Problem 2.1.1:

$$I_1 = \int \frac{x\,dx}{(x^2+z^2)^{3/2}}$$

where z is constant

This integral can be evaluated using **two** "u"-substitutions

- First subs $\boxed{u = x^2 \quad du = 2x\,dx}$

$$I_1 = \frac{1}{2}\int \frac{du}{(u+z^2)^{3/2}}$$

- second subs $\boxed{w = u + z^2 \quad dw = du}$

$$I_1 = \frac{1}{2}\int \frac{dw}{w^{3/2}} = \frac{1}{2}\int w^{-3/2}\,dw$$

Now integrate the polynomial

$$I_1 = \frac{1}{2}\left(\frac{1}{-1/2} w^{-1/2}\right) = -w^{-1/2} = -\frac{1}{\sqrt{w}}$$

then back substitute:

$$I_1 = -\frac{1}{\sqrt{u+z^2}} = \boxed{-\frac{1}{\sqrt{x^2+z^2}}}$$

problem 2.1.2:

this integral uses a "trig-sub"

$$I_2 = \int \frac{dx}{(x^2+z^2)^{3/2}}$$

$\int$ · substitution

$x = z\tan\theta$

$dx = z\sec^2\theta\, d\theta$

trig identities

$\tan^2\theta + 1 = \sec^2\theta$

$\sin\theta = \dfrac{x}{\sqrt{x^2+z^2}}$

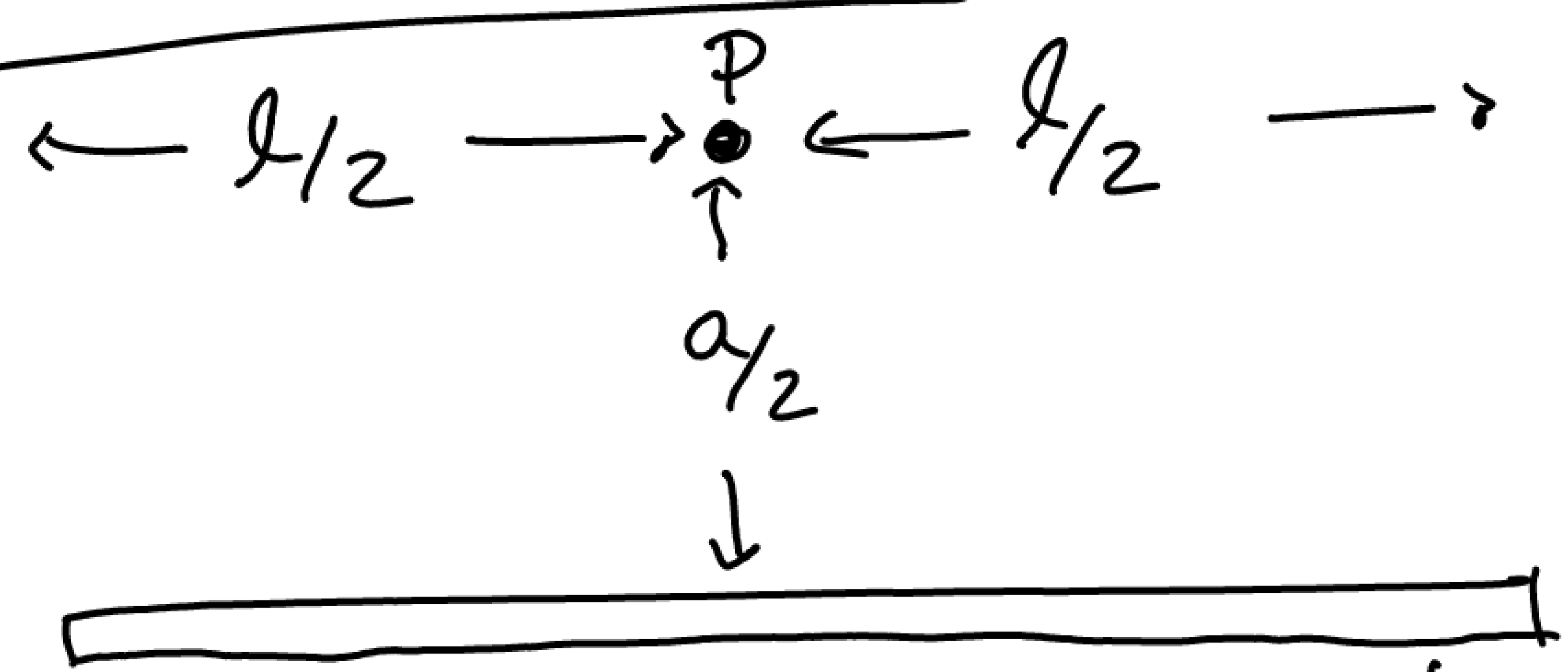
$$I_2 = \int \frac{z\sec^2\theta\, d\theta}{(z^2\tan^2\theta + z^2)^{3/2}} = \int \frac{z\sec^2\theta\, d\theta}{(z^2\sec^2\theta)^{3/2}}$$

$$= \int \frac{z\sec^2\theta\, d\theta}{z^3\sec^3\theta} = \frac{1}{z^2}\int \frac{d\theta}{\sec\theta} = \frac{1}{z^2}\int \cos\theta\, d\theta$$

$$= \frac{1}{z^2}\sin\theta = \boxed{\frac{x}{z^2\sqrt{x^2+z^2}}}$$

yikes this is a tricky integral

Problem 2.2.1: (Calculus Problems)

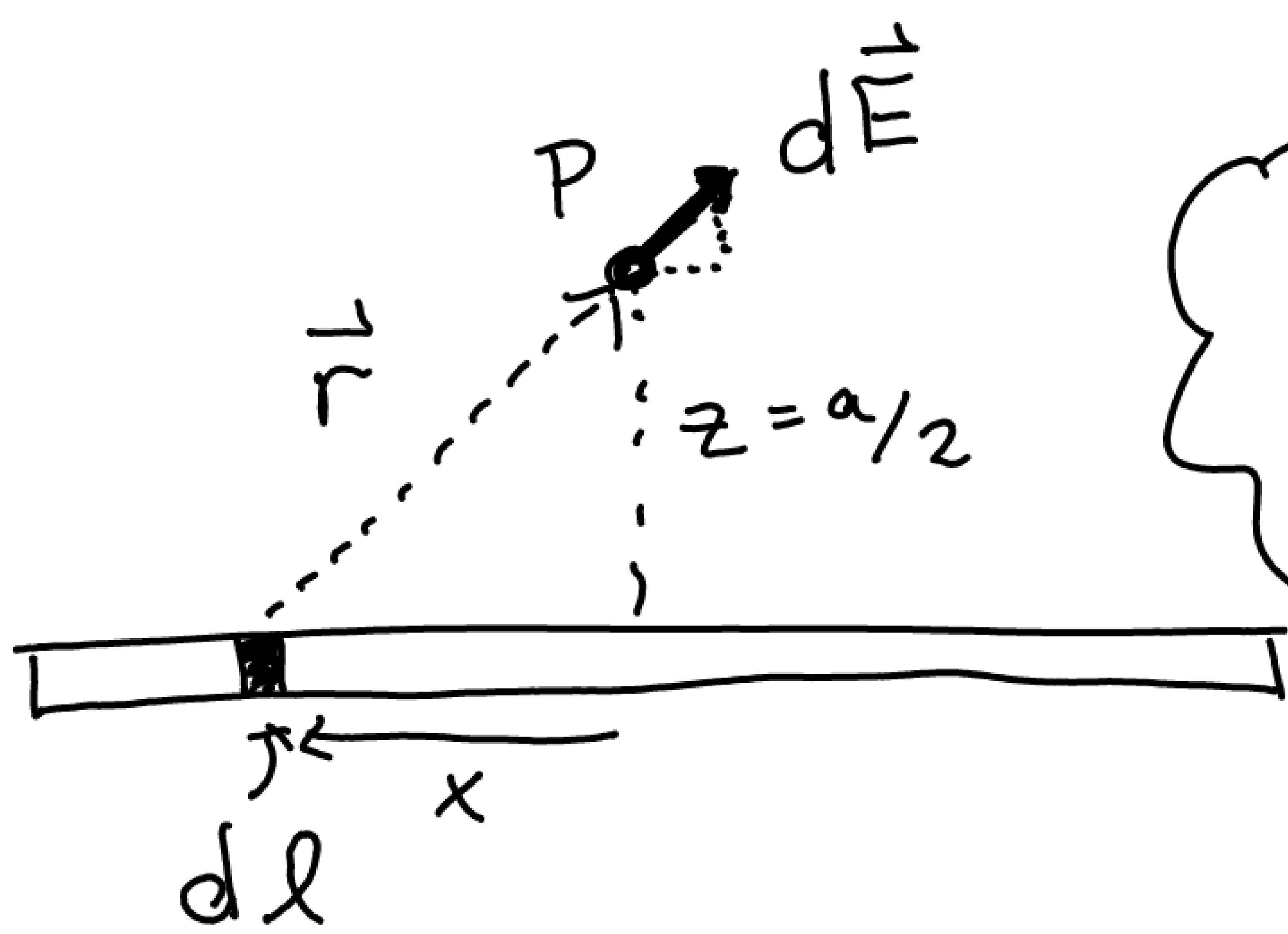


uniform charge q on rod

use the differential form of the $\vec{E}$-Field

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0} \int \frac{dq}{|\vec{r}|^2}\,\hat{r}$$

where $\vec{r}$ is the vector from: the infinitesimal chunk of charge to: the Point P



← Sum up all of the $d\vec{E}$

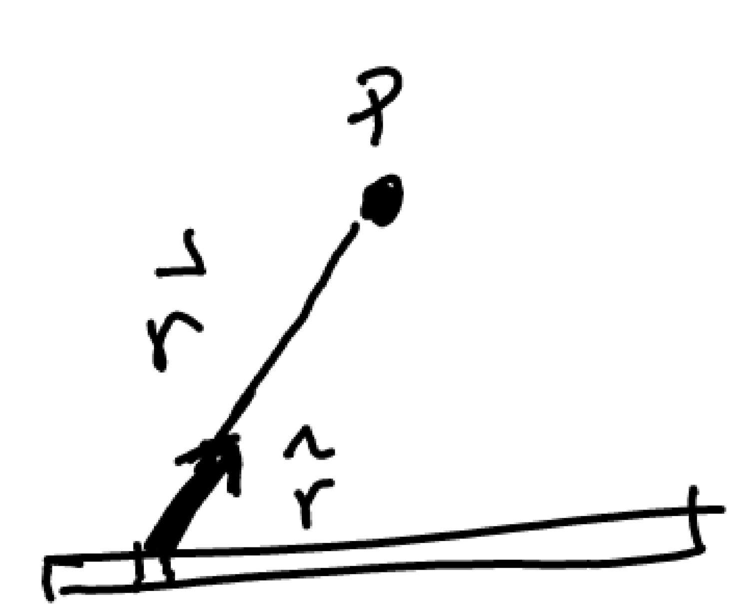
Note: the horizontal parts must cancel from symmetry

use "uniform linear density": $\lambda = \frac{q}{\ell}$

$dq = \lambda\, d\ell$

will need to use unit vectors:

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{x} + z\hat{z}}{\sqrt{x^2 + z^2}}$$



$$E_x = \frac{\lambda}{4\pi\varepsilon_0}\int_{-l/2}^{l/2}\frac{dx}{x^2+z^2}\,\frac{x}{\sqrt{x^2+z^2}}$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\int\frac{x\,dx}{(x^2+z^2)^{3/2}}$$

$\int$ substitution

$u = x^2+z^2$

$du = 2x\,dx$

$$= \frac{\lambda}{4\pi\varepsilon_0}\int\frac{du}{2u^{3/2}}$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\,\frac{1}{2}\int u^{-3/2}\,du$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\,\frac{1}{2}\left(-\frac{1}{1/2}\,u^{-1/2}\right)$$

$$= -\frac{\lambda}{4\pi\varepsilon_0}\,\frac{1}{\sqrt{x^2+z^2}}\Bigg|_{-l/2}^{l/2}$$

this is zero! $\left(\frac{l}{2}\right)^2 = \left(\frac{-l}{2}\right)^2$

$$= \frac{-\lambda}{4\pi\varepsilon_0}\left[\frac{1}{(l/2)^2+z^2} - \frac{1}{\left(\frac{-l}{2}\right)^2+z^2}\right] = 0$$

The horizontal field vanishes $\boxed{E_x = 0}$

$$E_z = \frac{\lambda}{4\pi\varepsilon_0} \int \frac{dx}{x^2+z^2} \frac{z}{\sqrt{x^2+y^2}}$$

$$= \frac{\lambda z}{4\pi\varepsilon_0} \int \frac{dx}{(x^2+z^2)^{3/2}}$$

<u>Note</u> this is different than E_x because the "z" term is constant and pulls out of the integral

<u>$\int$ substitution</u>

$$x = z\tan\theta$$

$$dx = z\sec^2\theta\, d\theta$$

<u>trig identities</u>

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\sin\theta = \frac{x}{\sqrt{x^2+z^2}}$$

$$E_z = \frac{\lambda z}{4\pi\varepsilon_0} \int \frac{z\sec^2\theta\, d\theta}{(z^2\tan^2\theta + z^2)^{3/2}}$$

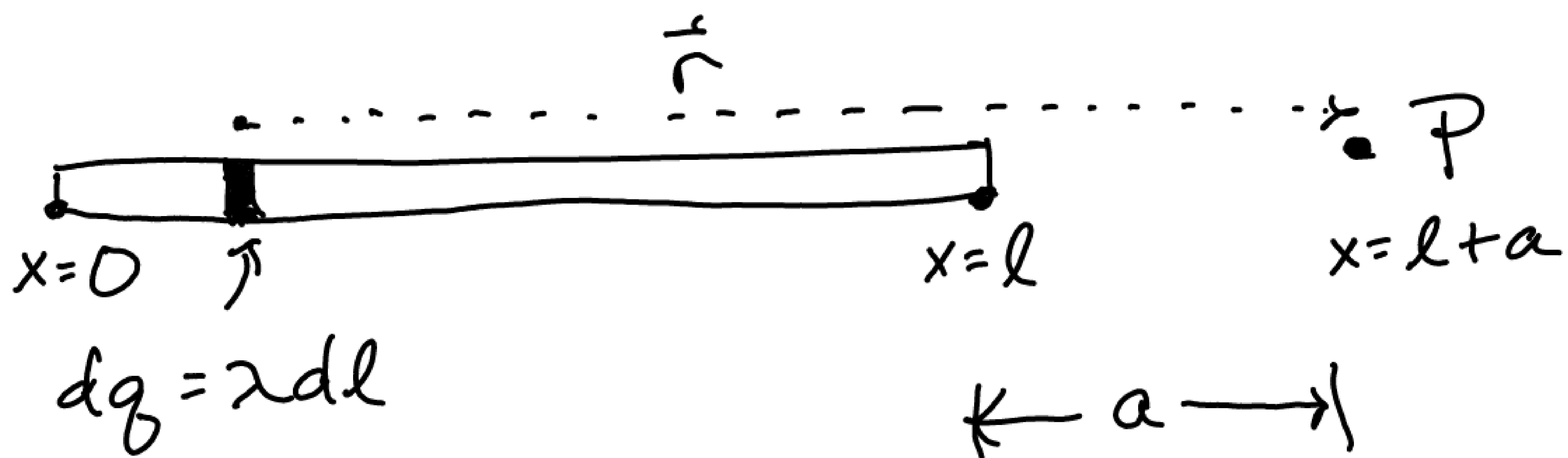
$$= \frac{\lambda z}{4\pi\varepsilon_0} \int \frac{1}{z^2}\cos\theta = \frac{\lambda z}{4\pi\varepsilon_0} \frac{1}{z^2}\sin\theta$$

$$= \frac{\lambda}{4\pi\varepsilon_0 z} \frac{x}{\sqrt{x^2+z^2}} \Bigg|_{-l/2}^{l/2}$$

$$= \frac{\lambda}{4\pi\varepsilon_0 z} \left[\frac{l/2}{\left(\frac{l}{2}^2+z^2\right)^{1/2}} - \frac{-l/2}{\left(l/2^2+z^2\right)^{1/2}} \right]$$

$$\boxed{\vec{E}_z = \frac{\lambda l}{4\pi\varepsilon_0 z} \frac{1}{\sqrt{(l/2)^2 + (a/2)^2}}} \leftarrow q = \lambda l$$

Problem 2.2.2



$\vec{r} = l+a-x$

$dq = \lambda dl$

$\lambda = q/l$

$dl = dx$

$$dE = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r^2}\,\hat{r}$$

$\vec{r} = (l+a-x, 0, 0)$

$\hat{r} = (1, 0, 0) = \hat{x}$

$$dE = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda\, dx}{(l+a-x)^2}\,\hat{x}$$

$u = l+a-x$

$du = -x$

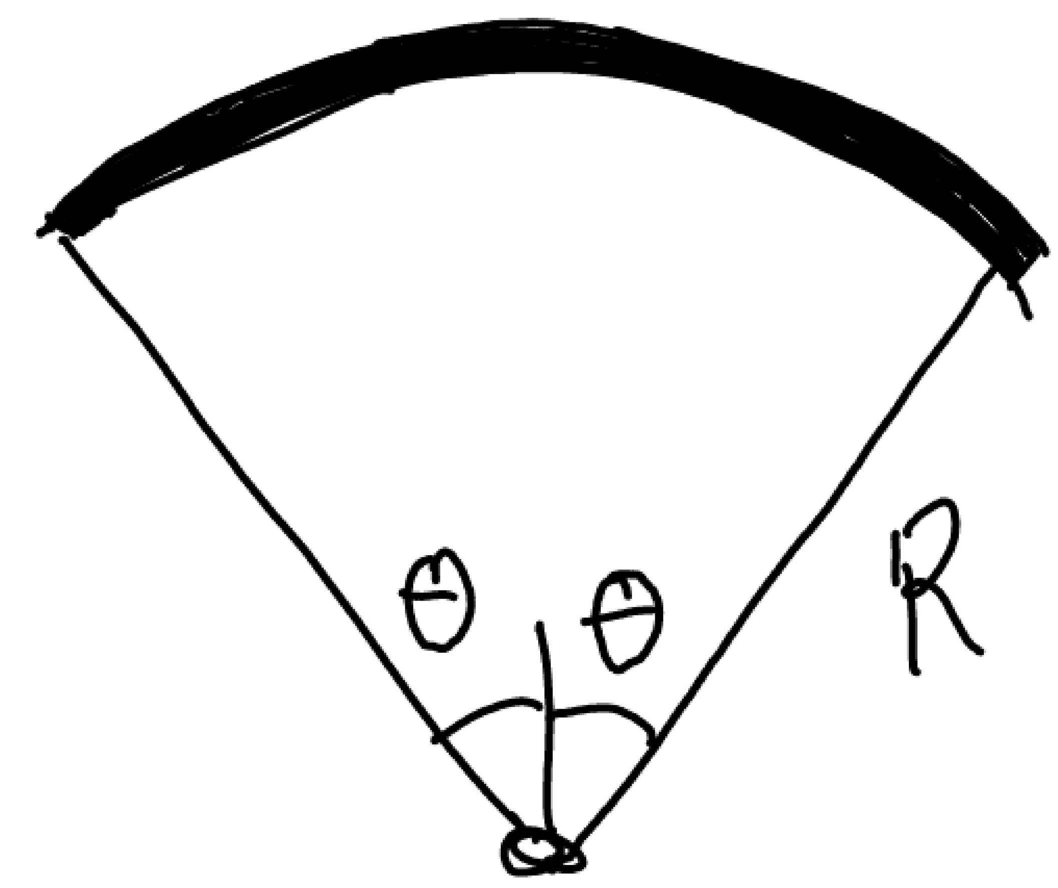
$$= \frac{\lambda\hat{x}}{4\pi\varepsilon_0}\int -\frac{du}{u^2}$$

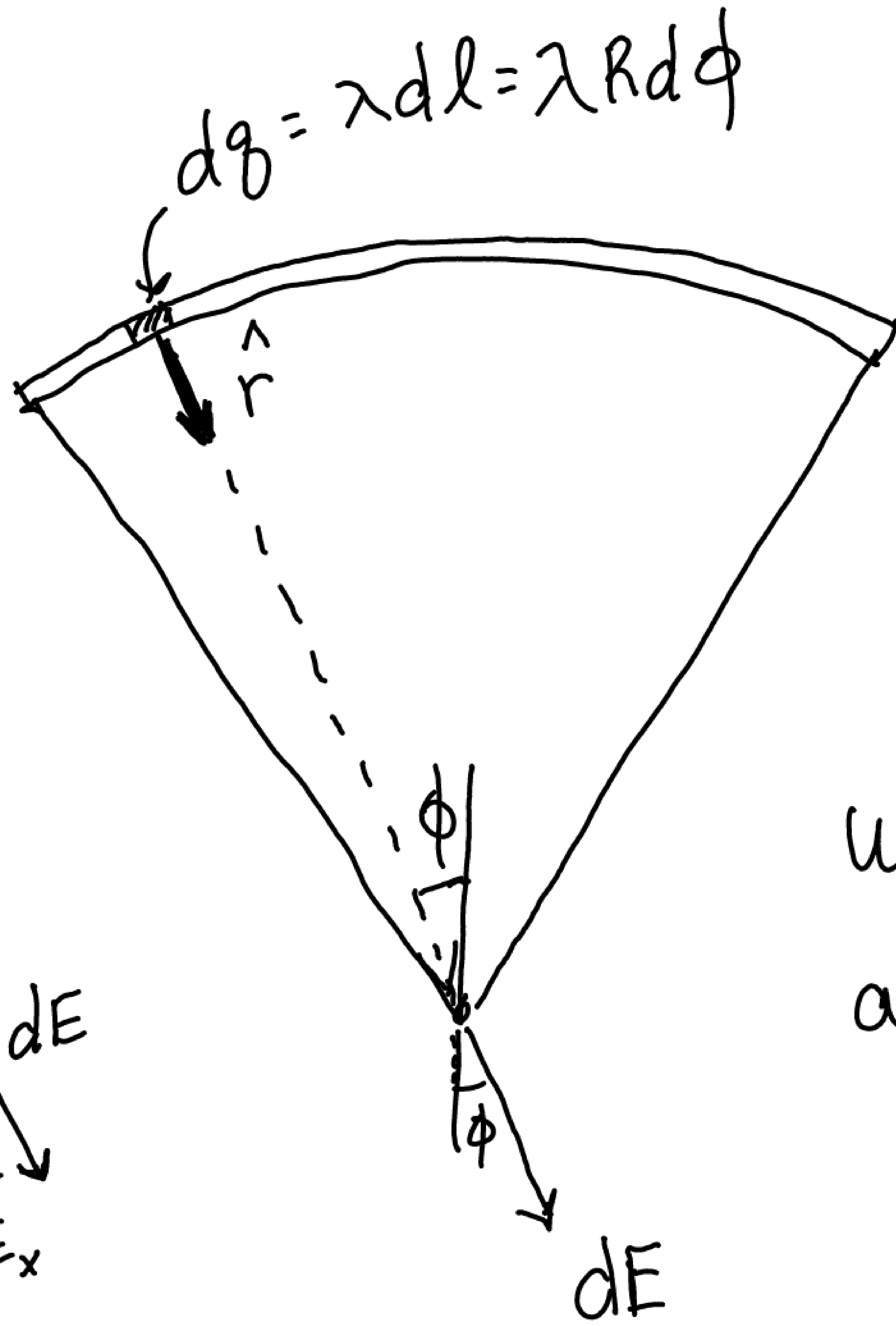
$$= \frac{\lambda\hat{x}}{4\pi\varepsilon_0}\left(\frac{1}{u}\right) = \frac{\lambda\hat{x}}{4\pi\varepsilon_0}\left(\left.\frac{1}{l+a-x}\right|_0^l\right)$$

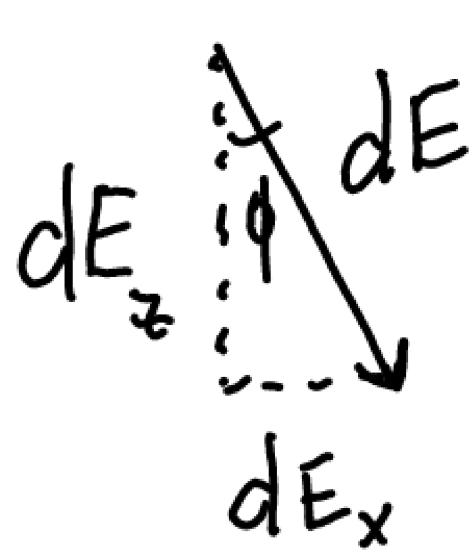
$$\boxed{\vec{E} = \frac{\lambda\hat{x}}{4\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{l+a}\right]}$$

Problem 2.3:

arc of charge







use ϕ as the variable and integrate from $(-\theta$ to $\theta)$

From symmetry: Horizontal part of dE cancels

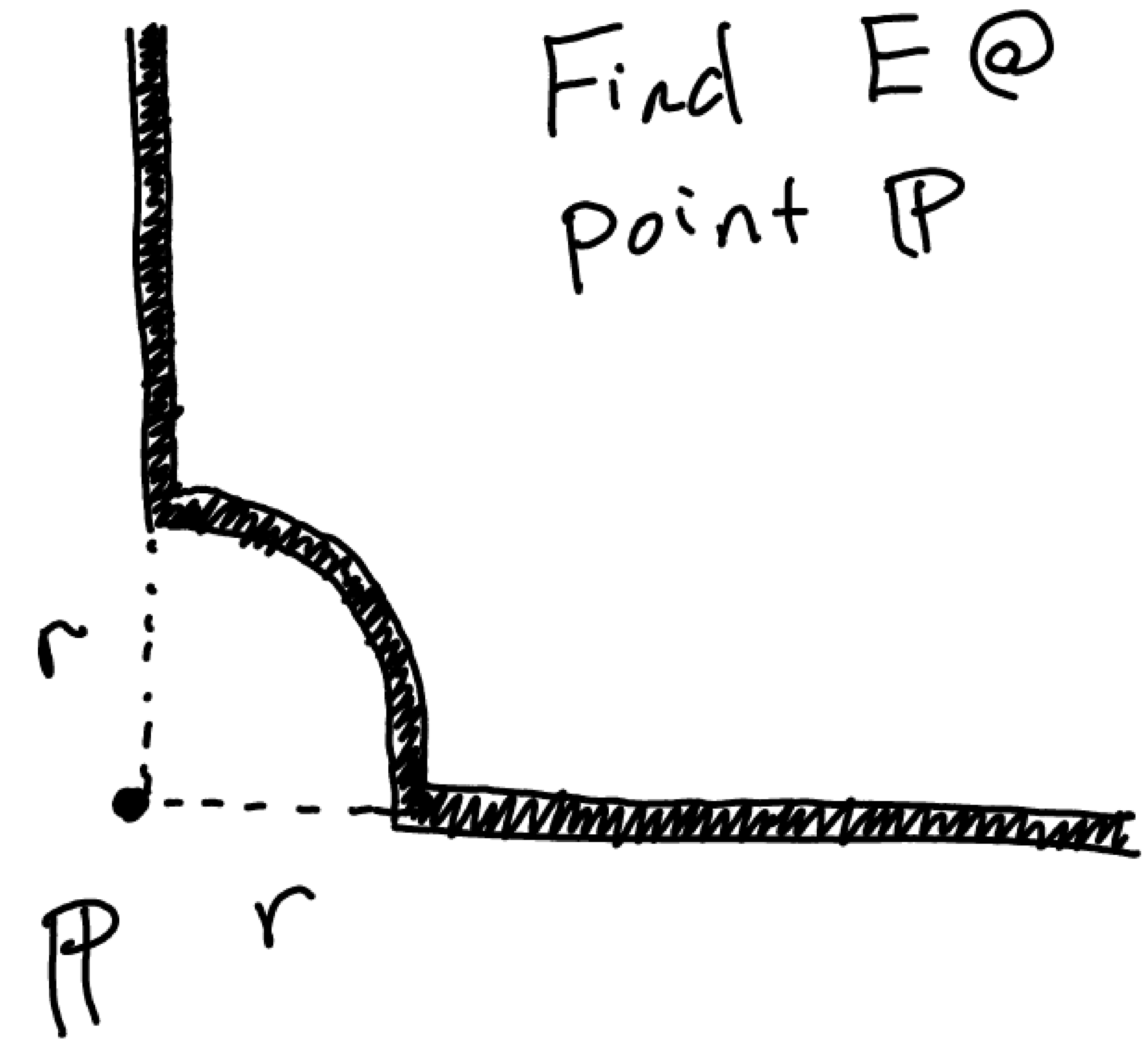
only have to worry about vertical part.

$$dE_z = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{|\vec{r}|^2}\hat{r} = \frac{1}{4\pi\varepsilon_0}\int_{-\theta}^{\theta}\frac{\lambda R\,d\phi}{R^2}\cos\phi$$

$$E_z = \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{R}\int_{-\theta}^{\theta}\cos\phi\,d\phi = \frac{\lambda}{4\pi\varepsilon_0 R}\left(\sin\phi\Big|_{-\theta}^{\theta}\right)$$

$$E_z = \frac{\lambda}{4\pi\varepsilon_0 R}\underbrace{\left(\sin\theta - \sin(-\theta)\right)}_{-\sin(-\theta) = \sin\theta} \boxed{= \frac{\lambda}{2\pi\varepsilon_0 R}\sin\theta}$$

Problem 2.4:



Find $\vec{E}$ @ point P $\Rightarrow$ use results from previous two problems

* superposition $\Rightarrow$ Add separate fields to construct total field

problem 2.2.2: $E_{line} = \dfrac{\lambda}{4\pi\varepsilon_0}\left[\dfrac{1}{a} - \dfrac{1}{\ell + a}\right]$

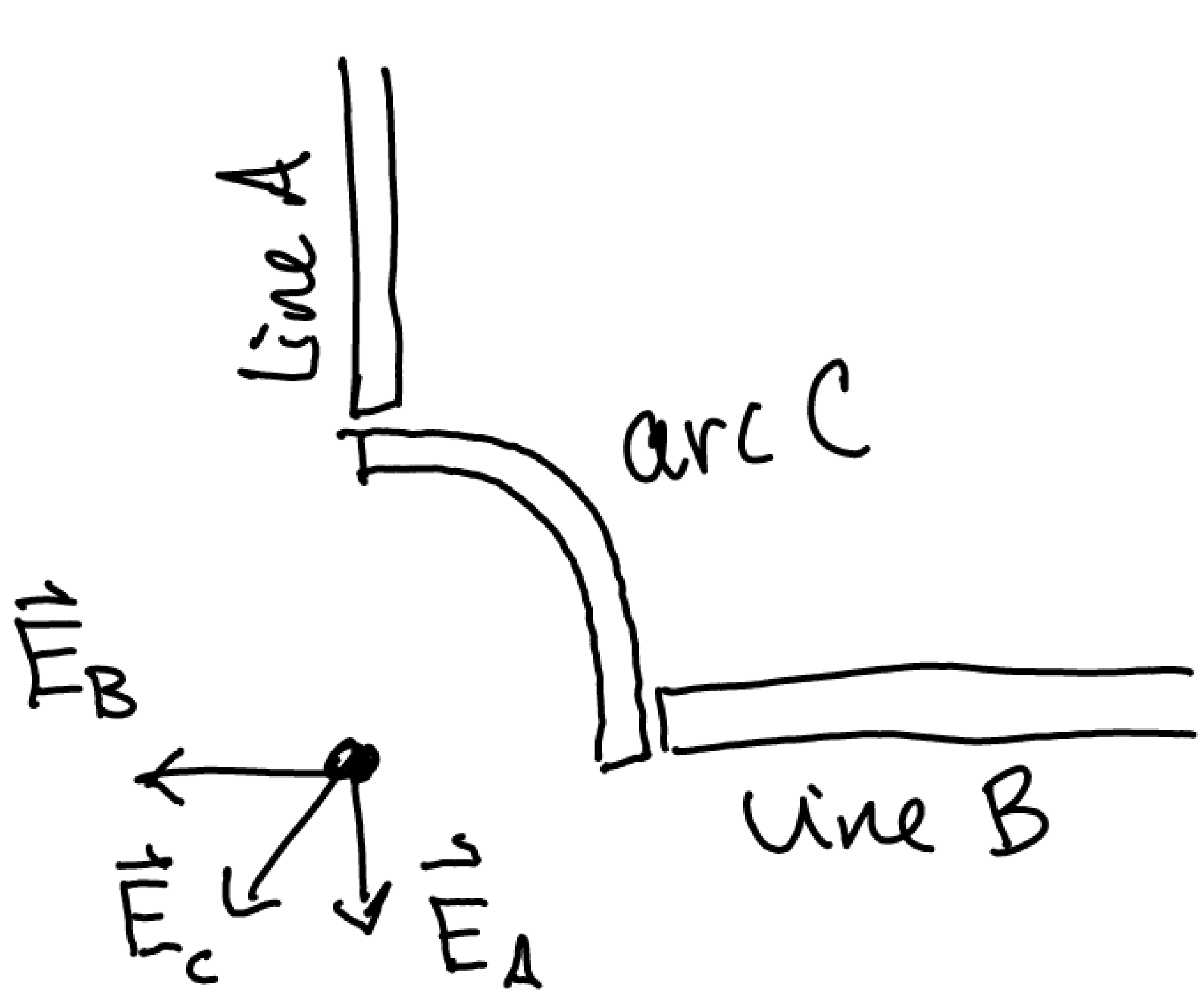
when $\ell \to \infty$, $\boxed{E_{line} = \dfrac{\lambda}{4\pi\varepsilon_0 r}}$

problem 2.3

$$E_{arc} = \frac{\lambda \sin\theta}{2\pi\varepsilon_0 r}$$

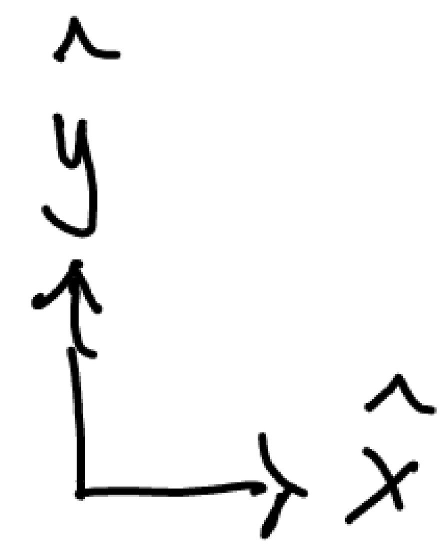
when $\theta = 45°$, $\boxed{E_{arc} = \dfrac{\lambda}{2\pi\varepsilon_0 r}\dfrac{1}{\sqrt{2}}}$

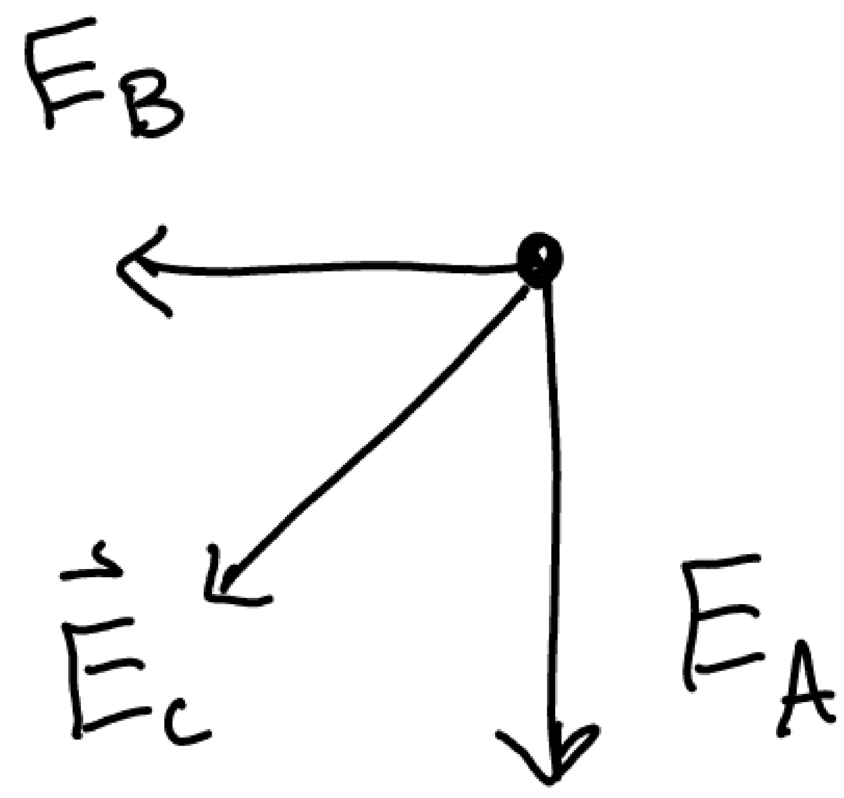
Superposition:



$$\vec{E}_{total} = \vec{E}_A + \vec{E}_B + \vec{E}_C$$

continues next page.





$$\vec{E}_A = \frac{-\lambda}{4\pi\varepsilon_0 r}\,\hat{y}$$

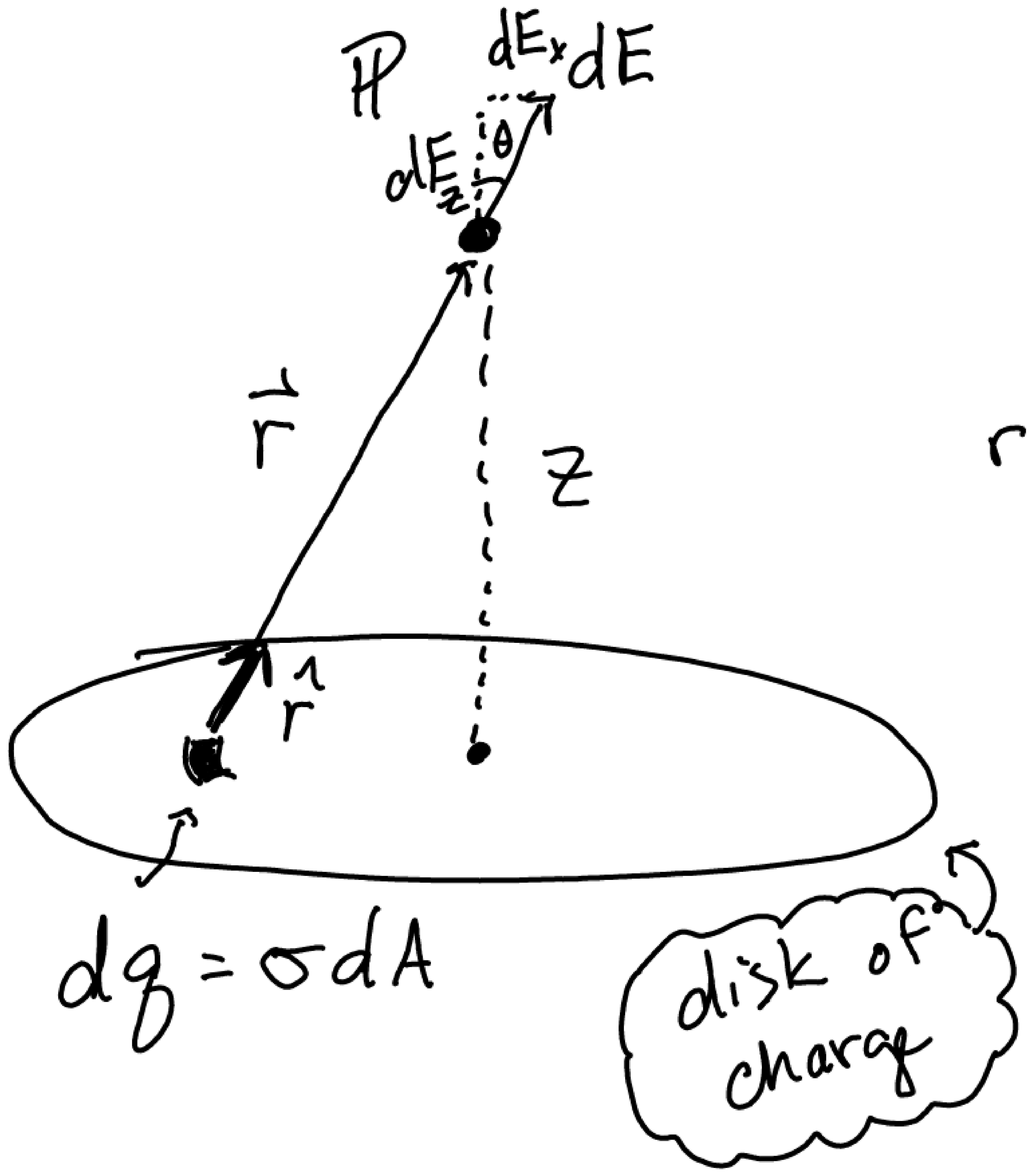
$$\vec{E}_B = -\frac{\lambda}{4\pi\varepsilon_0 r}\,\hat{x}$$

$$\vec{E}_c = \frac{-\lambda}{2\sqrt{2}\,\pi\varepsilon_0 r}\,\frac{\hat{x},\hat{y}}{\sqrt{1^2+1^2}}$$

$$= \frac{-\lambda}{4\pi\varepsilon_0 r}\,(\hat{x},\hat{y})$$

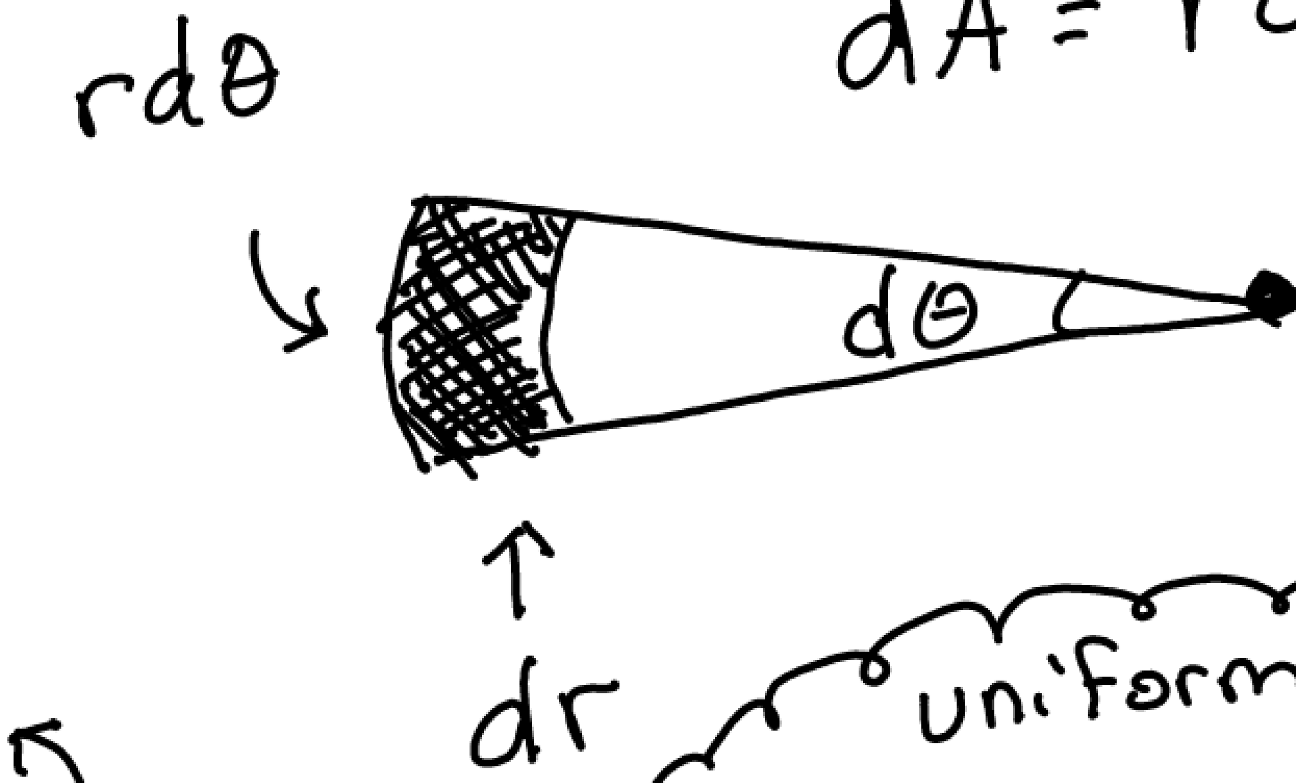
$$\vec{E}_{total} = \vec{E}_A + \vec{E}_B + \vec{E}_c$$

$$\boxed{= \frac{-\lambda}{2\pi\varepsilon_0 r}\,(\hat{x},\hat{y})}$$



Infinitesimal Area:

$dA = r\,dr\,d\theta$



uniform charge:

$\sigma = \frac{q}{A} \Rightarrow dq = \sigma dA$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{|\vec{r}|^2}\hat{r} = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{|\vec{r}|^3}\vec{r}$$

$$= \int_A \frac{\sigma\, dA}{|\vec{r}|^3}(r\cos\theta, r\sin\theta, z)$$

$$\vec{E} = (E_x, E_y, E_z) = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi} d\theta \int_0^R dr \left(\frac{\sigma\, r\,dr\,d\theta}{|\vec{r}|^3}\right)(r\cos\theta, r\sin\theta, z)$$

$$\vec{E}_x = \frac{\sigma}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^R \frac{r^2\cos\theta}{(r^2+z^2)^{3/2}}\,dr\,d\theta\,\hat{x} = 0$$

$$\vec{E}_y = \frac{\sigma}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^R \frac{r^2\sin\theta}{(r^2+z^2)^{3/2}}\,dr\,d\theta\,\hat{y} = 0$$

Both of these E-fields are zero as the θ integral is zero over the 360° rotation.

Symmetry.

$$\vec{E}_z = \frac{\sigma}{4\pi\varepsilon_0} \int_0^{2\pi} \int_0^R \frac{z r}{|\vec{r}|^3} \, d\theta \, dr \, \hat{z}$$

theta integral first
$\int_0^{2\pi} d\theta = 2\pi$

$$|\vec{E}_z| = \frac{2\pi\sigma z}{4\pi\varepsilon_0} \int_0^R \frac{r\,dr}{(r^2+z^2)^{3/2}}$$

subs:
$u = r^2$
$du = 2r\,dr$

subs 2:
$w = u + z^2$
$dw = du$

$$= \frac{\sigma z}{2\varepsilon_0} \frac{1}{2} \int_0^R \frac{du}{(u+z^2)^{3/2}} = \frac{\sigma z}{4\varepsilon_0} \int_0^R \frac{dw}{w^{3/2}}$$

$$= \frac{\sigma z}{2\varepsilon_0} \left(-\frac{1}{w^{1/2}} \right)$$

$\int w^{-3/2} dw$
$= \left(\frac{1}{-3/2+1}\right) w^{-3/2+1}$
$= -\frac{1}{2} w^{-1/2}$

$$= \frac{\sigma z}{2\varepsilon_0} \left(-\frac{1}{(u+z^2)^{1/2}} \right)$$

$$= -\frac{\sigma z}{2\varepsilon_0} \frac{1}{\sqrt{r^2+z^2}} \Bigg|_0^R$$

$$= -\frac{\sigma z}{2\varepsilon_0} \left(\frac{1}{\sqrt{R^2+z^2}} - \frac{1}{z} \right)$$

$$\boxed{E_z = -\frac{\sigma z}{2\varepsilon_0\sqrt{R^2+z^2}} + \frac{\sigma}{2\varepsilon_0}}$$

as $R \to \infty$ you get an important result:

Infinite plane:

$$E = \frac{\sigma}{2\varepsilon_0}$$