LU Decomposition

Chapter 3 - Linear Systems

- ## Chapter 3.1 - Linear Algebra Review

- Using the LU decomposition to compute the matrix inverse.

  - If A is $n \times n$ and we find an $n \times n$ matrix B such that $BA = I$, then $AB = I$ and B is unique

Proof: Show $Ax = 0$ has exactly one solution:

$$Ax = 0$$
$$BAx = B0 = 0$$
$$Ix = 0$$
$$x = 0$$

- columns of A are linearly independent.
- A^{-1} exists!

$$Ax = x_1 \operatorname{col}_1(A) + x_2 \operatorname{col}_2(A) + \cdots + x_n \operatorname{col}_n(A) = 0$$

$$Ax = 0 \iff x = 0$$

- Columns of A are linearly independent
- $T(x) = Ax$ is "one-to-one" and "onto"
- We say A is "invertible"

1 Show that if $BA = \underbrace{I}_{n\times n \text{ identity matrix}}$, then $AB = I$

Proof

$$A\underbrace{(BA)}_{I}x = Ax \quad (\text{since } BA = I)$$

$$(AB)Ax = Ax \quad (\text{associativity})$$

$$(AB - I)Ax = 0$$

True for any x, so we must have

$$AB - I = 0 \Rightarrow AB = I \quad \blacksquare$$

(2) Show that B^{-1} is unique, i.e there is only one B for which

$AB = BA = I$

Suppose

$$AB = AC = I, \quad B \neq C$$

Then

$$BAB = BAC \qquad BAB - BAC = 0$$

$$\underbrace{BA}_{I}(B - C) = B - C = 0$$

$$\Rightarrow B = C \quad * \quad \Rightarrow \quad \blacksquare$$

So B is the unique matrix for which

$$AB = BA = I \quad \Rightarrow \quad B = A^{-1}$$

or B is the inverse of A.

$$AA^{-1} = A^{-1}A = I$$

How can we compute the matrix inverse?

$$x = A^{-1} b$$

How can we compute A^{-1} ?

① Wolfram Alpha:

Apply row operations to $[A \,\vdots\, I]$

$$[A \,\vdots\, I] \sim [I \,\vdots\, A^{-1}] \quad \text{rref}$$

apply row operations

Gauss-Jordan elimination

② Use the LU decomposition

Problem:

Find a matrix B for which

$$AB = I$$

Then, from previous proof, $B = A^{-1}$.

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Find a matrix B for which

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Assume we have L and U for which

$LU = A$.

LU can be used to solve $Ax = b$

x, b are column vectors.

$$AB = A[b_1 \ b_2 \ b_3 \cdots b_n] = \underbrace{[Ab_1 \ Ab_2 \ Ab_3 \cdots Ab_n]}_{\text{columns of } AB}$$

$I = [e_1 \ e_2 \ e_3 \cdots e_n]$ where e_i is the i^{th} column of the identity matrix.

$e_i = [0, 0, 0, \cdots 1, 0, 0 \cdots 0]$

i^{th} position.

$$AB = A\begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_n \end{bmatrix} = \underbrace{\begin{bmatrix} Ab_1 & Ab_2 & Ab_3 & \cdots & Ab_n \end{bmatrix}}_{\text{columns of } AB}$$

$$I = \begin{bmatrix} e_1 & e_2 & e_3 & \cdots & e_n \end{bmatrix}$$

Equating columns of AB and I

we get:

$$\left.\begin{aligned} Ab_1 &= e_1 \\ Ab_2 &= e_2 \\ Ab_3 &= e_3 \\ &\vdots \\ Ab_n &= b_n \end{aligned}\right\} \text{n linear solves.}$$

Idea: factor first: $A = LU \sim \frac{2}{3}n^3$

Then solve $Ab_i = e_i$ using LU

$$\left.\begin{aligned} &\text{1 forward solve: } n^2 \\ &\text{1 back solve: } n^2 \end{aligned}\right\} 2n^2 \text{ for each } Ab_i = e_i$$

Total cost of computing A^{-1}:

$$\frac{2}{3}n^3 + 2n^3 = \frac{8}{3}n^3$$

To compute matrix inverse

$$\frac{8}{3}n^3 \text{ ops.}$$

x : multiply $A^{-1}b$: $\Theta(n^2)$ ops

$$x : \frac{8}{3}n^3 + n^2$$

$x = A^{-1}b$ is 4x the cost of solving $Ax=b$ directly.

`x = inv(A)*b` $\$\$\$$

To solve $Ax=b$ requires

$$x : \underbrace{\frac{2}{3}n^3}_{LU} + \underbrace{2n^2}_{\text{Forward + back solve}} =$$

`x = A\b` $\$$

`x = A^{-1}b`

$$AB = A\underbrace{[b_1 \; b_2 \; b_3 \cdots b_n]}_{\text{what are the } b_i\text{?}} = \underbrace{[e_1 \; e_2 \; e_3 \cdots e_n]}$$

$AB = I$ means:

$$Ab_1 = e_1$$
$$Ab_2 = e_2 \quad \ldots$$
$$Ab_3 = e_3$$
$$\vdots$$
$$Ab_n = e_n$$

Work:

$$\frac{2}{3}n^3 + 2n^3$$
$$= \frac{8}{3}n^3$$

work to get A^{-1}

Idea: Use LU to solve $Ab_i = e_i$, $i = 1, 2, \ldots n$

n solves.

1. Compute LU
2. for $i = 1:n$
   a) Solve $Ly = e_i$ (i^{th} column of identity matrix)
   b) Solve $Ub_i = y$

$\Rightarrow$ column i of B is b_i

<u>work</u> $\frac{2}{3}n^3$ to get LU; $\underbrace{2n^2 \text{ to get } b_i}_{n \text{ times}}$