— KWARC Blue Note* —
Examples and Counterexamples in OMDoc

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February 18, 2016

Abstract

Examples are a very important part of mathematical practice, they are used to strengthen or refute conjectures, to explain and to guide intuition about mathematical concepts. Given this, they have received surprisingly little attention in mathematical knowledge representation and mechanized reasoning.

In this note we look at examples in OMDoc, and the tension between examples as views in theory graphs and the practice of writing down examples in mathematical documents.

This note is part of the ongoing OMDoc2 language design [Koh13] and studies an important aspect in more detail.

*Inspired by the “blue book” in Alan Bundy’s group at the University of Edinburgh, KWARC blue notes, are documents used for fixing and discussing ϵ-baked ideas in projects by the KWARC group (see http://kwarc.info). Unless specified otherwise, they are for project-internal discussions only. Please only distribute outside the KWARC group after consultation with the author.

Contents

1 Introduction

Mathematical knowledge management has focused almost entirely on the "holy trinity" of statements: axiom/definition, theorem, proof. This neglects a type of statement that is at least as important for mathematical practice: examples. Alison Pease and Ursula Martin have a chart of the polymath discussion on a math olympiad problem and find that 33% of the dialogue turns in the proof is about examples. And they keep being strongly represented all across the 74 minute length of the proof endeavour [PM12]. In another study, Ursula records that about 15% of the questions on Mathoverflow are about examples for some property [MP13]. In this note, we are most interested in the structure and epistemic links of examples, so that we can model them adequately in OMDoc 2.

1.1 Examples as Mathematical Structures

In mathematical documents, examples come in various forms. We can distinguish them by epistemic level into the following categories:

1. *specific object-level*, e.g. "13 is a prime number", here the object "13" is an example the property of being a "prime number", i.e. the predicate "prime" evaluates true on "13" or in other words $13 \in \mathsf{prime}$. We observe that every example carries a proof obligation. We have to prove that the object we exhibit indeed has the desired property it exemplifies. Note that in many instances in the literatures the proofs are elided – presumably because they are considered trivial.
2. *generic object-level*, e.g. "All differentiable functions are continuous." here any object that is a differentiable function is an example for being continuous. In other words the set of differentiable functions is a subset of continuous functions. In this case, the proof obligation is more conspicuous, the ϵ/δ proof of this fact is one of the first in any elementary calculus course.[1] EdN:1
3. *specific theory-level* "$(\mathbb{N}, +)$ is a Monoid", here we interpret $(\mathbb{N}, +)$ as a mathematical "theory" given by the symbols $\mathbb{N}$ (natural numbers) and $+$ (addition). Here the relation given by the words "is a" has to be understood as "can be interpreted as" in the sense that there is a meaning-preserving mapping from the theory of monoids to the theory $(\mathbb{N}, +)$.
4. *generic theory level* A similar situation obtains with "Boolean Algebras are Complete Lattices", even though the phrase uses the plural as in generic object-level case. In both cases, we have theory morphisms that allow framing [KK09], we will elaborate on this in Section 3.

In general, an example states a satisfaction relation between objects and properties. At the object level, either between a specific object (we call it the **exemplar**) or a class of generic objects (which we will call the **exemplar class**[2]) and a property (which we call the **exemplified property**). Thus from an epistemological perspective, an example is an assertion of the form $p(e)$ or $\forall x E(x) \Rightarrow p(x)$, where p is the exemplified property, e the exemplar and E the exemplar class, or if we want to see things more set-theoretically: $e \in p$ or $E \subseteq p$. We call this assertion the **proof obligation** of the example. EdN:2

[1] EdNote: MK: this is related to the hyponomy relation, cf. [Koh14].
[2] EdNote: MK: I just made this up, is there any traditional name for this?

1.2 Polarity of Examples

Moreover we can distinguish counter-examples from supporting ones, we speak of the **polarity** of the example. The two polarities exist at all four levels we discussed above. To look at just the first, we can use 13 as an example for primality and 12 as a counter-example. In the latter case we have the proof obligation $13 \notin \mathsf{prime}$, i.e. the negation of the obligation of the positive example. Counter-examples are most

5. *at the statement-level*, most saliently as supporting evidence for or counter-example against conjectures. Here [3] EdN:3

1.3 Examples as Cognitive Devices

In the process of mathematical discovery, examples are used to strengthen conjectures, to show that concepts are non-vacuous or consistent, and to elucidate concepts.

Counterexamples are used to delineate the boundaries of possible theorems. By using counterexamples to show that certain conjectures are false, mathematical researchers avoid going down blind alleys and learn how to modify conjectures to produce provable theorems. [4] EdN:4

1.4 Examples as Narrative Structures

Examples can come in different syntactic forms in a mathematical discourse

1. *Embellished examples*[5] are given as special paragraphs marked up with special cues – prefixed by the emphasized keyword "Example", possibly numbered for reference, and sometimes even marked up in a special font family. EdN:5
2. Examples in the *narrative flow* are usually single sentences that state the exemplified property of the exemplar.
3. *Inline examples* just consist of a phrase, often an interjection like "for instance 13".

No matter the syntactic structure, they always have the structural parts (which may be elided) identified above.

2 Parallel Markup for Immediate Examples

Following [KI13], we will use OMDoc markup for the narrative aspects of examples and MMT markup for the formal ones. For object-level examples, we have the parallel markup in Figure 1: there is a narrative representation, which represents the structure of the presentation of the example as an (embellished) paragraph with a statement and a proof as the justification. In this simple case, the statement of the justifying assertion is identical to the example text, so we can just represent the justification as a proof. The content representation is just a special constant representing the proof obligation. We have indicated the parallels between content and narrative by cross-references (the <**meta**> elements with the o:formalizes relation) on the content side.

[3] EdNote: continue, we need examples for counterexamples, recount the analysis from the OMDoc book in a nutshell.

[4] EdNote: integrate Lakatos view on counterexamples from [Lak76, pp. 10-11]

[5] EdNote: MK: is there a good word for this?

narrative	content

narrative:

```
<definition for="?prim">
    A prime number is a natural number that
    has only 1 and itself as divisors.
</definition>
<example name="p13" for="prim">
  <assertion xml:id="p13.st">
   13 is a prime number.
  </assertion>
  <proof for="p13" name="p13.p">
    <CMP>13 is indivisible by 2, 3, and 4
      and 4^2 > 13.</CMP>
  </proof>
<example>
```

content:

```
<constant name="prim">...</constant>
<constant name="prim13">
  <meta rel="o:formalizes" resource="?p13"/>
  <type name="prim13.t">
     <meta rel="o:formalizes" resource="?p13.st"/>
     prim(13)
  </type>
  <definiens name="prim13.p">
    <meta rel="o:formalizes" resource="?p13.p"/>
    ...
  </definiens>
</constant>
```

Figure 1: A Simple Object-Level Example

The situation in Figure 1 is also simple, since the both the exemplar and the exemplified property naturally come from the same theory: natural numbers arithmetic (which we have omitted).

For the next level of complexity let us consider the generic object-level example[1] "All differentiable functions are continuous.", where the exemplified property "continuous" naturally lives in the theory cont, but the example invokes the concept of differentiable functions from the theory diff of differentiable functions. As we do not want to include diff into cont, we have to provide a separate theory cont-diff.ex for the example. This includes diff for the exemplar class and cont for the exemplified property and we get the markup in Figure 3.

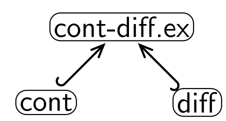


Figure 2: Structure

narrative	content

narrative:

```
<theory name="cont">
  ...
  <example name="diffcont.ex" for="continuous">
    <meta rel="o:verbalizes"
                resource="?cont-diff.ex"/>
     <uses from="?diff"/>
     <assertion name="cd.ex.p">
       All differentiable functions are continuous.
     </assertion>
     <proof for="#cd.ex.p">...</proof>
  <example>
  ...
</theory>
```

content:

```
<theory name="cont-diff.ex">
  <structure from="?diff"/>
  <structure from="?cont"/>
  <constant name="cd.ex.p">
    <type>C^1(R,R) ⊆ C^0(R,R)</type>
    <definiens>...</definiens>
  </constant>
</theory>
```

Figure 3: A Generic Object-Level Example

Here we made use of the o:verbalizes relation – which is the converse to o:formalizes – for statement-level parallel markup and thus resides in the narrative representation. For the

[1]Nothing hinges on the genericity here, we could have made a similar point with specific Mersenne numbers being prime, which would be specific object-level

remaining examples below, we assume one of the two representations, but omit them to avoid clutter.

The content side in Figure 3 shows us what is really happening from a structural perspective: The example object is a theory of its own, which inherits from theories cont and diff as shown in Figure 2. This structure is somewhat disguised in the narrative part: the example element implicitly introduces a scope (of visibility of symbols and assumptions in effect) that includes all from theory cont it is nested in and includes the material from theory diff via the uses directive.

The examples we have looked up so far are **immediate** in that they do not involve any structural substitutions: they correspond to simple set membership or subsumption.

3 Examples via Morphisms

We will now turn to examples that involve structural substitutions. We have already seen above that we best employ theory structures to model examples. So we make full use of OMDoc/MMT theory graphs [RK13] for the content structures. We model examples as views and make use of assignments in theory morphisms to account for structural substitutions.

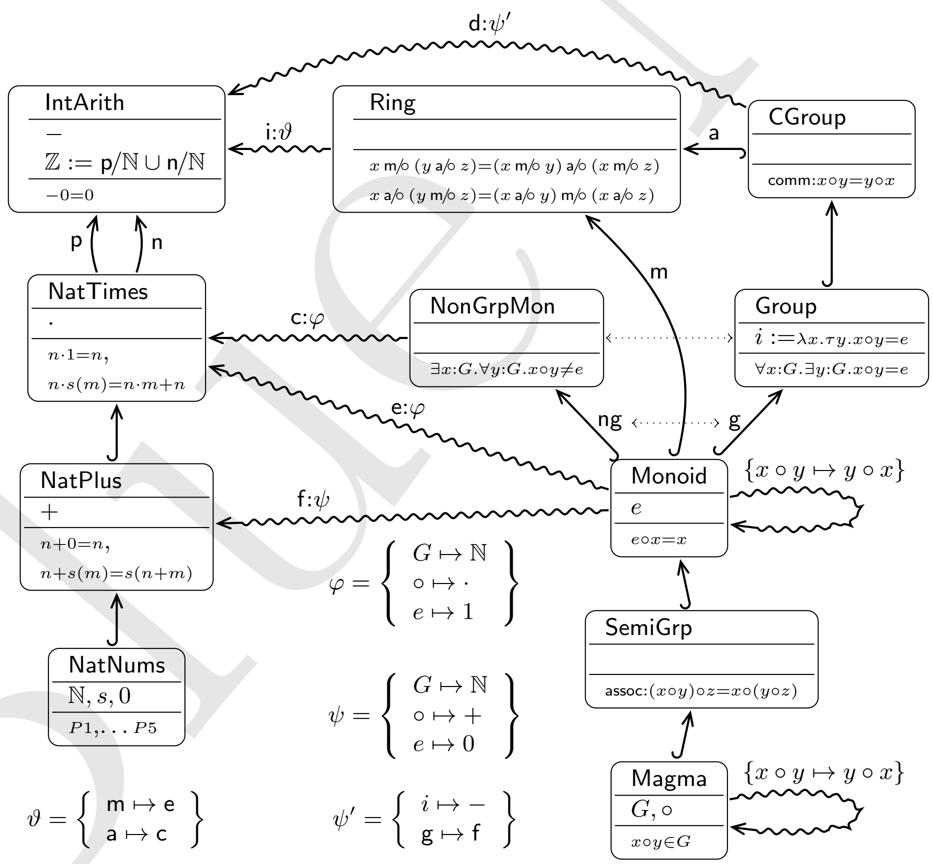


Figure 4: A Theory Graph with Examples

3.1 Examples as Theory Morphisms in Theory Graphs

For the monoids example above we will use the theory graph in Figure 4. There we have two examples for monoids: natural numbers with addition and multiplication, represented by the views f and e. The fact that e and f are views means that every axiom in source theory Monoid is provable in the respective target theories NatPlus and NatTimes.

narrative	content
<**example** name="natmon.ex"> <**uses** from="?NatPlus"/> <**meta** rel="o:verbalizes" **resource**="?natmon"/> <**assertion** name="natmon.ass">$\boxed{(\mathbb{N},+)}$ is a <**termref ref**="?Monoid?monoid">monoid</**termref**> with neutral element $\boxed{0}$. </**assertion**> <**proof** for="natmon.ex">...</**proof**> <**example**>	<**view** name="natmon" from="?Monoid" to="NatPlus"> <**conass** name="set">$\boxed{\mathbb{N}}$</**conass**> <**conass** name="op">$\boxed{+}$</**conass**> <**conass** name="neut">$\boxed{0}$</**conass**> <**conass** name="neutax">π</**conass**> </**view**>

Figure 5: A Theory-Level Example

The narrative markup is just as in the case of simple examples from Section 2, only that the content markup is in the form of a view, which clarifies the structural substitutions. This example shows nicely that the narrative structure does two things: it marks up the text in a mathematical document and it also gives a view that is present in the theory graph a didactic label as "suitable for exemplifying the concept of a monoid" (in a particular narrative situation). Other views from monoid might not be suitable examples – in the situation; e.g. since the reader does not know their targets, or if they are too trivial. For instance in Figure 4 we have an inclusion of Monoid into group, which by construction is also a theory morphism. But in an situation where groups have just been introduced as special monoids, this example might be considered too immediate to mention it.

3.2 Compiling Narrative Examples into Theory Graphs

Note that immediate examples can be "compiled" into a theory graph. For instance the one in Figure 1 directly corresponds to the theory graph in Figure 6. Background: the theory primes introduces the predicate prime for prime numbers based on natural number arithmetic, in this situation, we want to give an object-level example for the prediate prime, i.e. a numer $n \in \mathbb{N}$ that is prime. For this, we need to "abstract" the situation in theory primes into a general situation with a predicate in propset, which is extended to be non-empty (with a witness n) in neprop. Then we construct the push-out theory neprim on top of Figure 6 that instantiates S with $\mathbb{N}$ and p with prime. This is the natural source of an example view ψ that instantiates n to 13 and ne to the primality proof for it – see Figure 1 on the lower left.

Note that this construction is fully general the theories propset and neprop just formalize the intuition that this kind of example is concerned with showing the non-emptiness of a given property by exhibiting a witness and proving that the property holds on it.

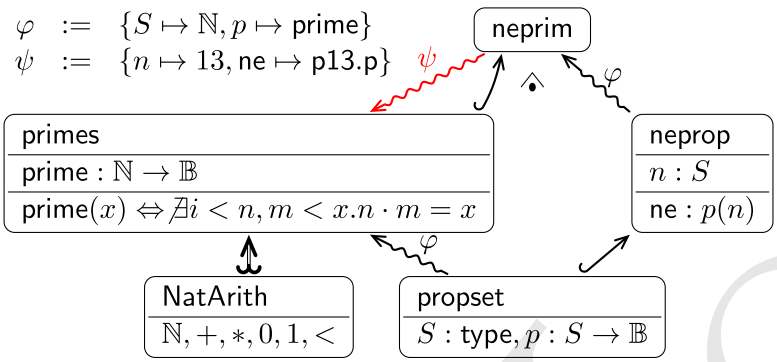
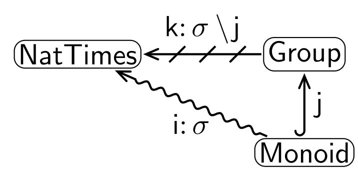


Figure 6: The Immediate Example from Figure 1 as a Theory Graph

3.3 Counterexamples in Theory Graphs

The case of counterexamples is a completely different matter; naturally, the theory morphism property forbids a direct interpretation. But if we add a new **complementarity relation** to theory graphs (the dotted arrows in Figure 4; either between theories or between extension inclusions), then we can characterize the counterexample relation as a composition of a complementarity link with a view. In our theory graph, the natural numbers with multiplication are a counter-example to group as evidenced by the dotted arrow (left to right) and the view c:φ.[6] EdN:6

But to understand the counter-example relation, we take a step back and develop a notion that is akin to the role the theory morphisms play for examples. Let us build our intuition about these first. If we have a morphism $A \xrightarrow{\sigma} B$, then we have $B \models \sigma(A)$ proof-theoretically or $\overline{\sigma(A)} \subseteq \overline{B}$, where $\overline{A}$ is the set of models of A.



The situation we want to model is one of mathematical complementarity, where we want to state that NatTimes is "not a Group" (but a Monoid). In terms of models this means that $\overline{\mathsf{NatTimes}} \subseteq \overline{\mathsf{Monoid}} \setminus \overline{\mathsf{Group}}$. In terms of theory presentations we want to have a morphism from group, such that NatTimes $\models \sigma(G)$, where G are those declarations that are in Group, but not in Monoid, i.e. those that have not been introduced by the theory morphism j[2]. We call such a mapping an **antimorphism** and depict it with a stricken-through arrow and write it as i: $\sigma \setminus$ j: i is the name, σ the assignment, and j the **kernel**[7] morphism. EdN:7

In Figure 4, we can see the complementarity relation depicted by the dotted bidirectional arrow as a pair of antimorphisms j: Id $\setminus$ g and k: Id $\setminus$ ng between NonMonGrp and Group. Antimorphisms compose with theory morphisms in the obvious ways: the set of antimorphisms is closed under extension with morphisms by post-composition and by pre-composition of kernel morphisms.

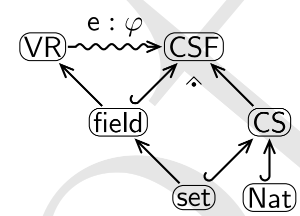
[6] EdNote: We probably need concrete syntax for this. The complementarity relation could be a meta-relation, and we could give the counterexample an attribute that points to it.

[2] In our case this is an inclusion, I guess that a structure would have worked just as well. I am not sure about the situation where j is a view, that should be explored further.

[7] EdNote: MK: I do not think that kernel is a good name, but I did not have any better at the moment.

3.4 Example-Only Theories

[8] EdN:8



In the previous sections, we claimed that "examples are views" (of various kinds), mainly because the motivating examples for examples, the target theories of the views were always interesting in their own right. But this is not always the case, consider for instance the case of the cartesian space F^n which forms a (an example of a) vector space if F is a field. Arguably the theory of "cartesian spaces of fields" is not "interesting" in its own right – though its instances $\mathbb{R}^n$ and $\mathbb{C}^n$ may be. We model this in the diagram on the right, where set is a base theory that only introduces a set S. The theory CS introduces the Cartesian space S^n. On the other hand set is used in field base the base set, and field is in turn used as the scalar field in the theory VR of vector spaces. In this situation, the example we have in mind can be modeled as the theory CSF together with the view e, which has the assignment

$$\varphi := \begin{cases} a \cdot (v_1, \ldots, v_n) & \mapsto & (a \cdot v_1, \ldots a \cdot v_n) \\ (v_1, \ldots, v_n) + (w_1, \ldots, w_n) & \mapsto & (v_1 + w_1, \ldots v_n \cdot w_n) \\ 0 & \mapsto & (0, \ldots, 0) \\ -(v_1, \ldots, v_n) & \mapsto & (- \cdot v_1, \ldots - \cdot v_n) \end{cases}$$

where $+$, $\cdot$, 0, and $-$ on the right hand side are the operations of the field F.

Note that incidentally, in this example the example target theory CSF arises as the push-out of the theories field and CS over the base set, which means that it is indeed "uninteresting" since it can be computed from the two morphisms between these three. Given theory expression operators on theories in MMT, (see e.g. [CO12] for a proposal in the context of MathScheme), this could be representing CSV as the "unnamed" theory expression $\mathsf{F} \uplus_{\mathsf{set}} \mathsf{CS}$, which would consequently not be presented in active documents except as an example.

It would be interesting to see whether all cases of "example-only theories" can be expressed as theory expressions.

In SMGloMwe can express example-only theories by module signatures and just not give them language bindings. That has the effect that they are not shown in narrative documents and only serve a role as (syntactic) target theories.

4 Induced Examples and Practical Matters

Another advantage of the theory graph realization of examples is that we can induce examples from the represented structures. Take for instance the situation in Figure 7. Here we have the example morphisms i and j for commutative groups (theory cgp). Composing them with the inclusion from monoid to cgp yields the two red views i′ and j′ that make $\mathbb{R}^+$ and $\mathbb{R}^*$ examples for a monoid. We call such example morphisms that are formed by composition **induced examples**. Note that induced examples

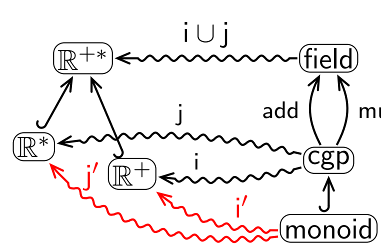


Figure 7: Induced Examples

[8] EdNote: MK: there must be a better name for such theories

are redundant in a theory graph since they can be computed from material in the remaining graph.

Obviously, the possibility of inducing examples suggests that examples should be specified as "high" in the theory graph as possible, since that maximize induced examples. However, this only seems to work along structures and inclusions as the construction with fields in Figure 7[9] shows. EdN:9

5 Flexiformal Theory Graphs and Narrative Examples

In SMGloMwe would split the example morphism into a signature and a language binding, where the language binding would say something like

> $\langle\mathbb{R},+,0,-\rangle$ is a commutative group, since [proofs of the cgroup axioms in $\langle\mathbb{R},+,0,-\rangle$]

In the purely formal setting of MMT, we can compute the composition morphisms like i'. But we in general cannot compute the language bindings for i', which would be something like

> $\langle\mathbb{R},+,0\rangle$ is a monoid, since [proofs of the monoid axioms in $\langle\mathbb{R},+,0\rangle$]

since natural language is opaque to the theory graph algorithms. What we can however compute is an explanation like

> $\langle\mathbb{R},+,0,-\rangle$ is a commutative group, since [proofs of the monoid axioms in $\langle\mathbb{R},+,0,-\rangle$] and commutative groups are monoids by definition.

by combining the language binding of i with an explanation for the inclusion from monoid to cgp. Note that this explanation directly corresponds to the structure of the composition that defines i'.

6 Conclusion

[10] EdN:10

We have presented an approach of modeling examples as theory morphisms in the OMDoc/MMT representation format for mathematical knowledge. It turns out that OMDoc-style parallel markup together with MMT theory morphisms allow to adequately capture the surface structure and the relational aspects of the meaning of examples.

We propose to extend OMDoc/MMT with the concept of antimorphisms capture the essence of counter-examples in a primitive that is commensurate with the design of MMT. The details of this proposal still have to be worked out, but the examples of counterexamples we have studied seem to bear out the conjecture that antimorphisms are for counter-examples what morphisms are for supporting examples.

I believe that examples and counter-examples form an important part of the mathematical domain knowledge that plays a great role in the formation of mathematical intuitions and the exploration of mathematical domains. Therefore any attempt to flexiformalize mathematical

[9]EdNote: MK: the graph does not work. $\mathsf{i}\cup\mathsf{j}$ is not the right graph. We seem to have R+* as the push out of i and j somehow; fix the graph and then argue why this precludes the top-down induction of i and j from the R+* example

[10]EdNote: Say something about [Rab12] that mediates between the theory-level and object-level representations of examples.

knowledge should take them seriously. In particular, universal digital mathematical libraries like the one called for in [Far11] or realized in [Ian+14; MH] should collect examples and counterexamples as first-class citizens – which the model proposed in this note allows to do transparently and conservatively.

In the future we want to study how the model of examples proposed here interacts with realms [CFK14], whether we can extend it to counter-examples for conjectures, and which mathematical practices with examples can be supported in active documents. Furthermore, there is clearly a relation between antimorphisms, the semantic antonym relation (see [Koh14] for a discussion of lexical relations and their relation with theory morphisms), and counter-examples, which we want to study in the future.

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