MATH 2P71: Intro to Combinatorics
Assignment 5: Graph Theory

Dennis Ideler

November 28, 2011

1. Draw all graphs on 5 nodes in which every node has degree at most 2. Assume that graphs are not labelled. Note: graphs where all nodes have the same degree, are *regular*.

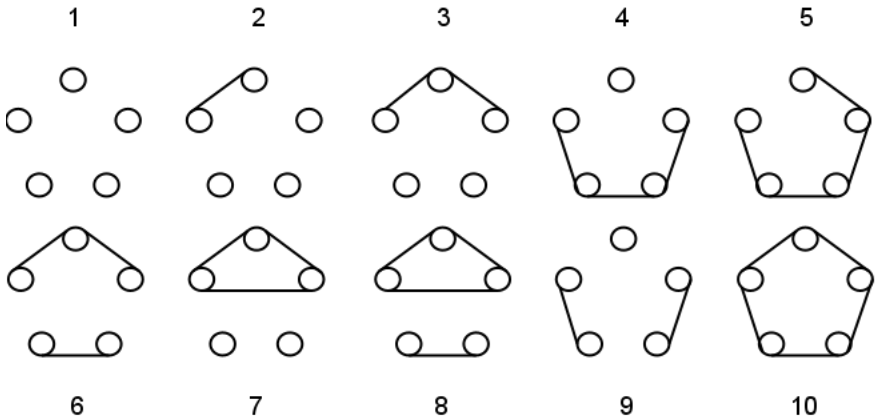

Figure 1: 10 graphs on 5 nodes in which every node has degree at most 2.

2. How many subgraphs does a 4-cycle have? Assume that the 4-cycles is labelled.

   The textbook states (as the answer to question 7.2.2) that the edgeless graph on n nodes has 2^n or $\sum_{k=0}^{n}\binom{n}{k}$ subgraphs, and that a 3-cycle (triangle) graph has 18 subgraphs. A 4-cycle graph will have $2^4 = 16$ edgeless subgraphs. We still need to count the subgraphs with edges.

	0 isolated nodes	1 isolated node	2 isolated nodes	3 isolated nodes	4 isolated nodes
0 edges	1	4	6	4	1
1 edge	4	8	4	—	—
2 edges	6	4	—	—	—
3 edges	4	—	—	—	—
4 edges	1	—	—	—	—

This can also be written as:

	0 isolated nodes	1 isolated node	2 isolated nodes	3 isolated nodes	4 isolated nodes
0 edges	$\binom{4}{0}$	$\binom{4}{1}$	$\binom{4}{2}$	$\binom{4}{3}$	$\binom{4}{4}$
1 edge	$\binom{4}{1}$	$\binom{4}{1}+\binom{4}{1}$	$\binom{4}{3}$	—	—
2 edges	$\binom{4}{2}$	$\binom{4}{3}$	—	—	—
3 edges	$\binom{4}{3}$	—	—	—	—
4 edges	$\binom{4}{4}$	—	—	—	—

In total, there's 47 subgraphs.

3. Let G be a connected graph with at least two vertices. Prove that it has a vertex such that if this vertex is removed (along with all edges incident with it), the remaining graph is connected.[1]

   To find the vertex that we can safely remove:

   (a) Find the longest path[2]
   (b) Remove endpoint node of longest path
   (c) Graph remains connected

   This works because every connected graph contains a spanning tree.[3] Our longest path cannot contain any repeated nodes, so it will be in the form of a spanning tree. In trees, the endpoint of the longest path will be a leaf node. As the lemma states, removing a leaf node (our endpoint) produces a tree, which by definition is connected (a tree is a connected acyclic graph).

   Lemma: If T is a tree and $n(T) \geq 2$ then T contains at least two leaves.
   $\therefore$ Deleting a leaf from a tree produces a tree.

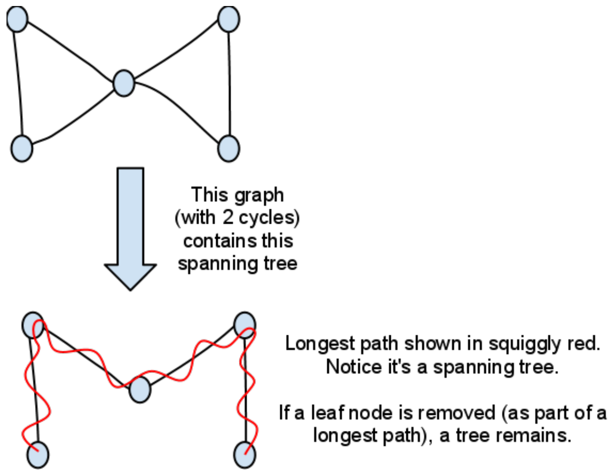

Figure 2: Example of how a connected graph with cycles can be reduced to a spanning tree which contains the longest path.

4. Does there exists a graph with the following degrees:

   (a) **0**, 2, 2, 2, 4, 4, **6**

   No, because there are 7 nodes, and if there exists a node with degree 6, it must be adjacent to every other node. Which means there cannot exist a node with degree 0, yet one is given.

   (b) 2, 2, **3**, **3**, 4, 4, **5**

   No. The Handshaking Lemma states that for any graph $G = (V, E)$, $\sum_{v \in V} deg(v) = 2|E|$ which means that the sum of degrees of each of the vertices is equal to the cardinality of the edge-set multiplied by two.

   From this we can derive:

   i. In every graph, the number of nodes with odd degree is even.

[1] The remaining graph can be disconnected, but there is *at least one* vertex that ensures it remains connected.

[2] The longest path problem is the problem of finding a simple path of maximum length in a given graph. A path is called simple if it does not contain any repeated vertices.

[3] To get a spanning tree out of a connected graph, remove an edge from any cycle that exists. Continue to do so until the subgraph has no cycles, thus a spanning tree remains.

ii. No graph of odd order is regular with odd degree.

The first statement says that our graph is impossible because the number of nodes with odd degree is odd (there are 3 of them).

5. Prove that if a tree has a node of degree d, then it has at least d leaves.

To prove this, we have to find the node in the tree with the highest degree, because any other node will have a lesser or equal degree and thus cannot have more leaves than the node with the highest degree. So we find the largest n-star which is a subtree.

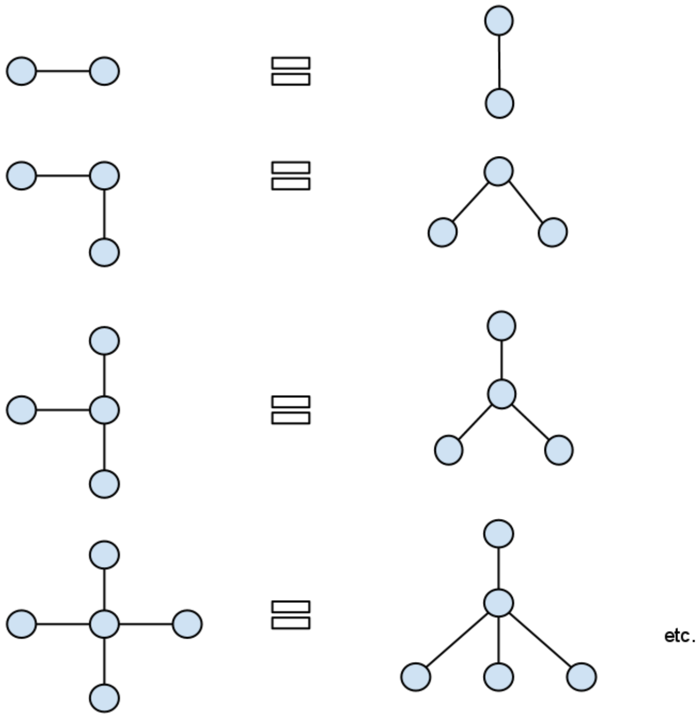

Figure 3: n-stars are trees (shown here as rooted trees).

An n-star graph has as many leaves as the degree of the centre node (i.e. $n-1$).
The tree lemma (mentioned earlier) states that we can remove leaves from the tree and it remains a tree. So we can do this until we reach the largest n-star which is a subtree. This subtree has a node of degree d (the centre node) and d leaves. So that means the supertree has at least d leaves since it contains this subtree (and possibly more).

6. Take an n-cycle, and connect two of its nodes at distance 2 by an edge. Find the number of spanning trees in this graph.

Loosely speaking, a spanning subgraph is a subgraph that contains the same vertex-set as the supergraph, $V(G') = V(G)$. A spanning tree is a connected acyclic spanning subgraph.

To make this problem easier we only consider labelled n-cycles.
Notice a pattern: $C_4 = 4 + \binom{2}{1} + \binom{2}{1} = 8$, $C_5 = 5 + \binom{3}{2} + \binom{3}{2} = 11$, $C_6 = 6 + \binom{4}{3} + \binom{4}{3} = 14$, etc.
There are $n + \binom{n-2}{n-3} + \binom{n-2}{n-3}$ spanning trees in this modified n-cycle graph, except for $n = 3$ which has 3 spanning trees.

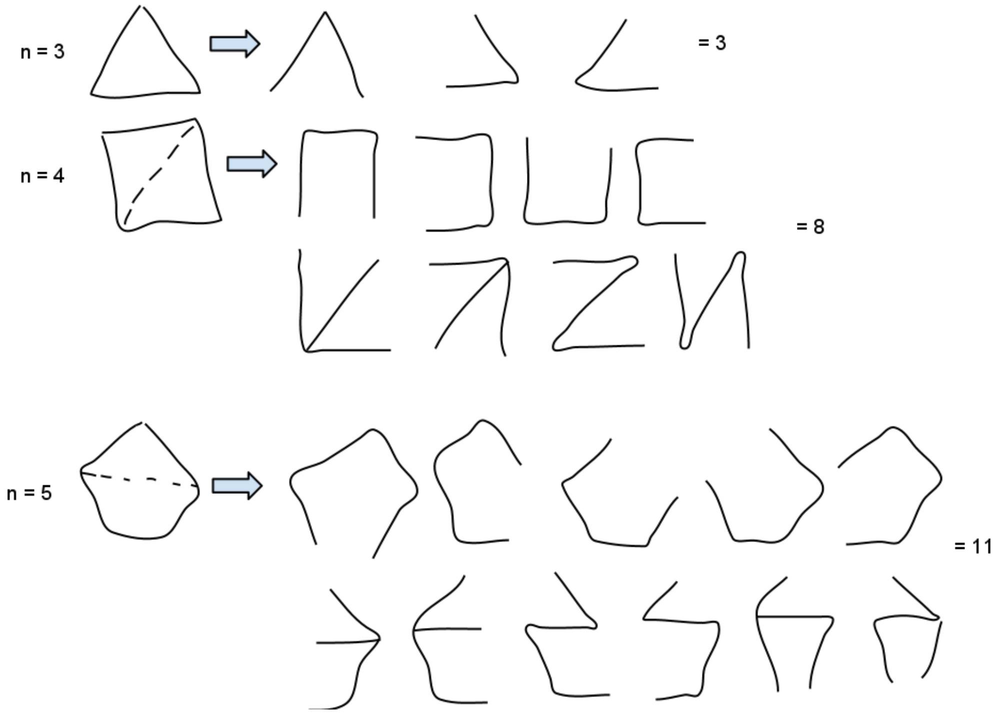

Figure 4: Sloppy drawings of all spanning subtrees for C_3 to C_5.