Gaussian Elimination

How can we solve

$$A\mathbf{x} = \mathbf{b}$$

?

Linear algebra

Typical linear system of equations :

$$\begin{aligned} 5x_1 - x_2 + 2x_3 &= 7 \\ -2x_1 + 6x_2 + 9x_3 &= 0 \\ -7x_1 + 5x_2 - 3x_3 &= 5 \end{aligned}$$

The variables x_1, x_2, and x_3 only appear as linear terms (no powers or products).

Linear algebra

Where do linear systems come from?

- Fitting curves to data
- Polynomial approximation to functions
- Computational fluid dynamics (“Poisson problem”)
- Network flow
- Circuit analysis
- Electrostatics
- Computer graphics (physics based modeling)

Solving linear systems

We are already familiar with at least one type of linear system

$$\begin{aligned} 3x_1 + 5x_2 &= 3 \\ 2x_1 - 4x_2 &= 1 \end{aligned}$$

The solution is the intersection of two lines represented by each equation. This solution is a point (x_1, x_2) that satisfies both equations *simultaneously.*

We could also view the solution as providing the correct combination of vectors (3,2) and $(5, -4)$ that give us (3,1).

$$x_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ -4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Linear algebra - a 2x2 system

We can *row-reduce* an augmented matrix to find the solution :

Use an elementary row operation to produce a "0" in the lower left corner.

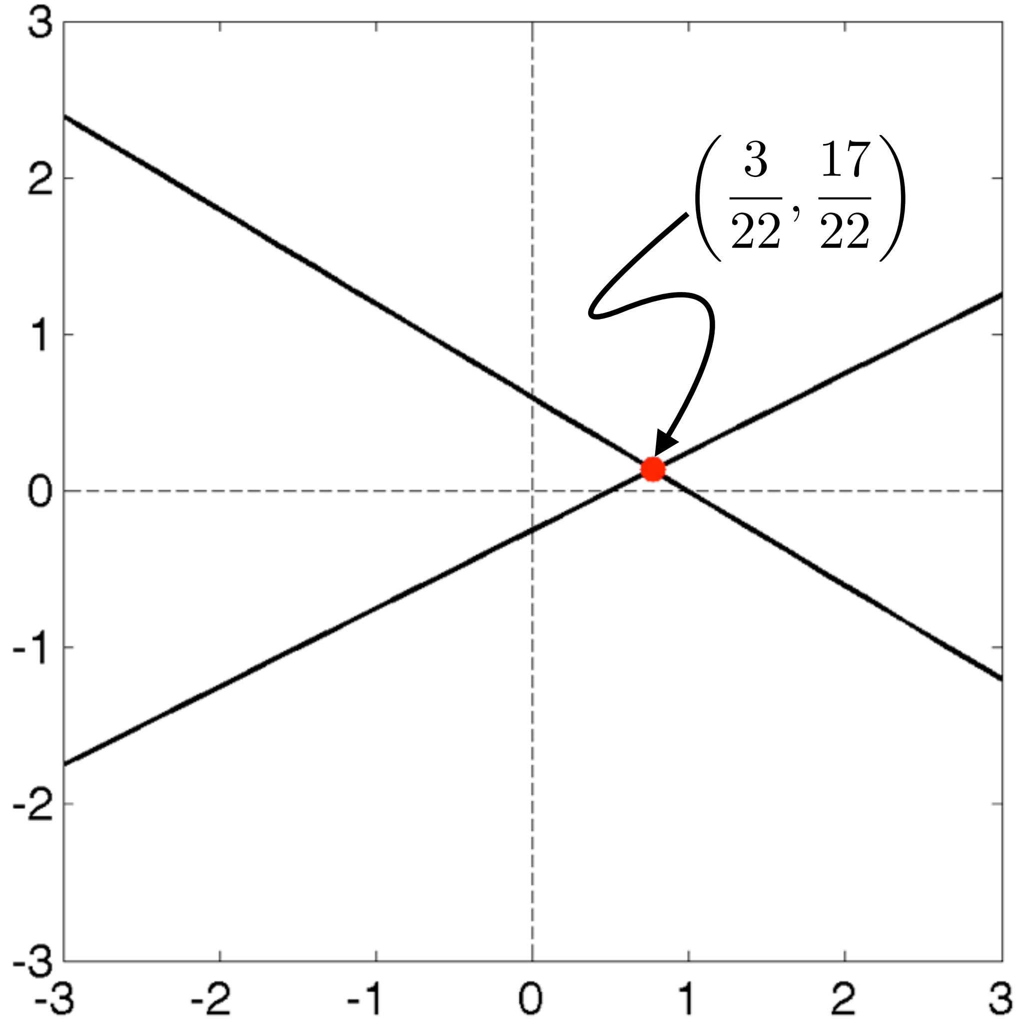
$$\left[\begin{array}{cc|c} 3 & 5 & 3 \\ 2 & -4 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 3 & 5 & 3 \\ 0 & -\frac{22}{3} & -1 \end{array}\right] \leftarrow (\text{eqn } 2) - \left(\frac{2}{3}\right)(\text{eqn } 1)$$

Use back-substitution to solve first for x_2 and then for x_1.

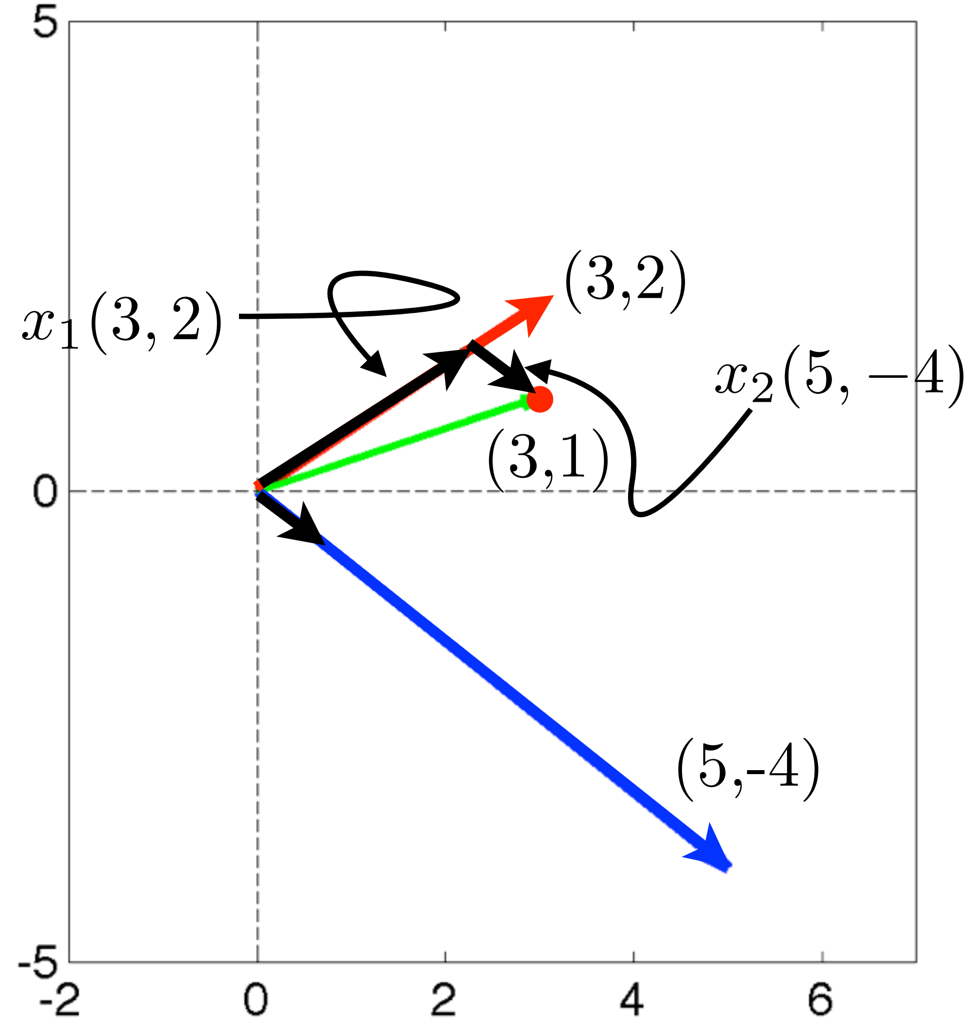
$$\begin{aligned} x_2 &= \left(\frac{-3}{22}\right)(-1) = \frac{3}{22} \\ x_1 &= \frac{1}{3}(3 - 5x_2) = \frac{1}{3}\left(3 - 5\left(\frac{-3}{22}\right)\right) = \frac{17}{22} \end{aligned}$$

Solution : $\quad x_1 = \frac{17}{22}, \qquad x_2 = \frac{3}{22}$

Linear algebra - a 2x2 system



Solution as the intersection of two lines



Solution as linear combination of vectors

Gaussian Elimination

$$\left[\begin{array}{rrr|r} 5 & -1 & 2 & 7 \\ -2 & 6 & 9 & 0 \\ -7 & 5 & -3 & 5 \end{array}\right]$$

Apply elementary row operations to the augmented matrix

$$\left[\begin{array}{rrr|r} 5 & -1 & 2 & 7 \\ 0 & \frac{28}{5} & \frac{49}{5} & \frac{14}{5} \\ 0 & \frac{18}{5} & -\frac{1}{5} & \frac{74}{5} \end{array}\right] \begin{array}{l} \\ \leftarrow (\textbf{eqn 2}) - \left(\frac{-2}{5}\right)(\textbf{eqn 1}) \\ \leftarrow (\textbf{eqn 3}) - \left(\frac{-7}{5}\right)(\textbf{eqn 1}) \end{array}$$

Pivots

Multipliers

$$\left[\begin{array}{rrr|r} 5 & -1 & 2 & 7 \\ 0 & \frac{28}{5} & \frac{49}{5} & \frac{14}{5} \\ 0 & 0 & -\frac{65}{10} & \frac{65}{5} \end{array}\right] \begin{array}{l} \\ \\ \leftarrow (\textbf{eqn 3}) - \left(\frac{9}{14}\right)(\textbf{eqn 2}) \end{array}$$

Solution : $x_1 = 3,\ x_2 = 4,\ x_3 = -2$

Gaussian Elimination

Apply elementary row operations to the augmented matrix to zero out entries below the diagonal and reduce the system to an upper triangular system.

$$\left[\begin{array}{rrr|r} 5 & -1 & 2 & 7 \\ -2 & 6 & 9 & 0 \\ -7 & 5 & -3 & 5 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 5 & -1 & 2 & 7 \\ 0 & \frac{28}{5} & \frac{49}{5} & \frac{14}{5} \\ 0 & \frac{18}{5} & -\frac{1}{5} & \frac{74}{5} \end{array}\right] \begin{array}{l} \\ \leftarrow (\textbf{eqn 2}) - \left(\frac{-2}{5}\right)(\textbf{eqn 1}) \\ \leftarrow (\textbf{eqn 3}) - \left(\frac{-7}{5}\right)(\textbf{eqn 1}) \end{array}$$

$$\left[\begin{array}{rrr|r} 5 & -1 & 2 & 7 \\ 0 & \frac{28}{5} & \frac{49}{5} & \frac{14}{5} \\ 0 & 0 & -\frac{65}{10} & \frac{65}{5} \end{array}\right] \begin{array}{l} \\ \\ \leftarrow (\textbf{eqn 3}) - \left(\frac{9}{14}\right)(\textbf{eqn 2}) \end{array}$$

Solve upper triangular system for x

$$\begin{bmatrix} 5 & -1 & 2 \\ 0 & \frac{28}{5} & \frac{49}{5} \\ 0 & 0 & -\frac{65}{10} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ \frac{14}{5} \\ \frac{65}{5} \end{bmatrix}$$

Notice that the right hand side is not the right hand side of the original system

Use back-substitution to solve for x :

$$\textbf{Step 1 :}\; x_3 = \left(\frac{-10}{65}\right)\left(\frac{65}{5}\right) = -2$$

Known from previous steps

$$\textbf{Step 2 :}\; x_2 = \left(\frac{5}{28}\right)\left(\frac{14}{5} - \frac{49}{5} x_3\right) = 4$$

$$\textbf{Step 3 :}\; x_1 = \left(\frac{1}{5}\right)(7 - (-1) x_2 - (2) x_3) = 3$$

Some notation

First pivot

$$\left[\begin{array}{ccc|c} 5 & -1 & 2 & 7 \\ -2 & 6 & 9 & 0 \\ -7 & 5 & -3 & 5 \end{array}\right]$$

Apply elementary row operations to the augmented matrix to zero out entries below the diagonal and reduce the system to an upper triangular system.

Second pivot

$$\left[\begin{array}{ccc|c} 5 & -1 & 2 & 7 \\ 0 & \frac{28}{5} & \frac{49}{5} & \frac{14}{5} \\ 0 & \frac{18}{5} & -\frac{1}{5} & \frac{74}{5} \end{array}\right] \quad \begin{array}{l} \\ \leftarrow (\textbf{eqn 2}) - \left(\frac{-2}{5}\right)(\textbf{eqn 1}) \\ \leftarrow (\textbf{eqn 3}) - \left(\frac{-7}{5}\right)(\textbf{eqn 1}) \end{array}$$

"Multipliers"

Third pivot

$$\left[\begin{array}{ccc|c} 5 & -1 & 2 & 7 \\ 0 & \frac{28}{5} & \frac{49}{5} & \frac{14}{5} \\ 0 & 0 & -\frac{65}{10} & \frac{65}{5} \end{array}\right] \quad \begin{array}{l} \\ \\ \leftarrow (\textbf{eqn 3}) - \left(\frac{9}{14}\right)(\textbf{eqn 2}) \end{array}$$

How do we implement this in a computer?

We need to introduce the use of matrices

Elimination using matrices

Write the system using matrices :

$$
\begin{aligned}
2x + 4y - 2z &= 2 \\
4x + 9y - 3z &= 8 \\
-2x - 3y + 7z &= 10
\end{aligned}
$$

$$
\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}
$$

Can we express elimination as matrix multiplication?

Elimination using matrices

Multiplication on the left by a row vector results in a row vector.

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 2 & 4 & 5 \\ 1 & 9 & -3 \\ -2 & 1 & 7 \end{bmatrix} =$$

Result is a row vector

$$a \begin{bmatrix} 2 & 4 & 5 \end{bmatrix} + b \begin{bmatrix} 1 & 9 & -3 \end{bmatrix} + c \begin{bmatrix} -2 & 1 & 7 \end{bmatrix}$$

Multiplication on the left can be thought of as a producing a *linear combination of rows.*

Elimination using matrices

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 2 & 4 & 5 \\ 1 & 9 & -3 \\ -2 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 9 & -3 \end{bmatrix}$$

$(a, b, c) = ?$ $\qquad (0, 1, 0)$

What about

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 2 & 4 & 5 \\ 1 & 9 & -3 \\ -2 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 5 & 12 \end{bmatrix}$$

$(a, b, c) = (1, 0, 1)$

We managed to eliminate the first coefficient

Elimination matrices

$$\begin{aligned} 2x + 4y - 2z &= 2 \\ 4x + 9y - 3z &= 8 \\ -2x - 3y + 7z &= 10 \end{aligned} \xrightarrow{\text{elimination}} \begin{aligned} 2x + 4y - 2z &= 2 \\ y + z &= 4 \\ 4z &= 8 \end{aligned}$$

$$\underset{A}{\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}} \quad \underset{E}{\xrightarrow[\text{multiplication?}]{\text{matrix}}} \quad \underset{U}{\begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}}$$

Find a matrix E so that $EA = U$

Elimination matrices : First step

$$\begin{aligned} 2x + 4y - 2z &= 2 \\ \boxed{\begin{matrix} 4x \\ -2x \end{matrix}}\ \begin{matrix} + 9y - 3z \\ - 3y + 7z \end{matrix} &\begin{matrix} = 8 \\ = 10 \end{matrix} \end{aligned} \xrightarrow{\text{elimination}} \begin{aligned} 2x + 4y - 2z &= 2 \\ y + z &= 4 \\ y + 5z &= 12 \end{aligned}$$

eliminate

$$\begin{bmatrix} 1 & 0 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix}$$

How do we express "(**equation 2**)$-(2)$(**equation 1**)"

How do we express "(**equation 3**)$-(-1)$(**equation 1**)"

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Elimination matrices : First step

How do we express "(**equation 2**)$-$(2)(**equation 1**)"

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix}$$

$-\ell_{21}$

How do we express "(**equation 3**)$-$(-1)(**equation 1**)"

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix}$$

$-\ell_{31}$

$E_{31}E_{21}$

"Elimination matrices"

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Elimination matrix : Second step

$$
\begin{aligned}
2x + 4y - 2z &= 2 \\
y + z &= 4 \\
\boxed{y} + 5z &= 12
\end{aligned}
\quad \xrightarrow{\text{elimination}} \quad
\begin{aligned}
2x + 4y - 2z &= 2 \\
y + z &= 4 \\
4z &= 8
\end{aligned}
$$

eliminate

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ ? & ? & ? \end{bmatrix}
\begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix}
=
\begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}
$$

How do we express "(**equation 3**)$-(1)$(**equation 2**)"

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Elimination matrix : Second step

How do we express "(**equation 3**)$-$(1)(**equation 2**)"

$$\underset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}} \underset{E_{31}E_{21}A}{\begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix}} = \underset{U}{\begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}}$$

$-\ell_{32}$

We found a sequence of matrices that will take A to U

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Elimination matrices

We have constructed *elimination matrices*

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{E_{32}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{E_{31}E_{21}} \underbrace{\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}}_{U}$$

Check

$$E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Product of elimination matrices is lower triangular

$$E_{32}E_{31}E_{21}A = U$$

U is an upper triangular matrix

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Inverses of elimination matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ -\ell_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (\textbf{row 1}) \\ (\textbf{row 2}) \\ (\textbf{row 3}) \end{bmatrix} = \begin{bmatrix} (\textbf{row 1}) \\ (\textbf{row 2}) - (\ell_{21})(\textbf{row 1}) \\ (\textbf{row 3}) \end{bmatrix}$$

E_{21}

$$\begin{bmatrix} X & X & X \\ X & X & X \\ X & X & X \end{bmatrix} \qquad \begin{bmatrix} X & X & X \\ 0 & X & X \\ X & X & X \end{bmatrix}$$

Find

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} (\textbf{row 1}) \\ (\textbf{row 2}) - (\ell_{21})(\textbf{row 1}) \\ (\textbf{row 3}) \end{bmatrix} = \begin{bmatrix} (\textbf{row 1}) \\ (\textbf{row 2}) \\ (\textbf{row 3}) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (\textbf{row 1}) \\ (\textbf{row 2}) - (\ell_{21})(\textbf{row 1}) \\ (\textbf{row 3}) \end{bmatrix} = \begin{bmatrix} (\textbf{row 1}) \\ (\textbf{row 2}) \\ (\textbf{row 3}) \end{bmatrix}$$

E_{21}^{-1}

$$(\ell_{21})(\textbf{row 1}) + (1)\left[(\textbf{row 2}) - (\ell_{21})(\textbf{row 1})\right] = (\textbf{row 2})$$

The inverse of the elimination matrix undoes the effect of elimination.

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Inverses of elimination matrices

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\ell_{31} & 0 & 1 \end{bmatrix}}_{E_{31}} \begin{bmatrix} (\textbf{row 1}) \\ (\textbf{row 2}) \\ (\textbf{row 3}) \end{bmatrix} = \begin{bmatrix} (\textbf{row 1}) \\ (\textbf{row 2}) \\ (\textbf{row 3}) - (\ell_{31})(\textbf{row 1}) \end{bmatrix}$$

$$\begin{bmatrix} X & X & X \\ 0 & X & X \\ X & X & X \end{bmatrix} \qquad \begin{bmatrix} X & X & X \\ 0 & X & X \\ 0 & X & X \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \ell_{31} & 0 & 1 \end{bmatrix}}_{E_{31}^{-1}} \begin{bmatrix} (\textbf{row 1}) \\ (\textbf{row 2}) \\ (\textbf{row 3}) - (\ell_{31})(\textbf{row 1}) \end{bmatrix} = \begin{bmatrix} (\textbf{row 1}) \\ (\textbf{row 2}) \\ (\textbf{row 3}) \end{bmatrix}$$

$$(\ell_{31})(\textbf{row 1}) + (1)\left[(\textbf{row 3}) - (\ell_{31})(\textbf{row 1})\right] = (\textbf{row 3})$$

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Inverses of elimination matrices

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\ell_{32} & 1 \end{bmatrix}}_{E_{32}} \begin{bmatrix} (\textbf{row 1}) \\ (\textbf{row 2}) \\ (\textbf{row 3}) \end{bmatrix} = \begin{bmatrix} (\textbf{row 1}) \\ (\textbf{row 2}) \\ (\textbf{row 3}) - (\ell_{32})(\textbf{row 2}) \end{bmatrix}$$

$$\boxed{\begin{bmatrix} X & X & X \\ 0 & X & X \\ 0 & \color{red}{X} & X \end{bmatrix}} \qquad \boxed{\begin{bmatrix} X & X & X \\ 0 & X & X \\ 0 & \color{red}{0} & X \end{bmatrix}}$$

upper triangular

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \ell_{32} & 1 \end{bmatrix}}_{E_{32}^{-1}} \begin{bmatrix} (\textbf{row 1}) \\ (\textbf{row 2}) \\ (\textbf{row 3}) - (\ell_{32})(\textbf{row 2}) \end{bmatrix} = \begin{bmatrix} (\textbf{row 1}) \\ (\textbf{row 2}) \\ (\textbf{row 3}) \end{bmatrix}$$

$$(\ell_{32})(\textbf{row 2}) + (1)\left[(\textbf{row 3}) - (\ell_{32})(\textbf{row 2})\right] = (\textbf{row 3})$$

Inverse of elimination matrices

What is the product

$$(E_{32}E_{31}E_{21})^{-1} = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} \quad ?$$

$$E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \ell_{31} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \ell_{32} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}$$

The multipliers used in elimination appear in L.

We can write A as the product of a lower triangular matrix and an upper triangular matrix.

$$E_{32}E_{31}E_{21}A = U$$

or

$$A = (E_{21}^{-1}E_{31}^{-1}E_{32}^{-1})U = LU$$

This is the LU decomposition of A.

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Try it!

Row reduce the matrix A to get an upper triangular matrix U. Along the way, *record* the multipliers ℓ_{ij} you use in a lower triangular matrix L. Check that you get $LU = A$.

$$A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix}$$

Numbers are all integers!

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \ell_{21} & 1 & 0 & 0 \\ \ell_{31} & \ell_{32} & 1 & 0 \\ \ell_{41} & \ell_{42} & \ell_{43} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} X & X & X & X \\ 0 & X & X & X \\ 0 & 0 & X & X \\ 0 & 0 & 0 & X \end{bmatrix}$$

Check that you get $LU = A$.

The end!

Cost of Gaussian elimination

The cost of eliminating entries below the 1^{st} pivot :

Pivot

$$\begin{bmatrix} X & X & X & X & X \\ X & X & X & X & X \\ X & X & X & X & X \\ X & X & X & X & X \\ X & X & X & X & X \end{bmatrix}$$

Each row operation costs n multiplications (including the cost of computing the multiplier) and $n-1$ subtractions.

$$\text{Total}: \ (n-1)(2n-1) = 2n^2 - 3n + 1 \approx 2n^2$$

Cost of elimination

The cost of eliminating entries below the 2^{nd} pivot :

$$\begin{bmatrix} X & X & X & X & X \\ 0 & X & X & X & X \\ 0 & \color{red}{X} & X & X & X \\ 0 & \color{red}{X} & X & X & X \\ 0 & \color{red}{X} & X & X & X \end{bmatrix}$$

To "zero out" the column below the second pivot, we must do approximately $2(n-1)^2 = 2(4)^2$ multiplications and subtractions.

Cost of elimination

The cost of eliminating entries below the 3^{rd} pivot :

$$\begin{bmatrix} X & X & X & X & X \\ 0 & X & X & X & X \\ 0 & 0 & X & X & X \\ 0 & 0 & \color{red}{X} & \boxed{X} & \boxed{X} \\ 0 & 0 & \color{red}{X} & \boxed{X} & \boxed{X} \end{bmatrix}$$

To "zero out" the column below the second pivot, we must do approximately $2(n-2)^2 = 2(3)^2$ multiplications and subtractions.

Cost of elimination

The cost of eliminating entries below the 3^{rd} pivot :

$$\begin{bmatrix} X & X & X & X & X \\ 0 & X & X & X & X \\ 0 & 0 & X & X & X \\ 0 & 0 & 0 & X & X \\ 0 & 0 & 0 & {\color{red}X} & \boxed{X} \end{bmatrix}$$

To "zero out" the column below the second pivot, we must do approximately $2(n-3)^2 = 2(2)^2$ multiplications and subtractions.

Cost of elimination

The total number of multiplications is then :

$$2\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \approx \frac{2}{3}n^3$$

we ignore lower order powers of n

We say that "elimination is an n^3 process".

This is consider "expensive" for a linear solve.

What about the back solve?

$$\begin{bmatrix} a_{11} & a_{12} & a_{a13} & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

One 'op' is a multiplication or divide; Ignore subtractions for now; we will add them in momentarily.

$$x_5 = b_5/a_{55} \quad \rightarrow \quad 1 \text{ op}$$

$$x_4 = (b_4 - a_{45}x_5)/a_{44} \quad \rightarrow \quad 2 \text{ ops}$$

$$x_3 = (b_3 - a_{34}x_x - a_{35}x_5)/a_{33} \quad \rightarrow \quad 3 \text{ ops}$$

$$x_2 = (b_2 - a_{23}x_3 - a_{24}x_x - a_{25}x_5)/a_{22} \quad \rightarrow \quad 4 \text{ ops}$$

$$x_1 = (b_1 - a_{12}x_2 - a_{13}x_3 - a_{14}x_x - a_{15}x_5)/a_{11} \quad \rightarrow \quad 5 \text{ ops}$$

What about the cost of the back solve?

Step k in the back solve requires k multiplications and k additions. So the total work for a back solve is :

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \approx \frac{1}{2}n^2$$

Ignore lower order powers of n, but pay attention to the coefficient of the highest order power of n.

or $2\left(\frac{1}{2}n^2\right) = n^2$ if we include subtractions, as well as divisions and multiplications.

We say that a back solve is an "order n^2" operation, which is considerably cheaper than the original elimination.

The LU decomposition

If we have more than one right hand side (as is often the case)

$$A\mathbf{x} = \mathbf{b}_i, \quad i = 1, 2, \ldots, M$$

We can use the LU decomposition to record the work involved carrying out the expensive $\mathcal{O}(n^3)$ elimination step and then use that factorization to solve for multiple right hand sides. This second solve step will only require $\mathcal{O}(n^2)$ work.

LU Decomposition

$$U = \begin{bmatrix} 5 & -1 & 2 \\ 0 & \frac{28}{5} & \frac{49}{5} \\ 0 & 0 & -\frac{65}{10} \end{bmatrix}$$

Store the multipliers in a lower triangular matrix :

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{5} & 1 & 0 \\ -\frac{7}{5} & \frac{9}{14} & 1 \end{bmatrix}$$

LU Decomposition

The product LU is equal to A :

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{5} & 1 & 0 \\ -\frac{7}{5} & \frac{9}{14} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 5 & -1 & 2 \\ 0 & \frac{28}{5} & \frac{49}{5} \\ 0 & 0 & -\frac{65}{10} \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} 5 & -1 & 2 \\ -2 & 6 & 9 \\ -7 & 5 & -3 \end{bmatrix}}_{A}$$

It does not cost us anything to store the multipliers. But by doing so, we can now solve many systems involving the matrix A.

Solution procedure given LU=A

How can we solve a system using the LU factorization?

$$A\mathbf{x} = \mathbf{b}$$

$$\begin{aligned}
&\textbf{Step 0:} \quad \text{Factor } A \text{ into } LU && \leftarrow \text{Row-reduction} \\
&\textbf{Step 1:} \quad \text{Solve } L\mathbf{y} = \mathbf{b} && \leftarrow \text{Forward substitution} \\
&\textbf{Step 2:} \quad \text{Solve } U\mathbf{x} = \mathbf{y} && \leftarrow \text{Back substitution}
\end{aligned}$$

For each right hand side, we only need to do n^2 operations. The expensive part is forming the original LU decomposition.

Cost of a matrix inverse

To solve using the matrix inverse A^{-1} to get $\mathbf{x} = A^{-1}\mathbf{b}$.
To get a column c_j of the matrix A^{-1}, we solve

$$Ac_j = e_j$$

for each column e_j of an identity matrix.

$$\text{total cost} \approx \frac{2}{3}n^3 + 2n^3 = \frac{8}{3}n^3 \quad \text{operations}$$

It costs about 4 times as much to multiply by the inverse as it does to solve the linear system using Gaussian elimination.
The cost of the matrix vector multiply $A^{-1}\mathbf{b}$ is n^2.

Row exchanges

What if we start with a system that looks like :

$$A = \begin{bmatrix} 0 & 0 & -\frac{65}{10} \\ 0 & \frac{28}{5} & \frac{49}{5} \\ 5 & -1 & 2 \end{bmatrix}$$

All we need to do is exchange the rows of A, and do the decomposition on

$$LU = PA$$

where P is a *permutation* matrix, i.e.

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Partial pivoting

We can also do row exchanges not just to avoid a zero pivot, but also to make the pivot as large as possible. This is called “partial pivoting”.

Find the largest pivot in the entire column, and do a row exchange.

$$\begin{bmatrix} X & X & X & X & X \\ X & X & X & X & X \\ X & X & X & X & X \\ X & X & X & X & X \\ X & X & X & X & X \end{bmatrix}$$

One can also do “full pivoting” by looking for the largest pivot in the entire matrix. But this is rarely done.

Top 10 algorithms

The matrix decompositions are listed as one of the top 10 algorithms of the 20th century

from *SIAM News*, Volume 33, Number 4

The Best of the 20th Century: Editors Name Top 10 Algorithms

1951: Alston Householder of Oak Ridge National Laboratory formalizes the **decompositional approach to matrix computations**.

The ability to factor matrices into triangular, diagonal, orthogonal, and other special forms has turnec out to be extremely useful. The decompositional approach has enabled software developers to produce flexible and efficient matrix packages. It also facilitates the analysis of rounding errors, one of the big bugbears of numerical linear algebra. (In 1961, James Wilkinson of the National Physical Laboratory in London published a seminal paper in the *Journal of the ACM*, titled "Error Analysis of Direct Methods of Matrix Inversion," based on the LU decomposition of a matrix as a product of lower and upper triangular factors.)