Assignment 9

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Download codes from

https://github.com/KUSUMAPRIYAPULAVARTY/assignment9

1 QUESTION

Prove that if two homogenous systems of linear equations in two unknowns have the same solutions, then they are equivalent.

2 SOLUTION

Let the two systems of homogenous equations be

$$\mathbf{A}\mathbf{x} = \mathbf{0} \tag{2.0.1}$$

$$\mathbf{B}\mathbf{y} = \mathbf{0} \tag{2.0.2}$$

We can write

$$\mathbf{C}\mathbf{A}\mathbf{x} = \mathbf{0} \tag{2.0.3}$$

$$\mathbf{D}\mathbf{B}\mathbf{y} = \mathbf{0} \tag{2.0.4}$$

where $\mathbf{C}$ and $\mathbf{D}$ are product of elementary matrices that reduce $\mathbf{A}$ and $\mathbf{B}$ into their reduced row echelon forms $\mathbf{R_1}$ and $\mathbf{R_2}$
(2.0.3) and (2.0.4) imply

$$\mathbf{R_1}\mathbf{x} = 0 \tag{2.0.5}$$

$$\mathbf{R_2}\mathbf{y} = 0 \tag{2.0.6}$$

Given that they have same solution, we can write

$$\mathbf{R_1}\mathbf{x} = 0 \tag{2.0.7}$$

$$\mathbf{R_2}\mathbf{x} = 0 \tag{2.0.8}$$

$$\implies (\mathbf{R_1} - \mathbf{R_2})\mathbf{x} = 0 \tag{2.0.9}$$

Note that for a solution to exist, $\mathbf{R_1}$ and $\mathbf{R_2}$ can be either of matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{or} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.10}$$

2.1 Case 1

Let us assume that the solution is unique. The unique solution is

$$\mathbf{x} = \mathbf{0} \tag{2.1.1}$$

Since they have the same solution, both $\mathbf{R_1}, \mathbf{R_2}$ must have their rank as 2.
So,

$$\mathbf{R_1} = \mathbf{R_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.1.2}$$

2.2 Case 2

Let us assume that (2.0.3),(2.0.4) have infinitely many solutions
So,

$$\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{B}) = 1 \tag{2.2.1}$$

equation (2.0.9) for solutions other than zero solution implies

$$\mathbf{R_1} = \mathbf{R_2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.2.2}$$

So, in both the cases, we have

$$\mathbf{R_1} = \mathbf{R_2} \tag{2.2.3}$$

$$\implies \mathbf{C}\mathbf{A} = \mathbf{D}\mathbf{B} \tag{2.2.4}$$

Since $\mathbf{C}, \mathbf{D}$ are product of elementary matrices, they are invertible.

$$\implies \mathbf{A} = \mathbf{C}^{-1}\mathbf{D}\mathbf{B} \tag{2.2.5}$$

$$\mathbf{B} = \mathbf{D}^{-1}\mathbf{C}\mathbf{A} \tag{2.2.6}$$

$$\text{Let } \mathbf{C}^{-1}\mathbf{D} = \mathbf{E} \tag{2.2.7}$$

where $\mathbf{E}$ is also a product of elementary matrices (2.2.5) and (2.2.6) hence become

$$\mathbf{A} = \mathbf{E}\mathbf{B} \tag{2.2.8}$$

$$\mathbf{B} = \mathbf{E}^{-1}\mathbf{A} \tag{2.2.9}$$

Hence the two systems of equations are equivalent.