A Riemann Solver for Wet/Dry Interfaces in Geoclaw

Synthesis Paper

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1 Introduction

A Riemann Problem is a specific intial value problem (Cauchy problem) of a partial differential equation (PDE) that consists of conservation equations combined with piecewise constant intial data which has a single discontinuity in the domain of interest as shown in equation(2) for a linear scalar advection equation (1).

$$q_t + aq_x = 0 \tag{1}$$

$$q(x,0) = \begin{cases} q_L, & \text{if } x \leq 0, \\ q_R, & \text{if } x > 0, \end{cases} \tag{2}$$

where q_R and q_L are two piecewise constant states separated by a discontinuity. Applying method of characteristics to the initial value problem (IVP) gives equation (3) of the trajectory characteristic curve.

$$x = x_o + at \tag{3}$$

Equation (3) is used to obtain the exact solution of the Riemann problem (equatio (4))

$$q(x,t) = \begin{cases} q_L, & \text{if } x - at \leq 0, \\ q_R, & \text{if } x - at > 0, \end{cases} \tag{4}$$

2 Wave Propagation Algorithm (WPA)

Computational diffculties arise due to shocks or dicontinuities. For instance, consider a PDE (equation (5)) which is a one dimensional (1D) conservation law, where q is the measure of the conserved quantity density and $f(q)$ is a flux function.

$$q_t(x,t) + f(q(x,t))_x = 0 \tag{5}$$

A discontinuity in q violates the PDE in the classical sence and only holds for the intergral conservation law (fundamental equation (6)). Therefore at grid points near discontinuites where PDEs don't hold, all classical finite difference methods breakdown, resorting to finite volume methods (FVM) which are based on (6).

$$\frac{d}{dx}\int_{x_1}^{x_2} q(x,t)dx = f(q(x_1,t)) - f(q(x_2,t)) \tag{6}$$

In FVM, the domain is broken down into grid cells. The approximate total integral of q over each grid cell is evaluated and updated at every time step by the grid cell edge fluxes. The determined numerical flux functions evaluate the cell averages over a certain volume, which are used to approximate the solutions within the cells [2].

The Riemann problem is a fundamental tool in the evolution of FVM. Taking two neighbouring grid cells: Q_{i-1} and Q_i to be cell averages, the information used to compute numerical flux and then modifying the cell average at each time step is obtained by solving the Riemann problem with $Q_{i-1} = q_L$ and $Q_i = q_R$. Obtaining the eigen values and vectors of the coefficient matrix of q_x of a linear hyperbolic system, easily solves its Riemann problem [2].

Consider $C_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ to be the i^{th} grid cell, the average value over the i^{th} interval at time t^n is numerically approximated by Q_i^n in equation (7).

$$Q_i^n \approx \frac{1}{\Delta x} \int_{C_i} q(x, t^n) dx \tag{7}$$

where $\Delta x = (x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}})$ is the cell length. Equation (7) agrees with function q at the mid-point of the interval to $O(\Delta x^2)$ only if q is smooth.

Equation (1) can be expressed in form of a linear hyperbolic system of the form:

$$q_t + A q_x = 0 \tag{8}$$

where $A \in \mathbb{R}^{m \times m}$ is a constant diagonalizable matrix with real eigenvalues (λ) such that:

$$A = R \wedge R^{-1} \tag{9}$$

where R is a matrices of right eigenvectors and $\wedge$ is a diagonal matrix of eigenvalues. Consider $w = R^{-1} q$, so that equation (8) can be reduced to a set of m decoupled equations as shown in equation (10).

$$w_t + \wedge w_x = 0 \tag{10}$$

The data (equation (12)) for equation (10) can be computed using the given data (equation (11)).

$$\begin{aligned} q(x, 0) &= q_o(x) \quad \text{for} \quad -\infty < x < -\infty && (11) \\ w_o(x) &\equiv R^{-1} q_o(x) && (12) \end{aligned}$$

The advection equation (13) with solution (14), is obtained from the p^{th} equation of (10)

$$\begin{aligned} w_t^p + \lambda^p w_x^p &= 0 && (13) \\ w^p(x, t) &= w_o^p(x - \lambda^p t) && (14) \end{aligned}$$

The solution ((15)) to equation (8) is obtained by combining all the computed components $w^p(x, t)$ into the vector $w(x, t)$ as shown in equation (15).

$$q(x, t) = R w(x, t) \tag{15}$$

Cosidering vector $q(x, t)$ as a linear combination of the right eigenvectors $(r_1, ..., r_m)$ at each point in space and time, and a superposition of waves moving at different velocities, equation (15), can be expressed as:

$$q(x, t) = \sum_{p=1}^{m} w^p(x, t) r^p \tag{16}$$

For the Riemann problem (equation (2)), q_L and q_R can be decomposed into:

$$q_L = \sum_{p=1}^{m} w_L^p(x,t) r^p \tag{17}$$

$$q_R = \sum_{p=1}^{m} w_R^p(x,t) r^p \tag{18}$$

Then Riemann data $w_o^p(x)$ of the p^{th} advection equation (13) is given by:

$$w_o^p(x) = \begin{cases} w_L^p, & \text{if } x < 0, \\ w_R^p, & \text{if } x > 0, \end{cases} \tag{19}$$

The discontinuity in (19) propagates with speed λ^p as shown in equation (20)

$$w^p(x,t) = \begin{cases} w_L^p, & \text{if } x - \lambda^p t < 0, \\ w_R^p, & \text{if } x - \lambda^p t > 0, \end{cases} \tag{20}$$

Since the solution is constant across the p^{th} characteristic, the solution jumps associated with the eigenvectors of matrix A being a scalar multiple of r^p (jump in q) is given by equation (21):

$$(w_R^p - w_L^p) r^p \equiv \alpha^p r^p \tag{21}$$

where α are coefficients of eigenvectors r. Then to solve the Riemann problem, the initial data (q_L, q_R) is taken and then the jump $(q_L - q_R)$ is decomposed into a set of eigenvectors of A:

$$q_R - q_L = \alpha^1 r^1 + ... + \alpha^m r^m \tag{22}$$

The linar system of equations (23) is solved to obtain vector α .

$$q_R - q_L = R\alpha \tag{23}$$

Equation (23), can be expressed as:

$$Q_i - Q_{i-1} = \sum_{p=1}^{m} \alpha_{i-\frac{1}{2}} r_{i-\frac{1}{2}} \tag{24}$$

The wave speed $s_{i-\frac{1}{2}}^p$ associated with the vector $r_{i-\frac{1}{2}}^p$, are preselected basing on the characteristic structure of the initial Riemann data [1]. Therefore the fluctuations $\mathcal{A}^+\Delta Q_{i-\frac{1}{2}}^n$ and $\mathcal{A}^-\Delta Q_{i+\frac{1}{2}}^n$ are defined by equations (25) and (26):

$$\mathcal{A}^+\Delta Q_{i-\frac{1}{2}}^n = \sum_{\{p:s_{i-\frac{1}{2}}^p > 0\}} s_{i-\frac{1}{2}}^p w_{i-\frac{1}{2}}^p \tag{25}$$

$$\mathcal{A}^-\Delta Q_{i+\frac{1}{2}}^n = \sum_{\{p:s_{i+\frac{1}{2}}^p < 0\}} s_{i+\frac{1}{2}}^p w_{i+\frac{1}{2}}^p \tag{26}$$

Combining equations (25) and (26), the first order 1D wave propagation method is given by equation (27)

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(\mathcal{A}^+\Delta Q_{i-\frac{1}{2}}^n + \mathcal{A}^- Q_{i+\frac{1}{2}}^n) \tag{27}$$

where $\Delta t = (t^{n+1} - t^n)$, $\mathcal{A}^{\pm}\Delta Q_{i\pm\frac{1}{2}}^n$ are fluctuations determined by the to the Riemann Problems at cell interfaces at $x_{i\pm\frac{1}{2}}$. The net updating contributions from the rightward and leftward moving waves into the grid cell C_i from the right and left interface are respectively given by $\mathcal{A}^+\Delta Q_{i-\frac{1}{2}}^n$ and $\mathcal{A}^- Q_{i+\frac{1}{2}}^n$[1]. In a standard conservative case ((5)), the sum of the left-going and right-going fluctuations should satisfy:

$$\mathcal{A}^+\Delta Q_{i-\frac{1}{2}}^n + \mathcal{A}^- Q_{i+\frac{1}{2}}^n = f(Q_i) - f(Q_{i-1}) \tag{28}$$

The second order accuracy is obtained by taking the correction terms into account as shown described in equation (29)

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(\mathcal{A}^+\Delta Q_{i-\frac{1}{2}}^n + \mathcal{A}^- Q_{i+\frac{1}{2}}^n) - \frac{\Delta t}{\Delta x}(\tilde{F}_{i+\frac{1}{2}}^n - \tilde{F}_{i-\frac{1}{2}}^n) \tag{29}$$

where $\tilde{F}_{i\pm\frac{1}{2}}^n$ are second order correction terms determined by the waves in the Riemann problems after approximating the average flux along $x = x_{i\pm\frac{1}{2}}$:

$$\tilde{F}_{i\pm\frac{1}{2}}^n = \frac{1}{2}\sum_{p=1}^{m} |s_{i\pm\frac{1}{2}}^p| \left(1 - \frac{\Delta t}{\Delta x}|s_{i\pm\frac{1}{2}}^p|\right) \tilde{w}_{i\pm\frac{1}{2}}^p \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(q(x_{i\pm\frac{1}{2}}, t))dt \tag{30}$$

Here $\tilde{w}_{i\pm\frac{1}{2}}^p$ depicts a limited version of the wave $w_{i\pm\frac{1}{2}}^p$, which is obtained after a comparision between $w_{i\pm\frac{1}{2}}^p$ and $w_{i\pm\frac{3}{2}}^p$ when $s^p > 0$.

3 Shallow Water Equations (SWE)

The SWE are a system of hyperbolic or parabolic (for viscous shear) PDEs governing the flow below a pressure surface in a fluid. They arise from the Navier-stokes equations and can be used to model a fluid in a channel of unit width, taking the vertical velocity negilible, and horizontal velocity roughly constant throughout any cross section of the channel [1].

Consider a small-amplitude waves in a fluid that is shallow relative to its wavelength. The conservation of momentum equation is written in terms of pressure, $p(x,t)$, (equation (31)) and the height field $h(x,t)$ (m), which breaks down into two equations (32) and (33).

$$p(x,t) = \frac{1}{2}\rho g h^2 (N/m^2) \tag{31}$$

$$h_t + (uh)_x = 0 \tag{32}$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0 \tag{33}$$

where hu measures the flow rate of water past a point, ρ (kg/m^3) is the constant density of the incompressible fluid, and $u(x,t)$ (m/s) is the horizontal velocity.

Combining the equations (32) and (33), forms a system of one-dimensional SWEs given by equation (34)

$$\begin{bmatrix} h \\ hu \end{bmatrix}_t + \begin{bmatrix} uh \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix}_x = 0 \tag{34}$$

Equation (34) can be written as a quasi-linear system as shown in equation (35):

$$\begin{bmatrix} h \\ hu \end{bmatrix}_t + \begin{bmatrix} 0 & 1 \\ -u^2 + gh & 2u \end{bmatrix} \begin{bmatrix} h \\ hu \end{bmatrix}_x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{35}$$

or more generally as:

$$\mathbf{q}_t + A\mathbf{q}_x = 0 \tag{36}$$

where $\mathbf{q}(x,t) = (h(x,t), hu(x,t))$ and A is a 2×2 matrix given by:

$$A = \begin{bmatrix} 0 & 1 \\ -u^2 + gh & 2u \end{bmatrix} \tag{37}$$

LeVeque and Pelanti in [3] proposed that equation (34) can be decomposed into equation (38):

$$\begin{bmatrix} H_i - H_{i-1} \\ HU_i - HU_{i-1} \\ \varphi(Q_i) - \varphi(Qi-1) \end{bmatrix}_t = \sum_{p=1}^{3} \alpha^p_{i-\frac{1}{2}} w^p_{i-\frac{1}{2}} \tag{38}$$

where the momentum flux $\varphi(q) = (hu^2 + \frac{1}{2}gh^2)$, $Q_i = (H_i, HU_i)^T$ represents the numerical solution of $q = (h, hu)^T$ in C_i, and $w^p_{i-\frac{1}{2}} \in \mathbb{R}^3$,$\forall\ p \in [1,3]$ with p a set of chosen independent vectors.

4 Reimann Problem for Wet/Dry States

Dry states are regions with zero water depth. In such states SWEs are not applicable, so we consider wet states adjacent to dry regions as shown in figure 1. This enables solving SWEs in wet states, right up the boundary between wet and dry states [4].

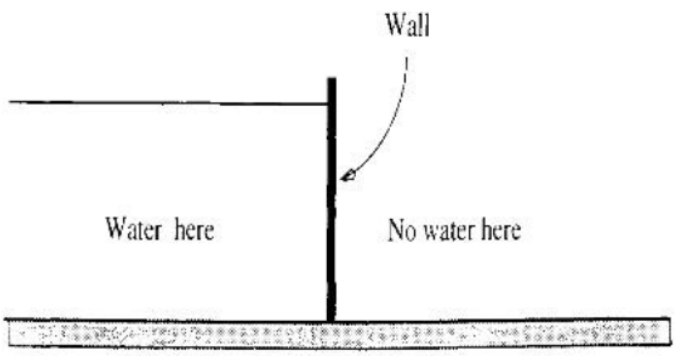

Figure 1: The Riemann problem with a dry bed (has no water) in one of the data state [4].

There is only one rarefaction in the exact Riemann solution, that connects wet to dry state. The developing wet/dry interface is in this case single edge of the rarefaction, making it possible to determine the interface propagating speed using the corresponding characteristics field of the Riemann invariants [1].

The wet/dry interface propagation speed is given by eqaution (39), since the right state in figure 1, is considered as the initial dry state, which also makes the rarefaction to be in the first characteristic field as shown in figure 2.(a).

$$s^3_{i-\frac{1}{2}} = \check{s}^+_{i-\frac{1}{2}} = \lambda^{-*}_{i-\frac{1}{2}}(0) = U_{i-1} - 2\sqrt{gH_{i-1}} \tag{39}$$

If the left state in figure 1, is considered as the initial dry state, then the wet/dry interface propagation speed is given by eqaution (39), and the rarefaction is in the second characteristic field as shown in figure 2.(b).

$$s^1_{i-\frac{1}{2}} = \check{s}^-_{i-\frac{1}{2}} = \lambda^{+*}_{i-\frac{1}{2}}(0) = U_i - 2\sqrt{gH_i} \tag{40}$$

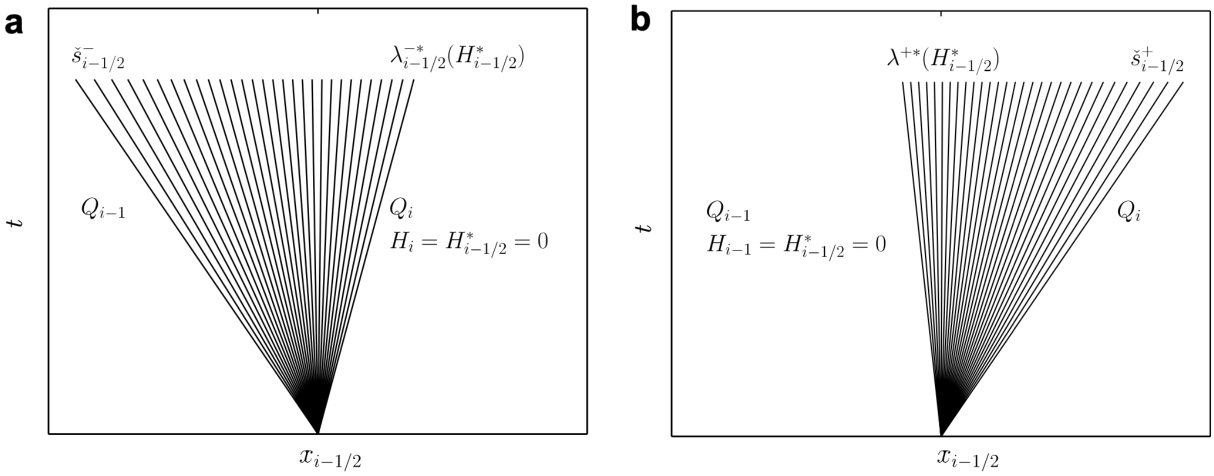

Figure 2: The characteristic struture of two dry intial state Riemann problem(dry bed) in the x-t plane [1].

5 Numerical Examples

References

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