1. Find the rate of Convergence of the Sequence

$$S_n = \frac{\sin(n)}{n}, \quad \text{as } n \to \infty$$

Since

$$-1 \le \sin(n) \le 1$$

diving through by n

$$-\frac{1}{n} \le \frac{\sin(n)}{n} \le 1/n \Rightarrow -\frac{1}{n} \le S_n \le 1/n$$

Taking the $\lim_{n\to\infty}$ through out we have,

$$-\lim_{n\to\infty} \frac{1}{n} \le \lim_{n\to\infty} S_n \le \lim_{n\to\infty} \frac{1}{n}$$

Since $\lim_{n\to\infty}\left(-\frac{1}{n}\right) = \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$, then by Sandwitch theorem, the $\lim_{n\to\infty} S_n = 0$

hence, $L = \lim_{n\to\infty} S_n = 0$

Rate of Convergence, β_n,

$$|S_n - L| \le \lambda |\beta_n|$$

$$\left|\frac{\sin(n)}{n} - 0\right| \le \frac{\sin(n)}{n} |1/n|$$

Therefore the rate of Convergence, β_n, is $O(1/n)$

2. Show that the sequence
$$S_n = \frac{1}{n^2}, \text{ converges linearly.}$$

For linear convergence,
$$\lim_{n\to\infty} \cancel{\left|\frac{S_{n+1} - S}{S_n - S}\right|} \quad \lim_{n\to\infty} \frac{|S_{n+1} - S|}{|S_n - S|^\alpha} = \lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^\alpha} = \lambda$$

where $\alpha = 1$ and $0 < \lambda \leq 1$.

therefore $S_n = \frac{1}{n^2}$, $S_{n+1} = \frac{1}{(n+1)^2}$

and, Limit, $S_1 = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{n^2}\right) = \frac{1}{\infty} = 0$

So,
$$\lim_{n\to\infty} \frac{|S_{n+1} - S|}{|S_n - S|^\alpha} = \lim_{n\to\infty} \frac{|S_{n+1}|}{|S_n|^\alpha} = \lim_{n\to\infty} \frac{1/(n+1)^2}{\left(\frac{1}{n^2}\right)^\alpha} = \cancel{\left(\frac{n^\alpha}{n+1}\right)^2}$$
$$= \lim_{n\to\infty} \left(\frac{n^\alpha}{n+1}\right)^2$$

If $\alpha = 1$, then $\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = 1 > 0$

Hence, the sequence converges linearly for $\alpha = 1$

3. $\lim_{x\to 0} \frac{e^x - \cos x - x}{x^2}$

Using taylor Series expansion

$e^x = 1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} e^{\xi}$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cos(\xi_1)$ for $0 < \varepsilon < 1$

then

$\lim_{x\to 0} \frac{e^x - \cos x - x}{x^2} = \lim_{x\to 0} 1 + \frac{x}{3!} + \frac{x^3}{5!} e^{\xi} + \frac{x^4}{6!} \cos(\xi_1)$

$$\underline{\underline{L = 1}}$$

Corresponding rate of Convergence, R_n,

$|S_n - L| \le \lambda |R_n|$, from definition

$\left| \frac{e^x - \cos x - x}{x^2} - 1 \right| = x/_{3!} + \frac{x^5}{5!} e^{\varepsilon} + \frac{x^4}{6!} \cos(\xi_1)$

$\left| \frac{e^x - \cos x - x}{x^2} - 1 \right| \le \frac{1}{3!} |x|$

The rate of Convergence is $O(x)$

4.

a). At $\cancel{K} = 5\cancel{4}$
$\cancel{X} = 2.2$

5(a) Analytical means

$$L = \lim_{x \to 0} \frac{\sqrt{1+x^2} - 1}{x^2} = \frac{1}{2}$$, where L is the limit

$$\left| f(x) - \frac{1}{2} \right| \le \frac{1}{8} x^2 \quad \text{—} ①$$

Comparing equation ① with the definition of rate of convergence

$$|f(x) - L| \le \lambda |\beta_n|$$

we conclude that $\beta_n = \cancel{x}^2$ have the rate of convergence is $O(x^2)$ as $x \to 0$

5b) from Remainder theorem

$F(x) = P(x) + R(x)$, where $R(x)$ is the remainder.

Using taylor series at $x = 0$

$$f(x) = \frac{1}{2} - \frac{x^2}{8} + \frac{x^4}{16} + O(\cancel{x}^6)$$

taking, $P(x) = \frac{1}{2}$

$$R(x) = -\frac{x^2}{8} + \frac{x^4}{16} + O(x^6)$$

from definition

$$|R_n(x)| \le M \frac{|x-a|^{n+1}}{(n+1)!}$$

therefore,

$$|R(x)| = \left| -\frac{x^2}{8} + \frac{x^4}{16} + O(x^6) \right|$$

$$|R(x)| \leq \frac{1}{8}|x^2|$$

Here β_n is x^2, rate of Convergence is $O(x^2)$

and $\lambda = 1/8$, which is a reasonable choice.

7b) from the ~~eq~~ question,

$$e_n f(x) = \left| S_n(x) - \frac{1}{1+x} \right|$$

Comparing with the definition of $|S_n - L| \leq \lambda |\beta_n|$

we Obtain, $L = \frac{1}{1+x}$, as ~~n→∞~~

for linear Convergence $\alpha = 1$, $0 < \lambda \leq 1$

$$\lambda = \lim_{n\to\infty} \left| \frac{e_{n+1}}{e_n} \right|$$

but using a geometric series

$$S_n = \frac{1-(-x)^{n+1}}{1+x}$$

So $e_n = \left| S_n(x) - \frac{1}{1+x} \right| = \left| \frac{1-(-x)^{n+1}}{1+x} - \frac{1}{1+x} \right|$

$$e_n = \frac{-(-x)^{n+1}}{1+x}, \quad e_{n+1} = \frac{-(-x)^{n+2}}{1+x}$$

$$\lambda = \cancel{\lim_{n\to\infty}} \left| \cancel{\frac{e_{n+1}}{e_n}} \right| \cancel{=} \quad , \text{for linear Convergence } \alpha = 1$$

$$\lambda = \lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^{\alpha}} = \lim_{n\to\infty} \left| \frac{e_{n+1}}{e_n} \right|$$

~~for linear~~

$$\lambda = \left| \frac{(-x)^{n+2}}{(-x)^{n+1}} \right| = |-x|$$

$$\lambda = x$$

Hence the Sequence Converges linearly since $\lambda = x$

and $x = 1/\pi$, therefore $0 < \lambda \leq 1$