2. Fictitious point method for Robin Boundary Conditions

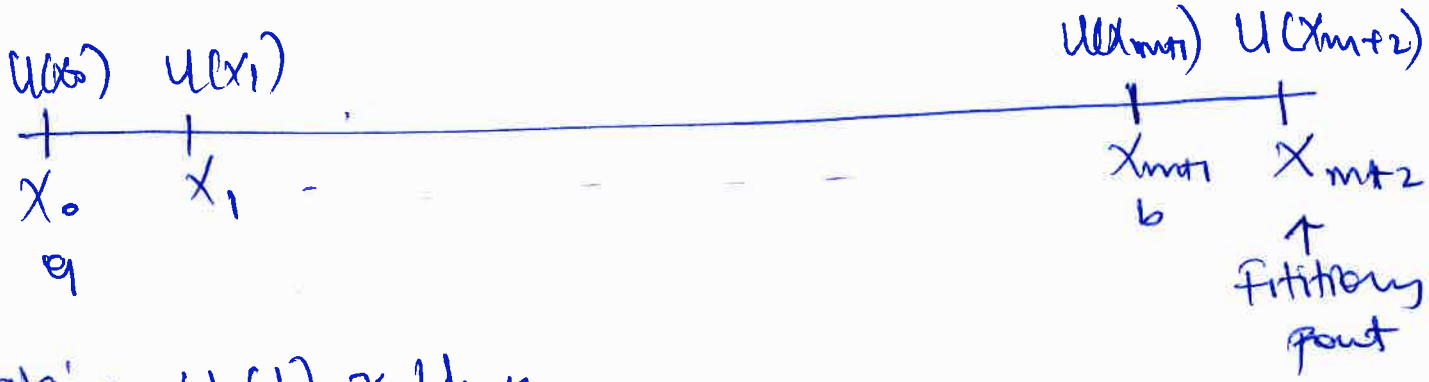
Consider

$$u'' = p(x)u' + q(x)u + r(x) \quad \text{--- ①}, \; x \in [a,b]$$

with mixed Boundary Conditions.

$$u(a) = \alpha \quad \text{and} \quad \beta_1 u(b) + \beta_2 u'(b) = \beta_3$$

Discretize equation ① with $m+1$ equally spaced Sub intervals.



taking $u(b) \approx u_{m+1}$

$u(b-h) \approx u_m$

$p_{m+1} = p(b)$

$q_{m+1} = q(b)$

$r_{m+1} = r(b)$

After discretization, at x_{m+1} equation ① becomes;

$$u''_{m+1} = p_{m+1}u'_{m+1} + q_{m+1}u_{m+1} + r_{m+1} \quad \text{——— ②}$$

From the Centered difference formula;

$$u''_{m+1} = \frac{u_{m+2} - 2u_{m+1} + u_m}{h^2}$$

where u_{m+2} is $u(x_{m+2})$, where for x_{m+2} a Fictitious point.

from the Boundary conditions; $\beta_1 u_{m+1} + \beta_2 u'_{m+1} = \beta_3$

we have

$$U'_{m+1} = \frac{\beta_3 - \beta_1 U_{m+1}}{\beta_2}$$

So

$$P_{m+1} U'_{m+1} = \frac{P_{m+1}}{\beta_2}\left(\beta_3 - \beta_1 U_{m+1}\right) \quad — ③$$

but U'_{m+1} can be discretised to),

$$U'_{m+1} = \frac{U_{m+2} - U_m}{2h} = \frac{\beta_3 - \beta_1 U_{m+1}}{\beta_2}$$

$$U_{m+2} = \frac{2h}{\beta_2}\left(\beta_3 - \beta_1 U_{m+1}\right) + U_m$$

from

$$U''_{m+1} = \frac{U_{m+2} - 2U_{m+1} + U_m}{h^2}$$

Substituting U_{m+2}, we obtain

$$U''_{m+1} = \frac{\frac{2h}{\beta_2}\left(\beta_3 - \beta_1 U_{m+1}\right) + U_m - 2U_{m+1} + U_m}{h^2}$$

Subsitituting U''_{m+1} and equation ③ into equation ②

we have

$$\frac{\frac{2h}{\beta_2}\left(\beta_3 - \beta_1 U_{m+1}\right) + 2U_m - 2U_{m+1}}{h^2} = \frac{P_{m+1}}{\beta_2}\left(\beta_3 - \beta_1 U_{m+1}\right) + q_{m+1} U_{m+1} + r_{m+1}$$

$$\frac{2h}{\beta_2}(\beta_3 - \beta_1 U_{m+1}) + 2U_m - 2U_{m+1} - q_{m+1} h^2 U_{m+1} +$$

$$\frac{p_{m+1}}{\beta_2} h^2 \beta_1 U_{m+1} = \frac{p_{m+1}}{\beta_2} h^2 \beta_3 + h^2 r_{m+1}$$

which simplifies too

$$2U_m + \left[-2 - h^2 q_{m+1} + \beta_1/\beta_2 \, h^2 p_{m+1} - 2h \, \beta_1/\beta_2\right] U_{m+1} =$$

$$-2h\frac{\beta_3}{\beta_2} + \frac{\beta_3}{\beta_2} h^2 p_{m+1} + h^2 r_{m+1}$$

$$-2U_m + \left[2 + h^2 q_{m+1} + (2 - h p_{m+1}) h \frac{\beta_1}{\beta_2}\right] U_{m+1} =$$

$$-h^2 r_{m+1} + (2 - h p_{m+1}) h \, \beta_3/\beta_2 \qquad \boxed{\cdot}$$

3 (a)

Show that the linear system (S) has a unique solution regardless of b.

from (1) $AU + \lambda w = 0$

multiplying through by w^T from the left hand side.

$$w^T(AU + \lambda w) = 0$$

$$w^T A U + w^T \lambda w = 0$$

Since w is an non zero vector with non zero entries and an eigen value of A then $w^T A = 0^T$, then,

$$w^T A U = 0 \Rightarrow w^T \lambda w = 0$$

This becomes

$\lambda w^T w = 0$, Since λ is a constant

Since w is non zero vector then $w^T w \neq 0$, therefore for $\lambda w^T w = 0$ then λ must be zero.

hence $\lambda = 0$

from (1): $AU + \lambda w = 0$

if $\lambda = 0$

$AU = 0$

(ii) $w^T U = 0$

If $AU = 0$, This means $U = \alpha e$ for some α

Using $w^T U = 0$, then substituting in $U = \alpha e$

$w^T \alpha e = 0$, Since α is a constant then

$\alpha w^T e = 0$.

but $w^T = \begin{bmatrix} \frac{1}{2} & 1 & \cdots & 1 & \frac{1}{2} \end{bmatrix}$

So,

$w^T e$, $e = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

So $w^T e = \begin{bmatrix} \frac{1}{2} & 1 & \cdots & 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

$w^T e = \frac{1}{2} + 1 + 1 \cdots 1 + 1 + \frac{1}{2}$

Since w and e are vectors of length $m+2$ ~~in~~

then $w^T e = 1 + 1 + 1 \cdots 1 + 1$,

we are suming 1 $m+1$ terms which reduces to

$$w^T e = \sum_{i=1}^{m+1} 1 = m+1 \neq 0$$

but $\alpha w^T e = 0$, hence for $\alpha w^T e$ to be zero then α must be zero because $m+1 \neq 0$.

therefore $\underline{\underline{\alpha = 0}}$

b)

Show that if $w^T b = 0$ in (5) then $\lambda = 0$.

from (5)

$$Au + \lambda w = b$$

multiplying through by w^T from the left hand side we have

$$w^T A u + w^T \lambda w = w^T b$$

If $w^T b = 0$ ~~than~~ and $Au = 0$ then

$$w^T \lambda w = 0 \Rightarrow \lambda w^T w = 0$$

Since w is an non zero vector then $w^T w \neq 0$,

~~therefore~~ for $\lambda w^T w$ to be zero then $\underline{\underline{\lambda = 0}}$

4. Neuman-Neuman Boundary Conditions and DST

a) Show that row j of system (2) simplifies to

$$\sum_{k=0}^{m+1}{}'' \hat{u}_k \left(2\cos\left(\frac{\pi k}{m+1}\right)-2\right)\cos\left(\frac{\pi j k}{m+1}\right) = h^2 \sum_{k=0}^{m+1}{}'' \hat{u}_k \cos\left(\frac{\pi j k}{m+1}\right)$$

from (2), we can conclude that the j^{th} row is given by

$$\frac{1}{h^2}\left(U_{j-1}-2U_j+U_{j+1}\right) = f_j$$

Starting for the case $1 \le j \le m$, we have

$$U_{j-1}-2U_j+U_{j+1} = 2\sum_{k=0}^{m+1}{}'' \hat{u}_k \cos\left(\frac{\pi(j-1)}{m+1}k\right) - 2\left(2\sum_{k=0}^{m+1}{}'' \hat{u}_k \cos\left(\frac{\pi j k}{m+1}\right)\right)$$
$$+ 2\sum_{k=0}^{m+1}{}'' \hat{u}_k \cos\left(\frac{\pi(j+1)}{m+1}k\right)$$

$$= 2\sum_{k=0}^{m+1}{}'' \hat{u}_k \cos\left(\frac{\pi(j-1)}{m+1}k\right) - 4\sum_{k=0}^{m+1}{}'' \hat{u}_k \cos\left(\frac{\pi j k}{m+1}\right)$$
$$+ 2\sum_{k=0}^{m+1}{}'' \hat{u}_k \cos\left(\frac{\pi(j+1)}{m+1}k\right)$$

$$= 2\sum_{k=0}^{m+1}{}'' \hat{u}_k \left[\cos\left(\frac{\pi(j-1)}{m+1}k\right) - 2\cos\left(\frac{\pi j k}{m+1}\right) + \cos\left(\frac{\pi(j+1)}{m+1}k\right)\right]$$

but $\cos\left(\frac{\pi(j-1)}{m+1}k\right) + \cos\left(\frac{\pi(j+1)}{m+1}k\right) = 2\cos\left(\frac{\pi j k}{m+1}\right)\cos\left(\frac{\pi k}{m+1}\right)$

$$= 2\sum_{k=0}^{m+1}{}'' \hat{u}_k \left[2\cos\left(\frac{\pi j k}{m+1}\right)\cos\left(\frac{\pi k}{m+1}\right) - 2\cos\left(\frac{\pi j k}{m+1}\right)\right]$$

Therefore;

$$U_{j-1} - 2U_j + U_{j+1} = 2\sum_{k=0}^{m+1}{}'' \hat{u}_k \left[2\cos\left(\frac{\pi k}{m+1}\right) - 2 \right] \cos\left(\frac{\pi j k}{m+1}\right)$$

So

$$U_{j-1} - 2U_j + U_{j+1} = h^2 f_j \quad ; \quad f_j = \sum_{k=0}^{m+1}{}'' \hat{f}_k \cos\left(\frac{\pi j k}{m+1}\right), \text{ becomes}$$

$$\sum_{k=0}^{m+1}{}'' \hat{u}_k \left(2\cos\left(\frac{\pi k}{m+1}\right) - 2 \right) \cos\left(\frac{\pi j k}{m+1}\right) = h^2 \sum_{k=0}^{m+1}{}'' \hat{f}_k \cos\left(\frac{\pi j k}{m+1}\right)$$

Checking for $j=0$.

$\cancel{U_{j-1}} - 2U_j + 2U_{j+1} = h^2 f_j$ becomes;

$\cancel{U_{0,1}} - 2U_0 + 2U_1 = h^2 \hat{f}_0$

$$-2U_0 + 2U_1 = -4\sum_{k=0}^{m+1}{}'' \hat{u}_k \cos\left(\frac{\pi k}{m+1}(0)\right) + 4\sum_{k=0}^{m+1}{}'' \hat{u}_k \cos\left(\frac{\pi k}{m+1}\right)$$

$$= 2\sum_{k=0}^{m+1}{}'' \hat{u}_k \left(2\cos\left(\frac{\pi k}{m+1}\right) - 2\cos\left(\frac{\pi k}{m+1}(0)\right)\right)$$

$$= 2\sum_{k=0}^{m+1}{}'' \hat{u}_k \left(2\cos\left(\frac{\pi k}{m+1}\right) - 2\right) \cos\left(\frac{\pi k}{m+1}(0)\right)$$

So $-2U_0 + 2U_1 = h^2 f_0$, becomes;

$$2\sum_{k=0}^{m+1}{}'' \hat{u}_k \left(2\cos\left(\frac{\pi k}{m+1}\right) - 2\right) \cos\left(\frac{\pi k}{m+1}(0)\right) = h^2 \sum_{k=0}^{m+1}{}'' \hat{f}_k \cos\left(\frac{\pi 0 k}{m+1}\right)$$

In case $j = m+1$

from (2), we have $\quad 2U_m - 2U_{m+1} = h^2 f_{m+1}$

but $U_m = U_{m+2}$, from (7)

then

$$2U_{m+2} - 2U_{m+1} = h^2 f_{m+1}$$

$$4\sum_{k=0}^{m+1}{}'' \hat{U}_k \cos\left(\frac{\pi(m+2)}{m+1}k\right) - 4\sum_{k=0}^{m+1}{}'' \hat{U}_k \cos\left(\frac{(m+1)k}{m+1}\right) = h^2 \sum_{k=0}^{m+1}{}'' \hat{f}_k \left(\frac{\pi(m+1)k}{m+1}\right)$$

This reduces to

$$2\sum_{k=0}^{m+1}{}'' \hat{U}_k \left[2\cos\left(\frac{\pi(m+2)k}{m+1}\right) - 2\cos\left(\frac{\pi(m+1)}{m+1}k\right)\right] = h^2 \sum_{k=0}^{m+1}{}'' \hat{f}_k \left(\frac{\pi(m+1)k}{m+1}\right)$$

but

$$\cos\left(\frac{\pi(m+2)}{m+1}k\right) = \cos\left(\frac{\pi(m+1)k}{m+1}\right)\cos\left(\frac{\pi k}{m+1}\right) - \sin\left(\frac{\pi(m+1)k}{m+1}\right)\sin\left(\frac{\pi k}{m+1}\right)$$

Since $\sin \pi k = 0$, for k, integer, then

$$\cos\left(\pi\frac{(m+2)}{m+1}k\right) = \cos\left(\frac{\pi(m+1)}{(m+1)}k\right)\cos\left(\frac{\pi k}{m+1}\right)$$

then;

$$2\sum_{k=0}^{m+1}{}'' \hat{U}_k \left[2\cos\left(\frac{\pi(m+1)}{m+1}k\right)\cos\left(\frac{\pi k}{m+1}\right) - 2\cos\left(\frac{\pi(m+1)}{m+1}k\right)\right] = h^2 \sum_{k=0}^{m+1}{}'' \hat{f}_k \left(\frac{\pi(m+1)k}{m+1}\right)$$

$$2\sum_{k=0}^{m+1}{}'' \hat{U}_k \left[2\cos\left(\frac{\pi k}{m+1}\right) - 2\right]\cos\left(\frac{\pi(m+1)}{m+1}k\right) = h^2 \sum_{k=0}^{m+1}{}'' \hat{f}_k \left(\frac{\pi(m+1)}{m+1}k\right).$$

b) from

$$\sum_{k=0}^{m+1}{}'' \hat{U}_k\left(2\cos\left(\frac{\pi k}{m+1}\right)-2\right)\cos\left(\frac{\pi j k}{m+1}\right) = h^2\sum_{k=0}^{m+1}{}'' \hat{f}_k \cos\left(\frac{\pi j k}{m+1}\right)$$

for $k=0$.

$$\sum_{\cancel{k\neq 0}}^{\cancel{m+1}} \hat{U}_0\left(2-2\right)\cos\left(\frac{\pi j(0)}{m+1}\right) = h^2 \cancel{\sum} \hat{f}_0$$

$0 \neq \hat{f}_0$, hence ~~un~~ undefined, therefore we start from $k=1$

$$\sum_{k=1}^{m+1}{}'' \hat{U}_k\left(2\cos\left(\frac{\pi k}{m+1}\right)-2\right)\cos\left(\frac{\pi j k}{m+1}\right) = h^2\sum_{k=1}^{m+1}{}'' \hat{f}_k \cos\left(\frac{\pi j k}{m+1}\right)$$

which reduces to

$$\hat{U}_k\left(2\cos\left(\frac{\pi k}{m+1}\right)-2\right) = h^2\hat{f}_k$$

hence
$$\hat{U}_k = \frac{h^2\hat{f}_k}{2\cos\left(\frac{\pi k}{m+1}\right)-2}$$

If $\hat{f}_0=0$, gives

$$\frac{1}{m+1}\left[\frac{1}{2}f_0 + \sum_{j=1}^{m} f_j \cos\left(\frac{\pi(0)j}{m+1}\right) + \frac{1}{2}f_{m+1}\right] = 0$$

$$\left[\frac{1}{2}f_0 + \sum_{j=1}^{m} f_j + \frac{1}{2}f_{m+1}\right] = 0 \quad \text{——— } (1)$$

~~then~~ for the discrete Compatibility Condition

$W^T b = W^T f = 0$.

$W^T = [\tfrac{1}{2}, 1, \ldots, 1, \tfrac{1}{2}]$

therefore;

$$W^T f = [\tfrac{1}{2}, 1, \dots, 1, \tfrac{1}{2}] \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{m+1} \end{bmatrix}$$

$$W^T f = \frac{1}{2} f_0 + f_1 + f_2 + \dots f_{m-1} + f_m + \tfrac{1}{2} f_{m+1}$$

but from (1), $\frac{1}{2} f_0 + \sum_{j=1}^{m} f_j + \frac{1}{2} f_{m+1} = 0$ for

$\hat{f}_0 = 0$, therefore;

$$W^T f = \frac{1}{2} f_0 + \sum_{j=1}^{m} f_j + \frac{1}{2} f_{m+1} = 0$$

Hence $\hat{f}_0 = 0$ Corresponds to $W^T b = W^T f = 0$

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c) - To Obtain the Solution for (2), the discrete Compatibility Condition $W^T f = 0$ must be satisfied.

- So since its satisfied on the right hand side, hence we can obtain the solution to (2)

- So for $W^T f = 0$ to be satisfied on the right hand means the non ~~zero~~ ~~eigen~~ vector ~~value~~ ~~eigen~~ is orthogonal to f, hence their dot product is zero.

Also explain how one makes the solution unique by fixing the arbitrary constant to U

let $\hat{u}_0$ be fixed to an arbitrary constant U i.e. $\hat{u}_0 = U$

but

$$\hat{u}_0 = \frac{1}{m+1}\left[\frac{1}{2}u_0 + \sum_{j=1}^{m} u_j + \frac{1}{2}u_{m+1}\right] = U$$

$$\frac{1}{2}u_0 + \sum_{j=1}^{m} u_j + \frac{1}{2}u_{m+1} = (m+1)U.$$

but $\frac{1}{2}u_0 + \sum_{j=1}^{m} u_j + \frac{1}{2}u_{m+1} = \left[\frac{1}{2} \ 1 \ \cdots \ 1 \ \frac{1}{2}\right]\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m+1}\end{bmatrix}$

$$= w^T u$$

then

$$\frac{1}{2}u_0 + \sum_{j=1}^{m} u_j + \frac{1}{2}u_{m+1} = w^T u = (m+1)U$$

So $w^T u = (m+1)U$ implies that the solution to (2) is unique.

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e) First of all they are mathematically equivalent since in both we are solving the same equation, and the conditions in both methods almost draw to the same conclusion.

In problem (3), we are interested more in the value of lambda, λ. if its zero ($\lambda = 0$) then the solution exists, and also the error gives some information

- As in problem (4) we see that the Poisson equation doesn't have a solution unless ~~a~~ the discrete compatibility condition is satisfied, and that the solution is unique if a_0 is fixed to u.
- So all these methods will draw to same equivalent solutions.