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Home work 4
Math 566

1. (Another way to solve the least Squares problem)

Let A be a real m-by-n matrix with $m \geq n$ and rank$(A) = n$, and $b \in \mathbb{R}^m$

a) Show that the X that minimizes $\|AX - b\|_2$ is given by the Solution to the Square linear System.

$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

Given $\mathcal{A} = \begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix}$

The 2-by-2 block Gaussian Elimination of $\mathcal{A}$ will be

$$\begin{array}{c} R_1 \\ R_2 \end{array} \left[\begin{array}{cc|c} I & A & b \\ A^T & 0 & 0 \end{array}\right] \xrightarrow[A^T R_1 - R_2 \to R_2]{R_1 \to R_1} \left[\begin{array}{cc|c} I & A & b \\ 0 & A^T A & A^T b \end{array}\right]$$

then

$$\begin{bmatrix} I & A \\ 0 & A^T A \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ A^T b \end{bmatrix}$$

$$Ir + AX = b$$
$$A^T A X = A^T b$$

$X = (A^T A)^{-1} A^T b$ is the Solution that minimizes $\|AX - b\|_2$.

then $Ir + AX = b$

$r + AX = b \Rightarrow r = b - AX$, which is the residual of $AX = b$.

b) Determine the 2-norm condition number of $\tilde{A}$ in terms of the singular values of A

let $A = U \Sigma V^T$ be full SVD of A,

Consider

$$\tilde{A} = \begin{bmatrix} I & U\Sigma V^T \\ V\Sigma^T U^T & 0 \end{bmatrix}$$

let

$$P = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}, \quad P \text{ is orthogonal}$$

$$P^T \tilde{A} P = \underbrace{\begin{bmatrix} I & \Sigma \\ \Sigma^T & 0 \end{bmatrix}}_{\tilde{A}}$$

Computing Eigenvalues of $\tilde{A}$ and relate those to singular values of A

$$|\tilde{A} - \lambda I| = 0$$

$$\begin{vmatrix} I - \lambda I & \Sigma \\ \Sigma^T & -\lambda I \end{vmatrix} = 0$$

Given a 2×2 block matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and, $D \in \mathbb{R}^{m\times m}$

If D is invertible then the

$$\det(M) = \det(A - BD^{-1}C)\det(D)$$

Since

$$(1-\lambda)I \in \mathbb{R}^{n\times n}$$

$$\Sigma \in \mathbb{R}^{n\times m}, \ \Sigma^T \in \mathbb{R}^{m\times n}$$

$$-\lambda I \in \mathbb{R}^{m\times m} \text{ and invertible}$$

then

$$\begin{vmatrix} I-\lambda I & \Sigma \\ \Sigma^T & -\lambda I \end{vmatrix} = \det\left((1-\lambda)I - \Sigma(-\lambda I)^{-1}\Sigma^T\right)\det(-\lambda I)$$

Since $\Sigma = \Sigma^T$

$$\det\left((1-\lambda)I + \lambda^{-1}\Sigma^2\right)\det(-\lambda I) = 0$$

$$\det\left(\lambda^{-1}(\lambda(1-\lambda)I + \Sigma^2)\right)\det(-\lambda I) = 0$$

$$\det(-\lambda I) = (-\lambda)^m$$

$$(-\lambda)^m(\lambda)^{-m}\det\left(\lambda(1-\lambda)I + \Sigma^2\right) = 0$$

$$(-1)^m\det\left(\lambda(1-\lambda)I + \Sigma^2\right) = 0$$

$$\det\left(\lambda(\lambda-1)I - \Sigma^2\right) = 0$$

$$\Sigma^2 = \begin{bmatrix} \sigma_1^2 & & & & & \\ & \sigma_2^2 & & & & \\ & & \ddots & & & \\ & & & \sigma_n^2 & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix}$$

$$\det\left(\lambda(\lambda-1) I - \Sigma^2\right) = 0$$

$$\left| \begin{bmatrix} \lambda(\lambda-1) & & & \\ & \lambda(\lambda-1) & & 0 \\ & & \ddots & \\ 0 & & & \lambda(\lambda-1) \end{bmatrix} - \begin{bmatrix} \sigma_1^2 & & & & & \\ & \sigma_2^2 & & & & 0 \\ & & \ddots & & & \\ & & & \sigma_n^2 & & \\ & 0 & & & 0 & \\ & & & & & \ddots & 0 \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} \lambda(\lambda-1) - \sigma_1^2 & & & & & \\ & \lambda(\lambda-1) - \sigma_2^2 & & & & \\ & & \ddots & & & \\ & & & \lambda(\lambda-1) - \sigma_n^2 & & \\ & & & & \lambda(\lambda-1) & \\ & & & & & \ddots & \\ & & & & & & \lambda(\lambda-1) \end{bmatrix} \right| = 0$$

$$\lambda(\lambda-1) - \sigma_1^2 = \cdots = \lambda(\lambda-1) - \sigma_n^2 = 0$$

and

$$\lambda(\lambda-1) = 0$$

Hence solving for λ,

$$\lambda^2 - \lambda - \sigma_i^2 = 0 \quad , \quad i = 1, 2, 3, \ldots, n$$

$$\lambda = \frac{1 \pm \sqrt{1 + 4\sigma_i^2}}{2}$$

$$\lambda^2 - \lambda = 0 \Rightarrow \lambda = 1 \text{ or } \lambda = 0$$

The 2-norm Condition number $K(A)$

$$K(A) = \frac{\|\text{maximum Eigen Value}\|}{\|\text{minimum Eigen Value}\|}$$

$$K(A) = \left| \frac{1 + \sqrt{1 + 4\sigma_1^2}}{1 - \sqrt{1 + 4\sigma_n^2}} \right|$$

$$K(A) = \frac{1 + \sqrt{1 + 4\sigma_1^2}}{-1 + \sqrt{1 + 4\sigma_n^2}}$$

c) How does the Condition number of A Compare to the Condition number of the normal Equation Matrix A^TA?

For the Equation Matrix, the Condition

$K(A^TA)$

$$K(A^TA) = \frac{\sigma_1^2}{\sigma_n^2}$$

where is different

$$K = \frac{1 + \sqrt{1 + 4\sigma_1^2}}{-1 + \sqrt{1 + 4\sigma_n^2}}$$

~~d) Compute the polynomial coefficients from problem 1 of the home work~~

4(c) Show that growth for the matrix A (but for the general n-by-n case) is exactly

$$g_n(A) = 2^{n-1}$$

The growth factor of the $n \times n$ matrix A is given by

$$\rho = \frac{\max_{i,j} |u_{ij}|}{\max_{i,j} |a_{ij}|}$$

with U the upper triangular matrix obtained from A by performing LU decomposition of A

u_{ij} and a_{ij} are respective entries of the upper triangular matrix U and matrix A.

Consider for $n=3$ $A \in \mathbb{R}^{3\times3}$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

Performing Gaussian Elimination on A, we obtain

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

The maximum Entry in U is 4, and in A is 1
So the growth factor.

$$\rho = \frac{4}{1} = 4 = 2^{3-1} = 2^{n-1}$$

for $n=4$, $A \in \mathbb{R}^{4\times 4}$

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix} \rightarrow \underset{L}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & -1 & -1 & 1 \end{bmatrix}} \times \underset{U}{\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 8 \end{bmatrix}}$$

$\max|u_{ij}| = 8$, $\max|a_{ij}| = 1$

$$\rho = 8/1 = 2^{4-1} = \underline{2^{n-1}}$$

Since its true for the base case, Assume its true for $n=k-1$, $A \in \mathbb{R}^{(k-1)\times(k-1)}$

$$A = \begin{bmatrix} 1 & 0 & \cdots & 1 \\ -1 & 1 & 0 \cdots & 1 \\ \vdots & & & \vdots \\ -1 & \cdots & -1 & +1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & 0 \cdots & 0 \\ \vdots & & & \vdots \\ & & & 0 \\ -1 & \cdots & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \\ & 1 & & & 2^1 \\ & & \ddots & & 2^2 \\ & & & & \vdots \\ & & & 1 & 2^{k-2} \end{bmatrix}$$

$\max|u_{ij}| = 2^{k-2}$
$\max|a_{ij}| = 1$

$$\rho = \frac{2^{k-2}}{1} = 2^{k-2} = 2^{(k-1)-1}$$

If its true for $n=k-1$, then its true for $n=k$, then the maximum Entry, ~~Max~~ of u_{ij}

The maximum $|u_{ij}| = 2^{k-1}$, and $\text{Max}|a_{ij}| = 1$

$$\rho = \frac{\text{Max}|u_{ij}|}{\text{Max}|a_{ij}|} = 2^{k-1}$$

Therefore the growth factor for $n \times n$ Matrix A is exactly 2^{n-1}

Hence proved by Induction.

3. (Unit Lower triangular Matrix)

Given a matrix $A \in \mathbb{C}^{m \times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & \ddots & & \vdots \\ a_{m1} & & & & a_{mn} \end{bmatrix}$$

The LU decomposition of A is given by $LU = A$ where:

L is a unit Lower triangular matrix with ones on the diagonal.

U is the Upper triangular Matrix.

- The multiplication, L_{ji}, accumulated into the Lower triangular part with a change of sign

$$L_{ji} = \frac{a_{ji}}{a_{ii}}, \quad j = i+1, \dots, n$$

$$L_i = \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ \vdots & 0 & \ddots & & \\ \vdots & \vdots & 1 & & \\ \vdots & \vdots & -L_{i+1,i} & 1 & \\ \vdots & \vdots & -L_{i+2,i} & 0 & \ddots \\ 0 & 0 & \vdots & \vdots & \\ & & -L_{n,i} & 0 & 1 \end{bmatrix}$$

Let

$$l_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ l_{i+1,i} \\ \vdots \\ l_{n,i} \end{bmatrix}$$

$$L_i = \overset{I}{\begin{bmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{bmatrix}} - \underbrace{\overset{l_i}{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ l_{i+1,i} \\ \vdots \\ l_{n,i} \end{bmatrix}} \overset{e_i'}{\begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & & & \\ & & 0 & & \\ & & 1 & \ddots & \\ & & 1 & & \\ \vdots & & \vdots & & \ddots \\ 0 & & 1 & & 0 \end{bmatrix}}}_{\text{Elementwise multiplication}}$$

$L_i = I - l_i e_i'$

Since L_i is unitary then $L_i L_i^{-1} = I$

$(I - L_i e_i')(I - l_i e_i')^{-1} = (I - l_i e_i')(I + l_i e_i')$

$= I \cdot I + I L_i e_i' - l_i e_i' I - l_i e_i' \cdot l_i e_i'$

$= I - l_i e_i' l_i e_i = I$

the element wise multiplication of $e_i' l_i = 0$, since the only zero entry in e_i, which is ith entry multiplies with zero entries in l_i,

Since non zero entries in L_i, start at $i+1$, for t
This implies that for any i,

$$L_i^{-1} = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & l_{i+1,i} & 1 & \\ & & \vdots & & \ddots \\ & & L_{n,i} & & & 1 \end{bmatrix}$$

Therefore mathematically the result is true