1. Show that

$$\frac{x_{n+1} - x_n}{x_n - x_{n-1}} \approx \lambda$$

Suppose that $x_{n+1} - \bar{x}$, $x_n - \bar{x}$ and $x_{n-1} - \bar{x}$

take

$$e_n = x_n - \bar{x} \quad \text{— ①}$$

$$e_{n+1} = x_{n+1} - \bar{x} \quad \text{— ②}$$

e_{n+1} can be expressed interms of e_n using:

$$e_{n+1} = \lambda e_n \quad \text{— ③}$$

Substituting ① and ② into ③ we obtain:

$$x_{n+1} - \bar{x} = \lambda (x_n - \bar{x}) \quad \text{— ④}$$

Similarly:

$$e_n = \lambda e_{n-1}$$

Obtaining

$$(x_n - \bar{x}) = \lambda (x_{n-1} - \bar{x}) \quad \text{— ⑤}$$

Subtracting ④ from ⑤

$$x_n - x_{n+1} = \lambda x_{n-1} - \lambda x_n$$

Simplifying to

$$\frac{x_{n+1} - x_n}{x_n - x_{n-1}} \approx \lambda$$

2. For Superlinear Convergence $\alpha=1, \lambda=0$, hence

$$\lim_{n\to\infty} \frac{|x_{n+1}-\bar{x}|}{|x_n-\bar{x}|} = 0$$

Consider $\lim_{n\to\infty} \frac{|x_{n+1}-\bar{x}|}{|x_n-\bar{x}|^{\alpha}} = \lambda$, for $\lambda \neq 0$ and $\alpha > 1$

So

$$\lim_{n\to\infty} \frac{|x_{n+1}-\bar{x}|}{|x_n-\bar{x}|} = \lim_{n\to\infty} \frac{|x_{n+1}-\bar{x}|}{|x_n-\bar{x}|^{(1-\alpha+\alpha)}}$$

$$= \lim_{n\to\infty} \frac{|x_{n+1}-\bar{x}|}{|x_n-\bar{x}|^{\alpha}\,|x_n-\bar{x}|^{1-\alpha}}$$

$$= \lim_{n\to\infty} \frac{|x_{n+1}-\bar{x}|}{|x_n-\bar{x}|^{\alpha}}\,|x_n-\bar{x}|^{\alpha-1}$$

Since $\lim_{n\to\infty} \frac{|x_{n+1}-\bar{x}|}{|x_n-\bar{x}|^{\alpha}} = \lambda$, then,

$$\lim_{n\to\infty} \frac{|x_{n+1}-\bar{x}|}{|x_n-\bar{x}|} = \lambda \lim_{n\to\infty} |x_n-\bar{x}|^{\alpha-1}$$

Since $\alpha > 1$, it means fast convergence therefore x_n tends quickly to $\bar{x}$, hence $\lim_{n\to\infty} |x_n-\bar{x}|^{\alpha-1} = 0$ as α increases. thus,

$$\lim_{n\to\infty} \frac{|x_{n+1}-\bar{x}|}{|x_n-\bar{x}|} = \lambda \cdot 0 = 0$$

which is Superlinear Convergence Since $\alpha=1$, and $\lambda=0$

3. Suppose that $\{x_n\}$ Converges Superlinearly to $\bar{x}$. Show that

$$\lim_{n\to\infty} \frac{|x_{n+1} - x_n|}{|x_n - \bar{x}|} = 1$$

$$\lim_{n\to\infty} \frac{|x_{n+1} - x_n|}{|x_n - \bar{x}|} = \lim_{n\to\infty} \frac{|x_{n+1} - \bar{x} + \bar{x} - x_n|}{|x_n - \bar{x}|}$$

$$\leq \lim_{n\to\infty} \frac{|x_{n+1} - \bar{x}|}{|x_n - \bar{x}|} + \lim_{n\to\infty} \frac{|x_n - \bar{x}|}{|x_n - \bar{x}|}$$

$$\lim_{n\to\infty} \frac{|x_{n+1} - x_n|}{|x_n - \bar{x}|} \leq 1 + \lim_{n\to\infty} \frac{|x_{n+1} - \bar{x}|}{|x_n - \bar{x}|}$$

for Super linear Convergence, $\lim_{n\to\infty} \frac{|x_{n+1} - \bar{x}|}{|x_n - \bar{x}|} = 0$

$$\lim_{n\to\infty} \frac{|x_{n+1} - x_n|}{|x_n - \bar{x}|} \leq 1 + 0$$

hence

$$\lim_{n\to\infty} \frac{|x_{n+1} - x_n|}{|x_n - \bar{x}|} = 1$$

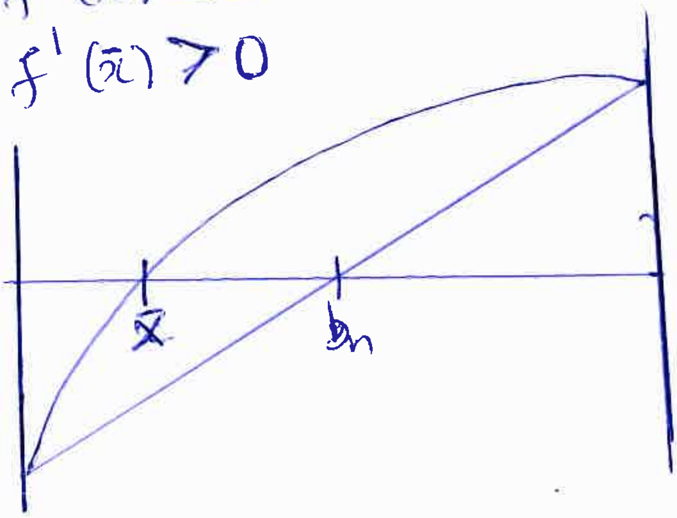
While developing root-finding schemes we are interested in how fast the Solution Converges to the root, So this makes us to be Interested in the error between the Solution and the root. So if the error be it all the possible Solutions near the root are the same, giving 1 after division, this means that the scheme approximates well the root. Therefore $|x_n - \bar{x}| = |x_{n+1} - x_n|$ gives more Information on how fast the Scheme will Converge to the ~~solution~~ root.

1. Show that $\lambda \approx \dfrac{Lf''(\bar{x})}{2f'(\bar{x}) + Lf''(\bar{x})}$ Satisfies $|\lambda|<1$.

Using the four possible Cases.

$$\lambda \approx \frac{L f''(\bar{x})}{2f'(\bar{x}) + Lf''(\bar{x})}, \text{ where } L = \begin{cases} b_n - \bar{x}, & b_n \text{ fixed} \\ a_n - \bar{x}, & a_n \text{ fixed.} \end{cases}$$

Case 1: $f''(\bar{x}) < 0$
$f'(\bar{x}) > 0$



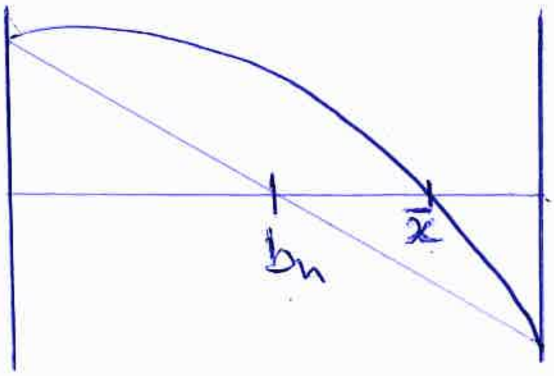
If b_n is fixed, $L = b_n - \bar{x}$.
from the diagram, $b_n - \bar{x} > 0$ and $f''(\bar{x}) < 0$
therefore $(b_n - \bar{x})(f''(\bar{x})) < 0$.
Since $f'(\bar{x}) > 0$, it follows that

$$2f'(\bar{x}) + (b_n - \bar{x})f''(\bar{x}) > (b_n - \bar{x})f''(\bar{x})$$

$$1 > \frac{(b_n - \bar{x})f''(\bar{x})}{2f'(\bar{x}) + (b_n - \bar{x})f''(\bar{x})}$$

$$1 > \left| \frac{(b_n - \bar{x})f''(\bar{x})}{2f'(\bar{x}) + (b_n - \bar{x})f''(\bar{x})} \right| = |\lambda|, \text{ hence } |\lambda| < 1$$

Case 2: $f''(\bar{x}) < 0$
$f'(\bar{x}) < 0$



If b_n is fixed, $L = b_n - \bar{x}$
from the diagram; $b_n - \bar{x} < 0$ and $f''(x) < 0$

$$|(b_n - \bar{x}) f''(\bar{x})| > 0$$

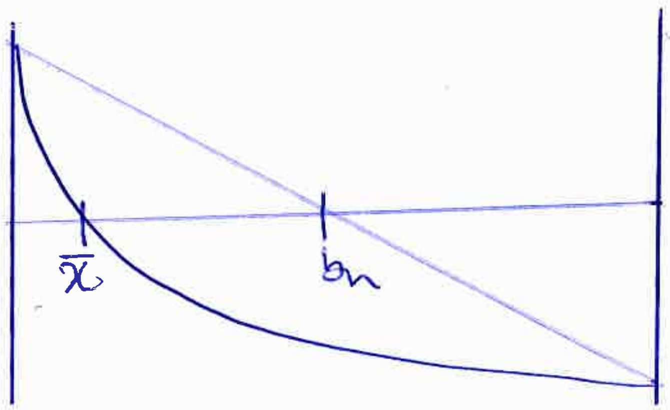
Since $f'(\bar{x}) < 0$, it follows that

$$|2f'(\bar{x}) + (b_n - \bar{x}) f''(\bar{x})| > |(b_n - \bar{x}) f''(\bar{x})|$$

$$1 > \left| \frac{(b_n - \bar{x}) f''(\bar{x})}{2f'(\bar{x}) + (b_n - \bar{x}) f''(\bar{x})} \right| = |\lambda|$$

hence $\underline{|\lambda| < 1}$

Case 3: $f''(\bar{x}) > 0$
$f'(\bar{x}) < 0$



fixing b_n, $(b_n - \bar{x}) > 0$

Since $f''(\bar{x}) > 0$

then $|(b_n - \bar{x}) f''(\bar{x})| > 0$

Since $f'(\bar{x}) < 0$, then it follows that

$$|2f'(\bar{x}) + (b_n - \bar{x}) f''(\bar{x})| > |(b_n - \bar{x}) f''(\bar{x})|$$

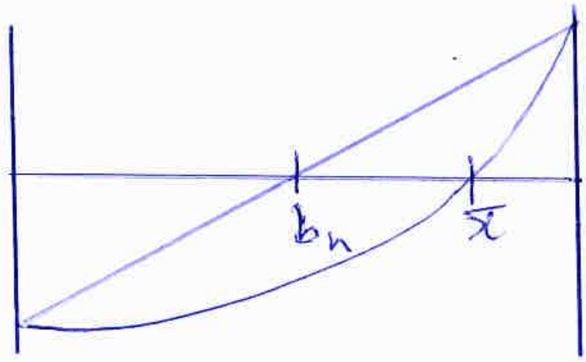
$$1 > \left| \frac{(b_n - \bar{x}) f''(\bar{x})}{2f'(\bar{x}) + (b_n - \bar{x}) f''(\bar{x})} \right| = |\lambda|$$

hence

$$\underline{|\lambda| < 1}$$

<u>Case 4:</u> $f''(\bar{x}) > 0$

$f'(x) > 0$



fixing b_n, $(b_n - \bar{x}) < 0$ and $f''(\bar{x}) > 0$

$(b_n - \bar{x}) f''(\bar{x}) \cancel{>} 0$

Since $f'(x) > 0$, it follows that

$$2f'(\bar{x}) + (b_n - \bar{x}) f''(\bar{x}) > (b_n - \bar{x}) f''(\bar{x})$$

$$1 > \left| \frac{(b_n - \bar{x}) f''(\bar{x})}{2f'(\bar{x}) + (b_n - \bar{x}) f''(\bar{x})} \right| = |\lambda|$$

Hence $\underline{1 > |\lambda|}$