1. fixed point algorithm

$$x_{k+1} = g(x_k)$$

$$g(x) = x$$

a) Show that analytically that for $g(x) = ax+b$, $|a|<1$ the fixed point iteration converges to the solution $\bar{x} = b/(1-a)$

for a fixed point iteration $|g'(x)| < 1$.

$$g(x) = ax+b \Rightarrow g'(x) = a$$
$$|g'(x)| = |a| < 1$$

If $|g'(x)| = |a| < 1$ is true then $g(x)$ has a unique solution, $g(\bar{x}) = \bar{x}$

$$a\bar{x} + b = \bar{x}$$

$$\bar{x} = \frac{b}{1-a}$$

hence $g(x)$ converges to the solution $\bar{x} = \frac{b}{1-a}$

Also for the fixed point iteration x_k

Suppose $|x_k - \bar{x}| = |g(x_{k+1}) - g(\bar{x})|$

Since $g(x_{k+1}) = x_{k+1}$ $g(x_{k-1}) = ax_{k-1} + b$

$$|x_k - \bar{x}| = |ax_{k-1} + b - \frac{b}{1-a}|$$

Since $g(\bar{x}) = \bar{x} = b/1-a$.

$$|x_k - \bar{x}| = \left| q x_{k-1} - \frac{qb}{1-q} \right|$$

$$\text{take } \bar{x} = \frac{b}{1-q}$$

$$|x_k - \bar{x}| = |q (x_{k-1} - \bar{x})|$$

$$|x_k - \bar{x}| \leq |q| |(x_{k-1} - \bar{x})|$$

$$\leq |q|^2 |x_{k-2} - \bar{x}|$$

$$|x_k - \bar{x}| \leq |q|^3 |x_{k-3} - \bar{x}|$$

$$\vdots$$

$$|x_k - \bar{x}| \leq |q|^k |x_0 - \bar{x}|$$

As $k \to \infty$, Since $|q| < 1$, then $|q|^k |x_0 - \bar{x}| \to 0$

therefore

$$|x_k - \bar{x}| \to 0$$

hence the fixed point iteration Converges to $\bar{x}$

$$x_k = \bar{x} = \frac{b}{1-q}$$

b) Show that for any fixed point problem, the error $e_{k+1} = x_k - \bar{x}$ Satisfies $e_{k+1} = q^k e_0$

Using the intermediate value theorem, It can be stated that

$$\frac{g(x_k) - g(\bar{x})}{x_k - \bar{x}} = g'(\xi)$$

$$g(x_k) - g(\bar{x}) = g'(\xi)(x_k - \bar{x})$$

but for fixed point Scheme

$$g(x_k) = x_{k+1}$$

$$g'(\xi) = a$$

$$g(\bar{x}) = \bar{x}$$

$$x_{k+1} - \bar{x} = a(x_k - \bar{x})$$

$$x_{k+1} - \bar{x} = a^2(x_{k-1} - \bar{x})$$

$$x_{k+1} - \bar{x} = a^3(x_{k-2} - \bar{x}) = a^4(x_{k-3} - \bar{x})$$

$$\vdots \qquad \vdots$$

$$x_{k+1} - \bar{x} = a^k(x_0 - \bar{x})$$

but $e_{k+1} = x_{k+1} - \bar{x}$, $e_0 = x_0 - \bar{x}$

$$\underline{e_{k+1} = a^k e_0}$$

c) Show that we can approximate the error using

$$e_{k+1} \approx \frac{a}{a-1}(x_{k+1} - x_k)$$

$$e_{k+1} = x_{k+1} - \bar{x}$$

Subtracting and adding x_k on the left hand side, we obtain

$$e_{k+1} = x_{k+1} - \bar{x} - x_k + x_k$$

$$e_{k+1} = x_{k+1} - x_k + x_k - \bar{x}$$

we have $e_{k+1} = q\,e_k. \Rightarrow e_k = \dfrac{q}{e_{k+1}}$, *

and $e_k = x_k - \bar{x}$

therefore $e_{k+1} = x_{k+1} - x_k + x_k - \bar{x}$ becomes

$e_{k+1} \approx x_{k+1} - x_k + e_k$

$e_{k+1} \approx x_{k+1} - x_k + \dfrac{q}{e_{k+1}}$

$e_{k+1} \approx \dfrac{q}{q-1}(x_{k+1} - x_k)$

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d) Show that for the linear problem $g(x) = qx + b$, the error estimate in problem 1c is exactly equal to the error you found in problem 1b.

$e_{k+1} \approx \dfrac{q}{q-1}(x_{k+1} - x_k)$

we know that $g(x) = x_{k+1} = q x_k + b$.

$e_{k+1} \approx \dfrac{q}{q-1}\left(g(x_k) - x_k\right) = \dfrac{q}{q-1}\left(q x_k + b - x_k\right)$

$e_{k+1} \approx q \cdot x_k + q\left(\dfrac{b}{q-1}\right)$, but $\bar{x} = \dfrac{b}{1-q}$

$e_{k+1} = q x_k - q\bar{x} = q(x_k - \bar{x})$

$e_{k+1} = q(x_k - \bar{x})$, take $x_k - \bar{x} = e_k$

$e_{k+1} = q\,e_k = q^2 e_{k-2} = q^3 e_{k-3} \cdots = q^k e_0$

there fore
$$e_{k+1} = q^k e_0$$

e) How many iterations does the fixed point algorithm require to solve $g(x) = \frac{1}{10}x + 1 = x$ to a tolerance of 10^{-8}?

from $g(x) = \frac{1}{10}x + 1$, $\Rightarrow a = \frac{1}{10}$, $b = 1$

tolerance, $\varepsilon = 10^8$

Using
$$|e_{k+1}| \leq \varepsilon$$

but $e_{k+1} = q^k e_0$

$$q^k e_0 \leq \varepsilon$$

Introducing log both sides

$$k \log a + \log|e_0| \leq \log \varepsilon$$

$$k \log\left(\frac{1}{10}\right) + \log|e_0| \leq \log 10^{-8}$$

$$-k \log 10 \leq -8 \log 10 - \log|e_0|$$

$$k \geq 8 + \log|(e_0)|$$

✱ The fixed point algorithm require atleast 8 iterations to solve $g(x) = \frac{1}{10}x + 1$

$$k \geq 8 + \log|(e_0)|$$

and thus depend on $\log|(e_0)|$

2] Steffensen's Method.

$$x_{k+1} = x_k - \frac{(g(x_k) - x_k)^2}{g(g(x_k)) - 2g(x_k) + x_k}$$

a) Show that analytically that for any $g(x)$, the iteration used in Steffensen's Method satisfies

$$\lim_{x \to \bar{x}} \left(x - \frac{(g(x_k) - x_k)^2}{g(g(x_k)) - 2g(x_k) + x_k} \right) = \bar{x}$$

Where $\bar{x}$ satisfies $g(\bar{x}) = \bar{x}$

Using Lahopital's Rule.

$$\lim_{x \to \bar{x}} \left(x - \frac{2(g'(x) - 1)(g(x) - \bar{x})}{g'(x) g'(g(x)) - 2g'(x) + 1} \right)$$

but $g(\bar{x}) = \bar{x}$

$$= \bar{x} - \frac{2(g'(\bar{x}) - 1)(\bar{x} - \bar{x})}{g'(\bar{x}) g'(g(\bar{x})) - 2g'(\bar{x}) + 1}$$

$$= \bar{x} - 0$$

$$= \underline{\underline{\bar{x}}}$$

b) Show analytically that Steffensen's iteration converges in one step for the fixed point problem.

$$g(x) = ax + b = x \quad , \text{ for } |a| < 1$$

from $x_{k+1} = x_k - \frac{(g(x) - x)^2}{g(g(x)) - 2g(x) + x}$

at $k=0$

$$x_1 = x_0 - \frac{(g(x_0) - x_0)^2}{g(g(x_0)) - 2g(x_0) + x_0}$$

for $g(x_0) = ax_0 + b$

$$x_1 = x_0 - \frac{(ax_0 + b - x_0)^2}{a(ax_0 + b) + b - 2(ax_0 + b) + x_0}$$

$$x_1 = x_0 - \frac{(a-1)(x_0) + b}{(a-1)}$$

$$x_1 = x_0 - x_0 + \frac{b}{a-1} = x_0 - x_0 - \frac{b}{a-1}$$

$$x_1 = \frac{b}{1-a} = \bar{x}$$

$$\underline{x_1 = \bar{x}}$$

hence it Converges in one step.

No.3   LU Decomposition

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -l_{21} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix}$$

a) Show that

$$E_{41} E_{31} E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -l_{21} & 1 & 0 & 0 \\ -l_{31} & 0 & 1 & 0 \\ -l_{41} & 0 & 0 & 1 \end{bmatrix}$$

$$E_{41} E_{31} E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -l_{41} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -l_{31} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -l_{21} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -l_{41} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -l_{21} & 1 & 0 & 0 \\ -l_{31} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_{41} E_{31} E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -l_{21} & 1 & 0 & 0 \\ -l_{31} & 0 & 1 & 0 \\ -l_{41} & 0 & 0 & 1 \end{bmatrix}$$

b) Choose multipliers l_{ij} so that applying $E_{41} E_{31} E_{21}$ to A zeros out the entries below a_{11}.

$$E_{41} E_{31} E_{21} A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -l_{21} & 1 & 0 & 0 \\ -l_{31} & 0 & 1 & 0 \\ -l_{41} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix}$$

Choose
$$l_{21} = -1$$
$$l_{31} = 2$$
$$l_{41} = -3$$

$$E_{41} E_{31} E_{21} A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 10 & 4 & -9 \\ 0 & -16 & -11 & 18 \end{bmatrix}$$

___

c) Show that the inverse of $E_{41} E_{31} E_{21}$ is

$$(E_{41} E_{31} E_{21})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & 0 & 1 & 0 \\ l_{41} & 0 & 0 & 1 \end{bmatrix}$$

$$\left(E_{41}\,E_{31}\,E_{21}\right)^{-1} = \frac{1}{\det\left(E_{41}\,E_{31}\,E_{21}\right)}\left(\text{Adjoint}\left(E_{41}\,E_{31}\,E_{21}\right)\right)$$

$$\det\left(E_{41}\,E_{31}\,E_{21}\right) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -l_{21} & 1 & 0 & 0 \\ -l_{31} & 0 & 1 & 0 \\ -l_{41} & 0 & 0 & 1 \end{vmatrix} = -1\begin{vmatrix} 1 & 0 & 0 \\ -l_{41} & 1 & 0 \\ -l_{31} & 0 & 1 \end{vmatrix} = \underline{\underline{1}}$$

the Adjoint $\left(E_{41}\,E_{31}\,E_{21}\right)$ is the transpose of the Cofactor Matrix $\left(E_{41}\,E_{31}\,E_{21}\right)$

$$\text{Cofactor Matrix} = \begin{bmatrix} 1 & l_{21} & l_{31} & l_{41} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Adjoint}\left(E_{41}\,E_{31}\,E_{21}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & 0 & 1 & 0 \\ l_{41} & 0 & 0 & 1 \end{bmatrix}$$

$$\left| E_{41}\,E_{31}\,E_{21} \right| = \frac{1}{1}\begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & 0 & 1 & 0 \\ l_{41} & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & 0 & 1 & 0 \\ l_{41} & 0 & 0 & 1 \end{bmatrix}$$

3(d) From the above steps, derive the LU decomposition of a general 4×4 matrix. Use these to derive the LU decomposition of the matrix in (3).

For U
$$E_{41} E_{31} E_{21} A = U_1$$

$$(E_{42} E_{32}) U_1 = U_2$$

$$E_{43} U_2 = U$$

So

$$\cancel{U = E_{43} E_{42} E_{41} E_{32} E_{31} E_{21} A}$$
$$U = E_{43} E_{42} E_{32} E_{41} E_{31} E_{21} A$$

& for L

$$(E_{41} E_{31} E_{21})^{-1} = L_1$$

$$(E_{42} E_{32})^{-1} L_1 = L_2$$

$$(E_{43})^{-1} L_2 = L$$

So

$$L = (E_{43} E_{42} E_{32} E_{41} E_{31} E_{21})^{-1}$$

Suppose we have $(AB)^{-1} = A^{-1} B^{-1}$

$$L = (E_{43} E_{42} E_{32} E_{41} E_{31} E_{21})^{-1} = (E_{43} E_{42} E_{32})^{-1} (E_{41} E_{31} E_{21})^{-1}$$

but $(E_{41} E_{31} E_{21})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ L_{21} & 1 & 0 & 0 \\ L_{31} & 0 & 1 & 0 \\ L_{41} & 0 & 0 & 1 \end{bmatrix}$

$$\left(E_{43}\, E_{42}\, E_{32}\right)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & L_{32} & 1 & 0 \\ 0 & L_{32}L_{43}+L_{42} & L_{43} & 1 \end{bmatrix}$$

$$L = \left(E_{43}\, E_{42}\, E_{32}\right)^{-1} \left(E_{41}\, E_{31}\, E_{21}\right)^{-1}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & L_{32} & 1 & 0 \\ 0 & L_{32}L_{43}+L_{42} & L_{43} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ L_{21} & 1 & 0 & 0 \\ L_{31} & 0 & 1 & 0 \\ L_{41} & 0 & 0 & 1 \end{bmatrix}$$

Let $L_{21} = a$ $L_{32} = d$

$L_{31} = b$ $L_{42} = e$

$L_{41} = c$ $L_{43} = f$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ da+b & d & 1 & 0 \\ fb+a(df+e)+c & df+e & f & 1 \end{bmatrix}$$

And $U = E_{43}\, E_{42}\, E_{32}\, E_{41}\, E_{31}\, E_{21}\, A$

48) Find the LU decomposition A the matrix in (3)

$$A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix}$$

$$A = LU$$

where

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

Using Gaussian Elimination method on A, we obtain an upper triangular Matrix U, as follows.

take $l_{21} = -1$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 + R_1} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 5 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix}$$

In this case $l_{21} = -1$

takes $l_{31} = -2$

$$\begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 5 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_2 + 2R_1} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 5 \\ 0 & -18 & -4 & -1 \\ -9 & 5 & -5 & 12 \end{bmatrix}$$

In this case $l_{31} = -2$

$$\begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & -18 & -4 & -1 \\ -9 & 5 & -5 & 12 \end{bmatrix} \xrightarrow{R_4 \leftarrow R_4 + 3R_1} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & -18 & -4 & -1 \\ 0 & -16 & -11 & 18 \end{bmatrix}$$

In this case, we took $L_{41} = -3$

for L_{32} take $L_{32} = 9$

$$\begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & -18 & -4 & -1 \\ 0 & -16 & -11 & 18 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - 9R_2} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & 5 & -19 \\ 0 & -16 & -11 & 18 \end{bmatrix}$$

take $L_{42} = 8$

$$\begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & 5 & -19 \\ 0 & -16 & -11 & 18 \end{bmatrix} \xrightarrow{R_4 \leftarrow R_4 - 8R_2} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & 5 & -19 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

take $L_{43} = -3/5$

$$\begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & 5 & -19 \\ 0 & 0 & -3 & 2 \end{bmatrix} \xrightarrow{R_4 \leftarrow R_4 + \frac{3}{5}R_3} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & 5 & -19 \\ 0 & 0 & 0 & \frac{-47}{5} \end{bmatrix}$$

Hence we Obtain ,U,

$$U = \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & 5 & -19 \\ 0 & 0 & 0 & -47/5 \end{bmatrix}$$

Corresponding to , L,

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 9 & 1 & 0 \\ -3 & 8 & -3/5 & 1 \end{bmatrix}$$

So A = LU

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 9 & 1 & 0 \\ -3 & 8 & -3/5 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & 5 & -19 \\ 0 & 0 & 0 & -\frac{47}{5} \end{bmatrix}$$

5. Jacobi Method

Assume that the following 2×2 matrix A is Strictly diagonally dominant

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

a) Show that $\rho(I - \Delta^{-1}A) < 1$

take $\Delta = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$

$$\Delta^{-1} = \begin{bmatrix} 1/a_{11} & 0 \\ 0 & 1/a_{22} \end{bmatrix}$$

$$\Delta^{-1}A = \begin{bmatrix} 1/a_{11} & 0 \\ 0 & 1/a_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & a_{12}/a_{11} \\ a_{21}/a_{22} & 1 \end{bmatrix}$$

then

$$I - \Delta^{-1}A = \begin{pmatrix} 0 & a_{12}/a_{11} \\ a_{21}/a_{22} & 0 \end{pmatrix}$$

the Spectral radius, ρ, is the maximum absolute value of the eigen values., So to compute the eigen values we use the Characteristic equation.

$$|(I - \Delta^{-1}A) - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & a_{12}/a_{11} \\ a_{21}/a_{22} & -\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^2 = \frac{a_{21}\,a_{12}}{a_{22}\,a_{11}}$$

Since a_{22} and a_{11} are diagonally dominant real entries then $a_{22}a_{11} > a_{21}a_{12}$, therefore

$$\lambda^2 = \frac{a_{21}a_{12}}{a_{22}a_{11}} < 1$$

$$\lambda = \pm\sqrt{\frac{a_{21}a_{12}}{a_{22}a_{11}}}$$

ρ is the largest $|\lambda|$, so

$$|\lambda| = \left|\sqrt{\frac{a_{21}a_{12}}{a_{22}a_{11}}}\right| < 1$$

So the So the Spectral radius $\rho(I - D^{-1}A) < 1$

Jacobi Iteration will Converge, Since for Jacobi, we take $M = D$, and we have Shown that

$$\rho(I - M^{-1}A) < 1,$$ hence the Iteration Converges

since according for Converge an Iteration Converges if and only if $\rho(I - M^{-1}A) < 1$

$$X_{k+1} = (I - M^{-1}A)X_k + M^{-1}b.$$

a) Show that If $M = A$, the Iteration Converges in one step.

If $M = A$

$$X_{k+1} = (I - A^{-1}A)X_k + A^{-1}b$$

but $A^{-1}A = I$

$$X_{k+1} = (I - I)X_k + A^{-1}b$$

$$X_{k+1} = A^{-1}b$$

but $A\bar{X} = b \Rightarrow \bar{X} = A^{-1}b$

therefore $X_{k+1} = \bar{X}$, have Converges in one step

b) Show that $Ae_k = -r_k$, where $r_k = b - AX_k$,

$$e_k = X_k - \bar{X},$$

and $\bar{x}$ solves $AX = b$ exactly

$$Ae_k = A(X_k - \bar{X}).$$

$$Ae_k = AX_k - A\bar{X}$$

we know that $A\bar{X} = b$

$$Ae_k = AX_k - b$$

$$Ae_k = -r_k \qquad \text{sinc } r_k = b - AX_k.$$

c) Show that the iteration for the error e_k is given by

$$e_{k+1} = (I - M^{-1}A) e_k$$

from

$$x_{k+1} = (I - M^{-1}A) x_k + M^{-1}b$$

Subtract $\bar{x}$ from both sides

$$x_{k+1} - \bar{x} = (I - M^{-1}A) x_k - \bar{x} + M^{-1}b$$

we know that $x_{k+1} - \bar{x} = e_{k+1}$

$$e_{k+1} = (I - M^{-1}A) x_k - \bar{x} + M^{-1}b$$

take $A\bar{x} = b$.

$$e_{k+1} = (I - M^{-1}A) x_k - \bar{x} + M^{-1}A\bar{x}$$

$$e_{k+1} = (I - M^{-1}A) x_k - (I - M^{-1}A)\bar{x}$$

$$e_{k+1} = (I - M^{-1}A) (x_k - \bar{x})$$

but $x_k - \bar{x} = e_k$

$$e_{k+1} = (I - M^{-1}A) e_k$$

---

d) Show that

$$\|e_k\| \leq \|I - M^{-1}A\|^k \|e_0\|$$

from $e_{k+1} = (I - M^{-1}A) e_k$

$$e_{k+1} = (I - M^{-1}A)(I - M^{-1}A) e_{k-1}$$

$$e_{k+1} = (I - M^{-1}A)^2 e_{k-1}$$

$$e_k = (I - M^{-1}A)^2 e_{k-2}$$

$$e_k = (I - M^{-1}A)^3 e_{k-3}$$

$$\vdots \qquad \vdots$$

$$e_k = (I - M^{-1}A)^k e_0$$

take the Modulus both sides, and then apply the triangular Inequality

$$\|e_k\| = \|(I - M^{-1}A)^k e_0\|$$

$$\|e_k\| \leq \|(I - M^{-1}A)\|^k \|e_0\|$$

---

② Show that

$$K \geq \frac{\log(\varepsilon)}{\log\|I - M^{-1}A\|}$$

from $\|e_k\| \leq \varepsilon$

~~but $\|e_k\| \leq \|(I - M^{-1}A)\|^k \|e_0\|$~~

but $e_k = (I - M^{-1}A)^k e_0$

$$\|(I - M^{-1}A)^k e_0\| \leq \varepsilon$$

taking log both sides we obtain

$$k \log\|I - M^{-1}A\| + \log\|e_0\| \leq \log \varepsilon$$

Since

$$k \log \| I - M^{-1}A \| \leq \log \varepsilon - \log \| e_0 \|$$

Since $\log \| I - M^{-1}A \| < 0$, then

$$k \geq \frac{\log \varepsilon - \log e_0}{\log \| I - M^{-1}A \|}$$

$$k \geq \frac{\log \varepsilon}{\log \| I - M^{-1}A \|} - \frac{\log \| e_0 \|}{\log \| I - M^{-1}A \|}$$

the term $\dfrac{\log(\varepsilon)}{\log \| I - M^{-1}A \|}$ dominates there for

$$k \geq \frac{\log(\varepsilon)}{\log \| I - M^{-1}A \|}$$

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f) Since $\rho(I - M^{-1}A)$ is the largest absolute value of the eigen values of $(I - M^{-1}A)$, and the number of iterations is given by

$$k \geq \frac{\log \varepsilon}{\log \| I - M^{-1}A \|} - \frac{\log \| e_0 \|}{\log \| I - M^{-1}A \|}$$

So for larger values of $\rho(I - M^{-1}A)$ implies the term

$$\frac{\log \|\varepsilon_0\|}{\log \|I - M^{-1}A\|}$$

is also large. This is because $\log \|I - M^{-1}A\|$ will be very small to reduce $\log \|\varepsilon_0\|$.

But again, as $\log \|I - M^{-1}A\|$ approaches to zero from the left side, 'It means that this term

$$\frac{\log \|\varepsilon_0\|}{\log \|I - M^{-1}A\|}$$

increases and eventually become undefined at $\log \|I - M^{-1}A\| = 0$, this case the solution no longer exists, so the the estimate iterations k will seem to underestima the actual number of k i.e.

$$K \geqslant \frac{\log(\varepsilon)}{\log \|I - M^{-1}A\|} + \text{a very big term}$$

after which becomes undefined.

9) Show analytically that the update to the residual r_{k+1} in the Conjugate Gradient Algorithm is equal to $b - Ax_k$

Suppose A is symmetric, positive definite

$$F(x) = \frac{1}{2} x^T A x - b^T x$$

So the direction of the greatest decrease of f is given by $-\nabla F(x_{k+1}) = r_{k+1}$ of the

Update to residual r_{k+1}

So taking the residual as the search direction we have

$$x_{k+1} = x_k + q_k r_k$$

$$-\nabla F(x_{k+1}) = r_{k+1}$$

$$-\nabla F(x_k + q_k r_k) = r_{k+1}$$

$$F(x_k + q_k r_k) = \frac{1}{2}(x_k + q_k r_k)^T A (x_k + q_k r_k) - b^T (x_k + q_k r_k)$$

$$\nabla F(x_k + q_k r_k) = \frac{dF}{dx_k} = \frac{1}{2}(2 A x_k) - b$$

$$\nabla F(x_k + q_k r_k) = A x_k - b$$

$$-\nabla F(x_k + q_k r_k) = b - A x_k = r_{k+1}$$

therefore $\underline{r_{k+1} = b - A x_k}$

8(b).

The directions P_k satisfy $P_{k+1}^T A P_k = 0$, iff P_{k+1} is A-Conjugate to P_k, with A a symmetric positive definite matrix.

Suppose A is positive symmetric definite, let P_k be an eigen vector of the matrix A, with corresponding eigen value λ_k then

$$A P_k = \lambda_k P_k \qquad \text{for } k=1, \ldots, n$$

So since $A \in \mathbb{R}^{n \times n}$ and is symmetric positive definite then $\exists$ n eigen vectors $P_1, \ldots, P_n$ which are mutually orthogonal ie $(P_i, P_j) = 0$ for $i \neq j$, hence

therefore

$$P_{k+1}^T A P_7 = (P_{k+1}, A P_k)$$
$$= (P_{k+1}, \lambda P_k)$$
$$= \lambda (P_{k+1}, P_k)$$

Since k is a positive integer, so $k+1 \neq k$ therefore;

$$P_{k+1}^T A P_k = \lambda (0) = 0$$

These directions P_k satisfy $P_{k+1}^T A P_7 = 0$ since they are mutually orthogonal, hence A-Conjugate to each other.