Assignment 9

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Download codes from

https://github.com/KUSUMAPRIYAPULAVARTY/assignment9

1 Question

Prove that if two homogenous systems of linear equations in two unknowns have the same solutions, then they are equivalent.

2 Solution

Let the two systems of homogenous equations be

$$\mathbf{A}\mathbf{x} = \mathbf{0} \tag{2.0.1}$$

$$\mathbf{B}\mathbf{x} = \mathbf{0} \tag{2.0.2}$$

$$\implies \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ \vdots & \vdots \\ A_{n1} & A_{n2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{2.0.3}$$

$$\text{and} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ \vdots & \vdots \\ B_{n1} & B_{n2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{2.0.4}$$

2.1 Case 1

Let us assume that the solution is unique. Since they have the same solution, both $\mathbf{A}, \mathbf{B}$ must have their rank as 2.

Let the reduced row echelon form of $\mathbf{A}$ be $\mathbf{R_1}$ and $\mathbf{B}$ be $\mathbf{R_2}$

$$\mathbf{R_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \tag{2.1.1}$$

$$\mathbf{R_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \tag{2.1.2}$$

Now the systems are equivalent if one of them can be obtained as a combination of others.Note that, reduced row echelon forms are obtained through combinations of rows of matrices implying they are linear combinations of the equations in the systems. Hence,If systems are equivalent, we need to be able to write

$$\mathbf{B} = \mathbf{E}\mathbf{R_1} \tag{2.1.3}$$

$$\tag{2.1.4}$$

$\mathbf{E}$ is a product of elementary matrices obtained as

$$\mathbf{E} = \mathbf{E_5}\mathbf{E_4}\mathbf{E_3}\mathbf{E_4}\mathbf{E_1} \tag{2.1.5}$$

$$\text{where } \mathbf{E_1} = \begin{pmatrix} 1 & \frac{B_{12}}{B_{11}} & 0 & \dots & 0 \\ 0 & 1 & 0\dots & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \tag{2.1.6}$$

$$\mathbf{E_2} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 1 & 0 & \dots & 1 \end{pmatrix} \tag{2.1.7}$$

$$\mathbf{E_3} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & B_{22} - \frac{B_{21}B_{12}}{B_{11}} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & B_{n2} - \frac{B_{n1}B_{12}}{B_{11}} \end{pmatrix} \tag{2.1.8}$$

$$\mathbf{E_4} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ B_{21} & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ B_{n1} & 0 & \dots & 1 \end{pmatrix} \tag{2.1.9}$$

$$\mathbf{E_5} = \begin{pmatrix} B_{11} & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \tag{2.1.10}$$

This indicates that $\mathbf{B}$ is obtained by linear combinations of $\mathbf{A}$

2.2 Case 2

Let us assume that (2.0.3),(2.0.4) have infinitely many solutions
So,

$$\text{either } \operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{B}) = 1 \tag{2.2.1}$$

$$\text{or } \operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{B}) = 0 \tag{2.2.2}$$

Rank zero indicates both $\mathbf{A}$ and $\mathbf{B}$ are null matrices and are equivalent.
If, rank of $\mathbf{A}$ = rank of $\mathbf{B}$ = 1
Row reduced echelon forms of $\mathbf{A}, \mathbf{B}$ become $\mathbf{R_1}, \mathbf{R_2}$

$$\mathbf{R_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \tag{2.2.3}$$

$$\mathbf{R_2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \tag{2.2.4}$$

Hence the same approach as in case 1 yields

$$\mathbf{B} = \mathbf{E}\mathbf{R_1} \tag{2.2.5}$$

where $\mathbf{E}$ is a product of elementary matrices. So, (2.0.4) can be expressed as linear combinations of (2.0.3) indicating that the two systems of equations are equivalent.