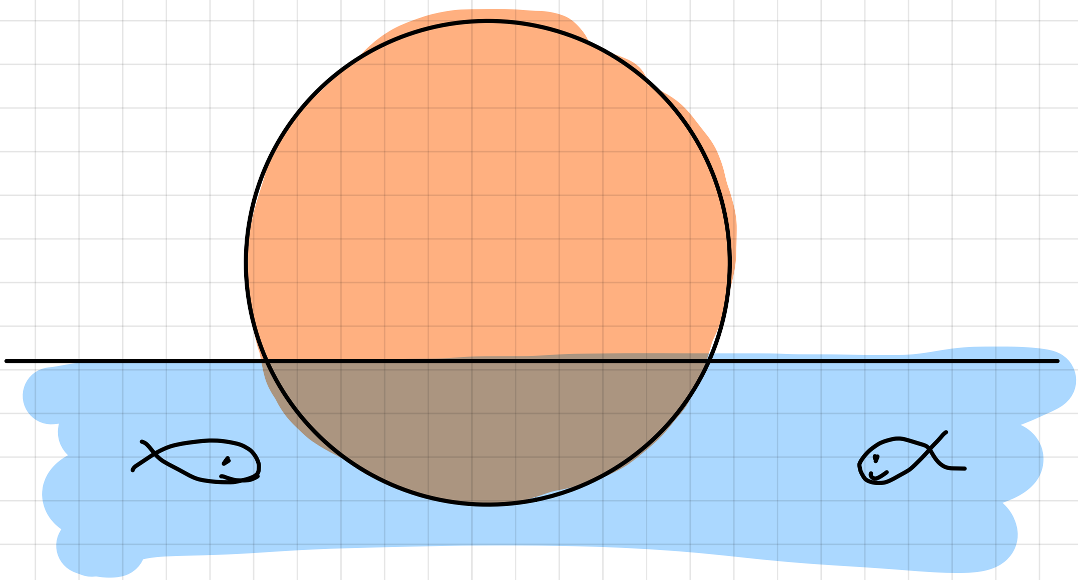
Rootfinding

Bisection
Method of False Position
Fixed Point Methods
Newton’s method
Accelerated Methods
Roots of Polynomials

Root Finding - Model Problem



How far does a sphere sink in a fluid?

Weight of fluid displaced by the sphere = weight of the sphere

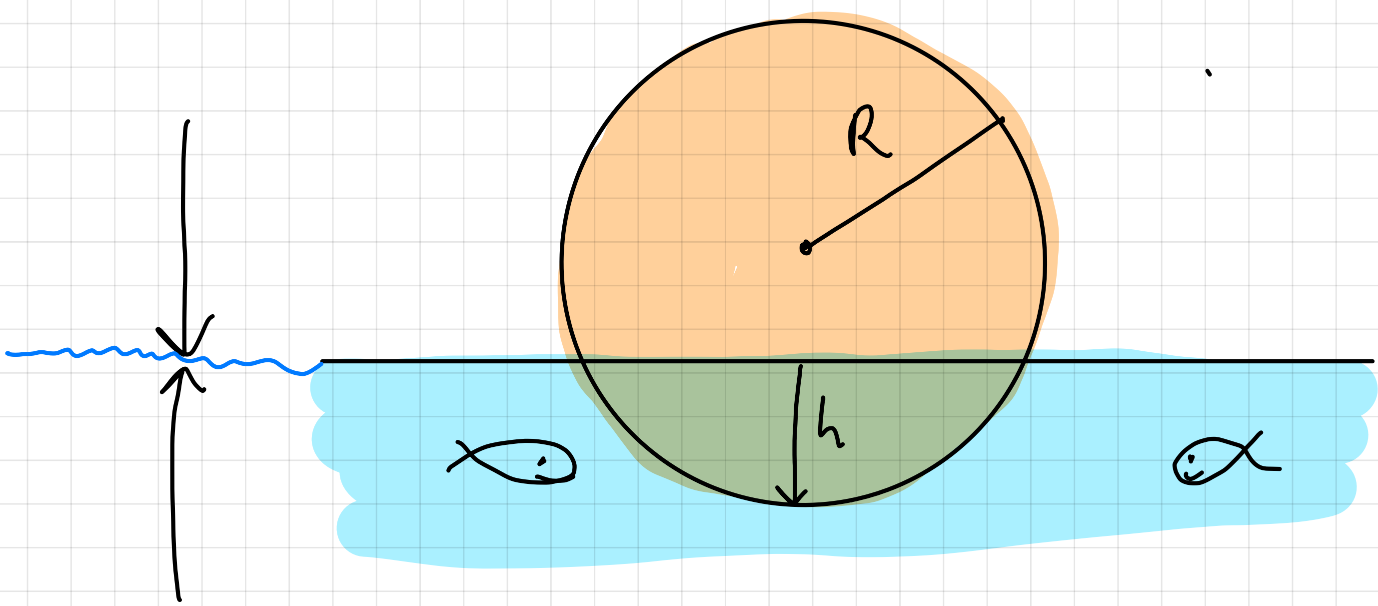
"Eureka!"
- Archimedes.

OR Force exerted on the fluid by the sphere has to equal force on the sphere by the fluid

Force exerted by the sphere on the fluid

balance

Force exerted by the fluid on the sphere



- Density of sphere : ρ_s g/cm^3
- Density of the fluid : ρ_f g/cm^3
- Radius of the sphere : R cm
- Acceleration due to gravity : g $\frac{cm}{s^2}$
- Volume of the sphere : (cm^3)

$$V_s = \frac{4}{3}\pi R^3$$

Volume of fluid displaced by the sphere:

spherical cap formula

$$V_d = \frac{\pi h^2}{3}(3R - h) = \pi h^2\left(R - \frac{h}{3}\right)$$

Force on the sphere: $\rho_f V_d g \uparrow$

Force on the fluid: $\rho_s V_s g \downarrow$

$$\underbrace{\rho_w\left(\pi h^2\left(R-\frac{h}{3}\right)\right)}_{\text{Force exerted by the fluid}} \cancel{g} = \underbrace{\frac{4}{3}\pi R^3 \rho_s}_{\text{Force exerted by the sphere}} \cancel{g}$$

$\Rightarrow$

$$\rho_f \cancel{\pi} h^2 R - \rho_f \cancel{\pi} \frac{h^3}{3} = \frac{4}{3}\cancel{\pi} R^3 \rho_s$$

or

$$\frac{\rho_w}{3}h^3 - \rho_w R h^2 + \frac{4}{3}R^3\rho_s = 0$$

$$\frac{\rho_w}{3}h^3 - \rho_w R h^2 + \frac{4}{3}R^3\rho_s = 0$$

Solve for h, the height of the sphere below the surface of the fluid. This is an example of a "non-linear scalar equation"

"non-linear" : unknown variable appears as powers of h: h^3, h^2

"scalar" : only one equation

To treat this mathematically we need to introduce "function" notation:

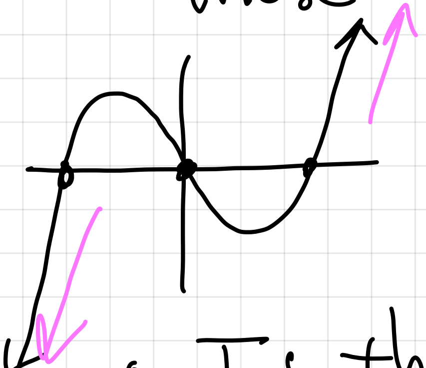
$$f(h) \equiv \frac{\rho_w}{3}h^3 - \rho_w R h^2 + \frac{4}{3}R^3\rho_s$$

Solve ✓ [g] [g] [g] g = grams

$$f(h) \equiv \frac{\rho_f}{3} h^3 - \rho_f R h^2 + \frac{4}{3} R^3 \rho_s = 0$$

This is a **non-linear root finding problem**

The "**roots**" of $f(h)$ are those values of h that satisfy



$$f(h) = 0$$

- Is there a solution? Yes - cubic equation will have at least one real root.
- How many roots are there?
- If there is more than one root which one do we choose?

- Will algebra work? Check Wolfram Alpha
  ↳ non-dimensionalize first.

Non-dimensionalize the problem:

$$f(h) \equiv \frac{\rho_f}{3} h^3 - \rho_f R h^2 + \frac{4}{3} R^3 \rho_s$$

Factor out

$$= \rho_f R^3 \left[\frac{1}{3}\left(\frac{h}{R}\right)^3 - \left(\frac{h}{R}\right)^2 + \frac{4}{3}\left(\frac{\rho_s}{\rho_w}\right) \right]$$

Define

$$g\left(\frac{h}{R}\right) = \frac{1}{\rho_f R^3} f(h)$$

$$= \frac{1}{3}\left(\frac{h}{R}\right)^3 - \left(\frac{h}{R}\right)^2 + \frac{4}{3}\left(\frac{\rho_s}{\rho_f}\right)$$

Let $x = \frac{h}{R}$ $\quad \beta = \frac{\rho_s}{\rho_f}$ be non-dimensional quantities.

$$g(x) = \frac{1}{3}x^3 - x^2 + \frac{4}{3}\beta$$

non-dimensional version

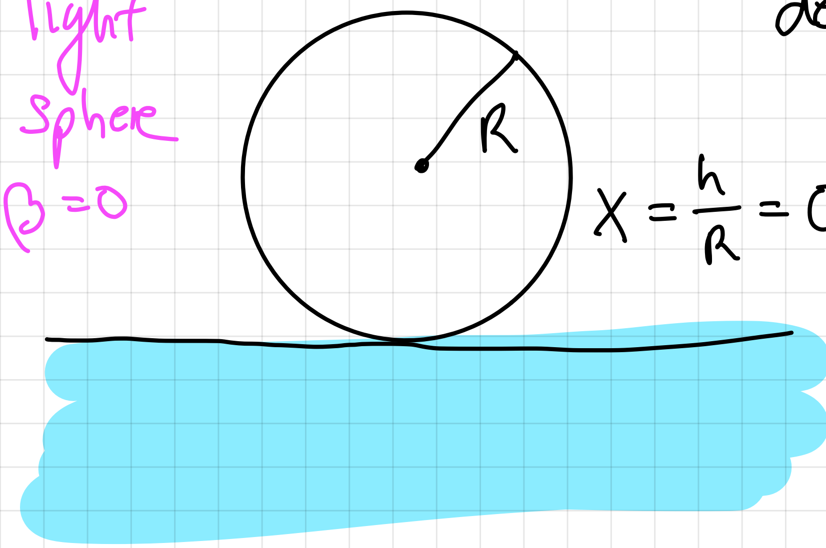
Submerged Sphere

$X = \frac{h}{R}$ Solve $f(x) = \frac{1}{3}x^3 - x^2 + \frac{4}{3}\beta = 0$

$\beta = \frac{\text{density of sphere}}{\text{density of fluid.}}$

light sphere

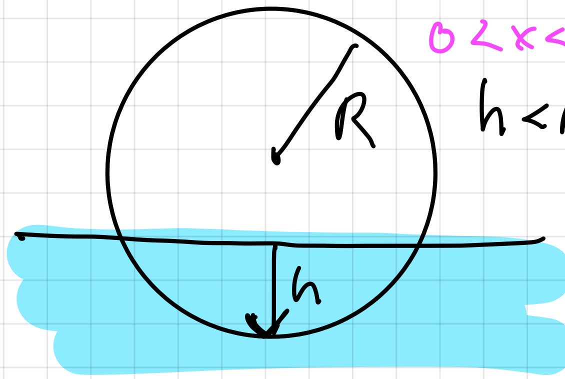
$\beta = 0$



$X = \frac{h}{R} = 0$

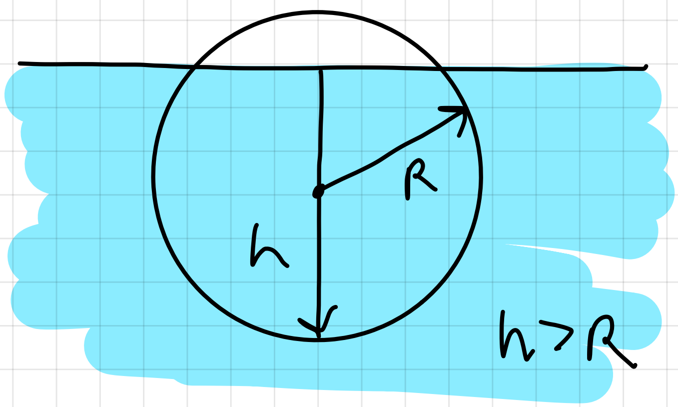
$0 < \beta < 1$

$0 < x < 1$



$h < R$

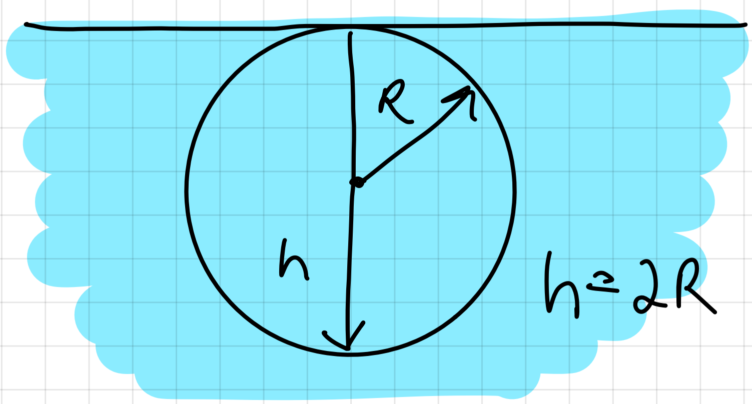
$0 < \beta < 1,\ 0 < x < 1$



$h > R$

heavy sphere $\beta = 1$

$X = 2$



$h = 2R$

$$0 \leq h \leq 2R$$

$$\Rightarrow \quad 0 \leq \frac{h}{R} \leq 2 \quad \Rightarrow \quad 0 \leq X \leq 2$$

Try solving this Wolfram Alpha:

wolframalpha.com

$$g(x) = \frac{1}{3}x^3 - x^2 + \frac{4}{3}\beta$$

"Wolfram Alpha"; $\beta = 0.1$

$\boxed{\text{solve } 1/3\,x\text{^}3 - x\text{^}2 + 4/3 * \underline{\underline{0.1}} = 0}$

or "B"

solve x^3/3 - x^2 + 4/3*0.1 = 0

Extended Keyboard   Upload

Input interpretation:

solve	$\frac{x^3}{3} - x^2 + \frac{4}{3} \times 0.1 = 0$

$\beta = 0.1$

Results:

$$x = 1 - \frac{1 + i\sqrt{3}}{2\sqrt[3]{\frac{4}{5} + \frac{3i}{5}}} + \frac{1}{2} i \sqrt[3]{\frac{4}{5} + \frac{3i}{5}} \left(\sqrt{3} + i\right)$$

$i = \sqrt{-1}$

$$x = 1 - \frac{1}{2}\sqrt[3]{\frac{4}{5} + \frac{3i}{5}} \left(1 + i\sqrt{3}\right) + \left(\frac{3}{10} + \frac{2i}{5}\right)\left(\frac{4}{5} + \frac{3i}{5}\right)^{2/3} \left(\sqrt{3} + i\right)$$

$$x = 1 + \frac{1}{\sqrt[3]{\frac{4}{5} + \frac{3i}{5}}} + \sqrt[3]{\frac{4}{5} + \frac{3i}{5}}$$

This is probably using Cardano's formula for the roots of a cubic. Let's not do this.

What about Matlab or Python?

Matlab - fzero would work

↳ provide a starting interval

roots : finds roots of polynomials

Python: SciPy

Discuss:

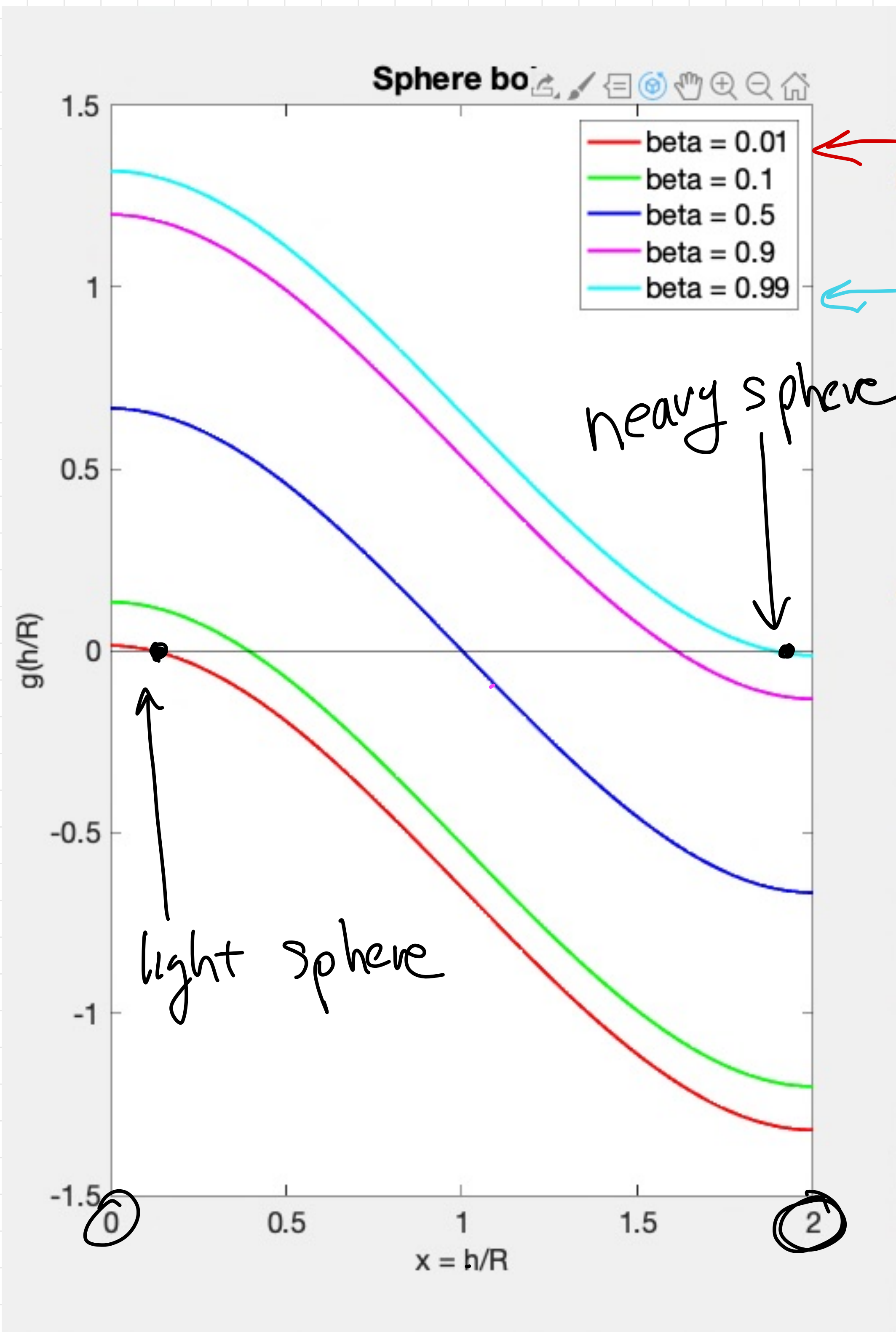
- location of roots
- location of physically meaningful roots
- Graph $g(x) = \frac{1}{3}x^3 - x^2 + \frac{4}{3}\beta$

for $\beta = 0.1$, $\beta = 0.5$, $\beta = .9$

$$g(x) = \frac{1}{3}h^3 - x^2 + \frac{4}{3}\beta$$

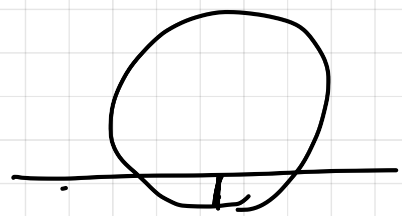
$$x = \frac{h}{R} \; ; \; \beta = \frac{\rho_s}{\rho_f} = \frac{\text{density of sphere}}{\text{density of fluid}}$$

x_r

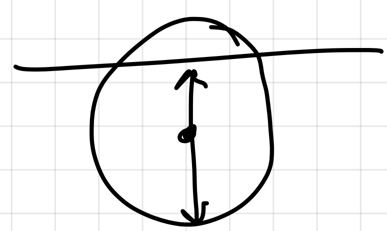


light sphere

heavy sphere



$x_r = \frac{h}{R} \ll 1$



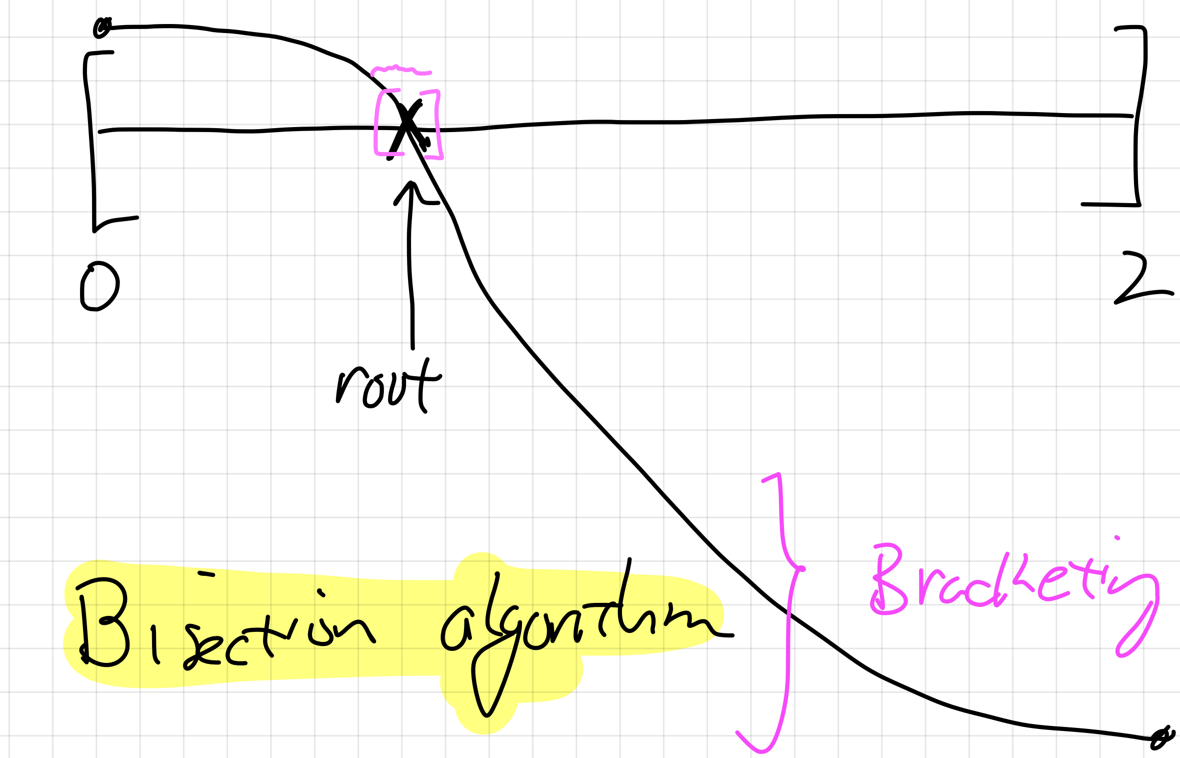
$x_r = \frac{h}{R} \approx 2$

What we know:

$g(x)$ has one root in $[0,2]$
(convincing graphical evidence)

How do we find it?

$\Rightarrow$ Bracketing method



Bisection algorithm

Bracketing method.