Homework #2

Math 465/565

A few notes about this homework.

- You will be graded on your presentation as well as the content of the work. This means that all of your plots should have axis labels, a title and legends, and any other annotation that makes it clear what you are demonstrating.
- You may submit your homework to BlackBoard. Please submit a PDF of your homework.
- Students in 465 may do the problems for 565 for extra credit.

1. Find the rate of convergence of the sequence

$$s_n = \frac{\sin(n)}{n} \tag{1}$$

as $n \to \infty$.

2. Show that the sequence

$$s_n = \frac{1}{n^2} \tag{2}$$

converges *linearly*.

3. Compute the limit and determine the corresponding rate of convergence for the function

$$\lim_{x \to 0} \frac{e^x - \cos x - x}{x^2} \tag{3}$$

4. The following Matlab code produces the output shown below

```
x = 1;
for k = 1:1074
    x = x/2;
    fprintf('%-7d %20.16e\n',k, x);
end
fprintf('\n');
fprintf('%10s %16.8e\n','realmin',realmin);
fprintf('%10s %16.8e\n','x',x);
```

Output

```
1       5.0000000000000000e-01
2       2.5000000000000000e-01
3       1.2500000000000000e-01
4       6.2500000000000000e-02
...............................
1071    3.9525251667299724e-323
1072    1.9762625833649862e-323
1073    9.8813129168249309e-324
1074    4.9406564584124654e-324

  realmin  2.22507386e-308
        x  4.94065646e-324
```

For the following, you are asked to show both the *binary* and 64-bit representation of the requested values. The binary representation should be of the form

$$(1 + b_1 2^{-1} + b_2 2^{-2} + b_3 2^{-3} + \ldots) \times 2^{\beta} \tag{4}$$

where $b_i = 0$ or 1, and β is an integer exponent. The 64-bit representation is a string of 64 0s and 1s.

(a) What is the value of iterate 52? Show the binary and 64-bit representations of this iterate. What is the name we often give to this number?

(b) Which iterate produces the value `realmin`? Show the binary and 64-bit representation of this number.

(c) Show the binary and 64-bit representations for the last value of x shown, and be sure to mention any IEEE floating point conventions that are involved. What is the name we give to these values smaller than `realmin`?

(d) Had the loop continued one more iteration, what would the resulting value of x be? Explain why.

(e) Using the Float-Toy (`http://evanw.github.io/float-toy/`)), give an example of a number less `realmin` that is not generated by the iterations above.

5. In class, we showed that

$$\lim_{x \to 0} \frac{\sqrt{1 + x^2} - 1}{x^2} = \frac{1}{2} \tag{5}$$

as $x \to 0$. We then argued that

$$\left| f(x) - \frac{1}{2} \right| \le \frac{1}{8} x^2 \tag{6}$$

for x close to 0, and concluced that the "rate of convergence" of $f(x)$ is $\mathcal{O}(x^2)$. To get to this conclusion, however, we simply disregarded all higher order terms in the Taylor Series expansion of $f(x)$, without justification.

Do the following to convince yourself, using graphic and analytic means, that the conclusion is justified.

(a) Provide convincing graphical evidence that $f(x)$ converges like $\mathcal{O}(x^2)$ as $x \to 0$.

- Plot $|f(x) - 1/2|$ and $x^2/8$ on the same graph to see the convergence behavior of $f(x)$.
- Use small enough values of x to really see the behavior near 0. In Matlab, for example, use

```
x = logspace(-3,-1,500)
```

  This creates a sequence of values x_i that are equally spaced in a logarithmic scale. That is, $x_i = 10^{p_i}$, where the $p_1 = -3$, $p_{500} = -1$ and the remaining p_i are equally spaced values between -3 and -1.
- To see the behavior clearly, use a log scale for the y-axis. In Matlab, you can set the scale of the y-axis using

```
set(gca,'yscale','log')
```

(b) (**Math 565.**) Use the Remainder Theorem to show formally that we are justified in claiming that the rate of convergence of $f(x)$ is $O(x^2)$. Is our choice of $\lambda = 1/8$ reasonable?

6. Recreate the plot you created for Problem 5, but this time use very small values of x, i.e.

```
x = logspace(-6,-3,500)
```

You should see very strange behavior for x less than about 10^{-4}. This behavior is the result of "catastrophic cancellation". To understand this, consider the following expression

$$g(\varepsilon) = \sqrt{1 + \varepsilon^2}, \qquad \varepsilon \ll 1 \tag{7}$$

(a) Show in finite precision arithmetic (roughly 16 decimal digits), we are justified in approximating $g(\varepsilon)$ as

$$g(\varepsilon) \approx 1 + \frac{1}{2}\varepsilon^2 \tag{8}$$

for $\varepsilon \lesssim 10^{-4}$. **Hint:** What is the size of the next term in the Taylor Series for $g(\varepsilon)$?

(b) Show the result of each operation needed to evaluate $f(\varepsilon)$, given by

$$f(\varepsilon) \approx \frac{(1 + \frac{1}{2}\varepsilon^2) - 1}{\varepsilon^2} \tag{9}$$

in finite precision arithmetic, for $\varepsilon = 10^{-4}$. Explain how "garbage" digits can appear in the calculation, resulting in the strange behavior you see in your plot of $f(x)$.

(c) For what value of ε do you expect to see almost no digits of accuracy in your evaluation of $f(\varepsilon)$?

(d) Recreate the plot from Problem 5 for these very small values, but this time use the Taylor Series approximation to $f(x)$. Include terms up to x^6.

7. (**Math 565**). Consider the sequence of partial sums

$$S_n(x) = 1 - x + x^2 - x^3 + \ldots + (-1)^n x^n = \sum_{k=0}^{n} (-1)^k x^k \tag{10}$$

for $|x| < 1$.

(a) Numerically generate values of $S_n(x)$ for $x = 1/\pi$. Show that the error sequence

$$e_n(x) = \left| S_n(x) - \frac{1}{1+x} \right| \tag{11}$$

converges linearly with asymptotic error constant x. Turn in your numerically generated sequence of values, and convincing numerical evidence that the sequence converges linearly.

(b) Show analytically that the sequence $S_n(x)$ converges linearly.