

# Math Notes

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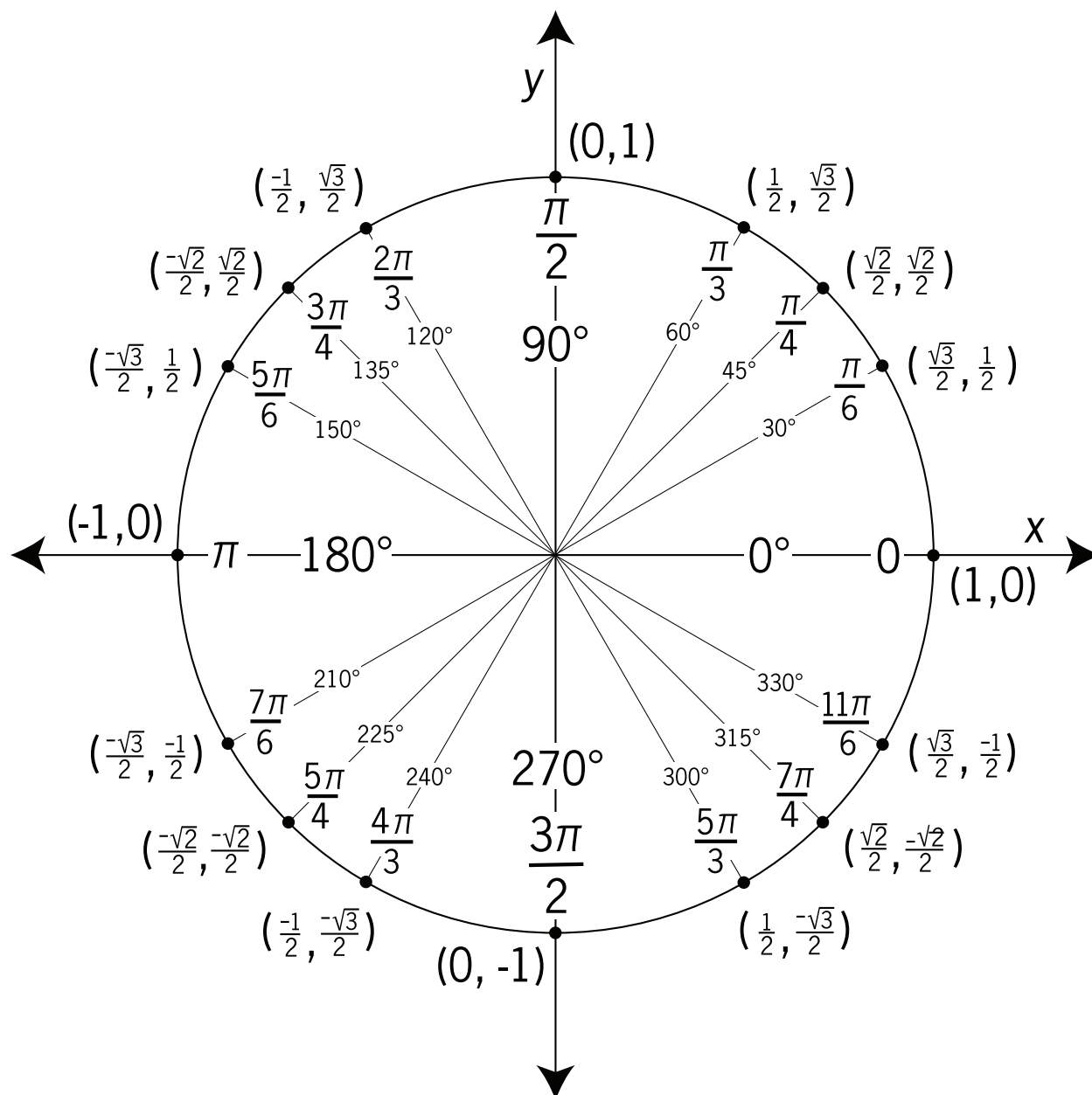
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# 1 Trigonometric Concepts

## 1.1 Unit Circle



Source: [https://etc.usf.edu/clipart/43200/43215/unit-circle7\\_43215.htm](https://etc.usf.edu/clipart/43200/43215/unit-circle7_43215.htm)

## 1.2 Basic Trigonometry

$$\begin{aligned}\sin(\theta) &= \frac{\textit{Opposite}}{\textit{Hypotenuse}} & \cos(\theta) &= \frac{\textit{Adjacent}}{\textit{Hypotenuse}} & \tan(\theta) &= \frac{\textit{Opposite}}{\textit{Adjacent}} \\ &= \frac{y}{1} = y & &= \frac{x}{1} = x & &= \frac{y}{x} \\ & & & & &= \frac{\sin(\theta)}{\cos(\theta)}\end{aligned}$$

$$\begin{aligned}\csc(\theta) &= \frac{1}{\sin(\theta)} & \sec(\theta) &= \frac{1}{\cos(\theta)} & \cot(\theta) &= \frac{\cos(\theta)}{\sin(\theta)}\end{aligned}$$

## 1.3 Pythagorean Theorem Identities

$$\begin{aligned}\sin^2\theta + \cos^2\theta &= 1 \\ \tan^2\theta + 1 &= \sec^2\theta \\ 1 + \cot^2\theta &= \csc^2\theta\end{aligned}$$

## 1.4 Half Angle Identities

$$\begin{aligned}\cos\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1 + \cos(\theta)}{2}} & \tan\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1 - \cos(\theta)}{2}} \\ \sin\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1 - \cos(\theta)}{2}} & &= \frac{\sin(\theta)}{1 + \cos(\theta)} \\ & & &= \frac{1 - \cos(\theta)}{\sin(\theta)}\end{aligned}$$

## 1.5 Double Angle Identities

$$\begin{aligned}\sin(2\theta) &= 2\sin\theta\cos\theta \\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta \\ \tan(2\theta) &= \frac{2\tan\theta}{1 - \tan^2\theta}\end{aligned}$$

## 1.6 Power Reduction Identities

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

$$\tan^2\theta = \frac{1 - \cos 2\theta}{1 + \cos 2\theta}$$

$$\csc^2\theta = \frac{2}{1 - \cos 2\theta}$$

$$\sec^2\theta = \frac{2}{1 + \cos 2\theta}$$

$$\cot^2\theta = \frac{1 + \cos 2\theta}{1 - \cos 2\theta}$$

## 2 Trigonometric Substitution

Expression	Substitution	Identity
$\sqrt{a^2 - b^2x^2}$	$x = \frac{a}{b}\sin\theta$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + b^2x^2}$	$x = \frac{a}{b}\tan\theta$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{b^2x^2 - a^2}$	$x = \frac{a}{b}\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$

## 3 Table of Basic Derivatives

$y$	$\frac{dy}{dx}$
$C$	0
$x$	1
$ax^2 + bx + c$	$2ax + b$
$x^n$	$nx^{n-1}$
$x^{-1}, \frac{1}{x}$	$-\frac{1}{x^2}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$\sqrt[n]{x}$	$\frac{1}{n\sqrt[n]{x^{n-1}}}$
$\ln(x)$	$\frac{1}{x}$
$\log_a(x)$	$\frac{1}{x\ln(a)}$
$a^x$	$a^x \ln(a)$
$e^x$	$e^x$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\frac{1}{\cos^2 x}, \sec^2 x$
$\cot(x)$	$-\frac{1}{\sin^2 x}, -\csc^2 x$
$\sec(x)$	$\tan(x)\sec(x)$
$\csc(x)$	$-\cot(x)\csc(x)$

## 4 Table of Basic Integrals

$f(x)$	$\int f(x)dx = F(x) + C$
$x^\alpha$ $\alpha \neq 0$	$\frac{x^{\alpha+1}}{\alpha+1} + C$
$\sin(kx)$	$-\frac{\cos(kx)}{k} + C$
$\cos(kx)$	$\frac{\sin(kx)}{k} + C$
$\sec^2(kx)$	$\frac{\tan(kx)}{k} + C$
$\csc^2(kx)$	$-\frac{\cot(kx)}{k} + C$
$e^{kx}$	$\frac{e^{kx}}{k} + C$
$x^{-1}, \frac{1}{x}$	$\ln x  + C$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + C$
$\frac{1}{1+x^2}$	$\tan^{-1} x + C$
$a^{kx}$	$\frac{1}{k\ln a} a^{kx} + C$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1} \frac{x}{a} + C$
$\frac{1}{\sqrt{a^2+x^2}}$	$\frac{1}{a} \tan^{-1}(x/a) + C$

## 5 Substitution Techniques

## 5.1 U Substitution

$$\int f(g(x))g'(x)dx = \int f(u)du = F(u) + C$$

where:  $u = g(x)$ ,  $du = g'(x)$

## 5.2 Integration by Parts

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

$$\int u dv = uv - \int v du$$

$$\int_a^b f(x)g'(x) = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)dx$$

# 6 Engineering Formulas

## 6.1 Spring Formulas

$$F = kx$$

$$W = \int_a^b kx dx$$

$F$  = Force (Newtons [N])

$k$  = spring constant (Newton meters  $^N/m$ )

$x$  = change in distance (meters [m])

$W$  = Work (Joules [J])

$a$  = initial length (meters [m])

$b$  = final length (meters [m])

## 6.2 Fluid Formulas

$$W = F \cdot d = \int F dx$$

$$V = \pi r^2 h (\text{apply to cylinders})$$

$$F = m \cdot a = V \cdot \rho$$

$$W = \text{Weight } ()$$

$$F = \text{Force (Newtons [N])}$$

$$d = \text{distance (meters [d])}$$

$$m = \text{mass (meters}^3[\text{m}^3])$$

$$a = \text{acceleration (meters per second}^2 [\text{m/s}^2])$$

$$\rho = \text{Something}$$

## 7 Method of Partial Fractions

$$\int \frac{P_n(x)}{Q_m(x)} dx \text{ when } m > n$$

$n$  and  $m$  are defined as the degree of the numerator and the denominator.

### 7.1 Decomposition Types

Type	Factor Example	Decomposition
Linear Factor	$(x - 4)$	$\frac{A}{x-4}$
Repeated Linear Factor	$(x - 4)^2$	$\frac{A}{(x+4)} + \frac{B}{(x+4)^2}$
Quadratic Irreducible Factor	$(x^2 + 4)$	$\frac{Ax+B}{x^2+4}$
Repeated Quadratic Irreducible Factor	$(x^2 + 4)^2$	$\frac{Ax+B}{(x^2+4)} + \frac{Cx+D}{(x^2+4)^2}$

### 7.2 Numerical Integration

#### 7.2.1 Midpoint Rule

$$\int_a^b f(x) dx \approx \sum_{i=1}^n f(\bar{x}_i) \Delta x$$

$$x_i^* = \frac{x_{i-1} + x_i}{2} = \bar{x}_i$$

$$x_i^* \in [x_{i-1}, x_i]$$

### 7.2.2 Trapezoidal Rule

$$\begin{aligned}\int_a^b f(x)dx &\approx \sum_{i=1}^n \frac{f(x_{i-1}) + f(x_i)}{2} \Delta x \\ &\approx \sum_{i=1}^n \frac{\Delta x}{2} [f(x_{i-1}) + f(x_i)]\end{aligned}$$

### 7.2.3 Error Approximation (Actual Error)

$$\begin{aligned}E_T &= \int_a^b f(x)dx - T_n \\ E_H &= \int_a^b f(x)dx - M_n\end{aligned}$$

### 7.2.4 Error Bounds

$$\begin{aligned}|f''(x)| &\leq K \text{ on } [b, a], f(x) = \frac{1}{x}, f''(x) = -\frac{2}{x^3} \\ \left| -\frac{2}{x^3} \right| &\leq 2 = K \\ |E_T| &\leq \frac{K(b-a)^3}{12n^2} = \frac{2(2-1)^3}{12 \cdot 5^2} = 0.0012 \\ |E_M| &\leq \frac{K(b-a)^3}{24n^2} = \frac{2(2-1)^3}{24 \cdot 5^2} = 0.0012 \\ E_T &\leq 0.6931 - 0.6956\end{aligned}$$

## 7.3 Improper Integrals

### 7.3.1 Type 1

$x \in [a, b]$ ,  $f(x)$  is continuous on  $[a, b)$  at  $x = b$   $f(x)$  discontinues

$$\int_a^\infty f(x) = \lim_{x \rightarrow \infty} \left( \int_a^b f(x)dx \right)$$

### 7.3.2 Type 2

$f(x)$  is continuous at  $x = a$ , continuous on  $(a, b]$

$$\int_a^b f(x)dx = \lim_{t \rightarrow a} \int_t^b f(x)dx$$



### 7.3.3 Type 3

$f(x)$  discontinues at  $x = c$ ,  $a < c < b$  everywhere else on  $[a, b]$   
 $f(x)$  is continuous

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

## 7.4 The Length of the Curve

$$x = x(t)$$

$$y = y(t)$$

$$a \leq t \leq b$$

$$S' = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}$$

## 8 Additional Resources

### Print

Calculus Study Guide: <https://mt-jfk.com/ap-calculus-study-guide.pdf>

### Video

The Organic Chemistry Tutor: <https://www.youtube.com/channel/UCeWpbFLzoYGPfuWUMFPSaoA>

Black Pen Red Pen: <https://www.youtube.com/user/blackpenredpen>