

WHY IS THE IDEAL CLASS GROUP A SHAFAREVICH GROUP?

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ABSTRACT. After a short discussion with Prof. David Helm.

Let K be a number field, G_K be the absolute Galois group of K . There is an exact sequence

$$1 \rightarrow K^\times / K^{\times M} \rightarrow \text{Sel}(K, \mu_M) \rightarrow \text{Cl}(K)[M] \rightarrow 0,$$

where Sel denotes the Selmer group with local conditions cut out by $O_{K_v}^\times / O_{K_v}^{\times M}$. This closely resembled the exact sequence one sees for the Selmer and Shafarevich groups of an elliptic curve.

We will see how to view $\text{Cl}(K)$ as a Shafarevich group. For intuition, $\text{Cl}(K)$ is the measure of failure of local-global principle for the ideal groups - locally, all ideals are principal, but given a collection of principal local ideals, the corresponding global ideal is principal iff its class in the class group is trivial.

We claim that there is an isomorphism

$$\text{Cl}(K) \simeq \text{III}(K, \overline{O}_K^\times) := \ker \left(H^1(K, \overline{O}_K^\times) \rightarrow \prod_v H^1(K_v, \overline{O}_{K_v}^\times) \right),$$

where $\overline{O}_K, \overline{O}_{K_v}$ denote the rings of integers of algebraic closures $\overline{K}, \overline{K}_v$ of K, K_v resp.

Lemma 0.1. *The map $\overline{K}^\times / \overline{O}_K^\times \rightarrow \oplus_v \overline{K}_v^\times / \overline{O}_{K_v}^\times$ induces an isomorphism*

$$(\overline{K}^\times / \overline{O}_K^\times)^{G_K} \rightarrow \oplus_v (\overline{K}_v^\times / \overline{O}_{K_v}^\times)^{G_{K_v}}.$$

Proof. The first map is clearly injective. The RHS is just $\oplus_v \mathbb{Q}$ under the valuation maps $K_v \rightarrow \mathbb{Z}$, and this product is G_{K_v} -invariant already because Galois elements preserve valuation. So we just need surjectivity of the original map. Take $(1/m_v)_v \in \oplus_v \mathbb{Q}$. Then take the ideal $\prod_v \mathfrak{p}_v^{m_v}$ in \overline{O}_K , and choose a Galois extension L/K in which this becomes principal, say $(x_m) = \prod p_v^{m_v}$. Then the class of $x_m \in \overline{K}_v^\times$ in $\oplus_v \overline{K}_v^\times / \overline{O}_{K_v}^\times$ is precisely $(1/m_v)_v$, and x_m is Galois invariant modulo units. \square

Taking G_K cohomology of the short exact sequence

$$1 \rightarrow \overline{O}_K^\times \rightarrow \overline{K}^\times \rightarrow \overline{K}^\times / \overline{O}_K^\times \rightarrow 1,$$

we get the exact sequence

$$1 \rightarrow K^\times / O_K^\times \rightarrow (\overline{K}^\times / \overline{O}_K^\times)^{G_K} \rightarrow H^1(K, \overline{O}_K^\times) \rightarrow 0.$$

We have a similar expression for K_v . Then by snake lemma on the diagram

$$\begin{array}{ccccccc} & & & 0 & \longrightarrow & \text{III}(K, \overline{O}_K^\times) & \\ & & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & K^\times / O_K^\times & \longrightarrow & (\overline{K}^\times / \overline{O}_K^\times)^{G_K} & \longrightarrow & H^1(K, \overline{O}_K^\times) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \oplus_v K_v^\times / O_{K_v}^\times & \longrightarrow & \oplus_v (\overline{K}_v^\times / \overline{O}_{K_v}^\times)^{G_{K_v}} & \longrightarrow & \oplus_v H^1(K_v, \overline{O}_{K_v}^\times) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & \text{Cl}(K) & \longrightarrow & 0 & & \end{array},$$

one immediately obtains

$$\mathrm{Cl}(K) \simeq \mathrm{III}(K, \overline{O}_K^\times).$$