In [1]:
```
%matplotlib notebook
%pylab
```

```
Using matplotlib backend: nbAgg
Populating the interactive namespace from numpy and matplotlib
```

Finite dimensional vector spaces

Vector space. A set of objects S form a *vector space* if

1. There is a vector $\mathbf{0} \in S$ such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$.
2. If $\mathbf{x}$ and $\mathbf{y}$ are in S, then $\mathbf{x} + \mathbf{y} \in S$. The set is *closed* under addition.
3. If $\alpha \in \mathcal{R}$ and $\mathbf{x} \in S$, then $\alpha\mathbf{x} \in S$. The set is *closed* under multiplication by a scalar.

Example.

- Euclidean space $\mathcal{R}^n$, with vector addition and multiplication by a scalar defined component-wise form a vector space.
- The subspace spanned by any finite set S of vectors in $\mathcal{R}^n$. For example, the set of vectors defined as

$$S = \text{span}\left\{\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\}$$

form a subspace.

Question. To the set of vectors (x, y) satisfying $2x + 3y = 5$ form a subspace of R^2 ?

Textbook. The notes for the next few sections will roughly follow the material in

> *Principles of Applied Mathematics : Transformation and Approximation*, by James Keener (CRC Press, 2018). Available for rent on Amazon.

Inner product spaces

If $\mathbf{x}, \mathbf{y} \in S$, then $\langle \mathbf{x}, \mathbf{y} \rangle$ is called an *inner product* if

1. $\langle \mathbf{x}, \mathbf{y} \rangle : S \times S \to \mathcal{R}$
2. $\langle \alpha \mathbf{x}, \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$
3. $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$
4. $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ and $\langle \mathbf{x}, \mathbf{x} \rangle = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$.

A vector space with an inner product is called an *inner product space*.

Vector norm

A vector $\mathbf{x} \in S$ has *norm* if there is a function $\|\cdot\| : S \to \mathcal{R}^+$ (nonnegative real numbers) such that

1. $\|\mathbf{x}\| > 0$ if $\mathbf{x} \neq 0$ and $\|\mathbf{x}\| = 0$ implies $\mathbf{x} = \mathbf{0}$.
2. $\|\alpha \mathbf{x}\| = \alpha \|\mathbf{x}\|$
3. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (Triangle inequality).

Induced norm

An obvious way to define an norm is to use the inner product : $\|\mathbf{x}\| \equiv \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$. This is called an *induced norm*.

Span, Basis and Dimension

The set S spanned by a finite set of vectors is the set of all possible linear combinations of vectors in the finite set. Suppose $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$. Then the set $\mathcal{S}$ spanned by $\mathcal{U}$ is given by

$$S = \text{span}\,\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\} = \left\{ \sum_{i=1}^{m} x_i \mathbf{u}_i \, : \, x_i \in \mathcal{R} \right\}$$

We can say that $\mathcal{U}$ *spans* $\mathcal{S}$.

A *basis* $\mathcal{B}$ for a vector space S is a minimal set of linearly independent vectors that span S.

The *dimension* of a vector space is the number of vectors in a basis.

Example. Consider all vectors $\mathbf{x} = (x, y, z)$ satisfying $3x + 2y - z = 0$. These all line on a line, and satisfy the matrix equation

$$\begin{bmatrix} 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

The vectors $\mathbf{x}$ satisfying the above form the *null space* of the matrix $A = [2, 3, -1]$,

$$\text{null}(A) = \text{span}\left\{ \begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The basis is formed by the two vectors $(-2/3, 1, 0)$ and $(1/3, 0, 1)$ and so the null space has dimenion 2.

Note. Even though the vectors in $\text{null}(A)$ are all in $\mathcal{R}^3$, the dimension of $\dim(\text{null}(A)) = 2$.

Euclidean space $\mathcal{R}^n$

Example. Eucliden space $\mathcal{R}^3$ has a canonical basis

$$\mathcal{R}^3 = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

and so has dimension 3.

Note. We cannot add any more vectors to the above basis and still have a linearly independent set. The space $\mathcal{R}^n$ is a *finite dimensional space* (even though $\mathcal{R}^n$ contains an infinite number of members.

In general, the dimension of $\mathbf{R}^n$ is n.

Norm in $\mathcal{R}^n$. The obvious norm in $\mathcal{R}^n$ is the usual *Cartesian product* or *dot* product. This induces a norm $\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$.

Example from Homework # 4.

Homework # 4 makes extensive use of proporties of the normed linear space $\mathcal{R}^3$. For example, to solve $A\mathbf{x} = \mathbf{b}$, we use an inner product to project $\mathbf{b}$ onto either a set of orthonormal eigenvectors of A, or onto $\text{Col}(A)$.

Assuming A is symmetric, we used the Spectral Theorem for Matrices to show that we can write

$$\mathbf{b} = \sum_{k=1}^{n} \langle \mathbf{b}, \mathbf{v}_k \rangle \mathbf{v}_k$$

for eigenvectors $\mathbf{v}_k$ of A.

To project onto the column space of A, we must first find a linear independent set of vectors that span the column space of A. If we assume this set is given by $\left{\mathbf a_1, \mathbf a_2, ..., \mathbf a_r\right}, we can write

$$\mathbf{b} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \ldots + c_r\mathbf{a}_r$$

To find each c_i , we can set up r equations of the form

$$\mathbf{a}_i \cdot \mathbf{b} = c_1(\mathbf{a}_1 \cdot \mathbf{a}_1) + c_2(\mathbf{a}_2 \cdot \mathbf{a}_i) + \ldots + c_r(\mathbf{a}_r \cdot \mathbf{a}_i)$$

and solve for the c_i .

Infinite dimensional vector spaces

To be continued!