Linear algebra
& Numerical Analysis

Orthogonalization

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Outline

- Orthogonal matrices
- Gram-Schmidt process
- Givens rotations
- Householder transformation

Euclidean inner product

- Euclidean inner product (dot product) is a mapping

$$\mathbf{u} \in \mathrm{R}^n,\ \mathbf{v} \in \mathrm{R}^n \longrightarrow (\mathbf{u}, \mathbf{v}) \in \mathrm{R}$$

- It is defined for two vectors **u**, **v** by

$$\boxed{(\mathbf{u}, \mathbf{v}) = \mathbf{u}_1\, \mathbf{v}_1 + \ldots + \mathbf{u}_n\, \mathbf{v}_n = \mathbf{u}^T\mathbf{v}}$$

- It satisfies for each **u**, **v**, **w** a $\alpha \in \mathrm{R}$ the following properties

$$(\mathbf{u}+\mathbf{v}, \mathbf{w}) = (\mathbf{u}, \mathbf{w}) + (\mathbf{v}, \mathbf{w})$$

$$(\alpha\mathbf{u}, \mathbf{v}) = \alpha(\mathbf{u}, \mathbf{v})$$

$$(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})$$

$$(\mathbf{u}, \mathbf{u}) > 0 \qquad \text{pro} \quad \mathbf{u} \neq \mathbf{o}$$

Orthogonal matrices

Square matrix **Q** satisfying

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$$

is called orthogonal matrix. This matrix has orthogonal columns and it holds

$$\mathbf{Q}^{-1} = \mathbf{Q}^T.$$

The columns form an orthogonal set of vectors, i.e.

$$(\mathbf{q}_i, \mathbf{q}_j) = \mathbf{q}_i^T \mathbf{q}_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

QR factorization

For an arbitrary matrix $\mathbf{A} \in \mathbb{R}^{m,n}$ there exist an orthogonal matrix $\mathbf{Q} \in \mathbb{R}^{m,m}$ and an upper triangular matrix $\mathbf{R} \in \mathbb{R}^{m,n}$, such that

$$\mathbf{A} = \mathbf{QR}.$$

QR factorization

m = n:

$$\left(\begin{array}{c|c|c|c} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{array}\right) = \left(\begin{array}{c|c|c|c} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \end{array}\right) \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ & r_{22} & & r_{2n} \\ & & \ddots & \vdots \\ & & & r_{nn} \end{pmatrix}$$

- The columns of **A** can be written as a linear combination of the columns of **Q**

$$\begin{aligned} \mathbf{a}_1 &= r_{11}\mathbf{q}_1, \\ \mathbf{a}_2 &= r_{12}\mathbf{q}_1 + r_{22}\mathbf{q}_2, \\ &\vdots \\ \mathbf{a}_n &= r_{1n}\mathbf{q}_1 + r_{2n}\mathbf{q}_2 + \cdots + r_{nn}\mathbf{q}_n \end{aligned}$$

Gram-Schmidt process

Example:

For matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Find the columns $\mathbf{q}_1, \ldots, \mathbf{q}_n$ and the elements of matrix **R.**

Gram-Schmidt process

$$a_1 = r_{11} \cdot q_1 \qquad (1a)$$

$$a_2 = r_{12} \cdot q_1 + r_{22} \cdot q_2 \qquad (1b)$$

$$a_3 = r_{13} \cdot q_1 + r_{23} \cdot q_2 + r_{33} \cdot q_3 \qquad (1c)$$

---

from (1a)

$$\|a\| = \sqrt{(a,a)}$$

$$v_1 = r_{11} \cdot q_1 \qquad q_1 = \frac{v_1}{\|v_1\|} = \frac{[1 \;\; 0 \;\; 1]^T}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$v_1 = a_1$$

$$\|v_1\| = \sqrt{r_{11}^2 \cdot \underbrace{(q_1, q_1)}_{1}} = r_{11}$$

$$r_{11} = \|v_1\| = \sqrt{(v_1, v_1)} = \sqrt{1+0+1} = \sqrt{2}$$

Gram-Schmidt process

from (18)

$$r_2 = r_{22} \cdot q_2$$

$$r_2 = a_2 - r_{12} \cdot q_1$$

since r_2 should be orthonormal to q_1

$$\Rightarrow (q_1, r_2) = 0$$

$$0 = (q_1, r_2) = (q_1, a_2 - r_{12} \cdot q_1) = (q_1, a_2) - r_{12} \underbrace{(q_1, q_1)}_{1} = 0$$

$$\Rightarrow r_{12} = (q_1, a_2) = \frac{1}{\sqrt{2}}(1 + 0 + 0) = \frac{1}{\sqrt{2}}$$

Gram-Schmidt process

$$r_2 = a_2 - r_{12}\cdot q_1 = \begin{bmatrix}1\\1\\0\end{bmatrix} - \underbrace{\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}}_{\frac{1}{2}}\cdot\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}1/2\\1\\-1/2\end{bmatrix}$$

$$r_{22} = \|r_2\| = \sqrt{(1/2)^2 + 1^2 + (-1/2)^2} = \sqrt{\frac{1}{4} + 1 + \frac{1}{4}} = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$$

rationalization

$$q_2 = \frac{r_2}{\|r_2\|} = \frac{2}{\sqrt{6}}\cdot\begin{bmatrix}1/2\\1\\-1/2\end{bmatrix} = \frac{1}{\sqrt{6}}\cdot\begin{bmatrix}1\\2\\-1\end{bmatrix}$$

Gram-Schmidt process

from (1c)

$v_3 = r_3 \cdot q_3$

$v_3 = a_3 - r_{13} \cdot q_1 - r_{23} \cdot q_2$ #

since v_3 should be orthogonal to q_1, q_2

$\Rightarrow (q_1, v_3) = 0$ and $(q_2, v_3) = 0$

$$0 = (q_1, v_3) = (q_1, a_3 - r_{13} \cdot q_1 \mp r_{23} \cdot q_2) = (q_1, a_3) - r_{13} \overset{1}{(q_1, q_1)} \cancel{- r_{23} \cdot \overset{0}{(q_1, q_2)}}$$

$$= (q_1, a_3) - r_{13} = 0 \Rightarrow r_{13} = (q_1, a_3) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}^T \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} (0 + 0 + 1) = \frac{1}{\sqrt{2}}$$

$$0 = (q_2, v_3) = (q_2, a_3 - r_{13} \cdot q_1 - r_{23} \cdot q_2) = (q_2, a_3) \cancel{- r_{13} \cdot \overset{0}{(q_2, q_1)}} - r_{23} \cdot \overset{1}{(q_2, q_2)}$$

$$= (q_2, a_3) - r_{23} = 0 \Rightarrow r_{23} = (q_2, a_3) = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}^T \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = (0 + 2 - 1) \frac{1}{\sqrt{6}} = \frac{1}{\sqrt{6}}$$

Gram-Schmidt process

$$r_3 = a_3 - r_{13} \cdot q_1 - r_{23} \cdot q_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \overset{-\frac{1}{2}}{-\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \overset{-\frac{1}{6}}{-\frac{1}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \frac{2}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$0 - \frac{3}{6} - \frac{1}{6} = \frac{2}{3}$$

$$1 - 0 - \frac{2}{6} = \frac{2}{3}$$

$$1 - \frac{3}{6} + \frac{1}{6} = \frac{2}{3}$$

$$r_{33} = \|r_3\| = \sqrt{\left(\frac{2}{3}\right)^2 (1^2 + 1^2 + 1^2)} = \sqrt{\frac{4}{\cancel{9}_{3}} \cdot \cancel{3}} = \frac{2}{\sqrt{3}}$$

Gram-Schmidt process

$$q_3 = \frac{r_3}{\|r_3\|} = \frac{\sqrt{3}}{\cancel{2}} \cdot \frac{\cancel{2}}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\sqrt{3}\,\sqrt{3}$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & 1/\sqrt{6} & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \qquad R = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ & \sqrt{6}/2 & 1/\sqrt{6} \\ & & 2/\sqrt{3} \end{bmatrix}$$

Gram-Schmidt process

Problem: Find the columns $\mathbf{q}_1, \ldots, \mathbf{q}_n$ and the elements of matrix **R**

Process:

- $$\mathbf{v}_1 = \mathbf{a}_1, \quad \mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \quad r_{11} = \|\mathbf{v}_1\|$$

$\boxed{||a||^2 = (a,a)}$

- Let us consider that we already know $\mathbf{q}_1, \ldots, \mathbf{q}_{j-1}$

$$\mathbf{a}_j = r_{1j}\mathbf{q}_1 + \cdots + r_{ij}\mathbf{q}_i + \cdots + r_{j-1,j}\mathbf{q}_{j-1} + r_{jj}\mathbf{q}_j \rightarrow \mathbf{v}_j$$

(*)

$$\Downarrow$$

$$\mathbf{v}_j = \mathbf{a}_j - r_{1j}\mathbf{q}_1 - \ldots - r_{ij}\mathbf{q}_i - \ldots - r_{j-1,j}\mathbf{q}_{j-1}$$

Gram-Schmidt process

- **Requirement:** $\mathbf{v}_j$ is orthogonal to $\mathbf{q}_1, \ldots, \mathbf{q}_{j-1}$

$\Downarrow$

for $k = 1, \ldots, j-1$

$$0 = (\mathbf{q}_k, \mathbf{v}_j) = (\mathbf{q}_k, \boxed{\mathbf{a}_j} - r_{1j}\mathbf{q}_1 - \ldots \boxed{- r_{ij}\mathbf{q}_i} - \ldots - r_{j-1,j}\mathbf{q}_{j-1})$$

- Since $(\mathbf{q}_k, \mathbf{q}_i) = 0$ for $k \neq i$

$$(\mathbf{q}_i, \mathbf{a}_j - r_{ij}\mathbf{q}_i) = (\mathbf{q}_i, \mathbf{a}_j) - r_{ij}\underbrace{(\mathbf{q}_i, \mathbf{q}_i)}_{=1} = 0 \quad \Rightarrow \quad r_{ij} = (\mathbf{q}_i, \mathbf{a}_j)$$

$$\mathbf{q}_j = \frac{\mathbf{v}_j}{\|\mathbf{v}_j\|} = \frac{\mathbf{v}_j}{r_{jj}} \quad \Rightarrow \quad r_{jj} = \|\mathbf{v}_j\|$$

$\boxed{\| v_j \|^2 = r_{jj}^2 \, q_j^T q_j = r_{jj}^2}$

Algorithm

```
function [Q,R] = my_gram_schmidt(A)
n = size(A,1); Q = zeros(n); R = zeros(n);
for j=1:n
    v=A(:,j);
    for i=1:j-1
        R(i,j)=Q(:,i)'*A(:,j);
        v=v-R(i,j)*Q(:,i);
    end
    R(j,j)= norm(v);
    Q(:,j)=v/R(j,j);
end
```

modify **A(:,j)** to **v** for more accuracy

$r_{ij} = (\mathbf{q}_i, \mathbf{a}_j)$

$\mathbf{v}_j = \mathbf{a}_j - r_{1j}\mathbf{q}_1 - \ldots - r_{ij}\mathbf{q}_i - \ldots - r_{j-1,j}\mathbf{q}_{j-1}$

$r_{jj} = \| v_j \|$

$\mathbf{q}_j = \frac{\mathbf{v}_j}{r_{jj}}$

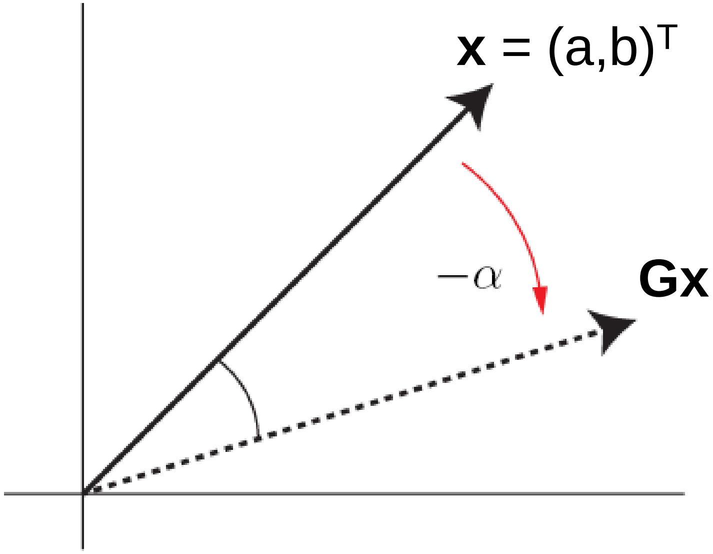
Remark: The Gram-Schmidt process can be stabilized by a small modification ➔ **modified Gram-Schmidt**, which gives the same result as the original formula in exact arithmetic and introduces smaller errors in finite-precision arithmetic.

Givens transformation

Let us consider **Givens matrix (rotation matrix)**

$$\mathbf{G} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

which rotates a vector $(a,b)^T$ in the *xy*-plane through an angle $-\alpha$ about the origin.



We will use a notation

$c = \cos\alpha$ a $s = \sin\alpha$

Example in Matlab: givens_rotation

Givens transformation

- We can use matrix **G** to zeroing elements. Let us consider that

$$\mathbf{G}^T\mathbf{x} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sqrt{a^2+b^2} \\ 0 \end{pmatrix}$$

- It is easy to see that

$$c = \frac{a}{\sqrt{a^2+b^2}} \quad \text{a} \quad s = \frac{-b}{\sqrt{a^2+b^2}}$$

Givens transformation

- Generalization for vectors of the order n

$$
\mathbf{G}_{ij} = \begin{array}{cc} \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & c & & s & & \\ & & & \ddots & & & \\ & & -s & & c & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} & \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \\ \end{array} \end{array}
$$

$$\mathbf{y} = \mathbf{G}_{ij}^T \mathbf{x}.$$

$$
y_k = \begin{cases} c x_i - s x_j, & k = i, \\ s x_i + c x_j, & k = j, \\ x_k, & k \neq i, j. \end{cases}
$$

- To zeroing the element in the j-th row we have

$$
c = \frac{x_i}{\sqrt{x_i^2 + x_j^2}}, \quad s = \frac{-x_j}{\sqrt{x_i^2 + x_j^2}}
$$

Givens QR method

- Form of the rotation matrix to zeroing element in the i-th row is

$$\mathbf{G}(i-1,i), \quad \text{where } c = \frac{x_{i-1}}{\sqrt{x_{i-1}^2 + x_i^2}}, \qquad s = \frac{-x_i}{\sqrt{x_{i-1}^2 + x_i^2}}$$

- Using the rotation matrices we will edit matrix **A**:

$$\mathbf{A} = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \xrightarrow{\mathbf{G}(2,3)^T} \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & * & * \end{pmatrix} \xrightarrow{\mathbf{G}(1,2)^T} \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \xrightarrow{\mathbf{G}(2,3)^T} \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} = \mathbf{R}.$$

$$\left\{ \mathbf{G}_p^T \ldots \mathbf{G}_1^T \mathbf{A} = \mathbf{R} \quad \Leftrightarrow \quad \mathbf{Q}^T \mathbf{A} = \mathbf{R} \right\}$$

$$\Downarrow$$

$$\mathbf{A} = \mathbf{QR}, \quad \mathbf{Q} = \mathbf{G}_1 \ldots \mathbf{G}_p.$$

Example

- **Example:**

$$\mathbf{A} = \begin{pmatrix} -1 & 4 & -1 \\ -2 & -1 & -11 \\ 2 & 10 & 2 \end{pmatrix}$$

- **Solution:**

To set **A(3,1) = 0**, we need to build matrix $\mathbf{G_1(2,3)}$

$$c = \frac{-2}{\sqrt{(-2)^2 + 2^2}} = \frac{-\sqrt{2}}{2}, \quad s = \frac{-2}{\sqrt{(-2)^2 + 2^2}} = \frac{-\sqrt{2}}{2}$$

$$\mathbf{G}(i-1, i)$$

$$c = \frac{x_{i-1}}{\sqrt{x_{i-1}^2 + x_i^2}}$$

$$s = \frac{-x_i}{\sqrt{x_{i-1}^2 + x_i^2}}$$

$$\mathbf{G}_1(2,3) = \begin{pmatrix} 1 & & \\ & -\sqrt{2}/2 & -\sqrt{2}/2 \\ & \sqrt{2}/2 & -\sqrt{2}/2 \end{pmatrix}$$

$$\mathbf{G(i\text{-}1,i)} = \begin{array}{c} \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & c & & s & & \\ & & & \ddots & & & \\ & & -s & & c & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \textbf{\textit{i-1}} \\ \\ \textbf{\textit{i}} \\ \\ \\ \end{array}$$

Example

$$\mathbf{A}_1 = \mathbf{G}_1(2,3)^T\mathbf{A} = \begin{pmatrix} -1 & 4 & -1 \\ 2\sqrt{2} & 11\sqrt{2}/2 & 13\sqrt{2}/2 \\ 0 & -9\sqrt{2}/2 & 9\sqrt{2}/2 \end{pmatrix}$$

- To set **$\mathbf{A_1(2,1) = 0}$**, we need to build matrix **$\mathbf{G_2(1,2)}$**

$$c = -1/3, \qquad s = -2\sqrt{2}/3 \qquad \mathbf{G}_2(1,2) = \begin{pmatrix} -1/3 & -2\sqrt{2}/3 & \\ 2\sqrt{2}/3 & -1/3 & \\ & & 1 \end{pmatrix}$$

$$\mathbf{A}_2 = \mathbf{G}_2(1,2)^T\mathbf{A}_1 = \begin{pmatrix} 3 & 6 & 9 \\ 0 & -9\sqrt{2}/2 & -3\sqrt{2}/2 \\ 0 & -9\sqrt{2}/2 & 9\sqrt{2}/2 \end{pmatrix}$$

Example

- To set $\mathbf{A_2(3,2) = 0}$, we need matrix $\mathbf{G_3(2,3)}$

$$c = -\sqrt{2}/2, \qquad s = -\sqrt{2}/2$$

$$\mathbf{G}_3(2,3) = \begin{pmatrix} 1 & & \\ & -\sqrt{2}/2 & \sqrt{2}/2 \\ & -\sqrt{2}/2 & -\sqrt{2}/2 \end{pmatrix}$$

$$\mathbf{A}_3 = \mathbf{G}_3(2,3)^T\mathbf{A}_2 = \begin{pmatrix} 3 & 6 & 9 \\ 0 & 9 & -3 \\ 0 & 0 & -6 \end{pmatrix} = \mathbf{R}$$

$$\mathbf{Q} = \mathbf{G}_1(2,3)\mathbf{G}_2(1,2)\mathbf{G}_3(2,3) = \begin{pmatrix} -1/3 & 2/3 & -2/3 \\ -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \end{pmatrix}$$

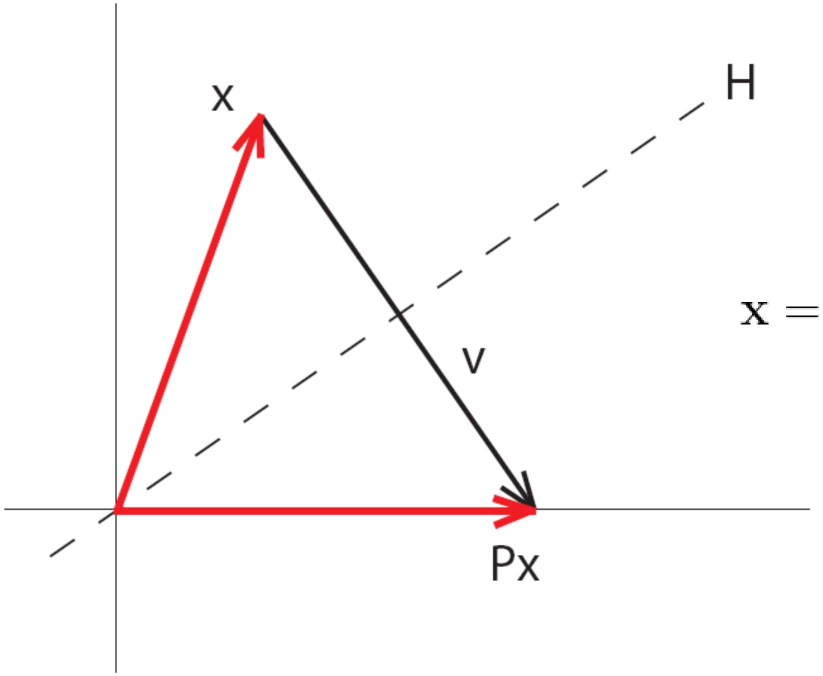
Algorithm

```
function [Q,R] = my_givens_QR(A)
n = size(A,1); Q=eye(n); R=A;
for j=1:n
    for i=n:(-1):j+1
        x=R(:,j);
        if norm([x(i-1),x(i)])>0
            c=x(i-1)/norm([x(i-1),x(i)]);
            s=-x(i)/norm([x(i-1),x(i)]);

            G=eye(n); G([i-1,i],[i-1,i])=[c,s;-s,c];
            R=G'*R;
            Q=Q*G;
        end
    end
end
```

Householder transformation

- For vector **x** we are able to find its reflection **Px** to axis x



$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} \xrightarrow{\mathbf{P}} \quad \mathbf{Px} = \begin{pmatrix} \|\mathbf{x}\| \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \|\mathbf{x}\|\mathbf{e}_1$$

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0)^T$$

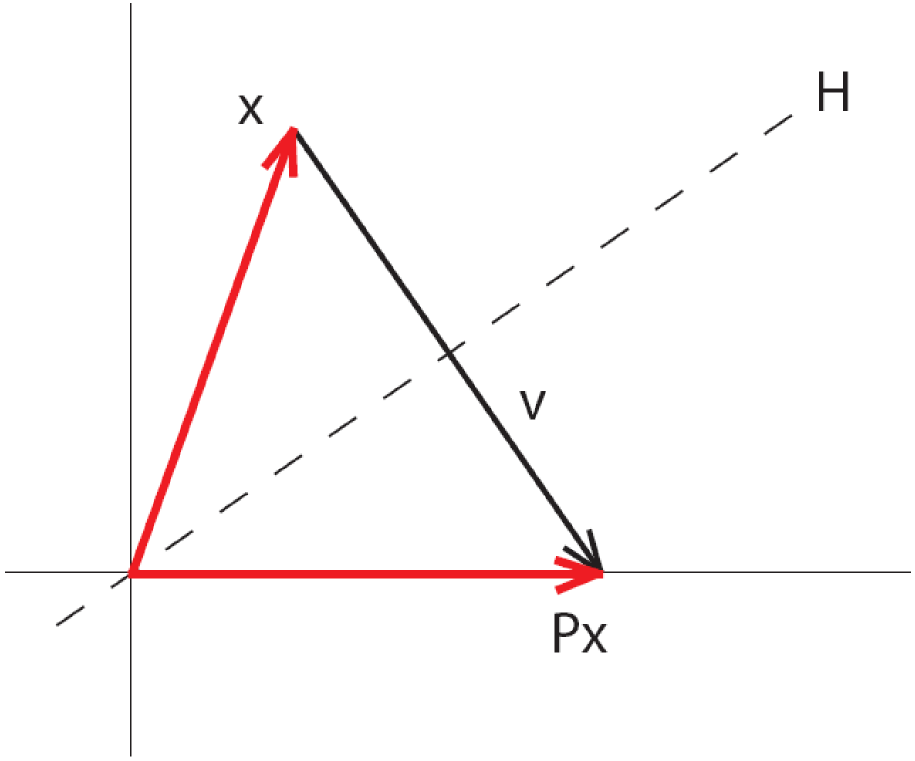
- Both vectors have the same length

$$\|\mathbf{Px}\| = [\mathbf{Px}]_1 = \|\mathbf{x}\|$$

Householder transformation

- **Px** is mirror image of **x** with axis H. H is orthogonal to vector

$$\mathbf{v} = \mathbf{P}\mathbf{x} - \mathbf{x} = \|\mathbf{x}\|\mathbf{e}_1 - \mathbf{x}$$

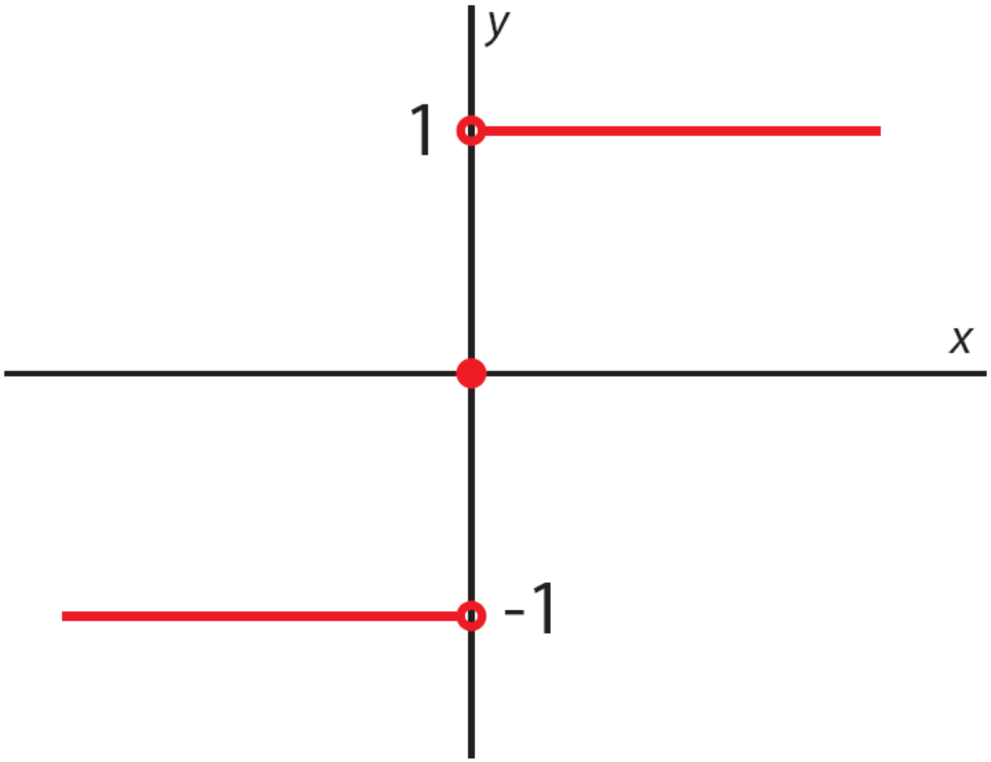


Reflection matrix:

$$\mathbf{P} = \mathbf{I} - 2\frac{\mathbf{v}\mathbf{v}^T}{\|\mathbf{v}\|^2} = \mathbf{I} - 2\frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}}$$

Function sign

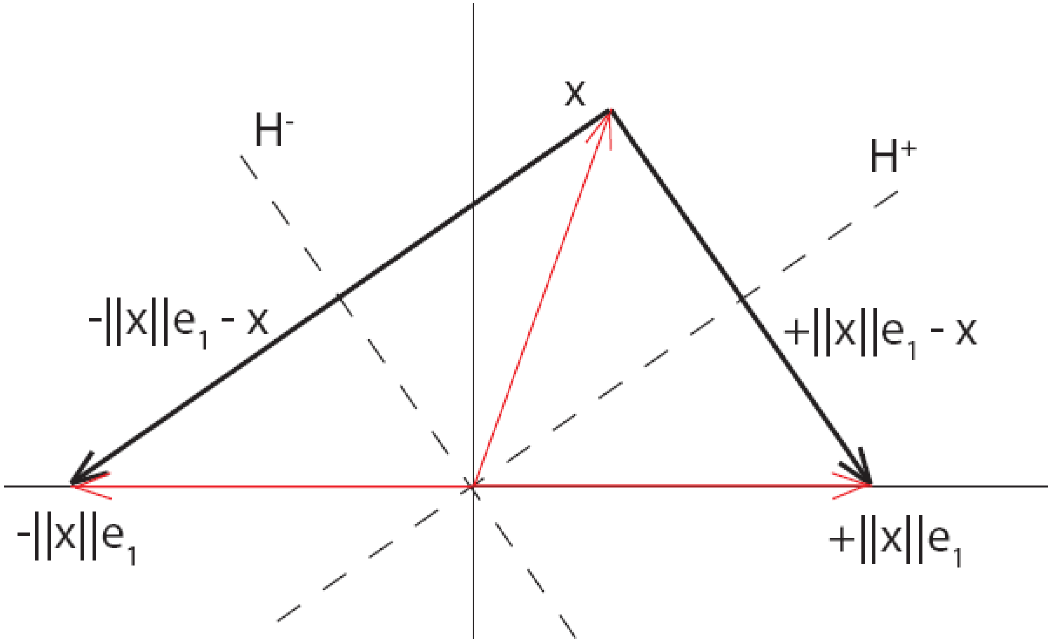
$$\text{sign}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$



Householder transformation

- The image of **x** is not unique: $z\|\mathbf{x}\|\mathbf{e}_1$ for all $|z| = 1$
- For numerical stability we choose $z = -\text{sign}(x_1)$

$$\boxed{\mathbf{v} = -\text{sign}(x_1)\|\mathbf{x}\|\mathbf{e}_1 - \mathbf{x}}$$



Householder QR method

- Using the reflections **P** we can modify matrix **A**

$$\mathbf{A} = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \xrightarrow{\mathbf{P}_1} \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \xrightarrow{\mathbf{P}_2} \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix} \xrightarrow{\mathbf{P}_3} \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{R}$$

$$\mathbf{P}_n \cdots \mathbf{P}_2 \mathbf{P}_1 \mathbf{A} = \mathbf{Q}^T \mathbf{A} = \mathbf{R}$$

$$\Downarrow$$

$$\mathbf{A} = \mathbf{Q}\mathbf{R}, \quad \mathbf{Q} = \mathbf{P}_1 \ldots \mathbf{P}_n$$

Example

- **Example:**

$$\mathbf{A} = \begin{pmatrix} -1 & 4 & -1 \\ -2 & -1 & -11 \\ 2 & 10 & 2 \end{pmatrix}$$

- **Solution:**

$$\mathbf{x} = (-1, -2, 2)^T, \quad \|\mathbf{x}\| = \sqrt{1+4+4} = 3, \quad \mathrm{sign}(x_1) = -1$$

$$\mathbf{v} = -\mathrm{sign}(x_1)\|\mathbf{x}\|\mathbf{e}_1 - \mathbf{x} = \|\mathbf{x}\|\mathbf{e}_1 - \mathbf{x} =$$

$$= \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$$

Example

$$\mathbf{P}_1 = \mathbf{I} - 2\frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}} = \begin{pmatrix} -1/3 & -2/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 2/3 & 1/3 & 2/3 \end{pmatrix}$$

$$\mathbf{R}_1 = \mathbf{P}_1\mathbf{A} = \begin{pmatrix} 3 & 6 & 9 \\ 0 & 0 & -6 \\ 0 & 9 & -3 \end{pmatrix}$$

$$\mathbf{x} = (0, 9)^T, \qquad \|\mathbf{x}\| = 9, \qquad \mathrm{sign}(x_1) = 1$$

$$\mathbf{v} = -\|\mathbf{x}\|\mathbf{e}_1 - \mathbf{x} = \begin{pmatrix} -9 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 9 \end{pmatrix} = \begin{pmatrix} -9 \\ -9 \end{pmatrix}$$

Example

$$\mathbf{P'}_2 = \mathbf{I} - 2\frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \Rightarrow \quad \mathbf{P}_2 = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & -1 \\ 0 & -1 & 0 \end{array}\right)$$

$$\mathbf{R}_2 = \mathbf{P}_2\mathbf{P}_1\mathbf{A} = \begin{pmatrix} 3 & 6 & 9 \\ 0 & -9 & 3 \\ 0 & 0 & 6 \end{pmatrix}$$

$$\mathbf{Q} = \mathbf{P}_1\mathbf{P}_2 = \begin{pmatrix} -1/3 & -2/3 & 2/3 \\ -2/3 & -1/3 & -2/3 \\ 2/3 & -2/3 & -1/3 \end{pmatrix}$$

Algorithm

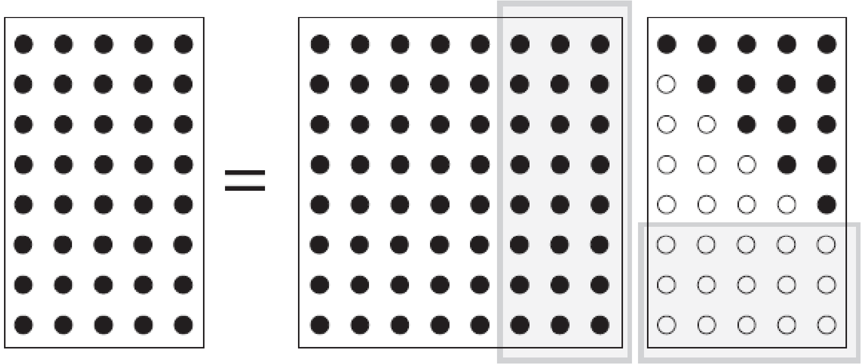
```
function [Q,R] = my_householder_QR(A)
n = size(A,1); Q=eye(n); R=A; I = eye(n);
for j=1:n-1
    x=R(j:n,j);
    v=-sign(x(1))*norm(x)*eye(n-j+1,1)-x;
    if norm(v)>0,
        v=v/norm(v);
        P=I; P(j:n,j:n)=P(j:n,j:n)-2*v*v';
        R=P*R;
        Q=Q*P;
    end
end
```

QR factorization

m > n:

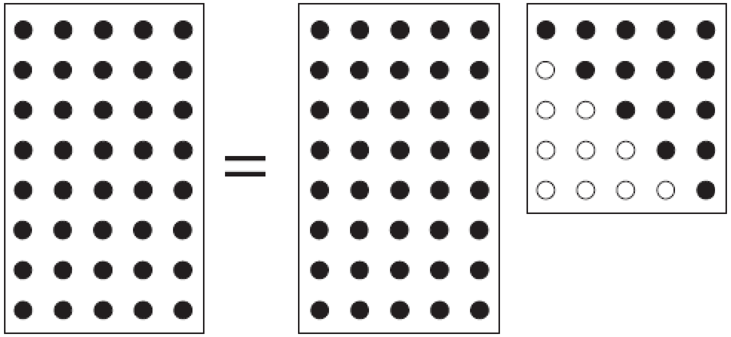
- **Full QR factorization**

$$A = QR$$



- **Reduced QR factorization**

$$A = \tilde{Q}\tilde{R}$$



Matlab function: `qr`

- `[Q,R] = qr(A),` where A is m-by-n, produces an m-by-n upper triangular matrix **R** and an m-by-m unitary matrix **Q** so that **A = Q*R**.

- `[Q,R] = qr(A,0)` produces the "economy size" decomposition. If m>n, only the first n columns of **Q** and the first n rows of **R** are computed. If m<=n, this is the same as `[Q,R] = qr(A)`

For more details see: `help qr`

References

- Gilbert W. Stewart, **Matrix Algorithms: Basic decompositions (available on Google books)**

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- **Gram-Schmidt in 9 Lines of MATLAB** http://ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010/related-resources/MIT18_06S10_gramschmidtmat.pdf