Math Notes

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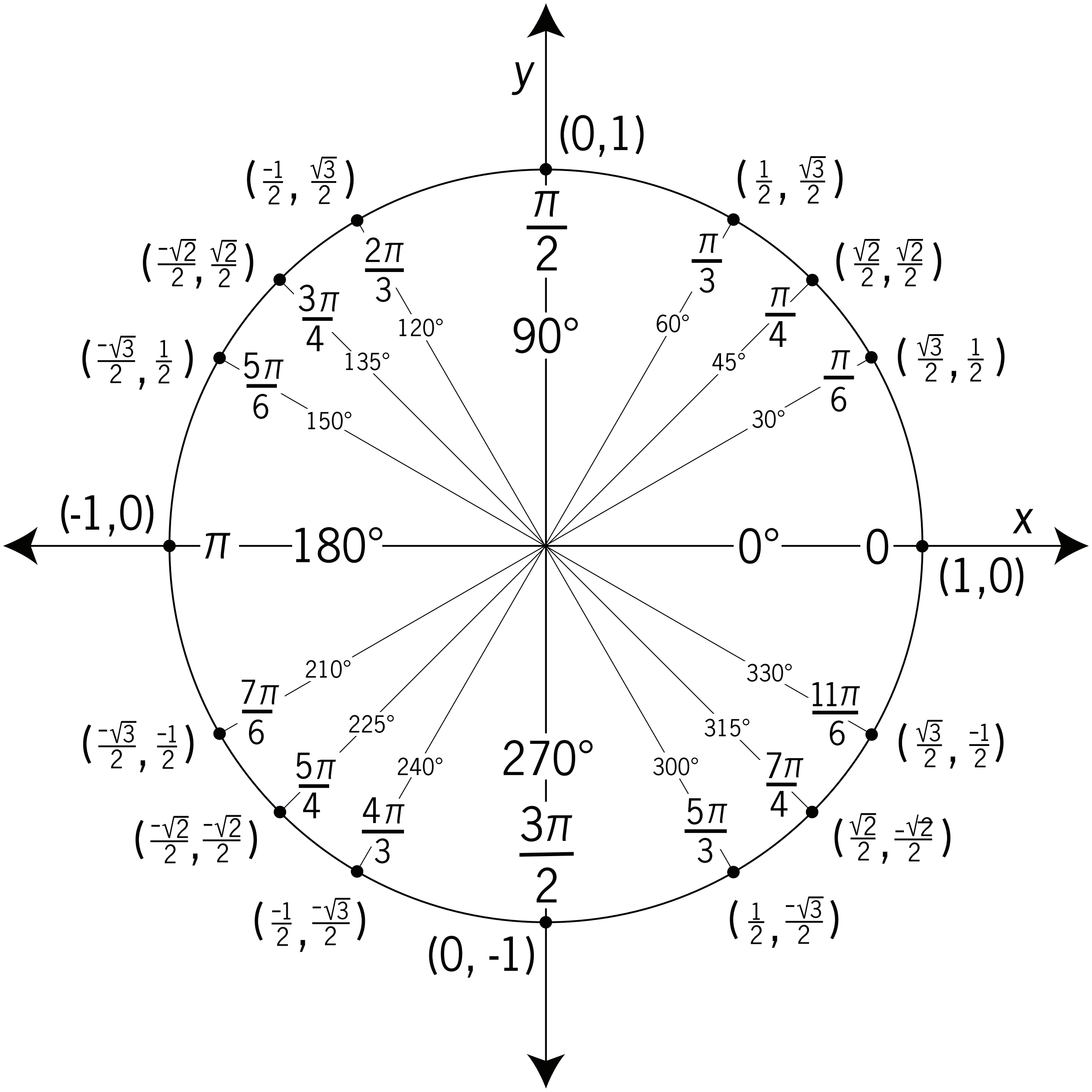
March 6, 2020

Contents

1 Trigonometric Concepts

1.1 Unit Circle



Source: https://etc.usf.edu/clipart/43200/43215/unit-circle7_43215.htm

1.2 Basic Trigonometry

$$
\begin{aligned}
sin(\theta) &= \frac{Opposite}{Hypotenuse} & cos(\theta) &= \frac{Adjacent}{Hypotenuse} & tan(\theta) &= \frac{Opposite}{Adjacent} \\
&= \frac{y}{1} = y & &= \frac{x}{1} = x & &= \frac{y}{x} \\
& & & & &= \frac{sin(\theta)}{cos(\theta)}
\end{aligned}
$$

$$
csc(\theta) = \frac{1}{sin(\theta)} \qquad sec(\theta) = \frac{1}{cos(\theta)} \qquad cot(\theta) = \frac{cos(\theta)}{sin(\theta)}
$$

1.3 Pythagorean Theorem Identities

$$
\begin{aligned}
sin^2\theta + cos^2\theta &= 1 \\
tan^2\theta + 1 &= sec^2\theta \\
1 + cot^2\theta &= csc^2\theta
\end{aligned}
$$

1.4 Half Angle Identities

$$
\begin{aligned}
cos(\frac{\theta}{2}) &= \pm\sqrt{\frac{1+cos(\theta)}{2}} & tan(\frac{\theta}{2}) &= \pm\sqrt{\frac{1-cos(\theta)}{2}} \\
sin(\frac{\theta}{2}) &= \pm\sqrt{\frac{1-cos(\theta)}{2}} & &= \frac{sin(\theta)}{1+cos(\theta)} \\
& & &= \frac{1-cos(\theta)}{sin(\theta)}
\end{aligned}
$$

1.5 Double Angle Identities

$$
\begin{aligned}
sin(2\theta) &= 2sin\theta cos\theta \\
cos(2\theta) &= cos^2\theta - sin^2\theta \\
&= 2cos^2\theta - 1 \\
&= 1 - 2sin^2\theta \\
tan(2\theta) &= \frac{2tan\theta}{1-tan^2\theta}
\end{aligned}
$$

1.6 Power Reduction Identities

$$sin^2\theta = \frac{1 - cos2\theta}{2} \qquad csc^2\theta = \frac{2}{1 - cos2\theta}$$
$$cos^2\theta = \frac{1 + cos2\theta}{2} \qquad sec^2\theta = \frac{2}{1 + cos2\theta}$$
$$tan^2\theta = \frac{1 - cos2\theta}{1 + cos2\theta} \qquad cot^2\theta = \frac{1 + cos2\theta}{1 - cos2\theta}$$

2 Trigonometric Substitution

Expression	Substitution	Identity
$\sqrt{a^2 - b^2x^2}$	$x = \frac{a}{b}sin\theta$	$1 - sin^2\theta = cos^2\theta$
$\sqrt{a^2 + b^2x^2}$	$x = \frac{a}{b}tan\theta$	$1 - tan^2\theta = sec^2\theta$
$\sqrt{b^2x^2 - a^2}$	$x = \frac{a}{b}sec\theta$	$sec^2\theta - 1 = tan^2\theta$

3 Table of Basic Derivatives

y	$\frac{dy}{dx}$
C	0
x	1
$ax^2 + bx + c$	$2ax + b$
x^n	nx^{n-1}
$x^{-1}, \frac{1}{x}$	$-\frac{1}{x^2}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$\sqrt[n]{x}$	$\frac{1}{n\sqrt[n]{x^{n-1}}}$
$ln(x)$	$\frac{1}{x}$
$log_a(x)$	$\frac{1}{xln(a)}$
a^x	$a^x ln(a)$
e^x	e^x
$sin(x)$	$cos(x)$
$cos(x)$	$-sin(x)$
$tan(x)$	$\frac{1}{cos^2x}, sec^2x$
$cot(x)$	$-\frac{1}{sin^2x}, -csc^2x$
$sec(x)$	$tan(x)sec(x)$
$csc(x)$	$-cot(x)csc(x)$

4 Table of Basic Integrals

$f(x)$	$\int f(x)dx = F(x) + C$
x^α $\alpha \neq 0$	$\frac{x^{\alpha+1}}{\alpha+1} + C$
$sin(kx)$	$-\frac{cos(kx)}{k} + C$
$cos(kx)$	$\frac{sin(kx)}{k} + C$
$sec^2(kx)$	$\frac{tan(kx)}{k} + C$
$csc^2(kx)$	$-\frac{cot(kx)}{k+C}$
e^{kx}	$\frac{e^{kx}}{k} + C$
$x^-1, \frac{1}{x}$	$ln\lvert x\rvert + C$
$\frac{1}{\sqrt{1-x^2}}$	$sin^{-1} + C$
$\frac{1}{1+x^2}$	$tan^{-1}x + C$
a^{kx}	$\frac{1}{klna}a^{kx} + C$
$\frac{1}{\sqrt{a^2-x^2}}$	$sin^{-1}\frac{x}{a} + C$
$\frac{1}{\sqrt{a^2+x^2}}$	$\frac{1}{a}tan^{-1}(x/a) + C$

5 Substitution Techniques

5.1 U Substitution

$$\int f(g(x))g'(x)dx = \int f(u)du = F(u) + C$$

$$\text{where: } u = g(x), du = g'(x)$$

5.2 Integration by Parts

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

$$\int udv = uv - \int vdu$$

$$\int_a^b f(x)g'(x) = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)dx$$

6 Engineering Formulas

6.1 Spring Formulas

$$F = kx$$

$$W = \int_a^b kxdx$$

$F =$ Force (Newtons [N])

$k =$ spring constant (Newton meters ${}^{N}/_{m}$)

$x =$ change in distance (meters [m])

$W =$ Work (Joules [J])

$a =$ initial length (meters [m])

$b =$ final length (meters [m])

6.2 Fluid Formulas

$$W = F \cdot d = \int F dx$$
$$V = \pi r^2 h \text{(apply to cylinders)}$$
$$F = m \cdot a = V \cdot \rho$$

$$W = \text{Weight ()}$$
$$F = \text{Force (Newtons [N])}$$
$$d = \text{distance (meters [d])}$$
$$m = \text{mass (meters}^3 [m^3])$$
$$a = \text{acceleration (meters per second}^2\ [^m/_{s^2}])$$
$$\rho = \text{Something}$$

7 Method of Partial Fractions

$$\int \frac{P_n(x)}{Q_m(x)} dx \text{ when } m > n$$

n and m are defined as the degree of the numerator and the denominator.

7.1 Decomposition Types

Type	Factor Example	Decomposition
Linear Factor	$(x-4)$	$\frac{A}{x-4}$
Repeated Linear Factor	$(x-4)^2$	$\frac{A}{(x+4)} + \frac{B}{(x+4)^2}$
Quadratic Irreducible Factor	(x^2+4)	$\frac{Ax+B}{x^2+4}$
Repeated Quadratic Irreducible Factor	$(x^2+4)^2$	$\frac{Ax+B}{(x^2+4)} + \frac{Cx+D}{(x^2+4)^2}$

7.2 Numerical Integration

7.2.1 Midpoint Rule

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(\overline{x_i})\Delta x$$
$$x_i* = \frac{x_{i-1} + x_i}{2} = \overline{x_i}$$
$$x_i* \in [x_{i-1}, x]$$

7.2.2 Trapezoidal Rule

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} \frac{f(x_{i-1}) + f(x_i)}{2}\Delta x$$
$$\approx \sum_{i=1}^{n} \frac{\Delta x}{2}[f(x_{i-1}) + f(x_i)]$$

7.2.3 Error Approximation (Actual Error)

$$E_T = \int_a^b f(x)dx - T_n$$
$$E_H = \int_a^b f(x)dx - M_n$$

7.2.4 Error Bounds

$$|f''(x)| \leq K \text{ on } [b,a] f(x) = \frac{1}{x}, f''(x) = -\frac{2}{x^3}$$
$$\left|-\frac{2}{x^3}\right| \leq 2 = K$$
$$|E_T| \leq \frac{K(b-a)^3}{12n^2} = \frac{2(2-1)^3}{12 \cdot 5^2} = 0.0012$$
$$|E_M| \leq \frac{K(b-a)^3}{24n^2} = \frac{2(2-1)^3}{24 \cdot 5^2} = 0.0012$$
$$E_T \leq 0.6931 - 0.6956$$

7.3 Improper Integrals

7.3.1 Type 1

$x \in [a,b]$, $f(x)$ is continuous on $[a,b)$ at $x = b$ $f(x)$ discontinues

$$\int_a^\infty f(x) = \lim_{x\to\infty}\left(\int_a^b f(x)dx\right)$$

7.3.2 Type 2

$f(x)$ dicontinues at $x = a$, continuous on $(a,b]$

$$\int_a^b f(x)dx = \lim_{t\to\ a} \int_t^b f(x)dx$$

7.3.3 Type 3

$f(x)$ discontinues at $x = c$, $a < c < b$ everywhere else on $[a, b]$
$f(x)$ is continuous

$$\int\limits_a^b f(x)dx = \int\limits_a^c f(x)dx + \int\limits_c^b f(x)dx$$

7.4 The Length of the Curve

$x = x(t)$ $\quad$ $y = y(t)$ $\quad$ $a \leq t \leq b$

$$S' = \int\limits_a^b \sqrt{(x'(t))^2 + (y'(t))^2}$$

8 Area of Revolution

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$dA = 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$1) \int\limits_a^b dA = A = 2\pi \int\limits_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$2) x = xb$$

9 Separable Equations

$$y'(x) = g(x)h(y), \frac{dy}{dx} = g(x)h(y)\Big| dx$$

$$dy = h(y)g(x)dx\Big| \frac{1}{h(y)}$$

$$\frac{1}{h(y)} dy = g(x)dx, H(y) = \frac{1}{h(y)} \longrightarrow H(y)dy = g(x)dx$$

$$\int H(y) = \int g(x)dx + C$$

Implicit Form of Solution	**Explicit Form of Solution**
$F(y) = G(x) + C$	$y = F^{-1}(G(x) + C)$

10 Parametric Equations of Curves

$$\begin{aligned} &x = x(t), y = y(t), \alpha \leq t \leq \beta \\ &= \varphi(x) y = (y(\varphi(x))) = f(x) \end{aligned}$$

$$t_0 = \begin{pmatrix} x_0 = x(t_0) \\ y_0 = y(t_0) \end{pmatrix}$$

$$t_1 = \begin{pmatrix} x_1 = x(t_1) \\ y_1 = y(t+1) \end{pmatrix}$$

$$t_2 = \begin{pmatrix} x_2 = x(t_2) \\ y_2 = y(t_2) \end{pmatrix}$$

11 Polar Coordinates

$$x = r\cos\theta y = r\sin\theta$$

Note: Angles are CounterClockwisw from the Positive X

12 Cartesian Coordinates

$$r^2 = x^2 + y^2 \tan\theta = \frac{y}{x}$$

13 Sequences

$$\begin{aligned} &a_n = f(n) \\ &a_1, a_2, a_3, \cdots, a_n, a_{n+1}, \cdots \\ &n \to a_n \\ &1 \to a, 2 \to a_2, \cdots \\ &\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, \ldots, a_n, \ldots\} = \{a_n\} \end{aligned}$$

<u>Discrete</u>

- can not take derivatice
- can take the limit

14 Curves in Polar Coordinates

$$\begin{aligned} &x = r\cos, y = r\sin\theta, r = f(\theta) \\ &x = x(t), u = y(t) \\ &\alpha \leq \theta \leq \beta \end{aligned}$$

15 Sequences (Continued)

$a_1, a_2, a_3, \ldots, a_n, a_{n+1}, \ldots$
if $\{a_n\}, \{b_n\}$ converges, then:

1. $\lim_{n\to\infty}(a_n \pm b_n) = \lim a_n \pm \lim b_n$
2. $\lim_{n\to\infty}(c \cdot a_n) = c \lim a_n$

3. $\lim_{n\to\infty}(a_n, b_n) = \lim a_n \cdot \lim b_n$

4. $\lim \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n}, b \neq 0$

5. $\lim_{n\to\infty}(a_n^p) = (\lim a_n)^p, a_n, p > 0$

16 Squeeze Theorem

for $n > n_0, a_n \leq b_n \leq c_n$
$\{a_n\}, \{b_n\}, \{c_n\} \to$ sequences
$\lim a_n = \lim c_n = L$, then
$\lim b_n = L$

1. $\lim_{n\to\infty} \frac{\lim}{n} = 0$

2. $\lim \sqrt[n]{n} = 1$

3. $\lim x^n = 0, |x| < 1$

4. $\lim x^{\frac{1}{n}}, x > 0 > 1$

5. $\lim(1 + \frac{x}{n})^n = e^x$

6. $\lim \frac{x^n}{n!} = 0, y$ any x

17 Series

$\{a_n\}_{n=1}^{\infty}, a_1 + a_2 + a3 + \cdots + a_n + \dots$
Infinite Series
$\sum_{n=1}^{\infty} a_n, \sum a_n$

18 Additional Resources

Print
Calculus Study Guide: https://mt-jfk.com/ap-calculus-study-guide.pdf

Video
The Organic Chemistry Tutor: https://www.youtube.com/channel/UCEWpbFLzoYGPfuWUMFPSaoA
Black Pen Red Pen: https://www.youtube.com/user/blackpenredpen