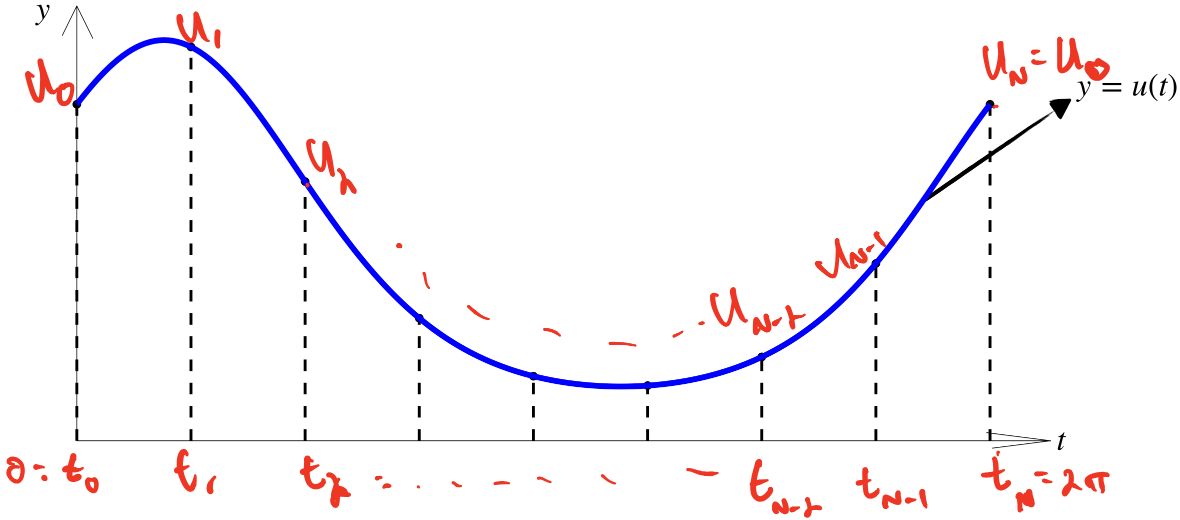
Discrete Fourier Transform (DFT)



Forward Transform

If N is odd:

$$\tilde{c}_k = \frac{1}{N}\sum_{j=0}^{N-1} u_j e^{-2\pi ijk/N},\ k = -\frac{N-1}{2},\ldots,\frac{N-1}{2}$$

If N is even:

$$\tilde{c}_k = \frac{1}{N}\sum_{j=0}^{N-1} u_j e^{-2\pi ijk/N},\ k = -\frac{N}{2},\ldots,\frac{N}{2}-1$$

Inverse Transform

If N is odd:

$$u_j = \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} \tilde{c}_k e^{2\pi ijk/N},\ j = 0,\ldots,N-1$$

If N is even:

$$u_j = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \tilde{c}_k e^{2\pi ijk/N},\ j = 0,\ldots,N-1$$

Switch to a more common notation: $\hat{u}_k$ for scaled Fourier coefficients and positive indicies $k = 0,1,\ldots,N-1$

Definition:

even N odd N

$$\hat{u}_k = \frac{1}{N}\tilde{c}_k, \qquad k = 0,1,\ldots,\frac{N-1}{2} \quad \text{or} \quad k = 0,1,\ldots,\frac{N}{2}-1$$

$$\hat{u}_{N+k} = \frac{1}{N}\tilde{c}_k, \qquad k = -1,-2,\ldots,-\frac{N-1}{2} \quad \text{or} \quad k = -1,-2,\ldots,-\frac{N}{2}$$

$$\hat{u} = \frac{1}{N}\left[\tilde{c}_0 \;\; \tilde{c}_1 \;\; \cdots \;\; \tilde{c}_{\frac{N-1}{2}} \;\; \tilde{c}_{-\frac{(N-1)}{2}} \;\; \tilde{c}_{-\frac{(N-1)}{2}+1} \;\; \cdots \;\; \tilde{c}_{-1}\right]$$

Forward Transform

Inverse Transform

$$\hat{u}_k = \sum_{j=0}^{N-1} u_j \underbrace{e^{-2\pi ijk/N}}_{\omega^{-jk}}, \;\; k = 0,\ldots,N-1$$

$$u_j = \frac{1}{N}\sum_{k=0}^{N-1} \hat{u}_k \underbrace{e^{2\pi ijk/N}}_{\omega^{jk}}, \;\; j = 0,\ldots,N-1$$

- Library: FFTW (Matlab, numpy, Julia)

DFT Matrix

Let $\omega = e^{2\pi i/N}$ (N^{th} root of unity: $\omega^N = 1$)

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-(N-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \cdots & \omega^{-2(N-1)} \\ \vdots & & & & \\ 1 & \omega^{-(N-1)} & \omega^{-2(N-1)} & \cdots & \omega^{-(N-1)^2} \end{bmatrix}}_{W_N} \underbrace{\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix}}_{\underline{u}} = \underbrace{\begin{bmatrix} \hat{u}_0 \\ \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_{N-1} \end{bmatrix}}_{\hat{\underline{u}}}$$

$$\hat{\underline{u}} = W_N \underline{u}$$

W_N is called the DFT matrix

Inverse of DFT matrix:

$$\underline{u} = W_N^{-1} \hat{\underline{u}}$$

Can see that: $W_N^{-1} = \frac{1}{N}\overline{W} = \frac{1}{N} W^*$

complex
Hermitean transpose

$$\overline{\omega^{-\ell}} = \omega^{\ell}$$

W is a unitary matrix up to scaling:

$$\frac{1}{N} W W^* = I$$

Fast Fourier Transform (FFT) :

A method for computing $\hat{\underline{u}}$ or $\underline{u}$ in $\mathcal{O}(N \log_2 N)$.

Discovered for case of N being a power of 2. → radix-2

History: Cooley & Tukey (1965)

Gauss (1805) → 28 years old.

Compute: $\hat{u}_k$, $k = 0, \ldots, N-1$

$$\hat{u}_k = \sum_{j=0}^{N-1} u_j \, \omega_N^{-jk} \quad , \qquad \omega_N = e^{2\pi i/N}$$

$k = 0, 1, \ldots, N-1$

<u>Step 1:</u> Split u_j depending on the parity of j

$\Rightarrow \quad x_j := u_{2j} \quad \& \quad y_j = u_{2j+1} \quad , \quad j = 0, 1, \dots, \frac{N}{2}-1$

$$\hat{u}_k = \sum_{j=0}^{\frac{N}{2}-1} \left(x_j \omega_N^{-2jk} + y_j \omega_N^{-(2j+1)k} \right), \quad k = 0, 1, \dots, N-1$$

Crucial observation:

$$\omega_N^{-2jk} = e^{-4\pi i jk/N} = e^{-2\pi i jk/(N/2)} = \omega_{N/2}^{-jk}$$

$$\omega_N^{-(2j+1)k} = \omega_N^{-2jk} \omega_N^{-k} = \omega_{N/2}^{-jk} \omega_N^{-k}$$

(In general: $\omega_N^{-pq} = \omega_{N/2}^{-p}$)

Step 2: Reduce computation to 2 DFTs of length $N/2$

$$\hat{u}_k = \underbrace{\sum_{j=0}^{N/2-1} x_j \, \omega_{N/2}^{-jk}}_{\substack{\text{DFT of length } \frac{N}{2} \\ := \hat{x}_k}} + \omega_N^{-k} \underbrace{\sum_{j=0}^{N/2-1} y_j \, \omega_{N/2}^{-jk}}_{\substack{\text{DFT of length } \frac{N}{2} \\ := \hat{y}_k}}$$

For $k = 0, \dots, \frac{N}{2}-1$:

$$\hat{u}_k = \hat{x}_k + \omega_N^{-k} \hat{y}_k$$

$$\hat{u}_{k+\frac{N}{2}} = \hat{x}_{k+\frac{N}{2}} + \omega_N^{-(k+\frac{N}{2})} \hat{y}_{k+\frac{N}{2}}$$

Note that $\omega_N^{-N/2} = e^{-2\pi i/N \cdot N/2} = e^{i\pi} = -1$

also $\hat{x}_{k+\frac{N}{2}} = \sum_{j=0}^{N/2-1} x_j \omega_{N/2}^{-j(k+\frac{N}{2})} = \sum_{j=0}^{\frac{N}{2}-1} x_j \omega_{N/2}^{-jk} \overbrace{\omega_{N/2}^{-j\frac{N}{2}}}^{=1} = \hat{x}_k$

So,

$$\left.\begin{aligned} \hat{u}_k &= \hat{x}_k + \omega_N^{-k} \hat{y}_k \\ \hat{u}_{k+\frac{N}{2}} &= \hat{x}_k - \omega_N^{-k} \hat{y}_k \end{aligned}\right\}$$ This called the "butterfly relationship"

$k = 0, 1, .., \frac{N}{2}-1$

<u>Computational cost:</u>

- Original DFT: N^2 multiplications, $N(N-1)$ additions:
  Total: $2N^2 - N = N(2N-1)$

- New method from above involving $\hat{x}_k$ & $\hat{y}_k$, $k=0,1,\dots,\frac{N}{2}-1$

→ 2 DFTs of length $\frac{N}{2}$ → $\hat{x}$ & $\hat{y}$

Cost: $2\left(\frac{N}{2}\left(2\frac{N}{2}-1\right)\right) = N^2 - N = N(N-1)$

→ Butterfly relationship.

$$\left.\begin{aligned}\hat{u}_k &= \hat{x}_k + \omega_N^{-k}\hat{y}_k \\ \hat{w}_{k+N/2} &= \hat{x}_k - \omega_N^{k}\hat{y}_k\end{aligned}\right]$$

Cost:
N add/subs, $\frac{N}{2}$ mult.

total: $N(N-1) + N + \frac{N}{2} = N^2 + \frac{N}{2} = N\left(N+\frac{1}{2}\right)$

- Thus then new method has a cost that is about half of the original method.

- We can repeat this process recursively until we have N transforms of length 1 to compute.

Ex N=8

$$
\begin{array}{ccccccccc}
x_0^{(0)} = u_0 & x_0^{(1)} & x_0^{(2)} & x_0^{(3)} = u_0 & \Longrightarrow & \hat{x}_0^{(3)} & \hat{x}_0^{(2)} & \hat{x}_0^{(1)} & \hat{u}_0 \\
\cline{4-6}
x_1^{(0)} = u_1 & x_1^{(1)} & x_1^{(2)} & y_0^{(3)} = u_4 & \Longrightarrow & \hat{y}_0^{(3)} & \hat{x}_1^{(2)} & \hat{x}_1^{(1)} & \hat{u}_1 \\
\cline{2-8}
x_2^{(0)} = u_2 & x_2^{(1)} & y_0^{(2)} & x_0^{(3)} = u_2 & \Longrightarrow & \hat{x}_0^{(3)} & \hat{y}_0^{(2)} & \hat{x}_2^{(1)} & \hat{u}_2 \\
\cline{4-6}
x_3^{(0)} = u_3 & x_3^{(1)} & y_1^{(2)} & y_0^{(3)} = u_6 & \Longrightarrow & \hat{y}_0^{(3)} & \hat{y}_1^{(2)} & \hat{x}_3^{(1)} & \hat{u}_3 \\
\hline
x_4^{(0)} = u_4 & y_0^{(1)} & x_0^{(2)} & x_0^{(3)} = u_1 & \Longrightarrow & \hat{x}_0^{(3)} & \hat{x}_0^{(2)} & \hat{y}_0^{(1)} & \hat{u}_4 \\
\cline{4-6}
x_5^{(0)} = u_5 & y_1^{(1)} & x_1^{(2)} & y_0^{(3)} = u_5 & \Longrightarrow & \hat{y}_0^{(3)} & \hat{x}_1^{(2)} & \hat{y}_1^{(1)} & \hat{u}_5 \\
\cline{2-8}
x_6^{(0)} = u_6 & y_2^{(1)} & y_0^{(2)} & x_0^{(3)} = u_3 & \Longrightarrow & \hat{x}_0^{(3)} & \hat{y}_0^{(2)} & \hat{y}_2^{(1)} & \hat{u}_6 \\
\cline{4-6}
x_7^{(0)} = u_7 & y_3^{(1)} & y_1^{(2)} & y_0^{(3)} = u_7 & \Longrightarrow & \hat{y}_0^{(3)} & \hat{y}_1^{(2)} & \hat{y}_3^{(1)} & \hat{u}_7
\end{array}
$$