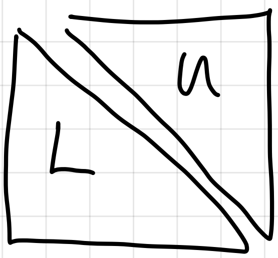
LU Decomposition Example

$$A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} = LU$$

$L = ?$ $\quad U = ?$

$$A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} \qquad LU =$$

$A =$ 

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix} \qquad E_{31} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -l_{31} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underbrace{(E_{43} E_{42} E_{32} E_{41} E_{31} E_{21})}_{\text{elementary matrices}} A = U$$

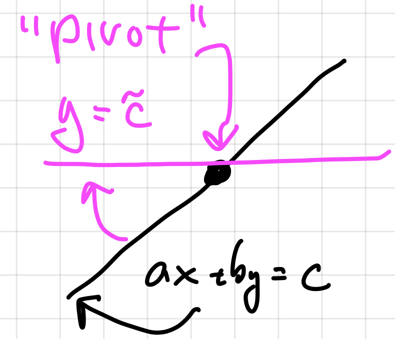
"Gaussian elimination in matrix form!

$$A = (E_{43} E_{42} E_{32} E_{41} E_{31} E_{21})^{-1} U$$

$$\underbrace{(E_{21}^{-1} E_{31}^{-1} E_{41}^{-1} E_{32}^{-1} E_{42}^{-1} E_{43}^{-1})}_{L} U$$

"pivot"

$$A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix}$$



$$\begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix}$$

Pivot Row

$R_2 \leftarrow R_2 - l_{21} R_1$

$R_3 \leftarrow R_3 - l_{31} R_1$

$R_4 \leftarrow R_4 - l_{41} R_1$

To get l_{21}: Need $a_{21} - l_{21} a_{11} = 0$

$$l_{21} = \frac{a_{21}}{a_{11}} = \frac{-3}{3} = -1$$

To get l_{31}: Need $a_{31} - l_{31} a_{11} = 0$

$$l_{31} = \frac{a_{31}}{a_{11}} = \frac{6}{3} = 2$$

To get l_{41}: Need $a_{41} - l_{41} a_{11} = 0$

$$l_{41} = \frac{a_{41}}{a_{11}} = \frac{-9}{3} = -3$$

$$L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix}$$

Pivot Row

$R_2 \leftarrow R_2 - \ell_{21} R_1$

$R_3 \leftarrow R_3 - \ell_{31} R_1$

$R_4 \leftarrow R_4 - \ell_{41} R_1$

$\ell_{21} = -1 \quad \ell_{31} = 2 \quad \ell_{41} = -3$

"pivots"

$$U_1 = \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 10 & 4 & -9 \\ 0 & -16 & -11 & 18 \end{bmatrix}$$

Pivot Row

$R_3 \leftarrow R_3 - \ell_{22} R_2$

$R_4 \leftarrow R_4 - \ell_{32} R_2$

$\ell_{22} = 10 - \ell_{22}(-2) = 0 \quad \ell_{22} = \frac{10}{-2} = -5$

$\ell_{32} = \frac{-16}{-2} = 8$

$$U_2 = \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -3 & 2 \end{bmatrix} \quad L_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & \ell_{43} & 1 \end{bmatrix}$$

$$U_2 = \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

Pivot row

$R_4 \leftarrow R_4 - l_{43} R_3$

$$l_{43} = \frac{-3}{-1} = 3$$

$$U_3 = \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$L_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

4 non-zero pivots $\quad [L, U] = \mathrm{lu}(A)$

LU costs $\sim \frac{2}{3} n^3$ work.

How do we solve $Ax = b$ using LU?

1. Factor A as LU
2. Write
$$L \underbrace{U x}_{y} = b$$

forward solve

3. Solve $Ly = b \Rightarrow y = L^{-1} b$

back solve

4. Solve $Ux = y \Rightarrow x = U^{-1} y$
5. $x = U^{-1} L^{-1} b \quad (LU)^{-1} b = A^{-1} b$