1. Derivation of Linear multistep (LMS) methods

$$U^{n+3} = U^{n+1} + \frac{k}{3}\left[7f(t_{n+2}, U^{n+2}) - 2f(t_{n+1}, U^{n+1}) + f(t_n, U^n)\right]$$

a) Derive this method using any technique you desire.

Consider

$$u(t_{n+3}) = u(t_{n+1}) + \int_{t_{n+1}}^{t_{n+3}} f(t, u(t))\,dt$$

$$u'(t) = f(t, u)$$

Consider a lagrange polynomial

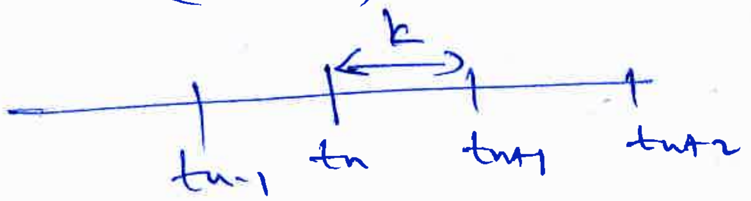
$$L_j(t) = \frac{(t-t_0)\cdots(t-t_{j-1})(t-t_{j+1})\cdots(t-t_n)}{(t_j-t_0)\cdots(t_j-t_{j-1})(t_j-t_{j+1})\cdots(t_j-t_n)}$$

$$\int_{t_{n+1}}^{t_{n+3}} f(t,u(t))\,dt = \sum_{j=0}^{n=2} \int_{t_{n+1}}^{t_{n+3}} L_j(t)\,dt\, f^{n+j}$$

$$\int_{t_{n+1}}^{t_{n+3}} f(t,u(t))\,dt = \int_{t_{n+1}}^{t_{n+3}} L_0(t)\,dt\, f^n + \int_{t_{n+1}}^{t_{n+3}} L_1(t)\,dt\, f^{n+1} + \int_{t_{n+1}}^{t_{n+3}} L_2(t)\,dt\, f^{n+2}$$

start with $\int_{t_{n+1}}^{t_{n+3}} L_0(t)\,dt\, f^n$

$$L_0(t) = \frac{(t-t_{n+1})(t-t_{n+2})}{(t_n-t_{n+1})(t_n-t_{n+2})}$$



, k is the time step

$$L_0(t) = \frac{(t-t_{n+1})(t-t_{n+2})}{2k^2}$$

~~let~~ ~~$/S/ = / \neq +1$~~

$$\int_{t_{n+1}}^{t_{n+3}} \left(\frac{(t-t_{n-1})(t-t_{n-2})}{2k^2} \right) dt \, f^{n}$$

let $s = t - t_{n-1} \Rightarrow ds = dt$

t	S
t_{n+1}	0
t_{n+3}	$2k$

and also

$t - t_{n-1} = S, \quad t - t_{n-2} = S - k$

So

$$\int_{t_{n+1}}^{t_{n+3}} \frac{(t-t_{n-1})(t-t_{n-2})}{2k^2} dt \, f^{n} = \int_{0}^{2k} \frac{S(S-k)}{2k^2} ds \, f^{n}$$

$$= \underline{\underline{\frac{k}{3} f^n}}$$

$$\int_{t_{n+1}}^{t_{n+3}} L_1(t) \, dt \, f^{n+1} = \int_{t_{n+1}}^{t_{n+3}} \frac{(t-t_n)(t-t_{n+2})}{-k^2} dt \, f^{n+1}$$

$$= \int_{0}^{2k} \frac{(S+k)(S-k)}{-k^2} ds \, f^{n+1} = \underline{\underline{\frac{-2k}{3}}} f^{n+1}$$

$$\int_{t_{n+1}}^{t_{n+3}} L_2(t) \, dt \, f^{n+2} = \int_{t_{n+1}}^{t_{n+3}} \frac{(t-t_n)(t-t_{n+1})}{2k^2} dt \, f^{n+2}$$

$$= \int_{0}^{2k} \frac{(S+k)S}{2k^2} ds \, f^{n+2} = \frac{7k}{3} f^{n+2}$$

So

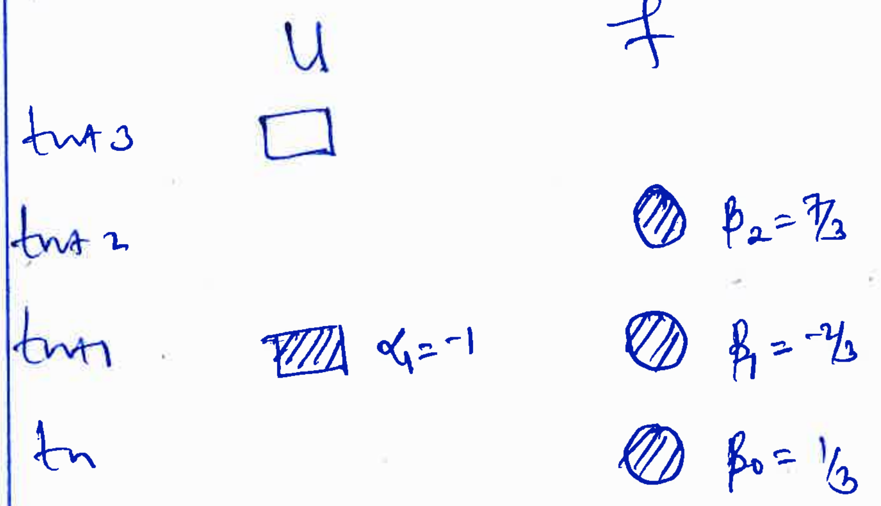
$$\int_{t_{n+1}}^{t_{n+3}} f(t, u(t))\,dt = \frac{k}{3} f^n - \frac{2k}{3} f^{n+1} + \frac{7k}{3} f^{n+2}$$

therefore

$$u(t_{n+3}) = u(t_{n+1}) + \frac{k}{3} f^n - \frac{2k}{3} f^{n+1} + \frac{7k}{3} f^{n+2}$$

$$U^{n+3} = U^{n+1} + \frac{k}{3}\left(7 f^{n+2} - 2 f^{n+1} + f^n\right)$$

b) Draw the stencil for this method.



c) Determine if this method is zero-stable.

For a method to be zero-stable, $|w| \leq 1$, so

$$\rho(w) = w^3 - w = 0$$
$$w(w^2 - 1) = 0$$
$$w = 0, \quad w = \pm 1$$

Its clear that $|w| \leq 1$, hence the method is zero stable.

d) Determine if this method is consistent.

For a method to be consistent $\rho(1)=0$ and $\rho'(1) = \sigma(1)$.

So

$$\sigma(w) = \tfrac{7}{3}w^2 - \tfrac{2}{3}w + \tfrac{1}{3}$$

$$\sigma(1) = 2$$

$$\rho(w) = w^3 - w \Rightarrow \rho'(w) = 3w^2 - 1$$

$$\rho'(1) = 3 - 1 = 2$$

So ~~$\sigma(w) =$~~ $\sigma(1) = \rho'(1)$ and $\rho(1) = 0$, hence it is consistent.

e) Determine if the method converges.

From Lax theorem, if the method is both stable and consistent then it converges, therefore our method converges.

f) Determine the order of accuracy of this method.

The local truncation error is given by

$$T(t) = C_0 + C_1 u'(t) + \frac{C_2}{2} u''(t) + \frac{C_3}{3!} k^2 u'''(t) + \cdots$$

$$C_q = \frac{1}{q!}\left[\alpha_1 + 2\alpha_2 + \cdots + r\alpha_r - \frac{1}{(q-1)!}(\beta_1 + 2\beta_2 + \cdots + r^{q-1}\beta_r\right.$$

Since the method is consistent, then $C_0 = 0$ and $C_1 =$

So

$$C_2 = \tfrac{1}{2}(1-9) - \left(-\tfrac{2}{3} + \tfrac{14}{3}\right) = 0$$

$$C_3 = \tfrac{1}{6}(27-1) - \tfrac{1}{2}\left(-\tfrac{2}{3} + \tfrac{28}{3}\right) = 0$$

$$C_4 = \frac{1}{24}(-1+81) - \frac{1}{6}\left(-\tfrac{2}{3} + \tfrac{56}{3}\right) = \underline{\underline{\tfrac{1}{3}}}$$

So

$$\tau(t) = \frac{C_4}{4!} k^3 u^{(iv)}(t) + \frac{C_5}{5!} k^4 u^{(v)}(t) + \cdots$$

$$\tau(t) = \frac{1}{72} k^3 u^{(iv)}(t) + \frac{C_5}{5!} k^4 u^{(v)}(t) + \cdots$$

therefore the order of Convergence, $\underline{\underline{p = 3}}$

2. Heat equation: Crank-Nicolson.

a) $u_t = \alpha u_{xx}$, $0 \le x \le 1$, $t \ge 0$

$u(x,0) = f(x)$, $u(0,t) = g_0(t)$, $u(1,t) = g_1(t)$

~~u_t~~ Using forward difference for u_t ~~and u_{xx}~~

$$u_t = \frac{U_j^{n+1} - U_j^n}{\Delta t}$$

also

$$u_{xx} = \frac{\nabla U_j^{n+1} + \nabla U_j^n}{2}$$

$$u_t = \alpha u_{xx}$$

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{\alpha}{2}\left[\nabla U_j^{n+1} + \nabla_j U_j^n\right]$$

$$= \frac{\alpha}{2h^2}\left[U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1} + U_{j-1}^n - 2U_j^n + U_{j+1}^n\right]$$

$$U_j^{n+1} - U_j^n = \frac{\Delta t \alpha}{2h^2}\left[U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1} + U_{j-1}^n - 2U_j^n + U_{j+1}^n\right]$$

let $r = \frac{\Delta t \alpha}{2h^2}$

$$-rU_{j-1}^{n+1} + (1+2r)U_j^{n+1} - rU_{j+1}^{n+1} = rU_{j-1}^n + (1-2r)U_j^n + rU_{j+1}^n$$

$$\begin{bmatrix} 1+2r & -r & & 0 \\ -r & 1+2r & -r & \\ & \ddots & \ddots & \ddots \\ & -r & 1+2r & -r \\ 0 & & -r & 1+2r \end{bmatrix} U^{n+1} = \begin{bmatrix} d_1 \\ \vdots \\ \vdots \\ d_m \end{bmatrix}$$

where

$$d_1 = rU_0^n + (1-2r)U_1^n + rU_2^n + rU_0^{n+1}$$

$$d_m = rU_{m-1}^n + (1-2r)U_m^n + rU_{m+1}^n + rU_{m+1}^{n+1}$$

$$d_j = rU_{j-1}^n + (1-2r)U_j^n + rU_{j+1}^n, \quad j = 2, 3, \ldots, m-1$$

3. Heat equation: BDF2

The BDF2 is given by

$$U'(t) = f(t, U(t))$$

$$U^{n+2} = \frac{4}{3}U^{n+1} - \frac{1}{3}U^n + \frac{k^2}{3}f^{n+2}$$

$$U^{n+1} = \frac{4}{3}U^n - \frac{1}{3}U^{n-1} + \frac{2}{3}kf^{n+1}$$

$$\frac{3}{2}U^{n+1} - 2U^{n} + \frac{1}{2}U^{n-1} = k f^{n+1}$$

Given $U_t = \alpha U_{xx}$,

then

$$U_t = \frac{\frac{3}{2}U_j^{n+1} - 2U_j^{n} + \frac{1}{2}U_j^{n-1}}{k}$$

So

$$\frac{\frac{3}{2}U_j^{n+1} - 2U_j^{n} + \frac{1}{2}U_j^{n-1}}{k} = \alpha \nabla U_j^{n+1}$$

$$\left(1 - \frac{2\alpha k}{3}\nabla\right) U_j^{n+1} = \frac{4}{3}U_j^{n} - \frac{1}{3}U_j^{n-1}$$

$$U_j^{n+1} - \frac{2\alpha k}{3}\nabla U_j^{n+1} = \frac{4}{3}U_j^{n} - \frac{1}{3}U_j^{n-1}$$

$$U_j^{n+1} - \frac{2\alpha k}{3}\left[U_{j}^{n+1} - 2U_j + U_{j+1}^{n+1}\right] = \frac{4}{3}U_j^{n} - \frac{1}{3}U_j^{n-1}$$

take $r = \frac{2\alpha k}{3}$

$$-r U_{j-1}^{n+1} + (1+2r)U_j^{n+1} - rU_{j+1}^{n+1} = \frac{4}{3}U_j^{n} - \frac{1}{3}U_j^{n-2}$$

4. Linear Stability analysis

$$u_t + u_{xxx} = 0 \quad , \quad 0 \le x \le 1, \quad t \ge 0$$

IC: $u(x,0) = g(x)$, BC: periodic

$$\frac{u(x,t+k) - u(x,t)}{k} + \frac{-\frac{1}{2}u(x-2h,t) + u(x-h,t) - u(x+h,t) + \frac{1}{2}u(x+2h,t)}{h^3} =$$

$$U_j^{n+1} - U_j^n = \frac{k}{h^3}\left[+\frac{1}{2}U_{j-2}^n - U_{j-1}^n + U_{j+1}^n + -\frac{1}{2}U_{j+2}^n\right]$$

let $\frac{k}{h^3} = r$

$$U_j^{n+1} - U_j^n = r\left[U_{j+1}^n - U_{j-1}^n + \frac{1}{2}\left(U_{j-2}^n - U_{j+2}^n\right)\right] \quad \text{--- (1)}$$

Using von Neumann stability analysis

$$U_j^n = \varepsilon^n e^{ikj\Delta x}$$

Therefore equation (1) becomes

$$\varepsilon^{n+1}e^{ikj\Delta x} - \varepsilon^n e^{ikj\Delta x} = r\left[\varepsilon^n e^{ik(j+1)\Delta x} - \varepsilon^n e^{ik(j-1)\Delta x} + \frac{1}{2}\left(\varepsilon^n e^{ik(j-2)\Delta x} - \varepsilon^n e^{ik(j+2)\Delta x}\right)\right]$$

$$\varepsilon - 1 = r\left[e^{ik\Delta x} - e^{-ik\Delta x} + \frac{1}{2}\left(e^{-2ik\Delta x} - e^{2ik\Delta x}\right)\right]$$

but $e^{ik\Delta x} - e^{-ik\Delta x} = 2i\sin(k\Delta x)$

$e^{-2ik\Delta x} - e^{2ik\Delta x} = -2i\sin(2k\Delta x)$

So

$$\varepsilon - 1 = r\left(2i\sin(k\Delta x) - i\sin(2k\Delta x)\right)$$

$$\mathcal{E}-1 = ir\left(2\sin(k\Delta x) - \sin(2k\Delta x)\right)$$

$$\mathcal{E} = 1 + ir\left(2\sin(k\Delta x) - \sin(2k\Delta x)\right)$$

but $\sin(2k\Delta x) = 2\sin k\Delta x \cos k\Delta x$

$$\mathcal{E} = 1 + 2ir\sin(k\Delta x)\left(1 - \cos(k\Delta x)\right)$$

$$\mathcal{E} = 1 - 2ir\sin(k\Delta x)\left(\cos(k\Delta x) - 1\right)$$

$$|\mathcal{E}| = \left|1 - 2ir\sin(k\Delta x)\left(\cos(k\Delta x) - 1\right)\right|$$

$$|\mathcal{E}| = 1 + 4r^2\sin^2(k\Delta x)\left(\cos(k\Delta x) - 1\right)^2$$

since $0 \le \sin^2(k\Delta x) \le 1$ and that

$$4r^2\sin^2(k\Delta x)\left(\cos(k\Delta x) - 1\right)^2 > 0$$

then

$$|\mathcal{E}| = 1 + \text{positive number} > 1$$

$$|\mathcal{E}| > 1$$

So, since $|\mathcal{E}| > 1$, then the scheme is unconditionally unstable. thus should never be used, since it will never converge.

b) $u_t + u_{xxx} = 0$

$$u_t = -u_{xxx}$$

$$u_t = \frac{u_j^{n+1} - u_j^n}{k} = \frac{u_j^{n+1} - u_j^n}{\Delta t}$$

$$u_{xx} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

$$U_{xx} = \frac{U_{j+1}^{n} - 2U_j^n + U_{j-1}^n}{\Delta x^2}$$

$$U_{xxx} = \frac{\partial}{\partial x}(U_{xx}) = \frac{1}{\Delta x^2}\left(\frac{\partial}{\partial x}U_{j+1}^n - 2\frac{\partial}{\partial x}U_j^n + \frac{\partial}{\partial x}U_{j-1}^n\right)$$

$$U_{xxx} = \frac{1}{\Delta x^2}\left(\frac{U_{j+2}^n - U_{j+1}^n}{\Delta x} - 2\left(\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x}\right) + \left(\frac{-U_{j-2}^n + U_{j-1}^n}{\Delta x}\right)\right)$$

$$U_{xxx} = \frac{1}{\Delta x^3}\left(U_{j+2}^n - U_{j+1}^n - U_{j+1}^n + U_{j-1}^n + U_{j-2}^n + U_{j-1}^n\right)$$

$$\cancel{= \frac{1}{\Delta x^3}\left(U_{j+2}^n - 2U_{j+1}^n + U_{j-2}^n\right)}$$

$$U_{xxx} = \frac{1}{\Delta x^3}\left(U_{j+2}^n - 2U_{j+1}^n + 2U_{j-1}^n - U_{j-2}^n\right)$$

Therefore $U_t = -U_{xxx}$

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = -\left(\frac{1}{\Delta x^3}\left(U_{j+2}^n - 2\left(U_{j+1}^n - U_{j-1}^n\right) - U_{j-2}^n\right)\right)$$

$$U_j^{n+1} - U_j^n = -\frac{\Delta t}{\Delta x^3}\left(U_{j+2}^n - 2\left(U_{j+1}^n - U_{j-1}^n\right) - U_{j-2}^n\right)$$

let $\frac{\Delta t}{\Delta x^3} = r$,

Using Von Neumann stability, let $U_j^n = \varepsilon^n e^{ikj\Delta x}$

$$\varepsilon^{n+1}e^{ikj\Delta x} - \varepsilon^n e^{ikj\Delta x} = -r\left(\varepsilon^n e^{ik(j+2)\Delta x} - 2\left(\varepsilon^n e^{ik(j+1)\Delta x} - \varepsilon^n e^{ik(j-1)\Delta x}\right) - \varepsilon^n e^{ik(j-2)\Delta x}\right)$$

$$\mathcal{E}-1 = -r\left(e^{2ik\Delta x} - 2\left(e^{ik\Delta x} - e^{-ik\Delta x}\right) - e^{-2ik\Delta x}\right)$$

$$\mathcal{E} = 1 - r\left(2i\sin(2k\Delta x) - 2\left(2i\sin(k\Delta x)\right)\right)$$

$$\mathcal{E} = 1 - 2ir\left(\sin 2k\Delta x - 2\sin k\Delta x\right)$$

$$\mathcal{E} = 1 - 2ir\left(2\cos k\Delta x \sin k\Delta x - 2\sin k\Delta x\right)$$

$$\mathcal{E} = 1 - 2ir\left(\cos k\Delta x - 1\right)\left(2\sin k\Delta x\right)$$

$$\mathcal{E} = 1 + 2ir\left(1 - \cos k\Delta x\right)\left(2\sin k\Delta x\right)$$

$$|\mathcal{E}| = \left|1 + 2ir\left(1-\cos k\Delta x\right)\left(2\sin k\Delta x\right)\right|$$

$$|\mathcal{E}| = 1 - 4r^2\left(1-\cos k\Delta x\right)^2\left(4\sin^2 k\Delta x\right)$$

Since $0 \leq \sin^2 k\Delta x \leq 1$ and that

$$4r^2\left(1-\cos k\Delta x\right)^2\left(4\sin^2 k\Delta x\right) > 0$$

$$|\mathcal{E}| = 1 - \text{positive value} < 1$$

$$|\mathcal{E}| < 1$$

Hence the Scheme is Conditionally Stable therefo
I propose this as an alternative Scheme.

$$U_t + U_{xxx} = 0$$

$$U_j^{n+1} - U_j^n = -r\left(U_{j+2}^n - 2\left(U_{j+1}^n - U_{j-1}^n\right) - U_{j-2}^n\right)$$

where $r = \dfrac{\Delta t}{\Delta x^3}$

$$u_j^{n+1} - u_j^n + r\left(u_{j+2}^n - 2\left(u_{j+1}^n - u_{j-1}^n\right) - u_{j-2}^n\right) = 0$$

$$u(x, t+k) - u(x,t) + r\Big[u(x+2h,t) - 2\big(u(x+h,t) - u(x-h,t)\big)$$

$$- u(x-2h,t)\Big) = 0$$

$$\frac{u(x,t+k) - u(x,t)}{\Delta t} + \frac{u(x+2h,t) - 2\left(u(x+h,t) - u(x-h,t)\right) - u(x-2h,t)}{\Delta x^3} =$$

$$0 \leq x \leq 1, \quad t \geq 0$$

IC: $u(x,0) = g(x)$, BC: periodic

5. Wave equation

a) Show that the two-way wave equation

$$u_{tt} = c^2 u_{xx}, \quad 0 \leq x < 2\pi, \ t > 0, \ u(x,0) = f(x),$$

$$u_t(x,0) = g(x),$$

Can be transformed into

$$u_t + v_x = 0$$

$$v_t + c^2 u_x = 0$$

let $u_t = -v_x$

$$u_{tt} = -\frac{\partial}{\partial t}(v_x) = c^2 u_{xx}$$

$$u_{tt} = \frac{\partial}{\partial t}(u_t), \quad 4$$

also $\frac{\partial}{\partial x}(v_t) = -\frac{\partial}{\partial x}(u_x)$

$$-\frac{\partial}{\partial x}(v_t) = \frac{\partial}{\partial x}(c^2 u_x)$$

$$-v_t = c^2 u_x \Rightarrow v_t = -c^2 u_x$$

hence

$$u_t + v_x = 0$$

$$v_t + c^2 u_x = 0$$

Given $q_t + A q_x = 0 \Rightarrow q_t = -A q_x$

from $u_t + v_x = 0$ and $v_t + c^2 u_x = 0$

$$A = \begin{pmatrix} 0 & 1 \\ c^2 & 0 \end{pmatrix}$$

$$q_t = F(q_x),$$

let $d_1 = q_j^n$

$$d_2 = q_j^n + \frac{k}{2}F(d_1) = q_j^n - \frac{k}{2}A(q_x)_j^n$$

$$d_3 = q_j^n + \frac{k}{2}F(d_2) = q_j^n + \frac{k}{2}F\left(q_j^n - \frac{k}{2}A(q_x)_j^n\right)$$

$$d_3 = \left(1 - \frac{kA}{2}\right)q_j^n + \frac{k^2A^2}{4}\left(q_x\right)_j^n$$

$$d_4 = q_j^n + \frac{k}{2}F(d_3)$$

$$d_4 = \left(1 - kA + \frac{k^2A^2}{2}\right)q_j^n - \frac{k^3A^3}{4}(q_x)_j^n$$

from

$$q_j^{n+1} = q_j^n + \frac{k}{6}\left(F(d_1) + 2F(d_2) + 2F(d_3) + F(d_4)\right)$$

then

$$F(d_1) = F(q_j^n) = -Aq_j^n$$

$$F(d_2) = F\left(q_j^n - \frac{k}{2}A(q_x)_j^n\right) = -A\left(q_j^n - \frac{kA}{2}(q_x)_j^n\right)$$

$$F(d_3) = -\left(A - \frac{kA^2}{2}\right)q_j^n - \frac{k^2A^2}{4}\left(q_x\right)_j^n$$

$$F(d_4) = -\left(A - kA^2 + \frac{k^2A^3}{2}\right)q_j^n - \frac{k^3A^4}{4}\left(q_x\right)_j^n$$

from

$$\eta_j^{n+1} = \eta_j^n + \frac{k}{6}\left(F(d_1) + 2F(d_2) + 2F(d_3) + F(d_4)\right)$$

$$\eta_j^{n+1} = \eta_j^n + \frac{k}{6}\Big[-A\eta_j^n + 2\Big(-A\eta_j^n + \frac{KA^2}{2}(\eta_x)_j^n\Big) + 2\Big(-\Big(A - \frac{KA^2}{2}\Big)\eta_j^n$$

$$- \frac{K^2A^2}{4}(\eta_x)_j^n\Big) + -\Big(A - KA^2 + \frac{K^2A^3}{2}\Big)\eta_j^n - \frac{K^3A^4}{4}(\eta_x)_j^n$$

$$\eta_j^{n+1} = \eta_j^n + \frac{k}{6}\Big[-3A\eta_j^n + KA^2(\eta_x)_j^n - 2A\eta_j^n + KA^2\eta_j^n - \frac{K^2A^2}{2}(\eta_x)$$

$$-A\eta_j^n + KA^2\eta_j^n - \frac{K^2A^3}{2}\eta_j^n - \frac{K^3A^4}{4}(\eta_x)_j^n\Big\}\Big]$$

$$\eta_j^{n+1} = \eta_j^n + \frac{k}{6}\Big[-6A\eta_j^n - KA^2(\eta_x)_j^n + 2KA^2\eta_j^n - \frac{K^2A^2}{2}(\eta_x)_j^n$$

$$- \frac{K^2A^3}{2}\eta_j^n - \frac{K^3A^4}{4}(\eta_x)_j^n\Big]$$