1. Fixed point algorithm

$$x_{k+1} = g(x_k)$$

$$g(x) = x$$

a) Show ~~that~~ analytically that for $g(x) = ax+b$, $|a|<1$, the fixed point iteration converges to the solution $\bar{x} = b/(1-a)$

For a fixed point iteration $|g'(x)| < 1$.

$$g(x) = ax+b \Rightarrow g'(x) = a$$

$$|g'(x)| = |a| < 1$$

If $|g'(x)| = |a| < 1$ is true then $g(x)$ has a unique solution, $g(\bar{x}) = \bar{x}$

$$a\bar{x} + b = \bar{x}$$

$$\bar{x} = \frac{b}{1-a}$$

hence $g(x)$ converges to the solution $\bar{x} = \frac{b}{1-a}$

Also for the fixed point iteration x_k

Suppose $|x_k - \bar{x}| = |g(x_{k-1}) - g(\bar{x})|$

since $g(x_{k+1}) = x_{k+1}$ $\quad g(x_{k-1}) = ax_{k-1}+b$

$$|x_k - \bar{x}| = \left| ax_{k-1} + b - \frac{b}{1-a} \right|$$

since $g(\bar{x}) = \bar{x} = b/1-a.$

$$|x_k - \bar{x}| = |q x_{k-1} - \frac{qb}{1-q}|$$

take $\bar{x} = \frac{b}{1-q}$

$$|x_k - \bar{x}| = |q(x_{k-1} - \bar{x})|$$

$$|x_k - \bar{x}| \le |q|\,|(x_{k-1} - \bar{x})|$$

$$\le |q|^2 |x_{k-2} - \bar{x}|$$

$$|x_k - \bar{x}| \le |q|^3 |x_{k-3} - \bar{x}|$$

$$\vdots \qquad \vdots$$

$$|x_k - \bar{x}| \le |q|^k |x_0 - \bar{x}|$$

As $k \to \infty$, since $|q| < 1$, then $|q|^k |x_0 - \bar{x}| \to 0$

therefore

$$|x_k - \bar{x}| \to 0$$

hence the fixed point iteration converges to $\bar{x}$

$$x_k = \bar{x} = \frac{b}{1-q}$$

b) Show that for any fixed point problem, the error $e_{k+1} = x_k - \bar{x}$ satisfies $e_{k+1} = q^k e_0$

Using the intermediate value theorem, it can be stated that

$$\frac{g(x_k) - g(\bar{x})}{x_k - \bar{x}} = g'(q)$$

$$g(x_k) - g(\bar{x}) = g'(\xi)\,(x_k - \bar{x})$$

but for fixed point Scheme

$$g(x_k) = x_{k+1}$$

$$g'(\xi) = a$$

$$g(\bar{x}) = \bar{x}$$

$$x_{k+1} - \bar{x} = a\,(x_k - \bar{x})$$

$$x_{k+1} - \bar{x} = a^2\,(x_{k-1} - \bar{x})$$

$$x_{k+1} - \bar{x} = a^3\,(x_{k-2} - \bar{x}) = a^4\,(x_{k-3} - \bar{x})$$

$$\vdots \qquad\qquad \vdots$$

$$x_{k+1} - \bar{x} = a^k\,(x_0 - \bar{x})$$

but $e_{k+1} = x_{k+1} - \bar{x}$, $e_0 = x_0 - \bar{x}$

$$\underline{\underline{e_{k+1} = a^k e_0}}$$

c) Show that we can approximate the error using

$$e_{k+1} \approx \frac{a}{a-1}\,(x_{k+1} - x_k)$$

$$e_{k+1} = x_{k+1} - \bar{x}$$

Subtracting and adding x_k on the left hand side, we obtain

$$e_{k+1} = x_{k+1} - \bar{x} - x_k + x_k$$

$$e_{k+1} = x_{k+1} - x_k + x_k - \bar{x}$$

we have $e_{k+1} = q e_k$. $\Rightarrow e_k = \frac{q}{e_{k+1}}$, *

and $e_k = x_k - \bar{x}$

therefore $e_{k+1} = x_{k+1} - x_k + x_k - \bar{x}$ becomes

$$e_{k+1} \approx x_{k+1} - x_k + e_k$$

$$e_{k+1} \approx x_{k+1} - x_k + \frac{q}{e_{k+1}}$$

$$e_{k+1} \approx \frac{q}{q-1}(x_{k+1} - x_k)$$

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d) Show that for the linear problem $g(x) = qx+b$, the error estimate in problem 1c is exactly equal to the error you found in problem 1b.

$$e_{k+1} \approx \frac{q}{q-1}(x_{k+1} - x_k)$$

we know that $g(x_k) = x_{k+1} = qx_k + b$.

$$e_{k+1} \approx \frac{q}{q-1}(g(x_k) - x_k) = \frac{q}{q-1}(qx_k + b - x_k)$$

$e_{k+1} \approx q x_k + q\left(\frac{b}{q-1}\right)$, but $\bar{x} = b/_{1-q}$

$$e_{k+1} = q x_k - q\bar{x} = q(x_k - \bar{x})$$

$e_{k+1} = q(x_k - \bar{x})$, take $x_k - \bar{x} = e_k$

$$e_{k+1} = q e_k = q^2 e_{k-2} = q^3 e_{k-3} \cdots = q^k e_0$$

there fore $e_{k+1} = a^k e_0$

e) How many iterations does the fixed point algorithm require to solve $g(x) = \frac{1}{10}x + 1 = x$ to a tolerance of 10^{-8}?

from $g(x) = \frac{1}{10}x + 1$, $\Rightarrow a = \frac{1}{10}$, $b = 1$

tolerance, $\varepsilon = 10^{-8}$

Using $|e_{k+1}| \leq \varepsilon$

but $e_{k+1} = a^k e_0$

$a^k e_0 \leq \varepsilon$

Introducing log both sides

$\cancel{a^k \log e_0}$ $k \log a + \log|e_0| \leq \log \varepsilon$

$k \log\left(\frac{1}{10}\right) + \log|e_0| \leq \log 10^{-8}$

$-k \log 10 \leq -8 \log 10 - \log|e_0|$

$k \geq 8 + \log|(e_0)|$

$\cancel{\therefore}$ The fixed point algorithm require atleast 8 iterations to solve $\cancel{\text{fx}}$ $g(x) = \frac{1}{10}x + 1$

$k \geq 8 + \log|(e_0)|$

and thus depend on $\log|(e_0)|$

2 Steffensen's Method.

$$x_{k+1} = x_k - \frac{(g(x_k) - x_k)^2}{g(g(x_k)) - 2g(x_k) + x_k}$$

a) Show ~~that~~ analytically that for any $g(x)$, the Iteration Used in Steffensen's Method Satisfies

$$\lim_{x \to \bar{x}} \left(x - \frac{(g(x_k) - x_k)^2}{g(g(x_k)) - 2g(x_k) + x_k} \right) = \bar{x}$$

Where $\bar{x}$ Satisfies $g(\bar{x}) = \bar{x}$

Using Lahopital's Rule.

$$\lim_{x \to \bar{x}} \left(x - \frac{2(g'(x) - 1)(g(x) - \bar{x})}{g'(x)g'(g(x)) - 2g'(x) + 1} \right)$$

but $g(\bar{x}) = \bar{x}$

$$= \bar{x} - \frac{2(g'(\bar{x}) - 1)(\bar{x} - \bar{x})}{g'(\bar{x})g'(g(\bar{x})) - 2g'(\bar{x}) + 1}$$

$$= \bar{x} - 0$$

$$= \bar{x}$$

b) Show analytically that Steffensen's iteration Converges in one Step for the fixed point problem.

$$g(x) = ax + b = x \quad , \text{for } |a| < 1$$

from $$x_{k+1} = x_k - \frac{(g(x) - x)^2}{g(g(x)) - 2g(x) + x}$$

at $k=0$

$$x_1 = x_0 - \frac{(g(x_0) - x_0)^2}{g(g(x_0)) - 2g(x_0) + x_0}$$

~~x_1~~ ~~A~~ $g(x_0) = ax_0 + b$

$$x_1 = x_0 - \frac{(ax_0 + b - x_0)^2}{a(ax_0 + b) + b - 2(ax_0 + b) + x_0}$$

$$x_1 = x_0 - \frac{(a-1)(x_0) + b}{(a-1)}$$

$$x_1 = x_0 - x_0 + \frac{b}{a-1} = x_0 - x_0 - b/a-1$$

$$x_1 = \frac{b}{1-a} = \bar{x}$$

$$\underline{\underline{x_1 = \bar{x}}}$$

hence it converges in one step.

No. 3 LU Decomposition

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -l_{21} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix}$$

a) Show that

$$E_{41}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -l_{21} & 1 & 0 & 0 \\ -l_{31} & 0 & 1 & 0 \\ -l_{41} & 0 & 0 & 1 \end{bmatrix}$$

$$E_{41}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -l_{41} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -l_{31} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -l_{21} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -l_{41} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -l_{21} & 1 & 0 & 0 \\ -l_{31} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_{41}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -l_{21} & 1 & 0 & 0 \\ -l_{31} & 0 & 1 & 0 \\ -l_{41} & 0 & 0 & 1 \end{bmatrix}$$

b) Choose multipliers l_{i1} so that applying $E_{41}E_{31}E_{21}$ to A zeros out the entries below a_{11}.

$$E_{41}E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -l_{21} & 1 & 0 & 0 \\ -l_{31} & 0 & 1 & 0 \\ -l_{41} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix}$$

Choose
$$l_{21} = -1$$
$$l_{31} = 2$$
$$l_{41} = -3$$

$$E_{41}E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 10 & 4 & -9 \\ 0 & -16 & -11 & 18 \end{bmatrix}$$

---

c) Show that the inverse of $E_{41}E_{31}E_{21}$ is

$$(E_{41}E_{31}E_{21})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & 0 & 1 & 0 \\ l_{41} & 0 & 0 & 1 \end{bmatrix}$$

$$(E_{41}E_{31}E_{21})^{-1} = \frac{1}{\det(E_{41}E_{31}E_{21})}\left(\text{Adjoint}(E_{41}E_{31}E_{21})\right)$$

$$\det(E_{41}E_{31}E_{21}) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -l_{21} & 1 & 0 & 0 \\ -l_{31} & 0 & 1 & 0 \\ -l_{41} & 0 & 0 & 1 \end{vmatrix} = -1\begin{vmatrix} 1 & 0 & 0 \\ -l_{41} & 1 & 0 \\ -l_{31} & 0 & 1 \end{vmatrix} = \underline{\underline{1}}$$

the Adjoint $(E_{41}E_{31}E_{21})$ is the transpose of the cofactor Matrix $(E_{41}E_{31}E_{21})$

$$\text{Cofactor Matrix} = \begin{bmatrix} 1 & l_{21} & l_{31} & l_{41} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Adjoint}(E_{41}E_{31}E_{21}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & 0 & 1 & 0 \\ l_{41} & 0 & 0 & 1 \end{bmatrix}$$

$$|E_{41}E_{31}E_{21}| = \frac{1}{1}\begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & 0 & 1 & 0 \\ l_{41} & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & 0 & 1 & 0 \\ l_{41} & 0 & 0 & 1 \end{bmatrix}$$

3(d) From the above steps, derive the LU decomposition of a general 4X4 Matrix. Use these to derive the LU decomposition of the matrix in (3).

For U

$$E_{41}E_{31}E_{21}A = U_1$$

$$(E_{42}E_{32})\,U_1 = U_2$$

$$E_{43}U_2 = U$$

So

$$\cancel{U = E_{43}E_{42}E_{41}E_{32}E_{31}E_{21}A}$$

$$U = E_{43}E_{42}E_{32}E_{41}E_{31}E_{21}A$$

~~L~~ for L

$$(E_{41}E_{31}E_{21})^{-1} = L_1$$

$$(E_{42}E_{32})^{-1}L_1 = L_2$$

$$(E_{43})^{-1}L_2 = L$$

So

$$L = (E_{43}E_{42}E_{32}E_{41}E_{31}E_{21})^{-1}$$

Suppose we have $(AB)^{-1} = A^{-1}B^{-1}$

$$L = (E_{43}E_{42}E_{32}E_{41}E_{31}E_{21})^{-1} = (E_{43}E_{42}E_{32})^{-1}(E_{41}E_{31}E_{21})^{-1}$$

but $(E_{41}E_{31}E_{21})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & 0 & 1 & 0 \\ l_{41} & 0 & 0 & 1 \end{bmatrix}$

$$(E_{43}E_{42}E_{32})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & L_{32} & 1 & 0 \\ 0 & L_{32}L_{43}+L_{42} & L_{43} & 1 \end{bmatrix}$$

$$L = (E_{43}E_{42}E_{32})^{-1}(E_{41}E_{31}E_{21})^{-1}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & L_{32} & 1 & 0 \\ 0 & L_{32}L_{43}+L_{42} & L_{43} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ L_{21} & 1 & 0 & 0 \\ L_{31} & 0 & 1 & 0 \\ L_{41} & 0 & 0 & 1 \end{bmatrix}$$

Let $L_{21} = a \qquad L_{32} = d$

$L_{31} = b \qquad L_{42} = e$

$L_{41} = c \qquad L_{43} = f$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ da+b & d & 1 & 0 \\ fb+a(df+e)+c & df+e & f & 1 \end{bmatrix}$$

And $U = E_{43}E_{42}E_{32}E_{41}E_{31}E_{21}A$

4a) Find the LU decomposition of the matrix in (3)

$$A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix}$$

$$A = LU$$

where

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

Using Gaussian Elimination method on A, we obtain an upper triangular Matrix U, as follows.

take $l_{21} = -1$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 + R_1} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 5 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix}$$

take $l_{31} = -2$ ~~In this case $L_{21} = -1$~~

$$\begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 5 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_2 + 2R_1} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 5 \\ 0 & -18 & -4 & -1 \\ -9 & 5 & -5 & 12 \end{bmatrix}$$

In ~~this case~~ $L_{31} = -2$

$$\begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & -18 & -4 & -1 \\ -9 & 5 & -5 & 12 \end{bmatrix} \xrightarrow{R_4 \leftarrow R_4 + 3R_1} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & -18 & -4 & -1 \\ 0 & -16 & -11 & 18 \end{bmatrix}$$

In this case we took $L_{41} = -3$

~~for L_{32}~~ take $L_{32} = 9$

$$\begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & -18 & -4 & -1 \\ 0 & -16 & -11 & 18 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - 9R_2} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & 5 & -19 \\ 0 & -16 & -11 & 18 \end{bmatrix}$$

take $L_{42} = 8$

$$\begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & 5 & -19 \\ 0 & -16 & -11 & 18 \end{bmatrix} \xrightarrow{R_4 \leftarrow R_4 - 8R_2} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & 5 & -19 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

take $L_{43} = -3/5$

$$\begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & 5 & -19 \\ 0 & 0 & -3 & 2 \end{bmatrix} \xrightarrow{R_4 \leftarrow R_4 + \frac{3}{5}R_3} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & 5 & -19 \\ 0 & 0 & 0 & -\frac{47}{5} \end{bmatrix}$$

Hence we obtain, U,

$$U = \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & 5 & -19 \\ 0 & 0 & 0 & -47/5 \end{bmatrix}$$

Corresponding to, L,

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 9 & 1 & 0 \\ -3 & 8 & -3/5 & 1 \end{bmatrix}$$

So $A = LU$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 9 & 1 & 0 \\ -3 & 8 & -3/5 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & 5 & -19 \\ 0 & 0 & 0 & -\frac{47}{5} \end{bmatrix}$$

5. Jacobi Method

Assume that the following 2×2 matrix A is strictly diagonally dominant

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

a) Show that $\rho(I - \Delta^{-1}A) < 1$

take $\Delta = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$

$$\Delta^{-1} = \begin{bmatrix} 1/a_{11} & 0 \\ 0 & 1/a_{22} \end{bmatrix}$$

$$\Delta^{-1}A = \begin{bmatrix} 1/a_{11} & 0 \\ 0 & 1/a_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & a_{12}/a_{11} \\ a_{21}/a_{22} & 1 \end{bmatrix}$$

then

$$I - \Delta^{-1}A = \begin{pmatrix} 0 & a_{12}/a_{11} \\ a_{21}/a_{22} & 0 \end{pmatrix}$$

the Spectral radius, ρ, is the maximum absolute value of the eigen values, so to compute the eigen values we use the characteristic equation.

$$\left|(I - \Delta^{-1}A) - \lambda I\right| = 0$$

$$\begin{vmatrix} -\lambda & a_{12}/a_{11} \\ a_{21}/a_{22} & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 = \frac{a_{21}a_{12}}{a_{22}a_{11}}$$

Since a_{22} and a_{11} are diagonally dominant ~~row~~ entries then $a_{22}a_{11} > a_{21}a_{12}$, therefore

$$\lambda^2 = \frac{a_{21}a_{12}}{a_{22}a_{11}} < 1$$

$$\lambda = \pm\sqrt{\frac{a_{21}\,a_{12}}{a_{22}\,a_{11}}}$$

ρ is the largest $|\lambda|$, so

$$|\lambda| = \left|\sqrt{\frac{a_{21}a_{12}}{a_{22}a_{11}}}\right| < 1$$

~~So th~~ So the Spectral radius $\rho(I - D^{-1}A) < 1$

Jacobi Iteration will Converge, since for Jacobi, we take $M = D$, and we have shown that

$\rho(I - M^{-1}A) < 1$, hence the Iteration Converges

Since according for ~~Converge~~ an Iteration Converges if and only if $\rho(I - M^{-1}A) < 1$

NO. 7

$$X_{k+1} = (I - M^{-1}A) X_k + M^{-1} b.$$

a) Show that if $M = A$, the iteration Converges in one step.

If $M = A$

$$X_{k+1} = (I - A^{-1}A) X_k + A^{-1} b$$

but $A^{-1}A = I$

$$X_{k+1} = (I - I) X_k + A^{-1} b$$

$$X_{k+1} = A^{-1} b$$

but $A\bar{X} = b \Rightarrow \bar{X} = A^{-1}b$

therefore $X_{k+1} = \bar{X}$, hence Converges in one step

b) Show that $Ae_k = -r_k$, where $r_k = b - AX_k$,

$e_k = x_k - \bar{x}$,

and $\cancel{\bar{x}}$ solves $AX = b$ exactly

$$Ae_k = A(X_k - \bar{x}).$$

$$Ae_k = AX_k - A\bar{x}$$

we know that $A\bar{x} = b$

$$Ae_k = AX_k - b$$

$$Ae_k = -r_k$$ since $r_k = b - AX_k$.

c) Show that the iteration for the error e_k is given by

$$e_{k+1} = (I - M^{-1}A)e_k$$

from

$$X_{k+1} = (I - M^{-1}A)X_k + M^{-1}b$$

Subtract $\bar{X}$ from both sides

$$X_{k+1} - \bar{X} = (I - M^{-1}A)X_k - \bar{X} + M^{-1}b$$

we know that $X_{k+1} - \bar{X} = e_{k+1}$

$$e_{k+1} = (I - M^{-1}A)X_k - \bar{X} + M^{-1}b$$

take $A\bar{x} = b$.

$$e_{k+1} = (I - M^{-1}A)X_k - \bar{X} + M^{-1}A\bar{x}$$

$$e_{k+1} = (I - M^{-1}A)X_k - (I - M^{-1}A)\bar{x}$$

$$e_{k+1} = [I - M^{-1}A](X_k - \bar{X})$$

but $X_k - \bar{X} = e_k$

$$e_{k+1} = (I - M^{-1}A)e_k$$

---

d) Show that

$$\|e_k\| \leq \|I - M^{-1}A\|^k \|e_0\|$$

from $e_{k+1} = (I - M^{-1}A)e_k$

$$e_{k+1} = (I - M^{-1}A)(I - M^{-1}A)e_{k-1}$$

$$e_{k+1} = (I - M^{-1}A)^2 e_{k-1}$$

$$e_k = (I - M^{-1}A)^2 e_{k-2}$$

$$e_k = (I - M^{-1}A)^3 e_{k-3}$$

$$\vdots \qquad \vdots$$

$$e_k = (I - M^{-1}A)^k e_0$$

take the Modulus both sides, and then apply the triangular inequality

$$\|e_k\| = \|(I - M^{-1}A)^k e_0\|$$

$$\|e_k\| \le \|(I - M^{-1}A)\|^k \|e_0\|$$

---

8) Show that

$$K \geq \frac{\log(\varepsilon)}{\log\|I - M^{-1}A\|}$$

from $\|e_k\| \le \varepsilon$

~~but $\|e_k\| \le \|(I - M^{-1}A)\|^k \|e_0\|$~~

but $e_k = (I - M^{-1}A)^k e_0$

$$\|(I - M^{-1}A)^k\| \|e_0\| \le \varepsilon$$

taking log both sides we obtain

$$k\log\|I - M^{-1}A\| + \log\|e_0\| \le \log \varepsilon$$

~~Since~~

$$k \log \|I - M^{-1}A\| \leq \log \varepsilon - \log \|e_0\|$$

Since $\log \|I - M^{-1}A\| < 0$, then

$$K \geq \frac{\log \varepsilon - \log e_0}{\log \|I - M^{-1}A\|}$$

$$K \geq \frac{\log \varepsilon}{\log \|I - M^{-1}A\|} - \frac{\log \|e_0\|}{\log \|I - M^{-1}A\|}$$

The term $\frac{\log(\varepsilon)}{\log \|I - M^{-1}A\|}$ dominates therefore

$$K \geq \frac{\log(\varepsilon)}{\log \|I - M^{-1}A\|}$$

f) Since $\rho(I - M^{-1}A)$ is the largest absolute value of the eigen values of $(I - M^{-1}A)$, and the number of iterations is given by

$$K \geq \frac{\log \varepsilon}{\log \|I - M^{-1}A\|} - \frac{\log \|e_0\|}{\log \|I - M^{-1}A\|}$$

So for larger values of $\rho(I - M^{-1}A)$ implies the term $\dfrac{\log \|e_0\|}{\log \|I - M^{-1}A\|}$ is also large. This is because $\log\|I - M^{-1}A\|$ ~~is~~ will be very small to reduce $\log\|e_0\|$.

But again, as $\log\|I - M^{-1}A\|$ approaches to zero from the left side, it means that this term $\dfrac{\log\|e_0\|}{\log\|I - M^{-1}A\|}$ increases and eventually become undefined at $\log\|I - M^{-1}A\| = 0$, this case the solution ~~no~~ no longer exists, so the the estimate iterations k will seem to underestimate the actual number of k i.e.

$$k \geq \frac{\log(\epsilon)}{\log\|I - M^{-1}A\|} + \text{a very big term}$$

after which become undefined.

9) Show analytically that the update to the residual r_{k+1} in the Conjugate Gradient Algorithm is equal to $b - Ax_k$

Suppose A is symmetric, positive definite

$$F(x) = \frac{1}{2} x^T A x - b^T x$$

So the direction of the greatest decrease of f is given by $-\nabla F(x_{k+1}) = r_{k+1}$ of the update to residual r_{k+1}

So taking the residual as the search direction we have
$$x_{k+1} = x_k + a_k r_k$$

$$-\nabla F(x_{k+1}) = r_{k+1}$$

$$-\nabla F(x_k + a_k r_k) = r_{k+1}$$

$$F(x_k + a_k r_k) = \frac{1}{2}(x_k + a_k r_k)^T A (x_k + a_k r_k) - b^T(x_k + a_k r_k)$$

$$\nabla F(x_k + a_k r_k) = \frac{dF}{dx_k} = \frac{1}{2}(2Ax_k) - b$$

$$\nabla F(x_k + a_k r_k) = Ax_k - b$$

$$-\nabla F(x_k + a_k r_k) = b - Ax_k = r_{k+1}$$

therefore $\underline{r_{k+1} = b - Ax_k}$