Elliptic Equations (Chapter 3)

General constant coefficient elliptic partial differential equation (PDE) in two-dimensions

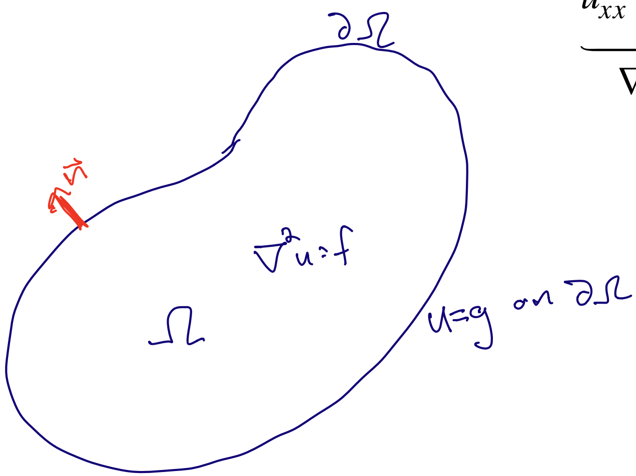
$$a_1 u_{xx} + a_2 u_{xy} + a_3 u_{yy} + a_4 u_x + a_5 u_y + a_6 u = f$$

where $a_2^2 - 4a_1 a_3 < 0$

Most common example is the Poisson equation:

$$\underbrace{u_{xx} + u_{yy}}_{\nabla^2 u} = f \quad , \text{ in } \Omega$$

(3-D $u_{xx} + u_{yy} + u_{zz} = f$)



(1) Boundary condition: $u(x,y) = g(x,y)$ on $\underbrace{\partial\Omega}_{\text{boundary}}$
(Dirichlet B.C)

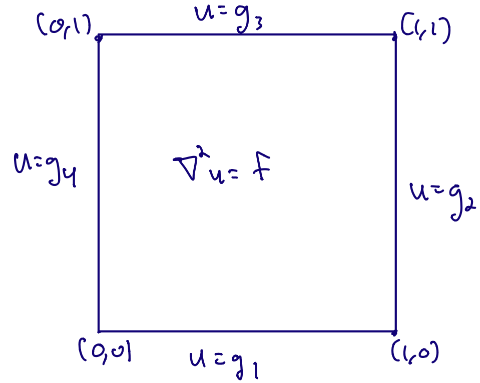
Other Boundary Conditions (B.C)

(2) $\vec{n} \cdot \nabla u = g(x,y)$ (Flux)
Neumann B.C.

(3) $\alpha_1 \vec{n} \cdot \nabla u + \alpha_2 u = g(x,y)$
Robin B.C.

Model:
(a) Steady-state temperature in Ω
(b) Deflection of a membrane or plate
(c) Electrical potential in Ω.

Model problem we focus on:



$$\nabla^2 u(x,y) = f(x,y),\ \Omega = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1\}$$

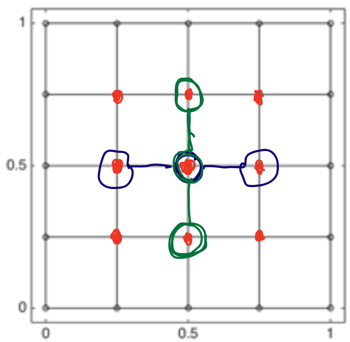
Boundary conditions

$$u(x,0) = g_1(x),\ u(1,y) = g_2(y),$$
$$u(x,1) = g_3(x),\ u(0,y) = g_4(y).$$

Grid: $(x_j, y_k) = (jh, kh),\ h = 1/(m+1),\ j,k = 0,1\ldots,m+1$

$$u_{jk} = u(x_j, y_k)\ f_{jk} = f(x_j, y_k)$$

Example with m=3



Approximate: $u_{xx} + u_{yy}$

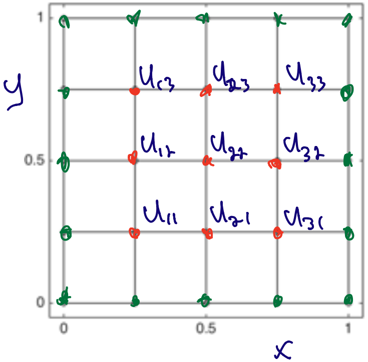
$$u_{xx}\Big|_{\substack{x=x_j\\y=y_k}} = \frac{u_{j-1,k} - 2u_{j,k} + u_{j+1,k}}{h^2} + \mathcal{O}(h^2)$$

$$u_{yy}\Big|_{\substack{x=x_j\\y=y_k}} = \frac{u_{j,k-1} - 2u_{j,k} + u_{j,k+1}}{h^2} + \mathcal{O}(h^2)$$

$$(u_{xx} + u_{yy})\Big|_{\substack{x=x_j\\y=y_k}} = \frac{u_{j-1,k} + u_{j,k-1} + u_{j+1,k} + u_{j,k+1} - 4u_{j,k}}{h^2} + \mathcal{O}(h^2)$$

(5-point stencil for Laplacian)

Example with $m=3$



Arrange the unknowns and knowns in a matrix:

$$U^h = \begin{bmatrix} u_{11} & u_{21} & u_{31} \\ u_{12} & u_{22} & u_{32} \\ u_{13} & u_{23} & u_{33} \end{bmatrix} \qquad F^h = \begin{bmatrix} f_{11} & f_{21} & f_{31} \\ f_{12} & f_{22} & f_{32} \\ f_{13} & f_{23} & f_{33} \end{bmatrix}$$

In general:

$$D^h_{2,x} = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & 0 \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & \ddots & \ddots & 1 \\ 0 & & & 1 & -2 \end{bmatrix}$$

We can then write the discretized equations as

$$\underbrace{\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}}_{h^2 D^h_{2,y}} \underbrace{\begin{bmatrix} u_{11} & u_{21} & u_{31} \\ u_{12} & u_{22} & u_{32} \\ u_{13} & u_{23} & u_{33} \end{bmatrix}}_{U^h} + \underbrace{\begin{bmatrix} u_{11} & u_{21} & u_{31} \\ u_{12} & u_{22} & u_{32} \\ u_{13} & u_{23} & u_{33} \end{bmatrix}}_{U^h} \underbrace{\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}}_{h^2 D^h_{2,x}} =$$

$$h^2 \underbrace{\begin{bmatrix} f_{11} & f_{21} & f_{31} \\ f_{12} & f_{22} & f_{32} \\ f_{13} & f_{23} & f_{33} \end{bmatrix}}_{F^h} - \underbrace{\begin{bmatrix} u_{10}+u_{01} & u_{20} & u_{30}+u_{41} \\ u_{02} & 0 & u_{42} \\ u_{03}+u_{14} & u_{24} & u_{34}+u_{43} \end{bmatrix}}_{U^h_{\text{bc}}}$$

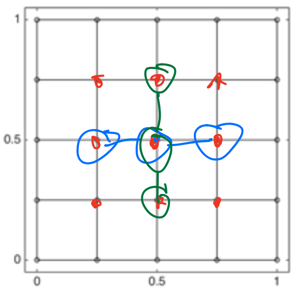
This gives the *matrix equation*:

$$D^h_{2,y} U^h + U^h D^h_{2,x} = F^h - \frac{1}{h^2} U^h_{\text{bc}}$$

(Sylvester equation)

More on this later

Example with m=3



$$U^h = \begin{array}{ccc} (1) & (2) & (3) \\ \end{array}$$

$$U^h = \begin{bmatrix} u_{11} & u_{21} & u_{31} \\ u_{12} & u_{22} & u_{32} \\ u_{13} & u_{23} & u_{33} \end{bmatrix} \qquad F^h = \begin{bmatrix} f_{11} & f_{21} & f_{31} \\ f_{12} & f_{22} & f_{32} \\ f_{13} & f_{23} & f_{33} \end{bmatrix}$$

Same thing $\underline{f}^h$

$\underline{u}^h = [u_{11}\ u_{12}\ u_{13}\ u_{21}\ u_{22}\ u_{23}\ u_{31}\ u_{32}\ u_{33}]^T$ (Column re-ordering)

(Matlab: If U is a matrix then U(:) is column reordering)

Alternatively, the unknowns can be re-ordered into a column vector to get a more standard linear system:

$$\underbrace{\begin{bmatrix} -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{bmatrix}}_{h^2 D_{xx}^h} \underbrace{\begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{21} \\ u_{22} \\ u_{23} \\ u_{31} \\ u_{32} \\ u_{33} \end{bmatrix}}_{\underline{u}^h} + \underbrace{\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}}_{h^2 D_{yy}^h} \underbrace{\begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{21} \\ u_{22} \\ u_{23} \\ u_{31} \\ u_{32} \\ u_{33} \end{bmatrix}}_{\underline{u}^h} = \underbrace{\begin{bmatrix} h^2 f_{11} - u_{01} - u_{10} \\ h^2 f_{12} - u_{02} \\ h^2 f_{13} - u_{03} - u_{14} \\ h^2 f_{21} - u_{20} \\ h^2 f_{22} \\ h^2 f_{23} - u_{24} \\ h^2 f_{31} - u_{41} - u_{30} \\ h^2 f_{32} - u_{42} \\ h^2 f_{33} - u_{43} - u_{34} \end{bmatrix}}_{\underline{f}^h}$$

This gives the linear system:

$$\underbrace{\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{bmatrix}}_{h^2(D_{xx}^h + D_{yy}^h)} \underbrace{\begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{21} \\ u_{22} \\ u_{23} \\ u_{31} \\ u_{32} \\ u_{33} \end{bmatrix}}_{\underline{u}^h} = \underbrace{\begin{bmatrix} h^2 f_{11} - u_{01} - u_{10} \\ h^2 f_{12} - u_{02} \\ h^2 f_{13} - u_{03} - u_{14} \\ h^2 f_{21} - u_{20} \\ h^2 f_{22} \\ h^2 f_{23} - u_{24} \\ h^2 f_{31} - u_{41} - u_{30} \\ h^2 f_{32} - u_{42} \\ h^2 f_{33} - u_{43} - u_{34} \end{bmatrix}}_{\underline{f}^h}$$

$$\boxed{A^h \underline{u}^h = \underline{f}^h}$$

The matrices can be constructed very conveniently using the Kronecker product:

$$D_{yy}^h = \underbrace{I_m}_{m\times m \text{ identity matrix}} \otimes D_{2,y}^h \qquad\qquad D_{xx}^h = (D_{2,x}^h)^T \otimes I_m$$

Properties of the Kronecker product and vec operators

Let $B \in \mathbb{R}^{m\times n}$ & $C \in \mathbb{R}^{p\times q}$ matrix then

$$B \otimes C = \begin{bmatrix} b_{11}C & b_{12}C & \cdots & b_{1n}C \\ b_{21}C & b_{22}C & & b_{2n}C \\ \vdots & \vdots & & \vdots \\ b_{m1}C & b_{m2}C & & b_{mn}C \end{bmatrix} \in \mathbb{R}^{mp\times nq}$$

vec operator:

$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ then $\text{vec}(A) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

$\underset{\sim}{a}_j \in \mathbb{R}^m$

General $A = \begin{bmatrix} \underset{\sim}{a}_1 \,|\, \underset{\sim}{a}_2 \,|\, \cdots \,|\, \underset{\sim}{a}_n \end{bmatrix}$ $\quad \text{vec}(A) = \begin{bmatrix} \underset{\sim}{a}_1 \\ \hline \underset{\sim}{a}_2 \\ \hline \vdots \\ \hline \underset{\sim}{a}_n \end{bmatrix} \in \mathbb{R}^{mn}$

$A \in \mathbb{R}^{m\times n}$

Property of Kronecker & vec operators:

$$\operatorname{vec}(ACB^T) = (B \otimes A)\operatorname{vec}(C)$$

(a) If $B = I$ then $\operatorname{vec}(AC) = (I \otimes A)\operatorname{vec}(C)$

(b) If $A = I$ then $\operatorname{vec}(CB^T) = (B \otimes I)\operatorname{vec}(C)$

Matrix equation:

$$D^h_{2,y} U^h + U^h D^h_{2,x} = F^h + {}^{-}\,{}^{h}_{bc}$$

Linear system:

$$\operatorname{vec}\left(D^h_{2,y} U^h + U^h D^h_{2,x}\right) = \operatorname{vec}\left(F^h + U^h_{bc}\right)$$

Solving the second-order finite difference equations for the Poisson equation using Gaussian elimination. See handout with matlab code and example.