Section 1.2 Convergence

Sequence: List of numbers

- Think of a sequence as a "function" on the integers

Example:

$$S(n) = \frac{1}{n}, \quad n = 1, 2, 3, 4, \dots$$

We usually use subscripts, though instead of the functional form:

$$S_n = \frac{1}{n}, \quad n = 1, 2, 3$$

What we are often interested in is how <u>fast</u> a sequence converges.

Examples:

$$S_n = \frac{1}{\log_2(n)} \quad S_n = \frac{1}{n}, \quad S_n = \frac{1}{n^2}, \quad S_n = \frac{1}{2^n}$$

all converge to 0, but how fast?

Definition: Limit

Definition: A sequence **converges** to the finite value L provided:

$$\lim_{n\to\infty} S_n = L < \infty$$

or

$$\lim_{n\to\infty} |S_n - L| = 0$$

If the sequence doesn't converge to a finite value, it is said to **diverge**.

Example

$$S_n = \frac{5n^2-2}{3n^2+n-1};$$

$$\lim_{n\to\infty} S_n = \frac{5}{3}$$

Definition: Rate of convergence

Let $\{S_n\}$ be a sequence that converges to a number L. If there exists a sequence $\{\beta_n\}$ that converges to zero and a positive constant λ, independent of n, such that

$$|S_n - L| \leq \lambda |\beta_n|$$

for sufficiently large values of n, the $\{S_n\}$ is said to converge to L with "rate of convergence" β_n.

Example: $|S_n - L| =$

① $$\left|\frac{5n^2-2}{3n^2+n-1} - \frac{5}{3}\right| = \frac{5n+1}{3(3n^2+n-1)}$$

② $$\frac{5n+1}{3(3n^2+n-1)} = \frac{5n}{3(3n^2+n-1)} + \frac{1}{3(3n^2+n-1)}$$

$$< \frac{5n}{9n^2} + \frac{1}{9n^2}$$ drop lower order terms in denominator.

$\frac{1}{n} < 1$

$$= \frac{1}{9}\left(\frac{5}{n} + \frac{1}{n^2}\right) = \frac{1}{9n}\left(5 + \frac{1}{n}\right) < \frac{6}{9n} = \frac{2}{3n}$$

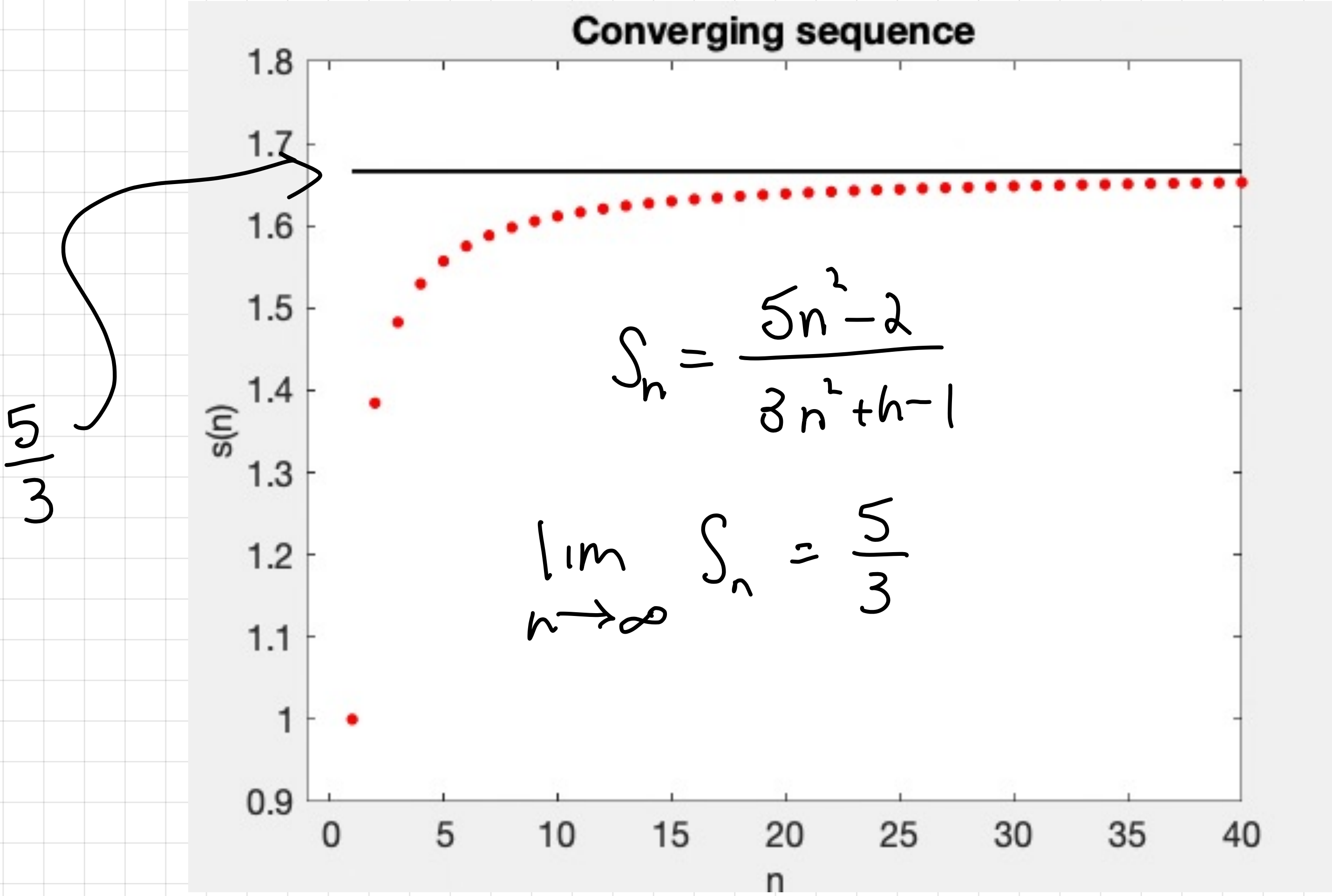
③

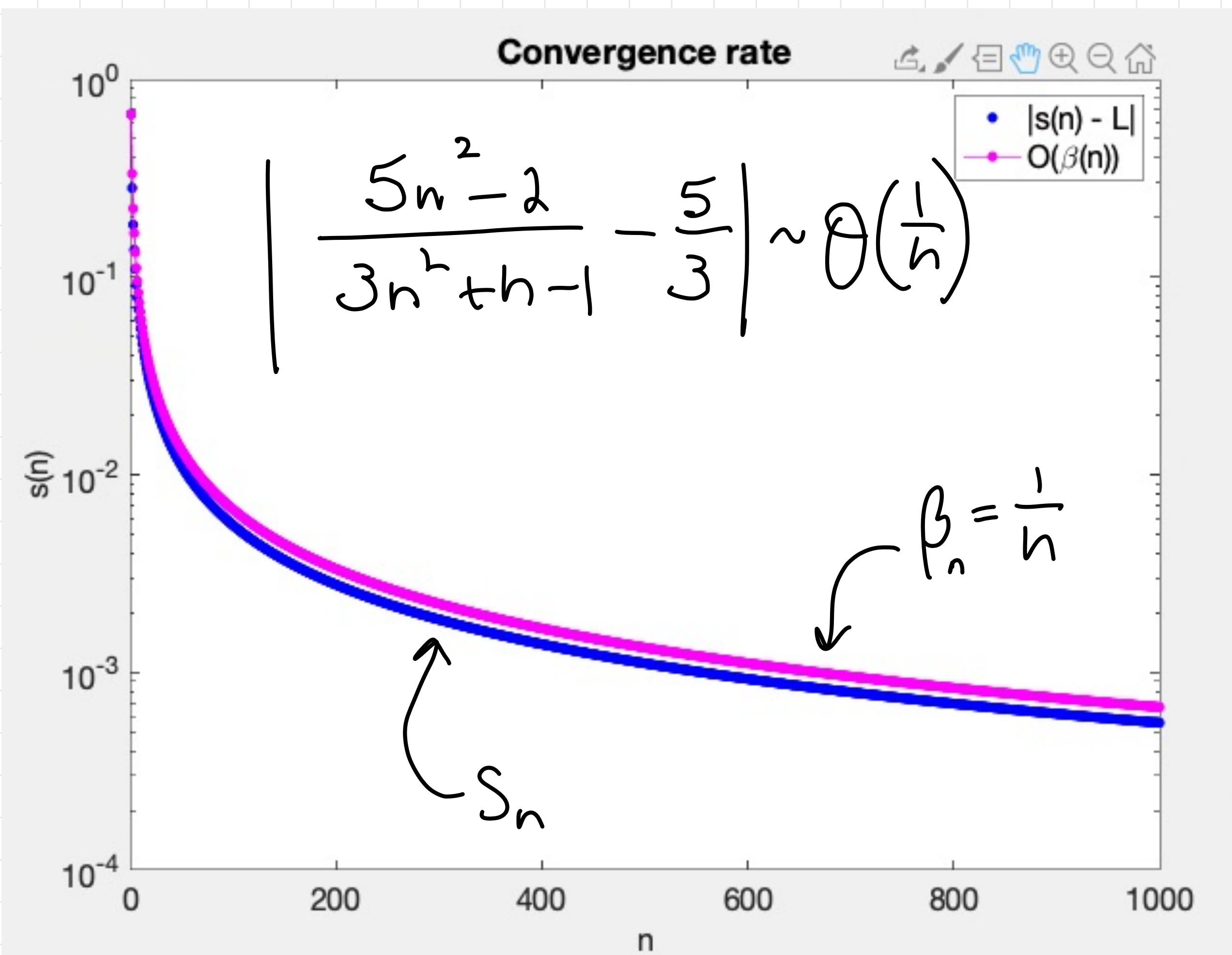
For large n, the terms look like

$$|S_n| < \underbrace{\frac{2}{3} \cdot \frac{1}{n}}_{\lambda \beta_n}$$

Rate of convergence : $\Theta\left(\frac{1}{n}\right)$ big "O" notation.

"converges like $\frac{1}{n}$"


Converging sequence
s(n)
n
$S_n = \frac{5n^2-2}{3n^2+n-1}$
$\lim_{n\to\infty} S_n = \frac{5}{3}$
$\frac{5}{3}$


Convergence rate
|s(n) - L|
O(β(n))
s(n)
n
$\left|\frac{5n^2-2}{3n^2+n-1} - \frac{5}{3}\right| \sim \Theta\left(\frac{1}{n}\right)$
$\beta_n = \frac{1}{n}$
S_n

Steps to finding rate of convergence of a sequence:

① Find limiting value L:

$$\lim_{n \to \infty} S_n = L$$

② Compute and algebraically manipulate

$$|S_n - L|$$

③ Find a sequence β_n and a constant λ so that

$$|S_n - L| < \lambda \beta_n$$

where β_n is some function of n. that goes to 0 and λ is a constant.

Example:

$$S_n = \sqrt{n+1} - \sqrt{n}$$

① $$\lim_{n\to\infty} S_n = \infty - \infty \quad ☹$$

Rationalize:

$$\frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0 \quad ☺$$

$L = 0$

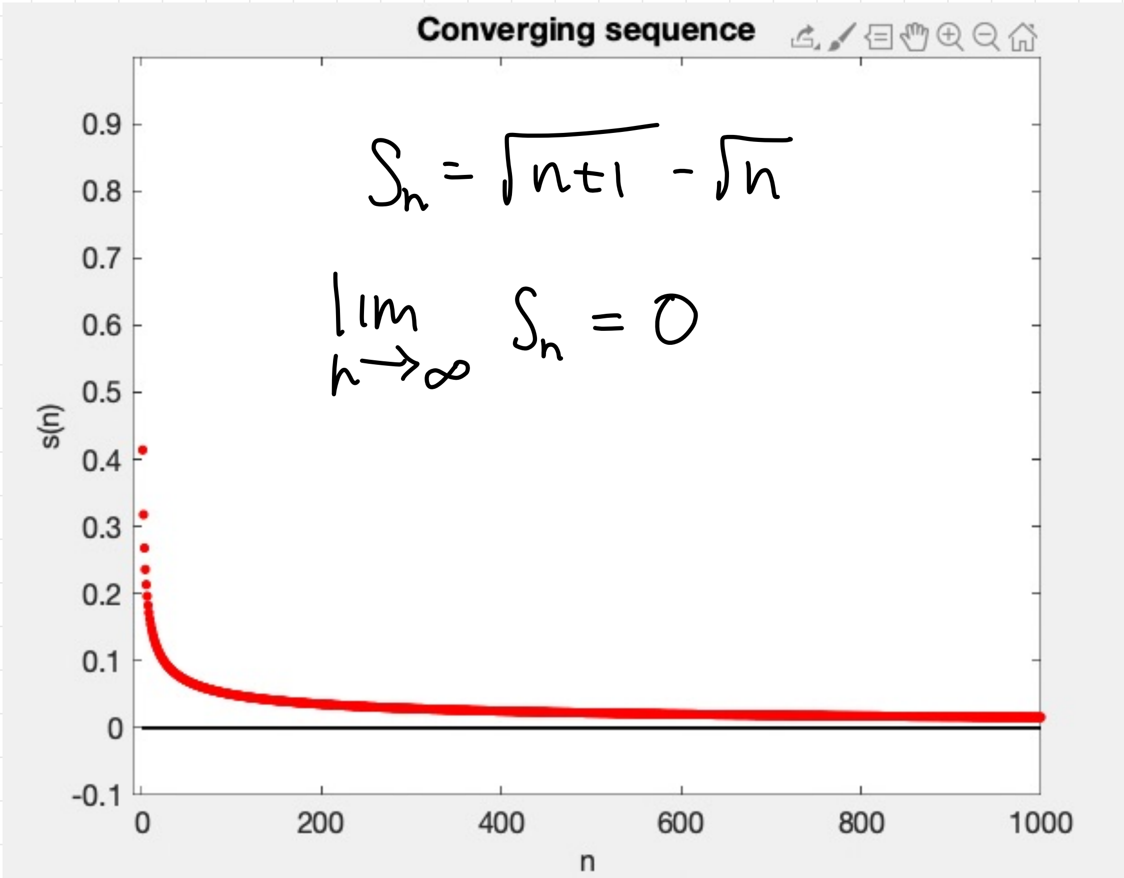
② $$\left| \frac{1}{\sqrt{n+1} + \sqrt{n}} - L \right| = \left| \frac{1}{\sqrt{n+1} + \sqrt{n}} \right|$$

$$\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$$

③ $$\frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}}$$

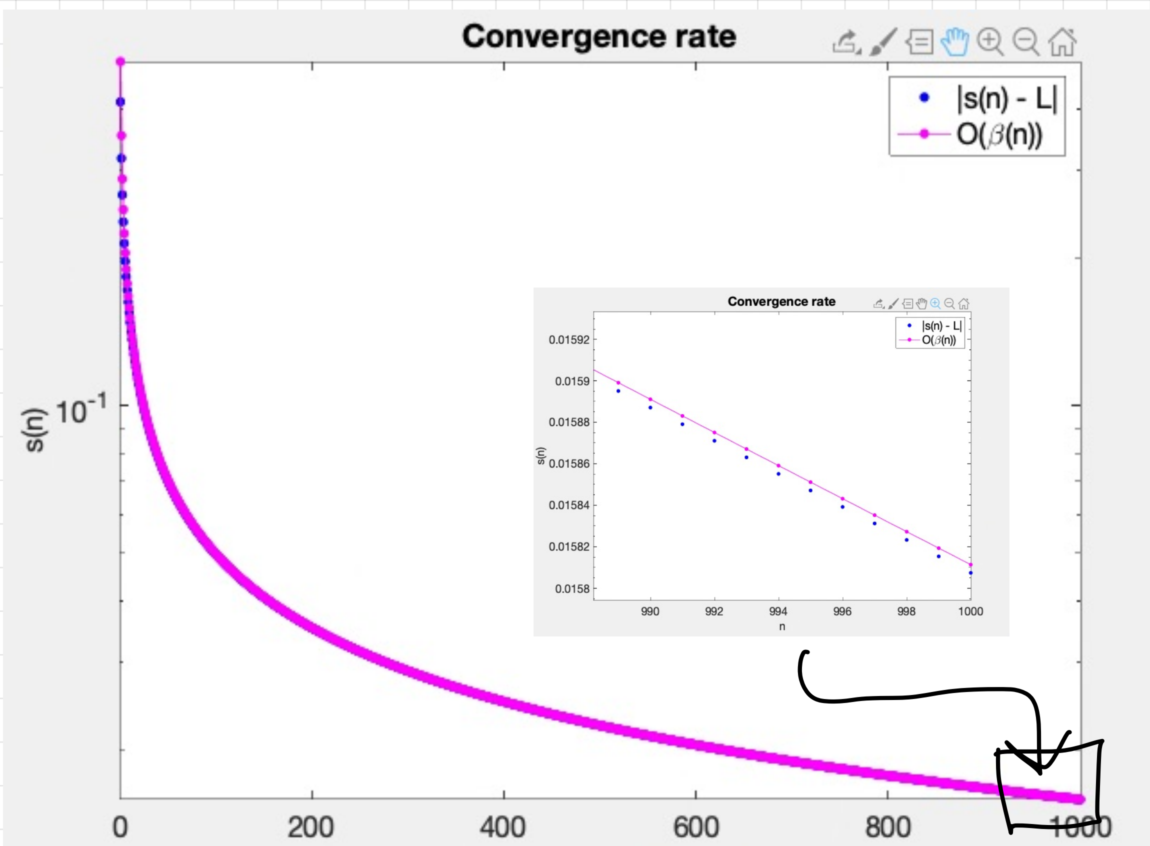
$$|S_n - L| < \frac{1}{2} \frac{1}{\sqrt{n}} \qquad \lambda = 2, \quad \beta_n = \frac{1}{\sqrt{n}}$$

"rate of convergence" is $\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$ ■



$$S_n = \sqrt{n+1} - \sqrt{n}$$

$$\lim_{n \to \infty} S_n = 0$$



Rate of convergence for functions

Definition

Let f be defined on an interval (a,b) that contains $x=0$, and suppose $\lim_{x\to 0} f(x) = L$. If there exists a function $g(x)$ for which

$$\lim_{x\to 0} f(x) = g(x)$$

and positive constant K, such that

$$|f(x) - L| \leq K|g(x)|$$

then $f(x)$ converges to L with rate of convergence of $O(g(x))$.

Example: $$f(x) = \frac{\cos(x) - 1}{x^2}$$

① Find limiting value

$$\lim_{x \to 0} \frac{\cos(x) - 1}{x^2}$$

Use Taylor series at $x = 0$:

Recall:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!}$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}$$

Remainder Theorem

ξ is rarely known.

$$f(x) = \underbrace{\sum_{n=0}^{N} \frac{f^{(n)}(0)x^n}{n!}}_{\substack{P_n(x) \\ \text{polynomial}}} + \underbrace{\frac{f^{(N+1)}(\xi)\,x^{N+1}}{(N+1)!}}_{\substack{R_n(x) \\ \text{remainder}}}$$

Example using Remander Theorem

$$\cos(x) = \underbrace{1 - \frac{x^2}{2!} + 0}_{P_3(x)} + \underbrace{\frac{F^{(4)}(\xi(x))}{4!}x^4}_{R_3(x)}$$

where $F^{(4)}(\xi(x)) = \cos(\xi(x))$

let $f(x) = \dfrac{\cos(x) - 1}{x^2}$

1 $$\lim_{x \to 0} \frac{\cos(x) - 1}{x^2} =$$

$$= \lim_{x \to 0} \frac{\left[1 - \frac{x^2}{2!} + \frac{\cos(\xi(x))}{4!}x^4\right] - 1}{x^2}$$

$$= \lim_{x \to 0} -\frac{1}{2!} + \frac{1}{3!}\cancel{\cos(\xi(x))}x^2 = -\frac{1}{2!}$$

② Compute

$L = -\frac{1}{2!}$

$$\left| \frac{\cos(x) - x}{x^2} + \frac{1}{2} \right| = \left| \frac{1}{4!} \cos(\xi(x))\, x^2 \right|$$

③ Find K, $g(x)$ so that

$$|f(x) - L| \leq K|g(x)|$$

$$\left| \frac{1}{4!} \cos(\xi(x))\, x^2 \right| \leq \frac{1}{4!} x^2, \text{ since } |\cos(\xi(x))| \leq 1$$

so $K = \frac{1}{24}$ $g(x) = x^2$

$\Rightarrow$ rate of convergence is $O(x^2)$

$|f(x) - L|$ "looks like" $\frac{1}{24} x^2$

near $x = 0$.

Don't confuse two ideas:

"converging to something"

and "converging like something"

"Converges to"

$$\lim_{n\to\infty} \frac{3n+2}{n} = \lim_{n\to\infty} 3+\frac{2}{n} = 3$$

So S_n "converges to" $L=3$

"Converges like"

$$\left|\underbrace{\frac{3n+2}{n}}_{S_n} - \underbrace{3}_{L}\right| = \left|\frac{2}{n}\right| \leq 2\frac{1}{n}$$

"converges like" $\frac{1}{n}$ ~ $\Theta(\frac{1}{n})$

"rate of converge" = "converges like"

Asymptotic order of convergence

Recall: Rate of convergence

$$\underbrace{|S_n - L|}_{\text{error}} \leq \lambda \underbrace{|\beta_n|}_{\text{rate}}$$

asymptotic error constant

Rate: sequence $\beta_n \to 0$. This tells us something about how the error behaves as $n \to \infty$.

$$\underset{\text{slow}}{\theta\left(\frac{1}{\sqrt{n}}\right)}, \quad \theta\left(\frac{1}{n}\right), \quad \underset{\text{fast}}{\theta\left(\frac{1}{n^2}\right)}$$

Define the error as:

$$e_n = S_n - L$$

The error behaves as:

$$\theta\left(\frac{1}{\sqrt{n}}\right), \quad \theta\left(\frac{1}{n}\right), \quad \theta\left(\frac{1}{n^2}\right)$$

In many cases, we can choose

$$\beta_n = |S_{n-1} - L|$$

Example: $S_n = \dfrac{1}{n}$

$$\lim_{n \to \infty} S_n = 0 = L$$

$$|S_{n+1} - L| = \frac{1}{n+1} < \underbrace{\frac{1}{n}}_{\beta_{n+1}} = \beta_n$$

Define $e_{n+1} = S_{n+1} - L$

$$e_n = S_n - L$$

What is the ratio

$$\lim_{n \to \infty} \frac{|e_{n+1}|}{|e_n|} = \lim_{n \to \infty} \frac{n}{n+1} = \underline{1}$$

This could tell us how quickly terms converge.

Definition: Order of Convergence

Definition: Let $\{S_n\}$ be a sequence that converges to a number L. Let $e_n = S_n - L$ for $n \geq 0$. If there exists positive constants λ and α such that

$$\lim_{n\to\infty} \frac{|S_{n+1} - L|}{|S_n - L|^{\alpha}} = \lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^{\alpha}} = \lambda$$

then $\{S_n\}$ is said to converge to L with "order of convergence" α.

$$|e_{n+1}| \approx \lambda |e_n|^{\alpha}$$

Types of convergence:

Typically α is an integer, 1, 2, 3, ...

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^{\alpha}} = \lambda$$

α: order of convergence
λ: asymptotic error constant

$\alpha = 1$:

$\lambda = 1$: sublinear convergence

$0 < \lambda < 1$: linear convergence

$\lambda = 0$: superlinear convergence.

$\alpha = 2$

$\lambda > 0$: quadratic convergence

$\alpha = 3$

$\lambda > 0$: cubic convergence.

Example:

$$S_{n+1} = \sqrt{n+2} - \sqrt{n+1}, \qquad L = 0$$

$$e_{n+1} = S_{n+1} - L = \sqrt{n+2} - \sqrt{n+1} = \frac{1}{\sqrt{n+2} + \sqrt{n+1}}$$

$$e_{n+1} = \frac{1}{\sqrt{n+2} + \sqrt{n+1}} < \frac{1}{\underbrace{\sqrt{n+1} + \sqrt{n}}_{\beta_n}} = e_n$$

So that

$$|e_{n+1}| < |e_n|$$

and

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|} = \lim_{n\to\infty} \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+2} + \sqrt{n+1}} = 1$$

$$\Rightarrow \quad \alpha = 1, \quad \lambda = 1$$

"sublinear convergence" (quite slow!)

$\blacksquare$

order of convergence from numerical data

$e_{n+1} = \lambda e_n$ linear slower

$e_{n+1} = \lambda e_n^2$ quadratic

$e_{n+1} = \lambda e_n^3$ cubic faster

	Linear	Quadratic	Cubic
e_1	0.5	0.5	0.5
e_2	0.25	0.125	0.0625
e_3	0.125	7.8125×10^{-3}	1.2207×10^{-4}
e_4	0.0625	3.0518×10^{-5}	9.0949×10^{-13}
e_5	0.03125	4.6566×10^{-10}	3.7616×10^{-37}
	slow		fast

Verify

linear

$$\frac{e_2}{e_1} = .5$$

$$\frac{e_3}{e_2} = .5$$

quadratic

$$\frac{e_2}{e_1^2} = .5$$

$$\frac{e_3}{e_2^2} = .5$$

cubic

$$\frac{e_2}{e_1^3} = .5$$

$$\frac{e_3}{e_2^3} = .5$$

We can view the order of convergence as a way to see how many digits of accuracy we can expect in each step.

$$e_{n+1} = \lambda e_n$$

Suppose $e_n = 10^{-3}$

Linear Convergence

$e_{n+1} = \lambda e_n$, $e_n = 10^{-3}$

$e_{n+1} = \lambda 10^{-3}$

Might get one digit more of accuracy with each step, Depends on size of λ.

$e_{n+1} \approx \lambda^{n+1} e_0$; convergence depends on value of λ.

Quadratic Convergence

$$e_{n+1} = \lambda e_n$$

$$e_n = 10^{-3}$$

$$e_{n+1} = \lambda 10^{-6}$$

double number of accurate digits.

Cubic Convergence

$$e_{n+1} = \lambda e_n^3$$

$$e_n = 10^{-3}$$

$$e_{n+1} = \lambda 10^{-9}$$

triple number of accurate digits!

Computing the order of convergence

Example

Computing $\sqrt{a}$

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

$$e_{n+1} = x_{n+1} - \sqrt{a} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) - \sqrt{a}$$

$$= \frac{1}{2}\left(\frac{x_n^2 - 2\sqrt{a}\,x_n + a}{x_n}\right)$$

$$= \frac{1}{2}\frac{\left(x_n - \sqrt{a}\right)^2}{x_n} = \frac{e_n^2}{2x_n}$$

very hard to get algorithms that converge quadratically!

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^2} = \lim_{n\to\infty} \frac{1}{2x_n} = \frac{1}{2\sqrt{a}}$$

Quadratic convergence with constant $\frac{1}{2\sqrt{a}}$

$\alpha = 2 \quad \lambda = \frac{1}{2\sqrt{a}}$

Example: Numerically approximating the order of convergence.

$$\left.\begin{array}{ll} e_0 & 3.7 \times 10^{-4} \\ e_1 & 1.2 \times 10^{-15} \\ e_2 & 1.5 \times 10^{-60} \end{array}\right\} \text{errors}$$

Guess: Look at # of digits is increasing by factor of 4 each time. $-4, -15, -60$

Numerically:

$$e_{n+1} = \lambda e_n^{\alpha}$$

$$\log(e_{n+1}) = \underline{\underline{\alpha}} \log(e_n) + \underline{\underline{\log(\alpha)}}$$

Find slope and intercept through points:

$$(\log(e_0), \log(e_1)) \ \& \ (\log(e_1), \log(e_2))$$

$\alpha \approx 3.90834$ $\lambda = 0.031034$

Possible source of confusion:

① Many sources use the term "rate of convergence" to mean the λ value in a linearly convergent sequence

$$|e_{n+1}| = \lambda |e_n|$$

Some sources call this the "rate of convergence"

② Other sources may (incorrectly) refer to the rate as the value of α.

Definition of order is more restrictive and in some sense not very discerning. For example,

$$\frac{1}{n}, \quad \frac{1}{n^2}, \quad \frac{1}{n^3}, \quad \frac{1}{2^n}, \quad \frac{1}{\log(k)}$$

all converge with order 1 (although they have different asymptotic constants.

So we will stick with our definition of rate:

$$|S_n - L| < \lambda |\beta_n|$$

rate of convergence $\Theta(\beta_n)$ $\blacksquare$

the end!