Some general results

Structure theory. M is a fg ~~torsion~~ Λ-mod.

1. Well-defined Λ-rank $\Rightarrow$ injection with torsion cokernel.
2. M_Γ finite iff M^Γ finite for M torsion.

Galois groups. F_∞/F be a $\mathbb{Z}_p$-extn, $F_n \subseteq F_\infty$ fixed by Γ_n.

M_∞/F Galois extn st. M_∞/F_∞, $\mathrm{Gal}(M_\infty/F_\infty)$ pro-p, abelian. $X := \mathrm{Gal}(M_\infty/F_\infty)$. Then X is Λ-mod and let M_n be max ab. extn of F_n inside M_∞. Then

$$\omega_n X = \mathrm{Gal}(M_\infty/M_n).$$

Cohomology. Γ procyclic, so cohomological dim $= 1$

$H^0(\Gamma, M) = M^\Gamma$, $H^1(\Gamma, M) = M_\Gamma$.

Then, for exact $0 \to A \to B \to C \to 0$: exact,

$$0 \to A^\Gamma \to B^\Gamma \to C^\Gamma \to A_\Gamma \to B_\Gamma \to C_\Gamma \to 0 : \text{ex.}$$

(alternatively, use snake lemma on

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow{\scriptstyle \sigma-1} & & \downarrow{\scriptstyle \sigma-1} & & \downarrow{\scriptstyle \sigma-1} & & \\ 0 & \to & A & \to & B & \to & C & \to & 0 \end{array}$$

.)

Cyclotomic units

Thm. M_n: max abelian p-ext of F_n unramified outside p. Then M_n/F_∞ is finite.

Pf. $U_n^1/E_n^1 \simeq Gal(M_n/L_n)$ and L_n^*/F_n is finite.
$\mathbb{Z}_p$-rank of $U_n^1 = [F_n:\mathbb{Q}]$, so we want $\mathbb{Z}_p$-rank $E_n^1 = [F_n:\mathbb{Q}]-1$.
– Leopoldt conjecture.

Def. $N_\infty(U_m^1)$, $N_\infty(K_m^\times)$.

Thm.
$$\alpha_{n,U}: (U_\infty^1)_{\Gamma_n} \longrightarrow N_\infty(U_n^1)$$
$$\alpha_{n,C}: (C_\infty^1)_{\Gamma_n} \longrightarrow C_n^1$$
Then ① $(U_\infty^1/C_\infty^1)^{\Gamma_n} = 0$ ② $(U_\infty^1/C_\infty^1)_{\Gamma_n} = N_\infty(U_n^1)/C_n^1$
and $(C_\infty^1)_{\Gamma_n} \to (U_\infty^1)_{\Gamma_n}$ induces an isomorphism
$\ker(\alpha_{n,U}) \simeq \ker(\alpha_{n,C}) \simeq \mathbb{Z}_p$, trivial Galois action.

Pf. (Very long, skip and do later).

Global Units

Thm. $\mathcal{L}^1(E_\infty^1)$ is principal in $\Lambda(G)$.

Prop (Iwasawa) X_∞ has no non-zero finite $\Lambda(\Gamma_0)$-~~sub~~module where $\Gamma_0 = Gal(F_\infty/F_0)$.
— use without proof.

Pf. $E_\infty^1 \xhookrightarrow{\mathcal{L}^1} \Lambda(G)$ has torsion cokernel so $rk_\Lambda(E_\infty^1)$ is well def, equal to 1. So $\exists\ E_\infty^1 \hookrightarrow \Lambda(G) \twoheadrightarrow Q$ for some finite module Q.
we want $Q=0 \iff Q_{\Gamma_0} = 0 \iff Q^{\Gamma_0} = 0$. (Nakayama lemma)

$0 \to E'_\infty \to \Lambda(G) \to Q \to 0$ gives

$0 \to Q^{\Gamma_0} \to (E'_\infty)_{\Gamma_0} \to \Lambda(G)_{\Gamma_0} \to Q_{\Gamma_0} \to 0$, $\Lambda(G)_{\Gamma_0}$ is free $\mathbb{Z}_p$-mod so Q^{Γ_0} is the $\mathbb{Z}_p$-torsion module of $(E'_\infty)_{\Gamma_0}$.

Now we show $(E'_\infty)_{\Gamma_0}$ has no torsion.

$0 \to E'_\infty \to U'_\infty \to \mathrm{Gal}(M_\infty/L_\infty) \to 0$: exact.

Then $0 \to \mathrm{Gal}(M_\infty/L_\infty)^{\Gamma_0} \to (E'_\infty)_{\Gamma_0} \to (U'_\infty)_{\Gamma_0} \to 0$: exact.

<u>$\mathrm{Gal}(M_\infty/L_\infty)^{\Gamma_0} = 0$</u> : $\mathrm{Gal}(M_\infty/L_\infty)^{\Gamma_0} \subseteq X_\infty^{\Gamma_0}$,
$(X_\infty)_{\Gamma_0} = \mathrm{Gal}(M_0/F_\infty)$ finite by Leopoldt conjecture, so $X_\infty^{\Gamma_0}$ finite, hence 0, so $\mathrm{Gal}(M_\infty/L_\infty)^{\Gamma_0} = 0$.

we can now get

$$\begin{tikzcd}
(C'_\infty)_{\Gamma_n} \arrow[r] \arrow[d, "\alpha_{n,C}"] & (E'_\infty)_{\Gamma_n} \arrow[r] \arrow[d, "\alpha_{n,E}"] & (U'_\infty)_{\Gamma_n} \arrow[d, "\alpha_{n,U}"] \\
C'_n \arrow[r] & N_\infty(E'_n) \arrow[r] & N_\infty(U'_n)
\end{tikzcd}$$

$(E'_\infty)_{\Gamma_n} \hookrightarrow (U'_\infty)_{\Gamma_n}$ because $(U'_\infty/E'_\infty)^{\Gamma_n} = 0$

$(C'_\infty)_{\Gamma_n} \hookrightarrow (U'_\infty)_{\Gamma_n}$ because $(U'_\infty/C'_\infty)^{\Gamma_n} = 0$ (old Thm).

So top maps are injective, induce map

$$\ker(\alpha_{n,C}) \simeq \ker(\alpha_{n,E}) \simeq \ker(\alpha_{n,U}) \simeq \mathbb{Z}_p.$$

<u>Prop</u>. $(E'_\infty/C'_\infty)_{\Gamma_n} \simeq N_\infty(E'_n)/C'_n$

<u>Pf</u>. $(E'_\infty/C'_\infty)_{\Gamma_n} \simeq (E'_\infty)_{\Gamma_n}/(C'_\infty)_{\Gamma_n} \simeq N_\infty(E'_n)/C'_n$.

<u>Thm</u> $Y_\infty^{\Gamma_n} \simeq E_n^1 / N_\infty(E_n^1)$.

<u>Pf</u>.

$$\begin{tikzcd}
0 \arrow[r] & (E_\infty^1)_{\Gamma_n} \arrow[r] \arrow[d, "\alpha"] & (U_\infty^1)_{\Gamma_n} \arrow[r] \arrow[d, "\alpha"] & \mathrm{Gal}(M_\infty/L_\infty)_{\Gamma_n} \arrow[r] & 0 \\
0 \arrow[r] & N_\infty(E_n^1) \arrow[r] & N_\infty(U_n^1) \arrow[r] & N_\infty(U_n^1)/N_\infty(E_n^1) \arrow[r] & 0
\end{tikzcd}$$

So $\mathrm{Gal}(M_\infty/L_\infty)_{\Gamma_n} \simeq N_\infty(U_n^1)/N_\infty(E_n^1)$: $\circledast$

$0 \to \mathrm{Gal}(M_\infty/L_\infty) \to X_\infty \to Y_\infty \to 0$ exact.

So, $0 \to Y_\infty^{\Gamma_n} \to \mathrm{Gal}(M_\infty/L_\infty)_{\Gamma_n} \to (X_\infty)_{\Gamma_n} \to (Y_\infty)_{\Gamma_n} \to 0$

$(X_\infty)_{\Gamma_n} = \mathrm{Gal}(M_n/F_\infty)$, $(Y_\infty)_{\Gamma_n} = \cancel{\mathrm{Gal}(L_n/F_n)}\ \mathrm{Gal}(L_n F_\infty/F_\infty)$

$$0 \to Y_\infty^{\Gamma_n} \to \mathrm{Gal}(M_\infty/L_\infty)_{\Gamma_n} \to \mathrm{Gal}(M_n/L_n F_\infty) \to 0$$

But Class field theory: $\mathrm{Gal}(M_n/L_nF_\infty) \simeq N_\infty(U_n^1)/E_n^1$

$Y_\infty^{\Gamma_n} \simeq E_n^1/N_\infty(E_n^1)$ by $\circledast$

For n large, $Y_\infty^{\Gamma_n}$ is finite, the largest finite $\Lambda(\Gamma_0)$ submodule of Y.

$\mathcal{L}'(E_\infty^1) = \alpha\Lambda(G)$ So $\alpha\beta = \zeta_p\theta(e,1)$

$\mathcal{L}'(C_\infty^1) = \zeta_p\theta(e,1)\Lambda(G)$ for some β.

There is canonical map $\tau: E_\infty^1/C_\infty^1 \simeq \Lambda(G)/\beta\Lambda(G)$.

This induces isom. $N_\infty(E_n^1)/C_n^1 \simeq R_n/\beta R_n$

Pf (First Thm)

Let Φ_∞ / K_∞ maximal abelian p-ext. Φ_n: max abelian p-extn of K_n. One has

$$\mathrm{Gal}(\Phi_\infty/K_\infty)_{\Gamma_n} = \mathrm{Gal}(\Phi_n/K_\infty)$$

$\wedge$ denotes p-adic completion.

$Z_\infty = \varprojlim \widehat{K_n^\times}$, inverse lim wrt norm. From local CFT,

$$Z_\infty \simeq \mathrm{Gal}(\Phi_\infty/K_\infty).$$

$$\widehat{N_\infty(K_n^\times)} \simeq \mathrm{Gal}(\Phi_n/K_\infty).$$

So natural map $(Z_\infty) \longrightarrow N_\infty(K_n^\times)$ induces isom: $(Z_\infty)_{\Gamma_n} \simeq \widehat{N_\infty(K_n^\times)}$.

Global side: W_n: subgroup of $F_n^\times$ gen by $\pm(\zeta_n^{-1/2} - \zeta_n^{1/2})$ and Galois conjugates. $W_n \subseteq N_\infty(K_n^\times)$, and valuation induces

$$0 \to D_n \to W_n \to \mathbb{Z} \to 0.$$

D_n: cyclotomic units. Take $\otimes \mathbb{Z}_p$, and get

$$0 \to D_n^1 \otimes \mathbb{Z}_p \to \widehat{W}_n \to \mathbb{Z}_p \to 0.$$

($D_n \otimes \mathbb{Z}_p = D_n^1 \otimes \mathbb{Z}_p$ since $[D_n : D_n^1]$ is prime to p)

But $D_n^1 \otimes \mathbb{Z}_p$ is isom. to its closure C_n^1 inside U_n^1: the $\mathbb{Z}_p$-rank of E_n^1 = $\mathbb{Z}$-rank of D_n^1 ~ Leopoldt conjecture.

$$\begin{tikzcd}
0 \arrow[r] & C_n^1 \arrow[r] \arrow[d, hook] & \widehat{W}_n \arrow[r] \arrow[d] & \mathbb{Z}_p \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & N_\infty(U_n^1) \arrow[r] & \widehat{N_\infty(K_n^\times)} \arrow[r] & \mathbb{Z}_p \arrow[r] & 0
\end{tikzcd}$$

(p-adic completion of $N_\infty(U_n^1) \hookrightarrow N_\infty(K_n^\times) \twoheadrightarrow \mathbb{Z}$)

By snake lemma, $\widehat{W}_n \hookrightarrow \widehat{N_\infty(K_n^\times)}$, and $N_\infty(U_n^1)/C_n^1 \simeq \widehat{N_\infty(K_n^\times)}/\widehat{W}_n$. Similarly $\widehat{K_n^\times}/\widehat{W}_n \simeq U_n^1/C_n^1$, so $Z_\infty/W_\infty \simeq U_\infty^1/C_\infty^1$, taking $\varprojlim$.

Now, $\ker \; Z_\infty \to \widehat{N_\infty(K_n^\times)} \to \widehat{N_\infty(K_n^\times)}/\widehat{W}_n$ is $W_\infty(\omega_n Z_\infty)$; so $N_\infty(U_n^1)/C_n^1 \simeq \widehat{N_\infty(K_n^\times)}/\widehat{W}_n \simeq (Z_\infty/W_\infty)_{\Gamma_n}$.

$U_\infty^1/C_\infty^1 \simeq \Lambda(G)/I(G)\zeta_p$, is elementary torsion module

$(U_\infty^1/C_\infty^1)_{\Gamma_n} = N_\infty(U_n^1)/C_n^1$ is finite because $\mathbb{Z}_p$-rank of C_n^1 = $\mathbb{Z}_p$-rank of E_n^1 = $\mathbb{Z}_p$-rank of $U_n^1 - 1 = [F_n:\mathbb{Q}]-1$.

So $(U_\infty^1/C_\infty^1)^{\Gamma_n}$ is finite but elementary $\Lambda(G)$-mod has no finite submodule. So $(U_\infty^1/C_\infty^1)^{\Gamma_n} = 0$.

This then shows $\ker(\alpha_{n,U}) \simeq \ker(\alpha_{n,C})$.

$0 \to U_\infty^1 \to Z_\infty \to \mathbb{Z}_p \to 0$. Γ_n-cohomology gives

$\mathbb{Z}_p \to (U_\infty^1)_{\Gamma_n} \to (Z_\infty)_{\Gamma_n} \to \mathbb{Z}_p \to 0$ exact $\iff$

$\mathbb{Z}_p \to (U_\infty^1)_{\Gamma_n} \to N_\infty(U_n^1) \to 0$. compare $\mathbb{Z}_p$-ranks to get inj. $\mathbb{Z}_p \hookrightarrow (U_\infty^1)_{\Gamma_n}$. ∎