Math 567 **Final Project:** **Due Dec. 16, 2020**
Grady Wright **Elastic Wave Equation**

Background. The 2D elastic wave equations are commonly used in seismology to study the propagation of seismic waves in the earth. The equations can be written in terms of the displacement $\mathbf{u} = (u_x, u_z)$ and stress $\boldsymbol{\tau} = \left[\begin{smallmatrix}\tau_{xx} & \tau_{xz}\\ \tau_{xz} & \tau_{zz}\end{smallmatrix}\right]$ as follows

$$
\begin{aligned}
\rho\frac{\partial^2 u_x}{\partial t^2} &= \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z},\\
\rho\frac{\partial^2 u_z}{\partial t^2} &= \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z},\\
\tau_{xx} &= (\lambda+2\mu)\frac{\partial u_x}{\partial x} + \lambda\frac{\partial u_z}{\partial z},\\
\tau_{zz} &= \lambda\frac{\partial u_x}{\partial x} + (\lambda+2\mu)\frac{\partial u_z}{\partial z},\\
\tau_{xz} &= \mu\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right),
\end{aligned}
$$

where λ and μ are called the Lamé parameters (μ is the shear modulus) and ρ is the density. All of these can vary spatially. Note here that subscripts involving x and z do not denote partial derivatives, but specify the directions associated with the variables. By letting

$$\frac{\partial u_x}{\partial t} = v_x \quad \text{and} \quad \frac{\partial u_z}{\partial t} = v_z \tag{1}$$

be the velocities in the x and z direction, respectively, we can write the elastic wave equations in first order form as

$$
\begin{aligned}
\rho\frac{\partial v_x}{\partial t} &= \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z}, &(2)\\
\rho\frac{\partial v_z}{\partial t} &= \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z}, &(3)\\
\frac{\partial \tau_{xx}}{\partial t} &= (\lambda+2\mu)\frac{\partial v_x}{\partial x} + \lambda\frac{\partial v_z}{\partial z}, &(4)\\
\frac{\partial \tau_{zz}}{\partial t} &= \lambda\frac{\partial v_x}{\partial x} + (\lambda+2\mu)\frac{\partial v_z}{\partial z}, &(5)\\
\frac{\partial \tau_{xz}}{\partial t} &= \mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right). &(6)
\end{aligned}
$$

In most applications, λ and μ are not specified, but are given implicitly in terms of the P-wave (pressure-wave) velocity, c_P, and S-wave (shear-wave) velocity, c_S, for the medium. These values are given as

$$c_P = \sqrt{\frac{\lambda+2\mu}{\rho}} \quad \text{and} \quad c_S = \sqrt{\frac{\mu}{\rho}}.$$

The Poisson ratio is $\nu = \frac{\lambda}{2(\lambda+\mu)}$ and for many types of rocks this value is close to 0.25. For most liquids, the S-wave velocity is zero, i.e. $c_S = 0$, making the Poisson ratio closer to $\nu = 0.5$.

Benchmark problem. In this problem, you are going to implement a numerical method to approximate the solution of the elastic eave equations using the first-order form (2)–(6). The set-up for the problem is as follows:

- The domain for the problem will be a square box of earth with $0 \le x \le 3000$ m and $0 \le z \le 3000$ m.
- The spatial grid resolution in both x and z will be 3 m, i.e. $h = 3$ m, which corresponds to $m = 1000$ grid cells.
- The solution should be simulated for $0 \le t \le 0.45$ sec.
- The density should be set to $\rho = 2200\ \text{kg/m}^3$ over the whole domain.
- The P-wave velocity c_P and S-wave velocity c_S should be set as follows

$$c_P(x,z) = \begin{cases} 1450\ \text{m/sec} & 1500\ \text{m} \le x \le 2100\ \text{m and } 1700\ \text{m} \ \le z \le 1800\ \text{m} \\ 3200\ \text{m/sec} & \text{otherwise} \end{cases}$$

$$c_S(x,z) = \begin{cases} 0\ \text{m/sec} & 1500\ \text{m} \le x \le 2100\ \text{m and } 1700\ \text{m} \ \le z \le 1800\ \text{m} \\ 1847.5\ \text{m/sec} & \text{otherwise} \end{cases}$$

  These values are meant to represent an isotropic medium with an underground rectangular water reservoir of length 600 m, height 100 m, and with left edge starting at 1500 m and top edge at 1700 m.

- Use periodic boundary conditions in both directions and zero initial conditions for the velocity and stress. The simulation will stop before any effects of the boundary are realized.
- You will simulate an underground explosion at a location (x_0, z_0), which can be realized by adding a source-time function $S(t)$ to the normal stress components τ_{xx} and τ_{zz}:

$$\tau_{xx} = (\lambda + 2\mu)\frac{\partial u_x}{\partial x} + \lambda\frac{\partial u_z}{\partial z} + \gamma S(t - t0)\int_0^L \int_0^L \delta(x - x0)\delta(z - z0)\,dxdz,$$
$$\tau_{zz} = \lambda\frac{\partial u_x}{\partial x} + (\lambda + 2\mu)\frac{\partial u_z}{\partial z} + \gamma S(t - t0)\int_0^L \int_0^L \delta(x - x0)\delta(z - z0)\,dxdz.$$

  where γ is a parameter controlling the pressure (measured in Pascals) generated by the explosion, $S(t)$ is unit-less and depends only on time, and δ is the Dirac delta function. Differentiating these equations in time gives

$$\frac{\partial \tau_{xx}}{\partial t} = (\lambda + 2\mu)\frac{\partial v_x}{\partial x} + \lambda\frac{\partial v_z}{\partial z} + \gamma S'(t - t0)\int_0^L \int_0^L \delta(x - x0)\delta(z - z0)\,dxdz, \tag{7}$$
$$\frac{\partial \tau_{zz}}{\partial t} = \lambda\frac{\partial v_x}{\partial x} + (\lambda + 2\mu)\frac{\partial v_z}{\partial z} + \gamma S'(t - t0)\int_0^L \int_0^L \delta(x - x0)\delta(z - z0)\,dxdz, \tag{8}$$

  which you will use in place of (4) and (5) above. For the source-time function, use the Ricker wavelet, which is defined as

$$S(t) = (1 - 2\pi^2 f_M^2 t^2)e^{-\pi^2 f_M^2 t^2}, \tag{9}$$

  where f_M is the peak frequency at time t. The parameters related to the explosion should be set as follows:

$$t_0 = 0.07\ \text{sec}, \quad (x_0, z_0) = (1500\ \text{m}, 1500\ \text{m}), \quad f_M = 16\ \text{Hz}, \quad \gamma = 5 \times 10^6\ \text{Pa}.$$

  To numerically implement the integrals of the Dirac-delta function in (7) and (8) (i.e. the point sources), you should use the smoothed out version:

$$\int_0^L \int_0^L \delta(x - x0)\delta(z - z0)\,dxdz \approx h^2\delta_h(x - x0)\delta_h(z - z0), \tag{10}$$

where h is the grid spacing and δ_h is the discrete delta function defined as

$$\delta_h(\xi) = \begin{cases} \dfrac{1}{4h}\left[1+\cos\left(\dfrac{\xi}{2h}\pi\right)\right] & |\xi| \le 2h, \\ 0 & |\xi| > 2h, \end{cases} \tag{11}$$

where h is the grid-spacing.

Discretization. You are free to try design your own numerical method for solving the first order system (2), (3), (6), (7), and (8), but I will describe one particularly effective way (and popular) way to do it. The idea is to use a staggered grid in space for the unknown variables. Figure 1 illustrates a good way to arrange the variables in space so that they are on different grids. With this

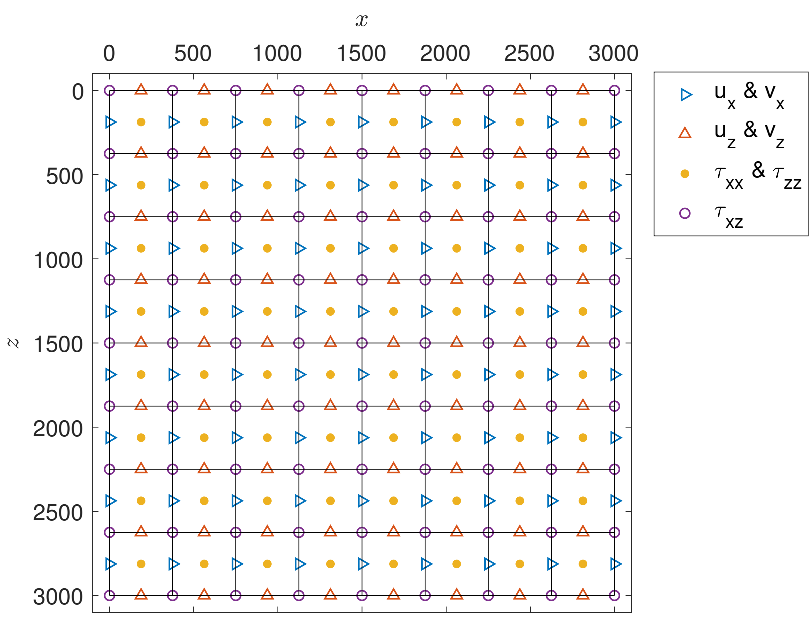


Figure 1: Example of a staggered-grid for the unknowns of the elastic wave equations for the model problem with $h = 500$ m. Note that for periodic boundary conditions, the unknowns at $x = 3000$ m are the same as they are for $x = 0$ m (the same is true for $z = 3000$ m).

arrangement one can compute all derivatives involved in the elastic wave equations using centered difference across one grid cell instead of two. For example, the unknown v_x in (2) is represented at the grid points $(x_i, z_{j+\frac{1}{2}}) = (ih, (j+\frac{1}{2})h)$, $i, j = 0, \ldots, m-1$ and we can discretize (2) in space using centered, second order accurate approximations as

$$\rho_{i,j+\frac{1}{2}} \frac{d}{dt}(v_x)_{i,j+\frac{1}{2}} = \frac{1}{h}\left((\tau_{xx})_{i+\frac{1}{2},j+\frac{1}{2}} - (\tau_{xx})_{i-\frac{1}{2},j+\frac{1}{2}}\right) + \frac{1}{h}\left((\tau_{xz})_{i,j+1} - (\tau_{xz})_{i,j}\right), \tag{12}$$

where periodicity is assumed when the indices are not in the range of $[0, m-1]$. This not only gives a method with theoretically smaller constants in the truncation error, but also is less susceptible to generating spurious oscillations at material discontinuities.

Staggering the variables in time can also be effective. The idea here is to let τ_{xx}, τ_{zz}, and τ_{xz} be approximated at the time-steps $t_n = nk$, $n = 0, 1, \ldots$ and to let v_x and v_z be approximated at the half time-steps $t_{n+\frac{1}{2}} = (n+\frac{1}{2})k$ (here k is the time-step). Combining the staggering in space described above with this staggering in time allows all time derivatives in the elastic wave equations to be approximated to second order accuracy using the leap-frog scheme. For example, we can discretize (12) in time as

$$(v_x)^{n+\frac{1}{2}}_{i,j+\frac{1}{2}} = (v_x)^{n-\frac{1}{2}}_{i,j+\frac{1}{2}} + \frac{k}{h\rho_{i,j+\frac{1}{2}}}\left((\tau_{xx})^n_{i+\frac{1}{2},j+\frac{1}{2}} - (\tau_{xx})^n_{i-\frac{1}{2},j+\frac{1}{2}} + (\tau_{xz})^n_{i,j+1} - (\tau_{xz})^n_{i,j}\right). \tag{13}$$

This staggering also gives a natural way to solve (1) for the displacements using leap-frog. If this scheme is implemented correctly then some standard stability analysis shows that it will be temporally stable provided the time-step is chosen as

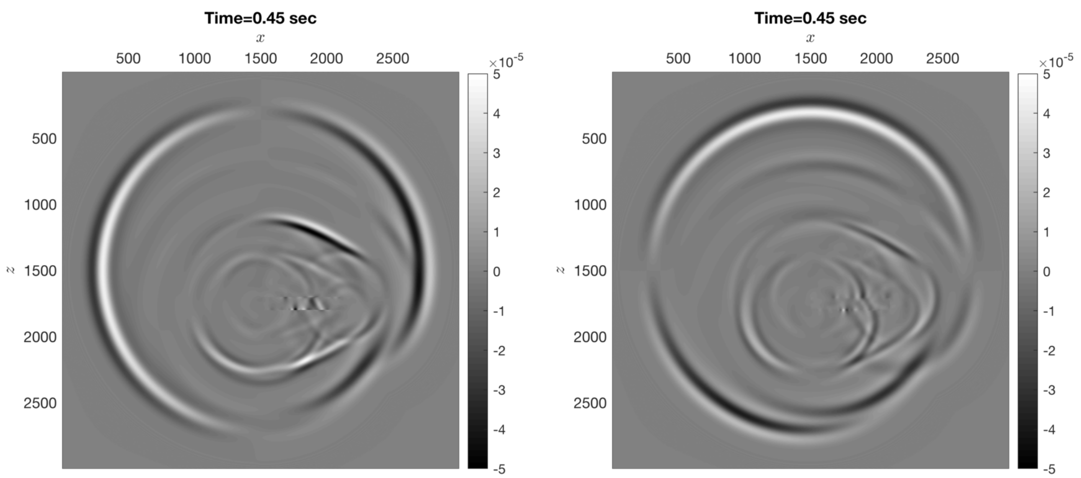
$$c_P \frac{k}{h} < \frac{1}{\sqrt{2}}.$$

A final note: one can use higher-order approximations to the space derivatives with relative ease for this problem since the boundary conditions are periodic. This will give much better resolution of the waves in the problem for the same h and make your professor much happier. In all my simulations, I used a 6th order accurate approximation of the spatial derivatives.

Tasks.

1. Write a code that numerically solves the elastic wave equations with the problem set-up given above. Your code should compute u_x, u_z, v_x, v_z, τ_{xx}, τ_{zz} and τ_{xz}. Describe the numerical method you implemented and e-mail me your complete code.

2. Produce plots of the displacements u_x and u_z at time $t = 0.45$ sec. If you do everything correctly, these plots should be similar to the ones shown in Figure 2. I produced these plots using `pcolor` in MATLAB.



(a) u_x (b) u_z

Figure 2: Displacements at $t = 0.45$ sec from my numerical simulation of the benchmark problem.

3. Produce a time-series plot of the displacements u_x and u_z measured at the grid point closest to $(x, z) = (1650 \text{ m}, 1410 \text{ m})$. If you do everything correctly, this plot should be similar to the one shown in Figure 3.

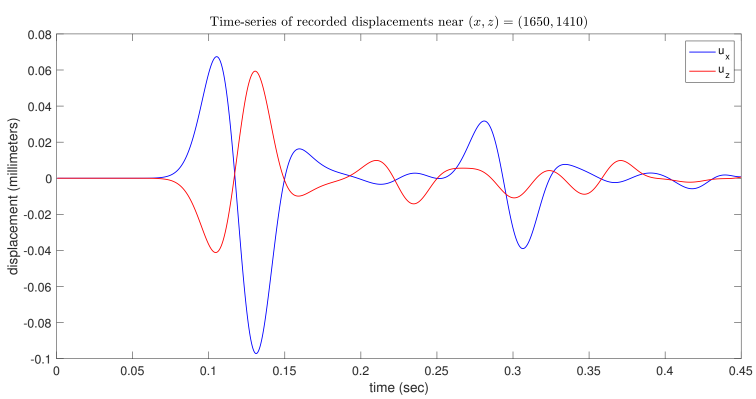


Figure 3: Time-series of the displacements measured at the grid point closest to $(x, z) = (1650 \text{ m}, 1410 \text{ m})$ from my numerical simulation of the benchmark problem.