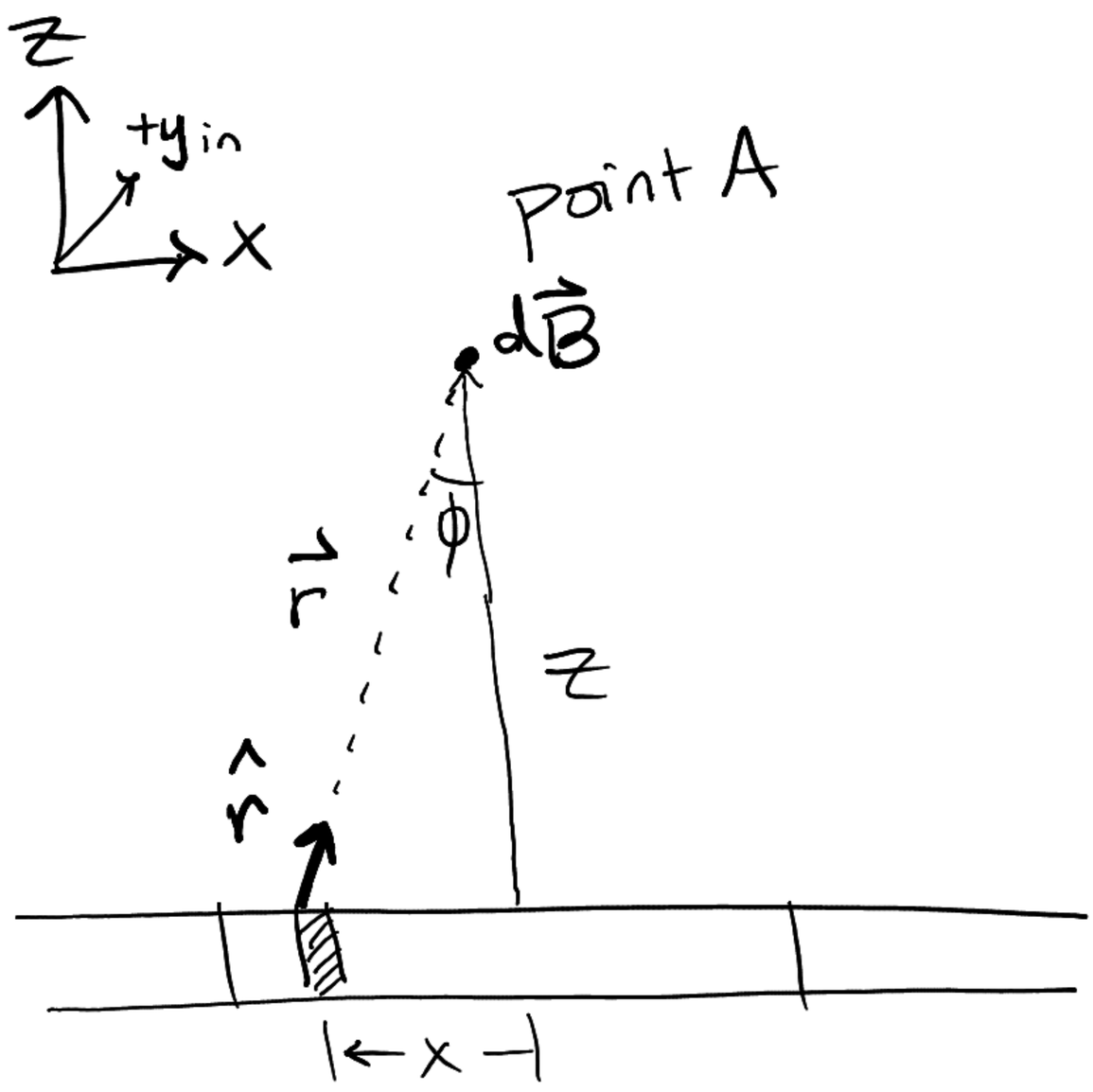
Problem 9.1: (calculus)



Biot-Savart:

$$d\vec{B} = \frac{\mu_0}{4\pi}\int \frac{I\,d\vec{l}\times\hat{r}}{r^2}$$

$$= \frac{\mu_0}{4\pi}\int \frac{I\,d\vec{l}\times\vec{r}}{r^3}$$

- $dl = (dx, 0, 0)$
- $\vec{r} = (x, 0, z)$
- $d\vec{l}\times\vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ dx & 0 & 0 \\ x & 0 & z \end{vmatrix} \begin{matrix} \hat{x} & \hat{y} \\ dx & 0 \\ x & 0 \end{matrix} = -z\,dx\,\hat{y}$

$$dB = \frac{\mu_0}{4\pi}\int \frac{I(-z\,dx)\hat{y}}{(x^2+z^2)^{3/2}}$$

$$= \frac{-\mu_0 I z\hat{y}}{4\pi}\int \frac{dx}{(x^2+z^2)^{3/2}} = \frac{-\mu_0 I z}{4\pi}\left(\frac{x}{z^2\sqrt{z^2+x^2}}\right)\Bigg|_{x=-l/2}^{x=+\frac{l}{2}}$$

trig substitution:

$x = z\tan\theta \qquad dx = z\sec^2\theta\,d\theta$

then evaluate

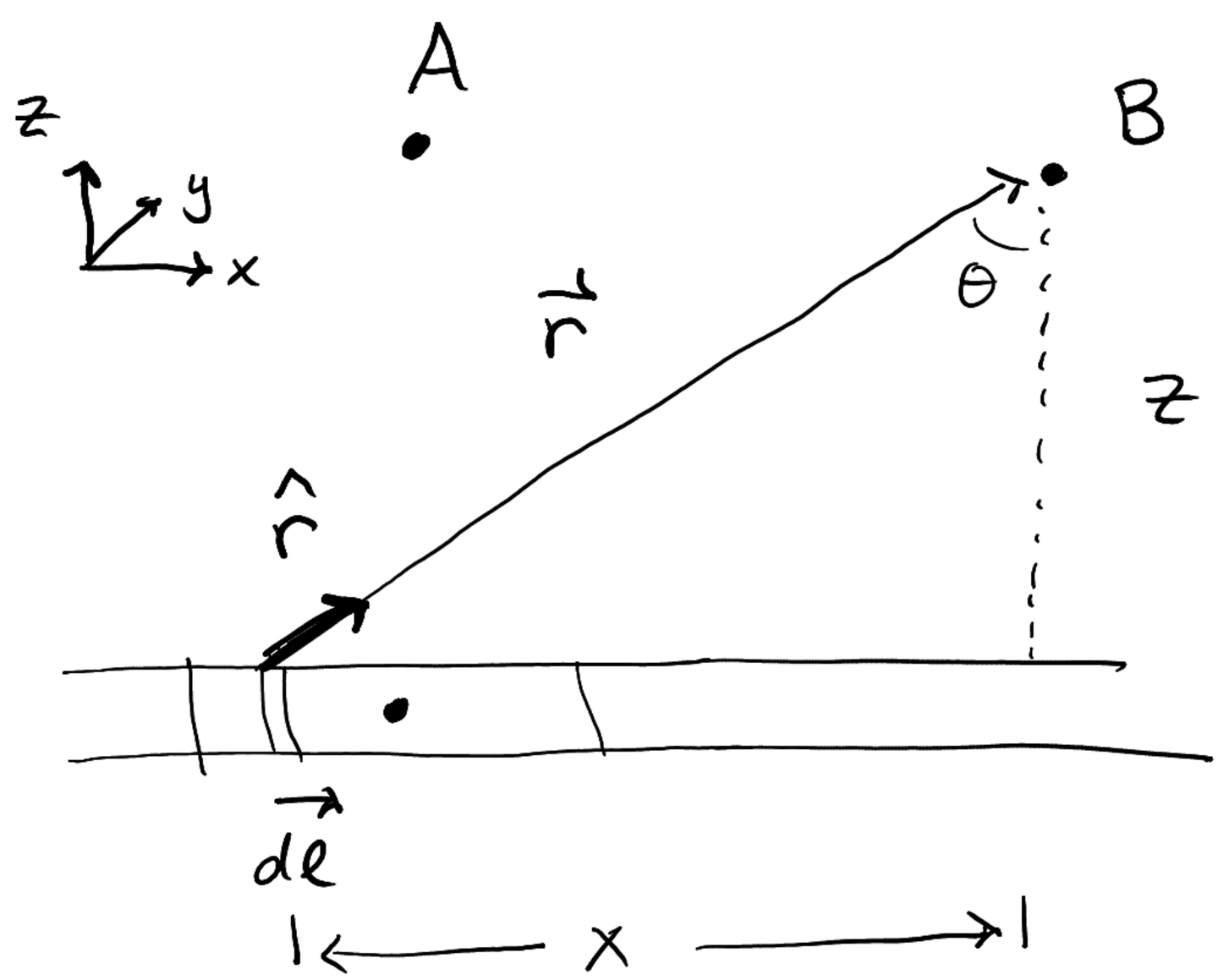
$$\vec{B}_A = \frac{-\mu_0 I \hat{y}}{4\pi z}\left[\frac{l/2}{\sqrt{z^2+(l/2)^2}} - \frac{(-l/2)}{\sqrt{z^2+(-l/2)^2}}\right]$$

$$= -\frac{\mu_0 I \hat{y}}{4\pi z}\left(\frac{l}{\sqrt{z^2+(l/2)^2}}\right)$$

evaluate when $l = 0.5\text{mm}$ & $z = 3\text{cm}$

$$B = \boxed{- \qquad \hat{y}\ \text{T}}$$

Now for Point B:



$$dB = \frac{\mu_0}{4\pi}\int \frac{I\, d\vec{l} \times \vec{r}}{r^3}$$

$d\vec{l} = (dx, 0, 0)$

$\vec{r} = (x, 0, z)$

this is the same setup. the part which changes are the limits of integration!

$$d\vec{\ell} \times \vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ dx & 0 & 0 \\ x & 0 & z \end{vmatrix} = -z\,dx\,\hat{y}$$

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{-z\,dx\,\hat{y}}{(x^2+z^2)^{3/2}}$$

same integral!
$x = z\tan\theta$
$dx = z\sec^2\theta\,d\theta$

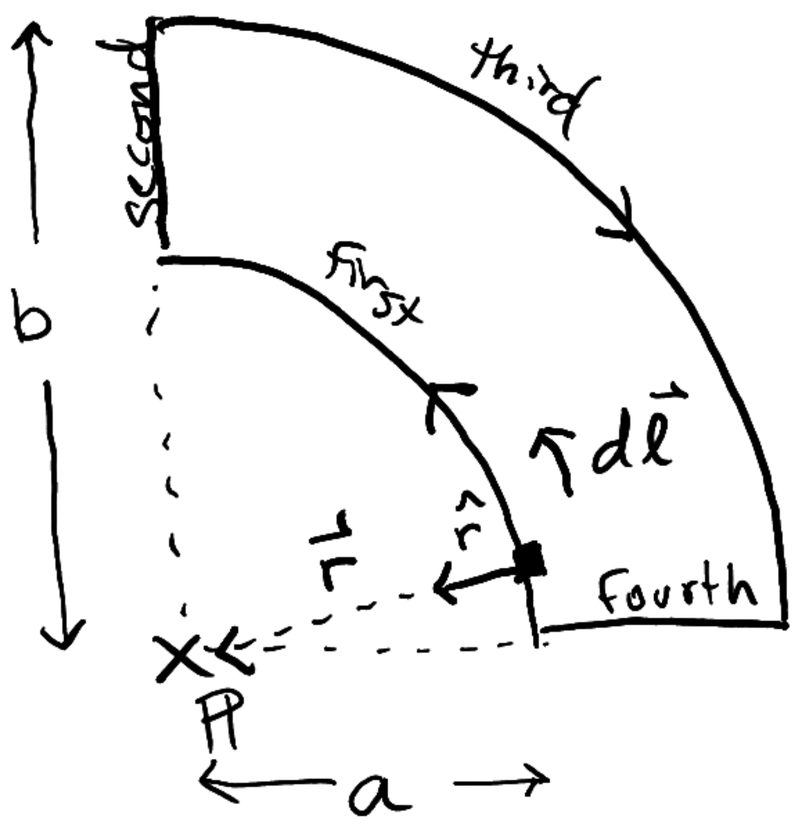
$$= -\frac{\mu_0 I z}{4\pi}\hat{y} \int_{-a-\ell/2}^{-a+\ell/2} \frac{dx}{(x^2+z^2)^{3/2}}$$

$$= -\frac{\mu_0 I\,\hat{y}}{4\pi z}\left(\frac{x}{\sqrt{z^2+x^2}} \Bigg|_{-a-\ell/2}^{-a+\ell/2} \right)$$

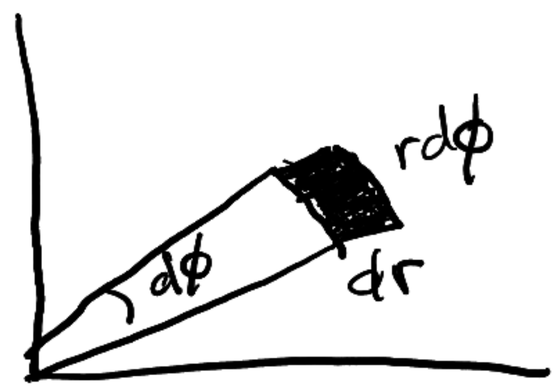
then evaluate this equation when $I = 10A$, $z = 3cm$, $a = 4cm$, and $\ell = 0.5mm$

$$\vec{B} = \boxed{\quad - \qquad \hat{y}\ [\text{Tesla}]}$$

Problem 9.2:



use polar coords:



(ρ, ϕ, z)

First the inside loop counterclockwise

$$d\vec{l} = (0, rd\phi, 0)$$

$$\vec{r} = (a, \phi, 0)$$

$$d\hat{l} \times \vec{r} = \begin{vmatrix} \hat{r} & \hat{\phi} & \hat{z} \\ 0 & rd\phi & 0 \\ -a & \phi & 0 \end{vmatrix} = +ard\phi\,\hat{z}$$

$$B = \frac{\mu_0}{4\pi}\int \frac{I\,d\vec{l}\times\vec{r}}{r^3} = \frac{\mu_0 I}{4\pi}\int_0^{\pi/2} \frac{ard\phi}{r^3}\,\hat{z}$$

$r = a$

$$= \frac{\mu_0 I}{4\pi a}\hat{z}\left(\phi\Big|_0^{\pi/2}\right) = \boxed{\frac{\mu_0 I}{8a}\hat{z}}$$

The <u>second</u> leg:

$$d\vec{l} = (d\rho,\ 0\ ,0)$$
$$\vec{r} = (-\rho,\ 0\ ,0)$$

$$d\vec{l} \times \vec{r} = \begin{vmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \\ dr & 0 & 0 \\ -r & 0 & 0 \end{vmatrix} = 0$$

the <u>third leg</u>: (similar to first)

$$d\vec{l} = (0, b\,d\phi, 0)$$
$$\vec{r} = (-b, \phi\ , 0)$$

$$d\vec{l} \times \vec{r} = \begin{vmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \\ 0 & +r\,d\phi & 0 \\ -b & \phi & 0 \end{vmatrix} = +\,b\,r\,d\phi\,\hat{z}$$

$$B = \frac{\mu_0}{4\pi}\int \frac{I\,d\vec{l} \times \vec{r}}{r^3} = \frac{\mu_0 I}{4\pi b}(-\hat{z})\int_0^{-\pi/2} d\phi$$

$$= -\frac{\mu_0 I}{8b}\hat{z}$$

<u>fourth leg</u>:

Zero: similar to second

the total magnetic field is then

$$\boxed{\vec{B} = \frac{\mu_0 I}{8}\left(\frac{1}{a} - \frac{1}{b}\right)\hat{z}}$$

<u>Note:</u> the direction of $\vec{B}$-field is fully encoded into Biot-Savart law, but is often easier to think about using the right hand rule:

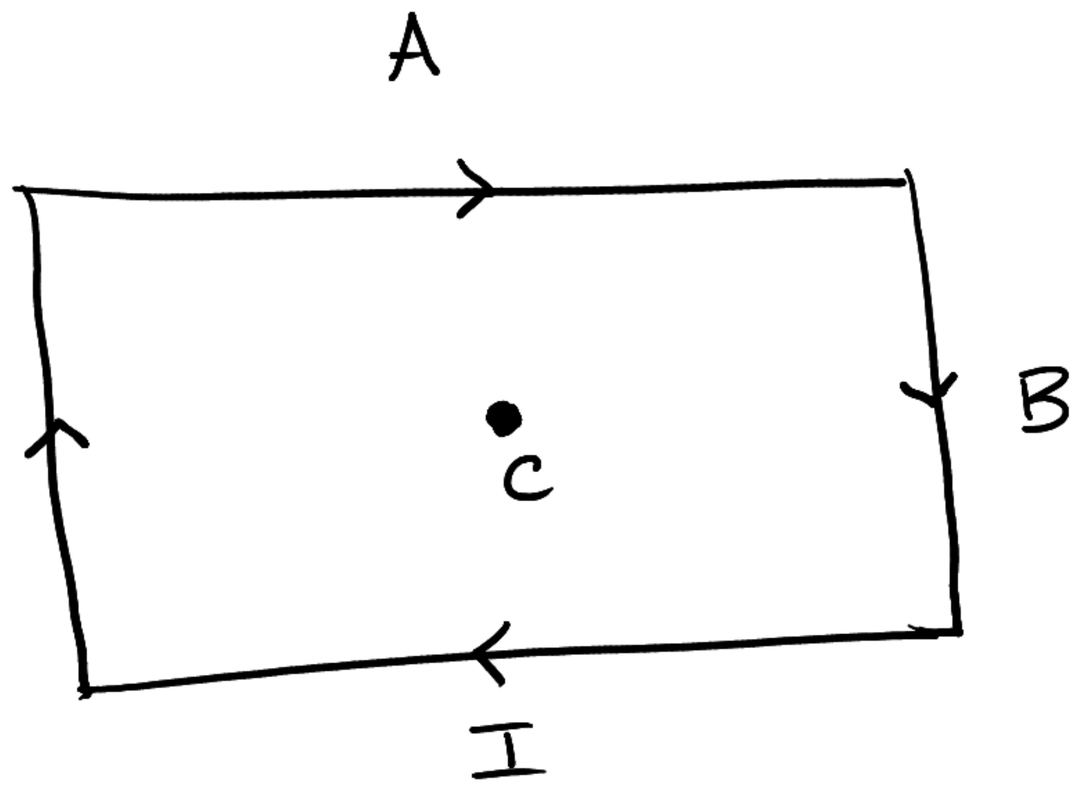
- thumb in direction of current $\vec{I}$
- fingers wrap in direction of $\vec{B}$

Use this to motivate the final solution

Inner arc : $\frac{\mu_0 I}{8a}$ <u>out of page</u> $(+\hat{z})$

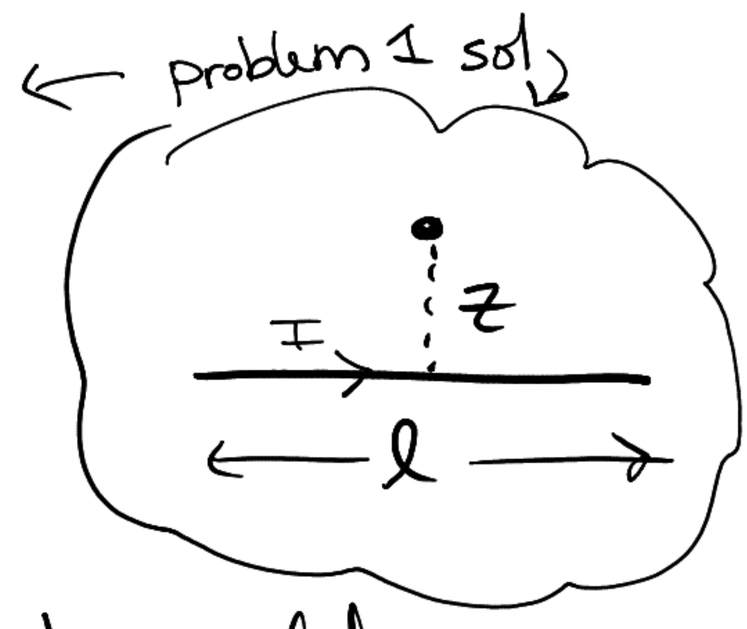
Outer arc : $\frac{\mu_0 I}{8b}$ <u>into page</u> $(-\hat{z})$

9.3



$\vec{B}$ - field will point into the page (RHR) and solutions are the same as problem 1

$$\vec{B} = \frac{\mu_0 I}{4\pi z}\left(\frac{\ell}{\sqrt{z^2+(\ell/2)^2}}\right)$$



This means there are 4 terms to add

- left & Right are copies
- top & Bottom are copies

$$B = 2B_{top} + 2B_{right}$$

$$= 2\left[\frac{\mu_0 I}{4\pi(B/2)}\frac{A}{\sqrt{(B/2)^2+(A/2)^2}} + \frac{\mu_0 I}{4\pi(A/2)}\frac{B}{\sqrt{(A/2)^2+(B/2)^2}}\right]$$

this simplifies down a bunch

$$B = 2\left[\frac{\mu_0 I}{4\pi(B/2)}\frac{A}{\sqrt{(B/2)^2+(A/2)^2}} + \frac{\mu_0 I}{4\pi(A/2)}\frac{B}{\sqrt{(A/2)^2+(B/2)^2}}\right]$$

$$= 2\left[\frac{\mu_0 I}{2\pi B}\frac{A}{\sqrt{\sim}}\left(\frac{A}{A}\right) + \frac{\mu_0 I}{2\pi A}\frac{B}{\sqrt{\sim}}\left(\frac{B}{B}\right)\right]$$

* Remember to find a common denominator

$$= \frac{\mu_0 I A^2}{\pi AB\sqrt{\sim}} + \frac{\mu_0 I B^2}{\pi AB\sqrt{\sim}}$$

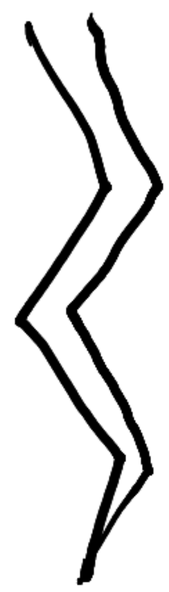
$$= \frac{\mu_0 I (A^2+B^2)}{\pi AB\sqrt{(A/2)^2+(B/2)^2}}$$

$$= \frac{\mu_0 I (A^2+B^2)}{2\pi AB\sqrt{A^2+B^2}}$$

$$\boxed{B = \frac{\mu_0 I}{2\pi AB}\sqrt{A^2+B^2}}$$ "Into page"

using right hand rule

9.4



$\downarrow I = 10^4$ Amps

use Amperes law: $|\vec{B}| = \dfrac{\mu_0 I}{2\pi r}$

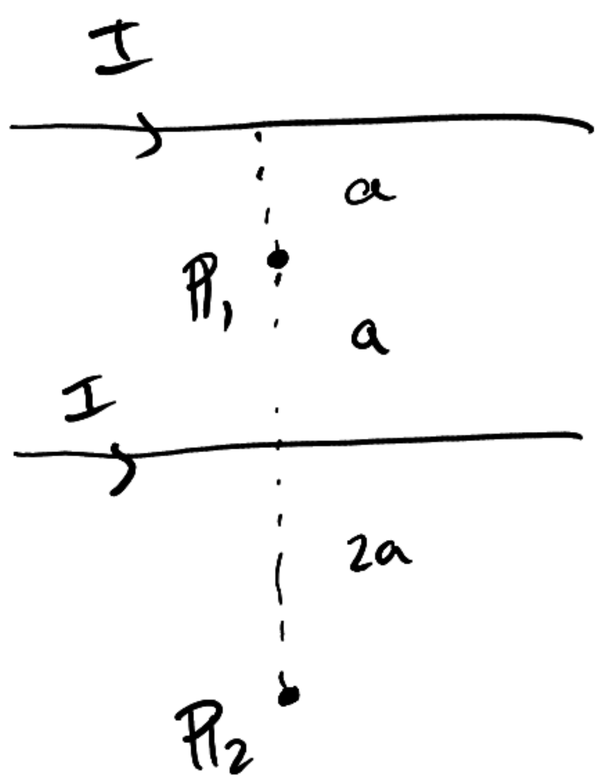
$$|B| = \frac{(4\pi E{-7})(10^4)}{2\pi(1)} = \boxed{0.02 \ T}$$

9.5 Amperes:

$$\oint_c \vec{B} \cdot d\vec{l} = \mu_0 I$$

$$B(2\pi r) = \mu_0 I$$

$$\boxed{B = \frac{\mu_0 I}{2\pi r}}$$



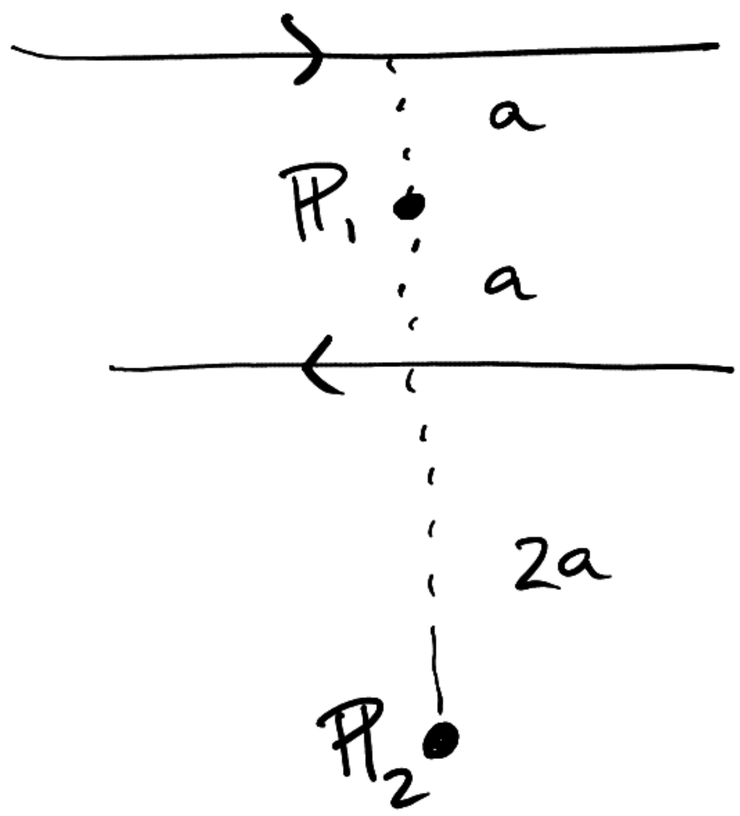
$$\vec{B}_{P_1} = \frac{\mu_0 I}{2\pi a}\hat{z} + \frac{\mu_0 I}{2\pi a}(-\hat{z})$$

$$\boxed{= 0}$$

$$\vec{B}_{P_2} = \frac{\mu_0 I}{2\pi(4a)}(-\hat{z}) + \frac{\mu_0 I}{2\pi(2a)}(-\hat{z})$$

$$\boxed{= \frac{3\mu_0 I}{8\pi a}(-\hat{z})}$$

9.6



Now B field @ P_1 will add instead of cancelling like previous problem

$$B_{P_1} = \frac{\mu_0 I}{2\pi a}(-\hat{z}) + \frac{\mu_0 I}{2\pi a}(-\hat{z})$$

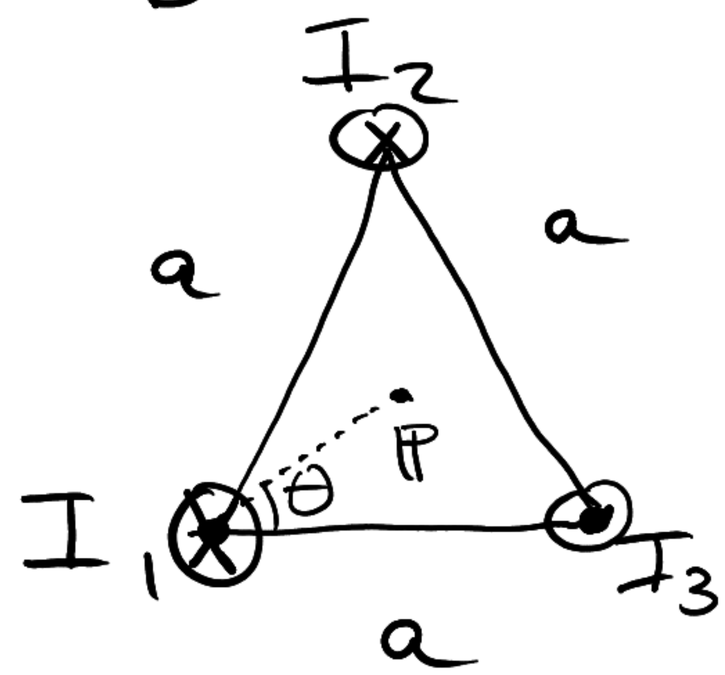
$$\boxed{\vec{B}_{P_1} = -\frac{\mu_0 I}{\pi a}\hat{z}}$$

$$B_{P_2} = \frac{\mu_0 I}{2\pi(4a)}(-\hat{z}) + \frac{\mu_0 I}{2\pi(2a)}(+\hat{z})$$

$$= -\frac{\mu_0 I}{2\pi(4a)} + \frac{2\mu_0 I}{2\pi(4a)} = \boxed{\frac{\mu_0 I}{8\pi a}(+\hat{z})}$$

9.7 Equilateral Triangle

Use Amperes Law (direction similar to Biot-Savart)



$$\vec{l}_1 = (0, 0, -z)$$

$$\vec{r}_1 = (a\cos\theta, a\sin\theta, 0)$$

hats

$$\hat{l}_1 \times \hat{r}_1 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & -1 \\ \cos\theta & \sin\theta & 0 \end{vmatrix} = \underline{\underline{\sin\theta\,\hat{x} - \cos\theta\,\hat{y}}}$$

$$\vec{l}_2 = (0, 0, -z)$$

$$r_2 = (0, -a/2, 0)$$

hats

$$\hat{l}_2 \times \hat{r}_2 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{vmatrix} = \underline{\underline{-\hat{x}}}$$

$$\vec{l}_3 = (0, 0, +z)$$

$$\vec{r}_3 = (-a\cos\theta, +a\sin\theta, 0)$$

$$\hat{l}_3 \times \hat{r}_3 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 1 \\ -\cos\theta & \sin\theta & 0 \end{vmatrix} = \underline{\underline{-\sin\theta\,\hat{x} - \cos\theta\,\hat{y}}}$$

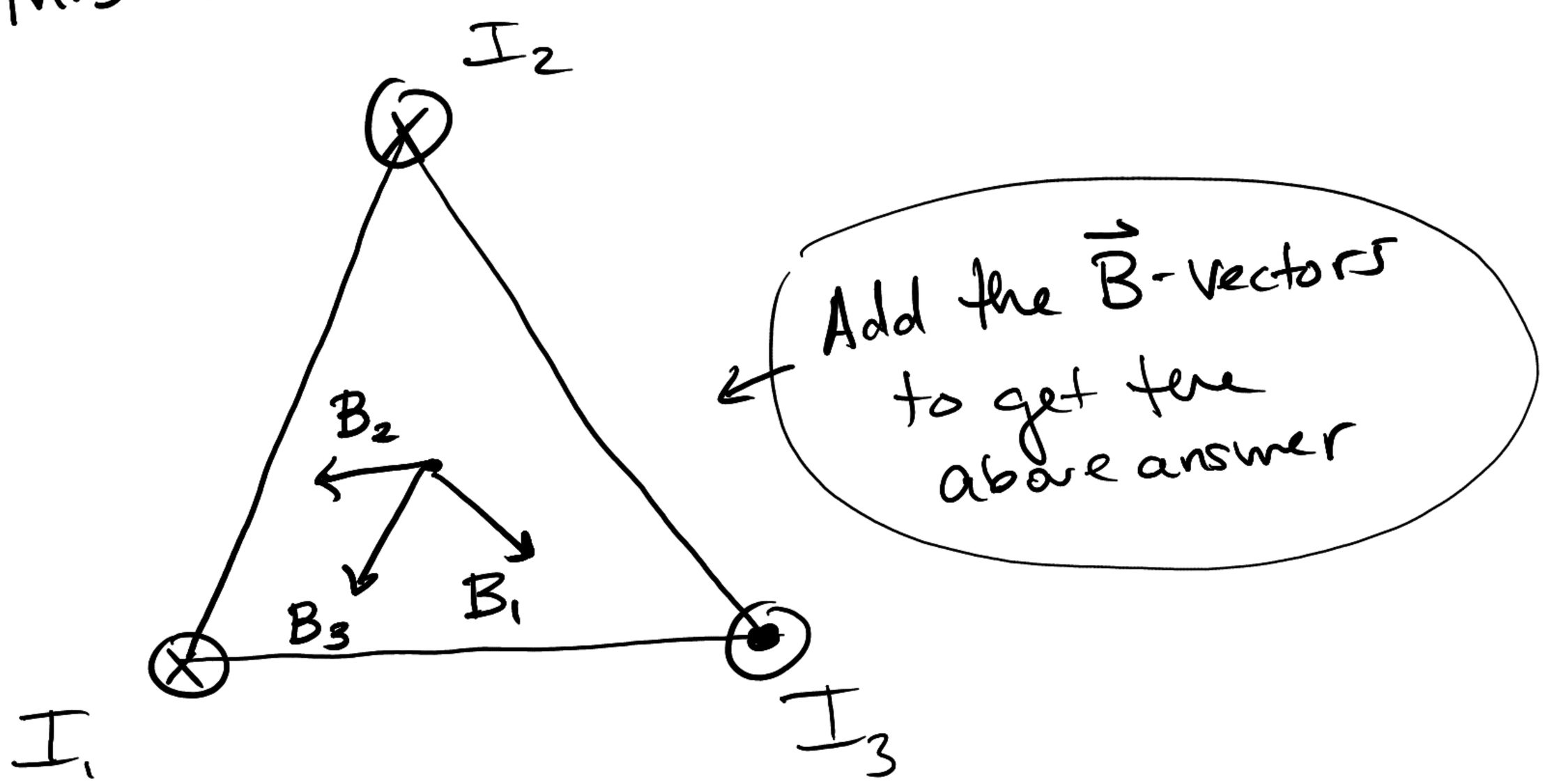
Now add the B fields as a vector
(all have same magnitude)

$$\frac{\mu_0 I}{2\pi(a/2)} \cdot \left[\cancel{\sin\theta}\,\hat{x} - \cos\theta\,\hat{y} - \hat{x} - \cancel{\sin\theta}\,\hat{x} - \cos\theta\,\hat{y} \right]$$

$$= \frac{\mu_0 I}{2\pi(a/2)} \left(-\hat{x} - 2\cos\theta\,\hat{y} \right)$$

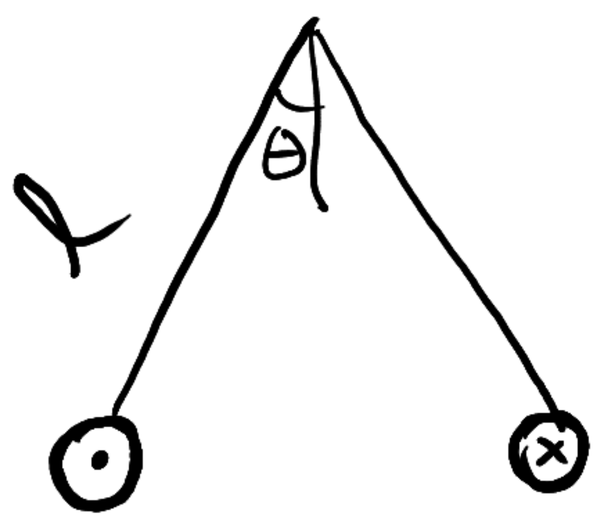
Then evaluate w/ $I = 20$ Amp
$a = 0.05$ m
$\theta = 30^\circ$

This can be solved like a FBD



Problem 9.8:

$$B = \frac{\mu_0 I}{2\pi r} \qquad F = I\vec{l} \times \vec{B}$$



Knowns:

$\frac{\text{mass}}{\text{length}} = 30\,\frac{g}{m} = 0.03\,\frac{kg}{m}$

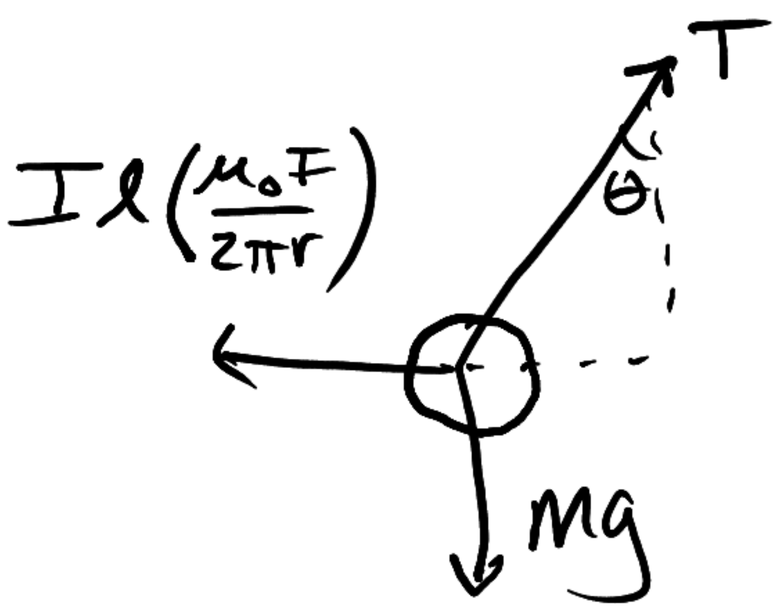
$\theta = 6^\circ$

$l = 5\,cm = 0.05\,m$

B-field is "upward" for both wires (use RHR)

Lorentz force then pushes wires apart.

FBD



$$\sum F_x = T\sin\theta - \frac{\mu_0 I^2 l}{2\pi r} = 0$$

$$\sum F_y = T\cos\theta - mg = 0$$

then solve the system for T & I

$$T = \frac{mg}{\cos\theta}$$

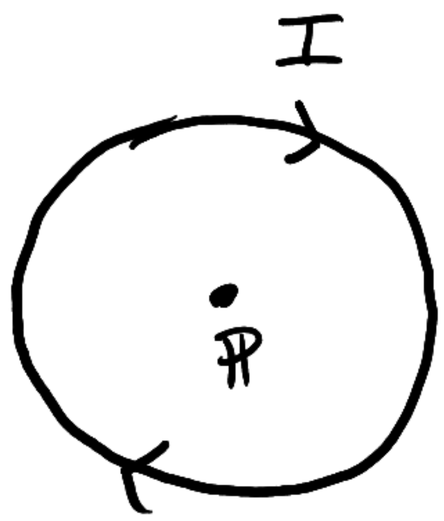
subs

$$mg\tan\theta = \frac{\mu_0 I^2 l}{2\pi r}$$

elim

$$\boxed{I = \quad A}$$

9.9:



- First derive the B-field using Biot-Savart

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{I\, d\vec{l} \times \vec{r}}{r^3}$$

$$d\vec{l} = (0, -R\,d\theta, 0)$$

$$\vec{r} = (-R, 0, 0)$$

$$d\vec{l} \times \vec{r} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{z} \\ 0 & -R\,d\theta & 0 \\ -R & 0 & 0 \end{vmatrix}$$

$$= -R^2\, d\theta\, \hat{z}$$

$$= \frac{\mu_0 I}{4\pi}\left(\int_0^{2\pi} \frac{-R^2\, d\theta\, \hat{z}}{R^3}\right)$$

$$= \frac{\mu_0 I}{4\pi R}\left(\int_0^{2\pi} d\theta\right)(-\hat{z})$$

$$\vec{B} = -\frac{\mu_0 I}{2R}\hat{z}$$

Knowns:

$I = 6\,A$

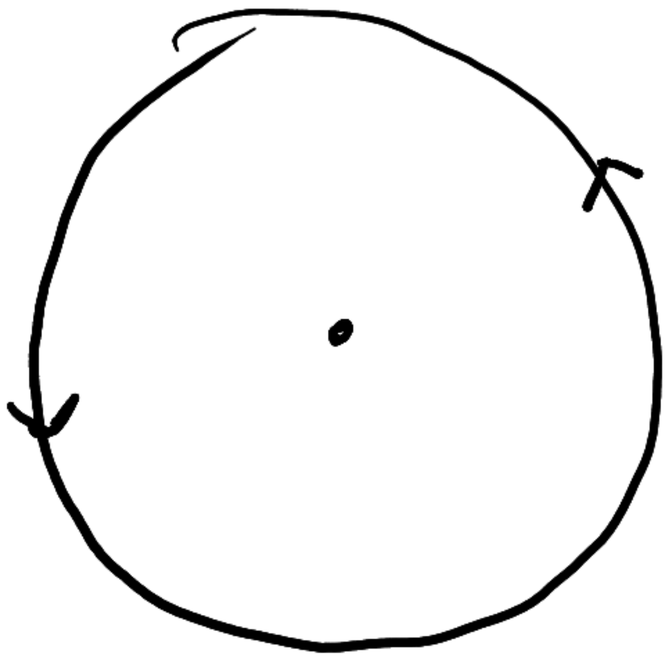
$B = 2E\text{-}4\,T$

$$B = \frac{\mu_0 I}{2R}$$

$$R = \frac{\mu_0 I}{2B} = \boxed{0.0188\ m}$$

9.10 Again Practice Biot-Savart

unit circle is positive in the counterclockwise direction



$$d\vec{l} = (0, +R\,d\theta, 0)$$

$$\vec{r} = (-R, 0, 0)$$

$$d\vec{l} \times \vec{r} = R^2 d\theta\, \hat{z}$$

$$B = \frac{\mu_0}{4\pi} \int \frac{I\, d\vec{l} \times \vec{r}}{r^3}$$

$$= \frac{\mu_0 I}{4\pi} \left(\int_0^{2\pi} \frac{R^2 d\theta\, \hat{z}}{R^3} \right)$$

$$B = \frac{\mu_0 I}{2R}$$

"20" turns means multiply this by 20

$$B = \frac{\mu_0 I}{2R}(20) = \frac{(4\pi E{-7})(0.5)}{2(0.1)}(20)$$

$$= \boxed{\qquad\qquad T}$$

9.11

use RHR to determine + direction

fingers in direction of loop
thumb points in direction of (+)

a) $\mu_0(+2)$

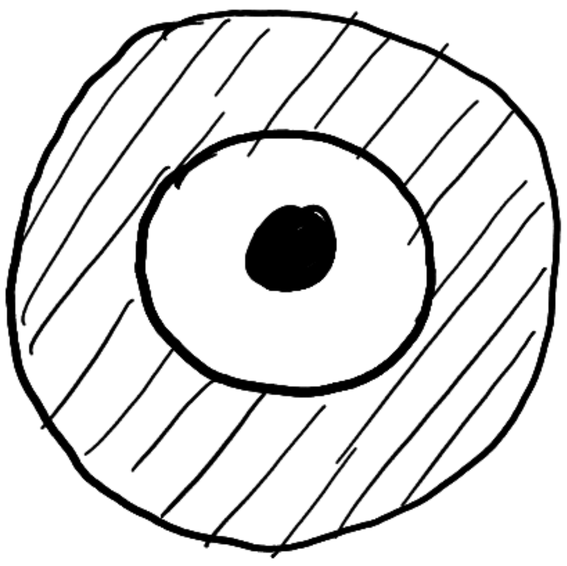
b) $\mu_0(+5-2+6)$

c) $\mu_0(0)$

d) $\mu_0(-2+4)$

e) $\mu_0(-3-2)$

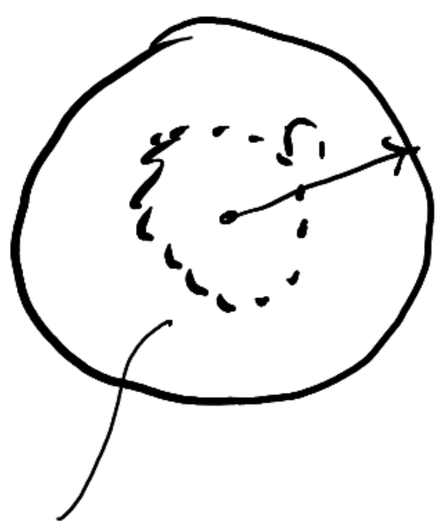
9.12: this is similar to Gauss
Make an "Amperian loop"
to enclose the current.



Inner wire: radius r_1

outer wire: Inside radius r_2
Outside radius r_3

Make an "Amperian" surface
First only the inner wire:



Amperian surface

$$\oint_c \vec{B} \cdot d\vec{l} = \mu_0 I_{in}$$

current is uniform:

$$\frac{I_{in}}{I} = \frac{\pi r^2}{\pi r_1^2}$$

$$I_{in} = I\left(r^2/r_1^2\right)$$

$$B(2\pi r) = \mu_0 I\left(r^2/r_1^2\right)$$

for $r < r_1$: $B = \frac{\mu_0 I}{2\pi r}\frac{r^2}{r_1^2} = \boxed{\frac{\mu_0 I r}{2\pi r_1^2}\hat{\theta}}$

for $r_1 < r < r_2$ (the airspace inbetween)

$I_{in} = I$ all current is enclosed from center wire!

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_{in}$$

$\underline{r_1 < r < r_2}$: $\boxed{\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\theta}}$

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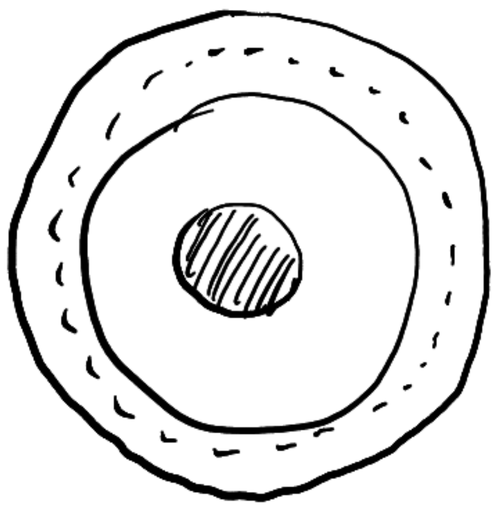
for $r_2 < r < r_3$ (inside outer wire)

- First find the enclosed current
  - $\hookrightarrow I_{r<r_1}$ is fully enclosed
  - $\hookrightarrow I_{r_2<r<r_3}$ is partially enclosed



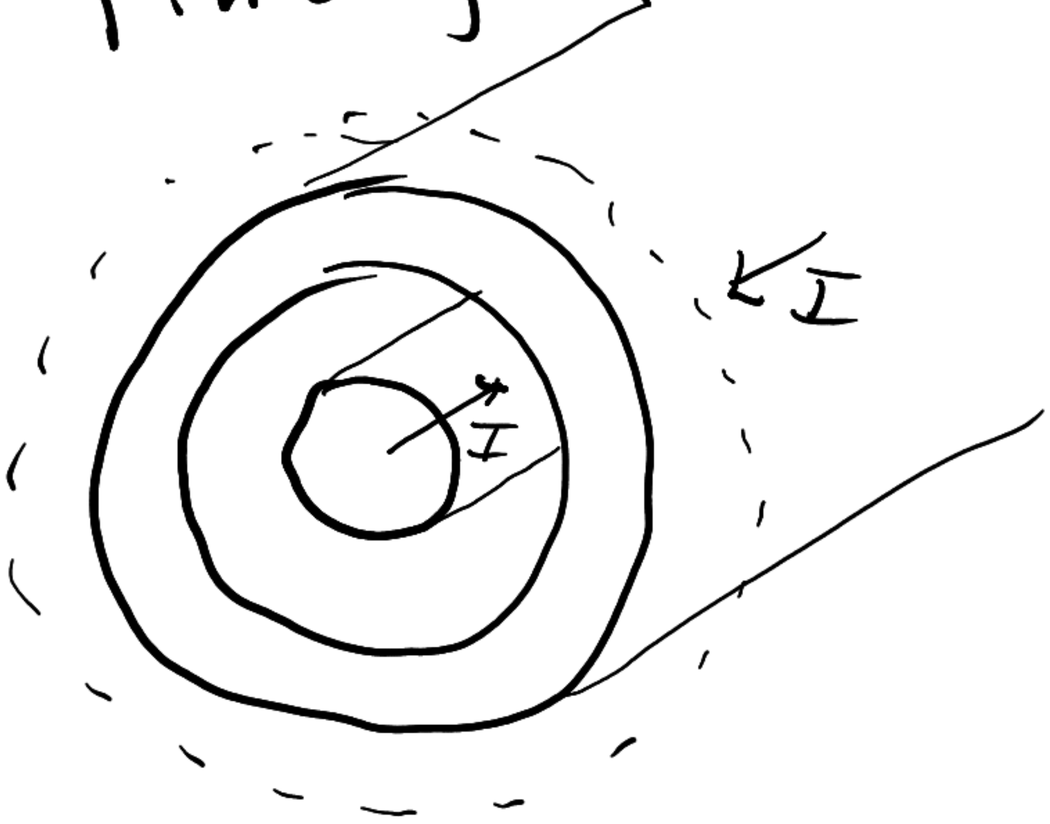
$$I_{enc} = I - I\left(\frac{r^2 - r_2^2}{r_3^2 - r_2^2}\right)$$

$$B \cdot dl = \mu_0 I_{enc}$$

$$A_{enclosed} = \frac{\pi r^2 - \pi r_2^2}{\pi r_3^2 - \pi r_2^2}$$

$\underline{r_2 < r < r_3}$: $\boxed{B = \frac{\mu_0}{2\pi r}\left(\frac{r^2 - r_2^2}{r_3^2 - r_2^2}\right)}$

Finally the last region



$$I_{enc} = I - I = 0$$

$\underline{r > r_3}$: $\boxed{B = 0}$

<u>Coaxial cables</u> are wonderful devices. Both $\vec{E}$ and $\vec{B}$ stay inside of the wire as long as the current / charge on inside is opposite to current/charge outside.