Elliptic Equations - Part II

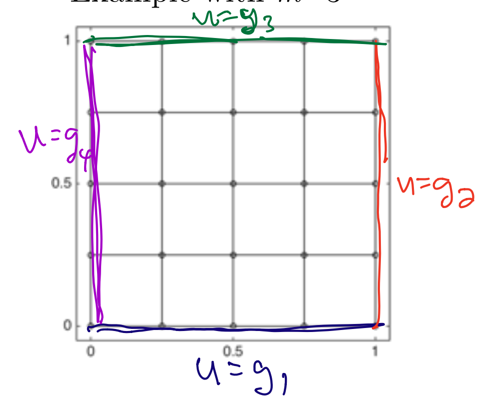
Model problem we focus on:

$$\nabla^2 u(x,y) = f(x,y),\ \Omega = \{(x,y)\,|\,0 \le x \le 1,\ 0 \le y \le 1\}$$

a b a b

$\nabla^2 u = u_{xx} + u_{yy}$

Example with m=3



Boundary conditions $\quad u(x,0) = g_1(x),\ u(1,y) = g_2(y),$
$\quad u(x,1) = g_3(x),\ u(0,y) = g_4(x).$

Grid: $\boxed{(x_j, y_k) = (jh, kh)},\ h = 1/(m+1),\ j,k = 0,1\ldots,m+1$

$$u_{jk} = u(x_j, y_k)\ f_{jk} = f(x_j, y_k)$$

General: $h = \frac{(b-a)}{m+1}$

Arrange the unknowns and knowns in a matrix:

$$U^h = \begin{bmatrix} u_{11} & u_{21} & u_{31} \\ u_{12} & u_{22} & u_{32} \\ u_{13} & u_{23} & u_{33} \end{bmatrix} \qquad F^h = \begin{bmatrix} f_{11} & f_{21} & f_{31} \\ f_{12} & f_{22} & f_{32} \\ f_{13} & f_{23} & f_{33} \end{bmatrix}$$

We can then write the discretized equations as

$$\underbrace{\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}}_{h^2 D^h_{2,y}} \underbrace{\begin{bmatrix} u_{11} & u_{21} & u_{31} \\ u_{12} & u_{22} & u_{32} \\ u_{13} & u_{23} & u_{33} \end{bmatrix}}_{U^h} + \underbrace{\begin{bmatrix} u_{11} & u_{21} & u_{31} \\ u_{12} & u_{22} & u_{32} \\ u_{13} & u_{23} & u_{33} \end{bmatrix}}_{U^h} \underbrace{\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}}_{h^2 (D^h_{2,x})^T} = h^2 \underbrace{\begin{bmatrix} f_{11} & f_{21} & f_{31} \\ f_{12} & f_{22} & f_{32} \\ f_{13} & f_{23} & f_{33} \end{bmatrix}}_{F^h} - \underbrace{\begin{bmatrix} u_{10}+u_{01} & u_{20} & u_{30}+u_{41} \\ u_{02} & 0 & u_{42} \\ u_{03}+u_{14} & u_{24} & u_{34}+u_{43} \end{bmatrix}}_{U^h_{\text{bc}}}$$

This gives the *matrix equation*:

$$D^h_{2,y} U^h + U^h (D^h_{2,x})^T = \underbrace{F^h - \frac{1}{h^2} U^h_{\text{bc}}}_{\widetilde{F}^h}$$

- Kronecker product

vec operator

Or the *linear system of equations*:

$$\underbrace{(I_m \otimes D^h_{2,y}}_{D_{yy}} + \underbrace{D^h_{2,x} \otimes I_m)}_{D_{xx}} \underline{u}^h = \underline{\tilde{f}}^h$$

$$
\underbrace{\frac{1}{h^2}\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{bmatrix}}_{A^h}
\underbrace{\begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{21} \\ u_{22} \\ u_{23} \\ u_{31} \\ u_{32} \\ u_{33} \end{bmatrix}}_{\underline{u}^h}
=
\underbrace{\begin{bmatrix} h^2 f_{11} - u_{01} - u_{10} \\ h^2 f_{12} - u_{02} \\ h^2 f_{13} - u_{03} - u_{14} \\ h^2 f_{21} \qquad\quad - u_{20} \\ h^2 f_{22} \\ h^2 f_{23} \qquad\quad - u_{24} \\ h^2 f_{31} - u_{41} - u_{30} \\ h^2 f_{32} - u_{42} \\ h^2 f_{33} - u_{43} - u_{34} \end{bmatrix}}_{\underline{f}^h}
$$

For a general m, A^h is an $m^2 \times m^2$ matrix

Three questions for this method.

1. Is the matrix A^h non-singular for all h?
2. Does the solution converge to the exact solution as $h \longrightarrow 0$?
3. Are there fast ways to solve the system?

Question 1: Is the matrix A^h non-singular for all h? → yes!

For 1-D problem: $D^h_{2,x} = D^h_{2,y} = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix} \rightarrow m \times m$ matrix

Eigenvalues $D^h_{2,x}$: $\lambda_n = -\frac{4}{h^2}\sin^2\left(\frac{n\pi}{2(m+1)}\right)$, $n = 1, \dots, m$

Eigenvectors $D^h_{2,x}$: $V^n_j = \sin\left(\frac{n\pi j}{m+1}\right)$, $n = 1, \dots, m$, $j = 1, 2, \dots, m$

It seems natural that the eigenvectors of A^h can be determined from the eigenvectors of $D_{2,x}$ & $D_{2,y}$

$$\underbrace{V^{n,\ell}_{j,k}}_{\text{eigenvector}} = \sin\left(\frac{n\pi j}{m+1}\right)\sin\left(\frac{\ell\pi k}{m+1}\right), \quad \begin{matrix} j,k = 1,2,\dots,m \\ n,\ell = 1,2,\dots,m \end{matrix}$$

Plug this into the matrix equation:

$$D^h_{2,y} V^{n,\ell} + V^{n,\ell} D^h_{2,x} = \operatorname{diag}\Big(\overbrace{-\frac{4}{h^2}\sin^2\left(\frac{n\pi}{2(m+1)}\right)}^{\lambda_n}\Big) V^{n,\ell}$$

$$+ V^{n,\ell}\operatorname{diag}\left(-\frac{4}{h^2}\sin^2\left(\frac{\ell\pi}{2(m+1)}\right)\right)$$

So the eigenvalues of A^h are:

$$\lambda_{n,\ell} = -\frac{4}{h^2}\left[\sin^2\left(\frac{n\pi}{2(m+1)}\right) + \sin^2\left(\frac{\ell\pi}{2(m+1)}\right)\right], \quad \begin{array}{l} n = 1, 2, \dots, m \\ \ell = 1, 2, \dots, m \end{array}$$

$$\boxed{h = \frac{1}{m+1}}$$

$$= -\frac{4}{h^2}\left[\sin^2\left(\frac{n\pi h}{2}\right) + \sin^2\left(\frac{\ell\pi h}{2}\right)\right]$$

From this we see that $\lambda_{n,\ell} \neq 0$ for any n, ℓ.

$\Rightarrow$ A^h is invertible.

2. Does the solution converge to the exact solution as $h \longrightarrow 0$?

Chapter 2: Consistency + Stability $\Rightarrow$ $\boxed{\text{converges}}$

Consistency: local truncation error $\longrightarrow 0$ as $h \to 0$
τ^h

$$\Rightarrow \quad A^h u^h_{exact} - \tilde{f}^h = \underbrace{\begin{bmatrix} O(h^2) \\ O(h^2) \\ \vdots \\ O(h^2) \end{bmatrix}}_{\tau^h} \Rightarrow \|\tau_h\|_2 \to 0$$

as $h \to 0$

Stability: $\lim_{h \to 0} \| (A^h)^{-1} \| = C < \infty$ (norm of the inverse of A^h is bounded)

$$\|(A^h)^{-1}\|_2 = \frac{1}{\text{minimum eigenvalue of } A^h} = \frac{1}{\min_{1 \le l, m \le m} |\lambda_{m,l}|} = \frac{1}{|\lambda_{1,1}|}$$

$$= \frac{1}{\frac{8}{h^2} \sin^2\left(\frac{\pi}{2} h\right)} \Rightarrow \lim_{h \to 0} \|(A^h)^{-1}\|_2 = \frac{1}{2\pi^2}$$

(so it's bounded)

3. Are there fast ways to solve the system?

$A^h \underline{u}^h = \underline{\tilde{f}}^h$

$\nabla^2 u = f$ on Ω

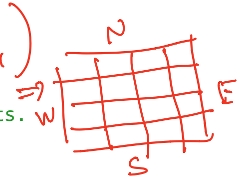
$u = g$ on $\partial\Omega$

Matlab code that uses Gaussian elimination:

```
function [u,x,y] = fd2poisson(ffun,gfun,a,b,m)

h = (b-a)/(m+1);   % Mesh spacing

[x,y] = meshgrid(a:h:b);   % Uniform mesh, including boundary points.

% Compute u on the boundary from the Dirichlet boundary condition
ub = zeros(m,m);
idx = 2:m+1;
idy = 2:m+1;
% West and East boundaries need special attention
ub(:,1) = feval(gfun,x(idy,1),y(idy,1));               % West
ub(:,m) = feval(gfun,x(idy,m+2),y(idy,m+2));           % East
% Now the North and South boundaries
ub(1,1:m) = ub(1,1:m) + feval(gfun,x(1,idx),y(1,idx)); % South
ub(m,1:m) = ub(m,1:m) + feval(gfun,x(m+2,idx),y(m+2,idx)); % North
% Convert ub to a vector using column reordering
ub = (1/h^2)*reshape(ub,m*m,1);

% Evaluate the RHS of Poisson's equation at the interior points.
f = feval(ffun,x(idy,idx),y(idy,idx));
% Convert f to a vector using column reordering
f = reshape(f,m*m,1);

% Create the D2x and D2y matrices

% Full matrix version.  This could be made much faster by using Matlab's
% sparse matrix functions (see "spdiags" for more details).
z = [-2;1;zeros(m-2,1)];
D2x = 1/h^2*kron(toeplitz(z,z),eye(m));
D2y = 1/h^2*kron(eye(m),toeplitz(z,z));

% Solve the system
u = (D2x + D2y)\(f-ub);

% Convert u from a column vector to a matrix to make it easier to work with
% for plotting.
u = reshape(u,m,m);

% Append on to u the boundary values from the Dirichlet condition.
u = [feval(gfun,x(1,1:m+2),y(1,1:m+2));...
    [feval(gfun,x(idy,1),y(idy,1)),u,feval(gfun,x(idy,m+2),y(idy,m+2))];...
     feval(gfun,x(m+2,1:m+2),y(m+2,1:m+2))];
```

f g

(for above problem a=0, b=1)



$\longrightarrow$ vec(ub)

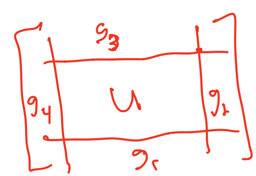
$\longrightarrow$ vec(f)

$\text{toeplitz}(z,z) = \begin{bmatrix} -2 & 1 & & 0 \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ 0 & & 1 & -2 \end{bmatrix}$

$\rightarrow D_{2,x} \otimes I_m$

$\rightarrow I_m \otimes D_{2,y}$

A^h f^h



Numerical results

```
% Script for testing fd2poisson over the square [a,b]x[a,b]
a = 0;
b = 1;
m = 19;  % Number of interior grid points in one direction

f = @(x,y) -5*pi^2*sin(pi*x).*cos(2*pi*y);  % Laplacian(u) = f
g = @(x,y) sin(pi*x).*cos(2*pi*y);          % u = g on Boundary
uexact = @(x,y) g(x,y);                     % Exact solution is g.

% Time the solution
tic
[u,x,y] = fd2poisson(f,g,a,b,m);
toc

h = (b-a)/(m+1);

% Plot error
figure, set(gcf,'DefaultAxesFontSize',8,'PaperPosition', [0 0 3.5 3.5]),
mesh(x,y,u-uexact(x,y)), colormap([0 0 0]),xlabel('x'),ylabel('y'),
zlabel('Error'), title(strcat('Error, h=',num2str(h)));
```

$\nabla^2 u = -5\pi^2 \sin(\pi x)\cos(2\pi y)$

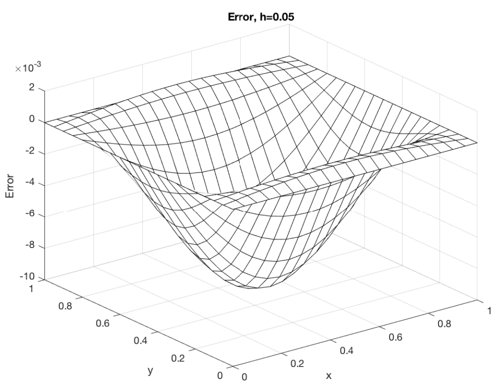
$\Omega = \{(x,y) \mid 0 \le x \le 1 \ , \ 0 \le y \le 1\}$

Exact solution:

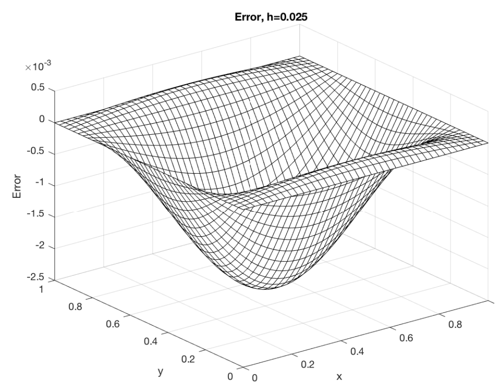
$u(x,y) = \sin(\pi x)\cos(2\pi y)$

Plots of the error for $m+1=20$, 40, 80

Total points: 400, 1600, 6400

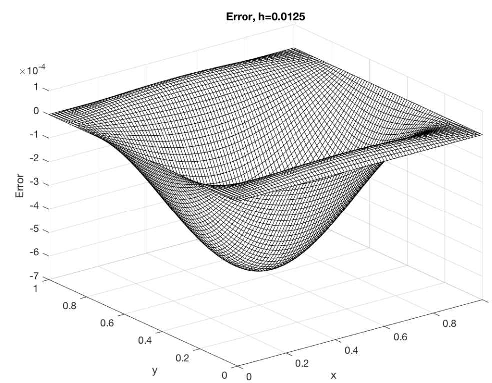


Elapsed time is 0.002523 seconds.



Elapsed time is 0.055546 seconds.

22 times more seconds than $m+1=20$



Elapsed time is 1.744623 seconds.

31 times more seconds than $m+1=40$

Computational cost of Gaussian elimination (GE) approach:

- For an $N\times N$ matrix GE takes $\mathcal{O}(N^3)$ operations to solve a system.

- For FD problem above: $N = m^2$

  Cost GE: $\mathcal{O}(m^6)$ (This is bad)

In 3-D: $N = m^3$

cost $\mathcal{O}(m^9)$

$m = 100 = 10^2 \rightarrow \mathcal{O}(10^{18})$

Increase m by factor of 2 the cost goes up by a factor of $2^6 = 64$

Sparse Gaussian elimination solvers:

$$
\begin{array}{c}
\qquad\quad T \qquad\qquad\quad I \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \\
\underbrace{\frac{1}{h^2}\left[\begin{array}{ccc|ccc|ccc}
-4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0\\
\hline
1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0\\
0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0\\
0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1\\
\hline
0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4
\end{array}\right]}_{\substack{A^h \\ m^2\times m^2}}
\underbrace{\begin{bmatrix} u_{11}\\ u_{12}\\ u_{13}\\ u_{21}\\ u_{22}\\ u_{23}\\ u_{31}\\ u_{32}\\ u_{33}\end{bmatrix}}_{\underline{u}^h}
=
\underbrace{\begin{bmatrix}
h^2 f_{11} - u_{01} - u_{10}\\
h^2 f_{12} - u_{02}\\
h^2 f_{13} - u_{03} - u_{14}\\
h^2 f_{21} \qquad\quad - u_{20}\\
h^2 f_{22}\\
h^2 f_{23} \qquad\quad - u_{24}\\
h^2 f_{31} - u_{41} - u_{30}\\
h^2 f_{32} - u_{42}\\
h^2 f_{33} - u_{43} - u_{34}
\end{bmatrix}}_{\underline{f}^h}
\end{array}
$$

Ex: $m = 3$

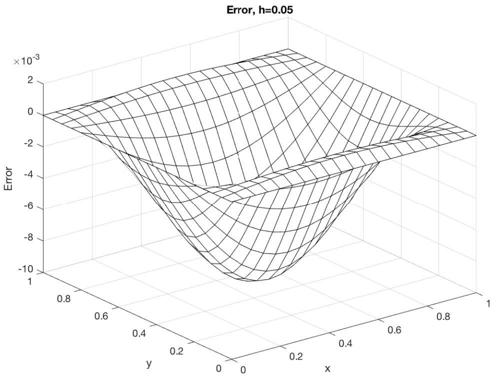
In general: $T = h^2 D^h_{2,x}$

$I = m\times m$ identity matrix

$$
A^h = \frac{1}{h^2}\begin{bmatrix}
T & I & & & & \\
I & T & I & & & \\
 & I & T & I & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & I & T & I\\
 & & & & I & T
\end{bmatrix}
$$

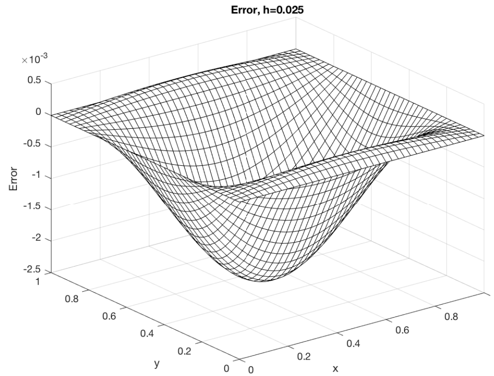
Block tridiagonal system.

Plots of the error for $m+1=20$, 40, 80



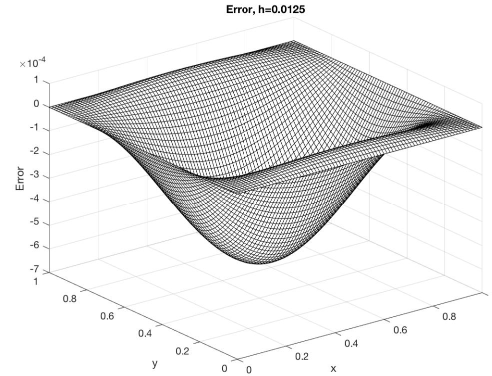
Dense: Elapsed time is 0.002523 seconds.

Sparse: Elapsed time is 0.000719 seconds.



Dense: Elapsed time is 0.055546 seconds.

Sparse: Elapsed time is 0.001734 seconds.



Dense: Elapsed time is 1.744623 seconds.

Sparse: Elapsed time is 0.008137 seconds.

Fast direct solvers based on the FFT:

Recall the matrix equation version of the problem:

$$D_{2,y}^h U^h + U^h (D_{2,x}^h)^T = \underbrace{F^h - \frac{1}{h^2} U_{\mathrm{bc}}^h}_{\widetilde{F}^h}$$

Let

$$V^h = \begin{bmatrix} \sin(\frac{\pi}{m+1}1) & \sin(\frac{\pi}{m+1}2) & \cdots & \cdots & \sin(\frac{\pi}{m+1}m) \\ \sin(\frac{\pi}{m+1}2) & \sin(\frac{\pi}{m+1}4) & \cdots & \cdots & \sin(\frac{\pi}{m+1}2m) \\ \sin(\frac{\pi}{m+1}3) & \sin(\frac{\pi}{m+1}6) & \cdots & \cdots & \sin(\frac{\pi}{m+1}3m) \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \sin(\frac{\pi}{m+1}m) & \sin(\frac{\pi}{m+1}2m) & \cdots & \cdots & \sin(\frac{\pi}{m+1}m^2) \end{bmatrix} \qquad \Lambda^h = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_m \end{bmatrix}$$

$$\lambda_j = -\frac{4}{h^2} \sin^2\left(\frac{\pi}{2(m+1)} j\right)$$