WHY IS THE IDEAL CLASS GROUP A SHAFAREVICH GROUP?

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Abstract. After a short discussion with Prof. David Helm.

Let K be a number field, G_K be the absolute Galois group of K. There is an exact sequence

$$1 \to K^\times / K^{\times M} \to \mathrm{Sel}(K, \mu_M) \to \mathrm{Cl}(K)[M] \to 0,$$

where Sel denotes the Selmer group with local conditions cut out by $O_{K_v}^\times / O_{K_v}^{\times M}$. This closely resembled the exact sequence one sees for the Selmer and Shafarevich groups of an elliptic curve.

We will see how to view $\mathrm{Cl}(K)$ as a Shafarevich group. For intuition, $\mathrm{Cl}(K)$ is the measure of failure of local-global principle for the ideal groups - locally, all ideals are principal, but given a collection of principal local ideals, the corresponding global ideal is principal iff its class in the class group is trivial.

We claim that there is an isomorphism

$$\mathrm{Cl}(K) \simeq Ш(K, \overline{O}_K^\times) := \ker\left(H^1(K, \overline{O}_K^\times) \to \prod_v H^1(K_v, \overline{O}_{K_v}^\times) \right),$$

where $\overline{O}_K, \overline{O}_{K_v}$ denote the rings of integers of algebraic closures $\overline{K}, \overline{K}_v$ of K, K_v resp.

Lemma 0.1. *The map $\overline{K}^\times / \overline{O}_K^\times \to \oplus_v \overline{K}_v^\times / \overline{O}_{K_v}^\times$ induces an isomorphism*

$$(\overline{K}^\times / \overline{O}_K^\times)^{G_K} \to \oplus_v (\overline{K}_v^\times / \overline{O}_{K_v}^\times)^{G_K}.$$

Proof. The first map is clearly injective. The RHS is just $\oplus_v \mathbb{Q}$ under the valuation maps $K_v \to \mathbb{Z}$, and this product is G_{K_v}-invariant already because Galois elements preserve valuation. So we just need surjectivity of the original map. Take $(1/m_v)_v \in \oplus_v \mathbb{Q}$. Then take the ideal $\prod_v \mathfrak{p}_v^{m_v}$ in $\overline{O}_K$, and choose an extension L/K in which this becomes principal, say $(x_m) = \prod p_v^{m_v}$. Then the class of $x_m \in \overline{K}_v^\times$ in $\oplus_v \overline{K}_v^\times / \overline{O}_{K_v}^\times$ is precisely $(1/m_v)_v$. $\square$

Taking G_K cohomology of the short exact sequence

$$1 \to \overline{O}_K^\times \to \overline{K}^\times \to \overline{K}^\times / \overline{O}_K^\times \to 1,$$

we get the exact sequence

$$1 \to K^\times / O_K^\times \to (\overline{K}^\times / \overline{O}_K^\times)^{G_K} \to H^1(K, \overline{O}_K^\times) \to 0.$$

We have a similar expression for K_v. Then by snake lemma on the diagram

$$\begin{tikzcd}
 & & 0 \arrow[r] \arrow[d] & Ш(K, \overline{O}_K^\times) \arrow[d] & \\
0 \arrow[r] & K^\times / O_K^\times \arrow[r] \arrow[d] & (\overline{K}^\times / \overline{O}_K^\times)^{G_K} \arrow[r] \arrow[d] & H^1(K, \overline{O}_K^\times) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \oplus_v K_v^\times / O_{K_v}^\times \arrow[r] \arrow[d] & \oplus_v (\overline{K}_v^\times / \overline{O}_{K_v}^\times)^{G_K} \arrow[r] \arrow[d] & \oplus_v H^1(K_v, \overline{O}_{K_v}^\times) \arrow[r] & 0 \\
 & \mathrm{Cl}(K) \arrow[r] & 0 & &
\end{tikzcd},$$

one immediately obtains

$$\mathrm{Cl}(K) \simeq Ш(K, \overline{O}_K^{\times}).$$