5. (Trigonometric interpolation)

$$x_j = \frac{2\pi j}{N} \quad , j = 0, 1, \cdots, N-1$$

Given

$$P_N(x) = \sum_{j=0}^{N-1} S_N(x - x_j)\, u(x_j)$$

where

$$S_N(t) = \begin{cases} \frac{1}{N} \sin\left(\frac{N}{2} t\right) \cot\left(\frac{t}{2}\right), & \text{If } N \text{ is even,} \\ \frac{1}{N} \sin\left(\frac{N}{2} t\right) \csc\left(\frac{t}{2}\right), & \text{If } N \text{ is odd.} \end{cases}$$

a) Verify that $P_N(x)$ actually interpolates the data.

Show that $P_N(x_k) = u(x_k)$ for $k = 0, 1, 2, \cdots, N-1$

We have to show that

$$S_N(x_j - x_k) = 0 \text{ when } k \neq j \text{ and } S_N(0) = 1$$

$$S_N(x_j - x_k) = \begin{cases} \frac{1}{N} \sin\left(\frac{N}{2}(x_j - x_k)\right) \cot\left(\frac{x_j - x_k}{2}\right), & \text{If } N \text{ is even} \\ \frac{1}{N} \sin\left(\frac{N}{2}(x_j - x_k)\right) \csc\left(\frac{x_j - x_k}{2}\right); & \text{If } N \text{ is odd.} \end{cases}$$

but $x_j = \frac{2\pi j}{N}$, $x_k = \frac{2\pi k}{N}$

$$S_N(x_j - x_k) = \begin{cases} \frac{1}{N} \sin(\pi(j-k)) \cot\left(\frac{\pi}{N}(j-k)\right), & \text{If } N \text{ is even,} \\ \frac{1}{N} \sin(\pi(j-k)) \csc\left(\frac{\pi}{N}(j-k)\right), & \text{If } N \text{ is odd.} \end{cases}$$

When $k \neq j$ and if N is even:

$$S_N(x_j - x_k) = \frac{1}{N} \sin(\pi(j-k)) \cdot \cot\left(\frac{\pi}{N}(j-k)\right)$$

Since j and k are integers and $N \neq 0$, then $\sin(\pi(j-k)) = 0$

hence

$$S_N(x_j - x_k) = 0$$

When $k \neq j$, if N is odd

$$S_N(x_j - x_k) = \frac{1}{N} \sin(\pi(j-k)) \csc\left(\frac{\pi}{N}(j-k)\right)$$

Since j and ~~k~~ are integers, then $\sin(\pi(j-k)) = 0$

hence

$$S_N(x_j - x_k) = 0$$

therefore

$$S_N(x_j - x_k) = \begin{cases} 0, & \text{if } N \text{ is even} \\ 0, & \text{if } N \text{ is odd.} \end{cases}$$

Also ~~to~~ show that $S_N(0) = 1$

If N is even

$$S_N(t) = \frac{1}{N} \sin\left(\frac{N}{2}t\right) \cot\left(t/2\right) = \frac{1}{N} \sin\left(\frac{N}{2}t\right) \frac{\cos(t/2)}{\sin(t/2)}$$

$$\lim_{t \to 0} S_N(t) = \frac{0}{0}$$

Using Lahopital's rule.

$$\lim_{t\to 0} S_N(t) = \frac{\frac{1}{N}\left(\frac{N}{2}\cos\left(\frac{N}{2}t\right)\cos(t/2) - \frac{1}{2}\sin(t/2)\sin\left(\frac{N}{2}t\right)\right)}{\frac{1}{2}\cos(t/2)}$$

As $t \to 0$ $\sin t/2 \, \sin\left(\frac{N}{2}t\right) \to 0$

$$\lim_{t\to 0} S_N(t) = \frac{2}{N}\cdot\frac{N}{2}\cos\left(\frac{N}{2}(0)\right) = \underline{\underline{1}}$$

If N is odd

$$S_N(t) = \frac{1}{N}\sin\left(\frac{N}{2}t\right)\csc(t/2) = \frac{\frac{1}{N}\sin\left(\frac{N}{2}t\right)}{\sin(t/2)}$$

$$\lim_{t\to 0} S_N(t) = \frac{0}{0}$$

Using Lahopitals Rule.

$$\lim_{t\to 0} S_N(t) = \frac{\frac{1}{N}\left(\frac{N}{2}\cos\left(\frac{N}{2}t\right)\right)}{\frac{1}{2}\cos(t/2)} = \frac{2}{N}\cdot\frac{N}{2} = 1.$$

$$\lim_{t\to 0} S_N(t) = \begin{cases} 1 \; ; & \text{If } N \text{ is even} \\ 1 \; , & \text{If } N \text{ is odd.} \end{cases}$$

Hence $P_N(x)$ actually interpolates the data.

5b) For the case of N being even show that

$$S_N'(x_k) = \begin{cases} 0, & \text{if } k=0 \\ \frac{1}{2}(-1)^k \cot\left(\frac{k\pi}{N}\right), & \text{if } k \neq 0 \end{cases}$$

~~to~~ from

$$S_N(t) = \begin{cases} \frac{1}{N}\sin\left(\frac{N}{2}t\right)\cot\left(t/2\right), & \text{if } N \text{ is even,} \\ \frac{1}{N}\sin\left(\frac{N}{2}t\right)\csc\left(t/2\right), & \text{if } N \text{ is odd.} \end{cases}$$

If N is even

$$S_N(t) = \frac{1}{N}\sin\left(\frac{N}{2}t\right)\cot\left(t/2\right)$$

taking $t = x_k$

$$S_N(x_k) = \frac{1}{N}\sin\left(\frac{N}{2}x_k\right)\cot\left(x_k/2\right)$$

$$S_N'(x_k) = \frac{d}{dx_k}S_N(x_k) = \frac{1}{N}\frac{d}{dx_k}\left[\sin\left(\frac{N}{2}x_k\right)\cot\left(\frac{x_k}{2}\right)\right]$$

$$= \frac{1}{N}\left[\cot\left(\cancel{x_k}/2\right)\left(\frac{N}{2}\cos\left(\frac{N}{2}\cancel{x_k}\right)\right) + \sin\left(\frac{N}{2}\cancel{x_k}\right)\left(\frac{-1}{2}\csc^2\left(\frac{\cancel{x_k}}{2}\right)\right)\right]$$

but $x_k = \frac{2\pi}{N}k$

$$S_N'(x_k) = \frac{1}{N}\left[\frac{N}{2}\cot\left(\frac{\pi}{N}k\right)\cos(\pi k) - \frac{1}{2}\sin(\pi k)\csc^2\left(\frac{\pi}{N}k\right)\right]$$

Since k is an integer $\sin(\pi k) = 0$

$$S_N'(x_k) = \frac{1}{N}\left[\frac{N}{2}\cot\left(\frac{\pi}{N}k\right)\cos(\pi k)\right]$$

$$S_N'(x_k) = \frac{1}{2}\cot\left(\frac{\pi}{N}k\right)\cos(\pi k)$$

If $k=0$

$$S_N'(x_k) = \frac{1}{2}\cos(\pi k)\frac{\cos\left(\pi/N\, k\right)}{\sin\left(\pi/N\, k\right)} = \frac{1}{0}$$, which is not good

Using Lahopital's Rule.

$$S_N'(x_k) = \frac{\frac{1}{2}\left[-\frac{\pi}{N}\sin\left(\frac{\pi}{N}k\right)\cos(\pi k) + -\pi\sin(\pi k)\cos\left(\frac{\pi}{N}k\right)\right]}{\frac{\pi}{N}\cos\left(\frac{\pi}{N}k\right)}$$

$$\lim_{k\to 0} S_N'(x_k) = \frac{0}{1} = \underline{\underline{0}}$$

If $k \neq 0$

$$S_N'(x_k) = \frac{1}{2}\cos(\pi k)\cot\left(\pi/N\, k\right).$$

$\cos(\pi k) \simeq (-1)^k$ for k integer.

$$S_N'(x_k) = \frac{1}{2}(-1)^k\cot\left(\frac{\pi}{N}k\right)$$

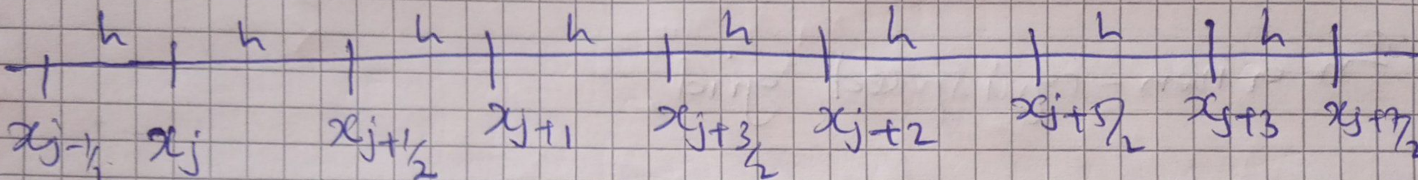
Therefore

$$S_N'(x_k) = \begin{cases} 0, & \text{if } k=0 \\ \frac{1}{2}(-1)^k\cot\left(\frac{\pi}{N}k\right), & \text{if } k\neq 0 \end{cases}$$

NO 2 (Finite-volume scheme)

a) For a uniform grid, from equation (1) in the question.

$$x_{j+1} - x_j = x_{j+3/2} - x_{j+1/2} = x_{j+1/2} - x_{j-1/2} = h$$

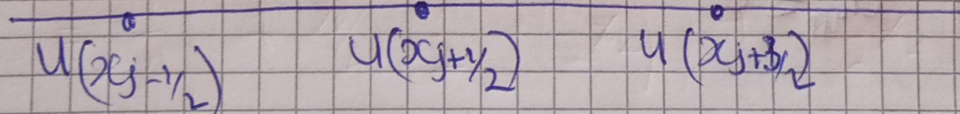


therefore

$$u''(x_{j+1/2}) \approx \frac{1}{h^2}\left[u(x_{j+3/2}) - u(x_{j+1/2}) - u(x_{j+1/2}) + u(x_{j-1/2})\right]$$

$$u''(x_{j+1/2}) \approx \frac{1}{h^2}\left[u(x_{j+3/2}) - 2u(x_{j+1/2}) + u(x_{j-1/2})\right]$$

$$h^2 u''(x_{j+1/2}) \approx u(x_{j+3/2}) - 2u(x_{j+1/2}) + u(x_{j-1/2}) \quad \text{— (1)}$$



hence a 3 point scheme.

In class, ~~we~~ for $u''(x_j)$, we obtained

$$h^2 u''(x_j) \approx u(\bar{x}-h) - 2u(\bar{x}) + u(\bar{x}+h) \quad \text{— (2)}$$

which is also a 3 point scheme.

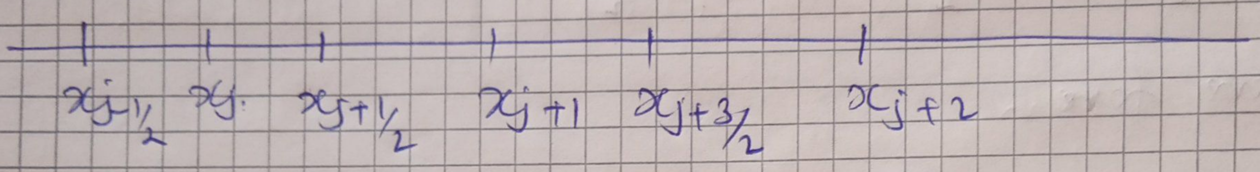
in equation (1), taking $\bar{x} = x_{j+1/2}$, we obtain

$h^2 u''(\bar{x}) \approx u(\bar{x}-h) - 2u(\bar{x}) + u(\bar{x}+h)$, which is

Similar to equation (2), hence the same stencil.

b) Show that on a non-equispaced grid, ~~Stenc~~ (1) is exact for $u=1$ and $u=x$, but not $u=x^2$

For a non-equispaced grid,



$x_{j+1/2} - x_{j-1/2} \neq x_{j+3/2} - x_{j+1/2} \neq x_j - x_{j+1} \neq h$.

For $u=1$

$\frac{d}{dx} u_{exact} = 0$, $u''_{exact} = 0$

Into (1), if $u=1$ then $u(x_{j+3/2}) - u(x_{j+1/2}) = 0$

and also $u(x_{j+1/2}) - u(x_{j-1/2}) = 0$, therefore

$\cancel{u''(x_{j+1/2})}$ $u''_{app} = 0$

Since

$u''_{exact} = u''_{approx}$, hence (1) is exact for $u=1$

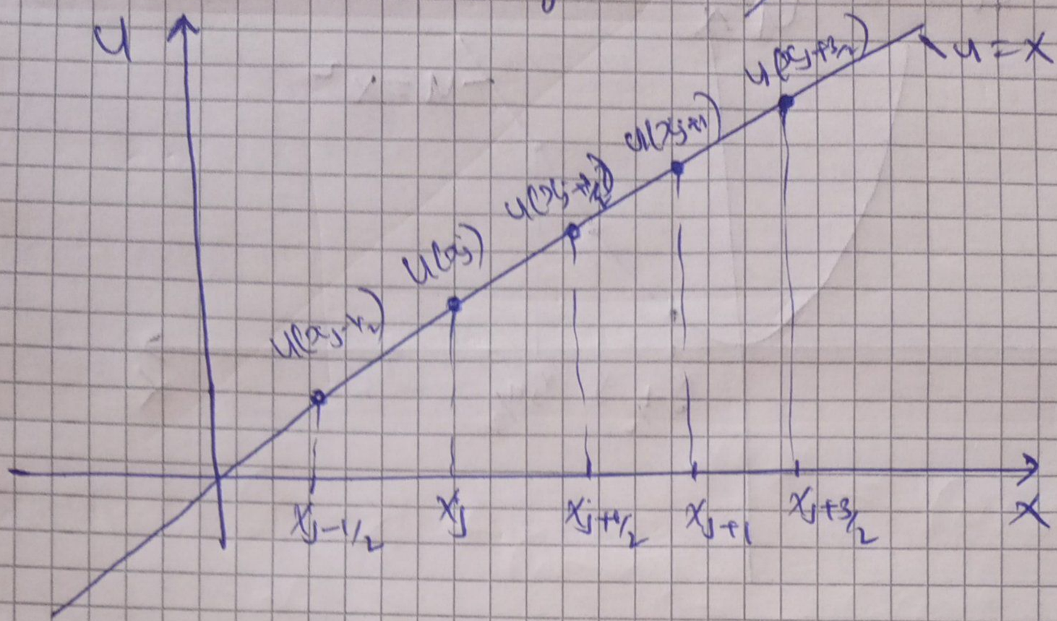
For $u = x$

Exact

$u'_{exact} = 1$, $u''_{exact} = 0$

Numerical

Since $u = x$, is a straight line,



then

$$\frac{u(x_{j+3/2}) - u(x_{j+1/2})}{x_{j+3/2} - x_{j+1/2}} = \frac{u(x_{j+1/2}) - u(x_{j-1/2})}{x_{j+1/2} - x_{j-1/2}}$$

therefore

$$u''_{approx} = \frac{1}{x_{j+1} - x_j}(0) = 0$$

Since $u''_{exact} = u''_{approx} = 0$. Hence, (1) is exact for $u = x$

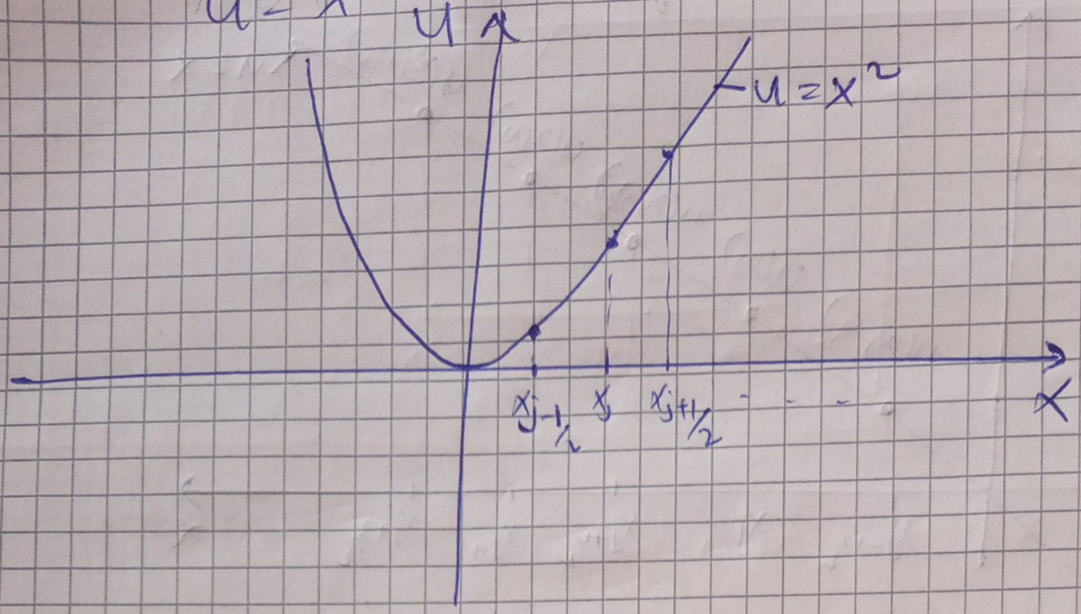
For $u = x^2$

$u'_{exact} = 2x$, $u''_{exact} = 2$

<u>Numerically</u>

Since $u = x^2$ is a curve, and that grid space points are not equal spaced,

$U = x^2$



Then,

$$\frac{u(x_{j+3/2}) - u(x_{j+1/2})}{x_{j+3/2} - x_{j+1/2}} \neq \frac{u(x_{j+1/2}) - u(x_{j-1/2})}{x_{j+1/2} - x_{j-1/2}}$$

therefore, $u''(x_{j+1/2}) \approx \frac{1}{x_{j+1} - x_j}\left[\frac{(x_{j+3/2})^2 - (x_{j+1/2})^2}{x_{j+3/2} - x_{j+1/2}} - \frac{(x_{j+1/2})^2 - (x_{j-1/2})^2}{x_{j+1/2} - x_{j-1/2}}\right]$

$u''(x_{j+1/2}) \neq 2$, Cannot equal to 2,

\# $u''_{approx} \neq u''_{exact}$, Therefore the technique is not Exact.

For the technique we discussed in class, the grid points are equally spaced, which means that

$$x_{j+1} - x_j = x_{j+3/2} - x_{j+1/2} = x_{j+1/2} - x_{j-1/2} = h$$

$$u''(x_{j+1/2}) \approx \frac{1}{h^2}\left[u(x_{j+3/2}) - 2u(x_{j+1/2}) + u(x_{j-1/2})\right]$$

therefore $u = 1$, $\bar{x} = x_{j+1/2}$

$u(x_{j+3/2}) = u(x_{j+1/2}) = u(x_{j-1/2}) = 1$, since its a straight line

$u(\bar{x}+h) = u(\bar{x}) = u(\bar{x}-h) = 1$

$$u''(\bar{x}) \approx \frac{1}{h^2}\left[u(\bar{x}+h) - 2u(\bar{x}) + u(\bar{x}-h)\right] = \frac{1}{h^2}(2-2)$$

$$u''(\bar{x}) \approx 0$$

which is an exact value.

For $u = x$

$u(\bar{x}+h) = \bar{x}+h$, $u(\bar{x}) = \bar{x}$, $u(\bar{x}-h) = \bar{x}-h$.

$$u''(\bar{x}) \approx \frac{1}{h^2}\left((\bar{x}+h) - 2\bar{x} + (\bar{x}-h)\right) = \frac{1}{h^2}(0) = 0$$

$u''(\bar{x}) = 0$, which is an exact value

For $u = x^2$

$u(\bar{x}+h) = (\bar{x}+h)^2 = \bar{x}^2 + h^2 + 2\bar{x}h$

$u(\bar{x}) = \bar{x}^2$

$u(\bar{x}-h) = (\bar{x}-h)^2 = \bar{x}^2 + h^2 - 2\bar{x}h$

$$u''(\bar{x}) \approx \frac{1}{h^2}\left(\cancel{\bar{x}^2} + \cancel{h^2} + \cancel{2\bar{x}h} - 2\bar{x}^2 + \cancel{\bar{x}^2} + \cancel{h^2} - \cancel{2\bar{x}h}\right) = \frac{1}{h^2}(2h^2)$$

$$u''(\bar{x}) \approx 2$$

which is an exact value.