1. Fixed point algorithm

$$x_{k+1} = g(x_k)$$
$$g(x) = x$$

a) Show ~~that~~ analytically that for $g(x) = ax + b$, $|a| < 1$, the fixed point iteration converges to the solution $\bar{x} = b/(1-a)$

For a fixed point iteration $|g'(x)| < 1$.

$$g(x) = ax + b \Rightarrow g'(x) = a$$
$$|g'(x)| = |a| < 1$$

If $|g'(x)| = |a| < 1$ is true then $g(x)$ has a unique solution, $g(\bar{x}) = \bar{x}$

$$a\bar{x} + b = \bar{x}$$
$$\bar{x} = \frac{b}{1-a}$$

hence $g(x)$ converges to the solution $\bar{x} = \frac{b}{1-a}$

Also for the fixed point iteration x_k

Suppose $|x_k - \bar{x}| = |g(x_{k-1}) - g(\bar{x})|$

since $g(x_{k-1}) = x_{k}$ $\quad g(x_{k-1}) = ax_{k-1} + b$

$$|x_k - \bar{x}| = \left| ax_{k-1} + b - \frac{b}{1-a} \right|$$

Since $g(\bar{x}) = \bar{x} = b/1-a$.

$$g(x_k) - g(\bar{x}) = g'(\xi)\,(x_k - \bar{x})$$

but for fixed point Scheme

$$g(x_k) = x_{k+1}$$

$$g'(\xi) = q$$

$$g(\bar{x}) = \bar{x}$$

$$x_{k+1} - \bar{x} = q\,(x_k - \bar{x})$$

$$x_{k+1} - \bar{x} = q^2\,(x_{k-1} - \bar{x})$$

$$x_{k+1} - \bar{x} = q^3\,(x_{k-2} - \bar{x}) = q^4\,(x_{k-3} - \bar{x})$$

$$\vdots \qquad \vdots$$

$$x_{k+1} - \bar{x} = q^k\,(x_0 - \bar{x})$$

but $e_{k+1} = x_{k+1} - \bar{x}$, $e_0 = x_0 - \bar{x}$

$$e_{k+1} = q^k e_0$$

c) Show that we can approximate the error using

$$e_{k+1} \approx \frac{q}{q-1}\,(x_{k+1} - x_k)$$

$$e_{k+1} = x_{k+1} - \bar{x}$$

Subtracting and adding x_k on the left hand side, we obtain

$$e_{k+1} = x_{k+1} - \bar{x} - x_k + x_k$$

$$e_{k+1} = x_{k+1} - x_k + x_k - \bar{x}$$

There fore $e_{k+1} = q^k e_0$

e) How many iterations does the fixed point algorithm require to solve $g(x) = \frac{1}{10}x + 1 = x$ to a tolerance of 10^{-8}?

from $g(x) = \frac{1}{10}x + 1$, $\Rightarrow q = \frac{1}{10}$, $b = 1$

tolerance, $\varepsilon = 10^{-8}$

Using $|e_{k+1}| \leq \varepsilon$

but $e_{k+1} = q^k e_0$

$q^k e_0 \leq \varepsilon$

Introducing log both sides

$\cancel{q^k \log e_0}$ $k \log q + \log e_0 \leq \log \varepsilon$

$k \log\left(\frac{1}{10}\right) + \log e_0 \leq \log 10^{-8}$

$-k \log 10 \leq -8 \log 10 - \log e_0$

$k \geq 8 + \log(e_0)$

~~k~~ The fixed point algorithm require atleast 8 iterations to solve ~~to~~ $g(x) = \frac{1}{10}x + 1$

$k \geq 8 + \log(e_0)$

and thus depend on $\log(e_0)$

at $k=0$

$$x_1 = x_0 - \frac{(g(x_0) - x_0)^2}{g(g(x_0)) - 2g(x_0) + x_0}$$

x_1 ~~/=/~~ $g(x_0) = a x_0 + b$

$$x_1 = x_0 - \frac{(a x_0 + b - x_0)^2}{a(a x_0 + b) + b - 2(a x_0 + b) + x_0}$$

$$x_1 = x_0 - \frac{(a-1)(x_0) + b}{(a-1)}$$

$$x_1 = x_0 - x_0 + \frac{b}{a-1} = x_0 - x_0 - b/a-1$$

$$x_1 = \frac{b}{1-a} = \bar{x}$$

$$x_1 = \bar{x}$$

hence it converges in one step.

b) Choose multipliers l_{ij} so that applying $E_{41}E_{31}E_{21}$ to A zeros out the entries below a_{11}.

$$E_{41}E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -l_{21} & 1 & 0 & 0 \\ -l_{31} & 0 & 1 & 0 \\ -l_{41} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix}$$

Choose
$l_{21} = -1$
$l_{31} = 2$
$l_{41} = -3$

$$E_{41}E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 10 & 4 & -9 \\ 0 & -16 & -11 & 18 \end{bmatrix}$$

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c) Show that the inverse of $E_{41}E_{31}E_{21}$ is

$$(E_{41}E_{31}E_{21})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & 0 & 1 & 0 \\ l_{41} & 0 & 0 & 1 \end{bmatrix}$$

3(d) From the above steps, derive the LU decomposition of a general 4X4 Matrix. Use these to derive the LU decomposition of the matrix in (3).

For U

$$E_{41} E_{31} E_{21} A = U_1$$

$$(E_{42} E_{32}) U_1 = U_2$$

$$E_{43} U_2 = U$$

So

$$\cancel{U = E_{43} E_{42} E_{41} E_{32} E_{31} E_{21} A}$$

$$U = E_{43} E_{42} E_{32} E_{41} E_{31} E_{21} A$$

for L

$$(E_{41} E_{31} E_{21})^{-1} = L_1$$

$$(E_{42} E_{32})^{-1} L_1 = L_2$$

$$(E_{43})^{-1} L_2 = L$$

So

$$L = (E_{43} E_{42} E_{32} E_{41} E_{31} E_{21})^{-1}$$

Suppose we have $(AB)^{-1} = A^{-1} B^{-1}$

$$L = [E_{43} E_{42} E_{32} E_{41} E_{31} E_{21})^{-1} = (E_{43} E_{42} E_{32})^{-1} (E_{41} E_{31} E_{21})^{-1}$$

but $(E_{41} E_{31} E_{21})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & 0 & 1 & 0 \\ l_{41} & 0 & 0 & 1 \end{bmatrix}$

4) Find the LU decomposition of the matrix in (3)

$$A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix}$$

$$A = LU$$

where

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

Using Gaussian Elimination method on A, we obtain an upper triangular Matrix U, as follows.

take $l_{21} = -1$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 + R_1} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 5 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix}$$

take $l_{31} = -2$ ~~In this case $L_{21} = -1$~~

$$\begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 5 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + 2R_1} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 5 \\ 0 & -18 & -4 & -1 \\ -9 & 5 & -5 & 12 \end{bmatrix}$$

~~In this case $L_{31} = -2$~~

Hence we obtain, U,

$$U = \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & 5 & -19 \\ 0 & 0 & 0 & -47/5 \end{bmatrix}$$

Corresponding to, L,

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 9 & 1 & 0 \\ -3 & 8 & -3/5 & 1 \end{bmatrix}$$

So $A = LU$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 9 & 1 & 0 \\ -3 & 8 & -3/5 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & 5 & -19 \\ 0 & 0 & 0 & \frac{-47}{5} \end{bmatrix}$$

Since a_{22} and a_{11} are diagonally dominant ~~Row~~ entries then $a_{22}a_{11} > a_{21}a_{12}$, therefore

$$\lambda^2 = \frac{a_{21}a_{12}}{a_{22}a_{11}} < 1$$

$$\lambda = \pm\sqrt{\frac{a_{21}\,a_{12}}{a_{22}\,a_{11}}}$$

ρ is the largest $|\lambda|$, so

$$|\lambda| = \left|\sqrt{\frac{a_{21}a_{12}}{a_{22}a_{11}}}\right| < 1$$

~~So th~~ So the spectral radius $\rho(I - D^{-1}A) < 1$

Jacobi Iteration will Converge, since for Jacobi, we take $M = D$, and we have shown that

$\rho(I - M^{-1}A) < 1$, hence the Iteration Converges

since according for Convergence an Iteration Converges if and only if $\rho(I - M^{-1}A) < 1$

c) Show that the iteration for the error e_k is given by

$$e_{k+1} = (I - M^{-1}A) e_k$$

from

$$x_{k+1} = (I - M^{-1}A) x_k + M^{-1}b$$

Subtract $\bar{x}$ from both sides

$$x_{k+1} - \bar{x} = (I - M^{-1}A) x_k - \bar{x} + M^{-1}b$$

we know that $x_{k+1} - \bar{x} = e_{k+1}$

$$e_{k+1} = (I - M^{-1}A) x_k - \bar{x} + M^{-1}b$$

take $A\bar{x} = b$.

$$e_{k+1} = (I - M^{-1}A) x_k - \bar{x} + M^{-1}A\bar{x}$$

$$e_{k+1} = (I - M^{-1}A) x_k - (I - M^{-1}A)\bar{x}$$

$$e_{k+1} = [I - M^{-1}A] (x_k - \bar{x})$$

but $x_k - \bar{x} = e_k$

$$e_{k+1} = (I - M^{-1}A) e_k$$

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d) Show that

$$\|e_k\| \leq \|I - M^{-1}A\|^k \|e_0\|$$

from $e_{k+1} = (I - M^{-1}A) e_k$

$$e_{k+1} = (I - M^{-1}A)(I - M^{-1}A) e_{k-1}$$

~~Since~~

$$k \log \|I - M^{-1}A\| \leq \log \varepsilon - \log \|e_0\|$$

Since $\log \|I - M^{-1}A\| < 0$, then

$$K \geq \frac{\log \varepsilon - \log e_0}{\log \|I - M^{-1}A\|}$$

$$K \geq \frac{\log \varepsilon}{\log \|I - M^{-1}A\|} - \frac{\log \|e_0\|}{\log \|I - M^{-1}A\|}$$

the term $\dfrac{\log(\varepsilon)}{\log \|I - M^{-1}A\|}$ dominates therefore

$$K \geq \frac{\log(\varepsilon)}{\log \|I - M^{-1}A\|}$$

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f) Since $\rho(I - M^{-1}A)$ ~~re~~ is the largest absolute value of the eigen values of $(I - M^{-1}A)$, and the number of iterations is given by

$$K \geq \frac{\log \varepsilon}{\log \|I - M^{-1}A\|} - \frac{\log \|e_0\|}{\log \|I - M^{-1}A\|}$$

9) Show analytically that the update to the residual r_{k+1} in the Conjugate Gradient Algorithm is equal to $b - Ax_k$

Suppose A is symmetric, positive definite

$$F(x) = \frac{1}{2} x^T A x - b^T x$$

So the direction of the greatest decrease of F is given by $-\nabla F(x_{k+1}) = r_{k+1}$ ~~of~~ the update to residual r_{k+1}

So taking the residual as the search direction we have

$$x_{k+1} = x_k + a_k r_k$$

$$-\nabla F(x_{k+1}) = r_{k+1}$$

$$-\nabla F(x_k + a_k r_k) = r_{k+1}$$

$$F(x_k + a_k r_k) = \frac{1}{2}(x_k + a_k r_k)^T A (x_k + a_k r_k) - b^T(x_k + a_k r_k)$$

$$\nabla F(x_k + a_k r_k) = \frac{dF}{dx_k} = \frac{1}{2}(2Ax_k) - b$$

$$\nabla F(x_k + a_k r_k) = Ax_k - b$$

$$-\nabla F(x_k + a_k r_k) = b - Ax_k = r_{k+1}$$

therefore $r_{k+1} = b - Ax_k$