Assignment 9

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Download codes from

https://github.com/KUSUMAPRIYAPULAVARTY/assignment9

1 Question

Prove that if two homogenous systems of linear equations in two unknowns have the same solutions, then they are equivalent.

2 Solution

Let the two systems of homogenous equations be

$$\mathbf{Ax} = \mathbf{0} \tag{2.0.1}$$

$$\mathbf{Bx} = \mathbf{0} \tag{2.0.2}$$

$$\implies \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ \vdots & \vdots \\ A_{n1} & A_{n2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{2.0.3}$$

$$\text{and} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ \vdots & \vdots \\ B_{n1} & B_{n2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{2.0.4}$$

2.1 Case 1

Let us assume that the solution is unique. Since they have the same solution, both $\mathbf{A}, \mathbf{B}$ must have their rank as 2.

Let the reduced row echelon form of $\mathbf{A}$ be $\mathbf{R_1}$ and $\mathbf{B}$ be $\mathbf{R_2}$

$$\mathbf{R_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \tag{2.1.1}$$

$$\mathbf{R_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \tag{2.1.2}$$

Consider $\mathbf{R_1}$
Performing elementary row operation
$R_1 \leftarrow R_1 + R_2 \times \frac{B_{12}}{B_{11}}$
using elementary matrix,

$$\mathbf{E_1} = \begin{pmatrix} 1 & \frac{B_{12}}{B_{11}} & 0 & \dots & 0 \\ 0 & 1 & 0\dots & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \tag{2.1.3}$$

$$\implies \mathbf{E_1R_1} = \begin{pmatrix} 1 & \frac{B_{12}}{B_{11}} \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \tag{2.1.4}$$

Performing the row operations

$$R_3 \leftarrow R_3 + R_2 \tag{2.1.5}$$

$$R_4 \leftarrow R_4 + R_2 \tag{2.1.6}$$

$$\vdots \tag{2.1.7}$$

$$R_n \leftarrow R_n + R_2 \tag{2.1.8}$$

Using a product of elementary matrices

$$\mathbf{E_2} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 1 & 0 & \dots & 1 \end{pmatrix} \tag{2.1.9}$$

$$\implies \mathbf{E_2}(\mathbf{E_1}\mathbf{R_1}) = \begin{pmatrix} 1 & \frac{B_{12}}{B_{11}} \\ 0 & 1 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix} \tag{2.1.10}$$

Performing the row operations

$$R_2 \leftarrow R_2 \times B_{22} - \frac{B_{21}B_{12}}{B_{11}} \tag{2.1.11}$$

$$\vdots \tag{2.1.12}$$

$$R_n \leftarrow R_n \times B_{n2} - \frac{B_{n1}B_{12}}{B_{11}} \tag{2.1.13}$$

Using a product of elementary matrices

$$\mathbf{E_3} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & B_{22} - \frac{B_{21}B_{12}}{B_{11}} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & B_{n2} - \frac{B_{n1}B_{12}}{B_{11}} \end{pmatrix} \tag{2.1.14}$$

$$\implies \mathbf{E_3}(\mathbf{E_2}(\mathbf{E_1}\mathbf{R_1})) = \begin{pmatrix} 1 & \frac{B_{12}}{B_{11}} \\ 0 & B_{22} - \frac{B_{21}B_{12}}{B_{11}} \\ \vdots & \vdots \\ 0 & B_{n2} - \frac{B_{n1}B_{12}}{B_{11}} \end{pmatrix} \tag{2.1.15}$$

Performing the row operations

$$R_2 \leftarrow R_2 + B_{21} \times R_1 \tag{2.1.16}$$

$$\vdots \tag{2.1.17}$$

$$R_n \leftarrow R_n + B_{n1} \times R_1 \tag{2.1.18}$$

Using a product of elementary matrices

$$\mathbf{E_4} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ B_{21} & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ B_{n1} & 0 & \dots & 1 \end{pmatrix} \tag{2.1.19}$$

$$\implies \mathbf{E_4}(\mathbf{E_3}(\mathbf{E_2}(\mathbf{E_1}\mathbf{R_1}))) = \begin{pmatrix} 1 & \frac{B_{12}}{B_{11}} \\ B_{21} & B_{22} \\ \vdots & \vdots \\ B_{n1} & B_{n2} \end{pmatrix} \tag{2.1.20}$$

Performing $R_1 \leftarrow R_1 \times B_{11}$ using elementary matrix

$$\mathbf{E_5} = \begin{pmatrix} B_{11} & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \tag{2.1.21}$$

$$\implies \mathbf{E_5}(\mathbf{E_4}(\mathbf{E_3}(\mathbf{E_2}(\mathbf{E_1}\mathbf{R_1})))) = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ \vdots & \vdots \\ B_{n1} & B_{n2} \end{pmatrix} = \mathbf{B} \tag{2.1.22}$$

$$\implies \mathbf{B} = \mathbf{E}\mathbf{R_1} \tag{2.1.23}$$

$$\text{where } \mathbf{E} = \mathbf{E_5}\mathbf{E_4}\mathbf{E_3}\mathbf{E_2}\mathbf{E_1} \tag{2.1.24}$$

is a product of elementary matrices.
This indicates that **B** is obtained by linear combinations of $\mathbf{R_1}$ which is a linear combination of system **A**.Hence **B** is obtained through linear combinations of **A**.

2.2 Case 2

Let us assume that (2.0.3),(2.0.4) have infinitely many solutions
So,

$$\text{either rank}(\mathbf{A}) = \text{rank}(\mathbf{B}) = 1 \tag{2.2.1}$$

$$\text{or rank}(\mathbf{A}) = \text{rank}(\mathbf{B}) = 0 \tag{2.2.2}$$

Rank zero indicates both **A** and **B** are null matrices and are equivalent.
If, rank of **A** = rank of **B** = 1
Row reduced echelon forms of $\mathbf{A}, \mathbf{B}$ become $\mathbf{R_1}, \mathbf{R_2}$

$$\mathbf{R_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \tag{2.2.3}$$

$$\mathbf{R_2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \tag{2.2.4}$$

Hence the same approach as in case 1 yields

$$\mathbf{B} = \mathbf{E}\mathbf{R_1} \tag{2.2.5}$$

where **E** is a product of elementary matrices. So, (2.0.4) can be expressed as linear combinations of

(2.0.3) indicating that the two systems of equations are equivalent.