Assignment 9

KUSUMA PRIYA
EE20MTECH11007

Download codes from

https://github.com/KUSUMAPRIYAPULAVARTY/assignment9

1 QUESTION

Prove that if two homogenous systems of linear equations in two unknowns have the same solutions, then they are equivalent.

2 SOLUTION

Let the two systems of homogenous equations be

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ \vdots & \vdots \\ A_{n1} & A_{n2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{2.0.1}$$

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ \vdots & \vdots \\ B_{n1} & B_{n2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{2.0.2}$$

$$\implies \mathbf{A}\mathbf{x} = \mathbf{0} \tag{2.0.3}$$

$$\mathbf{B}\mathbf{y} = \mathbf{0} \tag{2.0.4}$$

We can write

$$\mathbf{C}\mathbf{A}\mathbf{x} = \mathbf{0} \tag{2.0.5}$$

$$\mathbf{D}\mathbf{B}\mathbf{y} = \mathbf{0} \tag{2.0.6}$$

where $\mathbf{C}$ and $\mathbf{D}$ are product of elementary matrices that reduce $\mathbf{A}$ and $\mathbf{B}$ into their reduced row echelon forms $\mathbf{R_1}$ and $\mathbf{R_2}$
(2.0.5) and (2.0.6) imply

$$\mathbf{R_1}\mathbf{x} = 0 \tag{2.0.7}$$

$$\mathbf{R_2}\mathbf{y} = 0 \tag{2.0.8}$$

Given that they have same solution, we can write

$$\mathbf{R_1}\mathbf{x} = 0 \tag{2.0.9}$$

$$\mathbf{R_2}\mathbf{x} = 0 \tag{2.0.10}$$

$$\implies (\mathbf{R_1} - \mathbf{R_2})\mathbf{x} = 0 \tag{2.0.11}$$

Note that for a solution to exist, $\mathbf{R_1}$ and $\mathbf{R_2}$ can be either of matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \text{or} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \tag{2.0.12}$$

2.1 *Case 1*

Let us assume that the solution is unique. The unique solution is

$$\mathbf{x} = \mathbf{0} \tag{2.1.1}$$

Since they have the same solution, both $\mathbf{R_1}, \mathbf{R_2}$ must have their rank as 2.
So,

$$\mathbf{R_1} = \mathbf{R_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \tag{2.1.2}$$

2.2 *Case 2*

Let us assume that (2.0.3),(2.0.4) have infinitely many solutions
So,

$$\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{B}) = 1 \tag{2.2.1}$$

equation (2.0.11) for solutions other than zero solution implies

$$\mathbf{R_1} = \mathbf{R_2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \tag{2.2.2}$$

So, in both the cases, we have

$$\mathbf{R_1} = \mathbf{R_2} \qquad (2.2.3)$$

Hence the two systems of equations are equivalent.