a) Suppose

$$X_{k+1} - X_k = \alpha_k P_k \quad — (1)$$

The residual $r_{k+1} = r_k - \alpha_k A P_k$, where A is a positive symmetric definite matrix.

Using equation (1) and since α_k is a constant

$$r_{k+1} = r_k - A\,\alpha_k P_k = r_k - A\,(X_{k+1} - X_k)$$

$$r_{k+1} = r_k - A X_{k+1} - \cancel{A X_k}$$

$$r_{k+1} = r_k - A X_k - A X_{k+1}$$

but the residual $r_k = b - A X_k$, given $A X_k = b$

so $r_{k+1} = b - A X_{k+1}$

b) - The directions P_k satisfy $P_{k+1}^T A P_k = 0$, iff P_{k+1} is A-conjugate to P_k with A a positive symmetric definite Matrix.

- This means that P_{k+1} is mutually orthogonal to P_k, given that P_{k+1} and P_k are eigen vectors of A with corresponding eigen values λ_{k+1} and λ_k

- If this is true then

$$A P_k = \lambda_k P_k \quad \text{for } k = 1, \ldots, n$$

Since $A \in \mathbb{R}^{n \times n}$ and is symmetric positive definite then $\exists$ n eigen vectors $P_1, P_2, \ldots, P_n$ which are mutually orthogonal i.e. $(P_i, P_j) = 0$ if $i \neq j$

therefore

$$P_{k+1}^T A P_k = (P_{k+1}, A P_k)$$
$$= (P_{k+1}, \lambda P_k)$$
$$= \lambda (P_{k+1}, P_k)$$

Since k is a positive Integer, so $k+1 \neq k$,

$$P_{k+1}^T A P_k = \lambda (0) = 0$$

Hence the directions P_k, P_{k+1} satisfy $P_{k+1}^T A P_T = 0$

Since they are mutually orthogonal, hence A - Conjugate to each other.