1. Find the rate of Convergence of the Sequence

$$S_n = \frac{\sin(n)}{n}, \text{ as } n \to \infty$$

Since

$$-1 \leq \sin(n) \leq 1$$

diving through by n

$$-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq 1/n \Rightarrow -\frac{1}{n} \leq S_n \leq 1/n$$

Taking the $\lim\limits_{n\to\infty}$ through out we have,

$$-\lim_{n\to\infty} \frac{1}{n} \leq \lim_{n\to\infty} S_n \leq \lim_{n\to\infty} \frac{1}{n}$$

Since $\lim\limits_{n\to\infty} \left(-\frac{1}{n}\right) = \lim\limits_{n\to\infty} \left(\frac{1}{n}\right) = 0$, then by Sandwitch theorem, the $\lim\limits_{n\to\infty} S_n = 0$

hence, $L = \lim\limits_{n\to\infty} S_n = 0$

Rate of Convergence, β_n,

$$|S_n - L| \leq \lambda |\beta_n|$$

$$\left|\frac{\sin(n)}{n} - 0\right| \leq \sin(n) |1/n|$$

Therefore the rate of Convergence, β_n, is $O(1/n)$

2. Show that the Sequence

$$S_n = \frac{1}{n^2}, \text{ Converges linearly.}$$

For linear Convergence,

$$\lim_{n\to\infty} \cancel{\left|\frac{S_{n+1} - S}{S_n - S}\right|} \quad \lim_{n\to\infty} \frac{|S_{n+1} - S|}{|S_n - S|^\alpha} = \lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^\alpha} = \lambda$$

~~for~~ where $\alpha = 1$ and $0 < \lambda \leq 1$.

therefore $S_n = \frac{1}{n^2}$, $S_{n+1} = \frac{1}{(n+1)^2}$

and, Limit, $S_1 = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{n^2}\right) = \frac{1}{\infty} = 0$

So,

$$\lim_{n\to\infty} \frac{|S_{n+1} - S|}{|S_n - S|^\alpha} = \lim_{n\to\infty} \frac{|S_{n+1}|}{|S_n|^\alpha} = \lim_{n\to\infty} \frac{1/(n+1)^2}{\left(\frac{1}{n^2}\right)^\alpha} = \cancel{\left(\frac{n^\alpha}{n+1}\right)^2}$$

$$= \lim_{n\to\infty}\left(\frac{n^\alpha}{n+1}\right)^2$$

If $\alpha = 1$, then $\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 = 1 > 0$

Hence, the Sequence Converges linearly for $\alpha = 1$

3. $\lim_{x\to 0} \dfrac{e^x - \cos x - x}{x^2}$

Using taylor Series expansion

$$e^x = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} e^{\xi}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\cos(\xi_1) \quad \text{for } 0 < \varepsilon < 1$$

then

$$\lim_{x\to 0} \frac{e^x - \cos x - x}{x^2} = \lim_{x\to 0} 1 + \frac{x}{3!} + \frac{x^3}{5!} e^{\xi} + \frac{x^4}{6!}\cos(\xi_1)$$

$$\underline{\underline{L = 1}}$$

Corresponding rate of Convergence, β_n,

$$|S_n - L| \leq \lambda |\beta_n| \text{ , from definition}$$

$$\left| \frac{e^x - \cos x - x}{x^2} - 1 \right| = x/3! + \frac{x^5}{5!} e^{\varepsilon} + \frac{x^4}{6!}\cos(\xi_1)$$

$$\left| \frac{e^x - \cos x - x}{x^2} - 1 \right| \leq \frac{1}{3!} |x|$$

The rate of Convergence is $O(x)$

4.
a). At ~~$K = 54$~~
~~$X = 2.2$~~

5(a) Analytical means

$L = \lim_{x \to 0} \frac{\sqrt{1+x^2}-1}{x^2} = \frac{1}{2}$, where L is the limit

$$\left| f(x) - \frac{1}{2} \right| \leq \frac{1}{8} x^2, \quad — ①$$

Comparing equation ① with the definition of rate of convergence

$$\left| f(x) - L \right| \leq \lambda \left| \beta_n \right|$$

we conclude that $\beta_n = x^2$ have the rate of convergence is $O(x^2)$ as $x \to 0$

5b) from Remainder theorem

$F(x) = P(x) + R(x)$, where $R(x)$ is the remainder.

Using taylor series at $x = 0$

$$f(x) = \frac{1}{2} - \frac{x^2}{8} + \frac{x^4}{16} + O(x^6)$$

taking, $P(x) = \frac{1}{2}$

$$R(x) = -\frac{x^2}{8} + \frac{x^4}{16} + O(x^6)$$

from definition

$$\left| R_n(x) \right| \leq M \frac{|x-a|^{n+1}}{(n+1)!}$$

therefore,

$$|R(x)| = \left| -\frac{x^2}{8} + \frac{x^4}{16} + O(x^6) \right|$$

$$|R(x)| \leq \frac{1}{8}|x^2|$$

Here β_n is x^2, rate of Convergence is $O(x^2)$

and $\lambda = 1/8$, which is a reasonable choice.

7b) from the ~~eq~~ question,

$$e_n(x) = \left| S_n(x) - \frac{1}{1+x} \right|$$

Comparing with the definition of $|S_n - L| \leq \lambda |\beta_n|$

we obtain, $L = \frac{1}{1+x}$, as ~~n→∞~~

for linear Convergence $\alpha = 1$, $0 < \lambda \leq 1$

$$\lambda = \lim_{n\to\infty} \left| \frac{e_{n+1}}{e_n} \right|$$

but using a geometric series

$$S_n = \frac{1-(-x)^{n+1}}{1+x}$$

So $$e_n = \left| S_n(x) - \frac{1}{1+x} \right| = \left| \frac{1-(-x)^{n+1}}{1+x} - \frac{1}{1+x} \right|$$

$$e_n = \frac{-(-x)^{n+1}}{1+x}, \quad e_{n+1} = \frac{-(-x)^{n+2}}{1+x}$$

$\cancel{\lambda = \lim_{n\to\infty} \left|\frac{e_{n+1}}{e_n}\right| \neq}$, for linear Convergence $\alpha = 1$

$$\lambda = \lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^{\alpha}} = \lim_{n\to\infty} \left|\frac{e_{n+1}}{e_n}\right|$$

~~for linear~~

$$\lambda = \left|\frac{(-x)^{n+2}}{(-x)^{nn}}\right| = |-x|$$

$$\lambda = x$$

Hence the sequence Converges linearly since $\lambda = x$ and $x = 1/\pi$, therefore $0 < \lambda \leq 1$

In [1]:
```
%matplotlib notebook
%pylab
```

```
Using matplotlib backend: nbAgg
Populating the interactive namespace from numpy and matplotlib
```

4a)

At k=52, The value of the iterate,x, is x=2.220446049250313le-16

Binary: 1×2^{-52}

64-bit representation is,

0 | 0 1 1 1 1 0 0 1 0 1 1 | 0 0 0 ... 0 0 0

The number is called machine epsilon

b)

iterate: k=1022

Binary: 1×2^{-1022}

0 | 0 0 0 0 0 0 0 0 0 0 1 | 0 0 0 0 ... 0 0 0 0

c)

Binary: 1×2^{-1074}

IEEE floating point convention, Since numbers smaller than the realmin are called denormalised numbers. For denormalised numbers, exponent,E, is zero (E=0), so therefore from E=e+1022, then e=-1022. Also drop the assumed 1.

From the binary, for the mantisa e=-1074

E = e+1022=-52, hence b_{52} =1, bi=0 for i=1,2,...,51

so, the 64-bit representation is,

0 | 0 0 0 0 0 0 0 0 0 0 0 | 0 0 0 ... 0 0 1

d)

The resulting value of x would be: 0.0000000000000000e+00

This because of finite precision (16 digits) round off error, so 1×2^{-1075} needs more digits morethan the finite precision 16 digits. And also since there is no significant number within the 16 digits, so after the rounding off we end up with zero.

e)

The number less than the realmin: 4.146142e-316

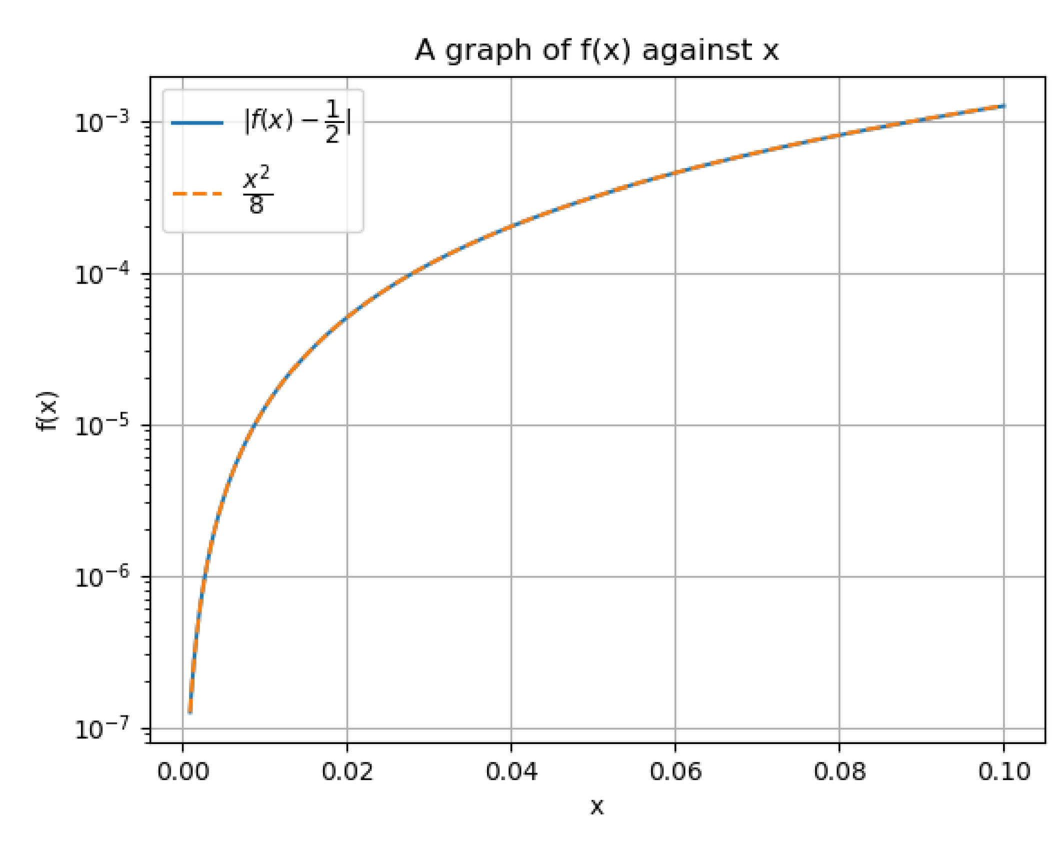
5.a)

Graphical means

In [4]:
```
x = logspace(-3,-1,500) #small enough values of x
f2=(x**2)/8

def f(x):
    return (abs(((sqrt(1+(x**2))-1)/(x**2))-(1/2)))
f1=f(x)

figure(1)
plot(x,f1,label='$|f(x)-\dfrac{1}{2}|$')
plot(x,f2,'--',label='$\dfrac{x^{2}}{8}$')
yscale('log')
title('A graph of f(x) against x')
ylabel('f(x)')
xlabel('x')
grid()
legend()
show()
```



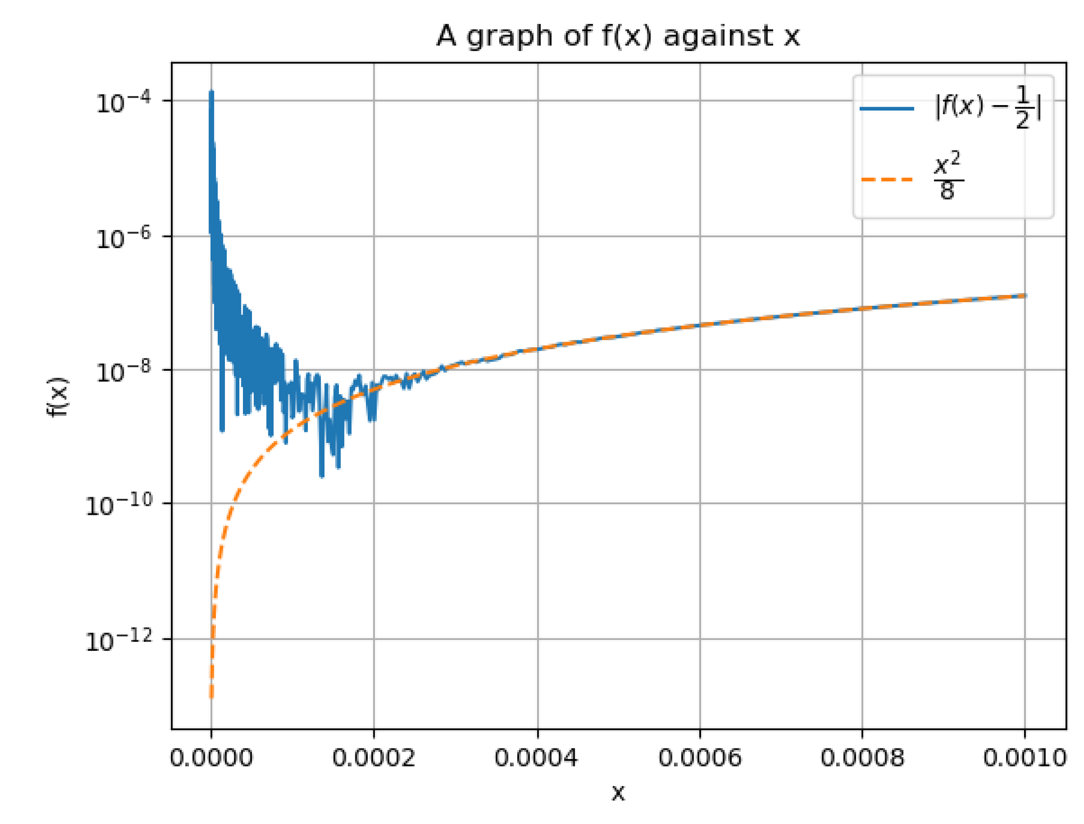
Since all the plotted functions coincide on the same ploting points for x close to zero in the format of the parabola $f(x) = kx^2$, where k is a positive constant, then f(x) converges like O(x^2) as x tends to zero.

N0.6

In [62]:
```
x1 = logspace(-6,-3,500)

f1=f(x1)
f2=(x1**2)/8

figure(2)
plot(x1,f1,label='$|f(x)-\dfrac{1}{2}|$')
plot(x1,f2,'--',label='$\dfrac{x^{2}}{8}$')
yscale('log')
#ylim(0.15, 1)
title('A graph of f(x) against x')
ylabel('f(x)')
xlabel('x')
grid()
legend()
show()
```



a)

The size of the next term is;

$\frac{\epsilon^4}{8}$=1.2500000000000003e-17

We are justified in approximating $g(\epsilon)$ as $1 + \frac{\epsilon^2}{2}$, since $\frac{\epsilon^4}{8}$ is less than the realmin, such numbers wen added to 1, we still obatin 1, meaning that even if we ignore such term, it doesnot matter much.

b)

In [7]:
```
ep=10**-4 #epsilon
```

In [8]:
```
a=0.5*ep**2
a
```

Out[8]:
```
5e-09
```

In [10]:
```
b=(a)+1
b
```

Out[10]:
```
1.000000005
```

In [11]:
```
c=b-1
c
```

Out[11]:
```
4.999999969612645e-09
```

In [12]:
```
d=ep**2
d
```

Out[12]:
```
1e-08
```

In [13]:
```
c/d
```

Out[13]:
```
0.4999999969612645
```

Ideally, b-1 would have given us 0.000000005, but instead we obtain c, this is because we are subtracting values that are very close in magnitude. And as we keep an applying operations to very small numbers in magnitude, roundoff error keep on accumulating resulting in catastropic loss of accuracy, hence gabbage digits appearing in the calculations.

c)

In [20]:
```
def f(x): #define function
    return ((1+0.5*x**2)-1)/x**2

e=[10**-3,10**-4, 10**-5, 10**-6, 10**-7,10**-8,10**-9,10**-10,10**-20]

for i in e:
    print(f(i))
```

```
$f(\epsilon)$=
0.500000000069889
0.4999999969612645
0.5000000413701854
0.5000444502911705
0.5107025913275721
0.0
0.0
0.0
0.0
```

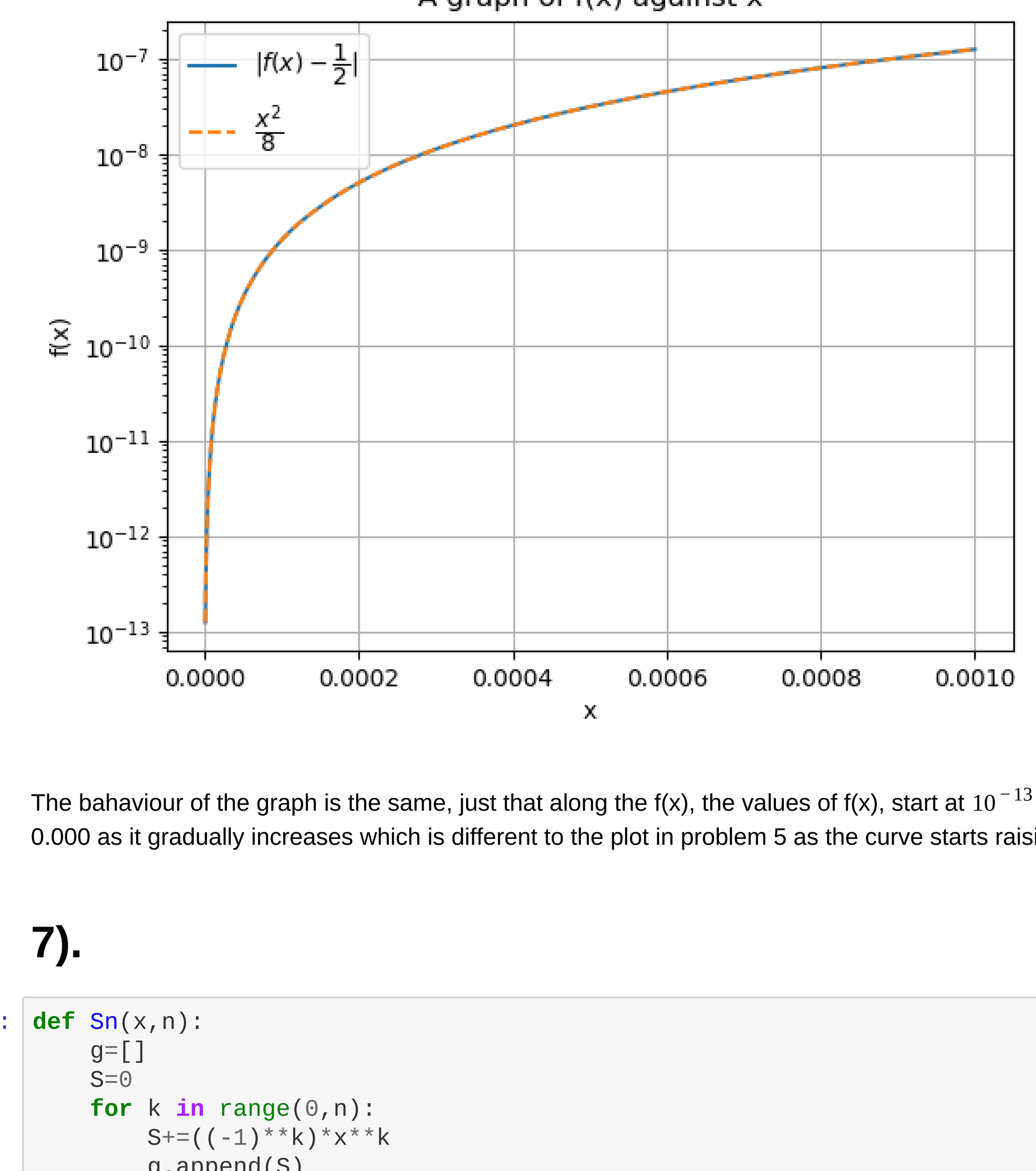
Its clear form the values of f(i) which are the values of $f(\epsilon)$, above in the code that at $\epsilon = 1 \times 10^{-8}$ the values of $f(\epsilon)$) becomes 0, no digits of accuracy in evaluation of $f(\epsilon)$

d).

In [70]:
```
x1 = logspace(-6,-3,500)
f2=(x1**2)/8

def fn(x):
    return (abs(0.5-(1/8)*x**2+(1/16)*x**4-(5/128)*x**6-0.5))
f3=fn(x1)

figure(3)
plot(x1,f3,label='$|f(x)-\dfrac{1}{2}|$')
plot(x1,f2,'--',label='$\dfrac{x^{2}}{8}$')
yscale('log')
#ylim(0.15, 1)
title('A graph of f(x) against x')
ylabel('f(x)')
xlabel('x')
grid()
legend()
show()
```



The bahaviour of the graph is the same, just that along the f(x), the values of f(x), start at 10^{-13} but is the curve leans alot on the line x= 0.000 as it gradually increases which is different to the plot in problem 5 as the curve starts raising at f(x)=10^{-7}.

7).

In [22]:
```
def Sn(x,n):
    g=[]
    S=0
    for k in range(0,n):
        S+=((-1)**k)*x**k
        g.append(S)
    return g

seq=Sn(1/pi,10)# Sequence at n=10 and x=1/pi
print('Sn(x) x=1/pi =',seq)
```

```
Sn(x) x=1/pi = [1.0, 0.6816901138162093, 0.783011297458547, 0.7507597630253475, 0.7610257452800319,
0.7577579816369785, 0.7587981431102744, 0.7584670494300968, 0.7585724398217503, 0.7585388930181782]
```

In [24]:
```
#Error sequence,en,
en = abs(array(seq) - (1/(1+(1/pi))))
print('en(x)=',en)
```

```
en(x)= [2.41453007e-01 7.68568792e-02 2.44643045e-02 7.78722997e-03
 2.47875229e-03 7.89011358e-04 2.51150115e-04 7.99435647e-05
 2.54468270e-05 8.09997660e-06]
```

In [28]:
```
#error ratios (Asymptotic error constatnt,lambda,)
g=[]
for j in range(len(en)-1):
    b=en[j+1]/en[j]
    g.append(b)
print('lambda=',g)
x=1/pi
print('x=',x)
```

```
lambda= [0.31830988618379086, 0.31830988618378975, 0.318309886183794, 0.31830988618377976, 0.31830988
61838175, 0.3183098861837569, 0.3183098861839281, 0.31830988618299383, 0.31830988618457917]
x= 0.3183098861837907
```

Since values of lambda are almost similar to the value of x, this means that the asymptotic error constant λ converges linearly with x.