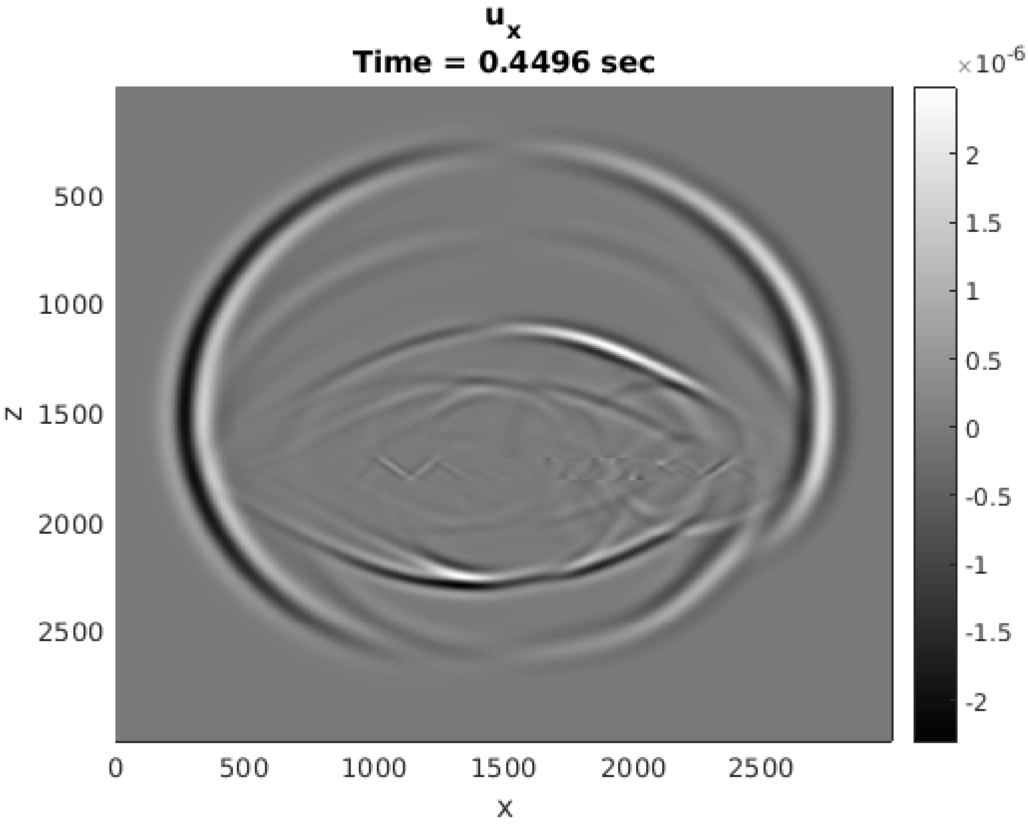
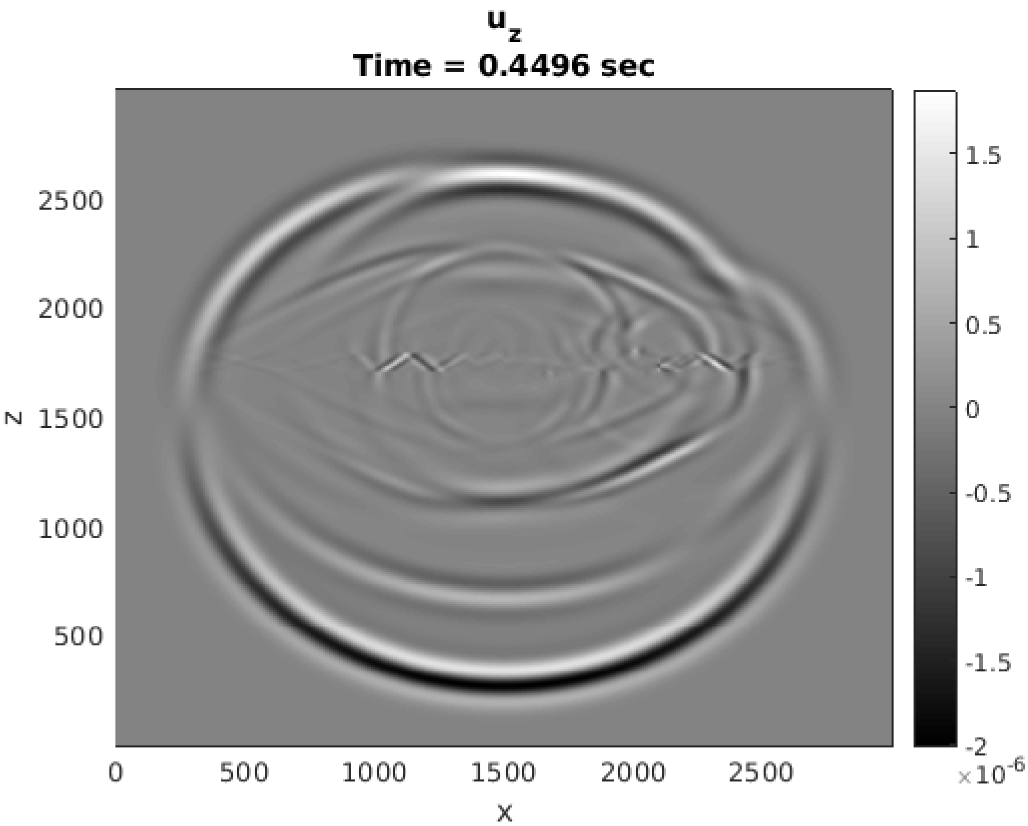
```
% This program uses 6th order approximation of spartial derivatives. It
% numerically solves the elastic wave equation.
% variables
% ---------
% h and k : grid spacing in space and time.
% tau : stress
% u_x and u_z : Displacement
% v_x and v_z : velocities associated with the directions
% rho : density
% mu and lambda : Lame parameters
% cp and cs : pressure wave and shear wave velocities

%Note: We used the Ricker wavelet to implement the source time function
%      Parameters related to explossion were set.
%      cp and cs were also set

%Comments: The results are cloer to what is expected

 clear all
 close all

m = 1000; N = 1000;  %grid cells

%density over the whole domain
rho = 2200; %kg/m^3

%space
a = 0; b = 3000;
h = (b-a)/m;

x = []; xn = [];
z = []; zn = [];
for i = 1:m
    x1 = (i-1)*h;   x = [x,x1];
    xn1 =  (i-0.5)*h; xn  = [xn,xn1];
    z1 = (i-1)*h;   z = [z,z1];
    zn1 =  (i-0.5)*h; zn  = [zn,zn1];
end

%meshes
[X,Z] = meshgrid(x,z);
[Xn,Zn] = meshgrid(xn,zn);
[Xnx,Znx] = meshgrid(x,zn);
[Xnz,Znz] = meshgrid(xn,z);

mu1  = zeros(m); mu2 = zeros(m);
lamb = zeros(m);
coef = zeros(m);

for i = 1:m
    lamb(i,:) = lambda(Xn(i,:),Zn(i,:));
    mu1(i,:) = mu(Xn(i,:),Zn(i,:));
    mu2(i,:) = mu(X(i,:),Z(i,:));
    coef(i,:) = lamb(i,:) + 2*mu1(i,:);
end

%Dx and Dz
s = [-5/2 -3/2 -1/2 1/2 3/2 5/2]; %stencil
w = weights(0,s,1); w1 = w(2,:);%weights

Dz = (1/h)*circulant([w1(4:end),zeros(1,m-6),w1(1:3)],1); %derivative operator 6th order
Dz = sparse(Dz); Dx = -Dz';
```

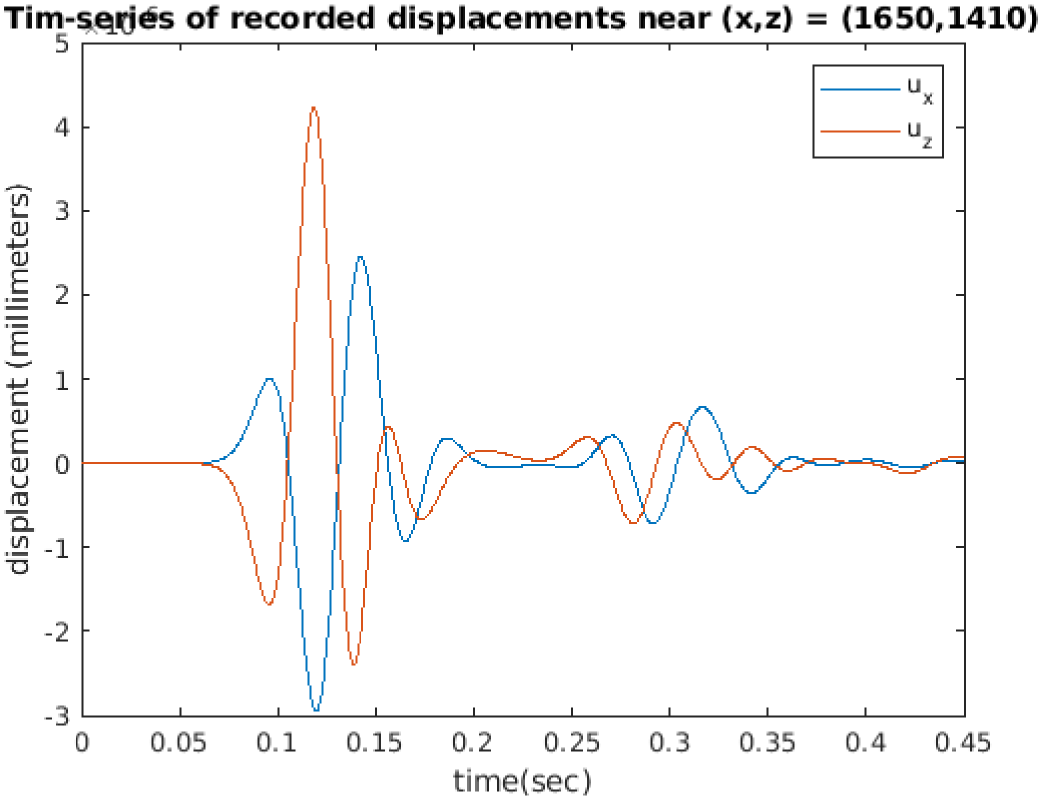
```
% Displacement initial values
v_x0 = zeros(m);
v_z0 = zeros(m);

% Stress intial Values
tau_xx0 = zeros(m);
tau_zz0 = zeros(m);
tau_xz0 = zeros(m);

%time
at =0; bt = 0.45;
k = (bt-at)/N; %timestep
t = at:k:bt;

tn = (0:N)*k;

% at (x,z) = (1650m, 1410m)
for j = 1:m
    if x(j) == 1650
        x_xp = j;
    end
    if z(j) == 1410
        z_xp = j;
    end
end

x0 = 1500; %m
z0 = 1500; %m

S1 = zeros(m,m); S2 = S1;
for i = 1:m
    S1(i,:) = Deltah(X(i,:)-x0,h);
    S2(i,:) = Deltah(Z(i,:)-z0,h);
end
explosion_site = h^2*(S1.*S2);

u_x = zeros(m,m); u_xp = zeros(N,1);
u_z = zeros(m,m); u_zp= zeros(N,1);

% computation of v_x, and v_z to the half time-level
v_x0 = v_x0 + 0.5*(k/rho)*(tau_xx0*Dx + Dz'*tau_xz0);
v_z0 = v_z0 + 0.5*(k/rho)*(tau_xz0*Dz + Dx'*tau_zz0);

%transpose of the differentiation matrix
D_z = Dz'; D_x = Dx';

%lambda and mu coefficients
%lc = k*lamb;

for n  = 2:N+1

    source = (Sourcetime(t(n)) - Sourcetime(t(n-1)))*explosion_site;

    %computation of tau_xx, tau_zz, tau_xz
    tau_xx0 = tau_xx0 + k*(coef.*(v_x0*Dz) + lamb.*D_z*v_z0) + source;
    tau_zz0 = tau_zz0 + k*(lamb.*(v_x0*Dz) + coef.*D_z*v_z0) + source;

    tau_xz0 = tau_xz0 + k*mu2.*(v_z0*(Dx) + D_x*v_x0);

    %computation of v_x, and v_z
    v_x0 = v_x0 + (k/rho)*(tau_xx0*Dx + D_z*tau_xz0);

    v_z0 = v_z0 + (k/rho)*(tau_xz0*Dz + D_x*tau_zz0);

    %computation of u_x and u_z
```

```
    u_x = u_x + k*v_x0; u_xp(n) = u_x(x_xp,z_xp);
    u_z = u_z + k*v_z0; u_zp(n) = u_z(x_xp,z_xp);

    if mod(n,10) == 0
        figure(1)
        p = pcolor(Xnx,Znx,u_x); set(p, 'EdgeColor', 'none');
        xlabel('x'); ylabel('z');
        colormap(gray(100)); colorbar;
        title({'u_x';['Time = ',sprintf('%.4f',t(n)),' sec']});
        set(gca,'YDir','Reverse')
        drawnow;
    end
    %snapnow
    %disp(['u_x']);

     if mod(n,10) == 0
        figure(2)
        p = pcolor(Xnx,Znx,u_z); set(p, 'EdgeColor', 'none');
        xlabel('x'); ylabel('z');
        colormap(gray(100)); colorbar;
        title({'u_z';['Time = ',sprintf('%.4f',t(n)),' sec']});
        drawnow;
     end
        %disp(['u_x']);


end

%time-series plot of ux and uz
figure(3)
plot(t, u_xp)
xlabel('time(sec)'); ylabel('displacement (millimeters)');
title('Tim-series of recorded displacements near (x,z) = (1650,1410)');
hold on
plot(t, u_zp)
legend('u_x','u_z');


%lambda
function la = lambda(x,z)

rho = 2200;
n = size(x,2);

for i = 1:n
    Cp(i) = CP(x(i),z(i));
    Cs(i) = CS(x(i),z(i));
end

la = rho*Cp.^2 - 2*rho*Cs.^2;
end

% mu
function muu = mu(x,z)

rho = 2200;
n = size(x,2);

for i = 1:n
    Cs(i) = CS(x(i),z(i));
end

muu = rho*Cs.^2;
end
```

```
% Discrete delta function
function delta = Deltah(ep,h)

n = size(ep,2);

for i = 1:n

    if abs(ep(i)) <= 2*h
        delta(i) = (1/(4*h))*(1+cos((ep(i)*pi)/(2*h)));
    elseif abs(ep(i)) > 2*h
        delta(i) = 0;
    end

end
end

% derivative of source-time function
function S = Sourcetime(t)
t0 = 0.07; %sec
fM = 16;   %Hz
gamma = 5*10^6; %Pa
S = gamma*(1 - 2*pi^2*fM^2*(t-t0)^2)*exp(-pi^2*fM^2*(t-t0)^2);
end


%pressure-wave velocity
function cp = CP(x,z)

if ((1500 <= x) && (x <= 2100)) && ((1700 <= z) && (z <= 1800))
    cp = 1450; %m/s
else
    cp = 3200; %m/s
end

end

%shear-wave velocity
function cs = CS(x,z)

if ((1500 <= x) && (x <= 2100)) && ((1700 <= z) && (z <= 1800))
    cs = 0;   %m/s
else
    cs = 1847.5; %m/s
end

end
```


$u_x$
Time = 0.4496 sec
z
x
$\times 10^{-6}$


$u_z$
Time = 0.4496 sec
z
x
$\times 10^{-6}$

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Title: Elastic Wave Equation

Background:

The propagation of seismic waves under the earth's surface is studied using 2D elastic wave equations. These equations are described by displacement $U=(U_x,U_z)$ and the stress $\tau = \begin{bmatrix} \tau_{xx} & \tau_{xz} \\ \tau_{xz} & \tau_{zz} \end{bmatrix}$ as follows.

$$\rho \frac{\partial^2 U_x}{\partial t^2} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z}$$

$$\rho \frac{\partial^2 U_z}{\partial t^2} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z}$$

$$\tau_{xx} = (\lambda + 2\mu)\frac{\partial U_x}{\partial x} + \frac{\partial \tau_{zz}}{\partial z},$$

$$\tau_{zz} = \lambda \frac{\partial U_x}{\partial x} + (\lambda+2\mu)\frac{\partial U_z}{\partial z},$$

$$\tau_{xz} = \mu\left(\frac{\partial U_z}{\partial x} + \frac{\partial U_x}{\partial z}\right).$$

where $\lambda$ and $\mu$ are lamé parameters.

$\mu$: shear modulus

$\rho$: density.

let $V_x = \frac{\partial U_x}{\partial t}$ and $V_z = \frac{\partial U_z}{\partial t}$

where $V_x$ and $V_z$ are the velocities in $x$ and $z$ directions respectively, The equations become

$$\rho \frac{\partial V_x}{\partial t} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} \quad \text{——— (2)}$$

$$\rho \frac{\partial V_z}{\partial t} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} \quad \text{——— (3)}$$

$$\frac{\partial \tau_{xx}}{\partial t} = (\lambda + 2\mu)\frac{\partial v_x}{\partial x} + \lambda \frac{\partial v_z}{\partial z} \quad \text{—— (4)}$$

$$\frac{\partial \tau_{zz}}{\partial t} = \lambda \frac{\partial v_x}{\partial x} + (\lambda + 2\mu)\frac{\partial v_z}{\partial z} \quad \text{—— (5)}$$

$$\frac{\partial \tau_{xz}}{\partial t} = \mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right) \quad \text{—— (6)}$$

P-wave velocity $C_P = \sqrt{\frac{\lambda + 2\mu}{\rho}}$ ~~and~~

S-wave velocity $C_S = \sqrt{\mu/\rho}$

Let the source time function: $S(t)$, be defined by the Ricker wavelet as

$$S(t) = (1 - 2\pi^2 f_M^2 t^2)\, e^{-\pi^2 f_M^2 t^2}$$

where
$f_M$ : Peak frequency at time $t$

The ~~book~~ underground explosion at a location $(x_0, z_0)$ is given by adding the source term to the normal stress components $\tau_{xx}$ and $\tau_{zz}$.

$$\tau_{xx} = (\lambda + 2\mu)\frac{\partial u_x}{\partial x} + \lambda \frac{\partial u_z}{\partial z} + \rho S(t - t_0)\int_0^L\int_0^L \delta(x - x_0)\,\delta(z - z_0)\,dx\,dz$$

$$\tau_{zz} = \lambda \frac{\partial u_x}{\partial x} + (\lambda + 2\mu)\frac{\partial u_z}{\partial z} + \rho S(t - t_0)\int_0^L\int_0^L \delta(x - x_0)\,\delta(z - z_0)\,dx\,dz$$

Differentiating the two equations with respect to time we obtain.

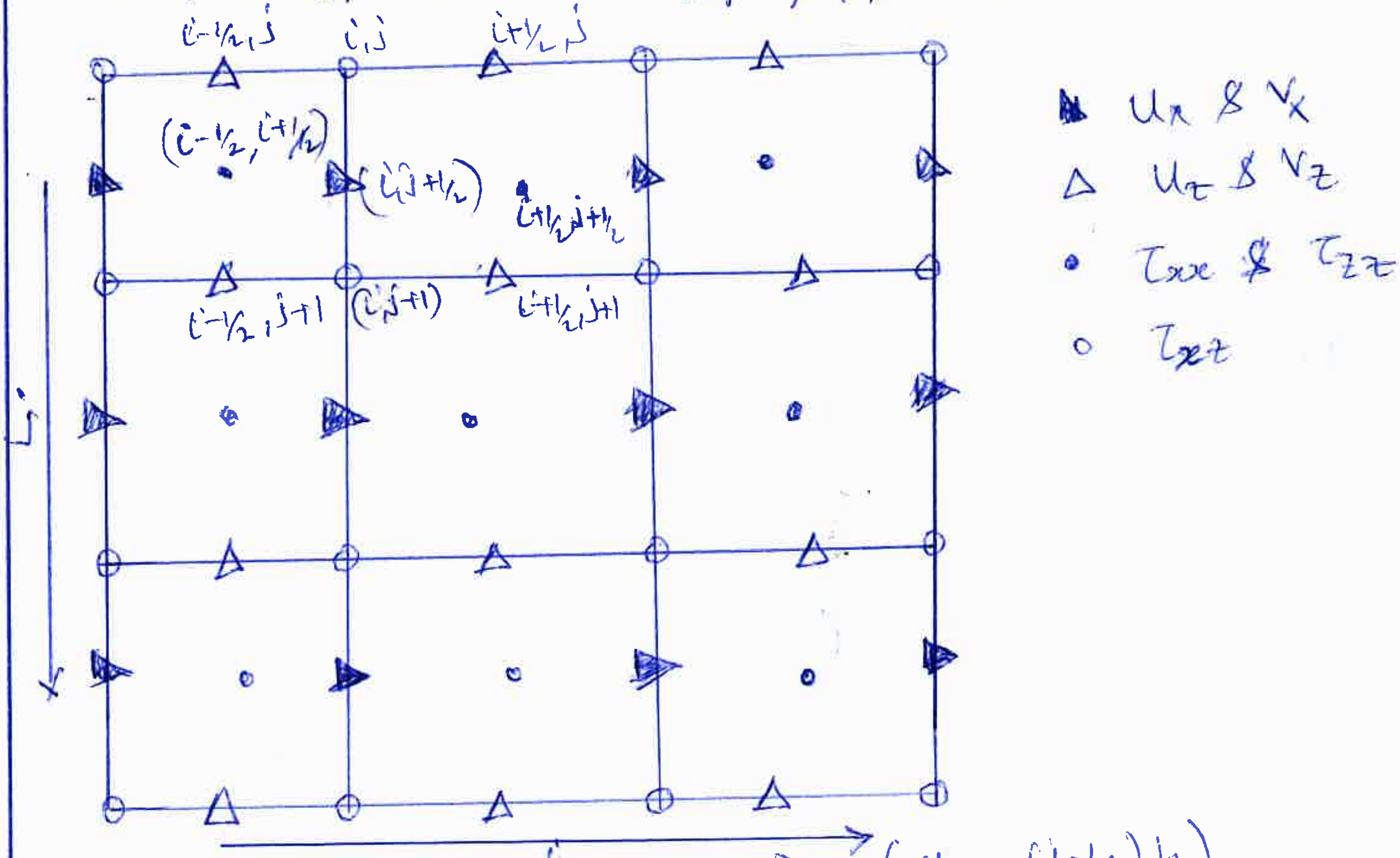
$$\frac{\partial \tau_{xx}}{\partial t} = (\lambda + 2\mu)\frac{\partial v_x}{\partial x} + \lambda \frac{\partial v_x}{\partial z} + \rho S'(t - t_0)\int_0^L\int_0^L \delta(x - x_0)\,\delta(z - z_0)\,dx\,dz \quad \text{—— (7)}$$

$$\frac{\partial \tau_{zz}}{\partial t} = \lambda \frac{\partial v_x}{\partial x} + (\lambda + 2\mu)\frac{\partial v_z}{\partial z} + \rho S'(t - t_0)\int_0^L\int_0^L \delta(x - x_0)\,\delta(z - z_0)\,dx\,dz \quad \text{—— (8)}$$

Discretization:

Consider a staggered grid in space for unknown variables. also take a grid point

$$(x_i, z_{j+1/2}) = (ih, (j+1/2)h), \; i,j = 0, \dots, m-1$$




Consider $v_x$ at $(x_i, z_{j+1/2}) = (ih, (j+1/2)h)$

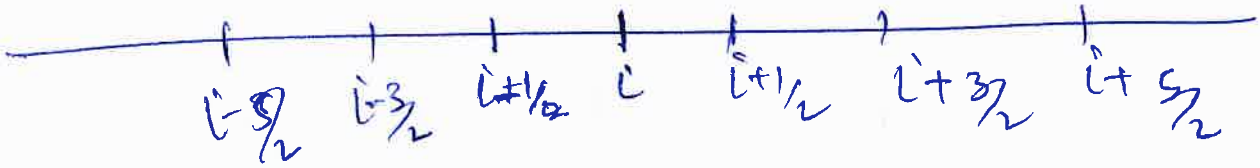
Equation (2) becomes

$$\rho_{i,j+1/2} \frac{d}{dt}(v_x)_{i,j+1/2} = \frac{1}{h}\left((\tau_{xx})_{i+1/2,j+1/2} - (\tau_{xx})_{i-1/2,j+1/2}\right) + \frac{1}{h}\left((\tau_{xz})_{i,j+1} - (\tau_{xz})_{i,j}\right) \quad \text{--- (12)}$$

Combining staggering in space and in time, we can approximate the elastic wave equations using the leap-frog scheme.

$$(v_x)^{n+1/2}_{i,j+1/2} = (v_x)^{n-1/2}_{i,j+1/2} + \frac{k}{h\rho_{i,j+1/2}}\left((\tau_{xx})^n_{i+1/2,j+1/2} - (\tau_{xx})^n_{i-1/2,j+1/2} + (\tau_{xz})^n_{i,j+1} - (\tau_{xz})^n_{i,j}\right) \quad \text{--- (13)}$$

Extending this to higher dimensions along x we Obtain a stencil




which gives us

$$\frac{1}{h}\left[-\tfrac{5}{2} \quad -\tfrac{3}{2} \quad -\tfrac{1}{2} \quad \tfrac{1}{2} \quad \tfrac{3}{2} \quad \tfrac{5}{2}\right]$$

as our stencil which we use in the weight function to compute the weights.

and along z we obtain.

$$\frac{1}{h}\left[-2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3\right]$$

The process is repeated for other elastic equations 3 – 6, and the corresponding discretizations are obtained.

~~Challenge~~