S-Λ duality (Sai).

k field, V fdvs over k, V^* dual vsp. ξ_i, x_i dual bases. Symmetric algebra

$$SV^* = \bigoplus S^i V^* = k[x_1 \cdots x_n].$$
$$\deg x_i = 1$$

Exterior algebra

$$\Lambda V = \bigoplus \Lambda^i V = k\{\xi_1 \cdots \xi_n\}/\langle \xi_i \xi_j = -\xi_j \xi_i \rangle$$
$$\deg \xi_i = -1.$$

These are graded algebras. We study graded modules over algebras: $M = \bigoplus M_j$, $x_i : M_j \to M_{j+1}$
$\xi_i : M_j \to M_{j-1}$.

The shift functor $(M\langle n \rangle)_j = M_{j-n}$.

$K^b(*\text{-grMod})$: homotopy category of bounded complexes of $*$-grMod. A complex M° in K^b is

$$M^\circ = \bigoplus_i M^i, \quad i: \text{homological grading}, \quad d: M^i \to M^{i-1}.$$

Another shift functor $[n]$ on K^b:

$$(M^\bullet[n])^i = (M^\bullet)^{i+n}.$$

Quotient by acyclic complexes, one obtains the derived category

$$D^b(*\text{-grMod}).$$

<u>Thm</u> (S-Λ duality).

$$D^b(\Lambda V\text{-gr Mod})_{fg} \cong D^b(SV^*\text{-grMod})_{fg}.$$

(fg: finitely generated cohomology).

<u>Pf outline.</u> 1. Shearing.

$\square : D^b(\Lambda V\text{-grMod})_{fg} \simeq D^b(\Lambda V[-1]\text{-grMod})_{fg}$

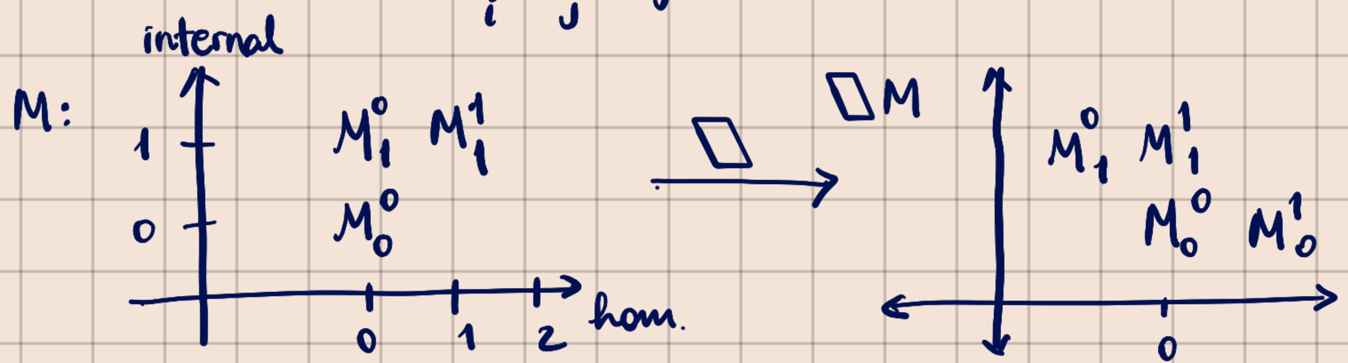
2. $F: D^b(\Lambda V[-1]\text{-grMod}) \xrightarrow{\cong} D^b(SV^*\text{-grMod})$, descends to fg.

(a) F well defined

(b) $G: D^b(SV^*\text{-grMod}) \longrightarrow D^b(\Lambda V[-1]\text{-grMod})$ is right adjoint of F, $F \dashv G$.

3. $F\square : D^b(\Lambda V\text{-grMod})_{fg} \xrightarrow{\simeq} D^b(SV^*\text{-grMod})_{fg}$.

I <u>Shearing</u>. If M is a complex of $*$-grMod, we can write $M = \bigoplus_i \bigoplus_j M_j^i$



$(\square M)_j^i = M_j^{i+j}$. Also, $d_{\square M}^i = (-1)^j d_M^{i+j}$.

We want new differential maps $M_j^{i+j} \longrightarrow M_{j-1}^{i+j-1}$.
So we modify our ring ΛV, in which ξ_i acts by degree $(1,-1)$.

DBZ: all objects in $D(\text{vsp})$, $\otimes$. There are ass. algebras in this category. (eg. ΛV). Shear $\square$ takes ass. alg to another ass algebra, modules over one to modules over the other. $\square$ $\Lambda V = \Lambda(V[-1])$.

$D^b(\Lambda V[-1]\text{-grMod})$ is complexes $M^\bullet$ of modules over $\Lambda V[-1]$ such that $d_M \xi_i + \xi_i d_M = 0$.

II. Functor F.

$$FM := SV^* \otimes_k M, \quad d^i_{FM} = \sum x_j \otimes \xi_j + 1 \otimes d^i_M \, ;$$

$$d^i_{FM}(f \otimes m) = \underline{\sum f(x_j) \otimes \xi_j(m) + f \otimes d^i_M(m).}$$

<u>DBZ Comments</u>

$V = \mathbb{C}$, $S = k[x] = \mathcal{O}(\mathbb{A}^1)$. S-module

$$S\text{-mod} \longrightarrow \text{Vect}, \quad M \longmapsto M \otimes_S k \quad \text{'fiber at 0'.}$$

Koszul duality "one can make the above functor into an equivalence". Multiple warnings!

fg module (gr S-mod) M cannot be supported away from 0.

$$\begin{tikzcd} & \text{End}(k)\text{-mod} \arrow[d] \\ S\text{-mod} \arrow[ur] \arrow[r] & \text{Vect} \end{tikzcd}$$

??

In the derived setting, the above is an equivalence. $\text{End}(k) = \Lambda^\square$.

A module $M^\bullet$ is fg in $D^b(\Lambda V[-1]\text{-grMod})$ if

$\bigoplus_i H^i(M)$ is fg over $\bigoplus_i H^i(\Lambda V[-1])$.

DG-algebras.

Andrei. F(free-mod) is the skyscraper. Connection to Koszul complex in alg geometry.

DBZ more comments: $\frac{1}{1-x} = 1+x+x^2+\cdots$

$$(1-x)(1+x+x^2+\cdots) = 1$$

comes from decategorification of Koszul complex.
Replace x with q. Think of q as dim of grdim = 1 vsp.
$k \oplus k\langle 1\rangle \oplus \cdots \quad = S(k\langle 1\rangle) = S$.
$k \oplus k\langle -1\rangle$ gives the $(1-q)$. "odd homological grading".
$\Lambda^\cdot \otimes S$ is quasi isom. to k : Koszul duality, so there is an equality of grdim: so $(1-q)(1+q+\cdots) = 1$.
(Andrei: two algebras, with reciprocal Hilbert series and some additional conditions means the algebras are Koszul dual!)

S-Λ Duality II (Sai)

Recall. $D^b(\Lambda V\text{-gr Mod}) \cong D^b(SV^*\text{-grMod})$

We constructed

$$D^b(\Lambda V[-1]\text{-gr Mod}) \xrightarrow{F} D^b(SV^*\text{-grMod}).$$

Here, $FM = SV^* \otimes_k M$, $d^i_{FM} = \sum x_j \otimes \xi_j + 1 \otimes d_M$

The adjoint.

$$GN = \mathrm{Hom}_k(\Lambda V[-1], N), \text{ and}$$

$$d^i_{GN}(\varphi)(\_) = \sum_j -x_j\, \varphi(\xi_j -) + d^i_N(\varphi(-)).$$

The unit map $\eta: id \to GF$.

$$\begin{aligned} \eta: \quad M &\longrightarrow \mathrm{Hom}_k(\Lambda V[-1], SV^* \otimes_k M) \\ m &\longmapsto (\lambda \mapsto 1 \otimes \lambda m). \end{aligned}$$

Adjunction $F \dashv G$ is $\otimes$-Hom adjunction. To descend to derived category, we show η induces isomorphism on cohomology.

Let $\varepsilon: SV^* \to k$ Augmentation map. $\varepsilon: SV^* \otimes M \to M$.
$e: k \to \Lambda V[-1]$.

Define $E_M: \mathrm{Hom}_k(\Lambda V[-1], SV^* \otimes M) \longrightarrow M$
(map of vector spaces, not Λ-mod) $\varphi \longmapsto \varepsilon(\varphi(e(1)))$.

So it suffices to show instead that E induces isomorphism on cohomology.

spectral sequence argument shows that we only need to check E_k is an isomorphism on cohomology.
"totalize away from M, spectral sequence degenerates."

$$E_k : \mathrm{Hom}(\Lambda V[-1], SV^*) \longrightarrow k = \mathrm{Hom}(\Lambda^0 V[-1], S^0 V^*).$$
$$\mathrm{Hom}(\Lambda V[-1], SV^*) \cong (\Lambda V[-1])^* \otimes_k SV^*.$$

V internal degree -1, V^* internal degree 1. Then $\Lambda V^*[1]$ is the dual of $\Lambda V[-1]$.

The basis of V^* is the basis $\{x_i\}$ but relabel to η_i.
$\xi_i \rightsquigarrow \partial/\partial\eta_i$.

$\Lambda V^*[1] \otimes_k SV^*$ is a complex with differentials given by $\sum \partial/\partial\eta_i \otimes x_i$.

$$\begin{tikzcd}
\Lambda^2 V^* \otimes SV^* \arrow[r] \arrow[d] & V^* \otimes SV^* \arrow[r] \arrow[d] & SV^* \arrow[d] \\
0 & 0 & k
\end{tikzcd}$$

<u>Lucky</u>: $\dim V = 1$.

observe isom. of cohomology.

$$\begin{array}{ccc}
\vdots & & \\
\oplus & & \vdots \\
\eta x^2 & \longrightarrow & \oplus \\
\oplus & & k x^2 \\
\eta x & \longrightarrow & \oplus \\
\oplus & & \\
\eta k & \longrightarrow & k x \\
 & & \oplus \\
 & & k
\end{array}$$

map $\partial/\partial\eta \otimes x$.

Let h be homotopy: $h = \sum \eta_i \otimes \partial/\partial x_i$. Then,

$$dh + hd = \sum \eta_i \,\partial/\partial\eta_i \otimes 1 + \sum 1 \otimes x_i\, \partial/\partial x_i.$$

The kernel of the chain map $dh + hd = |\alpha| \cdot \alpha$.

$$(dh + hd)(\alpha) = \text{tot} \cdot \deg(\alpha) \cdot \alpha.$$

$\deg(\alpha) = 0$ iff $\alpha \in \operatorname{span}(1 \otimes 1) \rightsquigarrow$ char 0 field.

The functor for the isomorphism

$$D^b(\Lambda V\text{-gr Mod}_{fg}) \xrightarrow{\sim} D^b(SV^*\text{-gr Mod})$$

is $F \circ \square = K$.

$$K(M\langle n\rangle) \cong KM[n]\langle n\rangle ; \; K(M[n]) = KM[n].$$

DBZ discussion.

Koszul Duality is a very very weird case of descent.

$$\begin{array}{ccc} X = \bigsqcup U_i & & \pi^* \mathcal{F} \\ \downarrow \pi & & \\ Y & & \mathcal{F} \end{array}$$

$\bigsqcup U_{ij} = X \times_Y X$

Descent: $Sh(Y) = Sh(X) +$ descent data.

Descent data: coaction of a coalgebra $\mathcal{O}(X \times_Y X)$.

$Y = V$, $\mathcal{O}(Y) = SV^*$. X: point. Descent data. What is $\bullet \times_V \bullet$? Functions on $X \times_Y X$ are $k \otimes_{SV^*} k$. This is a co-algebra. The descent data is a co-module for this co-algebra. Λ^*-comodule $= \Lambda$-module.
Derived language magic happen!

Equivariant cohomology. $X = \bullet \quad ES^1 = S^\infty$

(univ. circle bundle is contractible, free S^1 action) $\quad Y = BS^1 = \mathbb{CP}^\infty$

$ES^1 \times_{BS^1} ES^1 = S^1$

$S = H^*(BS^1) = \mathbb{C}[u]$, $|u| = 2$, equivariant cohomology.
$\Lambda = H_* S^1$, $\Lambda^* = H^* S^1$ co-algebra, Λ has a multiplication. Koszul duality: modules over S = comodules over Λ^*.

Homotopy theory. pointed space (X, x). Functor
Pointed spaces $\longrightarrow$ Groups (collection of paths)

$$X \qquad \Omega X \qquad \pi_0 \Omega X = \pi_1 X$$

homotopy class of paths $\leftrightarrow$ connected components of ΩX.

Functor Groups $\longrightarrow$ Pt. space: classifying space.
Connected pointed space X and Groups are equivalent under above functors.
Classifying space is the base of universal G-bundle.
ΩX: path $* \times_X *$ fiber product, derived sense
$* \wedge *$ (self intersection of pt). Eg. $\Omega_0 V = \operatorname{spec} \Lambda^* V$

The linearization of such equivalences can lead to more equivalences - Koszul duality.

$$\left\{ \begin{array}{l} \text{Rational homotopy category} \\ X \sim Y \text{ if } \pi_k \otimes \mathbb{Q} \text{ is an} \\ \text{isom.} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{dg } \mathbb{Q}\text{-lie alge-} \\ \text{bras.} \end{array} \right\}$$

"Ignore π_0, π_1".

$$\{\mathbb{Q}\text{-commutative ring}\}$$

X: topological space. The htpy gps shifted by 1 has a lie alg. structure ($\mathbb{Z}$-graded algebra).

$\pi_1 \circlearrowleft \pi_n$, $[\pi_1, \pi_1] \to \pi_1$ "commutator". There is a canonical map

$$\pi_m \times \pi_n \longrightarrow \pi_{m+n-1},$$

the Whitehead group. Eg S^2: $\pi_2 = \mathbb{Z}$, $\pi_3 = \mathbb{Z}$. The whitehead map $\mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$ is non-trivial; and π_3 is the Hopf fibration. $\pi_{*-1} X = \pi_* \Omega X$.

Projective Spaces. - Lachy. [BGG].

Let k be a field (noetherian base scheme S).

Let $\mathbb{P}^r = \mathrm{Proj}_k(A)$, $A = k[x_0 \cdots x_r] = \bigoplus_{i \geq -1} \Gamma(\mathbb{P}^r, \mathcal{O}(i)) = SV^*$

V^*: basis x_i

$$\mathrm{QCoh}(\mathbb{P}^r) \xleftarrow{\text{sheafification}} A\text{-grMod}$$

M be A-grMod. f homogeneous, $(M_f)_0$ is a module over $\mathrm{Spec}(A_f)_0 = D_+(f) \subseteq \mathbb{P}^r$. Glue $\widetilde{(M_f)_0}$ over $D_+(f)$.

Facts. $M \mapsto \tilde{M}$ is an exact functor.

$\tilde{A} = \mathcal{O}_{\mathbb{P}^r}$

$\widetilde{M}(n) = \widetilde{M} \otimes \mathcal{O}_{\mathbb{P}^n}(n)$

Now restrict to fg mod, coherent sheaf.

Fact. $\widetilde{(.)}$ has a kernel - eventually zero gr-Mod.

Serre subcategory - two out of three of some exact seq. is in the kernel then the third is in the kernel.

We can quotient by serre subcategory.

The graded analog of Global sections $\Gamma_\bullet$:

$$\Gamma_\bullet(\mathcal{F}) = \bigoplus_n \Gamma(\mathbb{P}^n, \mathcal{F}(n)) \text{ is } A\text{-grMod}.$$

$\Gamma_\bullet$ is an equivalence of cat. $\mathrm{Coh}(\mathbb{P}^r) \underset{\Gamma_\bullet}{\overset{\widetilde{(.)}}{\rightleftarrows}} \dfrac{A\text{-grMod}}{\ker \widetilde{(.)}}$

$$\underset{\mathrm{coh}(\mathbb{P}^r)}{D^b(\mathbb{P}^r_S)} = D^b(A\text{-grMod}_{fg})/\langle k\rangle \xrightarrow[\text{Koszul}]{\mathcal{K}} D^b(\Lambda V\,\text{gr-Mod}_{fg})/\langle \Lambda V\rangle$$

$$D^b(\mathbb{A}^{n+1}/\mathbb{G}_m) \simeq D^b_{\mathbb{G}_m}(\mathbb{A}^{n+1}) \simeq D^b(\Lambda V\text{-grMod}).$$

$\mathbb{P}^n = (\mathbb{A}^{n+1} - 0)/\mathbb{G}_m$. Removing point ~ killing sky-

-scraper. Skyscraper in Koszul goes to ΛV.

So $D^b(\mathbb{P}^n) \simeq D^b(\Lambda V\text{-gr Mod})$.
Recall $G(N) = \mathrm{Hom}_k(\Lambda V[-1], N)$ in $\mathcal{K}$.

Some lines on functions $\longrightarrow$ End(k)-mod.

① $\mathcal{O}_{\mathbb{P}^n} \xrightarrow{\Gamma_\circ} SV^* \xrightarrow{G} \mathrm{Hom}_k(\Lambda V[-1], N) \simeq k[0]\langle 0\rangle$.

$$(\Lambda^2 V, k) \quad (V,V) \xrightarrow{\sim} (k, \mathrm{Sym}^2 V^*) \quad \text{multiply by } x\,?$$
$$(V,k) \xrightarrow{\sim} (k, V^*)$$
$$(k,k)$$

② $\mathcal{O}_{\mathbb{P}^n}(i) \xrightarrow{\Gamma_\circ} SV^*\langle i\rangle \xrightarrow{\mathcal{K}} k\langle i\rangle$??
twist —— shift

Hyperplane H.
③ $\mathcal{O}(-1) \to \mathcal{O} \longrightarrow \mathcal{O}_H$ s.e.s. $\Gamma_\circ$ is exact, so
$\mathcal{O}_H \sim$ complex $\mathcal{O}(-1) \longrightarrow \mathcal{O}$.
$\Gamma_\circ(\mathcal{O}_H) \sim \Gamma_\circ(\mathcal{O}(-1) \to \mathcal{O}) \xrightarrow{G} \mathrm{Hom}_k(\Lambda V[-1], SV^*\langle -1\rangle \xrightarrow{\text{multiply } x} SV^*)$.

internal

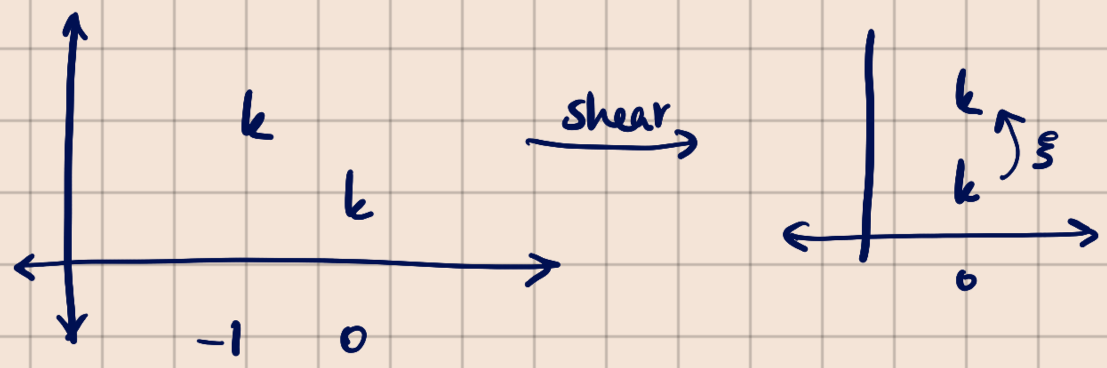
$\cdots$ $\mathrm{Hom}(k, V^*) \oplus \mathrm{Hom}(V, V^*) \to \mathrm{Hom}(k, S^2V^*)$

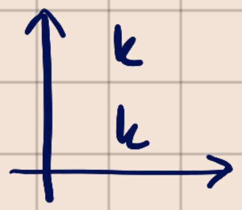
$\mathrm{Hom}(k,k) \oplus \mathrm{Hom}(V,k) \to \mathrm{Hom}(k, V^*)$

$\cdot x$

$\mathrm{Hom}(k,k)$

homological



So $\Gamma_\circ(\mathcal{O}_H) =$ 

with ΛV action.

some dual of $\mathcal{O}_H$ action.

DBZ: some comments on $\mathrm{Ext}^1(\ )$?

<u>DBZ</u>. Take $\Lambda = H^*(S^2)$ "cool ring". $D(\Lambda\text{-mod})$
$= \mathrm{Loc}(S^2)$ local systems on S^2. Λ has diff coh-grade.

$\mathrm{Ext}^1(k, k[1]) = \mathrm{Ext}^2_{\mathrm{locSys}}(k,k) = H^2(S^2)$ k: trivial local sys.

$k[1]$: non-trivial extn of k Hopf fibration.

$S^3 \xrightarrow{\pi} S^2$ circle bundle k const sh. $R\pi_* k$: k deg 0, k deg 1.

<u>Full Exceptional Collection.</u>

X be a smooth projective variety $/k$. Think $V = K_0(D^b(X))$

$D^b(X)$	V
$R\mathrm{Hom}(-,-)$.	$\langle , \rangle$ inner prod.
Collection	subset (!)
Full	Spanning
Exceptional	Orthonormal.
$K_0(D^b(X))$.	dim

Grothendieck group.

Full (strong) exceptional collections

<u>Def</u>. An ordered collection $\{E_i\}$.

- Full if $\{E_i\}$ generates $D^b(X)$. "smallest Δ-cat. containing E_i $= D^b(X)$."
- Exceptional if 1. $\mathrm{Hom}_{D^b(X)}(E_i, E_i[n]) = \begin{cases} k & n=0 \\ 0 & o/w \end{cases}$

  "normalized" "derived Schur's lemma".

  2. $\mathrm{Hom}(E_i, E_j[n]) = 0$ for $i>j$

  "orthogonality"
- Strong if $\mathrm{Hom}(E_i, E_j[n]) = 0$ for $n \neq 0, i<j$.

<u>Example</u> $X = \mathbb{P}^n$. Take $E_i = \mathcal{O}(i)$ for $i = 0, 1, \ldots, n$ is a full exceptional collection.

Exceptional.

$$\begin{aligned} R\mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j)[\ell]) &= \mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j)[\ell]) \\ &= \mathrm{Hom}(\mathcal{O}, \mathcal{O}(j-i)[\ell]) \\ &= H^\ell(\mathbb{P}^n, \mathcal{O}(j-i)). \end{aligned}$$

The last is a classical Čech cohomology computation concentrated at $0, n$.

<u>DBZ</u>: Easier to have exc. in top. If X is a stratified space. Contractible strata (eg. $\mathbb{P}^n$, flag var, Grassman- with Schubert stratification). $D_{cons}(X)$ constructible $sh(X)$ derived category. E_i : const sh on strata. The contractibility shows const sheaf on E_i inner product = cohomology trivial.

$$\mathrm{Hom}(\text{const}, sp) = \text{Global section }(sp)$$

Mirror symmetry : X alg. $\longrightarrow$ stratification on mirror of X. connects alg. exceptional collections to topological.

$$0 \to k_0 \to F \to k_0 \to 0$$

$$QC((\mathbb{A}^{n+1}/\mathbb{G}_m) \xrightarrow[\text{at } 0]{\text{fiber}} \underline{\mathrm{Ext}^*_{\mathbb{A}^{n+1}}(k_0, k_0)\text{-mod}} \simeq \underline{\Lambda V\text{-mod}}$$

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

k_0 : skyscraper at $0 \in \mathbb{A}^{n+1}$

SV^*-modules

k_0

$$QC(\mathbb{A}^{n+1}) \longrightarrow \mathrm{Ext}^*_{\mathbb{A}^{n+1}}(k_0, k_0)\text{-mod} = \Lambda\text{-mod}$$

$$QC(\mathbb{A}^{n+1}/\mathbb{G}_m)$$
$$\|$$
$$QC(\mathbb{A}^{n+1})^{\mathbb{G}_m} \longrightarrow \mathrm{gr}\,\Lambda\text{-mod}$$

($\mathbb{G}_m$-action = $\mathbb{Z}$-grading).

$\mathbb{G}_m \curvearrowright V = \bigoplus V_n$

$\mathbb{G}_m \curvearrowright V_n$

$z \qquad z^n$

Jiwoong words of wisdom.

$$QC(\mathbb{P}^n)$$
$$\|$$
$$QC(\mathbb{A}^{n+1}/\mathbb{G}_m - 0) \longrightarrow \mathrm{gr}\,\Lambda\text{-mod}/\langle \Lambda V \rangle$$
$$\|$$
$$QC(\mathbb{A}^{n+1}/\mathbb{G}_m)/\underbrace{QC_0(\mathbb{A}^{n+1}/\mathbb{G}_m)}_{\| \atop \langle k_0 \rangle}.$$

$\mathbb{P}^n$ - II (Lachy).

<u>Thm</u>. $\langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n) \rangle$ form a full strong exceptional collection for $D^b(\mathbb{P}^n)$.

(DBZ: for irred variety X, we cannot have
$\mathrm{Hom}(E_i, E_j) = 0$ for all $i \neq j$.
This breaks $D^b(X)$ into $\oplus$ of two categories.)

<u>Pf</u>. $R\mathcal{H}om(\mathcal{O}(i), \mathcal{O}(j)) = \mathcal{O}(j-i)$.

Fact:

$$\mathrm{Hom}_D(\mathcal{O}(i), \mathcal{O}(j)[\ell]) = H^\ell(\mathbb{P}^n, R\mathcal{H}om(\mathcal{O}(i), \mathcal{O}(j))).$$

$$H^\ell(\mathbb{P}^n, \mathcal{O}(j-i)) = \begin{cases} 0 & \forall \ell \text{ if } j-i \in [-1, -n+1] \\ k & i=j \text{ and } \ell = 0. \\ k^d & j-i \in [1, n-1] \; \underset{\ell=0}{d} = \binom{n+j-i}{n} \\ 0 & j-i \in [1, n-1], \; \ell \neq 0. \end{cases}$$

Symmetry in $R\mathrm{Hom}(\_,\_)$ is Serre Duality.
If canonical bundle is positive or trivial then it cannot have FEC. (DBZ)

If $\{E_i\} \subseteq D^b(X)$ is a strong FEC. Then $T = \bigoplus_i E_i$.
Then $\mathrm{End}(T) = \bigoplus_{i \leq j} \mathrm{Hom}_D(E_i, E_j)$.

$R\mathrm{Hom}(T, C)$ has $\mathrm{End}(T)$ action. so

$$R\mathrm{Hom}(T, \_) : D^b(X) \xrightarrow{\sim} D^b(\mathrm{End}(T)\text{-Mod})$$

$\mathrm{End}(T)$ is a finite dimensional algebra when $\{E_i\}$ is finite.

Eg. $X = \mathbb{P}^1$. $T = \mathcal{O} \oplus \mathcal{O}(1)$ $\quad \mathrm{Hom}(T, T) = \begin{tikzcd} \underset{\mathcal{O}}{\bullet} \arrow[r, bend left, "x_0"] \arrow[r, bend right, "x_1"'] & \underset{\mathcal{O}(1)}{\bullet} \end{tikzcd}$

Then $\mathrm{End}(T) = kQ$, Q: quiver algebra.

Then

$$D^b(\mathbb{P}^1) \simeq D^b(kQ\text{-mod}).$$

DBZ:

	E_1	E_2
E_1	k	$\mathrm{Hom}(E_1, E_2)$
E_2	0	k

$\mathrm{Hom}(E_1, E_2) \simeq kx_0 \oplus kx_1$.

$X = \mathbb{P}^2$. $T = \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$.

$$Q: \begin{tikzcd} \underset{\mathcal{O}}{\bullet} \arrow[r, bend left, "x_0"] \arrow[r, "x_1"] \arrow[r, bend right, "x_2"'] & \underset{\mathcal{O}(1)}{\bullet} \arrow[r, bend left, "x_0"] \arrow[r, "x_1"] \arrow[r, bend right, "x_2"'] & \underset{\mathcal{O}(2)}{\bullet} \end{tikzcd}$$

Relations: $\mathcal{O} \xrightarrow{x_1} \mathcal{O}(1) \xrightarrow{x_0} \mathcal{O}(2)$

$= \mathcal{O} \xrightarrow{x_0} \mathcal{O}(1) \xrightarrow{x_1} \mathcal{O}(2)$

Then, $D^b(\mathbb{P}^2) \simeq D^b(kQ\text{-mod})$

Note: what if FEC is not strong?

$T = \bigoplus E_i$, $\quad R\mathrm{End}(T) = \bigoplus_{i \le j} R\mathrm{Hom}(E_i, E_j)$.

These are dg-algebras.

$$R\mathrm{Hom}(T, -): D^b(X) \xrightarrow{\sim} D^b(R\mathrm{End}(T)\text{-mod}).$$

Here, $R\mathrm{End}(T)$ is a fd dg-algebra.

<u>Andrei Comments.</u>

Fullness: $\mathrm{End}(T)$ is a Koszul ring.

Procedure to get Koszul collection from strong FEC.

A sequence of mutations. Braid gp elt. of full length.

Projective $\xrightarrow[K.D.]{}$ Simple $\xrightarrow[K.D]{}$ Injective

Alternate choice of T — $\mathrm{End}(T)$ is Koszul. Then there is a Koszul duality.

Square of the (name) duality is Serre duality!

— Wandall, Reps of associative alg and coherent sheaves.

Discussions

$$T \longrightarrow X_*(T) \quad \text{covariant}$$

$$\{\text{Tori}\} \underset{\longrightarrow}{\overset{\longrightarrow}{\text{eq.}}} \{\text{Free ab groups}\}$$

$$T \longrightarrow X^*(T) \quad \text{contravariant}$$

T torus, $T^\vee$ dual torus, $T^\vee = \mathbb{G}_m \otimes X^*(T)$
$= \mathrm{Hom}(X_*(T), \mathbb{G}_m)$

Note $\pi_1(T) = X_*(T)$. so

$$T^\vee = \mathrm{Hom}(\pi_1(T) \longrightarrow \mathrm{Aut}(\mathbb{C})).$$

$T^\vee$ parameterizes local systems on T. There should be a canonical local system on $T \times T^\vee$.

$\mathbb{P}^n$ - Lachy

The set $\{\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n)\}$ is a FEC in $D^b(\mathbb{P}^n)$.

Fourier Mukai Transforms.

All functors have a non-derived version.
Let $\pi: X \to Y$ be a map of smooth proj-varieties.

Derived pushforward $R\pi_*$:

1) Resolve $\mathcal{F}^\bullet \overset{qis}{\simeq} \mathcal{I}^\bullet$ injective / flasque sheaf
2) $R\pi_*(\mathcal{F}^\bullet) := \pi_*(\mathcal{I}^\bullet)$

Derived pullback Lf^*:

1) Resolve $\mathcal{F}^\bullet$ by $\mathcal{L}^\bullet$ of locally free (not projective)
2) $Lf^*(\mathcal{F}^\bullet) = f^*(\mathcal{L}^\bullet)$. (loc. free is acyclic.

If underived functors are exact we don't need to resolve.

Projection formula.

Coh: $i_*(\mathcal{F} \otimes i^*\mathcal{E}) \simeq i_*\mathcal{F} \otimes \mathcal{E}$ if $\mathcal{E}$ is loc. free
D^b: $Ri_*(\mathcal{F}^\bullet \otimes Li^*\mathcal{E}) \simeq Ri_*\mathcal{F}^\bullet \overset{L}{\otimes} \mathcal{E}^\bullet$
Every $\mathcal{E}^\bullet$ is qis to bdd cplx of loc. free sheaves.
so there is

Flat base change.

$$\begin{tikzcd} X \times Y \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d] \\ X \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

$\mathcal{F} \to Y$ sheaf.
Assume g is flat for general case, here all over field.

Coh: $R^i f_* g^* \mathcal{F} \simeq \mathcal{O}_X \otimes H^i(Y, \mathcal{F})$
D^b : $Rf_* Lg^* \mathcal{F}^\bullet \simeq \mathcal{O}_X \overset{L}{\otimes}_k H^\bullet(Y, \mathcal{F})$

<u>Def</u>.

$$\begin{tikzcd} & X \times Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

$\mathcal{F}^\circ$ on X.

$\mathcal{P}^\bullet \in D^b(X \times Y)$
Fourier Mukai transform $\Phi_{\mathcal{P}}$

$$D^b(X) \xrightarrow{\Phi_{\mathcal{P}}} D^b(Y)$$
$$\mathcal{F}^\bullet \longmapsto q_*(\mathcal{P}^\bullet \otimes p_* \mathcal{F}^\bullet)$$

<u>Eg</u>. $X = Y$, $\mathcal{P} = \mathcal{O}_\Delta$. Then $\Phi_{\mathcal{O}_\Delta} = \mathrm{Id}$ on $D^b(X)$.

$x \in X$ $\quad \Phi_{\mathcal{P}}(k(x)) = \mathcal{P}|_{\{x\} \times Y}$

A be ab. variety, $A^\vee$ dual ab. variety. There is a Poincare bundle $\mathcal{P}$ on $A \times A^\vee$. The FMT gives an equivalence of derived catgories $D^b(A) \simeq D^b(A^\vee)$.
"If canonical bundle is positive (ample), negative then Φ is an isomorphism of D^b."

<u>Fact</u> Every isom of $D^b(X)$, $D^b(Y)$ is a FMT.

<u>Koszul Duality</u> $\quad (A, \mathfrak{m}, k)$: local noetherian ring.

Choose $f = f_1 \dots f_n \in \mathfrak{m}$ generators, minimal n.

$$K_\bullet(\underline{f}) := \cdots \Lambda^i A^n \to \Lambda^{i-1} A^n \to \cdots \to A^n \to A \to k \to 0)$$

$$e_i \longmapsto f_i$$

$\Lambda^2 A^n \longrightarrow A^n$: $\quad e_i \wedge e_j \longmapsto f_i e_j - f_j e_i$.

<u>DH Rmks.</u> $K_\bullet^{r-1}$: $K_\bullet(f_1, \dots, f_{r-1})$. Build complex

$$K_0 : A \xrightarrow{f_1} A$$

$$\begin{tikzcd} \vdots \arrow[d] & \vdots \arrow[d] \\ K_2 \arrow[r, "f_r"] \arrow[d] & K_2 \arrow[d] \\ K_1 \arrow[r, "f_r"] \arrow[d] & K_1 \arrow[d] \\ K_0 \arrow[r, "f_r"] & K_0 \end{tikzcd}$$

$K_\bullet^r = \mathrm{Tot}(\_)$,

$K_\bullet^r = \mathrm{Cone}(f_r : K_\bullet^{r-1} \to K_\bullet^{r-1})$

f_r are regular sequence then "exactness" of $K_\bullet(\underline{f})$ comes from inductive def.

$K_\bullet(\underline{f})$ is exact iff $H_1(K_\bullet(\underline{f})) = 0$

iff A is regular

iff $\dim A = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$.

<u>Globalizing.</u> X be smooth projective. $Z \overset{i}{\subseteq} X$ smooth, $Z = V(s)$. Here, $s \in \Gamma(X, \mathcal{E})$ for $\mathcal{E}$ a v. bundle of rank r. We want codim $V = r$. Then,

$$\cdots \longrightarrow \Lambda^2 \mathcal{E}^\vee \longrightarrow \mathcal{E}^\vee \longrightarrow \mathcal{O}_X \xrightarrow{i_*} \mathcal{O}_Z \longrightarrow 0$$

is exact! (Resolution of $\mathcal{O}_Z$ by locally free sheaves).

<u>Rmk.</u> For $Z = D$ divisor on sm. proj curve X, the above shows

$$0 \to \mathcal{O}_{-D} \to \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0 \quad \text{is exact.}$$

Eg. $\Delta \subseteq \mathbb{P}^n \times \mathbb{P}^n$ diagonal.

$$\mathbb{V}_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}(-1)) = \underbrace{\underline{\mathrm{Spec}}}_{\text{rel. spectrum}} \mathrm{Sym}^\circ(\mathcal{O}(-1)^\vee)$$

$$Bl_0\,\mathbb{A}^{n+1} \longrightarrow \mathbb{P}^n$$

$\mathcal{O}_{\mathbb{P}^n}(-1)$ is a sheaf on $\mathbb{V}_{\mathbb{P}^n}$. The fibers of line $l \subseteq \mathbb{P}(V) = \mathbb{P}^n$ in $\mathcal{O}(-1)$ is also l.

$$\begin{array}{c} \Omega(1) \\ | \\ \mathbb{P}^n \end{array}$$ the fibers of $l \subseteq V$ is $l^\perp$.

Euler Exact sequence

!

$$0 \longrightarrow \Omega_{\mathbb{P}^n}(1) \longrightarrow V^* \otimes \mathcal{O}_X \longrightarrow \mathcal{O}(1) \longrightarrow 0.$$

Taking fibers at l,

$$0 \longrightarrow l^\perp \longrightarrow V^* \longrightarrow l^* \longrightarrow 0.$$

(Aside: $\mathcal{O}(1)$ space: $\mathbb{P}^2 \setminus pt \longrightarrow \mathbb{P}^1$).

$$\begin{array}{ccc} (x, \varphi) & \overset{l}{\mathcal{O}(-1)} \, , \, \boxtimes \overset{l'}{\Omega(1)} & \text{bundle on } \mathbb{P}^n \times \mathbb{P}^n \\ \downarrow & \downarrow & \\ \varphi(x) & \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} & \end{array}$$

$\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} / m \simeq \mathcal{O}_\Delta$ when $l = l'$

<u>Thm</u>. $\langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n) \rangle$ is full in $\mathbb{P}^n$.

<u>Pf</u>. $\xi = \mathcal{O}(-1) \boxtimes \Omega(1)$. Then the Koszul sequen-ce is exact.

$$0 \to \Lambda^n \mathcal{E} \to \cdots \to \Lambda^2 \mathcal{E} \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \to \mathcal{O}_\Delta \to 0$$

Take $\mathcal{F}^\bullet \in D^b(\mathbb{P}^n)$. Apply FMT $Rq_*(\_ \overset{L}{\otimes} p^* \mathcal{F})$ for $\mathbb{P}^n \times \mathbb{P}^n \xrightarrow{p,q} \mathbb{P}^n$. So we have

$$0 \to \Phi_{\Lambda^n \mathcal{E}}(\mathcal{F}) \to \cdots \to \Phi_{\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n}}(\mathcal{F}^\bullet) \to \Phi_{\mathcal{O}_\Delta}(\mathcal{F}^\bullet) \to 0$$

is exact. Recall that $\Phi_{\mathcal{O}_\Delta}(\mathcal{F}^\bullet) = \mathcal{F}^\bullet$, so we prove

$\Phi_{\Lambda^i \mathcal{E}}(\mathcal{F})$ can be generated by $\mathcal{O}(i)$s. But

$$\begin{aligned}
\Phi_{\Lambda^i \mathcal{E}}(\mathcal{F}) &= Rq_*\left(\mathcal{O}(-i) \boxtimes \Omega^i(i) \overset{L}{\otimes} Lp^* \mathcal{F}^\bullet\right). \\
&= Rq_*\left(Lq^* \mathcal{O}(-i) \overset{L}{\otimes} Lp^* \Omega^i(i) \overset{L}{\otimes} Lp^* \mathcal{F}\right). \\
&= Rq_*\left(Lq^* \mathcal{O}(-i) \overset{L}{\otimes} Lp^*(\Omega^i(i) \overset{L}{\otimes} \mathcal{F})\right). \\
\text{(Projection)} \quad &= \mathcal{O}(-i) \overset{L}{\otimes} Rq_* Lp^*(\Omega^i(i) \otimes \mathcal{F}) \\
\text{(flat base ch)} \quad &= \mathcal{O}(-i) \otimes \mathcal{O}_{\mathbb{P}^n} \otimes H^\bullet(\mathbb{P}^n, \Omega^i(i) \otimes \mathcal{F}) \\
&\in \langle \mathcal{O}(-i) \rangle.
\end{aligned}$$

$\square$

Infinity

Def Δ (simplex category): category of non-empty finite totally ordered sets, maps: non increasing.

Objects $\mathbb{n} = \{0, \dots, n-1\}$ upto isomorphism.

A simplicial set is a presheaf X on Δ.
We call $X[\mathbb{n}+\mathbb{1}]$: n-simplices of X.

Face maps $\delta^i : \mathbb{n} \longrightarrow \mathbb{n}+\mathbb{1}$ "delete i"
Degeneracy maps $\sigma^i : \mathbb{n}+\mathbb{1} \longrightarrow \mathbb{n}$ "collapse i".
There are four relations between δ^i, σ^j, and these maps generate Δ. (a presentation of Δ).

Eg X be a top. space, Δ_n: standard n-simplex. We define $X[\mathbb{n}+\mathbb{1}] = \mathrm{Hom}(\Delta_n, X)$. Here, δ^i is literally a face map.

For $n \in \mathbb{N}$, we take $\Delta[n] := \mathrm{Hom}(\_, \mathbb{n}+\mathbb{1})$, the abstract n-simplex.
For X a top sp, $\mathrm{Hom}(\Delta[n], X) \simeq X[\mathbb{n}+\mathbb{1}]$ n-simplices of X.

Let X be a simplicial set. We make X a category.
Objects: 0-cells, morphism: 1-cells. Eg $x \in X^0$, then $\sigma^0 x \in$ 1-cell (degenerate 1-cell). "identity map on x."

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[dr, dotted, "\exists"'] & b \arrow[d, "g"] \arrow[dl, phantom, "\cong"] \\ & c \end{tikzcd}$$

"composition of morphisms" $a, b, c \in X^0$, $f, g \in X^1$. Define simplicial set $\Lambda^1[2]$:

$$\begin{tikzcd} 0 \arrow[r] \arrow[dr, "\cong"'] & 1 \arrow[d] \\ & 2 \end{tikzcd}$$

start with 2 simplex, remove 2-cell, delete codim 1-simplex opposite 1. (2-simplex is an isomorphism $0 \to 1 \to 2 \cong 0 \to 2$)

$$\begin{tikzcd} 0 \arrow[r] & 1 \arrow[d] \\ & 2 \end{tikzcd}$$

$\Lambda^1[2] \hookrightarrow \Delta[2]$.

For composition law, we must have an extension of $\Lambda^1[2] \longrightarrow X$ to $\Delta[2] \longrightarrow X$. More generally, for $n \geq 2$, $0 < k < n$ a map $\Lambda^k[n] \longrightarrow X$ must extend to $\Delta[n] \longrightarrow X$.

A simplex which satisfy these extension properties is called a quasi-category, $(\infty, 1)$-category.

"homotopy theory of topological spaces $\rightsquigarrow$ equivalences of simplicial sets" - For nice complexes, the notions agree.

Eg. $X[n+1] = \mathrm{Hom}(\Delta_n, X)$ for top space X is an infinity category.

Category of categories. "Inverse of a equivalence of categories means composite is isomorphic to identity. So we lose associativity of composition in cat of cat, hence we move to 2-category."

Eg. $\mathcal{C}$ category. "nerve of $\mathcal{C}$". $n \in [n+1] = X_0 \to X_1 \to \dots \to X_n$. Nerve is an inclusion of 1-category into an infinity category.

The filling is unique. "a simplex is determined by the 1-skeleton."

Morita Theory Let k be a field. Consider a k-alg A. Consider A-mod, the category of A-modules (left). Can we obtain A from the k-linear category A-mod?

No. Suppose $A\text{-mod} \xrightarrow{F} B\text{-mod}$ is an equivalence of categories. The functor is determined by FA. Then FA is a (B,A)-bimodule. F preserves colimits.

For $M = FA$, $N \in A$-mod, then $M \otimes_A N$ is a B-module. Furthermore, $M \otimes_A N$ is colimit preserving so this equals $F(N)$. One has

$$\text{Bimod}(B,A) \cong \underbrace{Pr^L}_{\text{colim. preserving.}}(A\text{-mod}, B\text{-mod}).$$

Morita cat.

$$\text{Mor} = \begin{cases} \text{0-cells: } k\text{-algebra } A \\ \text{1-cells: functor } A\text{-mod}, B\text{-mod} \equiv \text{Bimod}(B,A) \\ \text{2-cells: bimodule maps.} \end{cases}$$

Eg. $A = k$, $B = \text{Mat}_{n\times n}(k)$. k^n is a (B,A)-bimod giving an equivalence $A\text{-mod} \to B\text{-mod}$.

In general, $A\text{-mod} \xrightarrow{\cong} B\text{-mod}$ if $\exists\, M \in (B,A)$-bimod, $N \in (A,B)$-bimod such that $M \otimes_A N = B$; $N \otimes_B M = A$.
(See this for above example, $M = k^n$, $N = k^n$.

We want to generalize to symmetric monoidal categories. Relative tensor product $M \otimes_A N$ is the exact seq.

$$M \otimes_k A \otimes_k N \rightrightarrows M \otimes_k N \to M \otimes_A N \to 0$$

Tensor preserves colimits. This generalizes nicely

TQFTs

Category $\quad Bord_n^{fr}$: $\left\{\begin{array}{ll} \text{0-cells} & \text{mfd} \\ \text{1-cells} & \text{cobordisms} \\ \text{2-cells} & \text{cobordism b/w 1-cells} \\ \vdots & \\ \text{n-cells} & \text{cobordism b/w n-1 cells} \\ \text{n+1-cells} & \text{diffeomorphism} \\ \text{n+2-} & \text{isotopy} \\ \vdots & \text{higher isotopy.} \end{array}\right.$

fr: M - k-dim A framing of M is a trivialization of $TM \oplus \mathbb{R}^{n-k}$.
cobordism respects framing.

Sym monoidal structure is $\sqcup$.

<u>cobordism hypothesis.</u>

Perfect correspondence

$$\mathrm{Hom}(Bord_n^{fr}, \mathcal{C})^{\underset{\sim}{N}} \cong \{\text{Fully dualizable objects in } \mathcal{C}\}$$

make groupoid.

Dualizable is a finiteness condition.

<u>Eg</u> A vector space V is 1-dualizable if $\exists V^*$ and maps $ev: V \otimes V^* \longrightarrow \mathbb{C}$, $coev: \mathbb{C} \longrightarrow V \otimes V^*$. We want $V \xrightarrow{coev \otimes 1} V \otimes V^* \otimes V \xrightarrow{1 \otimes ev} V = id: V \to V$. Also $V \leftrightarrow V^*$ $ev \leftrightarrow coev$.

In a symmetric monoidal category, 1-dualizable is fully dualizable. "ev and coev must have adjoints for all levels."

Eg. A simple algebra / k is dualizable. A Morita category is a 2-category, and 1-dualizability is for free.
We introduce higher Morita categories.

Let $\mathcal{C}$ be an E_n-category with all colimits, presentable, $\otimes$ preserves colimits in each argument.
(E_n-category: n=1, monoidal cat, n=2: braided category,

$n = \infty$: sym monoidal category.)

Define $\mathrm{Alg}_{(k)}(\mathcal{C})$ by induction, $k \leq n$. $\mathrm{Alg}_{(0)}\mathcal{C} = \mathcal{C}$,

$$\mathrm{Alg}_{(k)}(\mathcal{C}) = \begin{cases} \text{ob: } \mathbb{E}_k\text{-alg in } \mathcal{C} \\ \mathrm{Hom}(A,B): \mathrm{Alg}_{(k-1)}(\mathrm{Bimod}(A,B)) \end{cases}$$

"$\mathbb{E}_k$-algebra: k different axes of multiplication."

$$\mathrm{Alg}_{(1)}\mathcal{C} = \begin{cases} \text{objects: } \mathbb{E}_1\text{-alg} \quad \text{"associative algebra"} \\ \underset{\text{1-cells}}{\mathrm{Hom}(A,B)}: \mathrm{Alg}_{(0)}(\mathrm{Bimod}(A,B)) = \mathrm{Bimod}(A,B). \\ \text{2-cells}: \text{ Bimodule maps.} \end{cases}$$

$\mathrm{Alg}_{(1)}\mathcal{C}$ is an $(\infty,2)$-category. Thm that all objects of $\mathrm{Alg}_{(1)}$ are 1-dualizable.

$$\mathrm{Alg}_{(2)}\mathcal{C} := \begin{cases} \text{objects: } \mathbb{E}_2\text{-alg} \quad \text{"commutative algebra" in } \mathcal{C}. \\ \underset{\text{1-cell}}{\mathrm{Hom}(A,B)}: \mathrm{Alg}_{(1)}(\mathrm{Bimod}(A,B)) = \mathrm{Alg}_{(1)}(A\otimes B\text{-mod}) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \mathrm{Alg}_{A\otimes B}. \\ \text{2-cells: } \quad \mathrm{Hom}(C,D) = \mathrm{Bimod}(C,D). \\ \text{3-cells: } \quad \text{Bimod maps.} \end{cases}$$

In $\mathrm{Alg}_{(2)}\mathcal{C}$, objects are 2-dualizable.

Equivariant Cohomology (Isaac Martin)

Quantum mechanics path integration $\int d\phi\, e^{iS(\phi)}$
Statistical mechanics $Z = \int d\phi\, e^{-\beta H(\phi)}$.

$$\int_{\mathbb{R}^n} g(x) e^{ikf(x)} dx \approx \sum_{\substack{x_0 \in \Delta \\ \text{crit pts}}} g(x_0) e^{ikf(x_0)} (f'' \text{ terms}) + O(k^{-n/2}).$$

Stationary phase approximation.

Setup. (M, ω), $2n$-dim compact symplectic mfd, $T = (S^1)^\ell$, $T \circlearrowright M$ symplectic action.

Moment map $\mu : M \to \mathrm{lie}(T)^\vee = \mathbb{R}^\ell$.

Thm (Noether) $f : C^\infty(M, \mathbb{R})$ invariant under T iff μ is constant along $\mathrm{grad}_\omega f$. "conservation law $\leftrightarrow$ symmetry."

component-wise exponentiation

we have an integral $\int_M e^{it\mu} \left(\frac{\omega^n}{n!}\right) \approx \sum_{p \in M^T} e^{-it\mu(p)} \frac{1}{(it)^m e(p)}$

is an equality (M^T = crit pts of μ).

Atiyah-Bott realized this as a localization of Equivariant cohomology.

Eq Coh. Let G be a topological group. G-Top : category of topological spaces with G-action.

We may try to define $H^*_G(X) = H^*(X/G)$. However, this is not a good definition unless $G \circlearrowright X$ is free.

So, one defines a new space $\tilde{X}$ with same htpy class as X, but now with a free action.

Def. Let $\mathbb{E}G$ be a (weakly) contractible G-space with free group action of G.

all π_i are 0

Set $$EG \times^G X = \frac{\mathbb{E}G \times X}{G}, \quad G \circlearrowright \mathbb{E}G \times X, \quad g(e,x) = (eg^{-1}, gx)$$

If G is Hausdorff, paracompact, $\mathbb{E}G$ exists.

Eg. $G = 1$, $\mathbb{E}G = 1$

$G = \mathbb{Z}$, $\mathbb{E}G = \mathbb{R}$, $H^*_G(pt) = H^*(\mathbb{R}/\mathbb{Z}) = \mathbb{Z}[t]/(t^2)$.

$G = \mathbb{Z}/2\mathbb{Z}$, $\mathbb{E}G = S^\infty$, $H^*_G(pt) = H^*(\mathbb{R}P^\infty)$

$G = S^1$, $\mathbb{E}G = (\mathbb{C}\setminus 0)^\infty$; $H^*_G(pt) = H^*(\mathbb{C}P^\infty) = \mathbb{Z}[t]$.

$G = T = (\mathbb{C}^*)^n$, $\mathbb{E}G = ((\mathbb{C}\setminus 0)^\infty)^n$, $H^*_G(pt) = \mathbb{Z}[t_1 \dots t_n]$

$H^*_G(pt) = H^*(\mathbb{E}G/G = \mathbb{B}G) = H^*(G, \mathbb{Z})$ group cohomology.

In algebraic geometry, we have other ways to define $\mathbb{E}G$ using approximation spaces $\mathbb{E}$ s.t. $\pi_n(\mathbb{E}) = 0$ for all $n < N$. (Equivariant Chow ?)

or using stacks. $H^*_G(X) = H^*([X/G])$.

or G-equivariant chain complexes. (Koszul Duality)

Properties

1) Let $\Lambda_G = H^*_G(pt)$. For
$$\begin{array}{c} X \\ \downarrow \\ pt \end{array} \rightsquigarrow \begin{array}{c} H^*_G(X) \\ \uparrow \\ H^*_G(pt) = \Lambda_G \end{array}$$

so all eq. coh has Λ_G-algebra structure.

2)
$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & EG \times^G X \arrow[d] \\ pt \arrow[r, hook] & BG \end{tikzcd} \rightsquigarrow \begin{tikzcd} H^*(X) & H^*_G(X) \arrow[l] \\ \mathbb{Z} \arrow[u] & H^*_G(pt) \arrow[l] \arrow[u] \end{tikzcd}$$

so $H^*_G \longrightarrow H^*$.

3) If $G \circlearrowright X$ freely then $H^*_G(X) = H^*(X/G)$.

Eg. $G = T = (\mathbb{C}^\times)^n$. Let X be a sm. proper n-dim variety/$\mathbb{C}$. (compact complex mfd) with T action.

$p \in X^T$ Then $T \circlearrowright T_pX$, so break T_pX into eigenspaces

$$T_pX = \bigoplus \mathbb{C}_{\lambda_i} \qquad \lambda_i \in X^*(T).$$

Eg: $T \circlearrowright \mathbb{A}^2 = X$ via scaling. $t \in T$, then $\varphi_t : X \to X$ is a map of varieties. On coord. rings,

$$(t_1, t_2) : f(x,y) \longmapsto f(t_1 x, t_2 x).$$

is a map of local rings, induces map of tangent spaces

$$\left(\frac{m}{m^2}\right)^\vee \longrightarrow \left(\frac{m}{m^2}\right)^\vee, \quad \text{matrix} \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}.$$

For $M = X^*(T)$, then $\mathrm{Sym}^* M = \mathbb{Z}[t_1, t_2] = \Lambda_T$.
This is not an accident!

Chern Classes

<u>Def</u>. A G-equivariant vb V on X is a vb V/X with $G \circlearrowright V$ acting linearly on fibers.

Chern class of eq. vb is a cohomology class.

$$c_k^G(V) = c_k(EG \times^G V) \in H_G^{2k}(V)$$

we note, $c_1(\mathcal{L} \otimes \mathcal{M}) = c_1(\mathcal{L}) + c_1(\mathcal{M})$.

Splitting formula. $V = L_1 \oplus \cdots \oplus L_r$, $c_k^G(V) = e_k(c_1(L_1), \ldots)$
the k^{th} symm. poly e_k.

<u>Eg</u> $M = X^*(T)$. $x \in M$, then $\mathbb{C}_x$ is a line bundle /pt.
$x_1, x_2 \in M$, $\mathbb{C}_{x_1+x_2} = \mathbb{C}_{x_1} \otimes \mathbb{C}_{x_2}$.

$$\begin{aligned} M &\longrightarrow \Lambda_T^2(pt) \\ x &\longmapsto c_1^T(\mathbb{C}_x) \end{aligned}$$ gives $\mathrm{Sym}^* M \simeq \Lambda_T$.

<u>Gysin hom</u> (pushfwd)

For $f: X \to Y$ proper morphism of complex mfd.

$$\begin{tikzcd} H^i(X) \arrow[r, dashed, "f_*"] \arrow[d, "PD"] & H^{i+2d}(Y) \arrow[d, "PD"] \\ H_{2x-i}(X) \arrow[r, "f_*"] & H_{2x-i}(Y) \end{tikzcd}$$

$d = x - y$, $x = \dim X$, $y = \dim Y$

Self intersection formula $f: X \hookrightarrow Y$ closed embedding.

Then, $f^* f_*(a) = c_d^G(N) \cdot a$, for $a \in H_G^*(X)$,
N normal bundle of X in Y: $TY|_X / TX$.

Eg self intersection for Tori. $T \circlearrowleft X$, $X^T = 1$. Let $i : pt \hookrightarrow X$ be the inclusion. Let $a \in \Lambda_T$. Then,
$i^* i_*(a) = c_d^T(T_pX) \cdot a$

Now, $T_pX = \bigoplus \mathbb{C}_{\lambda_i}$, then $c_d^T(T_pX) = \lambda_1 \dots \lambda_d$.

Main Thm (Atiyah-Bott localization). X dim d non-sing variety over $\mathbb{C}$. $T \circlearrowleft X$ with X^T finite. Then $\exists\, m \le \# X^T$ such that m classes of $H_T^*(X)$ restrict to a basis of $H^*(X)$. Let $i : X^T \hookrightarrow X$. There exists $S \subseteq \Lambda_T$ multiplicative set such that

$$i^* : H_T^* X \longrightarrow H_T^* X^T$$

is an isomorphism after localizing at S.

Pf. Let $n = \# X^T$. $H_T^*(X^T) = \Lambda_T^{\oplus n}$.
consider

$$\Lambda_T^{\oplus n} = H_T^*(X^T) \xrightarrow{i_*} H_T^*(X) \xrightarrow{i^*} H_T^*(X^T) = \Lambda_T^{\oplus n}.$$

We want to show this is an isomorphism after localizat-
-ion.
By lemma, for $a_i \in H_T^*(p_i)$, $i^* i_*(a_i) = c_d^T(T_{p_i}X) \cdot a_i$.

Then,

$$i^* i_*(a_1 \dots a_n) = \begin{pmatrix} c_d^T(T_{p_1}X) & & \\ & \ddots & \\ & & c_d^T(T_{p_n}X) \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

Let $c = \det i^* i_* = \prod c_d^T(T_{p_i}X)$. If c is invertible, then $i^* i_*$ is surjective

For S any mult. set containing c, one has

$$S^{-1}\Lambda_T^{\oplus n} \longrightarrow S^{-1}\Lambda_T^{\oplus n} \text{ is surjective.}$$

Hence, $S^{-1}i^*i_*$ is surj, so $S^{-1}i^*$ is surjective.

For injectivity, we use the assumption

$$\begin{tikzcd} EG \times^G X \arrow[d] & X \arrow[l, hook'] \\ BG \end{tikzcd}$$

LH is true for fiber bundles in general. $H^*(E) \simeq H^*(B) \otimes H^*(X)$, with $X \hookrightarrow E$, fibers X.
$$\begin{tikzcd} X \arrow[r, hook] & E \arrow[d] \\ & B \end{tikzcd}$$

Leray-Hirsch: $H^*(EG \times^G X) \cong H^*(BG) \otimes H^*(X)$ as $\mathbb{Z}$-modules. $H^*(X)$ is rank m free over Λ, so rk $H_T^*(X) = m$. But $m \leq n$, and localization is surjective so $m \geq n$. Hence $m = n$ and this is an isomorphism.

DBZ: $T \circlearrowright X$. Stratify X into pieces by rank of stab.
X^T: highest rank stab. Free action
Cohomology is built from strata. Trivial action "eq cohomology" $H^*(X^T) = H^*(X^T) \otimes \Lambda$
In the free piece, Λ acts trivially. In between, Λ acts via quotients. Localizing carefully can realize different strata.

Equivariant coh of compact conn. Lie group is the Weyl group invariants of the torus eq. cohomology.

$\mathcal{H}$: Hecke algebra = distributions on G supported in K.

$$\mathcal{H} \simeq \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{k}} A_K \quad , A_K : \text{finite measures on } K.$$

$\mathcal{U}\mathfrak{g}$: distribution supported at e $\quad \mathrm{Dist}_e(G)$, $C^\infty(G) \to \mathbb{C}$
$\mathcal{U}\mathfrak{k}$: $C^\infty(K) \to \mathbb{C}$, supported at e

$$\begin{aligned} \mathcal{U}\mathfrak{g} \times A_K &\longrightarrow C^\infty(G)^* \\ \varphi, \mu &\longmapsto f \mapsto \varphi(f) \cdot \int_K f \, d\mu \end{aligned}$$

Img supported on K, so $\mathcal{U}\mathfrak{g} \times A_K \to \mathcal{H}$.

$\mathcal{U}_\mathfrak{k}$ module structure on $\mathcal{U}\mathfrak{g}$ is convolution on $\mathrm{Dist}_e(G) \supseteq \mathrm{Dist}_e(K)$ (?) Here $\psi \circ \varphi(f) = \psi(f)\varphi(f)$
$\mathcal{U}_\mathfrak{k}$ module structure on A_K: $\psi\mu(f) = \psi(f) \cdot \mu(f)$.

so $\mathcal{U}\mathfrak{g} \times A_K \to \mathcal{H}$ is $\mathcal{U}_\mathfrak{k}$-bilinear. So

$$\mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{k}} A_K \longrightarrow \mathcal{H}.$$

Inverse map? $\xi \in C^\infty(G)^*$ support in K. Want
$\xi(f) = \varphi(f)\,\mu(f)$, $\quad \varphi$ action on Taylor series
$\quad\quad \mu$ action on K
$\varphi(f) =$ linear comb. of $f(e), f'(e), \dots, f^{(n)}(e), \dots$
$f : G \to \mathbb{C}$ such that $f', f'', \dots$ are 0 then
$\varphi(f) = f(e)$.

Equivariant Cohomology II – Isaac

Integration, Pushforward formula. $T = (\mathbb{C}^\times)^n$

Thm. $f: X \to Y$ is a T-equivariant map of complex manifolds. $P \subseteq X^T$ connected component, $\exists ! \, Q \subseteq Y^T$ conn-component $f(P) \subseteq Q$. For $u \in H_T^*(X)$, $u|_p$ is pullback along $i_p : P \to X$.
For any $u \in H_T^*(X)$,

$$f_*(u)|_Q = c_{top}^T(N_{Q/Y}) \cdot \sum_{\substack{P \subseteq X^T \\ f(P) \subseteq Q}} (f|_P)_* \left(\frac{u|_P}{c_{top}^T(N_{P/X})} \right)$$

$$\in C^{-1} H_T^* Q$$

Cor. For $\rho: X \to T$, X^T finite,

$$\rho_*(u) = \sum_{p \in X^T} \frac{u|_p}{c_{top}^T(T_p X)}.$$

For

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

cartesian fiber map, f, f' proper.

$$\begin{tikzcd} H^* X' \arrow[d, "f'_*"'] & H^*(X) \arrow[l, "g'^*"'] \arrow[d, "f_*"] \\ H^* Y' \arrow[r] & H^*(Y) \end{tikzcd}$$

commutes by general property of Gysin pushfwd

For
$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "p"'] & EG \times^G X \arrow[d] \\ pt \arrow[r] & BG \end{tikzcd}$$
one gets the diagram

$$\begin{tikzcd} H^* X & H_T^* X \arrow[l] \arrow[d] \\ \mathbb{R} & \Lambda = H^*(BT) \arrow[l] \end{tikzcd}$$
with $p_*: H^*X \to \mathbb{R}$.

p_*: only non-trivial on top cohomology class, corresponds to integration of top forms.

So this $p_*(u) = \sum$ ____ is called integration formula.

<u>Aside</u> (Enumerative geometry)

$$\# \text{objects satisfying fns} = \int_{[\text{moduli space}]} [\text{condition on objects}].$$

$$\# \text{lines via 2 points in } \mathbb{P}^2 = \int_{\mathbb{P}^{2*}} c_1(L_1) \cap c_1(L_2) = 1.$$

<u>Koszul Duality in Equivariant Cohomology.</u>

Let X be a real smooth mfd, G a cpt, conn Lie group (usually torus). Let $S = \mathrm{Sym}\, \mathfrak{g}^*$, $\Lambda = (\Lambda_* \mathfrak{g})^G$.

We look for some relations between $H_G^* X$, $H^* X$.

Eg. $S^1 \circlearrowright X$, $H^*_{S_1}(pt) = \mathbb{Z}[a] = \mathrm{Sym}\, \mathfrak{g}^*$, $\deg a = 2$ is a symmetric variable.

$H_*(S^1) = \mathbb{Z}[b] = \Lambda_\circ \mathbb{R}$, $\deg b = -1$, b ext. variable.

H^*X module over $H_*G \longleftrightarrow \Lambda$
$H^*_G X$ module over $H^*_G pt \longleftrightarrow S$.

Koszul Duality " $D^b(\Lambda\text{-mod}) \longleftrightarrow D^b(S\text{-mod})$",
$H^*X \longleftrightarrow H^*_G X$

<u>Λ-action on H^*X</u>

$(\Lambda_\circ \mathfrak{g})^G \simeq H_*G$ as algebras.

For $\mu: G \times X \to X$, one gets a sweep action of Λ on $C_*(X)$ (chains on X):

$\sigma \in C_i X$, $\xi \in C_j X$ then $\sigma \times \xi \in C_{i+j}(G \times X)$.

$$\sigma \cdot \xi = \begin{cases} \mu_*(\sigma \times \xi) & \text{if } \dim = i+j \\ 0 & \text{otherwise.} \end{cases}$$

One can extend this action of $H_*(G)$ on $C_*(X)$.
$\sigma_1, \ldots, \sigma_d \in H_*(G)$ represented by $\omega_1 \cdots \omega_d \in C_*(G)$
$u = \sigma_{i_1} \cdots \sigma_{i_k}$, $u \cdot \xi$ acts by $\mu_*(\omega_{i_1} \cdots \omega_{i_k} \cdot \xi)$, action map $\mu: G \times G \times \cdots \times G \times X \to X$.

<u>DBZ</u>: $G \times X \to X$ gives $H^*(X) \longrightarrow H^*(G) \otimes H^*(X)$, and one obtains an action of $\Lambda_\circ$ on $H^*(X)$.

Dualize above action to $\Lambda \circlearrowleft C^*(X)$, this descends to cohomology.

Eg. (Failure of KD).

$S^1 \circlearrowleft S^3$ trivially.

$$H^*_{S^1}(S^3) \cong H^*(S^3) \otimes H^*(BS^1).$$

Here $\Lambda = \Lambda\mathbb{R}$ acts by 0 outside degree 0.

$S^1 \circlearrowleft S^3$ Hopf action $H^*_{S^1}(S^3) \simeq H^*(S^2)$, and Λ acts by 0 again.

Note $[S^1] \in H_*(S^1)$ fundamental class.
$[S^1]\cdot[pt] = [S^1 \cdot pt] = [S^1] = [\partial\beta] = [S^2]$
$[S^1]\cdot[S^2] = [S^3]$. What is this ?

Isaac III .

Setup. G connected compact lie group. G acts on X, a compact connected real manifold.

Def. For $\mathfrak{g} = \text{lie } G$, there is a differential on $\Lambda\mathfrak{g}$:

$$\partial_L(x_0 \wedge \cdots \wedge x_n) = \sum (-1)^{i+j} [x_i, x_j] \wedge \cdots \wedge x_n$$

Milner Moore theorem   Hopf algebra comul

$P = \{ x \in \Lambda\mathfrak{g} \mid \Delta_*(x) = x \otimes 1 + 1 \otimes x \}$, $\Delta : \Lambda\mathfrak{g} \to \Lambda\mathfrak{g} \otimes \Lambda\mathfrak{g}$.

If $x \in P$, odd degree, $\partial_L x = 0$. Then $P \hookrightarrow (\Lambda\mathfrak{g})^G$.

DBZ $U\mathfrak{g}$ has a comultiplication which is cocommutative.

$U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g}$, $x \mapsto 1 \otimes x + x \otimes 1$ for $x \in \mathfrak{g}$, extend to $U\mathfrak{g}$ with compatibility condition.

Milner Moore: H cocommutative Hopf algebra ("homology of top gp G", Δ comes from cup product). A primitive element of H is $P = \{ a \in H \mid \Delta(a) = a \otimes 1 + 1 \otimes a \}$. Then P is a lie algebra and $H = \mathcal{U}(P)$.

For G simply connected $H_*(G) = \Lambda\mathfrak{g}$ and the defn falls into M.M. thm.

$H_*(\mathfrak{g})$ is calculated from the complex Λ.

$$\Lambda_\bullet = \Lambda P \cong (\Lambda\mathfrak{g})^G \cong H_*(\mathfrak{g}, \mathbb{R}) \cong H_*(G, \mathbb{R}). \quad \text{(odd deg)}$$

$S_\bullet = \mathrm{Sym}(\tilde{P}^*)$. $\tilde{P}$ = grading shifted by 1 on P.

$= H^*_G(pt) = H^*(BG)$.

Koszul Duality.

$$\begin{tikzcd} D_+(S\text{-mod}) \arrow[r, bend right, "t = k \otimes_S -"'] & D_+(\Lambda\text{-mod}) \arrow[l, bend right, "h = \mathrm{Hom}_\Lambda(k, -)"'] \end{tikzcd}$$

"men of god"   "men of war"

Recall $S^1 \circlearrowleft S^3$ in two ways. We elaborate later.

<u>Sweep action.</u> $\mu: G \times X \longrightarrow X$ action map. Then, $\mu_*: C_\bullet(G \times X) \longrightarrow C_\bullet(X)$. We will give action of $C_\bullet(G)$ on $C_\bullet(X)$:

$$f: D^i \longrightarrow G, \quad j: D^j \longrightarrow G \qquad i, j\text{-chain in } G, X.$$

$$f \cdot g = \begin{cases} \mu_*(f \times g) & \text{if } \dim(\mu(f(D^i), g(D^j))) = i+j \\ 0 & \text{otherwise} \end{cases}$$

Action of $\Lambda_\bullet$ on $C_\bullet(X, \mathbb{R})$.

Let $\{\bar{x}_i\} \subseteq P$ be a basis, $x_i \in C_\bullet(G)$ For an

$$u = \bar{x}_{i_1} \circ \cdots \circ \bar{x}_{i_n}, \quad f \in C_\bullet(X)$$

$$u \cdot f = x_{i_1} \cdot \cdots \cdot [x_{i_n} \cdot f].$$

Dualizing gives an action $C_\bullet(G) \circlearrowleft C^\bullet(X)$.

<u>Example</u> ① $S^1 \circlearrowleft S^3$ trivial action.

$$H^*_{S_1}(S^3) \cong H^*(BS^1 \otimes S^3) = S \otimes H^*(S^3).$$

② $S^1 \circlearrowleft S^3$ Hopf action.

not isom. as S-mod.

$$H^*_{S^1}(S^3) \cong H^*(S^3/S^1) = H^*(S^2) \ncong S \otimes_{\mathbb{R}} H^*(S^3)$$

The Λ-action on $H^*(S^3)$ is the same in both cases.

one can check this on cochains.

Equivariant Chains.

Traditionally, $H^\bullet_G(X) = H^\bullet(EG \times^G X)$.
Using chains, take $\ell = \dim G$.

$$C_i^G(X) = \left\langle \begin{array}{c} C \to X \\ G\text{-eq.} \end{array} \middle| \begin{array}{l} \dim C = i + \ell \\ G \circlearrowright C \text{ freely} \\ C \text{ compact; oriented bd} \end{array} \right\rangle$$

$$H^i_G(X) = H^i\left(C_i^G(X), \partial\right).$$

Koszul Duality.

$$\begin{tikzcd} & D^f_+(S) \arrow[d, shift left, "t"] \\ D^b_{G,c}(pt) \arrow[ur] \arrow[dr] & D^f_+(\Lambda) \arrow[u, shift left, "h"] \end{tikzcd}$$

t, h exchange the complexes $C_\bullet(X)$ and $C_\bullet^G(X)$.
h, t form an equivalence of categories.
f here is for finitely generated

One can define $D^b_{G,c}(pt)$, c: constructible.

DBZ comments. If $G \circlearrowright X$, we can ask $H^*(X)$, $H^*_G(X)$.
In the level of spaces, $G \circlearrowright X \rightsquigarrow$ $\begin{array}{c} EG \overset{G}{\times} X \\ \downarrow \\ BG \end{array}$ X bundle $/BG$
Conversely,
$\begin{array}{c} Z \\ \downarrow \\ BG \end{array}$ bundle $\rightsquigarrow Z \times_{BG} EG$

KD exchanges the two construction.

$$H^*(X \times^G EG) \otimes_{H^*(BG)} k = H^*(X)$$

in the derived category

$$X \times^G EG = Z$$

$$H^*(Z) = \mathrm{Hom}_G(k, H^*(X)).$$

On cochains,

$$C^\circ(Z) = R\mathrm{Hom}_G(k, C^\circ(X))$$
$$C^\circ(X) = C^\circ(Z) \otimes^L_S k$$

More generally $\mathcal{F}$ a G-eq sheaf on X.

$$C_\circ(G) \circlearrowleft C^\circ(X, \mathcal{F})$$
$$C^\circ(BG) \circlearrowleft C^\circ_G(X, \mathcal{F}).$$

Koszul Rings Pt 1 (John Michael)

k base field, char $k \neq 2$. V/k fdvsp. V^* dual space. $\dim V = n$.

$A = \bigoplus_{i \geq 0} A_i$ be positively graded ring. We say A is Koszul if

- A_0 is semisimple.
- A_0 considered as a graded A-mod admits a graded projective resolution

$$\to P^2 \to P^1 \to P^0 \to A \to 0$$

with $P^i = A \cdot P^i_i$.

Eg: SV^*, ΛV are Koszul: $SV^* = \bigoplus \mathrm{Sym}^i V^*$

$$\to SV^* \otimes \Lambda^2 V \to SV^* \otimes V \to SV^* \to k$$
$$\to \Lambda V \otimes S^2 V^* \to \Lambda V \otimes V^* \to \Lambda V \to k$$

Notation: A-Mod : modules, Ext, Hom
A-mod : graded mod, ext, hom.

Alternate def. A graded mod M over graded ring is called pure of weight m if $M = M_{-m}$.
Let A be a gr ring with A_0 semisimple. Any simple gr mod over A is pure, any pure grmod is semi-simple.

Lemma. Let A be a positively graded ring, A_0 semi-simple. Let $M, N \in A$-mod be pure of wt m, n. Then $\mathrm{ext}^i_A(M,N) = 0$ for $i > m-n$.

Prop. A_0 is semisimple. TFAE

(1) A is Koszul

(2) For any two pure A-mod M, N of wt m, n, then $\text{ext}^i_A(M,N) = 0$ unless $i = m-n$.

(3) $\text{ext}^i_A(A_0, A_0\langle n\rangle) = 0$ unless $i = n$.

Pf. (2) $\Rightarrow$ (3), (1) $\Rightarrow$ (2) is easy. (3) $\Rightarrow$ (1) ?

Fact. Opposed ring A^{opp} of a Koszul ring A is also Koszul

Quadratic Rings

A quadrating ring is a positively graded ring $A = \bigoplus_{i \geq 0} A_i$ such that A_0 is semisimple and A is generated over A_0 by A_1 with relations of degree 2.

$$T_{A_0} A_1 = A_0 \oplus A_1 \oplus A_1 \otimes_{A_0} A_1 \oplus A_1 \otimes_{A_0} A_1 \otimes_{A_0} A_1 \oplus \cdots$$

$$T_{A_0} A_1 \longrightarrow A \quad , \quad A \simeq T_{A_0} A_1/(R) \text{ for } R \subseteq A_1 \otimes_{A_0} A_1.$$

Let k be semisimple ring, $A_0 = k$. Let $\otimes = \otimes_k$.
(

Prop. TFAE.

(1) $\text{ext}^1_A(k, k\langle n\rangle) \neq 0$ only for $n = 1$

(2) A is generated by A_1 over k.

Pf. Let $I = \bigoplus_{i \geq 1} A_i$. Then $0 \to I \to A \to k \to 0$ is exact.

suppose (2). $I = A \cdot A_1$. $\hom(I, k\langle n\rangle) = 0$ for $n \neq 1$.
I is projective.

Thm. Suppose A is gen by A_1/k. If $\mathrm{ext}_A^2(k, k\langle n\rangle) \neq 0 \Rightarrow n = 2$ then A is quadratic.

Cor. A Koszul ring is quadratic.

Koszul complex (k fixed semisimple ring). V is a k-bimodule. $R \subseteq V \otimes V$ sub bimodule. $A = T_k V/(R)$ corresponding quadratic ring. $A_1 = V$

Koszul complex of A:

$$\cdots \to K^2 \xrightarrow{d} K^1 \xrightarrow{d} K^0 \to 0, \quad K^i \in A\text{-mod}$$

$K_0^0 = k$, $K_1^1 = V$, $K_2^2 = R$, $K_3^3 = V \otimes R \cap R \otimes V$,
$K_4^4 = V \otimes V \otimes R \cap V \otimes R \otimes V \cap R \otimes V \otimes V$, ...

$K^i = A \otimes K_i^i$, $\quad d: a \otimes v_1 \otimes \cdots \otimes v_n \mapsto a v_1 \otimes \cdots \otimes v_n$.

Thm. $A = T_k V/(R)$ quadratic. A is Koszul iff its Koszul complex is a resolution of k.

Koszul Rings JMC

"Koszul rings are closest we get to semisimplicity for $\mathbb{Z}$-graded rings."

Quadratic dual rings, Koszul complex

Graded ring over k (semisimple ring) $A = \bigoplus A_i$, $A_0 = k$.
A is left finite iff all A_i are finitely generated as left A_0-modules. Right finiteness.

For A left finite quadratic over k, $A = T_k V/(R)$,
T_k : tensor algebra.

The left dual $A^!$ is right finite quadratic ring $/k$.

$$A^! = T_k V^* /(R^\perp), \quad R \subseteq V \otimes V$$

$R^\perp$: $f \in V^* \otimes V^*$ $f(R) = 0$.
Right dual ${}^!A$ is

$${}^!A = T_k {}^*V /(R^\perp)$$

Then

$$({}^!A)^! = A \text{ if } A \text{ right finite}$$
$${}^!(A^!) = A \text{ if } A \text{ left finite}$$

For $A = T_k V/(R)$ left finite quadratic over k, the Koszul complex is

$$\cdots \to A \otimes {}^*(A_2^!) \to A \otimes {}^*A_1^! \to A$$

$d : A \otimes {}^*A_i^! \to A \otimes {}^*A_{i-1}^!$.

$$\mathrm{Hom}_{-k}(A_i^!, A) \longrightarrow \mathrm{Hom}_{-k}(A_{i-1}^!, A).$$

(If $R=0$, $A=T_k V$, $A^! = k\oplus V^* = \mathrm{Ext}^*_A(k,k)$.)

$\mathrm{Hom}_{k-}(V,V) = V^* \otimes V$, $\mathrm{id}_V = \sum {}^\vee v_\alpha \otimes v_\alpha$

$df(a) := \sum f(a\, {}^\vee v_\alpha) v_\alpha$. Coordinate free description is

$A\otimes V \to A$, $A_i^! \otimes k \to A_i^! \otimes V^* \otimes V \to A_{i-1}^! \otimes V$

$d: \mathrm{Hom}_{-k}(A^!_{i+1}, A) \longrightarrow \mathrm{Hom}_{-k}(A^!_{i+1}\otimes V, A) \longrightarrow \mathrm{Hom}(A^!_i, V)$

Recall $K^i = A\otimes {}^*A^!_i$. $K^i_i = {}^*A^!_i$ (?)

<u>Prop</u>. Let A be a left finite Koszul ring. Then $A^!$ is Koszul.

Recall from Sai: SV^*, ΛV duality. Consider double complex $SV^*\otimes \Lambda V$. $SV^* = A$, $A^! = \Lambda V$.

For $V\in$ mod-k, ${}^\circledast V \in k$-mod is defined as

$${}^\circledast V_i = {}^* V_{-i} = \mathrm{Hom}_{-k}(V_{-i}, k).$$

<u>Quadratic Dual and Cohomology.</u>

Let R be a ring, $M\in R$-Mod. Let $P^\bullet, Q^\bullet$ be two complex--es of R-Mod. Define $\mathcal{H}om^\bullet_R(P^\bullet, Q^\bullet)$ internal Hom of complexes

$$\mathcal{H}om^i_R(P^\bullet, Q^\bullet) = \bigoplus_n \mathrm{Hom}(P^n, Q^{n-i})$$

Differential maps d of degree -1 on P, Q gives diff on

$\mathrm{Hom}^\bullet_R(P^\bullet, Q^\bullet)$. In particular, consider $\mathrm{End}^\bullet_R(P^\bullet)$, and one has a DGA structure on

$$H^* \mathrm{End}^\bullet_R(P^\bullet) = \bigoplus_i H^i \mathrm{End}^\bullet_R(P^\bullet).$$

If $P^\bullet \twoheadrightarrow M$ is projective resolution of $M \in R\text{-Mod}$ then

$$H^\circ \mathrm{End}^\bullet_R(P^\bullet) = \mathrm{Ext}^\bullet_R(M, M).$$

<u>Thm</u>. Let A be a left finite Koszul ring over k, then $\mathrm{Ext}^\bullet_k(A, A) = (A^!)^{op}$ canonically.

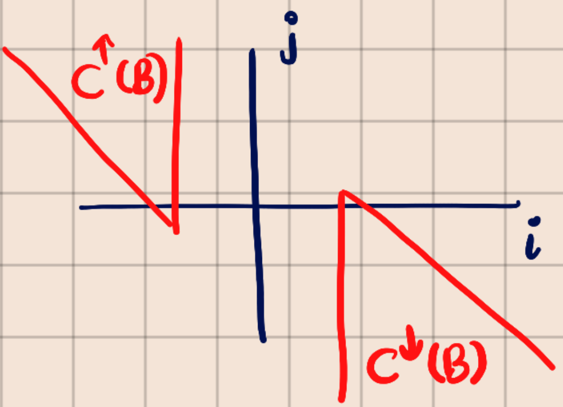
<u>Thm</u>. For A as before, $E(A) = (A^!)^{op}$ is also left finite, and $E^2(A) \simeq A$ canonically.

Koszul Duality (JMC)

Let $B = \bigoplus_{j \geq 0} B_j$ be a positively graded ring. $C(B)$ is the homotopy category of complexes in B-mod. For $M \in C(B)$,

$$M : \cdots \longrightarrow M^i \xrightarrow{\partial} M^{i+1} \xrightarrow{\partial} \cdots \; ; \; M^i = \bigoplus_j M^i_j .$$

$C^\uparrow(B)$ (and $C^\downarrow(B)$) is a full subcategory of $C(B)$.
$\mathrm{Ob}(C^\uparrow(B)) =$ M so that $M^i_j = 0$ if $i \gg 0$ or $i+j \ll 0$.
$\mathrm{Ob}(C^\downarrow(B)) =$ M so that $M^i_j = 0$ if $i \ll 0$ or $i+j \gg 0$.



Define $D^\uparrow(B)$, $D^\downarrow(B)$ as the localization of $C^\uparrow(B)$, $C^\downarrow(B)$ at quasi-isomorphisms.

<u>Thm</u>. Let A be a left finite Koszul ring. There is an equivalence of triangulated categories

$$D^\downarrow(A) \simeq D^\uparrow(A^!).$$

<u>Pf Sketch</u>. Construct functor $DF : D^\downarrow(A) \longrightarrow D^\uparrow(A^!)$. For $M \in C^\downarrow(A)$, construct $FM = A^! \otimes M = \bigoplus_{i,\ell} A^!_\ell \otimes M^i$

$$= \bigoplus \mathrm{Hom}_A(A \otimes {}^*A^!_\ell, M^i).$$

FM has anti-commuting differentials coming from the Koszul complex and M. For basis v_α of $V = A_1$,

$m \in M^i_j, a \in A^!_\ell$: $d'(a \otimes m) = (-1)^{i+j} \sum a \check{v}_\alpha \otimes v_\alpha m$

$$d''(a \otimes m) = a \otimes \partial(m).$$

$d' + d''$ is the total differential of FM (double complex grade ℓ, i), $d' + d''$ on $\mathrm{Tot}(FM) \in C(A^!)$.

$$\mathrm{Tot}\, FM^p_q = \bigoplus_{\substack{p = i+j \\ q = \ell - j}} A^i_\ell \otimes M^i_j .$$

If $M \in C^{\downarrow}(A) \Rightarrow FM \in C^{\uparrow}(A^!)$. $F(\text{acyclic})$ is acyclic, so F induces $DF : D^{\downarrow}(A) \to D^{\uparrow}(A^!)$.

Construct functor $DG : D^{\uparrow}(A^!) \to D^{\downarrow}(A)$.

For $N \in C^{\uparrow}(A^!)$, GN is a bigraded k-module

$$(GN)_{\ell, i} = \mathrm{Hom}_k (A_{-\ell}, N^i).$$

For $N \in C^{\uparrow}(A^!)$, $GN \in C^{\downarrow}(A)$. $G(\text{acyclic})$ is acyclic.

$$(GN)^p_q = \bigoplus_{\substack{p = i+j \\ q = \ell - j}} \mathrm{Hom}_k (A_{-\ell}, N^i_j).$$

So there is map $DG : D^{\uparrow}(A^!) \to D^{\downarrow}(A)$.

Next we show F, G form an adjoint pair. Then construct an equivalence $DF \circ DG \simeq id$, and an equivalence $id \simeq DG \circ DF$. DF, DG are the duality functors

$$K: D^{\downarrow}(A) \longrightarrow D^{\uparrow}(A^!).$$

<u>Thm</u>. Let A be a left finite Koszul ring over k.
Then

1. $K: D^{\downarrow}(A) \longrightarrow D^{\uparrow}(A^!)$, and canonical isomorphism $K(M[1]) \simeq (KM)[1]$ is an equivalence of triangulated categories. (here $[1]$ is cohomological shift by 1)

2. $K(M\langle n\rangle) \simeq (KM)[-n]\langle -n\rangle$ canonical isomorphism.
"The standard t-structures do not correspond."

3. For any $p \in k$, $K(A_0 p) = A^! p$, $K(A^{\circledast} p) = A_0^! p$
where $A^{\circledast} = \bigoplus_{\ell} A_{\ell}^{\circledast} \in A\text{-Mod-}k$

"$A^! p$ is a proj. cover of $A_0 p$ as a module over $A^!$, contragradient dual (?). Simple modules $\longrightarrow$ projective
Simple modules $\longrightarrow$ injective.

DBZ

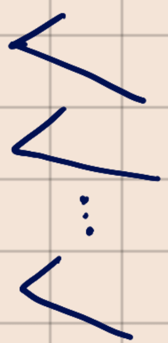
A:

$A = \bigoplus_i \mathbb{C}$

simple mod. S_i

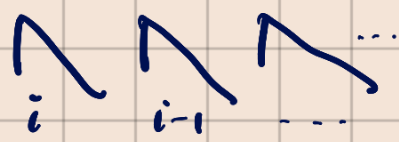
$A = \bigoplus P_i$

$A^!$:



simples $S_i \xrightarrow{K} P_i^!$,
simples $S_i^! \xrightarrow{K} P_i^{*}$.

$A^{\circledast}$



<u>Thm</u>. (SV^*, ΛV setting). Let A be a Koszul ring$/k$, and A be a fg left, right k-mod. So $A_i = 0$ for $i \gg 0$, $A^!$ is

left noetherian. K induces an eq. of triangulated cat.

$$K: \underset{\substack{| \\ \text{fg, bdd graded module}}}{D^b}(A\text{-mod}) \longrightarrow D^b(A^!\text{-mod})$$

Pf. $D_e^{\downarrow}(A) \subseteq D^{\downarrow}(A)$, $D_e^{\downarrow}$: full subcat, objects with finite nonvanishing cohomology groups, which are also fg A-modules. One can show $D_e^{\downarrow}(A) \simeq D^b(A\text{-mod})$ is an equivalence.
$A^! = KA_0$. $A^!$ has finite homological dimension.

Category $\mathcal{O}$ - Ryan

Def. Category $\mathcal{O}$: full subcat of $U\mathfrak{g}$-modules, objects M such that

① M is fg $U\mathfrak{g}$-mod

② M is a weight module $\mathfrak{h}$ s.s. $\bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda = M$

③ M is locally $\mathfrak{n}$-finite.

It is closed under ker, $\oplus$, quotients.

Cor. Every $M \in \mathcal{O}$ has max vector, $v^+ \in M$ st $\mathfrak{n} v^+ = 0$ of weight $\lambda \in \mathfrak{h}^*$. M gen by one vector then M is a highest wt module.

Ex. Standard objects ("highest wt") are Verma modules $\lambda \in \mathfrak{h}^*$,

$$M(\lambda) = U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_\lambda ,$$

$\mathbb{C}_\lambda$ rep of $\mathfrak{b}$ inflated from $\mathfrak{h}$.

we can show $M(\lambda) = U\mathfrak{g}/I$, $I = (h-\lambda, U\mathfrak{g}\,\mathfrak{n})$

"$\mathfrak{h}$ acts by λ, $\mathfrak{n}$ acts trivially".

Verma modules : universal wrt being highest wt module with highest wt λ.

Ex. Every Verma module $M(\lambda)$ has max proper submodule $N(\lambda)$, and unique irreducible quotient $L(\lambda) = M(\lambda)/N(\lambda)$.

Fact. Cat $\mathcal{O}$ is finite length, and multiplicities are well defined. $[M : L(\lambda)]$.

<u>Question</u> What is $[M(\lambda) : L(\mu)]$?

Let $Z_{\mathfrak{g}}$ be center of $\mathcal{U}\mathfrak{g}$. $Z_{\mathfrak{g}}$ act on $M(\lambda)$ by scalars, so we define central character as the alg. hom

$$\chi_\lambda : Z_{\mathfrak{g}} \longrightarrow \mathbb{C} \qquad zv = \chi_\lambda(z) v$$

For $[M(\lambda) : L(\mu)] \neq 0$, $\chi_\lambda = \chi_\mu$.

Eg ($SL_2\mathbb{R}$ Verma module: $M_0 = \underset{-8}{\circ}\ \underset{-6}{\circ}\ \underset{-4}{\circ}\ \underset{-2}{\circ}\ \underset{0}{\circ}$

$M_{-2} \subseteq M_0$ is max submod.

Here, $\chi_0 = \chi_{-2}$. (more generally, $M_\lambda \supseteq M_{-2-\lambda}$)

"-1 is the center of the universe"

1 is the half sum of positive roots. So the center of the universe in general is $-\rho$, where

$$\rho = \tfrac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$

<u>Dot Action</u>. $\lambda \in \mathfrak{h}^*$, $w \in W$. Then, define dot action

$$w \cdot \lambda = w(\lambda + \rho) - \rho.$$

Orbits are called linkage classes. λ, μ are in the same linkage class iff $\chi_\lambda = \chi_\mu$.

We may decompose $\mathcal{O}$ into $\bigoplus_{[\lambda]} \mathcal{O}_{\chi_\lambda}$, χ_λ: generalized central characters.

There are no Ext between $\mathcal{O}_{\chi_\lambda}$ for different $[\lambda]$.

<u>Thm</u> (Verma, BGG). The foll. are equivalent.

(a) $[M(\lambda) : L(\mu)] \neq 0$

(b) $M(\mu) \hookrightarrow M(\lambda)$

(c) μ strongly linked to λ, $\mu = s_\alpha \cdot \lambda < \lambda$, $<$ is the Bruhat order.

Eg. Projective covers. $P(\lambda)$ for $L(\lambda)$ "smallest projective object covering $P \twoheadrightarrow L$."
Every proj. object in $\mathcal{O}$ admits a std filtration, a filtration where successive quotients are Verma mod.

BGG reciprocity: $[P(\lambda) : M(\mu)] = [M(\mu) : L(\lambda)]$.

(What are these $P(\lambda)$ for $SL_2\mathbb{R}$?)

Def. Let $\mathfrak{p}$ be a parabolic subalgebra containing $\mathfrak{b}$, Levi $\mathfrak{l}$. $\mathfrak{p}$ corresponds to $I \subseteq \Delta$. The parabolic subcategory $\mathcal{O}^{\mathfrak{p}}$ is the full subcategory of $U\mathfrak{g}$-mod, objects satisfy

① M is finitely generated

② as $U\mathfrak{l}$-mod, M is ss over fd modules

③ M is locally $\mathfrak{u}$-finite, $\mathfrak{u}$ the nilradical of $\mathfrak{p}$.

$\mathcal{O}^{\mathfrak{p}} \subseteq \mathcal{O}$. Rmk. $(\mathfrak{g}, B)$-mod $\sim \mathcal{O}$ because ②

$(\mathfrak{g}, P)$-mod $\sim \mathcal{O}^{\mathfrak{p}}$

② $\Rightarrow$ old ②

②,③ $\Rightarrow$ old ③

Fact. W_I the Weyl group of I. Then, $M \in \mathcal{O}$ is in $\mathcal{O}^{\mathfrak{p}}$ iff the weights $\Pi(M)$ of M are stable under W_I.

Koszul Duality

Let $\lambda \in \mathfrak{h}^*$ be an integral but possibly singular weight. (singular wrt dot action). λ determines a block $\mathcal{O}_{\chi_\lambda}$. $I = \{\alpha \mid \alpha \cdot \lambda = \lambda\}$.

Let W^I be the set of min length right coset reps of W_I in W.
Let $L^\lambda = \bigoplus_{w \in W^I} L(w,\lambda)$. $P^\lambda = \bigoplus P(w,\lambda)$

consider $\mathcal{O}_0^{\mathfrak{p}}$, define $L^{\mathfrak{p}}$, $P^{\mathfrak{p}}$ param. by W/W_I.

<u>Thm</u>. (Koszul duality) There is an algebra isom.

$$\mathrm{End}_{\mathcal{O}} P^\lambda \simeq \mathrm{Ext}^\bullet_{\mathcal{O}^{\mathfrak{p}}}(L^{\mathfrak{p}}, L^{\mathfrak{p}}).$$

$$\mathrm{End}_{\mathcal{O}} P^{\mathfrak{p}} \simeq \mathrm{Ext}^\bullet_{\mathcal{O}}(L^\lambda, L^\lambda).$$

and the two algebras are Koszul, and they are Koszul dual.

(Eg. what happens for $\lambda = 0$?)