Ordinary Differential Equations Reference Sheet

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Contents

1 Table of Integrals

1.1 Table of Laplace Transforms

This table summarizes the general properties of Laplace transforms and the Laplace transforms of particular functions derived in Chapter 4.

Function	Transform	Function	Transform
$f(t)$	$F(s)$	e^{at}	$\frac{1}{s-a}$
$af(t)+bg(t)$	$aF(s)+bG(s)$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$f'(t)$	$sF(s)-f(0)$	$\cos kt$	$\frac{s}{s^2+k^2}$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$	$\sin kt$	$\frac{s}{s^2-k^2}$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$	$\cosh kt$	$\frac{s}{s^2-k^2}$
$\int_0^t f(\tau)d\tau$	$\frac{F(s)}{s}$	$\sinh kt$	$\frac{k}{s^2-k^2}$
$e^{at}f(t)$	$F(s-a)$	$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
$u(t-a)f(t-a)$	$e^{-as}F(s)$	$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$
$\int_0^t f(t)g(t-\tau)d\tau$	$F(s)G(s)$	$\frac{1}{2k^3}(\sin kt-kt\cos kt)$	$\frac{1}{(s^2+k^2)^2}$
$tf(t)$	$-F'(s)$	$\frac{t}{2k}\sin kt$	$\frac{s}{(s^2+k^2)^2}$
$t^nf(t)$	$(-1)^nF^{(n)}(s)$	$\frac{1}{2k}(\sin kt+kt\cos kt)$	$\frac{s^2}{(s^2+k^2)^2}$
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)d\sigma$	$u(t-a)$	$\frac{e^{-as}}{s}$
$f(t)$, period p	$\frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)dt$	$\delta(t-a)$	e^{-as}
1	$\frac{1}{s}$	$(-1)^{[\![t/a]\!]}$ (square wave)	$\frac{1}{s}\tanh\frac{as}{2}$
t	$\frac{1}{s^2}$	$[\![\frac{t}{a}]\!]$ (staircase)	$\frac{e^{-as}}{s(1-e^{-as})}$
t^n	$\frac{n!}{s^{n+1}}$		
$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$		
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$		

1.2 ELEMENTARY FORMS

1. $\int udv = uv - \int vdu$
2. $\int u^n du = \frac{1}{n+1}u^{n+1} + C$ if $n \neq -1$
3. $\int \frac{du}{u} = ln|u| + C$
4. $\int e^u du = e^u + C$
5. $\int a^u du = \frac{a^u}{\ln a} + C$
6. $\int \sin u du = -\cos u + C$
7. $\int \cos u du = \sin u + C$
8. $\int \sec^2 u du = \tan u + C$
9. $\int \csc^2 u du = -\cot u + C$
10. $\int \sec u \tan u du = \sec u + C$
11. $\int \csc u \cot u du = -\csc u + C$
12. $\int \tan u du = \ln|\sec u| + C$
13. $\int \cot u du = \ln|\sin u| + C$
14. $\int \sec u du = \ln|\sec u + \tan u| + C$
15. $\int \csc u du = \ln|\csc u - \cot u| + C$
16. $\int \frac{du}{\sqrt{a^2+u^2}} = \sin^{-1}\frac{u}{a} + C$

17. $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

18. $\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C$

1.3 TRIGONOMETRIC FORMS

19. $\int \sin^2 u du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$

20. $\int \cos^2 u du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$

21. $\int \tan^2 u du = \tan u - u + C$

22. $\int \cot^2 u du = -\cot u - u + C$

23. $\int \sin^3 u du = -\frac{1}{3}(2+\sin^2 u)\cos u + C$

24. $\int \cos^3 u du = \frac{1}{3}(2+\cos^2 u)\sin u + C$

25. $\int \tan^3 u du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$

26. $\int \cot^3 u du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$

27. $\int \sec^3 u du = \frac{1}{2}\sec u\tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$

28. $\int \csc^3 u du = -\frac{1}{2}\csc u\cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$

29. $\int \sin au \sin bu du = \frac{sin(a-b)u}{2(a-b)} - \frac{sin(a+b)u}{2(a+b)} + C$ if $a^2 \neq b^2$

30. $\int \cos au \cos bu du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)}C$ if $a^2 \neq b^2$

31. $\int \sin au \cos bu du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$ if $a^2 \neq b^2$

32. $\int \sin^n u du = -\frac{1}{n}\sin^{n-1} u\cos u + \frac{n-1}{n}\int \sin^{n-2} u du$

33. $\int \cos^n u du = \frac{1}{n}\cos^{n-1} u\sin u + \frac{n-1}{n}\int \cos^{n-2} u du$

34. $\int tan^n u du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u du$ if $n \neq 1$

35. $\int \cot^n u du = -\frac{1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u du$ if $n \neq 1$

36. $\int \sec^n u du = \frac{1}{n-1}\sec^{n-2} u\tan u + \frac{n-2}{n-1}\int \sec^{n-2} u du$ if $n \neq 1$

37. $\int \csc^n u du = -\frac{1}{n-1}\csc^{n-2} u\cot u + \frac{n-2}{n-1}\int \csc^{n-2} u du$ if $n \neq 1$

38. $\int u\sin u du = \sin u - u\cos u + C$

39. $\int u\cos u du = cosu + usinu + C$

40. $\int u^n \sin u du = -u^n\cos u + n\int u^{n-1}\cos u du$

41. $\int u^n \cos u - n\int u^{n-1}\sin u du$

1.4 FORMS INVOLVING $\sqrt{u^2 \pm a^2}$

42. $\int \sqrt{u^2 \pm a^2} du = \frac{u}{2}\sqrt{u^2 \pm a^2} \pm \frac{a^2}{2}\ln|u + \sqrt{u^2 \pm a^2}| + C$

43. $\int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln|u + \sqrt{u^2 \pm a^2}| + C$

1.5 FORMS INVOLVING $\sqrt{a^2-u^2}$

44. $\int \sqrt{a^2-u^2}du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

45. $\int \frac{\sqrt{a^2-u^2}}{u}du = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

1.6 EXPONENTIAL AND LOGARITHMIC FORMS

46. $\int ue^u du = (u-1)e^u + C$

47. $\int u^n e^u du = u^n e^u - n\int u^{n-1}e^u du$

48. $\int u^n \ln u du = \frac{u^{n+1}}{n+1}\ln u - \frac{u^{n+1}}{(n+1)^2} + C$

49. $\int e^{au}\sin bu du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu) + C$

50. $\int e^{au}\cos bu du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu) + C$

1.7 INVERSE TRIGONOMETRIC FORMS

51. $\int \sin^{-1} u du = u\sin^{-1} u + \sqrt{1-u^2} + C$

52. $\int \tan^{-1} u du = u\tan^{-1} u - \frac{1}{2}\ln(1+u^2) + C$

53. $\int \sec^{-1} u du = u\sec^{-1} u - \ln|u + \sqrt{u^2-1}| + C$

54. $\int u\sin^{-1} u du = \frac{1}{4}(2u^2-1)\sin^{-1} u + \frac{u}{4}\sqrt{1-u^2} + C$

55. $\int u\tan^{-1} u du = \frac{1}{2}(u^2+1)\tan^{-1} u - \frac{u}{2} + C$

56. $\int u\sec^{-1} u du = \frac{u^2}{2}\sec^{-1} u - \frac{1}{2}\sqrt{u^2-1} + C$

57. $\int u^n \sin^{-1} u du = \frac{u^{n+1}}{n+1}\sin^{-1} u - \frac{1}{n+1}\int \frac{u^{n+1}}{\sqrt{1-u^2}}du$ if $n \neq -1$

58. $\int u^n \tan^{-1} u du = \frac{u^{n+1}}{n+1}\tan^{-1} u - \frac{1}{n+1}\int \frac{u^{n+1}}{1+u^2}du$ if $n \neq -1$

59. $\int u^n \sec^{-1} u du = \frac{u^{n+1}}{n+1}\sec^{-1} u - \frac{1}{n+1}\int \frac{u^{n+1}}{\sqrt{u^2-1}}du$ if $n \neq -1$

1.8 OTHER USEFUL FORMULAS

60. $\int_0^\infty u^n e^{-u} du = \Gamma(n+1) = n! \qquad (n \geqq 0)$

61. $\int_0^\infty e^{-au^2} du = \frac{1}{2}\sqrt{\frac{\pi}{a}} \qquad (a > 0)$

62. $\int_0^{\pi/2} \sin^n u du = \int_0^{\pi/2} \cos^n u du = \begin{cases} \frac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n}\cdot\frac{\pi}{2} & \text{if } n \text{ is an even integer and } n \geqq 2 \\ \frac{2\cdot 4\cdot 6\cdots(n-1)}{3\cdot 5\cdot 7\cdots n} & \text{if } n \text{ is an odd integer and } n \geqq 3 \end{cases}$