Numerical Integration

$$I(f) = \int_a^b f(x)\,dx$$

$$\approx \sum_{i=0}^{N-1} f(\bar{x}_i)\,h$$

Numerical Quadrature

Problem: Approximate

Definite Integral

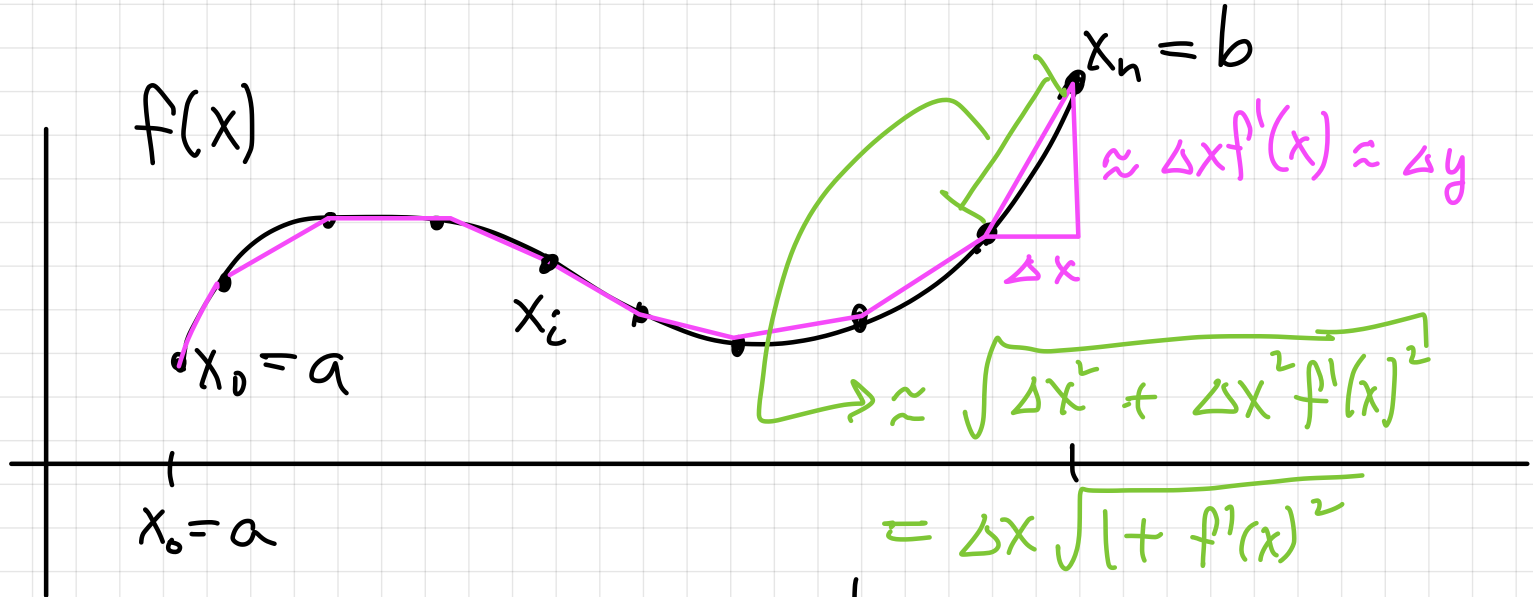
$$\int_a^b f(x)\,dx \approx I(f)$$

a number!

numerical approximation.

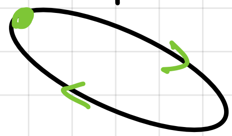
① Some integrals have no closed form:

Arc-length calculations, for example:

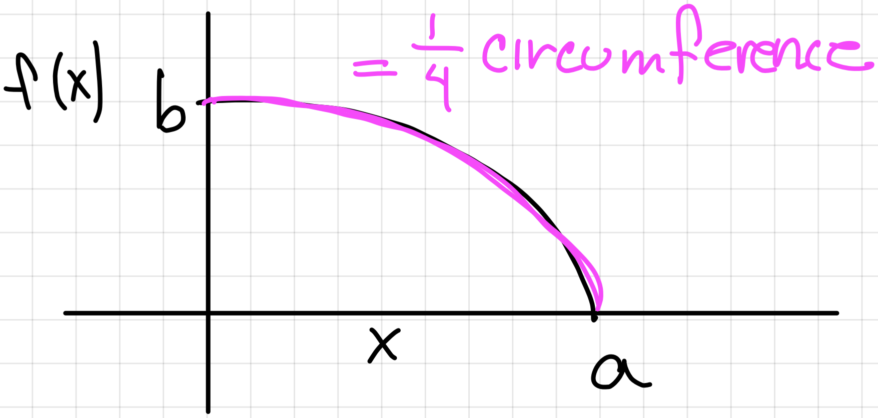


$$\lim_{\Delta x \to 0} \sum_{k=0}^{n} \sqrt{1+f'(x_i)^2}\,\Delta x = \int_a^b \sqrt{1+f'(x)^2}\,dx$$

Example: Circumference of an ellipse



$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$



$$\left(\frac{b}{a}\right)^2 x^2 + y^2 = b^2$$

$$y^2 = b^2 - \left(\frac{b}{a}\right)^2 x^2$$

$$y = b\sqrt{1 - \left(\frac{x}{a}\right)^2}$$

$$\boxed{f(x) = b\sqrt{1 - \left(\frac{x}{a}\right)^2}}$$

Arclength of quarter ellipse:

$$f'(x) = \frac{b}{2}\left(1 - \left(\frac{x}{a}\right)^2\right)^{-1/2}\left(-2\left(\frac{x}{a}\right)\cdot\frac{1}{a}\right) = \frac{b}{a}\,\frac{-\left(\frac{x}{a}\right)}{\sqrt{1-\left(\frac{x}{a}\right)^2}}$$

arc-length:

$$L = \int_0^a \sqrt{1 + \left(\frac{b}{a}\right)^2 \frac{(x/a)^2}{1 - \left(\frac{x}{a}\right)^2}}\, dx$$

Trig. substitution: $\frac{x}{a} = \sin\theta \qquad dx = a\cos\theta\, d\theta$

some algebra! . . . . $$h = a\int_0^{\pi/2}\sqrt{1-\left(1-b^2/a^2\right)\sin^2\theta}\; d\theta$$

$$E\left(\sqrt{1-\frac{b^2}{a^2}}\right)$$

Wolfram Alpha

Input: integral sqrt(1- ...) from t=0 to t=pi/2

$$a\int_0^{\frac{\pi}{2}}\sqrt{1-\left(1-\left(\frac{b}{a}\right)^2\right)\sin^2(t)}\; dt$$

$$a\int_0^{\frac{\pi}{2}}\sqrt{1-\left(1-\left(\frac{b}{a}\right)^2\right)\sin^2(t)}\; dt$$

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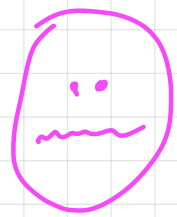
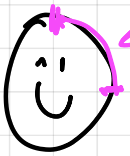
Standard computation time exceeded...



$a=b$?

$h = \frac{\pi}{2} a$



$\frac{a\pi}{2}$

Complete elliptic integral of the second kind [edit]

The **complete elliptic integral of the second kind** E is defined as

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\theta}\, d\theta = \int_0^1 \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\, dt,$$

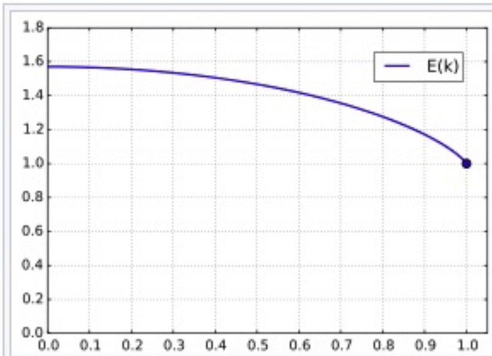
or more compactly in terms of the incomplete integral of the second kind $E(\varphi,k)$ as

$$E(k) = E\left(\tfrac{\pi}{2},k\right) = E(1;k).$$

For an ellipse with semi-major axis a and semi-minor axis b and eccentricity $e=\sqrt{1-b^2/a^2}$, the complete elliptic integral of the second kind $E(e)$ is equal to one quarter of the circumference c of the ellipse measured in units of the semi-major axis a. In other words:

$$c = 4aE(e).$$

circumference of an ellipse.



Plot of the complete elliptic integral of the second kind $E(k)$

Wikipedia: Elliptic integral.

Problem

Approximate

$$I(f) = \int_a^b f(x)\,dx$$

Idea 1. Approximate $f(x)$ by a polynomial

Integrate the polynomial approximation.

Example:

$$f(x) = \sqrt{1+x^2} \approx 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 \cdots$$

$$\int_0^1 f(x)\,dx \approx \int_0^1 dx + \frac{1}{2}\int_0^1 x^2\,dx - \frac{1}{8}\int_0^1 x^4\,dx + \frac{1}{16}\int_0^1 x^6\,dx$$

$$= 1 + \frac{1}{6} - \frac{1}{40} + \frac{1}{112} \cdots = 1.15059523\ldots$$

Wolfram Alpha

$$\int_0^1 \sqrt{1+x^2}\,dx = \frac{1}{2}\left(\sqrt{2} + \sinh^{-1}(1)\right) \approx 1.14779357469632$$

If we integrate over smaller interval, we get much better accuracy:

$$\int_0^{0.125} f(x)\,dx \approx \int_0^{0.125} dx + \frac{1}{2}\int_0^{0.125} x^2\,dx - \frac{1}{8}\int_0^{0.125} x^4\,dx + \frac{1}{16}\int_0^{0.125} x^6\,dx \ldots$$

$$= 0.125 + \frac{1}{6}(.125)^3 - \frac{1}{40}(.125)^5 + \frac{1}{112}(.125)^7 + \ldots$$

$$= 0.125325|0303722$$

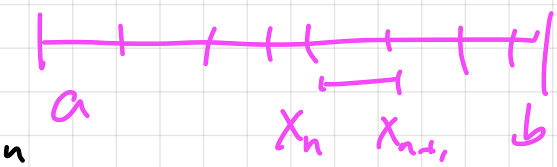
$$\int_0^{\frac{1}{8}} \sqrt{1+x^2}\,dx = \frac{1}{128}\left(\sqrt{65} + 64\sinh^{-1}\left(\frac{1}{8}\right)\right) \approx 0.1253247|62119304$$

With two terms in the Taylor series:

$$0.125 + \frac{1}{6}(.125)^3 \approx 0.125325|5208$$

Idea (2) Break up integral into many small pieces; use lower order polynomials.

$$\int_a^b f(x)\,dx = \sum_{n=1}^{N} \int_{x_{n-1}}^{x_n} f(x)\,dx \qquad x_0 = 0 \quad x_N = 1$$



Focus on approximating $\int_{x_{n-1}}^{x_n} f(x)\,dx$

- We will write this problem as

$$\int_a^b f(x)\,dx \qquad [a,b] \text{ is a small interval.}$$

- To make approximation general, we won't use Taylor series directly. Instead, we will use Lagrange polynomials.

Example: Approximate $f(x)$ on $[a,b]$

using values $(a, \underset{y_0}{f(a)})$, $(b, \underset{y_1}{f(b)})$

$$f(x) \approx p_1(x) = \ell_0(x) f(a) + \ell_1(x) f(b)$$

$$\ell_0(x) = \frac{(x-b)}{a-b} \qquad \ell_1(x) = \frac{(x-a)}{b-a}$$

$$p_1(x) = \frac{(b-x)}{b-a} f(a) + \frac{(x-a)}{b-a} f(b)$$

$$= \frac{1}{b-a}\Big((b-x)f(a) + (x-a)f(b)\Big)$$

$$\int_a^b p_1(x)\,dx = \frac{1}{b-a}\left\{ f(a)\int_a^b (b-x)\,dx + f(b)\int_a^b (x-a)\,dx \right\}$$

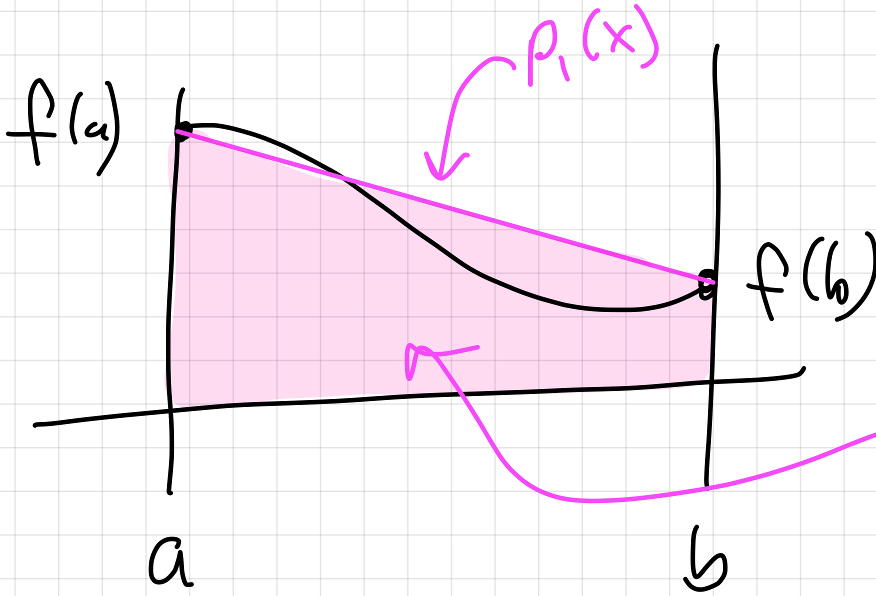
some algebra

$$\ldots = (b-a)\left(\frac{f(a)+f(b)}{2}\right)$$

Trapezoidal Rule

$$= (b-a)\left(\frac{f(a)+f(b)}{2}\right)$$

Trapezoidal Rule



area $= (b-a)\left(\frac{f(a)+f(b)}{2}\right)$

Area of trapezoid: $\underbrace{(\text{base})}_{b-a}\underbrace{(\text{average of heights})}_{\frac{1}{2}(f(a)+f(b))}$

Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx (b-a)\,\frac{f(a)+f(b)}{2} \equiv T(f)$$

$n=1$

Closed Newton-Cotes formula

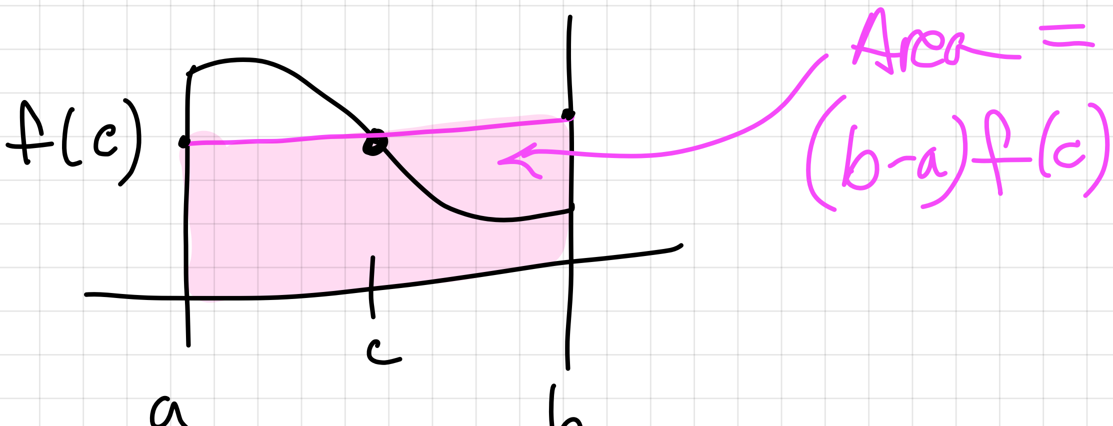
Another approximation: using $(c, f(c))$

$$f(x) \approx p_0(x) = \ell_0(x) f(c), \quad c = \frac{a+b}{2}$$

$$\ell_0(x) = 1$$

$$p_0(x) = f(c)$$

$$\int_a^b p_0(x)\,dx = f(c)\int_a^b dx = (b-a)f(c)$$



$\Rightarrow$ Midpoint Rule: $\displaystyle\int_a^b f(x)\,dx \approx (b-a)f(c) = M(f)$

Open Newton Cotes formula

Another approximation: $(a, f(a))$, $(c, f(c))$, $(b, f(b))$

$$f(x) \approx p_2(x) = \ell_0(x) f(a) + \ell_1(x) f(c) + \ell_2(x) f(b)$$

$h = b - a$

$$\ell_0(x) = \frac{(x-c)(x-b)}{(a-c)(a-b)} = \frac{2}{h^2}(x-c)(x-b)$$

(use: $a - c = -\frac{h}{2}$

$$\ell_1(x) = \frac{(x-a)(x-b)}{(c-a)(c-b)} = \frac{-4}{h^2}(x-a)(x-b)$$

$$\ell_2(x) = \frac{(x-a)(x-c)}{(b-a)(b-c)} = \frac{2}{h^2}(x-a)(x-c)$$

$$\int_a^b \ell_0(x)\,dx = \frac{2}{h^2}\left[-\frac{1}{12}(a-b)^3\right] = \frac{1}{6}h$$

check!

$$\int_a^b \ell_1(x)\,dx = -\frac{4}{h^2}\left(\frac{1}{6}(a-b)^3\right) = \frac{2}{3}h$$

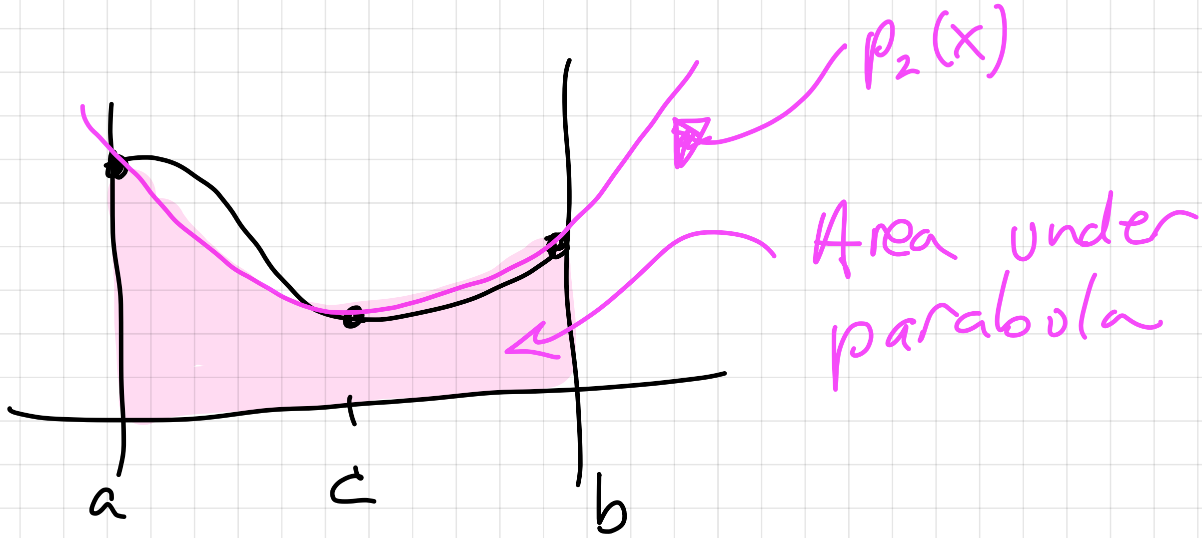
$$\int_a^b \ell_2(x)\,dx = \frac{2}{h^2}\left(-\frac{1}{12}(a-b)^3\right) = \frac{1}{6}h$$

, $h = (b-a)$

(cont.)

$$\int_a^b p_2(x)\,dx = \frac{h}{6}f(a) + \frac{2}{3}h f(c) + \frac{h}{6}f(b)$$

$$= \frac{h}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$



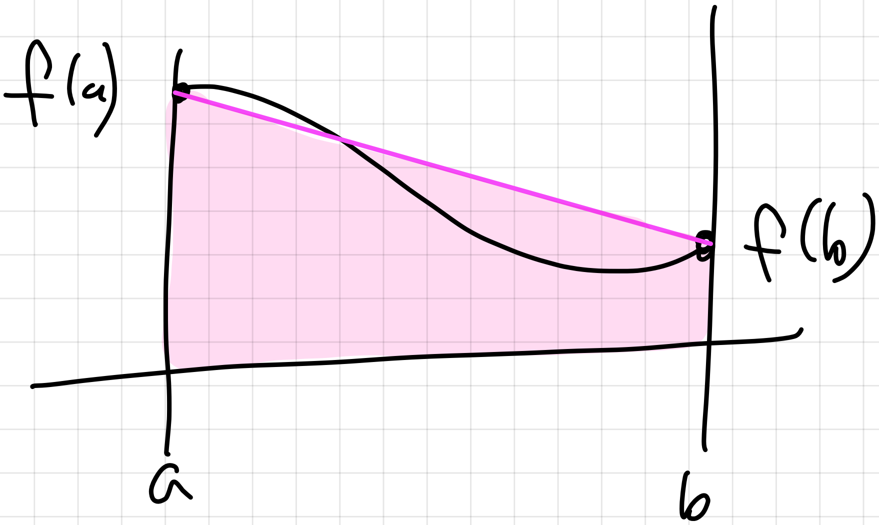
Simpson's Rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] = S(f)$$

closed Newton-Cotes formula of order $n=2$.

Summary

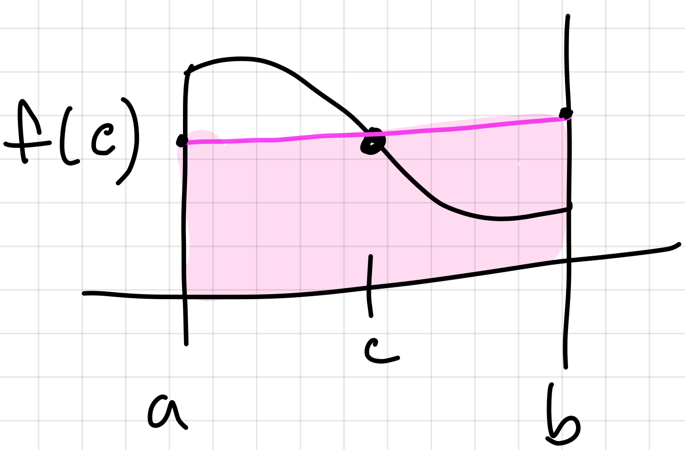
Closed Newton-Cotes Formulas



Trapezoidal Rule

$n=1$

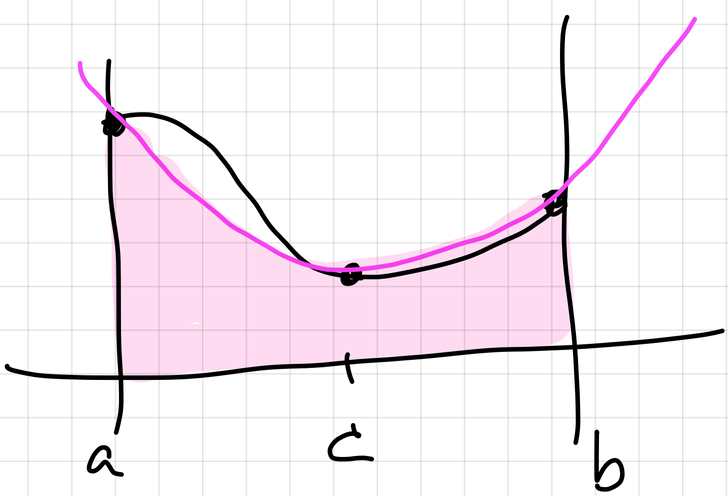
$$T(f) = (b-a)\,\frac{f(a)+f(b)}{2}$$



Midpoint Rule

$n=0$

$$M(f) = (b-a)\,f\left(\frac{a+b}{2}\right)$$

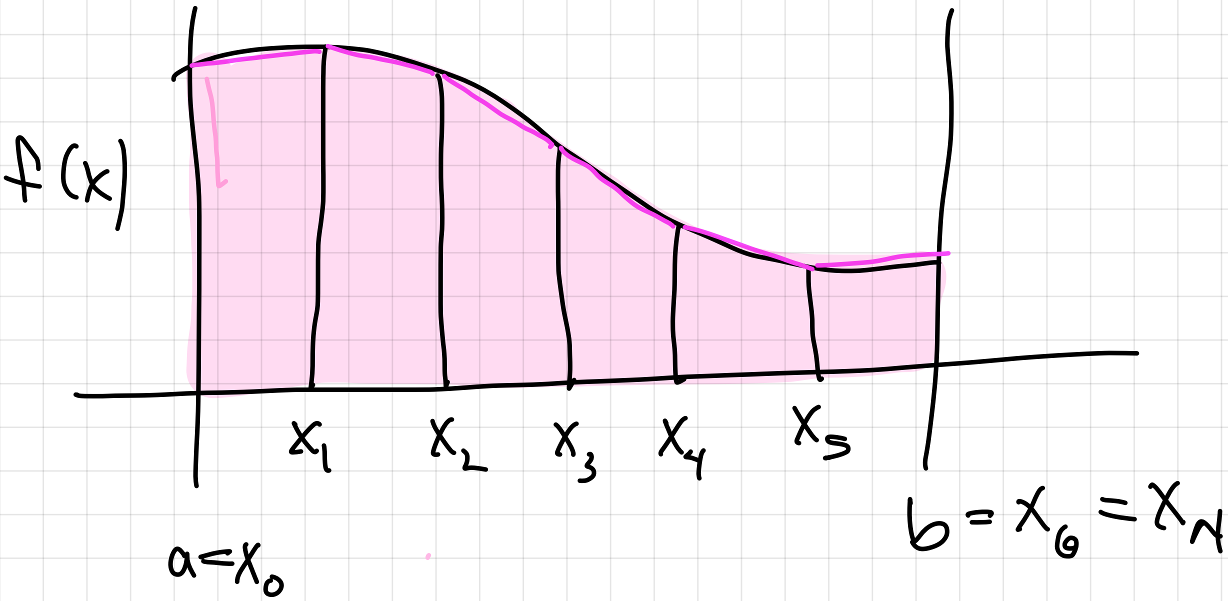


Simpson's Rule

$n=2$

$$S(f) = \frac{h}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Composite Rules



- Subdivide $[a,b]$ into intervals
- Write integral as sum over all intervals.

$$\int_a^b f(x)\,dx = \sum_{n=0}^{N-1} \int_{x_n}^{x_{n+1}} f(x)\,dx$$

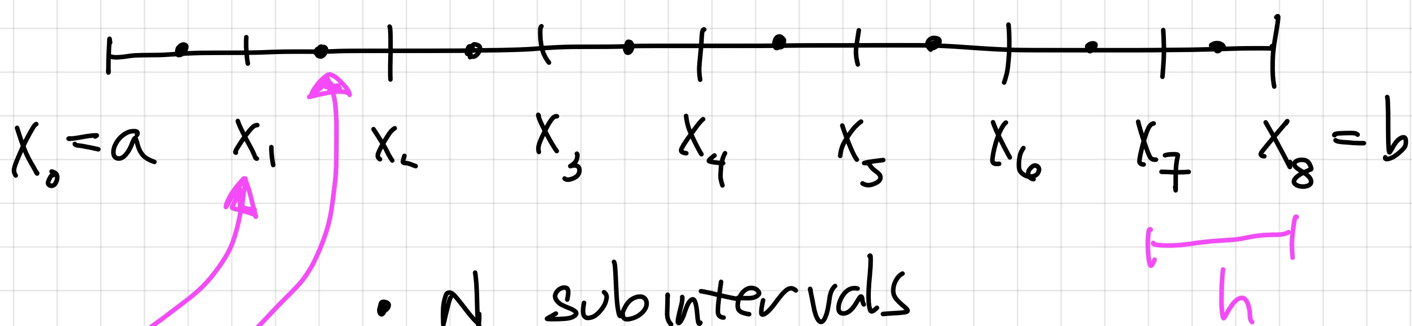
- Apply quadrature rule to each interval

Midpoint Composite Rule

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} f(x)\,dx \approx \sum_{i=0}^{N} f(\bar{x}_i)(x_{i+1} - x_i)$$

If we assume equispaced points, then

$$x_i = a + ih \quad , \quad h = \frac{b-a}{N} \qquad N=8$$



- N subintervals
- $x_i = a + ih$ (edges)
- $\bar{x}_i = a + (i + \frac{1}{2})h$ (cell centers)

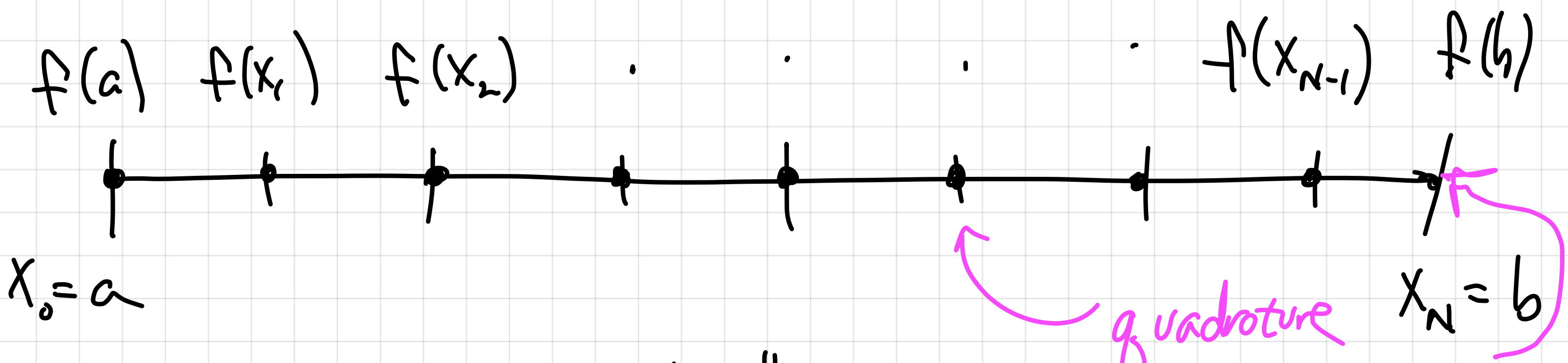
$$\int_a^b f(x)\,dx \approx h \sum_{i=0}^{N-1} f(\bar{x}_i) \quad \text{(Midpoint)}$$

Trapezoidal Composite Rule

$$\int_a^b f(x)\,dx = \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} f(x)\,dx$$

$$\approx h \sum_{i=0}^{N-1} \frac{f(x_i) + f(x_{i+1})}{2}$$

$$= \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{N-1} f(x_i) + f(b) \right]$$



"Quadrature weights":

$$w_i = \begin{cases} \frac{h}{2} & i = 0,\ i = N \\ h, & 0 < i < N \end{cases}$$

$$\approx h \sum_{i=0}^{N-1} \frac{f(x_i) + f(x_{i+1})}{2}$$

$$\frac{f(x_0)+f(x_1)}{2} + \frac{f(x_1)+f(x_2)}{2} + \frac{f(x_2)+f(x_3)}{2}$$

$i=0$ $\quad$ $i=1$ $\quad$ $i=2$

$$\cdots \quad \frac{f(x_{N-1})+f(x_N)}{2}$$

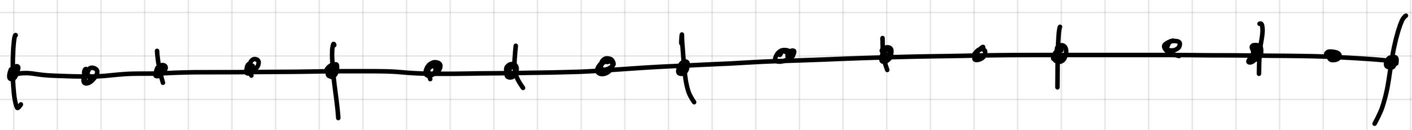
$i=N-1$

$$= \frac{1}{2}f(x_0) + \underbrace{f(x_1) + f(x_2) + \cdots + f(x_{N-1})}_{\sum_{i=1}^{N-1} f(x_i)} + \frac{1}{2}f(x_N)$$

Simpsons Rule

$$\int_a^b f(x)\,dx = \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} f(x)\,dx$$

$$\approx \frac{h}{6} \sum_{i=0}^{N-1} \left[f(x_i) + 4f(\bar{x}_i) + f(x_{i+1}) \right]$$



Use both edge and cell-centered nodes.

$$= \frac{h}{3} \underbrace{\left[\sum_{i=0}^{N-1} \frac{f(x_i) + f(x_{i+1})}{2} \right]}_{\text{trapezoidal}} + \frac{2h}{3} \underbrace{\left[\sum_{i=0}^{N-1} f(\bar{x}_i) \right]}_{\text{midpoint}}$$

$$= \frac{1}{3} \left[T + 2M \right]$$