

Extending the Zeta Function to the p -adic Integers

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Introduction and Motivation

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Definition: The Riemann zeta function $\zeta(s)$ is defined on $s \in \mathbb{C}$ with $\Re(s) > 1$ as

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$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

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Clausen: If $p-1 \nmid k$ and if $k \equiv k' \pmod{(p-1)p^N}$, then

$$(1-p^{k-1})\frac{B_k}{k} \equiv (1-p^{k'-1})\frac{B_{k'}}{k'} \pmod{p^{N+1}}.$$

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von Staudt: If $p-1 \mid k$ and k is even (or $k=1$ and $p=2$), then

$$pB_k \equiv -1 \pmod{p}.$$

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- Triangle inequality for ultrametric space:

$$|a_1 + a_2 + \cdots + a_n|_p \leq \max\left(|a_1|_p, |a_2|_p, \dots, |a_n|_p\right).$$

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$$\left| n^s - n^{s'} \right|_p = \left| 1 - n^{qp^N} \right|_p = \left| 1 - (1 + mp)^{qp^N} \right|_p \leq \frac{1}{p^{N+1}}.$$

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- For $1 < n \pmod{p} < p$, if we restrict the input s to be some fixed residue $s_0 \pmod{p-1}$, then n^s is continuous on S_{s_0} .

p -adic distributions

***p*-adic distributions**

$$a + p^N \mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x - a|_p \leq p^{-N}\} = a + (p^N)$$

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Definition: Let X and Y be topological spaces. A map $f : X \rightarrow Y$ is called *locally constant* if every point $x \in X$ has a neighborhood U such that $f(U)$ is a single element in Y .

p -adic Distributions

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Definition: A *p*-adic distribution μ on X is a linear map over \mathbb{Q}_p from the set of locally constant functions on X to \mathbb{Q}_p . If $f : X \rightarrow \mathbb{Q}_p$ is locally constant, instead of writing $\mu(f)$ for the value of μ at f , we usually write $\int f\mu$.

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Example: The Haar distribution μ_{Haar} is defined by

$$\mu_{\text{Haar}}(a + (p^N)) = \frac{1}{p^N}.$$

p -adic Distributions

Proposition: Every map μ from the set of intervals contained in X to \mathbb{Q}_p for which

$$\mu(a + (p^N)) = \sum_{b=1}^{p-1} \mu(a + bp^N + (p^{N+1}))$$

whenever $a + (p^N) \subseteq X$, extends uniquely to a p -adic distribution on X .

Bernoulli Distributions

Definition: Consider the function $f(t) = \frac{t}{e^t - 1}$. Then we define the k th Bernoulli number to be the $k!$ times the coefficient of t^k in the power series of f , i.e.

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

Bernoulli Distributions

Definition: Consider

$$\frac{te^{xt}}{e^t - 1} = \left(\sum_{k=0}^{\infty} B_k \frac{t^k}{k!} \right) \left(\sum_{k=0}^{\infty} \frac{(xt)^k}{k!} \right).$$

We define the k th Bernoulli polynomial $B_k(x)$ by

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}.$$

Bernoulli Distributions

Definition: Fix $0 \leq a < p^N$, and define the k th Bernoulli distribution by

$$\mu_{B,k}(a + (p^N)) = p^{N(k-1)} B_k \left(\frac{a}{p^N} \right)$$

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$$\mu_{k,\alpha}(U) = \mu_{B,k}(U) - \alpha^{-k} \mu_{B,k}(\alpha U).$$

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$$\mu_{k,\alpha}(U) = \mu_{B,k}(U) - \alpha^{-k} \mu_{B,k}(\alpha U).$$

Proposition: $|\mu_{1,\alpha}(U)|_p \leq 1$ for all compact-open $U \subseteq \mathbb{Z}_p$.

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$$\mu_{1,\alpha}(a + (p^N)) = \frac{1}{\alpha} \left\lfloor \frac{\alpha a}{p^N} \right\rfloor + \frac{1/\alpha - 1}{2}.$$



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$$|\mu_{1,\alpha}(U)|_p \leq \max_{I \subseteq U} |\mu_{1,\alpha}(I)|_p \leq 1.$$

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$$|\mu_{1,\alpha}(U)|_p \leq \max_{I \subseteq U} |\mu_{1,\alpha}(I)|_p \leq 1.$$

Thus $\mu_{1,\alpha}$ is a measure. ■

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Proposition: Let d_k be the least common denominator of the coefficients of $B_k(x)$. Then

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Using the previous two results, we can show that every $\mu_{k,\alpha}$ is a measure.

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Definition: Let μ be a p -adic measure on X , and let $f : X \rightarrow \mathbb{Q}_p$ be a continuous function. For each N , pick a set $\{x_{a,N}\}$, where $x_{a,N} \in a + (p^N)$. Define the N th Riemann sum to be

$$S_{N,\{x_{a,N}\}} = \sum_{\substack{0 \leq a < p^N \\ a + (p^N) \subseteq X}} f(x_{a,N}) \mu(a + (p^N)).$$

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$$\int f \mu := \lim_{N \rightarrow \infty} S_{N,\{x_{a,N}\}}$$

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Proposition: If $f : X \rightarrow \mathbb{Q}_p$ is continuous and bounded on X by A , and if $|\mu(U)|_p \leq B$ for all compact-open $U \subseteq X$, then

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Proposition: If $f, g : X \rightarrow \mathbb{Q}_p$ are two continuous functions such that $|f(x) - g(x)|_p \leq \varepsilon$ for all $x \in X$, and $|\mu(U)|_p \leq B$ for all compact-open $U \subseteq X$, then

$$\left| \int f \mu - \int g \mu \right|_p \leq \varepsilon B.$$

Interpolation

Proposition: Let X be a compact-open subset of \mathbb{Z}_p . Then

$$\int_X 1 \mu_{k,\alpha} = k \int_X x^{k-1} \mu_{1,\alpha}.$$

Interpolation

Proof: From one of the earlier propositions, we have

$$\mu_{k,\alpha}(a + (p^N)) \equiv ka^{k-1}\mu_{1,\alpha}(a + (p^N)) \pmod{p^{N-\nu_p(d_k)}}.$$



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Picking N large enough so that X is the union of intervals of the form $a + (p^N)$, we have

$$\begin{aligned} \int_X 1\mu_{k,\alpha} &= \lim_{N \rightarrow \infty} \sum_{\substack{0 \leq a \leq N \\ a + (p^N) \subseteq X}} \mu_{k,\alpha}(a + (p^N)) \\ &\equiv \lim_{N \rightarrow \infty} \sum_{\substack{0 \leq a \leq N \\ a + (p^N) \subseteq X}} ka^{k-1}\mu_{1,\alpha}(a + (p^N)) \pmod{p^{N-\nu_p(d_k)}}. \end{aligned}$$

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The last equation is the Riemann sum for $k \int x^{k-1} \mu_{1,\alpha}$, so letting $N \rightarrow \infty$ yields $\int_X 1\mu_{k,\alpha} = k \int x^{k-1} \mu_{1,\alpha}$, as desired. ■

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is continuous. Fix $s_0 \in \{0, 1, \dots, p-2\}$. Since $\varphi(p^{N+1}) = (p-1)p^N$, Euler's theorem tells us

$$x^{k-1} \equiv x^{k'-1} \pmod{p^{N+1}} \Rightarrow \left| x^{k-1} - x^{k'-1} \right|_p \leq \frac{1}{p^{N+1}}.$$

Interpolation

$$\left| \int_{\mathbb{Z}_p^\times} x^{k-1} \mu_{1,\alpha} - \int_{\mathbb{Z}_p^\times} x^{k'-1} \mu_{1,\alpha} \right|_p \leq \frac{1}{p^{N+1}}.$$

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$$\int_{\mathbb{Z}_p^\times} x^{k-1} \mu_{1,\alpha} = \frac{1}{k} \int_{\mathbb{Z}_p^\times} 1 \mu_{k,\alpha}.$$

Interpolation

$$\begin{aligned}\frac{1}{k} \int_{\mathbb{Z}_p^\times} 1 \mu_{k,\alpha} &= \frac{1}{k} \mu_{k,\alpha}(\mathbb{Z}_p^\times) \\ &= \frac{1}{k} (\mu_{B,k}(\mathbb{Z}_p^\times) - \alpha^{-k} \mu_{B,k}(\alpha \mathbb{Z}_p^\times))\end{aligned}$$

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Interpolation

$$(1 - p^{k-1}) \left(-\frac{B_k}{k} \right) = \frac{1}{\alpha^{-k} - 1} \int_{\mathbb{Z}_p^\times} x^{k-1} \mu_{1,\alpha}.$$

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Definition: For positive integers k , we define

$$\zeta_p(1 - k) = (1 - p^{k-1}) \left(-\frac{B_k}{k} \right) = (1 - p^{k-1}) \zeta(1 - k).$$

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as $|x^{k-1}|_p \leq 1$ for all $x \in \mathbb{Z}_p^\times$ and $|\mu_{1,\alpha}(U)|_p \leq 1$ for any compact-open U .