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Homework 6

MATH 566

2. Companion Matrix.

a) Write down the Companion Matrix A for $p(z) = z^2 + c_1 z + c_0$ and show that $p(z) = \det(zI - A)$ for general Coefficients.

$$p(z) = z^2 + c_1 z + c_0$$

$$A = \begin{bmatrix} 0 & 1 \\ -c_0 & -c_1 \end{bmatrix}$$

- Since the roots of $p(z)$ can be computed by solving for the eigen values of its companion matrix.

- Suppose z is a root of $p(z) = 0$, then we can show that ~~the~~ z, is an eigen value of A with eigen vector $\begin{pmatrix} 1 \\ z \end{pmatrix}$

So

$$A\begin{pmatrix} 1 \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -c_0 & -c_1 \end{pmatrix}\begin{pmatrix} 1 \\ z \end{pmatrix}$$

$$= \begin{pmatrix} z \\ -c_0 - c_1 z \end{pmatrix}$$

In the last row of $A\begin{pmatrix} 1 \\ z \end{pmatrix}$, taking $p(z) = 0$

~~in~~

$$A\begin{pmatrix} 1 \\ z \end{pmatrix} = \begin{pmatrix} z \\ z^2 \end{pmatrix} = z\begin{pmatrix} 1 \\ z \end{pmatrix}$$

$$A\begin{pmatrix} 1 \\ z \end{pmatrix} = z\begin{pmatrix} 1 \\ z \end{pmatrix}$$

let $\frac{1}{z} = x$

$$Ax = zx \Rightarrow (A - zI)x = 0$$

Since matrix $(A - zI)$ is singular for an eigen value z, then the determinant of $(A - zI)$ is zero:

$$\det(zI - A) = 0$$

$$p(z) = \det(zI - A)$$

where $z^2 = -c_0 - c_1 z$

5) Sylvester equation.

(a) Show how the Jordan normal forms can be used to solve (2)

$$AX - XB = C$$

$$A = S_A^{-1} \Lambda_A S_A \quad , \text{ and}$$

$$B = S_B^{-1} \Lambda_B S_B$$

$$S_A^{-1} \Lambda_A S_A X - X S_B^{-1} \Lambda_B S_B = C$$

introduce S_A from the ~~right~~ left

$$S_A S_A^{-1} \Lambda_A S_A X - S_A X S_B^{-1} \Lambda_B S_B = S_A C$$

$$\Lambda_A S_A X - S_A X S_B^{-1} \Lambda_B S_B = S_A C$$

Introduce S_B^{-1} from the right

$$\Lambda_A S_A X S_B^{-1} - S_A X S_B^{-1} \Lambda_B S_B S_B^{-1} = S_A C S_B^{-1}$$

$$\Lambda_A S_A X S_B^{-1} - S_A X S_B^{-1} \Lambda_B = S_A C S_B^{-1}$$

take $\hat{C} = S_A C S_B^{-1}$

$\hat{X} = S_A X S_B^{-1}$

$$\Lambda_A \hat{X} - \hat{X} \Lambda_B = \hat{C} \quad \text{—— (1)}$$

We need to obtain jth column entries of $\Lambda_A \hat{X}_j$, $\Lambda_B(j,j)$ and $\hat{C} = S_A C S_{Bj}^{-1}$

where I is an identity matrix.

therefore equation (1) becomes

$$\Lambda_A \hat{X}_j - \Lambda_B(j,j) I \hat{X}_j = S_A C S_{Bj}^{-1}$$

$$(\Lambda_A - \Lambda_B(j,j) I) \hat{X}_j = S_A C S_{Bj}^{-1}$$

where $\hat{X}_j = S_A X S_{Bj}^{-1} \Rightarrow X = S_A^{-1} \hat{X} S_{Bj}$

the solution exists iff $\Lambda_A \neq \Lambda_B(j,j) I$

Hence $X_j = S_A^{-1} \hat{X} S_{Bj}$ is the solution to the Sylvester Equation.