tag{2.1.5}$$

$$\mathbf{E_4} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & -A_{32} + \frac{A_{31}A_{12}}{A_{11}} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & -A_{n2} + \frac{A_{n1}A_{12}}{A_{11}} & 0 & \dots & 1 \end{pmatrix} \tag{2.1.6}$$

$$\mathbf{E_5} = \begin{pmatrix} 1 & \frac{-A_{21}}{A_{11}} & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \tag{2.1.7}$$

$$\implies \mathbf{R_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \tag{2.1.8}$$

Finding the reduced row echelon form of $\mathbf{B}$,

$$\mathbf{R_2} = \mathbf{DB} \tag{2.1.9}$$

where $\mathbf{D}$ is a product of elementary matrices

$$\mathbf{D} = \mathbf{E'_5 E'_4 E'_3 E'_2 E'_1} \tag{2.1.10}$$

$$\mathbf{E'_1} = \begin{pmatrix} \frac{1}{B_{11}} & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \tag{2.1.11}$$

$$\mathbf{E'_2} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -B_{21} & 1 & 0 & \dots & 0 \\ -B_{31} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ -B_{n1} & 0 & 0 & \dots & 1 \end{pmatrix} \tag{2.1.12}$$

$$\mathbf{E'_3} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{B_{22}-\frac{B_{21}B_{12}}{B_{11}}} & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \tag{2.1.13}$$

$$\mathbf{E'_4} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & -B_{32} + \frac{B_{31}B_{12}}{B_{11}} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & -B_{n2} + \frac{B_{n1}B_{12}}{B_{11}} & 0 & \dots & 1 \end{pmatrix} \tag{2.1.14}$$

$$\mathbf{E'_5} = \begin{pmatrix} 1 & \frac{-B_{21}}{B_{11}} & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \tag{2.1.15}$$

$$\implies \mathbf{R_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \tag{2.1.16}$$

Therefore, (2.1.1) and (2.1.9) imply

$$\mathbf{CA} = \mathbf{DB} \tag{2.1.17}$$

Since $\mathbf{C}$ and $\mathbf{D}$ are product of elementary matrices and all the elementary matrices are invertible,

$$\mathbf{C^{-1}} = \mathbf{E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1}} \tag{2.1.18}$$

$$\mathbf{D^{-1}} = \mathbf{E_1'^{-1} E_2'^{-1} E_3'^{-1} E_4'^{-1} E_5'^{-1}} \tag{2.1.19}$$

$$\implies \mathbf{A} = (\mathbf{C^{-1}D})\mathbf{B} \tag{2.1.20}$$

$$\mathbf{B} = (\mathbf{D^{-1}C})\mathbf{A} \tag{2.1.21}$$

Hence the equations in one system can be obtained through elementary operations on equations of other system.Hence the systems (2.0.3) and (2.0.4) are equivalent.

2.2 *Case 2*

Let us assume that (2.0.3),(2.0.4) have infinitely many solutions
So,

$$\text{either rank}(\mathbf{A}) = \text{rank}(\mathbf{B}) = 1 \tag{2.2.1}$$

$$\text{or rank}(\mathbf{A}) = \text{rank}(\mathbf{B}) = 0 \tag{2.2.2}$$

If, rank of $\mathbf{A}$ = rank of $\mathbf{B}$ = 1
Row reduced echelon forms of $\mathbf{A}, \mathbf{B}$ become $\mathbf{R_1}, \mathbf{R_2}$

$$\mathbf{R_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \tag{2.2.3}$$

$$\mathbf{R_2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \tag{2.2.4}$$

Hence the same approach as in case 1 yields equivalence of two systems.
Rank zero indicates both $\mathbf{A}$ and $\mathbf{B}$ are null matrices and hence the systems (2.0.3) and (2.0.4) are equivalent.