Math Notes

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Contents

1 First Order Differential Equations

1.1 Separable Differential Equations

$$\begin{aligned}
\frac{dy}{dx} &= G(x) \cdot H(y) \Big| \cdot dx \\
dy &= G(x) \cdot H(y) dx \Big| \cdot \frac{1}{H(y)} \\
\frac{dy}{H(y)} &= G(x) dx \\
\int \frac{dy}{H(y)} &= \int G(x) dx \\
h(y) + C_1 &= g(x) + C_2 \\
h(y) &= g(x) + C
\end{aligned}$$

Example

$$
\begin{aligned}
\frac{dy}{dx} &= y \sin x \\
dy &= y \sin x (dx) \\
\frac{dy}{y} &= \sin x dx \\
\int y^{-1} dy &= \int \sin x dx \\
\ln |y| + C &= -\cos x + C \\
\ln |y| &= -\cos x + C \\
y &= e^{-\cos x + C} \\
&= e^{-\cos x} e^{C} \\
&= \frac{1}{e^{\cos x}} \cdot e^{C} \\
&= \frac{e^{C}}{e^{\cos x}} \\
& D = e^{C} \\
&= \frac{D}{e^{\cos x}}
\end{aligned}
$$

1.2 Linear First Order Differential Equations Homogeneous Differential Equation

$y'(x) + p(x)y(x) = q(x)$
$p(x)$, $q(x)$ given $q(x) = 0$, then $y' + p(x)y = 0$

- Linear because all terms are to the power of 1

$$
\begin{aligned}
y' + y^2 = 0 &\rightarrow \text{non-linear} \\
y' + y = 0 &\rightarrow \text{linear}
\end{aligned}
$$

1.3 Method of Integrating Factor

$\rho = e^{\int p(x)dx}$

$$
\begin{aligned}
y' e^{\int p(x)dx} + P(x) y e^{\int p(x)dx} &= q(x) e^{\int p(x)dx} \\
\frac{d}{dx}(y \cdot e^{\int p(x)dx}) &= y' e^{\int p(x)dx} \cdot \frac{d}{dx}(\int p(x)dx) \\
&= y' e^{\int p(x)dx} + y e^{\int p(x)dx} p(x) \\
\frac{d}{dx}(y \cdot e^{\int p(x)dx}) &= q(x) \cdot e^{\int p(x)dx} \Big| \cdot dx \int \\
\int \frac{d}{dx}(y e^{\int p(x)dx}) dx &= \int q(x) \cdot e^{\int p(x)dx} \\
y \cdot e^{\int p(x)dx} &= \int (q(x) e^{\int p(x)dx}) dy + e \cdot \frac{1}{e^{\int p(x)dx}} \\
y &= (\int (q(x) \cdot e^{\int p(x)dx}) dx + C) e^{-\int p(x)dx}
\end{aligned}
$$

1.3.1 Method of Substitution

1.

$$
\begin{aligned}
y' &= f(ax+by+c) \\
& a, b, c \text{given constants} \\
& f \text{given functions} \\
u &= ax+by+c \\
\frac{du}{dy} &= \frac{d}{dx}(ax+by+c) \\
\frac{du}{dy} &= a + b\frac{dy}{dx} + 0 \\
& \rightarrow \frac{dy}{dx} = \frac{\frac{du}{dx}-a}{b} \\
\frac{\frac{du}{dx}-a}{b} &= f(u) \\
\frac{du}{dx} &= bf(u)+a \Big| \cdot dx \\
du &= (bf(u)+a)dx \\
\frac{du}{bf(u)+a} &= dx \\
\int \frac{du}{bf(u)+a} &= \int dx \\
F(u) &= x + C \\
F(ax+by+C) &= x + C
\end{aligned}
$$

Example

$$
\begin{aligned}
\frac{dy}{dx} &= (x+y+3)^2 \\
u &= x+y+3 \\
\frac{du}{dx} &= \frac{d}{dx}(x+y+3) \\
\frac{du}{dx} &= 1 + \frac{dy}{dx} \\
\frac{dy}{dx} &= \frac{du}{dx} - 1 \\
\frac{du}{dx} - 1 &= (u)^2 \\
\frac{du}{dx} &= 1 + (u)^2 \Big| \cdot dx \\
du &= (1+u^2)dx \Big| \frac{1}{1+u^2} \\
\frac{du}{1+u^2} &= dx \\
\int \frac{du}{1+u^2} &= \int dx \\
\tan^{-1} u &= x + C \\
\tan(\tan^{-1} u) &= \tan(x+C) \\
u &= \tan(x+C), \qquad\qquad x+y+3 = \tan(x+C) \\
y &= \tan(x+C) - x - 3
\end{aligned}
$$

2.

$$y' = f(\frac{y}{x})$$

$$u = \frac{y}{x}, y = u \cdot x, \frac{dy}{dx} = \frac{du}{dx}x + 1 \cdot u$$

$$\frac{du}{dx} \cdot x + u = f(u) \to \frac{du}{dx}x = f(u) - u \,\Big|\, \cdot dx$$

$$du \cdot x = (f(u) - u)dx \,\Big|\, \frac{1}{x(f(u) - u)}$$

$$\frac{du}{f(u) - u} = \frac{dy}{x}$$

$$F(u) = \ln|x| + C$$

$$F(\frac{y}{x}) = \ln|x| + C$$

$$
\begin{aligned}
2xy\frac{dy}{dx} &= 4x^2 + 3y^2 \\
2xy\frac{dy}{dx} &= 4x^2 + 3y^2 \;\Big|\; \frac{1}{x^2} \\
2\frac{y}{x} \cdot \frac{dy}{dx} &= 4 + 3(\frac{y}{x})^2 \\
\frac{dy}{dx} &= \frac{4 + 3(\frac{y}{x})}{2(\frac{y}{x})} \\
u &= \frac{y}{x}, y = u \cdot x, \frac{dy}{dx} = \frac{du}{dx}x + u \\
\frac{du}{dx} &= \frac{4 + 3u^2}{2u} - u \\
&= \frac{4 + 3u^2 - 2u^2}{2u} \\
&= \frac{4 + u^2}{2u} \\
x \cdot \frac{du}{dx} &= \frac{4 + u^2}{2u} \;\Big|\; \cdot dx \\
x \cdot du &= \frac{4 + u^2}{2u} \cdot dx \;\Big|\; \cdot \frac{1}{x \cdot \frac{4 \cdot u^2}{2u}} \\
\int \frac{2u}{4 + u^2} du &= \int \frac{dx}{x} \\
z &= 4 + u^2 \\
dz &= 2udu \\
\int \frac{dz}{z} &= \int \frac{dx}{x} \\
\ln|z| &= \ln|x| + C \\
e^{\ln|z|} &= e^{\ln|x| + C} = e^{\ln|x|}e^C \\
|z| &= |x|e^C \\
z &= e^C \cdot x = \pm e^C \cdot x \\
A &= \pm e^C \\
z &= Ax \\
4 + u^2 &= Ax \\
4 + (\frac{y}{x})^2 &= Ax \qquad \text{(general solution)} \\
(\frac{y}{x})^2 &= Ax - 4 \\
\frac{y}{x} &= \pm\sqrt{Ax - 4} \\
y &= \pm x\sqrt{Ax - 4} \qquad \text{(explicit form)}
\end{aligned}
$$

Two Types of U-Substitution

1. $y' = f(ax + by + c)$
2. $y' = f(\frac{x}{y}$

1.4 Exact Equations

$$dF = M(x,y)dx + N(x,y)dy = 0$$
$$dF(x,y) = 0$$
$$dF(x,y) = 0$$
$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$$
$$[\frac{\partial F}{\partial x} = N(x,y)][\frac{\partial F}{\partial y} = M(x,y)]$$
$$\frac{\partial}{\partial y}[\frac{\partial F}{\partial x} = N(x,y)], \frac{\partial}{\partial x}[\frac{\partial F}{\partial y} = M(x,y)]$$
$$\frac{\partial^2 F}{\partial y dx} = \frac{\partial N}{\partial y}, \frac{\partial^2 F}{dx dy} = \frac{\partial M}{\partial x}$$
$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$$

$$\frac{\partial F}{\partial x} = N \Big| \cdot dx \int$$
$$\int \frac{\partial F}{\partial x} = \int N(x,y)dx$$
$$F(x,y) = \int N(x,y)dx + g(y)$$
$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\int N(x,y)dx + g'(y) = M(x,y)$$
$$\Rightarrow g(y)$$
$$F(x,y) = \int N(x,y)dx + g(y)$$

(equation from earlier step)

Example

$$\begin{aligned}
(6xy-y^3)dx+(4y+3x^2-3xy^2)dx &= 0 \\
M(x,y)+N(x,y) &= 0 \\
\frac{\partial M}{\partial y}(6xy-y^3) &= 6x-3y^2
\end{aligned}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(4y+3x^2-3xy^2) = 0+6x-3y^2$$

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

$$\begin{aligned}
\frac{\partial F}{\partial x} &= 6xy-y^3, \frac{\partial F}{\partial y} = 4y+3x^2-3xy^2 \\
\int \frac{\partial F}{\partial x}dx &= \int (6xy-y^3)dx \\
F(x,y) &= 6\frac{x^2}{2}\cdot y - y^3x + g(y) \\
\frac{\partial F}{\partial y} &= 3x^2\cdot 1 - x\cdot 3y^2 + g'(y) \\
&\text{Remember: } \frac{\partial F}{\partial y} = 4y+3x^2-3xy^2 \\
3x^2-3xy^2+g'(y) &= 4y+3x^2-3xy^2 \\
g'(y) &= 4y \\
\int g'(y)dy &= \int 4ydy \\
g(y) &= \frac{4y^2}{2}+C \\
&\text{Remember: } F(x,y) = 6\frac{x^2}{2}\cdot y - y^3x + g(y) \\
F(x,y) &= 3x^2\cdot y - y^3x + 2y^2 + C
\end{aligned}$$

$$\int dF = 0, F(x,y) = C_1$$

$$\begin{aligned}
3x^3y-y^3x+2y^2 &= C_1 \\
y(x)
\end{aligned}$$

2 Bernoulli Equation

$y'(x)+P(x)y(x) = Q(x)y^n(x)$
P, Q-given, $n \neq 1$

linear $y'+p(x)y = q(x)$ by integrating factor $\rho = e^{\int p(x)dx}$

dividing by y^n to obtain

$$
\begin{aligned}
y^{-n}y' + P(x)y^{1-n} &= Q(x) \\
z &= y^{1-n}, (z(x))' = (y(x)^{1-n})' \\
\frac{dz}{dx} &= (1-n)y^{1-n-1}\frac{dy}{dx} \\
&= (1-n) \cdot y^{-n}\frac{dy}{dx} \\
y^{-n}y'(x) &= \frac{dz}{dx} \cdot \frac{1}{1-n}
\end{aligned}
$$

Example

$$y' - \frac{3}{2x}y = \frac{2x}{y}$$
$$P(x) = -\frac{3}{2x}, Q = 2x, n = -1$$

$$y' - \frac{3}{2x}y = \frac{2x}{y} \;\Big|\; \cdot y$$
$$z = y^{1-n}$$
$$yy' - \frac{3}{2x}y^2 = 2x$$
$$z = y^2, z'(x) = 2y \cdot y'(x)$$
$$\rightarrow y \cdot y'(x) = \frac{z(x)}{z}$$

$$\frac{z'(x)}{z} - \frac{3}{2x}z = 2x$$
$$\rightarrow z'(x) - \frac{3}{2}z = 4x$$

$$\rho = e^{\frac{-3}{2}dx}$$
$$= e^{-e \ln x}$$
$$= e^{\ln x^{-3}}$$
$$= x^{-3}$$

$$z'(x) - \frac{3}{x}z = 4x \;\Big|\; \cdot \rho = x^{-3}$$
$$x^{-3} \cdot z'(x) - 3x^{-4}z(x) = 4x^{-2}$$
$$check \rightarrow \frac{d}{dx}(x^{-3} \cdot z) = 4x^{-2}$$

$$\frac{d}{dx}(x^{-3} \cdot z) = z'x^{-3} - 3x^{-4}z$$
$$= x^{-3}z' + z(-3) \cdot x^{-4}$$

$$\int \frac{d}{dx}(x^{-3}z)dx = \int 4x^{-2}dx$$
$$x^{-3}z = -\frac{4}{x} + C$$
$$z = -4x^2 + Cx^3$$

$$y^2 = 4x^2 + Cx^3$$
$$y = \pm\sqrt{-4x^2 + Cx^3}$$

3 Second Order Ordinary Differential Equations

$F(y'', y', y, x) = 0, y = y(x)$

3.1 Type I

$F(y''(x), y'(x), x) = 0$
Function of x

$$
\begin{aligned}
y'(x) = P(x), y''(x) = (y'(x))' &= P'(x) \\
&\rightarrow F(p'(x), p(x), x) = 0 \\
&\rightarrow p(x) = f(x, C_1) \\
&\quad y' = f(x, C_1) \Big| \cdot dx \int \\
&\rightarrow \int y'(x)dx = \int f(x, C_1)dx \\
&\quad y(x) = \int f(x, C_1)dx + C_2
\end{aligned}
$$

Example

$$
\begin{aligned}
xy'' + 2y' &= 6x, y(x) = ? \\
xp'(x) + 2p &= 6x
\end{aligned}
$$

$$
y' + p(x)y = q(x) \Big| \qquad \cdot\rho = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2
$$

$$
\begin{aligned}
p'(x)x^2 + \frac{2x^2}{x}p(x) &= 6x^2 \\
\frac{d}{dx}(x^2 \cdot p(x)) &= p'(x)x^2 + 2xp \\
&\rho \cdot p(x) \\
\frac{d}{dx}(p(x), y(x)) &= (p(x), y(x))' \\
\frac{d}{dx}(x^2 \cdot p(x)) &= 6x^2 \Big| \cdot dx \int \\
\int \frac{d}{dx}(x^2 \cdot p(x))dx &= \int 6x^2 dx \\
x^2 p &= \frac{6x^2}{3} + C,
\end{aligned}
\qquad
\begin{aligned}
p(x) &= 2x + \frac{C_1}{x^2} \\
&= y'(x)
\end{aligned}
$$

$$
\begin{aligned}
&\rightarrow \int y'(x)dx = \int (2x + C_1 x^{-2})dx \\
&\quad y(x) = \frac{2x^2}{2} - \frac{C_1}{y} + C_2
\end{aligned}
$$

3.2 Type II

$F(y''(x), y'(x), y(x)) = 0$
Function of y

$$\begin{aligned} y'(x) = p(y), y'' &= (y'(x))' \\ &= (P(y))'' \\ &= \frac{dP}{dy} \cdot \frac{dy}{dx} \\ &= \frac{dp}{dy} p \end{aligned}$$

Example

$$\begin{aligned} yy''(x) &= (y'(x))^2 \\ y\frac{dp}{dy} \cdot p &= p^2, p \neq 0 \\ y\frac{dp}{dy}, \frac{dp}{p} &= \frac{dy}{y}, \int \frac{dp}{p} = \int \frac{dy}{y} \\ &\to \ln|p| = \ln|y| + C_1 \\ &\quad e^{\ln|p|} = e^{\ln|y| + C_1} = e^{\ln|x|} \cdot e^{C_1} \\ &\quad |p| = |y| \cdot e^{C_1}, p = \pm e^{C_1} \cdot y \\ &\qquad A_1 = \pm e^{C_1} \\ &\quad p = A_1 y \\ &\to y'(x) = A_1 y \\ &\quad \frac{dy}{dx} = A_1 y \\ &\quad \frac{dy}{y} = A_1 dx \\ &\quad \int \frac{dy}{y} = \int A_1 dx \\ &\to \ln|y| = A_1 x + C_2 \\ &\quad e^{\ln|y|} = e^{A_1 x + C_2} \\ &\quad |y| = e^{A_1 x} e^{C_2} \\ &\to y = \pm e^{C_2} e^{A_1 x} \\ &\qquad A_2 = \pm e^{C_2} \\ &\quad y = A_2 e^{A_1 x} \end{aligned}$$

3.3 Inital Value Problem

(1) $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$
<u>Then</u> If $f(x, y), \frac{\partial f}{\partial y}$ (continnuous function) on $R[a \leq x \leq b, c \leq y \leq d]$ there exists wuch interval $I \in [a, b]$ also $x_0 \in I$, where the inital value problem (1) has unique solution

Rectangle (R) is designated by the chosen value (can be very large or small)

$$\begin{gathered} y'' + p(x)y' + g(x)y = f(x), x \in (a, b) \\ p(x), q(x), f(x) - given y(x)? \\ y(x_0) = y_0, y'(x_0) = y_1 \end{gathered}$$

Theorem 1 if $p(x), q(x), f(x)$
- continuous on $(a, b), x_0 \in (a, b)$ then (2) has unique solution

4 Linear Second Order Differential Equations

4.1 Initial Value Problem

$$y'' + p(x)y' + q(x)y = f(x), x \in (a, b)$$
$$p(x), q(x), f(x) - \text{given} y(x)?$$
$$y(x_0) = y_0, y'(x_0) = y_1$$

4.1.1 Theorem 1

if $p(x), q(x), f(x)$
- continuous on $(a, b), x_0 \in (a, b)$ then (2) has unique solution

$$f(x) \equiv 0, y'' + p(x)y' + g(x)y = 0$$
$$\text{homogeneous equation}$$

$$0 = \text{homogeneous equation}$$
$$0 \neq \text{non homogeneous equation}$$

$$y_1(x), y_2(x) - \text{solutions of (3)}$$
$$y(x) = C_1y_1(x) + C_2y_2(x)$$
$$- \text{is also solution of (3) linear combination, where } C_1, C_2 - \text{some constant}$$

$$(C_1y_1 + C_2y_2)'' + p(x)(C_1y_1 + C_2y_2) = 0$$
$$C_1y_1'' + C_2y_2 + p(x)(C_1y_1' + C_2y_2') + q(x)(C_1y_1 + C_2y_2) = 0$$
$$C_1(y_1'' + p(x)y_1' + q(x)y_1) + C_2(y_2'' + p(x)y_2' + q(x)y_2) = 0$$

$$y_1'' + p(x)y_1' + q(x)y_1 = 0$$
$$y_2'' + p(x)y_2' + q(x)y_2 = 0$$

Definition $y_1(x), y_2(x)$ - linear independent on (a, b) if $C_1y_1(x) + C_2y_2(x) = 0$ if $C_1 = C_2 = 0$

$$y_1 = \left(-\frac{C_2}{C_1}\right)y_2(x) \qquad y_1 = K \cdot y_2$$
$$\frac{y_1}{y_2}K$$

Example 1
$y_1 = \sin x, y_1 = \cos x$
$\frac{\sin x}{\cos x} \neq K$
linear independent $\rightarrow$ ratio is not equal to constant
Example 2
$y_1 = \sin 2x, y_2 = \sin x \cos y$
$\frac{y_1}{y_2} = \frac{\sin 2x}{\sin x \cos x} = \frac{2 \sin x \cos x}{\sin x \cos x} = 2$

5 Second Order Linear Differential Equations

1. $y'' + p(x)y' + q(x)y = f(y)$
   - non homogeneous equation
2. $y(x_0) = y_0, y'(x_0) = y_1$
   - inital conditions
3. 1 + 2 → I.V.P. - has unique solution when p(x), q(x), f(x) - continuous

5.1 Homogeneous Equation

$$(3) f(x) = 0 \to y'' + p(x)y' + q(x)y = 0$$

$$\begin{aligned} y_1(x), y_2(x)&\text{- linear independent} \\ &\text{if } C_1 y_1(x) + C_2 y_2(x) = 0 \\ &\text{if } C_1 = c_2 = 0 \\ \\ &y = \frac{C_2}{C_1} y_2, k = \frac{C_2}{C_1} \\ &y_1 = k y_2 \end{aligned}$$

Theorem

If $y_1(x), y_2(x)$ - are solutions of equation (3) then if (1) y_1, y_2 - linear independent on (a, b) then wronskian:

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0 \text{for all } a < x < b$$

(2) y_1, y_2 - linear dependent, then $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 0$ for all x.

Theorem

If $y_1(x), y_2(x)$ - linear independent solutions of (3), then $y = C, \; y_1 + C_2 y_2(x)$ -generalsolution of (3), where C_1, C_2 - arbitrary constant

5.2 Solving Initial Value Problem (3) + (2)

1. find particular $y_1(x)$, $y_2(x)$
   linear independence
2. Set up general solution of (3)
   - $y = C_1 y_1 + C_2 y_2$
3. Satisfy Initial Condition (2)
   - 1.e subset solution $y = C_1 y_1 + C_2 y_2$

$$\left.\begin{aligned} y(x_0) = C_1 y_1(x_0) + C_2 y_2(x_0) = y_0 \\ y'(x_0) = C_1 y_1'(x_0) + C_2 y_2'(x_0) = y_1 \end{aligned}\right\} C_1, C_2$$

$$\left\{\begin{aligned} a_{11} C_1 + a_{12} C_2 = d_1 \\ a_{21} C_1 + a_{22} C_2 = d_2 \end{aligned}\right| C_1, C_2$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$$

$a_{11}, a12, a21, a22, d_1, d_2$ are given constants

6 Method of Elimination

C_1, C_2 -?

$$\begin{cases} 5C_1 + 3C_2 = 1 \\ C_1 - 2C_2 = 8 \end{cases}$$

$$\begin{cases} 5C_1 + 3C_2 = 1 \\ C_1 - 2C_2 = 8 \end{cases} \Bigg| \cdot (-5)$$

$$(+) \begin{cases} 5C_1 + 3C_2 = 1 \\ -5C_1 + 10C_2 = 8 \end{cases}$$

$$0 + 13C_2 = -39$$

$$\to 13C_2 = -39$$

$$C_2 = -3$$

$$C_1 - 2(-3) = 8$$

$$C_1 = 2$$

7 Method of Substitution

$$\begin{aligned} 5(2C_2 + 8) + 3C_2 &= 1 \\ 13C_2 + 40 &= 1 \\ 13C_2 &= -39 \\ C_2 &= -3 \\ \\ C_1 &= 2C_2 + 8 \\ &= 2(-3) + 8 = 2 \\ C_1 - 2 \cdot (-3) &= 8 \\ C_1 &= 2 \end{aligned}$$

8 Homogeneous Equation with Constant Coefficients

$ay'' + by' + cy = 0$; a, b, c - given constant

$$\begin{aligned} y &= e^{rx}, r - \text{constant} \\ y' &= e^{rx} \cdot r \\ y'' &= e^{rx} \cdot r \cdot r = e^{rx} \cdot r^2 \end{aligned}$$

$$\begin{aligned} a(e^{rx}r^2) + b(e^{rx}r) + c(e^{rx}) = 0 \,\Big|\, \frac{1}{e^{rx}} \\ ar^2 + br + c = 0 \\ r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

$$\begin{aligned} (1) \quad r = r_1, r = r_2\text{- distinct } r_1 \neq r_2 \\ y_1 = e^{r_1 x}, y_2 = e^{r_2 x} \\ \text{-linear independent} \end{aligned}$$

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{vmatrix} \\ &= r_2 e^{r_1 x} e^{r_2 x} - r e^{r_1 x} e^{r_2 x} \\ &= C_1 e^{r_1 x} + C_2 e^{r_2 x} \\ &\quad \text{- general solution} \end{aligned}$$

Example 1

$$\begin{aligned} y'' - 5y' + by = 0 \qquad y = e^{rx} \\ r^2 - 5r + 6 = 0 \\ r_1 = 3, r_2 = 2 \\ y_1 = e^{3x}, y_2 = e^{2x} \end{aligned}$$

Example 2

$$\begin{aligned} y'' + 2y' = 0 \leftarrow y = e^{rx} \\ r^2 + 2r = 0, r(r+2) = 0 \\ r = 0, r = -2 \\ y_1 = e^{0 \cdot x} = 1, y_2 = e^{-2x} \\ \rightarrow y = C_1 + C_2 \cdot e^{-2x} \end{aligned}$$

$$
\begin{aligned}
ay`` + by` + cy = 0 \leftarrow e^{rx} = y \\
ar^2 + br + c = 0 \\
r = r_1, r = r_2, r_1 = r_2 \\
a(r - r_1)^2 = 0 \\
a(r^2 - 2r_1 r + ar^2) = 0 \\
b = (-2ar), c = (ar^2) \\
y_1(x) = e^{r_1 x}, y_2(x) = e^{r_1 x} x \\
y = C_1 e^{r_1 x} + C_2 e^{r_2 x}
\end{aligned}
$$

9 Euler's Equation

$ax^2 y''(x) + bxy'(x) + cy(x) = 0$

Example 2

$$
\begin{aligned}
x^2 y'' + xy' - y &= 0 \\
v &= \ln x, y(v) \\
y'(x) &= \frac{dy}{dv} \cdot \frac{dv}{dx} = \frac{dy}{dv} \cdot \frac{1}{x} \\
y''(x) &= \frac{x}{dx}(y'(x)) \\
&= \frac{d}{dx}\left(\frac{dy}{dv} \cdot \frac{1}{x}\right) \\
&= \frac{d}{dx}\left(\frac{dy}{dv}\right)\frac{1}{x} + \frac{dy}{dv}\left(-\frac{1}{x^2}\right) \\
&= \frac{d}{dv}\left(\frac{dy}{dv}\right)\frac{dv}{dx} \cdot \frac{1}{x} - \frac{1}{x^2} \cdot \frac{dy}{dx} \\
&= \frac{d^2}{dv} \cdot \frac{1}{x} \cdot \frac{1}{x} - \frac{1}{x} \cdot \frac{dy}{dv}
\end{aligned}
$$

$$
\begin{aligned}
x^2\left(\frac{d^2y}{dy^2} \cdot \frac{1}{x} - \frac{dy}{dv} \cdot \frac{1}{x^2}\right) + x\left(\frac{dy}{dv} \cdot \frac{1}{x}\right) - y = 0 \\
\frac{d^2y}{dy^2} - \frac{dy}{dv} + \frac{dy}{dv} - y \\
\frac{d^2y}{dv^2} - y = 0 \leftarrow y(v) = e^{rv} \\
r^2 - 1 = 0, r_1 = 1, r_2 = -1
\end{aligned}
$$

$$
\begin{aligned}
y &= C_1 e^v + C_2 e^{-v} \\
&= C_1 e^v + C_2 e^{\ln x} + C_2 e^{-\ln x} \\
&= C_1 x + \frac{C_2}{x} = y
\end{aligned}
$$

10 Higher Order Differential Equation

$y^{(n)} + p_1(n) y^{(n-1)} + \cdots + p_{n-1}(x) y'(x) + p_n(x) y = f(x)$
$p_1, p_2, p_3, \ldots, p_n$

f-continuous function on (a, b)

if $f(x) \neq 0$ then equation is non-homogeneous
if $f(x) = g(x)$ then equation is homogeneous

$y_(x), y_2(x), \ldots, y_n(x) - n$
particular linear independent solutions of homogeneous equations

then $y(x) = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n$
$C_1, C_2, \ldots, C_n$ - constants

linear independent

$$
\begin{aligned}
W[y_1, y_2, \ldots, y_n] &\neq 0 \\
W[y_1, y_2, \ldots, y_n] &= \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1' & y_2' & y_3' & \cdots & y_n' \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{n-3} & y_3^{n-4} & \cdots & y_n^{(n-1)} \end{vmatrix} \\
&\neq 0
\end{aligned}
$$

$S_1, S_2, S_3, S_4, \ldots, S_n$
- linear independent
if $C_1 f_1(x) + C_2 f_2(x) + \cdots + C_n f_n(x) = 0$
if $C_1 = C_2 = \cdots = C_n = 0$

Problem 3

$$
\begin{aligned}
f(x) = 0, g(x) = \sin x, \ln(x) = e^x \\
C_1 f(a) + C_2 g(x) + C_3 e^x = 0 \\
C_x \cdot 0 + C_2 \sin x + C_3 e^x = 0 \\
C_1 = 100, C_2 = 0, C_3 = 0
\end{aligned}
$$

$$
\begin{aligned}
&\text{Determinate:} \\
W[0, \sin x, e^x] &= \begin{vmatrix} 0 & \sin x & e^x \\ 0 & \cos x & e^x \\ 0 & -\sin x & e^x \end{vmatrix} \\
&\equiv 0
\end{aligned}
$$

Problem 17

$$
\begin{aligned}
y^{(3)} - 3y'' + 3y' - y = 0 \\
y(0) = 2, y'(0) = 0, y''(0) = 0 \\
y_1 = e^x, y_2 = x \cdot e^x, y_3 = x^2 e^x
\end{aligned}
$$

general solution

$$\begin{aligned}
y &= C_1y_1 + C_2y_2 + C_3y_3 \\
&= C_1e^x + C_2xe^x + C_3x^2e^x \\
y(0) &= C_1e^0 + C_20e^0 + C_30^2e^0 \\
&= 2
\end{aligned}$$

$$\begin{aligned}
y'(x) &= C_1e^0 + C_2(e^x + xe^x) + C_3(2xe^x + x^2e^x) \\
y'(0) &= C_1e^0 + C_2(e^0 + 0) + C_3(0 + 0) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
y''(x) &= C_1e^x + C_2(e^x + e^x + e^x) + C_3(2e^x + 2xe^x + 2xe^x + x^2e^x) \\
y''(0) &= C_1e^0 + C_2(e^0 + e^0 + 0) + C_3(3e^0 + 0 + 0 + 0) \\
&= 0
\end{aligned}$$

$$\begin{cases} C_1 = 2 \\ C_1 + 2 = 0 \\ C_1 + 2C_2 + 2C_3 = 0 \end{cases}$$

$$\begin{cases} C_2 = -C_1 = -2 \\ 2C_3 = -C_1 - 2C_2 = -2 - 2(-2) = 2 \\ C_3 = 1 \end{cases}$$

$$y = 2e^x - 2xe^x + x^2e^x$$

(1) $y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1} \cdot y' + p_ny = 0$
homogeneous equation

general solution of (1): $y = C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n$
$y_1, y_2, \ldots, y_3$
-linear independent particular solutions of (1)

(2) $y(x_0) = y_0, y'(x_0) = y_1, \cdots, y^{(n-1)} = y_{n-1}$
-initial conditions
(1) + (2) -Initial Value Problem
(3) $y^{(n)} + p_1(x)y^{(n+1)} + \cdots + p_{n-1}(x)y' + p_n(x) \cdot y = f(x)$
$f(x) \neq 0$ non-homogenous equation

$\rightarrow$ General Solution of (3):

$$\begin{aligned}
y &= C_1y_1 + C_2y_2 + \cdots + C_ny_n + y_p \\
&= (\text{General Solution of (1)}) + (\text{Particular Solution of (3)}) \\
y_c &= C_1y_1 + C_2y_2 + \cdots + C_ny_n \\
&\quad \text{Complementary Solution of (3)}
\end{aligned}$$

$$
\begin{aligned}
y &= y_c + y_p \\
&= (\text{General Solution of } (1)) + (\text{Particular Solution of } (3))
\end{aligned}
$$

$\rightarrow$Initial Value Problem for (3): (3) + (2)

Example 1

$$
\begin{aligned}
y'' - y &= 12x \\
y(0) = 5, y'(0) &= 7
\end{aligned}
$$

(1) homogeneous equation:

$$
\begin{aligned}
y'' - 4y = 0 &\rightarrow y = e^{rx} \\
r^2 \dot{e}^{rx} - 4 = 0 &\,\Big|\, \frac{1}{e^{rx}} \\
r^2 - 4 = 0 & \qquad\qquad\qquad\qquad r = r_1 = 2 \\
& \qquad\qquad\qquad\qquad r = r_2 = -2
\end{aligned}
$$

$$
\begin{gathered}
y_1 = e^{2x}, y_2 = e^{-2x} \\
\rightarrow y = C_1 e^{2x} + C_2 e^{-2x} \\
\text{general solution}
\end{gathered}
$$

(2) non-homogeneous

$$\begin{aligned} y &= y_c + y_p & \qquad\qquad y_p = 3x \\ y_c &= C_1 e^{2x} + C_2 e^{-2x} \\ y_p &= 3x \to 0 - 4(3x) \equiv -12x \\ y &= C_1 e^{2x} + C_2 e^{-2x} + 3x \\ &\text{General Solution } = C_1 e^{2x} + C_2 e^{-2x} \\ &\text{Particular Solution } = 3x \end{aligned}$$

$$\begin{aligned} y(0) &= C_1 e^0 + C_2 e^0 + 3 \cdot 0 = 5 \\ y(0) &= 2C_2 e^0 - 2C_2 e^0 + 3 = 7 \end{aligned}$$

$$\begin{aligned} C_1 + C_2 &= 5 \\ 2C_1 - 2C_2 + 3 &= 7 \end{aligned}$$

$$\begin{aligned} C_1 + C_2 &= 5 \\ C_1 - C_2 &= 2 \end{aligned}$$

$$2C_1 + 0 = 7$$

$$\begin{aligned} C_1 &= \frac{7}{2} \\ C_2 &= 5 - C_1 \\ &= 5 - \frac{7}{2} = \frac{3}{2} \end{aligned}$$

$$y = \frac{7}{2} e^{2x} + \frac{3}{2} e^{-2x} + 3x$$

11 Higher Order Linear Differential Equations

(1) $y^{(1)} + p_1 y^{(n-1)} + \cdots + p_{n-1} y' + p_n y = f(x)$
- non homogeneous equation
general solution y of (1) is $y = y_c + y_p$,
y_p - any particular solution of (1)
y_c - complementary solution which is the general solution of homogeneous equation [(1) if $f(x) \equiv 0$]
$y_c = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n$, where $y_1, y_2, \cdots, y_n$
-linear independent solutions of homogeneous equations

Example 1

$$\begin{aligned} y'' - 4y &= -12x, y_p = 3x \\ y_C &=? \to y'' - 4y = 0 \\ y &= C_1 y_1 + C_2 y_2 = C_1 e^{2x} + C_2 y e^{-2x} \\ y &= y_C + y_P = C_1 e^{2x} + C_2 e^{-2x} + 3x \end{aligned}$$

(2) $y(x_0) = y_0, y'(x_0) = y_1, y''(x_0) = y_2, \ldots, y^{(n-1)}(x_0) = y_{n-1}$
problem (1) + (2) - Initial Value Proplem (IVP) for (1)
$y' = f(x, y), y(x_0) = y_1$
$y'' + p_1 y' + p_2 y = f(x), y(x_0) = 0, y'(x_0) = y_1$

12 Linear Equation with Constant Coefficient

$$
\begin{aligned}
&a_0, a_1, a_2, \ldots, a_n \\
y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1} y' + a_n y& \\
&= \sum_{i=1}^{n} a_{n-1} y^{(i)} = 0 \\
&y = e^{rx} \\
r^n e^{rx} + a_1 r^{n-1} e^{rx} + a_2 r^{n-2} e^{rx} + \cdots + a_{n-1} r e^{rx} + a_n e^{rx} &= 0 \\
\frac{r^n e^{rx} + a_1 r^{n-1} e^{rx} + a_2 r^{n-2} e^{rx} + \cdots + a_{n-1} r e^{rx} + a_n e^{rx}}{e^{rx}} &= 0 \\
r^n + a_1 r^{n-1} + a_2 r^{n-2} + \cdots + a_{n-1} r + a_n = 0& \\
\text{-characteristic equation}& \\
r = r_1 (r - r_1)&
\end{aligned}
$$

12.1 Rules for Characteristic Equation

(1) roots are n distinct real numbers

$$
\begin{aligned}
&r = r_1, r = r_2, \cdots, r = r_n \\
&\to y_1 = e^{r_1 x}, y_2 = e^{r_2 x}, y_3 = e^{r_3 x}, \cdots, y_n = e^{r_n x} \\
&y_n = e^{r_n x} \\
&y = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \cdots + C_n e^{r_n x} \\
&= \sum_{z=1}^{n} C_1 e^{r_1 x}
\end{aligned}
$$

(2) $r = r_s, r = r_s, \cdots, r = r_s$ for k times

$$
\begin{aligned}
&y_1 = e^{r_2 x}, y_2 = x \cdot e^{r_s \cdot x}, y_3 = x^2 \cdot e^{r_s x}, \ldots, y_k = x^{k-1} e^{r_s x} \\
&\text{multiplicity of } k
\end{aligned}
$$

Example 2

$$
\begin{aligned}
y^{(3)} + 3y'' + 3y' + y = 0 &\leftarrow y e^{rx} \\
r^3 e^{rx} + 3r^2 e^{rx} + 3r e^{rx} + e^{rx} &= 0 \\
\frac{r^3 e^{rx} + 3r^2 e^{rx} + 3r e^{rx} + e^{rx}}{e^{rx}} &= \frac{0}{e^{rx}} \\
r^3 + 3r^2 + 3r + 1 &= 0
\end{aligned}
$$

General Rule of Algebraic Polynomials

- root is such number that divides all values as an integer

Long Division
$(r+1)^3 = 0$
$(r+1)(r+1)(r+1) = 0$
$r = r_1 = -1, r = r_2 = -1, r = r_3 = -1$

One Repeated Root of Multiplicity 3
$y_1 = e^{-x}, y_2 = xe^{-x}, y_3 = x^2e^{-x}$
$y = C_1e^{-x} + C_2e^{-x}x + C_3e^{-x}x^2$

Example 3 (P. 134 #26)

$$\begin{aligned}
y^{(3)} + 10y'' + 25y' &= 0; y(0) = 3, y'(0) = 4, y''(0) = 5 \\
&\to y = e^{rx} \\
r^3e^{ex} + 10r^2e^{rx} + 25re^{rx} &= 0 \\
\frac{r^3e^{ex} + 10r^2e^{rx} + 25re^{rx}}{e^{rx}} &= \frac{0}{e^{rx}} \\
\to r^3 + 10r^2 + 25r &= 0 \\
r(r^2 + 10r + 25) &= 0 \\
r(r+5)^2 &= 0 \\
&\text{repeated root of multiplicity 2} \\
r &= r_1 = 0, r = r_2 = -5, r = r_3 = -5 \\
y_1 &= e^0, y_2 = e^{-5x}, y_3 = e^{-5x}x \\
y &= C_1 + C_2e^5x + C_3e^{-5x}x
\end{aligned}$$

$$\begin{aligned}
y'(x) &= C_2(-5)e^{-5x} + C_3[e^{-5x} - 5xe^{-5x}] \\
y''(x) &= C_225e^{-5x} + C_3[-5e^{-5x} - 5e^{-5x} + 25xe^{-5x}]
\end{aligned}$$

Initial Conditions

$$\begin{cases}
y(0) = C_2 + C_2e^0 + C_3 \cdot 0 = 3 \\
y'(0) = -5C_2e^0 + C_3[e^0 - 0] = 4 \\
y''(0) = 25C_2e^0 + C_3[-5e^02 + 0] = 5
\end{cases}$$

$$\begin{cases}
C_1 + C_2 = 3 \\
-5C_2 + C_3 = 4 \\
25C_2 - 10C_3 = 5
\end{cases}$$

$$\begin{aligned}
0 - 5C_3 &= 25 \\
25C_2 &= 5 + 10C_3 = 5 - 50 \\
C_2 &= \frac{1}{5} - 10 = C_1\frac{4}{1} \\
C_1 &= 3 - C_2
\end{aligned}$$

13 Complex Roots of Charatistics Equations

Imaginary Number $\to \sqrt{-1} = i$
Complex Number $\to 3 \pm 4\sqrt{-1} = 3 \pm 4i$

Real Number $\rightarrow \text{Re}[3 \pm 4i] = 3$
Imaginary Number $\rightarrow \text{Re}[3 \pm 4i] = \pm 4$

Complex Number Plane

$$\begin{aligned} ay'' + by' + cy = 0, y = e^{rx} \\ ar^2 + br + c = 0 \\ r_{1,2} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}, b^2 - 4ac = 0 \\ r = \alpha \pm i\beta, \sqrt{-1} = i \\ y_1 = e^{(\alpha + i\beta)x}, y_2 = e^{(\alpha - i\beta)x} \end{aligned}$$

13.1 Euler Formula

$$\begin{aligned} e^{i\beta x} &= \cos(\beta x) + i\sin(\beta x) \\ e^{-i\beta x} &= \cos(\beta x) - i\sin(\beta x) \end{aligned}$$

$$\begin{aligned} y_1 &= e^{(\alpha+\beta i)x} & &= e^{\alpha x} e^{\beta i} \\ &= e^{\alpha x}(\cos \beta x + i \sin \beta x) \end{aligned}$$

$$\begin{aligned} y = u(x) + iw(x) \\ a(u + iw)'' + b(u + iw)' + c(u + w) \equiv 0 \\ (a''bu + cu) + (aw'' + bw' + cw)i \equiv 0 \\ 0 + 0 = 0 \end{aligned}$$

$$\begin{aligned} y_1 = e^{\alpha x} \cos \beta x, y_1 = e^{\alpha x} \sin \beta x \\ y = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x \end{aligned}$$

14 Method of Undetermined Coefficients

$$\begin{aligned} (r^2 + 4)(ar^2 + br + c) = 0 \\ &\equiv 6r^4 + 5r^3 + 25r^2 + 20r + 4 \\ &\equiv ar^4 + br^3 + cr^2 + 4ar^2 + 4br + 4c \end{aligned}$$

$$\begin{aligned} r^4 &: 6 = a \\ r^3 &: 5 = b \\ r^2 &: 25 = c + 4a \\ r &: 20 = 4b \\ r^0 &: 4 = 4C \rightarrow c = 1 \end{aligned}$$

$$(r^2+4)(6r^2+5r+1)=0$$

15 Non-Homogeneous Equation

$y^{(n)} + p_1 y^{n-1} + \cdots + p_{n-1}y' + p_n y = f(x)$
$f(x) \neq 0$
general solution $y = y_c + y_p$
y_c - general solution of homogeneous ($f(x) \equiv 0$) equation
y_p - any particular solution of non-homogeneous equation

16 Method of Undetermined Coefficients

$$\begin{aligned}(1) \quad f(x) &= \sum_{k-0} nA_k x^k \\ \to y_p &= \sum_{k=0}^{n} B_k x^k\end{aligned}$$

Example 1

$$\begin{aligned} y'' + 3y' + 4y &= 3x+2 \\ \to y_p &= Ax+B \\ (A+B)'' + 3(Ax+B)' + 4(Ax+B) &= 3x+2 \\ 0 + 3A + 4(Ax+B) &= 3x+2 \end{aligned}$$

$$\begin{aligned} x : 4A = 3 \to A &= \frac{3}{4} \\ x^0 : 3A + 4B &= 2 \\ B = \frac{2-3A}{4} &= \frac{2-3-\frac{3}{4}}{4} \\ &= -\frac{1}{16} \end{aligned}$$

$$\begin{aligned} (2) \quad f(x) &= a\sin kx + b\cos kx \\ y_p &= A\sin kx + B\cos kx \\ &A, Bundetermined \end{aligned}$$

Example 2

$$
\begin{aligned}
y' - 2y &= 2\cos x \\
\nwarrow y_p &= A\sin x + B\cos x \\
3(A\sin x + B\cos x)'' + (A\sin x + B\cos x) &= 2\cos x \\
3(-A\sin x + B\cos x) &= 2\cos x
\end{aligned}
$$

$$
\left.\begin{aligned}
\cos x: -B + A - 2B = 2 \\
\sin x: -3A - B - 2A = 0
\end{aligned}\right\}
$$

$$
\begin{aligned}
A - 5B &= 2 \\
-5 - B &= 0
\end{aligned}
$$

$$
\begin{aligned}
B &= -5A \\
A + 25 &= 2 \\
A &= \frac{1}{13}
\end{aligned}
$$

$$
\begin{aligned}
B &= -\frac{5}{13} \\
y_p &= \frac{1}{13}\sin x - \frac{5}{13}\cos x
\end{aligned}
$$

$$
(3) \quad f(x) = e^{px}\sum_{k=0}^{n} A_k x^k, y_p = e^{px}\sum_{k=0}^{n} B_k x^k
$$

$$
B_k \text{ - undetermined}
$$

Example 3

$$
\begin{aligned}
y'' - 4y &= 2e^{3x}, y_p = Be^{3x} \\
(Be^{3x})'' - 4Be^{3x} &= 2e^{3x} \\
9Be^{3x} - 4Be^{3x} &= 2e^{3x} \\
\rightarrow 9B - 4B &= 2, B = \frac{2}{5}
\end{aligned}
$$

Example 4

$$
\begin{aligned}
y'' - 4y &= 3x^2e^{3x} \\
y_p &= (Ax^2 + B_x + C)e^{3x}
\end{aligned}
$$

Example 5

$$\begin{aligned} y'' - 4y &= 2 \cdot e^{2x} \\ \nwarrow y_p &= A \cdot e^{2x} \\ (A \cdot e^{2x})'' - 4Ae^{2x} &= 2e^{2x} \\ 4Ae^{2x} - 4Ae^{2x} &= 2e^{2x} \\ 0 &= 2 \cdot e^{2x} \end{aligned}$$

$$\begin{aligned} y'' - 4y &= 0 \\ r^2 - 4 &= 0 \\ r &= \pm 2 \\ y_p &= A_x \cdot e^{2x} \end{aligned}$$

$$\begin{aligned} y^{(n)} + \cdots + p_n y &= f(x) \\ y &= y_c + y_p \end{aligned}$$

(1) $f(x) = \sum_{k=0}^{n} a_k x^n, y_p = \sum_{n=0}^{n} A_k + x^k$ when $r = 0$ is not a root of characteristic equation
$\rightarrow$ n-degree polynomial

$y_p = \left(\sum_{k=0}^{n} a_k x^k \right) x^m$
if $r = p$ is a root of characteristic equation of multiplicity m

(2) $f(x) = e^{px} \sum_{k=0}^{n} a_k x^k, y_p = e^{px} \sum_{k=0}^{n} A_k x^k$ if $r = p$ is not a root of characteristic equation

$y_p = \left(e^{px} \sum_{k=0}^{n} A_k x^k \right) x^m$ if $r = p$ is a root of characteristic equation of multiplicity m

(3) $f(x) = e^{px}(p(x) \cos qx + Q_m(x) \sin(qx))$,
$y_p = e^p x(\overline{p_k}(x) \cos(qx) + \overline{Q_k}(x) \sin(qx))$ where $k = mx(n, m)$, if $r \neq p + iq$

$y_p = e^{px}(\overline{p_k}(x) \cos(qx) + \overline{Q_k}(x) \sin(qx))x^m$ of multiplicity m, then

$y^{(n)} + \cdots + p_n y = f(x)$
(1) $f(x) = e^{px}(p_n(x) \cos qx + Q_m(x) \sin qx)$
$y_p = e^{px}(\overline{p_k}(x) \cos qx + \overline{Q_k}(x) \sin qx)x^m$
$k = max(n, m)$

$y_p = e^{px}(\overline{p_k} \cos qx + \overline{Q_k}(x) \sin qx)x^m$
$\rightarrow$ if $p + iq = r$ - root of characteristic equation of multiplicity "m"

(2) $f(x) = e^{px}, P_k(x)$
$y_p = e^{px} \cdot Q_k(x), p = 0$

$y_p = e^{px} Q_n(x) x^m$
if $p = r$ -root of characteristic equation of multiplicity, "m"

if $f(x)$ ith sum of the above functions,

$$\begin{aligned} y^{(n)} + \cdots + p_n y = f_1(x) + f_2(x) &\to y_p = (y_1)_p + (y_2)_p \\ (1) \quad y_1^{(k)} + \cdots + p_k y_1 = f_1(x) &\to (y_1)_p \\ (2) \quad y_2^{(k)} + \cdots + p_k y_2 = f_2(x) &\to (y_2)_p \end{aligned}$$

(1) $\quad f(x) = e^{px}[p_k(x)\sin(qx) + Q_m(x)\cos(qx)]$
$y_p = e^{px}[\overline{p_k}(x)\sin qx + \overline{Q_k}(x)\cos(qx)]x^m$

(2) $\quad f(x) = e^{px} \cdot P_k(x), p = r, m$
$y = e^{px}\frac{1}{P_k(x)} \cdot x^m$

Problem 25

$$y'' + 3y' + 2y = xe^{-x} - e^{2x}$$

$$y_p = (y_1)_p = (y_2)_p$$

$$\begin{aligned} 1. \quad y_1'' + 3y_1' + 2y_1 = x \cdot e^{-x} &= f_1(x) \\ (y_1)_p &= (ax+b)e^{-x} \end{aligned}$$

$$\begin{aligned} 2. \quad y_2'' + 3y_2' + 2y_2 = xe^{-2x} &= f_2(x) \\ (y_2)_p &= (cx+d)e^{-2} \end{aligned}$$

$$\begin{aligned} y = e^{rx} \to y'' + 3y' + 2y &= 0 \\ r^2 + 3r + 2 &= 0 \\ (r+1)(r+2) &= 0 \\ m = 1, \quad r &= -1 \\ r &= -2 \end{aligned}$$

Example 1

$$\begin{aligned} (y_1)_p &= (ax+b)e^{-x} \\ &= (ax^2+bx)e^{-x} \\ (y_1)p' &= (2ax+b)e^{-x} - (ax^2+bx)e^{-x} \\ (y_1)p'' &= 2ae^{-x} - e^{-x}(2a+b) - (2ax+b)e^{-x} + e^{-x}(ax^2+b) \\ &= 2ae^{-x} - 2e^{-x}(2ax+b) + e^{-x}(ax^2+bx) \\ &= 2ae^{-x} - 2e^{-x}(ax+b) + e^{-x}(ax^2+bx) + 3[(2ax+b)e^{-x} - (ax^2+bx)e^{-x}] + 2(ax^2+bx)e^{-x} \\ &= x \cdot e^{-x} \end{aligned}$$

$$\begin{aligned} x^2 &: a - a = 0 \\ x &: -2a + b + ba - 3b + 2b = 1 \\ x^0 &: 2a - 2b + 3b = 0 \end{aligned}$$

$$4a = 1 \to a = \frac{1}{4}$$
$$2a + b = 0 \to b = -2a = -\frac{1}{2}$$
$$(y_1)_p = (\frac{1}{4}x - \frac{1}{2})e^{-x} \cdot x$$

$$\begin{aligned} y_p &= (y_1)_p + (y_2)_p \\ &= (ax^2 + bx)e^{-x} + (cx + d)e^{-2x} \cdot x \end{aligned}$$

17 Method of Variation of Parameters

$y'' + P(x)y' + Q(x)y = f(x), y_p = ?$

<u>1 step</u> We solve $y'' + P(x)y' + Q(x)y = 0 \quad ^{(2)}$ we find solutions $y_1(x), y_2(x)$

- linear independent

$y = C_1 y_1(x) + C_2 y_2(x)$ - general (2) solution of (2)

<u>2 step</u> We'll search for solution $y_p \quad ^{(1)}$ of (1) as $y_p = u_1(x) \cdot y_1(x) + y_2(x) y_2(x)$

$$\begin{aligned} y_p' &= u_1'(x) \cdot y_1(x) + u_1(x) \cdot y_1'(x) + u_2'(x) y_2(x) + u_2(x) \cdot y_2'(x) \\ y_p' &= u_1(x) y_1'(x) + u_2(x) y_2'(x) + u_1' y_1(x) + u_2'(x) y_2(x) \end{aligned}$$

Add restrictions
$u_1'(x) y_1(x) + u_2'(x) y_2(x) = 0$

$$\begin{aligned} y_p' &= u_1(x) y'(x) + u_2(x) y_2'(x) \\ (y_p)'' &= u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 \cdot y_2'' \end{aligned}$$

$$y_p' = u_1(x) y_1'(x) + u_2(x) y_2'(x)$$

from (1) we have:

$$\begin{aligned} f(x) &= u_1' y_1' + u_1 \cdot y_2'' + u_2' y_2' + u_2 y_2'' + P(x)(u_1 y_1' + u_2 y_2') + Q(x)(u_1 y_1 + u_2 y_2) \\ y_p'' &= u_1' y_1' + u_1 \cdot y_2'' + u_2' y_2' + u_2 y_2'' \\ y_p' &= u_1 y_1' + u_2 y_2' \\ y_p &= u_1 y_1 + u_2 y_2 \end{aligned}$$

$$\begin{aligned} u_1[y_1'' + p y_1' + Q y 1] + u_2[y_2'' + P \cdot y_2' + Q y_2] + u_1' y_1' + u_2' y + 2' &= f(x) \\ y_1'' + p y_1' + Q y 1 &= 0 \\ y_2'' + P \cdot y_2' + Q y_2 &= 0 \end{aligned}$$

$$(3) \to u_1' = \int \phi(x) dx y_1 + \int \psi(x) dx \cdot y_2(x)$$

$\nwarrow$ parameter solution of (1)

Example 1
$y'' + y = \tan x$

$$\begin{aligned} \underline{\text{1 step}} \quad & y'' + y = 0 \leftarrow y = e^{rx} \\ \rightarrow & r^2 + 1 = 0, r_1, \\ \rightarrow & y_1 = \cos x, y_2 = \sin x \\ & y = C_1 \cos x + C_2 \sin x \end{aligned}$$

<u>2 step</u> $\quad y_p = u_1(x) \cdot \cos x + u_2(x) \sin x$ system (3) in this case is, following

$$\begin{cases} u_1' \cos x + u_2' \sin x = 0 \\ u_1'(\sin x) + u_2'(\cos x) = \tan x \end{cases} \quad \Big| \begin{matrix} \sin x \\ \cos x \end{matrix} (+)$$

$$0 + u_2'(\sin^2 x + \cos^2 x) = \tan x \cos x$$

$$
\begin{aligned}
u_2' &= \sin x \\
u_1' &= -u_2' \cdot \frac{\sin x}{\cos x} \\
&= -\frac{\sin x \sin x}{\cos x}
\end{aligned}
$$

$$
\begin{aligned}
u_2 &= \int u_2' dx \\
&= \int \sin x dx \\
&= -\cos x
\end{aligned}
$$

$$
\begin{aligned}
u_1 &= \int u_1' dx \\
&= \int \frac{\sin^2 x}{\cos x} dx \\
&= -\frac{\sin^2 x}{\cos^2 x} \cos x dx \\
dv &= \cos x dx
\end{aligned}
$$

$$
\begin{aligned}
v &= \sin x, dv = \cos x dx \\
&= -\int \frac{v^2 dv}{1 - v^2} \\
&= -\int \frac{(v^2 - 1 + 1)}{v^2 - 1} dv \\
&= \int dv + \int \frac{dv}{v^2 - 1} \\
&\quad (v-1)(v+1) = v^2 - 1 \\
&= \int dv + \int \frac{dv}{(v-1)(v+1)} \\
&= v + \int \frac{1}{2} \left[\frac{1}{v-1} - \frac{1}{v+1} \right] dv
\end{aligned}
$$

$$
\begin{aligned}
u_1 &= v + \frac{1}{2}[\ln|v-1| - \ln|v+1|] \\
&= \sin x + \frac{1}{2} \ln \left| \frac{\sin x - 1}{\sin x + 1} \right|
\end{aligned}
$$

$$
\begin{aligned}
y_p &= u_1 y_1 + u_2 y_2 \\
&= \left(\sin + \frac{1}{2} \ln \left| \frac{\sin x - 1}{\sin x + 1} \right| \right) \cos x - \cos x \sin x
\end{aligned}
$$

18 Variation of Parameters

$$
\begin{aligned}
y'' + P(x)y' + Q(x)y &= f(x) \\
y_p &= ?
\end{aligned}
$$

Step 1

$$y'' + P(x)y' + Q(x) = 0, y_1(x), y_2(x)$$
$$y_c = C_1 y_1(x) + C_2 y_2(x)$$

Step 2

$$y_p = u_1(x)y_1(x) + u_2(x)y_2$$
$$\left\{\begin{matrix} u_1'(x) \cdot y_1(x) + u_2'(x) \cdot y_2(x) = 0 \\ u_1'(x)y_1'(x) + u_2'(x)y_2'(x) = f(x) \end{matrix}\right\}$$
$$u_1'(x) = \rho(x), u_2'(x) = \psi(x)$$

Problem 49

$y'' - 4y' + 4y = 2e^{2x}$

Step 1

$$y'' - 4y' + 4y = 0$$
$$y = e^{rx}$$
$$r^2 - 4r + 4 = 0$$
$$(r-2)^2 = 0$$
$$r_1 = 2, r_2 = 2$$
$$\rightarrow y_1 = e^{2x}, y_2 = e^{2x} \cdot x$$

Step 2

$$
\begin{aligned}
y_p &= u_1(x) \cdot y_1(x) + u_2(x) y_2 \\
&= u_1(x) \cdot e^{2x} + u_2(x) e^{2x} x \\
&\begin{cases} u_1' e^{2x} + u_2' e^{2x} = 0 \\ u_1' \cdot 2e^{2x} + u_2'(x)(2e^{2x}x + e^{2x}) = 2e^{2x} \end{cases} \qquad \Big| \, \frac{1}{e^{2x}} \\
&\begin{cases} u_1' + u_2' x = 0 \\ 2u_1' + u_2'(2x+1) = 0 \end{cases} \\
0 + u_2'(2x - 2x - 1) &= -2 \\
\\
u_1' &= 2 \\
\\
u_1' &= -2 \\
u &= \int u_1'(x) dx \\
&= -2 \int x dx \\
&= \frac{-2x}{2} \\
\\
u_2 &= \int u_2'(x) dx \\
&= \int 2dx \\
&= 2x \\
y_p &= -x^2 e^{2x} + 2xe^{2x} x \\
&= x^2 e^{2x}
\end{aligned}
$$

19 Linear Systems of First Order Differential Equations

$$
\begin{aligned}
&y_1(t), y_2(t), \cdots, y_n(t) - \text{unknown} \\
&\frac{dy_1}{dt} = p_{11} y_1 + p_{12} y_2 + \cdots + p_{1n} y_n + f_1(t) \\
&\frac{dy_2}{dt} = p_{21} y_1 + p_{22} y_2 + \cdots + p_{2n} y_n + f_2(t) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\frac{dy_n}{dt} = p_{n1} y_1 + p_{n2} y_2 + \cdots + p_{nn} y_n + f_n(t)
\end{aligned}
$$

$p_{2i}(t)$ -given
$f_i(t)$ -given

$$
Y'(t) = \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix}, Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n' \end{bmatrix}, A = \begin{bmatrix} p_{11} & p_{12} & \cdots & y_{1n} \\ p_{21} & p_{22} & \cdots & y_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ p_{1n} & p_{2n} & \cdots & y_{nn} \end{bmatrix}, F = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}
$$

$y(0) = y_{10}, y_2(0) = y_{20}, \cdots, y_n(0) = y_{n0}$
Initial Conditions

20 Systems of 2 Equations for $x(t), y(t)$

$$\frac{dx}{dt} = a_{11}x + a_{12}y + f(t)$$
$$\frac{dy}{dt} = a_{21}x + a_{22}y + f(t)$$

$$Y = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, Y' = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, F = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \\ &= \begin{bmatrix} a_{11}x + a_{12}y \\ a_{12}x + a_{22}y \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \\ &= \begin{bmatrix} a_{11}x + a_{12}y + f_1 \\ a_{12}x + a_{22}y + f_2 \end{bmatrix} \end{aligned}$$

$$x' = a_{11} + a_{12}y + f_1$$
$$y' = a_{22} + a_{22}y + f_2$$

Example
$ay'' + by' + cy = t^2$
$\rightarrow$ second order differential equation
$y' = x$

$$ax' + bx + cy = t^2$$
$$x' = \frac{-bx - cy + t^2}{a}$$
$$\begin{cases} x' = -\frac{bx}{a} - \frac{c}{a}y + \frac{t^2}{a} \quad x(t) \\ y' = x \end{cases}$$

21 Method of Elimination

Example 1

$$
\begin{aligned}
x' &= -2xy, y' = \frac{1}{2}x \\
&\downarrow \\
(x'(t)) &= (-2y)' \leftarrow \text{derivative with respect to } t \\
x''(t) &= -2y'(t) \\
y'(t) &= -\frac{x''(t)}{2}
\end{aligned}
$$

$$
\begin{aligned}
-\frac{x''(t)}{2} &= \frac{1}{2}x \\
\rightarrow x''(t) + x &= 0 \leftarrow xe^{rt} \\
r^2 + 1 &= 0, r_{1,2} = \pm 2 \\
x_1 &= \cos t, x_2 = \sin t
\end{aligned}
$$

$$
\begin{aligned}
x &= C_1x_1 + C_2x_2 \\
&= C_1 \cos t + C_2 \sin t \\
y &= -\frac{x'(t)}{2} \\
&= -\frac{1}{2}(-C_1 \sin t + C_2 \cos t) \\
&= \frac{C_1}{2} \sin t + \frac{C_2}{2} \cos t
\end{aligned}
$$

Using Initial Conditions

$$
\begin{aligned}
x(0) &= C_1 \cos 0 + C_2 \sin 0 = 0 \\
y(0) &= \frac{C_1}{2} \sin 0 - \frac{C_2}{2} \cos 0 = 0
\end{aligned}
$$

$$
\begin{aligned}
C_1 &= 2 \\
C_2 &= 0
\end{aligned}
$$

$$
\begin{aligned}
x &= 2 \cos t \\
y &= \frac{2}{2} \sin t
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{x}{2}\right)^2 &= (\cos t)^2 \\
\underline{(y)^2 = (\sin t)^2} \\
\left(\frac{x}{2}\right)^2 + y^2 &= \cos^2 t + \sin^2 t = 1
\end{aligned}
$$

$$
\frac{y^2}{2^2} + \frac{y^2}{2^2} = 1
$$

Example 2

$$(1)x'(t) = 4x - 3y$$
$$(2)y'(t) = 6x - 7y$$

$$\text{from } (2)x = \frac{y'(t) + 7y}{6}$$
$$x'(t) = \frac{y'' + 7y'}{6}$$
$$\text{from } (1)\frac{y'' + 7y'}{6} = 4\frac{y' + 7y}{6} - 3y$$

$$y'' + 7y' = 4y' + 28y - 18y$$
$$y'' + 3y = 0 \leftarrow y = e^{rt}$$
$$r^2 + 3r - 10 = 0$$
$$r_1 = 2, r_2 = -5$$

$$y_1 = e^{2t}$$
$$y_2 = r^{-5t}$$

$$y = C_1e^{2t} + C_2e^{-5t}$$

$$x = \frac{2C_1e^{2t} - 5C_2e^{-5t} + 7(C_1e^{2t} + C_2e^{-5t})}{6}$$
$$= \frac{9C_1e^{2t}}{6} + \frac{2C_2e^{-5t}}{6}$$
$$= x$$

$$y = C_1e^{2t} + C_2e^{-5t}$$

22 System of first order Differential Equations

$$\overline{X}' = \overline{A} \cdot \overline{X} \to \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

solutions

$$\begin{cases} x' = a_{11}x + a_{12}y \\ y' = a_{21}x + a_{22}y \end{cases}$$

$$\begin{cases} x' = C_1x_1 + C_2x_2 \\ y' = C_1y_1 + C_2y_2 \end{cases}$$

$$\overline{X} = C_1\overline{X_1} + C_2\overline{X_2}$$

$$\text{where } \overline{X} = \begin{bmatrix} x \\ y \end{bmatrix}, \overline{X_1} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \overline{X_2} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$\overline{X_1}, \overline{X_2}$ particular solution of system 1
$\overline{X_1}, \overline{X_2}$ - linear independent or $C_1\overline{X_1} + C_2\overline{X_2} = 0$, only if $C_1 = C_2 = 0$

$$W = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \neq 0$$

Example 1

$$\overline{X'} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \overline{X}(t)$$
$$\overline{X_1} = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}, \overline{X_2} \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$$
$$\overline{X_1}, \overline{X_2} - \text{linear independent}$$

Substitute $\overline{X_1}$, into the system gives

$$\begin{bmatrix} (3e^{2t})' \\ (2e^{2t})' \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$$
$$\begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix} = \begin{bmatrix} 4 \cdot 3 \cdot e^{2t} + (-3) \cdot 3 \cdot e^{2t} \\ 6 \cdot 2 \cdot e^{2t} + (-7) \cdot 2 \cdot e^{2t} \end{bmatrix}$$
$$= \begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix}$$

Substitute $\overline{X_2}$ into the system:

$$\begin{bmatrix} (e^{-5t})' \\ (3e^{-5t})' \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$$
$$\begin{bmatrix} -5e^{-5t} \\ -15e^{-5t} \end{bmatrix} = \begin{bmatrix} 4e^{-5t} + (-3)3e^{-5t} \\ 6e^{-5t} + (-7)3e^{-5t} \end{bmatrix}$$
$$= \begin{bmatrix} -5e^{-5t} \\ -15e^{-5t} \end{bmatrix}$$

$$W = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = \begin{vmatrix} 3e^{2t} & e^{-5t} \\ 2e^{2t} & 3e^{-5t} \end{vmatrix}$$
$$= 3e^{2t} \cdot 3e^{-5t} - 2e^{2t} \cdot e^{-5t}$$
$$= 9e^{-3t} - 2e^{-3t}$$
$$= 7e^{-3t} \neq 0$$

Linear Independent

$$\overline{X} = C_1 \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} + \begin{bmatrix} C_2 e^{-5t} \\ 3e^{-5t} \end{bmatrix}$$
$$x = 3C_1 e^{2t} + C_2 e^{-5t}$$
$$y = 2C_1 e^{2t} + 3C_2 e^{-5t}$$

23 Method of Eigenvalues

$y'' + ay' + b = 0, y = e^{rt}$

$$\overline{X'} = \overline{A} \cdot \overline{X} \leftarrow \overline{X} = \overline{V} \cdot e^{\lambda t}, \overline{V} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$
\begin{aligned}
\overline{V} \cdot \lambda e^{\lambda t} &= \overline{A} \cdot \overline{V} \\
\rightarrow \overline{AV} - \lambda \overline{V} &= 0 \\
\overline{V} &= \overline{IV}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \\
\overline{A} \cdot \overline{V} - \lambda I \overline{V} &= 0 \leftarrow \text{ (cannot subtract constant from matrix)} \\
(\overline{A} - \lambda I) \cdot \overline{V} &= 0 \\
|(\overline{A} - \lambda I)| &= 0 \leftarrow \text{ (infintely many solutions)} \\
\rightarrow \left| \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| &= 0 \\
\rightarrow \left| \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} \right| &= 0 \\
\rightarrow \left| \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{12} & a_{22} - \lambda \end{bmatrix} \right| &= 0 \\
(a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} &= 0 \\
\lambda^2 - \lambda(a_{11} + a_{22}) + a_{11}a_{12} - a_{12}a_{22} &= 0
\end{aligned}
$$

(2) Two distinct roots of characteristic equations $\lambda = \lambda_1$, $\lambda = \lambda_2$, $\lambda_1 \neq \lambda_2$ substitute λ in $(\overline{A} \rightarrow I)\overline{V} = 0$

$$
\begin{aligned}
\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \right) \begin{bmatrix} a \\ b \end{bmatrix} &= 0 \\
\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} &= 0 \\
\begin{cases} (a_{11} - \lambda) + a_{12}b = 0 \\ a_{21}a + (a_{22} - \lambda_1)b = 0 \end{cases}
\end{aligned}
$$

Problem 13

$$
\begin{aligned}
&\begin{cases} x' = 2x + 4y + 3 \cdot e^t \\ y' = 5x - y - t^2 \end{cases} \\
\overline{X} &= \begin{bmatrix} x \\ y \end{bmatrix}, \overline{F} \begin{bmatrix} 3e^t \\ -t \end{bmatrix} \\
\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 2 & 4 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3e^t \\ -t^2 \end{bmatrix} \\
\overline{X'} &= \begin{bmatrix} 2 & 4 \\ 5 & -1 \end{bmatrix} \cdot \overline{X} + \overline{F} \\
\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 2x + 4y \\ 5x - y \end{bmatrix} + \begin{bmatrix} 3e^t \\ -t^2 \end{bmatrix} \\
&= \begin{bmatrix} 2x + 4y + 3e^t \\ 5x - y - t^2 \end{bmatrix} \\
&\begin{cases} x' = 2x + 4y + 3e^t \\ y' = 5x - y - t^2 \end{cases}
\end{aligned}
$$

24 Method of Eigenvalues

$$\overline{X'} = \overline{A}\cdot\overline{X} \leftarrow \overline{X} = \overline{V}e^{\lambda t}, \overline{V} = \begin{bmatrix} a \\ b \end{bmatrix} \leftarrow \text{constant vector}$$

$$\overline{V}(e^{\lambda t})' = \overline{A}\cdot\overline{V}e^{\lambda t}$$

$$\lambda\overline{V}\cancel{e^{\lambda t}} = \overline{A}\cdot\overline{V}\cancel{e^{\lambda t}}$$

$$\rightarrow \overline{AV} - \lambda\overline{V} = 0$$

$$\rightarrow \overline{AV} - \lambda\overline{IV} = 0$$

$$\overline{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\rightarrow (\overline{A} - \lambda\overline{I}) - \overline{V} = \overline{0}$$

$$\overline{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\rightarrow \Bigg(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Bigg) \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} (a_{11} - \lambda)a + a_{12}b \\ a_{21}a + (a_{22} - \lambda)b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} (a_{11} - \lambda)a + a_{12}b = 0 \\ a_{21}a + (a_{22} - \lambda)b = 0 \end{cases} \quad (a, b)?$$

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0$$

$$(a_{11} - \lambda)(a_{22} - \lambda) - a_{21}a_{12} = 0$$

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{21}a_{12} = 0$$

$$(1)\lambda = \lambda_1, \lambda = \lambda_2, \lambda_1 \neq \lambda_2$$
$$(2)\lambda_1 = \lambda_2$$
$$(3)\lambda_1, \lambda_2 - \text{ complex}$$

$$\overline{X_1} = \overline{V_1}e^{\lambda t}$$

Problem 3

$$\begin{cases} x_1' = 3x_1 + 4x_2 & x_1(0) = 1 \\ x_2' = 3x_1 + 2x_2 & x_2(0) = 1 \end{cases}$$

$$x_1(t), x_2(t)?$$

$$\overline{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, A = \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix}$$

$$\rightarrow \overline{X'} = \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix}\overline{X}$$

substitute $\overline{X} = \begin{bmatrix} a \\ b \end{bmatrix} e^{\lambda t}$ after determining λ

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0$$
$$\begin{vmatrix} 3 - \lambda & 4 \\ 3 & 2 - \lambda \end{vmatrix} = 0$$
$$(3 - \lambda)(2 - \lambda) - 3 - 4 = 0$$
$$\lambda^2 - 5\lambda - 6 = 0$$
$$\lambda = \lambda_1 = 6, \qquad \lambda = \lambda_2 = -1$$

$$\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$
$$\begin{cases} (a_{11} - \lambda)a + a_{12}b = 0 \\ a_{21}a + (a_{22} - \lambda)b = 0 \end{cases}$$
$$\downarrow$$
$$\begin{cases} (3 - \lambda)a + 4b = 0 \\ 3a + (2 - \lambda)b = 0 \end{cases}$$

(2) $\lambda = \lambda_1 = b$

$$\begin{cases} (3 - 6)a + 4b = 0 \\ 3a + (2 - 6)b = 0 \end{cases}$$
$$\begin{cases} -3a + 4b = 0 \\ 3a - 4b = 0 \end{cases}$$

Test for valid equations, expect unlimited solutions
Chose any value for a and b

$$a = 4, b = 3$$
$$\rightarrow \begin{bmatrix} 4 \\ 3 \end{bmatrix} \rightarrow \overline{X_1} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} e^{\lambda_1 t}$$
$$= \begin{bmatrix} 4 \\ 3 \end{bmatrix} e^{6t}$$

(3) $\lambda = \lambda_2 = -1$

$$\begin{cases} (3 - (-1))a + 4b = 0 \\ 3a + (3 - (1))b = 0 \end{cases}$$
$$\begin{cases} 4a + 4b = 0 \\ 3a + 3b = 0 \end{cases}$$

Test for valid equation

choose values for a and b

$$\begin{aligned}
a &= -1, a = 1 \\
\overline{X_2} &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{\lambda_2 t} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} \\
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= C_1 \begin{bmatrix} 4 \\ 3 \end{bmatrix} e^{6t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} \\
&= \begin{bmatrix} 4C_1 e^{6t} \\ 3C_1 e^{6t} \end{bmatrix} + \begin{bmatrix} C_2 e^{-t} \\ -C_2 e^{-t} \end{bmatrix} \\
&= \begin{bmatrix} 4C_1 e^{6t} + C_2 e^{-t} \\ 3C_1 e^{6t} - C_2 e^{-t} \end{bmatrix} \\
&\begin{cases} x_1 = 4C_1 e^{6t} + C_2 e^{-t} \\ x_2 = 3C_1 e^{6t} - C_2 e^{-t} \end{cases}
\end{aligned}$$

Using Initial Conditions

$$\left.\begin{array}{l} x_1(0) = 4C_1 e^0 + C_2 e^0 = 1 \\ x_2(0) = 3C_1 e^0 - C_2 e^0 = 1 \end{array}\right\}$$

$$\begin{cases} 4C_1 + C_2 = 1 \\ 3C_1 - C_2 = 1 \end{cases} \quad \oplus$$

$$\begin{aligned}
7C_1 + 0 &= 2 \\
C_1 &= \frac{2}{7}
\end{aligned}$$

$$\begin{aligned}
4\left(\frac{2}{7}\right) + C_2 &= 1 \\
C_2 &= 1 - 4 \cdot \frac{2}{7} \\
&= -\frac{1}{7}
\end{aligned}$$

$$\begin{aligned}
x_1 &= \frac{8}{7} e^{6t} + \left(-\frac{1}{7}\right) e^{-t} \\
x_2 &= \frac{6}{7} e^{6t} + \frac{1}{7} e^{-t}
\end{aligned}$$

25 Eigenvalue Method to $\overline{X'}$

$$\begin{aligned}
\overline{X'} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \overline{X} \\
\overline{X'} &= \begin{bmatrix} a \\ b \end{bmatrix} e^{\lambda t}
\end{aligned}$$

Characteristic Equation

$$\begin{aligned}
&\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0 \\
&\to \lambda = \lambda_1, \lambda = \lambda_2
\end{aligned}$$

System of Equations for "a" and "b"

$$\begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$\to \begin{cases} (a_{11}-\lambda)a + a_{21}b = 0 \\ a_{21}a + (a_{22}-\lambda)b = 0 \end{cases}$$

$$\to \begin{bmatrix} a \\ b \end{bmatrix}$$

2. λ - complex number
$\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$

$$\begin{cases} (a_{11}-\lambda)a + a_{12}b = 0 \\ a_{21} \cdot a + (a_{22}-\lambda_1)b = 0 \end{cases}$$

$$\begin{cases} (a_{11}-\alpha-i\beta)a + a_{12}b = 0 \\ a_{21}a + (a_{22}-\alpha-i\beta)b = 0 \end{cases}$$

$$\to \begin{pmatrix} a = a_1 + ia_2 \\ b = b_1 + ib_2 \end{pmatrix}$$

$$\overline{X} = \begin{bmatrix} a_1 + ia_2 \\ b_1 + ib_2 \end{bmatrix} e^{(\alpha+i\beta)}t$$

Euler's Formula

$$\begin{aligned}
\to & e^{(\alpha+i\beta)} \\
&= e^{\alpha t}(\cos\beta t \pm i\sin\beta t) \\
\overline{X} &= \begin{bmatrix} a_1 + ia_2 \\ b_1 + ib_2 \end{bmatrix} e^{\alpha t}(\cos\beta t \pm i\sin\beta t) \\
&= \begin{bmatrix} (a_1 + ia_2)(\cos\beta t + i\sin\beta t) \\ (b_1 + ib_2)(\cos\beta t + i\sin\beta t) \end{bmatrix} e^{\alpha t} \\
&= \begin{bmatrix} a_1\cos\beta t + i(a_1\sin\beta t + a_2\cos\beta t) - a_2\sin\beta t \\ b_1\cos\beta t + i(b_1\sin\beta t + b_2\cos\beta t) + b_2\sin\beta t \end{bmatrix} e^{\alpha t} \\
&= \begin{bmatrix} e^{\alpha t}(a_1\cos\beta t - a_2\sin\beta t) \\ e^{\alpha t}(b_1\cos\beta t - b_2\sin\beta t) \end{bmatrix} + i\begin{bmatrix} e^{\alpha t}(a_1\sin\beta t + a_2\cos\beta t) \\ e^{\alpha t}(b_1\sin\beta t + b_2\cos\beta t) \end{bmatrix} \\
\overline{X} &= \overline{X_1} + i\overline{X_2} \\
\overline{X_1} &= \mathrm{Re}[\overline{X}], \overline{X_2} = \mathrm{Im}[\overline{X}]
\end{aligned}$$

- Particular Solutions

Example 1

$$\begin{aligned}
\frac{dx_1}{dt} &= 4x_1 - 3x_2 \\
\frac{dx_2}{dt} &= 3x_1 + 4x_2 \\
\to A &= \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} \\
&= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}
\end{aligned}$$

Characteristic Equation

$$\begin{vmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(4-\lambda) - 3\cdot(-3) = 0$$

$$\rightarrow (\lambda-4)^2 + 9 = 0$$

$$(\lambda-4)^2 = -9$$

$$\lambda - 4 = \pm 3i$$

$$\lambda_1 = 4+3i, \lambda_2 = 4-3i$$

$$\text{for } \begin{bmatrix} a \\ b \end{bmatrix} : \begin{bmatrix} 4-\lambda_2 & -3 \\ 3 & 4-\lambda_2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$\begin{cases} 4-(4(4-3i))\cdot a + (-3)b = 0 \\ 3a + (4(4-3i))b = 0 \end{cases}$$

$$\begin{cases} 3ia - 3b = 0 \\ 3a + 3ib = 0 \end{cases}$$

$$\begin{cases} ia - b = 0 \\ a + ib = 0 \end{cases}$$

$$\rightarrow ia - b = 0$$

$$ia - b = 0$$

$$a = i, b = -1$$

$$\begin{aligned}
x &= \begin{bmatrix} i \\ -1 \end{bmatrix} e^{(4-3i)t} \\
&= \begin{bmatrix} i \\ -1 \end{bmatrix} e^{4y}(\cos 3t - i\sin 3t) \\
&= \begin{bmatrix} i(\cos 3t - i\sin 3t)e^{4t} \\ -1(\cos 3t - i\sin 3t)e^{4t} \end{bmatrix} \\
&= \begin{bmatrix} (i\cos 3t + \sin 3t)e^{4t} \\ (-\cos 3t + i\sin 3t)e^{4t} \end{bmatrix} \\
\overline{X_1} &= \mathrm{Re}[\overline{X}] \\
&= \begin{bmatrix} \sin 3t\cdot e^{4t} \\ -\cos 3t \cdot e^{4t} \end{bmatrix} \\
\overline{X_2} &= \mathrm{Im}[\overline{X}] \\
&= \cancel{i} \begin{bmatrix} \cos 3t\cdot e^{4t} \\ \sin 3t \cdot e^{4t} \end{bmatrix} \\
&= \begin{bmatrix} \cos 3t\cdot e^{4t} \\ \sin 3t \cdot e^{4t} \end{bmatrix} \\
\overline{X} &= C_1\overline{X_1} + C_2\overline{X_2} \\
&= C_1 \begin{bmatrix} \sin 3te^{4t} \\ -\cos 3te^{4t} \end{bmatrix} + C_2 \begin{bmatrix} \cos 3te^{4t} \\ \sin 3te^{4t} \end{bmatrix} \\
&= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
x_1 &= C_1 \sin 3te^{4t} + C_2\cos 3te^{4t} \\
x_2 &= -C_1\cos 3te^{4t} + C_2 \sin 3te^{4t}
\end{aligned}$$