1. Fixed point algorithm

a) For a fixed point iteration $|g'(x)| < 1$

so

$$g(x) = ax + b \Rightarrow g'(x) = a$$

$$|g'(x)| = |a| < 1$$

If this is true then $g(x)$ at $\bar{x}$ has a unique solution, $g(\bar{x}) = \bar{x}$

$$g(\bar{x}) = a\bar{x} + b = \bar{x} \Rightarrow \bar{x} = \frac{b}{1-a}$$

Suppose $|x_k - \bar{x}| = |g(x_{k-1}) - g(\bar{x})|$, for a fixed point iteration

$$x_{k+1} = g(x), \quad g(\bar{x}) = \bar{x} = \frac{b}{1-a}$$

$$|x_k - \bar{x}| = |ax_{k-1} + b - \bar{x}|$$

$$|x_k - \bar{x}| = \left|ax_{k-1} + b - \frac{b}{1-a}\right| = \left|ax_{k-1} - \frac{ab}{1-a}\right|$$

$$= \left|a\left(x_{k-1} - \frac{b}{1-a}\right)\right| = |a(x_{k-1} - \bar{x})|$$

$$|x_k - \bar{x}| = |a(x_{k-1} - \bar{x})|$$

Using triangular inequalities.

$$|x_k - \bar{x}| \leq |a|\,|x_{k-1} - \bar{x}| = |a|^2|x_{k-2} - \bar{x}|$$

$$\vdots$$

$$|x_k - \bar{x}| \leq |a|^k|x_0 - \bar{x}|$$

As $k \to \infty$, since $|a| < 1$, then $|a|^k|x_0 - \bar{x}| \to 0$

therefore $|x_k - \bar{x}| \to 0$, hence the fixed point iteration converges to $\bar{x}$.

$$x_k = \bar{x} = \frac{b}{1-a}$$

b) Using the Intermediate value Theorem, It can be stated that

$$\frac{g(x_k) - g(\bar{x})}{x_k - \bar{x}} = g'(\xi)$$

$$g(x_k) - g(\bar{x}) = g'(\xi)(x_k - \bar{x})$$

for a fixed point Scheme

$$g(x_k) = x_{k+1}$$

$$g(\bar{x}) = \bar{x}$$

$$g'(\xi) = a$$

So

$$x_{k+1} - \bar{x} = a(x_k - \bar{x}) = a^2(x_{k-2} - \bar{x})$$

$$\vdots$$

$$x_{k+1} - \bar{x} = a^k(x_1 - \bar{x})$$

$$x_{k+1} - \bar{x} = a^{k+1}(x_0 - \bar{x})$$

but $x_{k+1} - \bar{x} = e_{k+1}$, $x_0 - \bar{x} = e_0$

$$e_{k+1} = a^{k+1} e_0$$

Hence satisfied.

c) $e_{k+1} \approx \frac{a}{a-1}(x_{k+1} - x_k)$

$e_{k+1} = x_{k+1} - \bar{x}$

Subtracting and adding x_k on the ~~f~~ left hand Side, we obtain

$e_{k+1} = x_{k+1} - \bar{x} - x_k + x_k$

$e_{k+1} = x_{k+1} - x_k + x_k - \bar{x}$

we have $e_{k+1} = a e_k \Rightarrow e_k = \frac{a}{e_{k+1}}$ and $e_k = x_k - \bar{x}$

therefore

~~$e_{k+1} = x_{k+1} + x_k - \bar{x}$ becomes~~

~~$e_{k+1} \approx x_{k+1} - x_k$~~

$e_{k+1} = x_{k+1} - x_k + x_k - \bar{x}$ becomes,

$e_{k+1} \approx x_{k+1} - x_k + e_k$

$e_{k+1} \approx x_{k+1} - x_k + \frac{a}{e_{k+1}}$

$e_{k+1} \approx \frac{a}{a-1}(x_{k+1} - x_k)$

d) $e_{k+1} \approx \frac{a}{a-1}(x_{k+1} - x_k)$

we know that $g(x_k) = x_{k+1} = a x_k + b$

$e_{k+1} \approx \frac{a}{a-1}(g(x_k) - x_k) = \frac{a}{a-1}(a x_k + b - x_k)$

$e_{k+1} \approx a x_k + a\left(\frac{b}{a-1}\right)$, but $\bar{x} = b/1-a$

$e_{k+1} = a x_k - a\bar{x} = a(x_k - \bar{x})$

take $x_k - \bar{x} = e_k$

$e_{k+1} = a e_k = a^2 e_{k-1}$

$e_{k+1} = a^3 e_{k-2}$

$\vdots \qquad \vdots$

$e_{k+1} = a^k e_{1}$

$e_{k+1} = a^{k+1} e_0$

hence exactly equal to the true error.

e). from $g(x) = \frac{1}{10}x + 1$, $\Rightarrow a = 1/10$, $b = 1$

tolerance, $\epsilon_q = 10^{-8}$

Using $|e_{k+1}| \le \epsilon_q$, but $e_{k+1} = a^{k+1} e_0$

$|a^{k+1} e_0| \le \epsilon_q$

Using logarithms, $\log|a^{k+1}| + \log|e_0| \le \log \epsilon_q$

$(k+1)\log|a| + \log|e_0| \le \log \epsilon_q$

$(k+1)\log|a| \le \log \epsilon_q - \log|e_0|$

$k\log|a| \le \log \epsilon_q - \log|e_0| - \log|a|$

but $a = 1/10 = 10^{-1}$

$-k\log 10 \le \log \epsilon_q - (\log|e_0| + \log 10^{-1})$

$k \ge \frac{8\log 10 + \log|e_0| - \log 10}{\log 10} = 8 + \log|e_0| - 1$

$k \ge 7 + \log|e_0|$

Hence it requires atleast 7 iterations, and this depends on $\log|e_0|$