Fixed Point Iteration

Fixed Point Iteration

Idea: Solve $f(x)=0$ by finding "fixed points" of a related function $g(x)$.

Example: Solve $f(x) = \cos(x) - x = 0$

Convert to "fixed point problem"

$$\left.\begin{array}{l} \underbrace{\cos(x)}_{g(x)} = x \\ g(x) = x \end{array}\right\} \quad g(x) = x \iff f(x) = \cos(x) - x = 0$$

To find fixed points of $g(x)$, we use a simple iteration scheme

$$\boxed{x_{k+1} = g(x_k)}$$

Stop when $|x_{k+1} - x_k| < \varepsilon$. Then

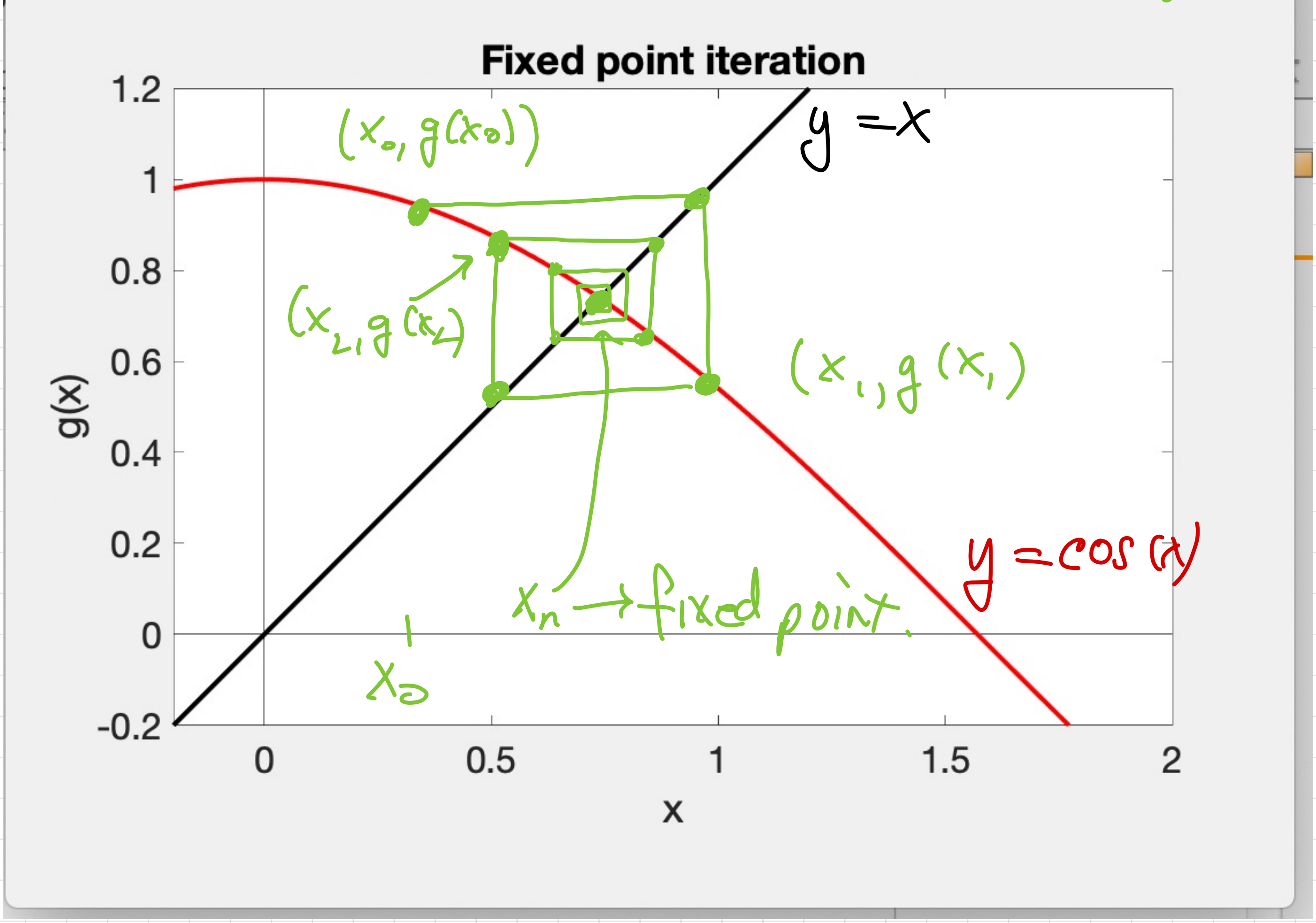
$$g(x_{k+1}) \approx x_k$$

or

$$\cos(x_{k+1}) \approx x_k$$

Fixed Point Iteration

$x_0 = 0.25$
$x_1 = g(0.25)$
$x_2 = g(x_1)$



Graphical evidence suggests the iteration converge, at least in this case

$$x_{k+1} = g(x_k)$$

"looks like it converges for $g(x) = \cos(x)$"

Fixed Point Algorithm

```
function fixed_point()

g = @(x) cos(x);

tol = 1e-5;
xk = 0.1;
kmax = 100;
for k = 1:kmax
    xkp1 = g(xk);
    if abs(xkp1-xk)< tol
        fprintf('Tolerance achieved\n');
        xroot = xkp1;
        break;
    end
    xk = xkp1;
end
fprintf('\n');
fprintf('Root is %24.16f\n',xkp1);
fprintf('Number of iterations : %d\n',k);

end
```

% specify $g(x)$ NOT $f(x)$

$g(x) = x$

✓ • Only one function call per iteration

✓ • Simple stopping criteria

✗ • No guarantee of convergence.

$g(x) = \cos(x) = x$ **Convergence**

K | X_K | $|X_{k+1} - X_k|$ ← stop when this is small!

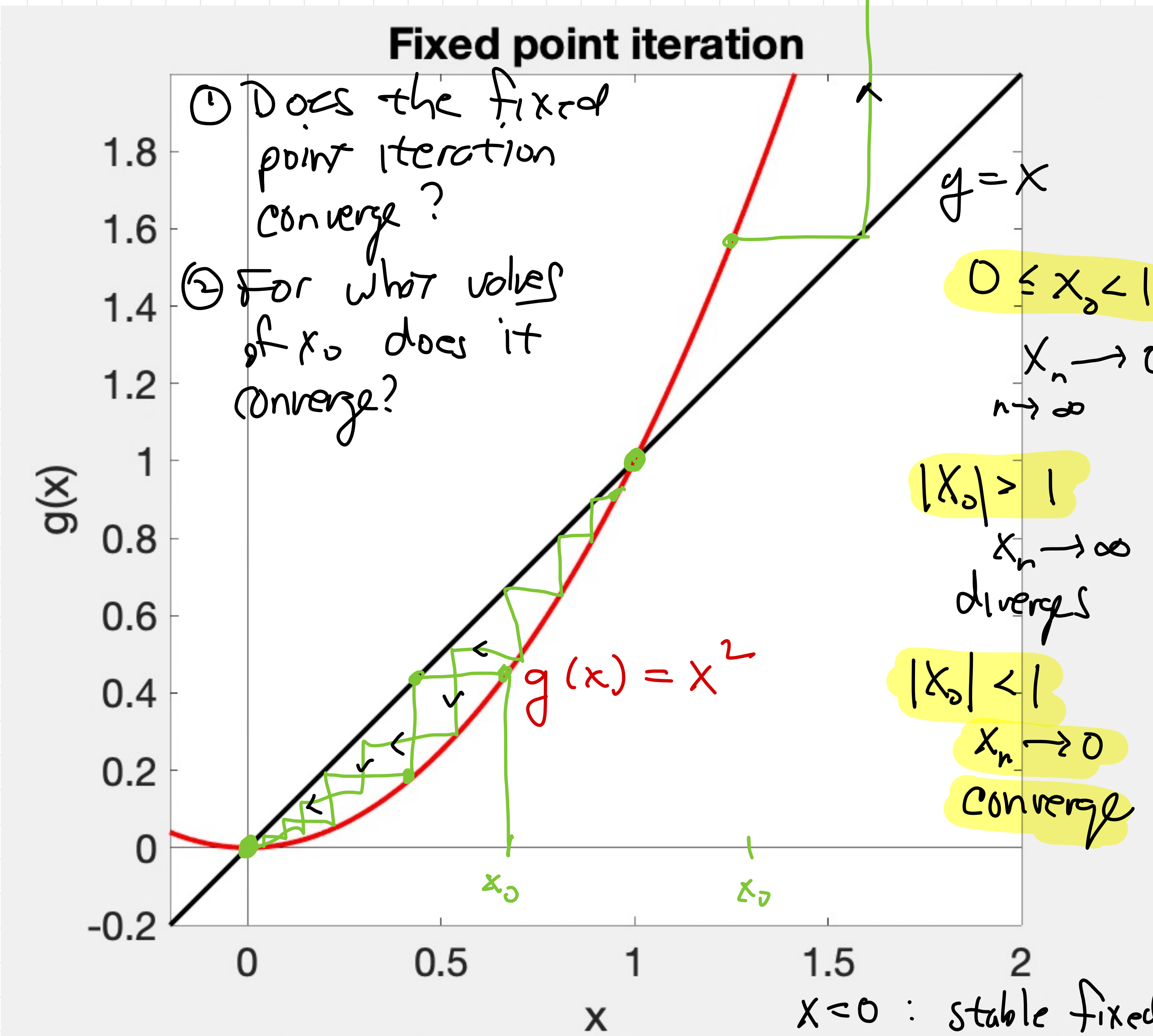
```
>> fixed_point
     1   0.9950041652780257   8.9500e-01
     2   0.5444993958277885   4.5050e-01
     3   0.8553867058793604   3.1089e-01
     4   0.6559266636704799   1.9946e-01
     5   0.7924831019448094   1.3656e-01
     6   0.7020792679906702   9.0404e-02
     7   0.7635010336918855   6.1422e-02
     8   0.7224196362389732   4.1081e-02
     9   0.7502080588752906   2.7788e-02
    10   0.7315470320442240   1.8661e-02
    11   0.7441418423459107   1.2595e-02
    12   0.7356694383362791   8.4724e-03
    13   0.7413816704611964   5.7122e-03
    14   0.7375362104631451   3.8455e-03
    15   0.7401276192037985   2.5914e-03
    16   0.7383825006298149   1.7451e-03
    17   0.7395582524973968   1.1758e-03
    18   0.7387663516682054   7.9190e-04
    19   0.7392998307427002   5.3348e-04
    20   0.7389404933450101   3.5934e-04
    21   0.7391825566401676   2.4206e-04
    22   0.7390195041168074   1.6305e-04
    23   0.7391293401735582   1.0984e-04
    24   0.7390553541540397   7.3986e-05
    25   0.7391051924212390   4.9838e-05
    26   0.7390716209439578   3.3571e-05
    27   0.7390942351761125   2.2614e-05
    28   0.7390790019941593   1.5233e-05
    29   0.7390892632654888   1.0261e-05
    30   0.7390823511572751   6.9121e-06
Tolerance achieved

Root is        0.7390823511572751
Number of iterations : 30
```

tolerance set to 10^{-5}

?

- Convergence depends on function!

Convergence?



① Does the fixed point iteration converge?

② For what values of x_0 does it converge?

$0 \le x_0 < 1$

$x_n \to 0$ as $n \to \infty$

$|x_0| > 1$

$x_n \to \infty$ diverges

$|x_0| < 1$

$x_n \to 0$ converge

$x = 0$: stable fixed point

$x = 1$: unstable fixed point

Fixed points:

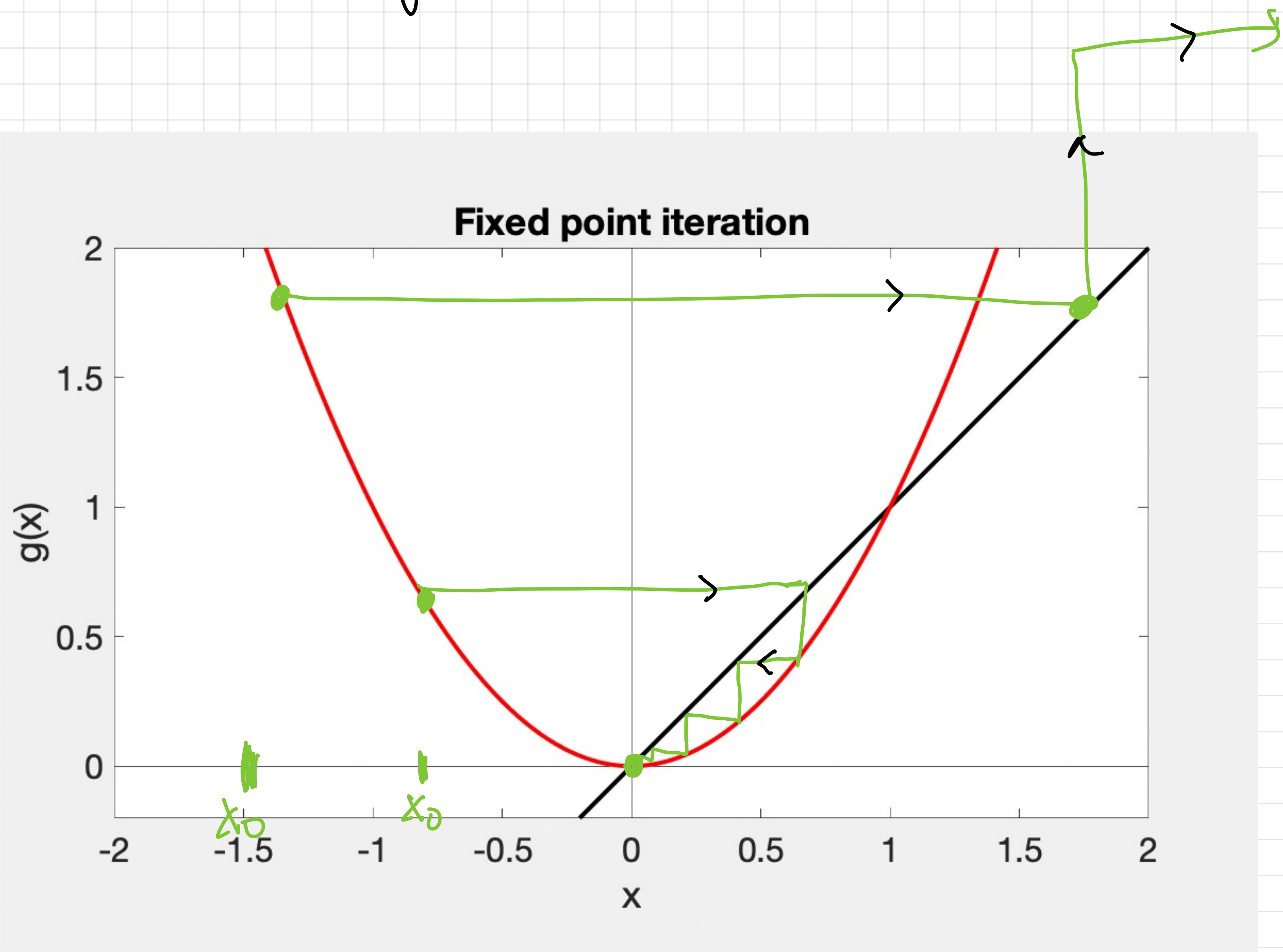
$$g(x) = x^2$$

$$g(x) = x$$

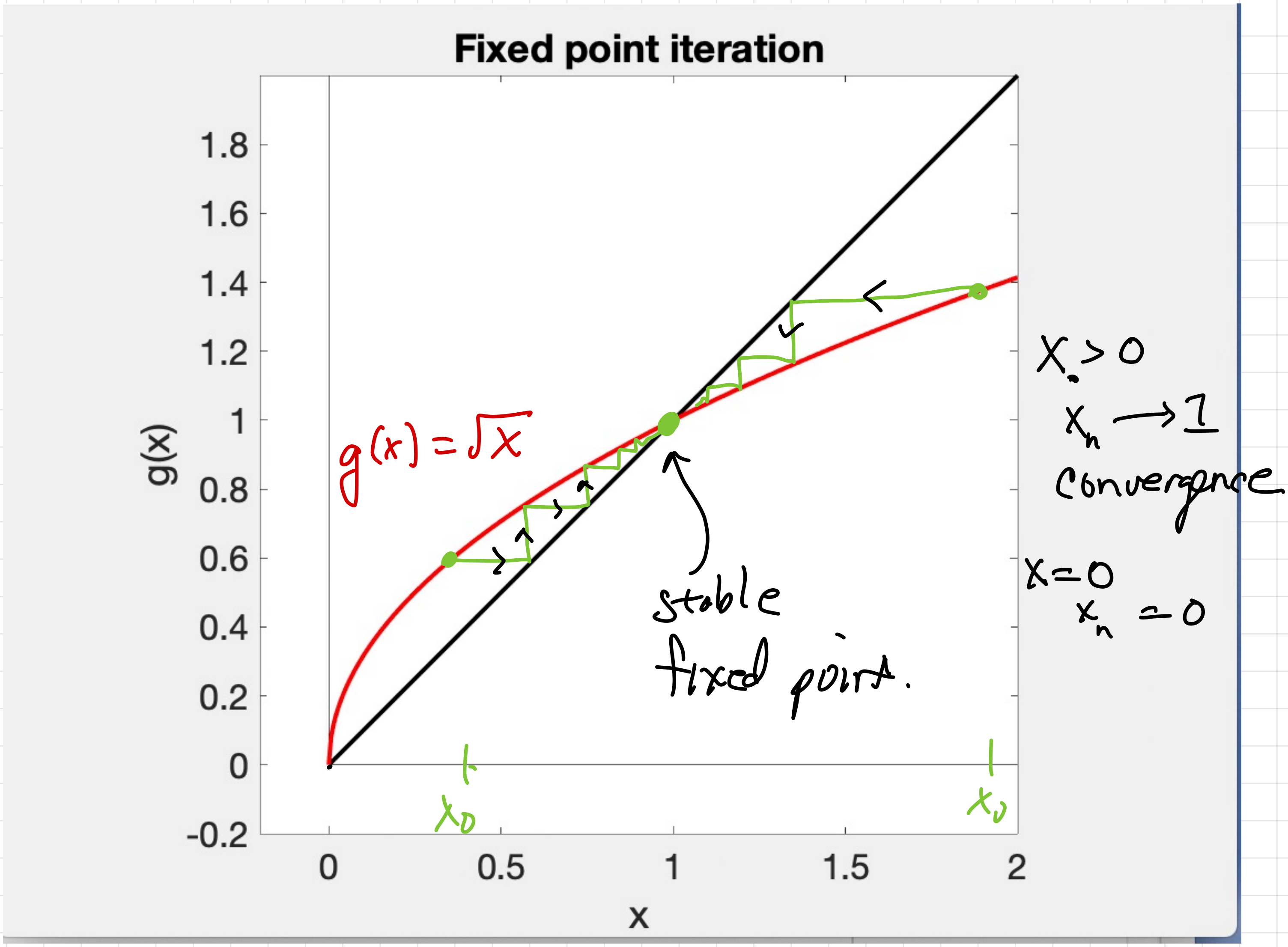
$x^2 = x \qquad x^2 - x = 0 \qquad x(x-1) = 0$

$x = 0, \quad x = 1$

$g(x) = x^2$



Convergence of fixed point



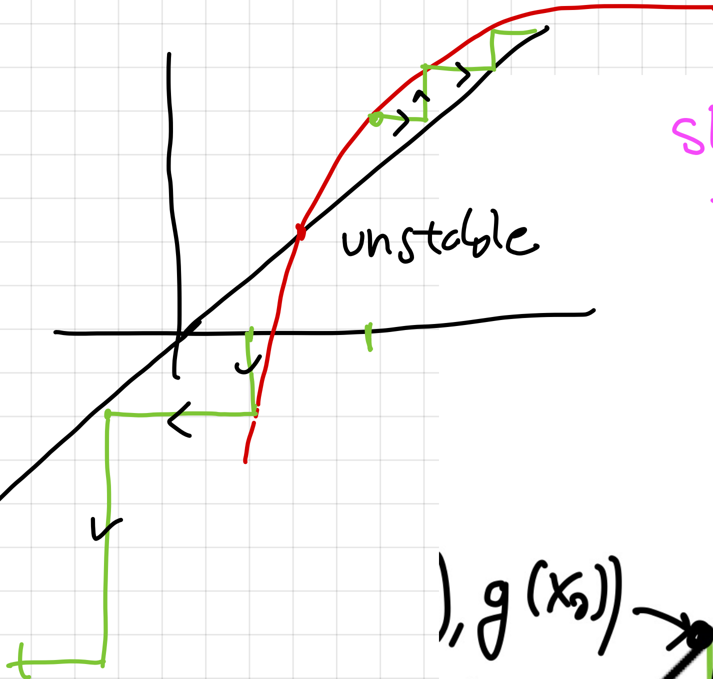
$x = 1$: stable fixed point

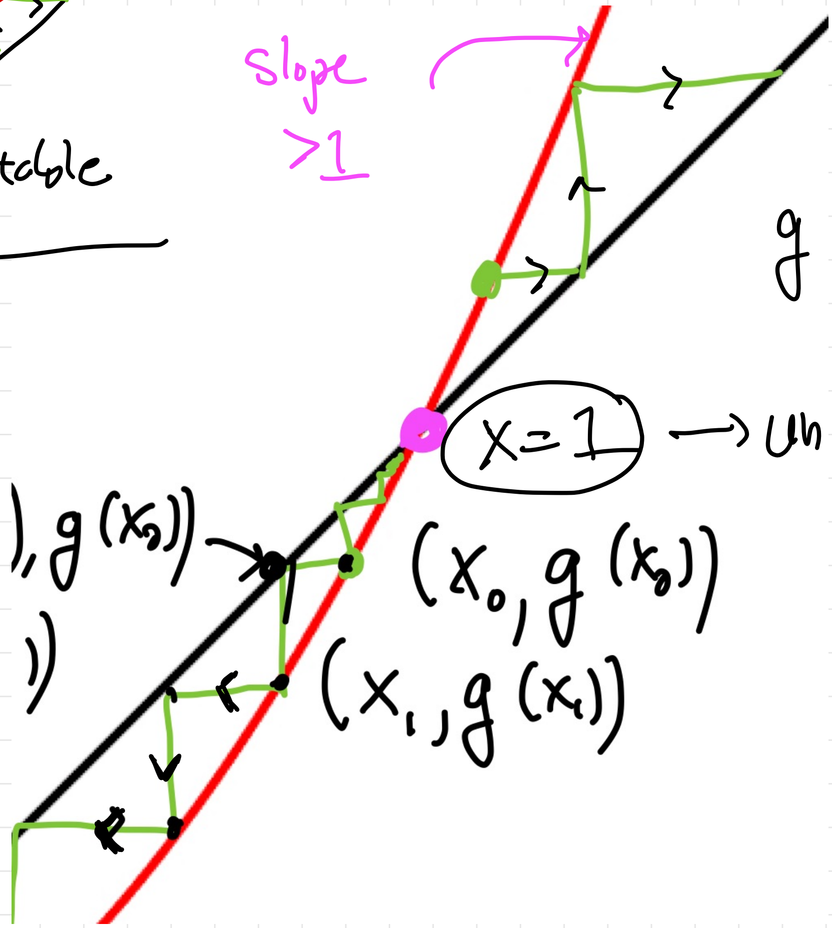
$x = 0$: unstable fixed points

$$g(x) = \sqrt{x}$$

$$g(x) = x = \sqrt{x}$$

$$x^2 = x \Rightarrow x = 0 \text{ or } x = 1$$



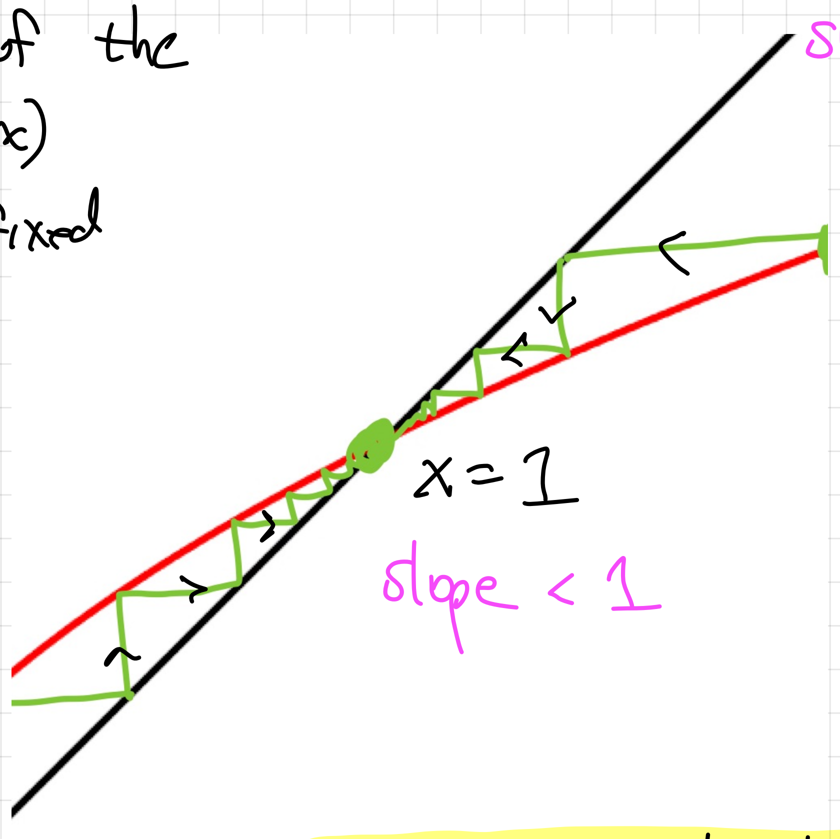


$g(x) = x^2$

$f''(1) > 0$

- concavity of the function $g(x)$ near the fixed point?

  $f''(1)$?



$g(x) = \sqrt{x}$

$f''(1) < 0$

Condition: Need $|g'(x)| < 1$ near the fixed point.

Example $f(x) = \cos(x) - x = 0$

Choose $g(x) = \cos(x)$

Solve $g(x) = x$

contains the fixedpoint.

Check: $|g'(x)| = |\sin(x)| < 1$ on $(0, \frac{\pi}{2})$

$\Rightarrow$ converges to a root in $(0, \frac{\pi}{2})$

Example $f(x) = x^3 + x^2 - 3x + 3 = 0$

$$3x = x^3 + x^2 + 3$$

$$x = \frac{1}{3}(x^3 + x^2 + 3) = g(x)$$

☹

① Can we find an interval (a,b) containing the fixed point?

② Is $|g'(x)| < 1$ in the interval?

① Graph function to get an idea where (a,b)

② Graph the derivative $\Rightarrow$ find $|g'(x)| < 1$

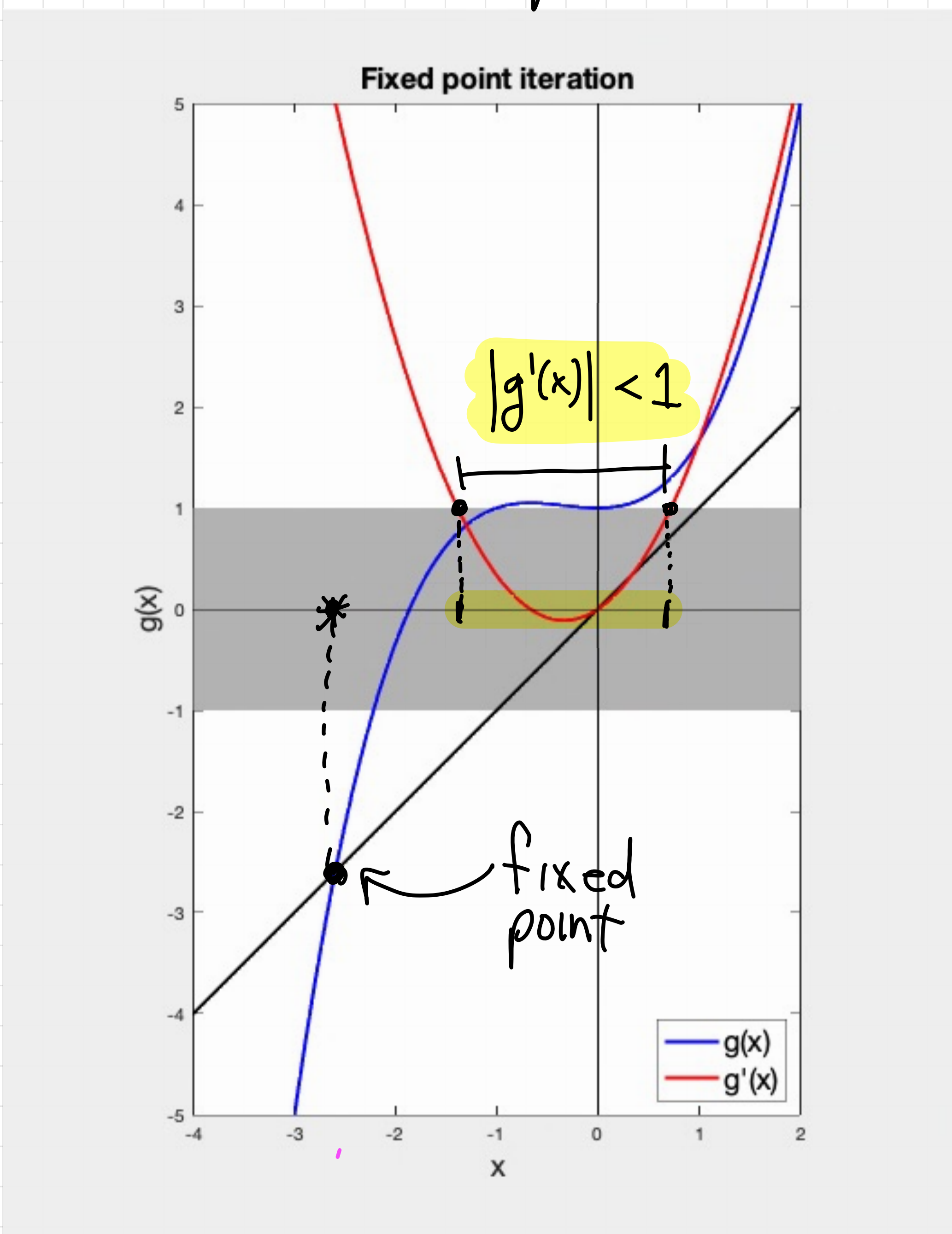
$$f(x) = x^3 + x^2 - 3x + 3 = 0; \quad \bar{x} \in (-2,-1)$$

Try: $$x = \frac{1}{3}(x^3 + x^2 + 3) \equiv g(x)$$

$$g'(x) = x^2 + \frac{2}{3}x$$

Fixed point is outside region where $|g'(x)| < 1$

Iteration will not converge for this choice of $g(x)$.



Example: $f(x) = x^3 + x^2 - 3x + 3 = 0$

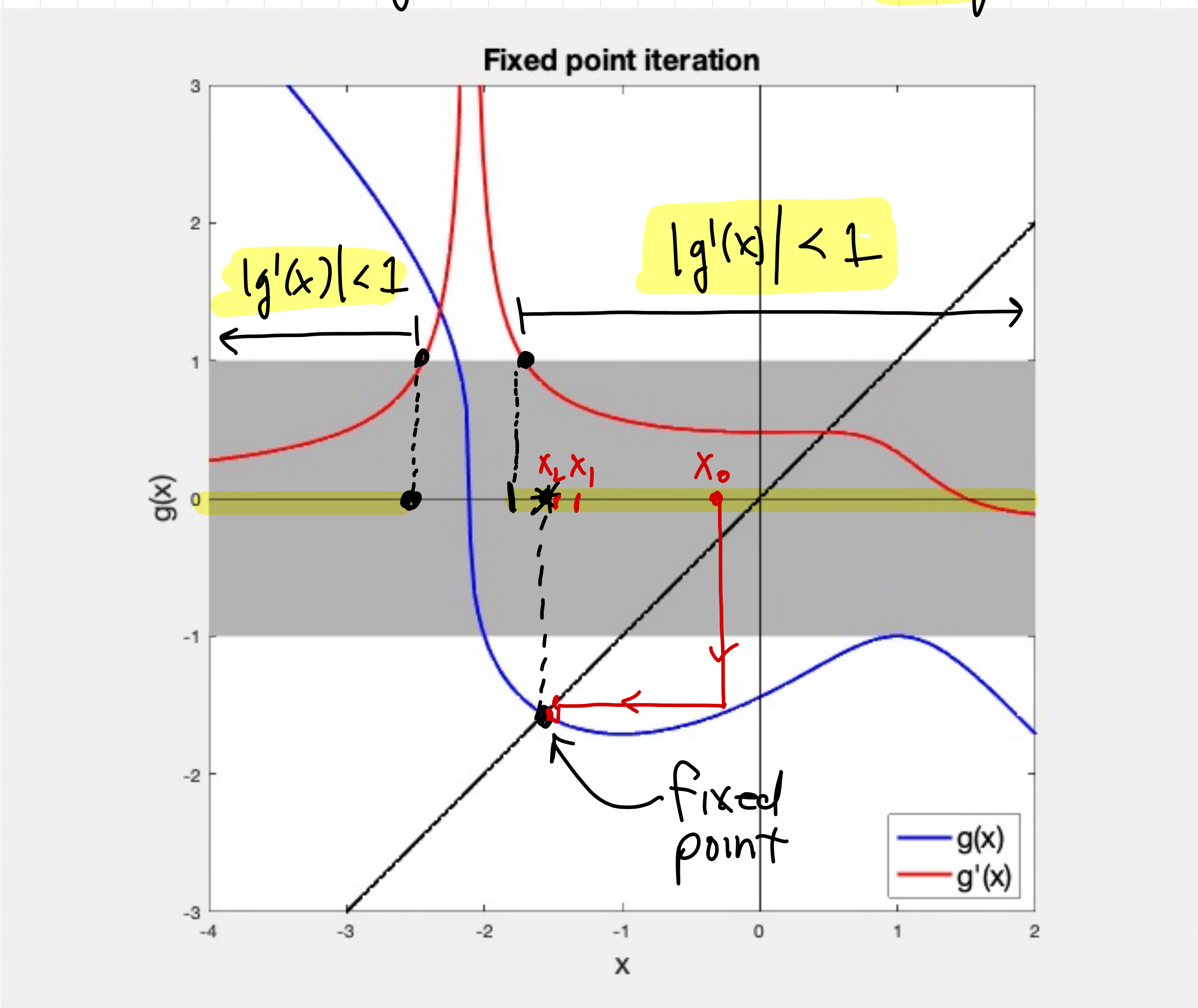
Try:

$$x = \sqrt[3]{-x^2 + 3x - 3} \equiv g(x)$$

$$= \left(-x^2 + 3x - 3\right)^{1/3}$$

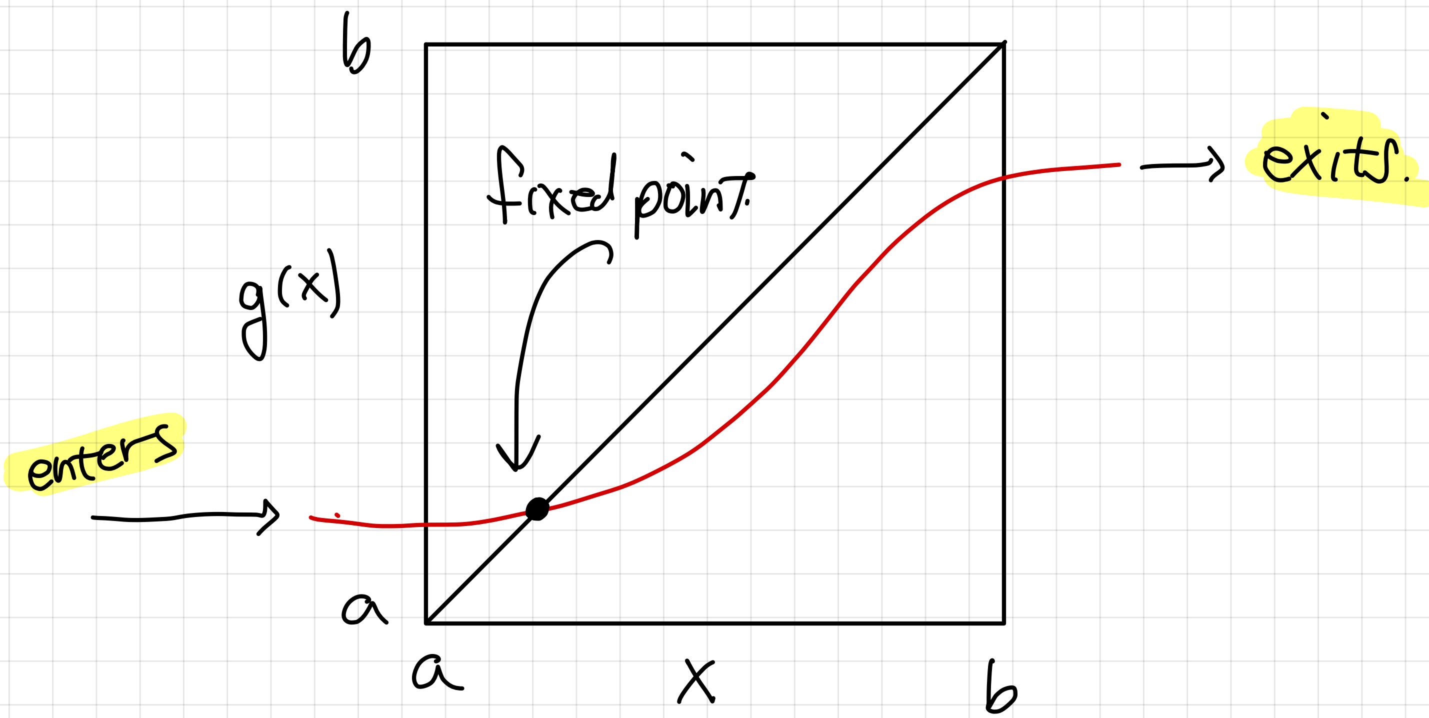
$$g'(x) = \frac{1}{3}\left(-x^2 + 3x - 3\right)^{-2/3}(-2x + 3)$$

This choice of $g(x)$ leads to a convergent iteration.



Theory

Suppose $g(x)$ enters the box $[a,b]\times[a,b]$ at the left edge and exits at the right edge. Then $[a,b]$ contains a fixed point of $g(x)$:



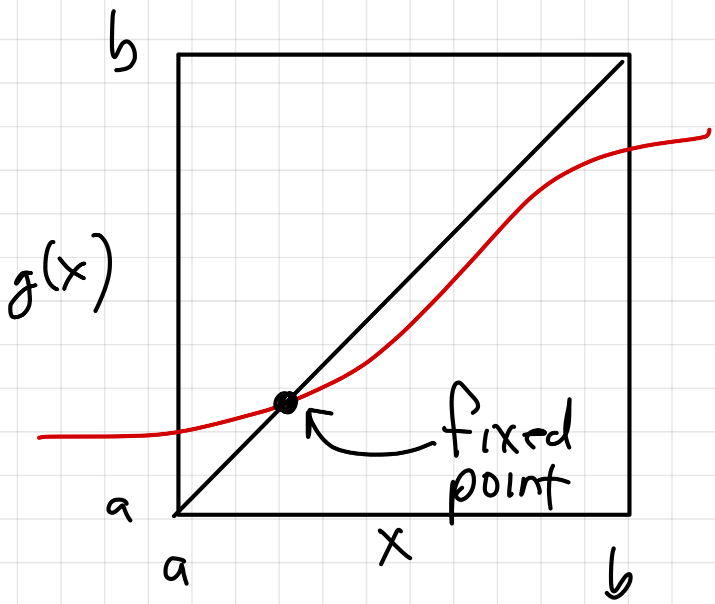
$g:[a,b]\to[a,b]$, $g(x)$ is continuous.

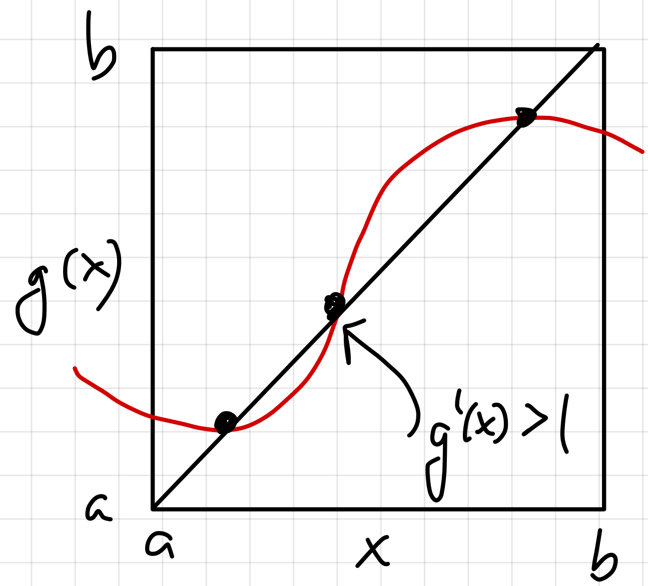
$\Rightarrow g(x)$ has a fixed point in $[a,b]$

Theory - continued

When is the fixed point unique?

Consider two cases:





Unique fixed point

$$|g'(x)| \leq k < 1$$

on $[a,b]$

multiple fixed points

$g'(x) > 1$ at some intervals in $[a,b]$

Convergence rate (without proof)

Suppose $g(x): [a,b] \to [a,b]$, and we have $|g'(x)| \le K < 1$ in $[a,b]$ for some K. Let x_n be the sequence generated by the iteration:

$$x_{n+1} = g(x_n)$$

Then the sequence converges to a unique fixed point $\bar{x}$ in $[a,b]$, and

$$|x_n - \bar{x}| \le \underbrace{\frac{|x_1 - x_0|}{1-K}}_{\lambda} \underbrace{K^n}_{\beta_n}$$

So the convergence rate is $O(K^n)$

The smaller the derivative in $[a,b]$, the faster the convergence.

Order of Convergence

Show

$$\lim_{n\to\infty} \frac{|x_{n+1}-\bar{x}|}{|x_n-\bar{x}|} = \lambda \quad , \lambda \in (0,1)$$

Assume $g'(\bar{x}) \neq 0$. Then

$$|x_{n+1}-\bar{x}| = |g(x_n) - g(\bar{x})| \qquad (1)$$

since $x_{n+1} = g(x_n)$ and $\bar{x} = g(\bar{x})$. Expand $g(x_n)$ in a Taylor series about $\bar{x}$.

Then

$$g(x_n) = g(\bar{x}) + g'(c)(x_n - \bar{x})$$

where $c \in [x_n, \bar{x}]$. Then (1) becomes

$$|x_{n+1}-\bar{x}| = |g'(c)||x_n-\bar{x}|$$

As $x_n \to \bar{x}$, we have $c \to \bar{x}$, so that

$$\lim_{n\to\infty} \frac{|x_{n+1}-\bar{x}|}{|x_n-\bar{x}|} = |g'(\bar{x})| < 1$$

asymp. error constant

$\Rightarrow$ linear convergence ∎

Order of Convergence, continued.

Suppose $g'(\bar{x}) = g''(\bar{x}) = \dots g^{(p-1)}(\bar{x}) = 0$

but $g^{(p)}(\bar{x}) \neq 0$.

Then, in the Taylor series expansion, we have

$$g(x_n) = g(\bar{x}) + \frac{1}{p!} g^{(p)}(c)(x_n - \bar{x})^p$$

From this we can show that

$$\frac{|x_{n+1} - \bar{x}|}{|x_n - \bar{x}|^p} = \frac{1}{p!} |g^{(p)}(\bar{x})|$$

asymptotic error constant.

Order of convergence p, with asymptotic error constant

$$\frac{1}{p!} |g^{(p)}(\bar{x})|$$

Note:

We do not need $\frac{1}{p!} |g^{(p)}(\bar{x})| < 1$

Comparison

Bisection, MFP, Fixed Point Iteration,

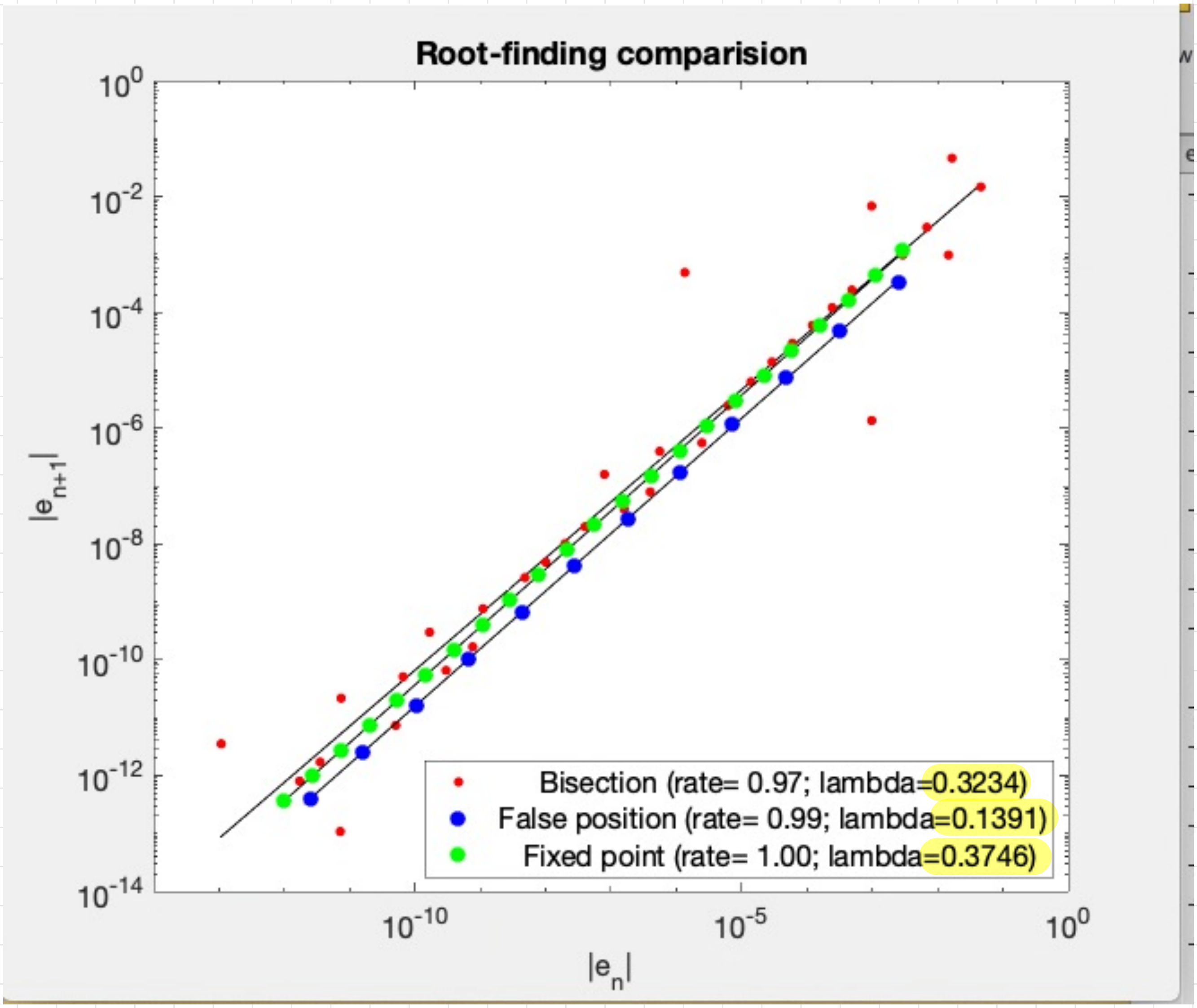
Similarites

- Only require one function call per iteration.
- linear convergence (in most cases)

Differences

- Bisection, MFP require a starting interval
- FP requires an initial guess.
- FP may not always converge.
- For some choices of $g(x)$, FP convergence can be faster than linear for FP

Bisection, MFP, Fixed Point



$$f(x) = \frac{1}{3}x^3 - x^2 + \frac{4}{3}\beta, \quad \beta = 0.1$$

Fixed point: $g(x) = f(x) + x = x$