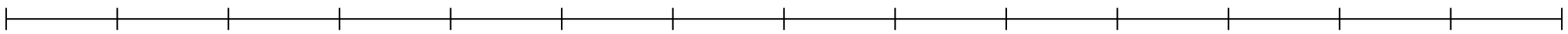
Boundary Value Problems - Part IV

Two Neumann boundary conditions (Section 2.12)

Equation: $u''(x) = f(x),\ 0 < x < 1$ Boundary conditions: $u'(0) = \sigma_0,\ u'(1) = \sigma_1$

Extra condition for uniqueness: $\int_0^1 u(x)\,dx = \gamma$, for some γ.



See homework 3 problem for more detail.

$$\underbrace{\frac{1}{h^2}\begin{bmatrix} -2 & 2 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & -2 & 1 & \ddots & & & \vdots \\ 0 & 1 & -2 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 1 & -2 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 2 & -2 \end{bmatrix}}_{A^h} \underbrace{\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_m \\ u_{m+1} \end{bmatrix}}_{\underline{u}^h} = \underbrace{\begin{bmatrix} f_0 + \frac{2}{h}\sigma_0 \\ f_1 \\ f_2 \\ \vdots \\ \vdots \\ f_m \\ f_{m+1} - \frac{2}{h}\sigma_1 \end{bmatrix}}_{\underline{f}^h},$$

- Singular linear system: BVP does not have a unique solution

• Existence of solutions:

$$\int_0^1 u''(x)\,dx = \int_0^1 f(x)\,dx \iff u'(1)-u'(0) = \int_0^1 f(x)\,dx$$

$$\iff \underline{\underline{\sigma_1 - \sigma_0 = \int_0^1 f(x)\,dx.}}$$

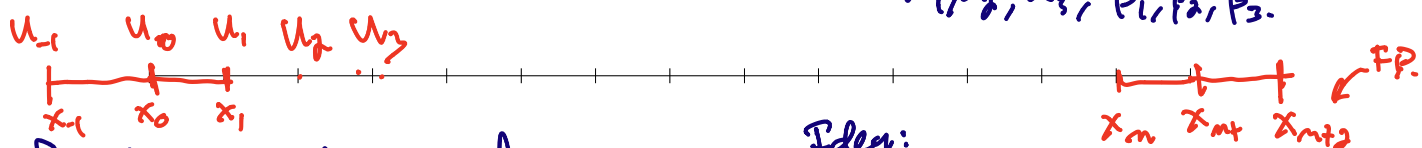
Compatibility condition.

• If this is satisfied then there exists a solution that is unique up to an additive constant: $u(x)+c$.

General boundary conditions: Robin conditions

Equation: $u''(x) = f(x)$, $0 < x < 1$ Boundary conditions: $\alpha_1 u(0) + \alpha_2 u'(0) = \alpha_3$ and $\beta_1 u(1) + \beta_2 u'(1) = \beta_3$

Given $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$.



Use fictitious point method:

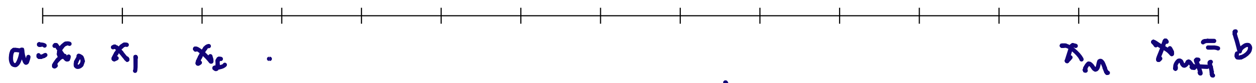
$$\alpha_1 U_0 + \frac{\alpha_2}{2h}(U_1 - U_{-1}) = \alpha_3$$

Idea: Solve for U_{-1} and use in the approximation

$$\frac{1}{h^2}(U_{-1} - 2U_0 + U_1) = f_0$$

General Linear two-point boundary value problem (Section 2.15)

Equation: $u''(x) = p(x)u'(x) + q(x)u(x) + r(x), \;\; a < x < b$ Boundary conditions: $u(a) = \alpha \;\; u(b) = \beta$



$$x_j = a + jh, \quad h = \frac{b-a}{m+1}$$

Idea: Write down discrete approximations for each term in the boundary value problem at the grid points

$$\begin{bmatrix} u''(x_1) \\ u''(x_2) \\ u''(x_3) \\ \vdots \\ \vdots \\ u''(x_{m-1}) \\ u''(x_m) \end{bmatrix} \approx \underbrace{\frac{1}{h^2} \begin{bmatrix} -2 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & -2 & 1 & \ddots & & & \vdots \\ 0 & 1 & -2 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 1 & -2 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & -2 \end{bmatrix}}_{D_2^h} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ \vdots \\ u_{m-1} \\ u_m \end{bmatrix}}_{\underline{u}^h} + \frac{1}{h^2} \begin{bmatrix} \alpha \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \beta \end{bmatrix}$$

$$
\begin{bmatrix} p(x_1)u'(x_1) \\ p(x_2)u'(x_2) \\ p(x_3)u'(x_3) \\ \vdots \\ \vdots \\ p(x_{m-1})u'(x_{m-1}) \\ p(x_m)u'(x_m) \end{bmatrix}
\approx
\underbrace{\begin{bmatrix}
p(x_1) & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & p(x_2) & 0 & \ddots & & & \vdots \\
0 & 0 & p(x_3) & 0 & \ddots & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & & \ddots & 0 & p(x_{m-1}) & 0 \\
0 & \cdots & \cdots & \cdots & 0 & 0 & p(x_m)
\end{bmatrix}}_{P^h}
\underbrace{\frac{1}{2h}\begin{bmatrix}
0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\
-1 & 0 & 1 & \ddots & & & \vdots \\
0 & -1 & 0 & 1 & \ddots & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & & \ddots & -1 & 0 & 1 \\
0 & \cdots & \cdots & \cdots & 0 & -1 & 0
\end{bmatrix}}_{D_1^h}
\underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ \vdots \\ u_{m-1} \\ u_m \end{bmatrix}}_{\underline{u}^h}
+\frac{1}{\cancel{h^2}\,2h}
\begin{bmatrix} -p(x_1)\alpha \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ p(x_m)\beta \end{bmatrix}
$$

$$
\begin{bmatrix} q(x_1)u(x_1) \\ q(x_2)u(x_2) \\ q(x_3)u(x_3) \\ \vdots \\ \vdots \\ q(x_{m-1})u(x_{m-1}) \\ q(x_m)u(x_m) \end{bmatrix}
\approx
\underbrace{\begin{bmatrix}
q(x_1) & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & q(x_2) & 0 & \ddots & & & \vdots \\
0 & 0 & q(x_3) & 0 & \ddots & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & & \ddots & 0 & q(x_{m-1}) & 0 \\
0 & \cdots & \cdots & \cdots & 0 & 0 & q(x_m)
\end{bmatrix}}_{Q^h}
\underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ \vdots \\ u_{m-1} \\ u_m \end{bmatrix}}_{\underline{u}^h}
$$

$$
\begin{bmatrix} r(x_1) \\ r(x_2) \\ r(x_3) \\ \vdots \\ \vdots \\ r(x_{m-1}) \\ r(x_m) \end{bmatrix} = \underline{r}^h
$$

Discrete system:

$$D_2^h \underline{u}^h = P^h D_1^h \underline{u}^h + Q^h \underline{u}^h + \underbrace{\underline{r}^h - \left(\frac{1}{h^2} + \frac{p(x_1)}{2h}\right)\alpha \underline{e}_1 - \left(\frac{1}{h^2} - \frac{p(x_m)}{2h}\right)\beta \underline{e}_m}_{\underline{f}^h}$$

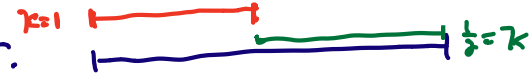
$$\underbrace{\left(D_2^h - P^h D_1 - Q^h\right)}_{A^h} \underline{u}^h = \underline{f}^h \iff \boxed{A^h \underline{u}^h = \underline{f}^h}$$

$$A^h = \begin{bmatrix} -\frac{2}{h^2} - q(x_1) & \frac{1}{h^2} - \frac{p(x_1)}{2h} & & & 0 \\ \frac{1}{h^2} + p(x_2) & -\frac{2}{h^2} - q(x_2) & \frac{1}{h^2} - \frac{p(x_2)}{2h} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ 0 & & & \frac{1}{h^2} + p(x_m) & -\frac{2}{h^2} - q(x_m) \end{bmatrix}$$

Theorem

Suppose that p, q, and r are continuous and $q(x) \geq 0$ on $[a, b]$. Then the system $A^h \underline{u}^h = \underline{f}^h$ has a unique solution if $h < 2/L$, where $L = \max\limits_{a \leq x \leq b} |p(x)|$.

Variable coefficient diffusion equation



material parameter.

Equation: $(\kappa(x)u'(x))' = f(x)$, $a < x < b$, $\kappa(x) > 0$ Boundary conditions: $u(a) = \alpha$ $u(b) = \beta$

Idea 1: Expand derivatives via the chain rule and discretize

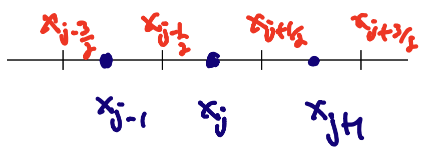
$$\Rightarrow \quad \kappa(x)\,u''(x) + \kappa'(x)\,u'(x) = f(x)$$

$$\Rightarrow \quad u''(x) = \underbrace{-\frac{\kappa'(x)}{\kappa(x)}}_{p(x)}\,u'(x) + \underbrace{\frac{f(x)}{\kappa(x)}}_{r(x)}$$

$q(x) = 0$

Idea 2: Discretize the problem directly: staggered differencing

$h = \dfrac{b-a}{m+1}$, $x_j = a + jh$



$j = 1, 2, \ldots, m.$

Approximate $\kappa(x)u'(x)$ at the 1/2 integer grid points:

$$\kappa(x)u'(x)\Big|_{x=x_{j-1/2}} = \kappa(x_{j-1/2})\left(\frac{-u_{j-1}+u_j}{h}\right) + O(h^2)$$

$$\kappa(x)u'(x)\Big|_{x=x_{j+1/2}} = \kappa(x_{j+1/2})\left(\frac{u_{j+1}-u_j}{h}\right) + O(h^2)$$

$$\Rightarrow \left(k(x)u'(x)\right)'\Big|_{x=x_j} = \frac{1}{h}\left[\underbrace{k(x_{j+1/2})}_{k_{j+1/2}}\left(\frac{u_{j+1}-u_j}{h}\right) - \underbrace{k(x_{j-1/2})}_{k_{j-1/2}}\left(\frac{-u_{j-1}+u_j}{h}\right)\right] + O(h^2)$$

$$= \frac{1}{h^2}\left[k_{j-1/2}u_{j-1} - \left(k_{j-1/2}+k_{j+1/2}\right)u_j + k_{j+1/2}u_{j+1}\right] + O(h^2)$$

Linear system:

$$\frac{1}{h^2}\begin{bmatrix} -(k_{1/2}+k_{3/2}) & k_{3/2} & & & 0 \\ k_{3/2} & -(k_{3/2}+k_{5/2}) & k_{5/2} & & \\ \ddots & \ddots & \ddots & & \\ 0 & k_{m-3/2} & -(k_{m-3/2}+k_{m-1/2}) & k_{m-1/2} & \\ & & k_{m-1/2} & -(k_{m-1/2}+k_{m+1/2}) & \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{m-1} \\ u_m \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{m-1} \\ f_m \end{bmatrix}$$

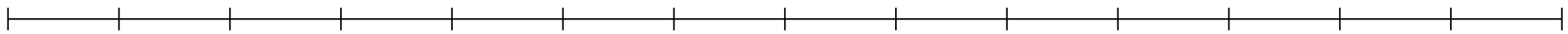
Higher-order finite difference (FD) methods (Section 2.20)

Model Problem:

Equation: $u''(x) = f(x),\ \ 0 < x < 1$ Boundary conditions: $u(0) = \alpha,\ \ u(1) = \beta$

Ideas for getting fourth-order accuracy

1) Wide FD stencils



$$\frac{1}{h^2}\begin{bmatrix} -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} \end{bmatrix}\begin{bmatrix} u(x_{j-2}) \\ u(x_{j-1}) \\ u(x_j) \\ u(x_{j+1}) \\ u(x_{j+2}) \end{bmatrix} = f(x_j) + \mathcal{O}(h^2),\ \ j = 2, \dots, m-1$$

$$\frac{1}{h^2}\begin{bmatrix} \frac{5}{6} & -\frac{5}{4} & -\frac{1}{3} & \frac{7}{6} & -\frac{1}{2} & \frac{1}{12} \end{bmatrix}\begin{bmatrix} u(x_0) \\ u(x_1) \\ u(x_2) \\ u(x_3) \\ u(x_4) \\ u(x_5) \end{bmatrix} = f(x_1) + \mathcal{O}(h^2) \quad \text{and} \quad \frac{1}{h^2}\begin{bmatrix} \frac{1}{12} & -\frac{1}{2} & \frac{7}{6} & -\frac{1}{3} & -\frac{5}{4} & \frac{5}{6} \end{bmatrix}\begin{bmatrix} u(x_{m-4}) \\ u(x_{m-3}) \\ u(x_{m-2}) \\ u(x_{m-1}) \\ u(x_m) \\ u(x_{m+1}) \end{bmatrix} = f(x_m) + \mathcal{O}(h^2)$$

2) Compact (Implicit) FD stencils (also called “deferred correction”)