In [1]:
```
%matplotlib notebook
%pylab
```

```
Using matplotlib backend: nbAgg
Populating the interactive namespace from numpy and matplotlib
```

Potential Theory (Part III)

Our focus has been on the boundary value problem in a domain Ω
$$\nabla^2 u = 0, \mathbf{x} \in \Gamma$$

subject to either Neumann or Dirichlet boundary conditons and where Γ is the boundary of Ω.

From Green's Second Identity, we derived an expression for $u(\mathbf{x})$ as a distribution of sources and dipoles
$$u(\mathbf{x}) = \int_\Gamma u(\mathbf{y})\frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial n} d\mathbf{y} + \int_\Gamma \frac{\partial u(\mathbf{y})}{\partial n} G(\mathbf{x}, \mathbf{y}) d\mathbf{y}$$

Some properties of the potential u

The properties of the solution $u(\mathbf{x})$ as stated above.

- If $u(\mathbf{x}) = 0$ on Γ. Then we have that u can be written as a **distribution of sources** only :

$$u(\mathbf{x}) = \int_{\partial\Omega} \frac{\partial u(\mathbf{y})}{\partial n} G(\mathbf{x}, \mathbf{y}) d\mathbf{y}$$

This is the "single layer potential" and the solution is *continuous* at $\partial\Omega$, but the normal derivative will exhibit a jump equal to the the strength of the source. The solution is "ridge-like" along Γ. The distribution of sources acts to "poke" the solution impose the correct Dirichlet boundary condition.

- If $\partial u/\partial n = 0$ along Γ, then the solution can be written as a **distribution of dipoles**

$$u(\mathbf{x}) = \int_{\partial\Omega} u(\mathbf{y}) \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial n} d\mathbf{y}$$

and is a "double layer potential". The solution exhibits a "tear" along Γ. The dipole acts to "rip" the solution along Γ in order to impose the correct Neumann boundary condition. In this case, the normal derivative is continuous at Γ, but the solution is not.

- **Mean value theorem.** Suppose we consider Green's Second Identity again, but this time on a spherical domain Γ_1 of radius R contained completely within a region in which $\nabla^2 u = 0$. Let $G(\mathbf{x}, \mathbf{y})$ be the free-space Green's function in 3d, i.e.

$$G(\mathbf{x}, \mathbf{y}) = -\frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} = -\frac{1}{4\pi r}$$

Then, we have

$$\begin{aligned}
u(\mathbf{x}) &= \int_{\Gamma_1} u(\mathbf{y}) \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial n} d\mathbf{y} - \int_{\Gamma_1} \frac{\partial u(\mathbf{y})}{\partial n} G(\mathbf{x}, \mathbf{y}) d\mathbf{y} \\
&= \int_{\Gamma_1} u(\mathbf{y}) \frac{\partial}{\partial \rho}\left(-\frac{1}{4\pi\rho}\right) \mathbf{d}\rho \\
&= \frac{1}{4\pi R^2} \int_{\Gamma_1} u(\mathbf{y}) d\mathbf{y}
\end{aligned}$$

The value of a harmonic function u at $\mathbf{x}$ is the *average* of the values it takes on any sphere surrounding the point $\mathbf{x}$.

Homework. Work through the details of the above! Hint: Use Gauss's Theorem.

- **Maximum Principle.** From the above, it follows directly that the maximum or minimum values of the harmonic function on Γ must occur on the boundary Γ.

Integral equations

The expression for $u(\mathbf{x})$ in terms of a distribution of sources and dipoles motivated the idea that we could write $u(\mathbf{x})$ as a distribution of either sources or dipoles of *unknown* strength

$$u(\mathbf{x}) = \int_\Gamma \mu(\mathbf{y}) \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial n} d\mathbf{y}$$

Imposing boundary conditions $u = g$ on Γ leads to

$$g(\mathbf{x}) = \frac{\mu(\mathbf{x})}{2} + \int_{\Gamma \backslash \mathbf{x}} \mu(\mathbf{y}) \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial n} d\mathbf{y}$$

This expression is the basis for an alternative approach to solving Poisson's problem numerically.

We could have also chosen a distribution of dipoles, or any combination of sources and dipoles. But the expression above leads to more practical solutions.

Project idea Explore the Fast Multipole Method for solving the integral equation above.

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