Title: Elastic Wave Equation

Background:

The propagation of seismic waves under the earth's surface is studied using 2D elastic wave equations. These equations are described by displacement $U = (U_x, U_z)$ and the stress $\tau = \begin{bmatrix} \tau_{xx} & \tau_{xz} \\ \tau_{xz} & \tau_{zz} \end{bmatrix}$ as follows.

$$\rho \frac{\partial^2 U_x}{\partial t^2} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z}$$

$$\rho \frac{\partial^2 U_z}{\partial t^2} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z}$$

$$\tau_{xx} = (\lambda + 2\mu) \frac{\partial U_x}{\partial x} + \frac{\partial \tau_{zz}}{\partial z},$$

$$\tau_{zz} = \lambda \frac{\partial U_x}{\partial x} + (\lambda + 2\mu) \frac{\partial U_z}{\partial z},$$

$$\tau_{xz} = \mu \left(\frac{\partial U_z}{\partial x} + \frac{\partial U_x}{\partial z} \right).$$

Where λ and μ are lamé parameters.
μ: shear modulus
ρ: density.

Let $V_x = \frac{\partial U_x}{\partial t}$ and $V_z = \frac{\partial U_z}{\partial t}$

where V_x and V_z are the velocities in x and z directions respectively. The equations become

$$\rho \frac{\partial V_x}{\partial t} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} \quad \text{———} (2)$$

$$\rho \frac{\partial V_z}{\partial t} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} \quad \text{———} (3)$$

$$\frac{\partial \tau_{xx}}{\partial t} = (\lambda + 2\mu)\frac{\partial v_x}{\partial x} + \lambda \frac{\partial v_z}{\partial z} \quad \text{——} (4)$$

$$\frac{\partial \tau_{zz}}{\partial t} = \lambda \frac{\partial v_x}{\partial x} + (\lambda + 2\mu)\frac{\partial v_z}{\partial z} \quad \text{——} (5)$$

$$\frac{\partial \tau_{xz}}{\partial t} = \mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right) \quad \text{——} (6)$$

P-wave velocity $C_P = \sqrt{\frac{\lambda + 2\mu}{\rho}}$ ~~and~~

S-wave velocity $C_S = \sqrt{\mu/\rho}$

Let the source time function: $s(t)$, be defined by the Ricker wavelet as

$$s(t) = (1 - 2\pi^2 f_M^2 t^2)\, e^{-\pi^2 f_M^2 t^2}$$

where f_M : peak frequency at time t

The ~~back~~ underground explosion at a location (x_0, z_0) is given by adding the source term to the normal stress components τ_{xx} and τ_{zz}.

$$\tau_{xx} = (\lambda + 2\mu)\frac{\partial u_x}{\partial x} + \lambda \frac{\partial u_z}{\partial z} + \rho s(t - t_0) \int_0^L\int_0^L \delta(x - x_0)\,\delta(z - z_0)\,dx\,dz$$

$$\tau_{zz} = \lambda \frac{\partial u_x}{\partial x} + (\lambda + 2\mu)\frac{\partial u_z}{\partial z} + \rho s(t - t_0) \int_0^L\int_0^L \delta(x - x_0)\,\delta(z - z_0)\,dx\,dz$$

Differentiating the two equations with respect to time we obtain.

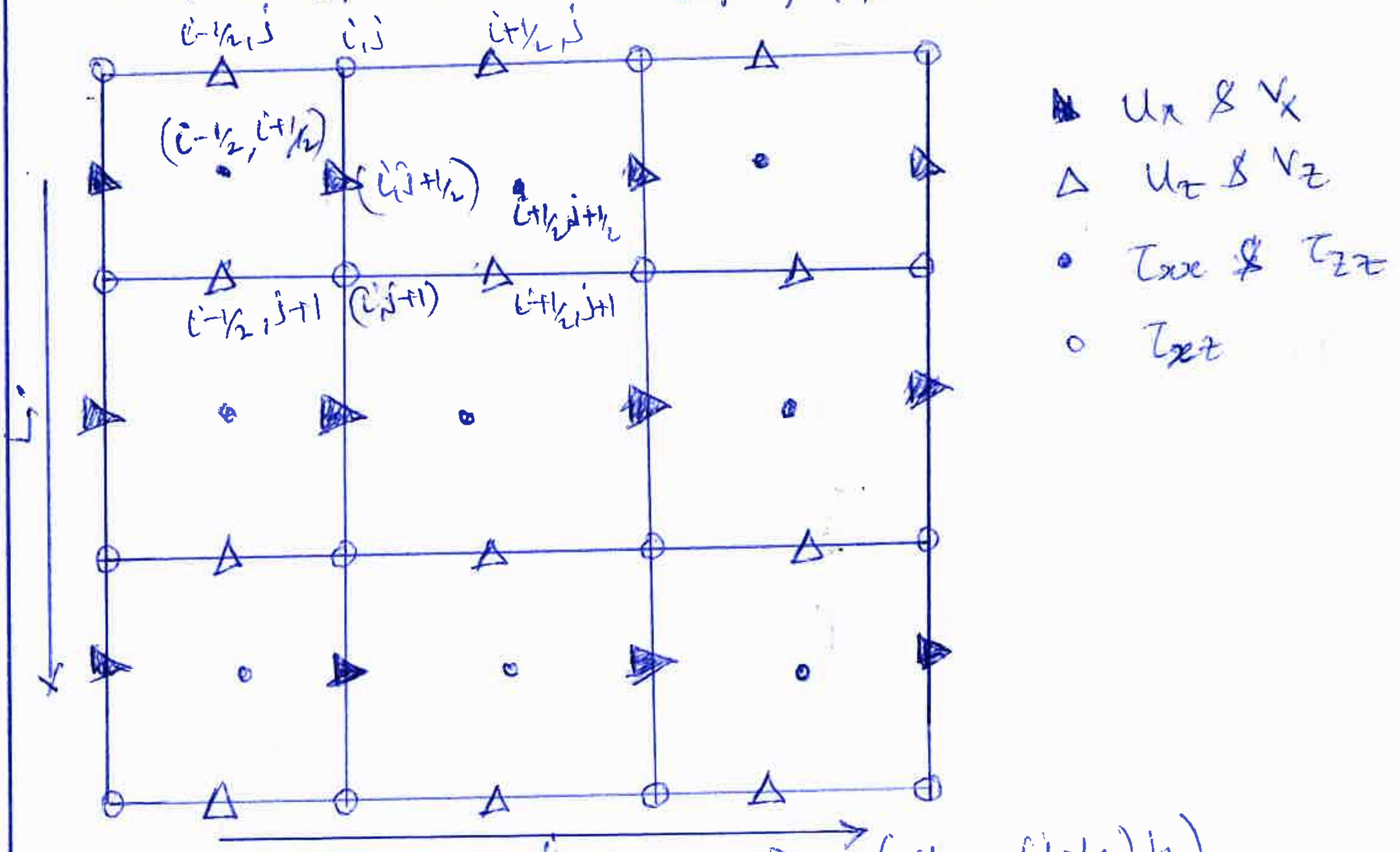
$$\frac{\partial \tau_{xx}}{\partial t} = (\lambda + 2\mu)\frac{\partial v_x}{\partial x} + \lambda \frac{\partial v_z}{\partial z} + \rho s'(t - t_0) \int_0^L\int_0^L \delta(x - x_0)\,\delta(z - z_0)\,dx\,dz \quad \text{——} (7)$$

$$\frac{\partial \tau_{zz}}{\partial t} = \lambda \frac{\partial v_x}{\partial x} + (\lambda + 2\mu)\frac{\partial v_z}{\partial z} + \rho s'(t - t_0) \int_0^L\int_0^L \delta(x - x_0)\,\delta(z - z_0)\,dx\,dz \quad \text{——} (8)$$

Discretization:

Consider a staggered grid in space for unknown variables. also take a grid point

$$(x_i, z_{j+1/2}) = (ih, (j+1/2)h), \quad i,j = 0, \ldots, m-1$$



Consider v_x at $(x_i, z_{j+1/2}) = (ih, (j+1/2)h)$

Equation (2) becomes

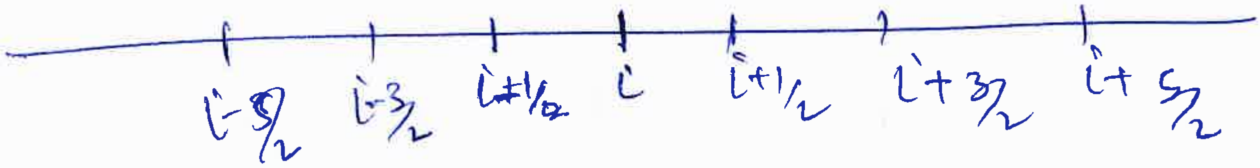
$$\rho_{i,j+1/2} \frac{d}{dt}(v_x)_{i,j+1/2} = \frac{1}{h}\left((\tau_{xx})_{i+1/2,j+1/2} - (\tau_{xx})_{i-1/2,j+1/2}\right) + \frac{1}{h}\left((\tau_{xz})_{i,j+1} - (\tau_{xz})_{i,j}\right) \quad \text{--- (12)}$$

Combining staggering in space and in time, we can approximate the elastic wave equations using the leap-frog scheme.

$$(v_x)^{n+1/2}_{i,j+1/2} = (v_x)^{n-1/2}_{i,j+1/2} + \frac{k}{h\rho_{i,j+1/2}}\left((\tau_{xx})^n_{i+1/2,j+1/2} - (\tau_{xx})^n_{i-1/2,j+1/2} + (\tau_{xz})^n_{i,j+1} - (\tau_{xz})^n_{i,j}\right) \quad \text{--- (13)}$$

Extending this to higher dimensions along x

Obtain a stencil



which gives us

$$\frac{1}{h}\left[-\frac{5}{2} \quad -\frac{3}{2} \quad -\frac{1}{2} \quad \frac{1}{2} \quad \frac{3}{2} \quad \frac{5}{2}\right]$$

as our stencil which we use in the weight function to compute the weights.

and along z we obtain.

$$\frac{1}{h}\left[-2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3\right]$$

The process is repeated for other elastic equations 3 – 6, and the corresponding discretizations are obtained.

~~Challenge~~