Math Notes

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Contents

1 First Order Differential Equations

1.1 Separable Differential Equations

$$\begin{aligned}
\frac{dy}{dx} &= G(x) \cdot H(y) \,\Big|\, \cdot dx \\
dy &= G(x) \cdot H(y) dx \,\Big|\, \cdot \frac{1}{H(y)} \\
\frac{dy}{H(y)} &= G(x) dx \\
\int \frac{dy}{H(y)} &= \int G(x) dx \\
h(y) + C_1 &= g(x) + C_2 \\
h(y) &= g(x) + C
\end{aligned}$$

Example

$$\begin{aligned}
\frac{dy}{dx} &= y\sin x \\
dy &= y \sin x(dx) \\
\frac{dy}{y} &= \sin x dx \\
\int y^{-1} dy &= \int \sin x dx \\
\ln|y| + C &= -\cos x + C \\
\ln|y| &= -\cos x + C \\
y &= e^{-\cos x + C} \\
&= e^{-\cos x} e^{C} \\
&= \frac{1}{e^{\cos x}} \cdot e^{C} \\
&= \frac{e^{C}}{e^{\cos x}} \\
& \quad D = e^{C} \\
&= \frac{D}{e^{\cos x}}
\end{aligned}$$

1.2 Linear First Order Differential Equations Homogeneous Differential Equation

$y'(x) + p(x)y(x) = q(x)$
$p(x)$, $q(x)$ given $q(x) = 0$, then $y' + p(x)y = 0$

- Linear because all terms are to the power of 1

$$\begin{aligned}
y' + y^2 = 0 &\rightarrow \text{non-linear} \\
y' + y = 0 &\rightarrow \text{linear}
\end{aligned}$$

1.3 Method of Integrating Factor

$\rho = e^{\int p(x)dx}$

$$\begin{aligned}
y' e^{\int p(x)dx} + P(x) y e^{\int p(x)dx} &= q(x) e^{\int p(x)dx} \\
\frac{d}{dx}(y \cdot e^{\int p(x)dx}) &= y' e^{\int p(x)dx} \cdot \frac{d}{dx}(\int p(x)dx) \\
&= y' e^{\int p(x)dx} + y e^{\int p(x)dx} p(x) \\
\frac{d}{dx}(y \cdot e^{\int p(x)dx}) &= q(x) \cdot e^{\int p(x)dx} \Big| \cdot dx \int \\
\int \frac{d}{dx}(y e^{\int p(x)dx}) dx &= \int q(x) \cdot e^{\int p(x)dx} \\
y \cdot e^{\int p(x)dx} &= \int (q(x) e^{\int p(x)dx}) dy + e \cdot \frac{1}{e^{\int p(x)dx}} \\
y &= (\int (q(x) \cdot e^{\int p(x)dx}) dx + C) e^{-\int p(x)dx}
\end{aligned}$$

1.3.1 Method of Substitution

1.

$$
\begin{aligned}
y' &= f(ax+by+c) \\
&a, b, c \text{given constants} \\
&f \text{given functions} \\
u &= ax+by+c \\
\frac{du}{dy} &= \frac{d}{dx}(ax+by+c) \\
\frac{du}{dy} &= a + b\frac{dy}{dx} + 0 \\
&\rightarrow \frac{dy}{dx} = \frac{\frac{du}{dx}-a}{b} \\
\frac{\frac{du}{dx}-a}{b} &= f(u) \\
\frac{du}{dx} &= bf(u) + a \, \Big| \cdot dx \\
du &= (bf(u)+a)dx \\
\frac{du}{bf(u)+a} &= dx \\
\int \frac{du}{bf(u)+a} &= \int dx \\
F(u) &= x + C \\
F(ax+by+C) &= x + C
\end{aligned}
$$

Example

$$
\begin{aligned}
\frac{dy}{dx} &= (x+y+3)^2 \\
u &= x+y+3 \\
\frac{du}{dx} &= \frac{d}{dx}(x+y+3) \\
\frac{du}{dx} &= 1 + \frac{dy}{dx} \\
\frac{dy}{dx} &= \frac{du}{dx} - 1 \\
\frac{du}{dx} - 1 &= (u)^2 \\
\frac{du}{dx} &= 1 + (u)^2 \, \Big| \cdot dx \\
du &= (1+u^2)dx \, \Big| \, \frac{1}{1+u^2} \\
\frac{du}{1+u^2} &= dx \\
\int \frac{du}{1+u^2} &= \int dx \\
\tan^{-1} u &= x + C \\
\tan(\tan^{-1} u) &= \tan(x+C) \\
u &= \tan(x+C), \qquad x+y+3 = \tan(x+C) \\
y &= \tan(x+C) - x - 3
\end{aligned}
$$

2.

$$
\begin{aligned}
y' &= f(\frac{y}{x}) \\
u &= \frac{y}{x}, y = u \cdot x, \frac{dy}{dx} = \frac{du}{dx}x + 1 \cdot u \\
\frac{du}{dx} \cdot x + u &= f(u) \rightarrow \frac{du}{dx}x = f(u) - u \;\Big|\; \cdot dx \\
du \cdot x &= (f(u) - u)dx \;\Big|\; \frac{1}{x(f(u) - u)} \\
\frac{du}{f(u) - u} &= \frac{dy}{x} \\
F(u) &= \ln|x| + C \\
F(\frac{y}{x}) &= \ln|x| + C
\end{aligned}
$$

$$
\begin{aligned}
2xy\frac{dy}{dx} &= 4x^2+3y^2 \\
2xy\frac{dy}{dx} &= 4x^2+3y^2 \;\Big|\; \frac{1}{x^2} \\
2\frac{y}{x}\cdot\frac{dy}{dx} &= 4+3(\frac{y}{x})^2 \\
\frac{dy}{dx} &= \frac{4+3(\frac{y}{x})}{2(\frac{y}{x})} \\
u &= \frac{y}{x}, y = u\cdot x, \frac{dy}{dx} = \frac{du}{dx}x+u \\
\frac{du}{dx} &= \frac{4+3u^2}{2u} - u \\
&= \frac{4+3u^2-2u^2}{2u} \\
&= \frac{4+u^2}{2u} \\
x\cdot\frac{du}{dx} &= \frac{4+u^2}{2u} \;\Big|\; \cdot dx \\
x\cdot du &= \frac{4+u^2}{2u}\cdot dx \;\Big|\; \cdot\frac{1}{x\cdot\frac{4\cdot u^2}{2u}} \\
\int\frac{2u}{4+u^2}du &= \int\frac{dx}{x} \\
z &= 4+u^2 \\
dz &= 2udu \\
\int\frac{dz}{z} &= \int\frac{dx}{x} \\
\ln|z| &= \ln|x|+C \\
e^{\ln|z|} &= e^{\ln|x|+C} = e^{\ln|x|}e^C \\
|z| &= |x|e^C \\
z &= e^C\cdot x = \pm e^C\cdot x \\
A &= \pm e^C \\
z &= Ax \\
4+u^2 &= Ax \\
4+(\frac{y}{x})^2 &= Ax \qquad \text{(general solution)} \\
(\frac{y}{x})^2 &= Ax-4 \\
\frac{y}{x} &= \pm\sqrt{Ax-4} \\
y &= \pm x\sqrt{Ax-4} \qquad \text{(explicit form)}
\end{aligned}
$$

Two Types of U-Substitution

1. $y' = f(ax+by+c)$
2. $y' = f(\frac{x}{y}$

1.4 Exact Equations

$$dF = M(x,y)dx + N(x,y)dy = 0$$
$$dF(x,y) = 0$$
$$dF(x,y) = 0$$
$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$$
$$[\frac{\partial F}{\partial x} = N(x,y)][\frac{\partial F}{\partial y} = M(x,y)]$$
$$\frac{\partial}{\partial y}[\frac{\partial F}{\partial x} = N(x,y)], \frac{\partial}{\partial x}[\frac{\partial F}{\partial y} = M(x,y)]$$
$$\frac{\partial^2 F}{\partial y dx} = \frac{\partial N}{\partial y}, \frac{\partial^2 F}{dx dy} = \frac{\partial M}{\partial x}$$
$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$$

$$\frac{\partial F}{\partial x} = N \Big| \cdot dx \int$$
$$\int \frac{\partial F}{\partial x} = \int N(x,y)dx$$
$$F(x,y) = \int N(x,y)dx + g(y)$$
$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\int N(x,y)dx + g'(y) = M(x,y)$$
$$\Rightarrow g(y)$$
$$F(x,y) = \int N(x,y)dx + g(y)$$

(equation from earlier step)

Example

$$(6xy - y^3)dx + (4y + 3x^2 - 3xy^2)dx = 0$$
$$M(x, y) + N(x, y) = 0$$
$$\frac{\partial M}{\partial y}(6xy - y^3) = 6x - 3y^2$$
$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(4y + 3x^2 - 3xy^2) = 0 + 6x - 3y^2$$

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

$$\frac{\partial F}{\partial x} = 6xy - y^3, \frac{\partial F}{\partial y} = 4y + 3x^2 - 3xy^2$$
$$\int \frac{\partial F}{\partial x}dx = \int (6xy - y^3)dx$$
$$F(x, y) = 6\frac{x^2}{2} \cdot y - y^3x + g(y)$$
$$\frac{\partial F}{\partial y} = 3x^2 \cdot 1 - x \cdot 3y^2 + g'(y)$$
$$\text{Remember: } \frac{\partial F}{\partial y} = 4y + 3x^2 - 3xy^2$$
$$3x^2 - 3xy^2 + g'(y) = 4y + 3x^2 - 3xy^2$$
$$g'(y) = 4y$$
$$\int g'(y)dy = \int 4ydy$$
$$g(y) = \frac{4y^2}{2} + C$$
$$\text{Remember: } F(x, y) = 6\frac{x^2}{2} \cdot y - y^3x + g(y)$$
$$F(x, y) = 3x^2 \cdot y - y^3x + 2y^2 + C$$

$$\int dF = 0, F(x, y) = C_1$$

$$3x^3y - y^3x + 2y^2 = C_1$$
$$y(x)$$

2 Bernoulli Equation

$y'(x) + P(x)y(x) = Q(x)y^n(x)$
P, Q-given, $n \neq 1$

linear $y' + p(x)y = q(x)$ by integrating factor $\rho = e^{\int p(x)dx}$

dividing by y^n to obtain

$$
\begin{aligned}
y^{-n}y' + P(x)y^{1-n} &= Q(x) \\
z &= y^{1-n}, (z(x))' = (y(x)^{1-n})' \\
\frac{dz}{dx} &= (1-n)y^{1-n-1}\frac{dy}{dx} \\
&= (1-n)\cdot y^{-n}\frac{dy}{dx} \\
y^{-n}y'(x) &= \frac{dz}{dx}\cdot\frac{1}{1-n}
\end{aligned}
$$

Example

$$y' - \frac{3}{2x}y = \frac{2x}{y}$$
$$P(x) = -\frac{3}{2x}, Q = 2x, n = -1$$

$$y' - \frac{3}{2x}y = \frac{2x}{y} \;\Big|\; \cdot y$$
$$z = y^{1-n}$$
$$yy' - \frac{3}{2x}y^2 = 2x$$
$$z = y^2, z'(x) = 2y \cdot y'(x)$$
$$\to y \cdot y'(x) = \frac{z(x)}{z}$$

$$\frac{z'(x)}{z} - \frac{3}{2x}z = 2x$$
$$\to z'(x) - \frac{3}{2}z = 4x$$

$$\begin{aligned}\rho &= e^{\frac{-3}{2}dx}\\ &= e^{-e\ln x}\\ &= e^{\ln x^{-3}}\\ &= x^{-3}\end{aligned}$$

$$z'(x) - \frac{3}{x}z = 4x \;\Big|\; \cdot\rho = x^{-3}$$
$$x^{-3} \cdot z'(x) - 3x^{-4}z(x) = 4x^{-2}$$
$$check \to \frac{d}{dx}(x^{-3} \cdot z) = 4x^{-2}$$

$$\begin{aligned}\frac{d}{dx}(x^{-3} \cdot z) &= z'x^{-3} - 3x^{-4}z\\ &= x^{-3}z' + z(-3) \cdot x^{-4}\end{aligned}$$

$$\int \frac{d}{dx}(x^{-3}z)dx = \int 4x^{-2}dx$$
$$x^{-3}z = -\frac{4}{x} + C$$
$$z = -4x^2 + Cx^3$$

$$y^2 = 4x^2 + Cx^3$$
$$y = \pm\sqrt{-4x^2 + Cx^3}$$

3 Second Order Ordinary Differential Equations

$F(y'', y', y, x) = 0, y = y(x)$

3.1 Type I

$F(y''(x), y'(x), x) = 0$
Function of x

$$\begin{aligned} y'(x) = P(x), y''(x) = (y'(x))' &= P'(x) \\ &\to F(p'(x), p(x), x) = 0 \\ &\to p(x) = f(x, C_1) \\ &\quad y' = f(x, C_1) \, \Big| \cdot dx \int \\ &\to \int y'(x)dx = \int f(x, C_1)dx \\ &\quad y(x) = \int f(x, C_1)dx + C_2 \end{aligned}$$

Example

$$\begin{aligned} xy'' + 2y' &= 6x, y(x) = ? \\ xp'(x) + 2p &= 6x \end{aligned}$$

$$y' + p(x)y = q(x) \, \Big| \qquad \cdot\rho = e^{\int \frac{2}{x} dx} = e^{2\ln x} = x^2$$

$$\begin{aligned} p'(x)x^2 + \frac{2x^2}{x}p(x) &= 6x^2 \\ \frac{d}{dx}(x^2 \cdot p(x)) &= p'(x)x^2 + 2xp \\ &\quad \rho \cdot p(x) \\ \frac{d}{dx}(p(x), y(x)) &= (p(x), y(x))' \\ \frac{d}{dx}(x^2 \cdot p(x)) &= 6x^2 \, \Big| \cdot dx \int \\ \int \frac{d}{dx}(x^2 \cdot p(x))dx &= \int 6x^2 dx \\ x^2 p &= \frac{6x^2}{3} + C, \end{aligned}$$

$$\begin{aligned} p(x) &= 2x + \frac{C_1}{x^2} \\ &= y'(x) \end{aligned}$$

$$\begin{aligned} \to \int y'(x)dx &= \int (2x + C_1 x^{-2})dx \\ y(x) &= \frac{2x^2}{2} - \frac{C_1}{y} + C_2 \end{aligned}$$

3.2 Type II

$F(y''(x), y'(x), y(x)) = 0$
Function of y

$$\begin{aligned} y'(x) = p(y), y'' &= (y'(x))' \\ &= (P(y))'' \\ &= \frac{dP}{dy} \cdot \frac{dy}{dx} \\ &= \frac{dp}{dy} p \end{aligned}$$

Example

$$\begin{aligned} yy''(x) &= (y'(x))^2 \\ y\frac{dp}{dy} \cdot p &= p^2, p \neq 0 \\ y\frac{dp}{dy}, \frac{dp}{p} &= \frac{dy}{y}, \int \frac{dp}{p} = \int \frac{dy}{y} \\ &\to \ln|p| = \ln|y| + C_1 \\ &\quad e^{\ln|p|} = e^{\ln|y|+C_1} = e^{\ln|x|} \cdot e^{C_1} \\ &\quad |p| = |y| \cdot e^{C_1}, p = \pm e^{C_1} \cdot y \\ &\qquad A_1 = \pm e^{C_1} \\ &\quad p = A_1 y \\ &\to y'(x) = A_1 y \\ &\quad \frac{dy}{dx} = A_1 y \\ &\quad \frac{dy}{y} = A_1 dx \\ &\quad \int \frac{dy}{y} = \int A_1 dx \\ &\to \ln|y| = A_1 x + C_2 \\ &\quad e^{\ln|y|} = e^{A_1 x + C_2} \\ &\quad |y| = e^{A_1 x} e^{C_2} \\ &\to y = \pm e^{C_2} e^{A_1 x} \\ &\qquad A_2 = \pm e^{C_2} \\ &\quad y = A_2 e^{A_1 x} \end{aligned}$$

3.3 Inital Value Problem

(1) $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$
Then If $f(x, y), \frac{\partial f}{\partial y}$ (continnuous function) on $R[a \leq x \leq b, c \leq y \leq d]$ there exists wuch interval $I \in [a, b]$ also $x_0 \in I$, where the inital value problem (1) has unique solution

Rectangle (R) is designated by the chosen value (can be very large or small)

$$\begin{gathered} y'' + p(x)y' + g(x)y = f(x), x \in (a, b) \\ p(x), q(x), f(x) - giveny(x)? \\ y(x_0) = y_0, y'(x_0) = y_1 \end{gathered}$$

4 Complex Roots of Charatistics Equations

Imaginary Number $\rightarrow \sqrt{-1} = i$
Complex Number $\rightarrow 3 \pm 4\sqrt{-1} = 3 \pm 4i$
Real Number $\rightarrow \mathrm{Re}[3 \pm 4i] = 3$
Imaginary Number $\rightarrow \mathrm{Re}[3 \pm 4i] = \pm 4$

Complex Number Plane

$$
\begin{aligned}
ay'' + by' + cy = 0, y = e^{rx} \\
ar^2 + br + c = 0 \\
r_{1,2} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}, b^2 - 4ac = 0 \\
r = \alpha \pm i\beta, \sqrt{-1} = i \\
y_1 = e^{(\alpha + i\beta)x}, y_2 = e^{(\alpha - i\beta)x}
\end{aligned}
$$

4.1 Euler Formula

$$
\begin{aligned}
e^{i\beta x} &= \cos(\beta x) + i\sin(\beta x) \\
e^{-i\beta x} &= \cos(\beta x) - i\sin(\beta x)
\end{aligned}
$$

$$
\begin{aligned}
y_1 &= e^{(\alpha + \beta i)x} && = e^{\alpha x} e^{\beta i} \\
&= e^{\alpha x}(\cos\beta x + i\sin\beta x)
\end{aligned}
$$

$$
\begin{aligned}
y &= u(x) + iw(x) \\
a(u + iw)'' + b(u + iw)' + c(u + w) &\equiv 0 \\
(a''bu + cu) + (aw'' + bw' + cw)i &\equiv 0 \\
0 + 0 &= 0
\end{aligned}
$$

$$
\begin{aligned}
y_1 = e^{\alpha x}\cos\beta x, y_1 = e^{\alpha x}\sin\beta x \\
y = C_1 e^{\alpha x}\cos\beta x + C_2 e^{\alpha x}\sin\beta x
\end{aligned}
$$

5 Method of Undetermined Coefficients

$$
\begin{aligned}
(r^2 + 4)(ar^2 + br + c) = 0 \\
&\equiv 6r^4 + 5r^3 + 25r^2 + 20r + 4 \\
&\equiv ar^4 + br^3 + cr^2 + 4ar^2 + 4br + 4c
\end{aligned}
$$

$$
\begin{aligned}
r^4 &: 6 = a \\
r^3 &: 5 = b \\
r^2 &: 25 = c + 4a \\
r &: 20 = 4b \\
r^0 &: 4 = 4C \rightarrow c = 1
\end{aligned}
$$

$$(r^2 + 4)(6r^2 + 5r + 1) = 0$$

6 Non-Homogeneous Equation