Brian KYANJO Homework #5

1. Lagrange polynomials and the Barycentric formula

a) Assume (x_k, y_k), $k=0,1,\dots,N$

Write $P_2(x)$ for $N=2$ using the Lagrange Interpolation formula.

Given

$$L_j(x) = \frac{\prod_{k=0,k\neq j}^{n} (x-x_k)}{\prod_{k=0,k\neq j}^{n} (x_j - x_k)} \quad \text{——— (1)}$$

$$P_2(x) = \sum_{j=0}^{N} L_j(x)\, y_j = L_0(x) y_0 + L_1(x) y_1 + L_2(x) y_2 \quad \text{——— (2)}$$

from Equation (1)

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

therefore Equation (2) becomes

$$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

b) How many operations are required to evaluate $P_2(x)$

Since the highest term in $P_2(x)$ will be of power 2, then the number of operations for a second order polynomial (quadratic) will be given by N^2, hence $2^2 = \underline{4 \text{ operations}}$.

c) The second Barycentric form is given by

$$P_N(x) = \frac{\sum_{j=0}^{N} \frac{w_j}{x - x_j} y_j}{\sum_{j=0}^{N} \frac{w_j}{x - x_j}}$$

where weights are computed as

$$w_j = \frac{1}{\prod_{\substack{k=0 \\ k \neq j}}^{N} (x_j - x_k)}, \quad j = 0, 1, \ldots, N.$$

For $N=2$,

$$P_2(x) = \frac{\sum_{j=0}^{N=2} \frac{w_j}{x - x_j} y_j}{\sum_{j=0}^{N=2} \frac{w_j}{x - x_j}}, \quad w_j = \frac{1}{\prod_{\substack{k=0 \\ k \neq j}}^{2} (x_j - x_k)}$$

$$P_2(x) = \frac{\frac{w_0}{x - x_0} y_0 + \frac{w_1}{x - x_1} y_1 + \frac{w_2}{x - x_2} y_2}{\frac{w_0}{x - x_0} + \frac{w_1}{x - x_1} + \frac{w_2}{x - x_2}}$$

$$P_2(x) = \frac{w_0 y_0 (x-x_1)(x-x_2) + w_1 y_1 (x-x_0)(x-x_2) + w_2 y_2 (x-x_0)(x-x_1)}{w_0 (x-x_1)(x-x_2) + w_1 (x-x_0)(x-x_2) + w_2 (x-x_0)(x-x_1)}$$

Since $w_j = \frac{1}{\prod\limits_{\substack{k=0 \\ k\neq j}}^{2} (x_j - x_k)}$, $w_0 = \frac{1}{(x_0-x_1)(x_0-x_2)}$

$$w_1 = \frac{1}{(x_1-x_0)(x_1-x_2)}$$

$$w_2 = \frac{1}{(x_2-x_0)(x_2-x_1)}$$

therefore $P_2(x)$ becomes

$$P_2(x) = \frac{\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2}{\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}}$$

substituting for L_j, $j = 0, 1, 2$, we get

$$P_2(x) = \frac{L_0 y_0 + L_1 y_1 + L_2 y_2}{L_0 + L_1 + L_2}$$

but $\sum\limits_{j=0}^{2} L_j = 1$

$$P_2(x) = L_0 y_0 + L_1 y_1 + L_2 y_2$$

which is the exact form of equation (2)

d) Only 2 operations are required.

e) For general N, The Lagrange polynomial is given by

$$P_N(x) = \sum_{j=1}^{N} f_j \cdot \prod_{\substack{k=1 \\ k \neq j}}^{N} \frac{x - x_k}{x_j - x_k} \quad — ①$$

Equation can be converted into.

$$P_N(x) = \prod_{k=1}^{N} (x - x_k) \cdot \sum_{j=1}^{N} \frac{f_j}{x - x_j} \prod_{\substack{k=1 \\ k \neq j}}^{N} \frac{1}{x_j - x_k} \quad — ②$$

at N points Equation ② can be evaluate as

$$O\left(N \cdot \log\left(\frac{1}{\epsilon}\right)\right)$$

This requires $O(N^2)$ operations to be evaluated, which is not the same case for the Barycentric form

- The ~~IBa~~ $P_N(x) = \sum_{j=0}^{N} y_j \cdot \prod_{\substack{k=0 \\ k \neq j}}^{N} \frac{(x_j - x_k)}{x - x_j}$

$$\overline{\sum_{j=0}^{N} \prod_{\substack{k=0 \\ k \neq j}}^{N} \left(\frac{x_j - x_k}{x - x_j}\right)} \quad , j = 0, 1, \ldots$$

$P_N(x)$ Can also be evaluate as

$$\cancel{O\left(N \cdot \log\left(\frac{1}{\epsilon}\right)\right)} \approx O(N^2) \text{ operations}$$

The Barycentric Form is more efficient.

$$P_N(x) = \frac{\sum_{j=0}^{N} y_j \cdot \prod_{\substack{k=0 \\ k \neq j}}^{N} \left(\frac{x_j - x_k}{x - x_j} \right)}{\sum_{j=0}^{N} \prod_{\substack{k=0 \\ k \neq j}}^{N} \left(\frac{x_j - x_k}{x - x_j} \right)}$$

this can be evaluated as $\frac{O(N \cdot \log(\ldots))}{O(N \cdot)}$

which becomes $\frac{O(N^2)}{O(N)} \approx O(N)$

Hence Barycentric form requires N op $O(N)$ operations

Therefore: The Barycentric form is more efficient

NO. 2 (b)

(ii) Why does the newton Iteration Converge in one Step?

Given
$$d^{k+1} = d^k - J^{-1} F(d^k)$$

Since the Jacobian J is given by the Hessian of $F(d)$ i.e. $J = HF(d)$,
For the newton Iteration F is Strictly convex and has a unique strict global minimizer d^*,

$$A d^* = -b$$

where A is an $n \times n$ positive symmetric matrix
$$F(d) = a + b \cdot d + \frac{1}{2} d \cdot A d,$$

~~F(d*)~~ The newton Iteration approximates $F(d^*)$ quadratically near d^k, therefore for any initial guess $d^{(0)}$, the Newton's ~~...~~ Iteration applied to $F(d)$ Converges to d^* in one step.