The HLLC Riemann Solver

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CONTENTS

- The HLLC approximation to the solution of the Riemann problem.
- Detailed presentation for the Euler equations (1D, 3D).
- Wave speed estimates.
- Ilustrative results.
- Extensions and very ambitious applications of HLLC

References:

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General initial-boundary value problem (IBVP)

$$\left.\begin{array}{ll}\text{PDEs:} & \partial_t \mathbf{Q} + \partial_x \mathbf{F}(\mathbf{Q}) = \mathbf{0} \; , \;\; x \in [a,b] \; , \;\; t > 0 \; , \\ \text{ICs:} & \mathbf{Q}(x,0) = \mathbf{Q}^{(0)}(x) \; , \;\; x \in [a,b] \; , \\ \text{BCs:} & \mathbf{Q}(a,t) = \mathbf{B}_L(t) \; , \;\; \mathbf{Q}(b,t) = \mathbf{B}_R(t) \; , \;\; t \geq 0 \; . \end{array}\right\} \tag{1}$$

- $\mathbf{Q}(x,t)$: conserved variables.
- $\mathbf{F}(\mathbf{Q})$: physical flux.
- $\mathbf{Q}^{(0)}(x)$: initial condition.
- $\mathbf{B}_L(t)$ and $\mathbf{B}_R(t)$: boundary conditions.
- Finite volume method:

$$\mathbf{Q}_i^{n+1} = \mathbf{Q}_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{i+\frac{1}{2}} - \mathbf{F}_{i-\frac{1}{2}}\right) \tag{2}$$

- Numerical flux:

$$\mathbf{F}_{i+\frac{1}{2}} = \mathbf{F}_0 = \frac{1}{\Delta t}\int_0^{\Delta t} \mathbf{F}(\mathbf{Q}(_{i+\frac{1}{2}}, t))dt \; . \tag{3}$$

Godunov's method for a system

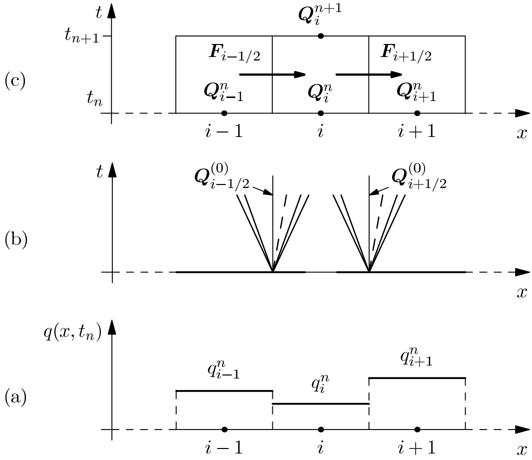


Fig. 1. (a) integral averages give piece-wise constant data
(b) structure of solutions of Riemann problems at intercell boundaries
(c) finite volume formula to update averages using numerical fluxes.

Godunov's flux

- To define scheme (2) we seek suitable approximations to integral (3) to obtain the **numerical flux**.
- The Godunov upwind flux $\mathbf{F}_{i+\frac{1}{2}}$ is computed from (3), making use of the solution $\mathbf{Q}_{i+\frac{1}{2}}(x/t)$ of the local Riemann problem

$$\left.\begin{array}{ll} \text{PDEs:} & \partial_t \mathbf{Q} + \partial_x \mathbf{F}(\mathbf{Q}) = \mathbf{0}\ , \\ & \\ \text{ICs:} & \mathbf{Q}(x,0) = \left\{ \begin{array}{lll} \mathbf{Q}_L \equiv \mathbf{Q}_i^n & \text{if} & x < 0\ , \\ & & \\ \mathbf{Q}_R \equiv \mathbf{Q}_{i+1}^n & \text{if} & x > 0\ . \end{array} \right. \end{array} \right\} \tag{4}$$

- The Godunov flux (1959) becomes

$$\mathbf{F}_{i+\frac{1}{2}} = \mathbf{F}(\mathbf{Q}_{i+\frac{1}{2}}(0))\ . \tag{5}$$

- $\mathbf{Q}_{i+\frac{1}{2}}(0)$: *Godunov state*: $\mathbf{Q}_{i+\frac{1}{2}}(x/t)$ evaluated at interface $x/t = 0$.
- Note use of local coordinates.

Local coordinates

To deal with the local Riemann problems at each interface one changes to local coordinates as follows:

$$\left.\begin{array}{lll} \bar{x} = x - x_{i+\frac{1}{2}} & , & \bar{t} = t - t^n \; , \\ x \in [x_i, x_{i+1}] & , & t \in [t^n, t^{n+1}] \; , \\ \bar{x} \in [-\frac{\Delta x}{2}, \frac{\Delta x}{2}] & , & \bar{t} \in [0, \Delta t] \; . \end{array}\right\} \qquad (6)$$

We then use (x, t) to mean the local coordinates $(\bar{x}, \bar{t})$.

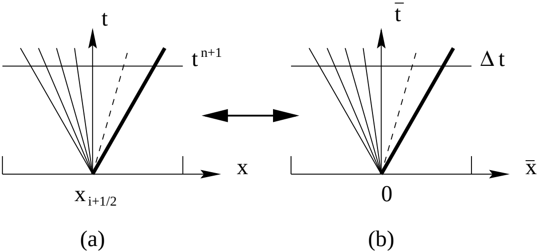


Fig. 2. Correspondence between the global (a) and local (b) frames of reference for the solution of the Riemann problem.

Approximate Riemann solvers: issues

- Godunov's method is the most accurate monotone scheme (monotone for the scalar case)
- For systems, also true, but the accuracy depends crucially on the **Riemann solver**
- Exact solver is the best. But note (i) complexity and (ii) computational expense
- Computational expense **not excessive** for well-known systems
- Approximate Riemann solvers can then be used but with great care
- Approximate Riemann solvers are required to be:
  - **Complete:** their **wave model** contains all characteristic fields of the exact Riemann problem
  - **Non-linear.** Linearised Riemann solvers have various defects and are thus to be avoided
- Simplest Riemann solver: Rusanov solver. Its wave model contains one wave
- Alternative to upwinding are **centred fluxes**, eg FORCE.

Background: the HLL Riemann solver

We seek an approximation to

$$\mathbf{F}_0 = \frac{1}{T}\int_0^T \mathbf{F}(\mathbf{Q}(0,t))dt \tag{7}$$

following Harten, Lax and van Leer (1983). See Fig. 3.

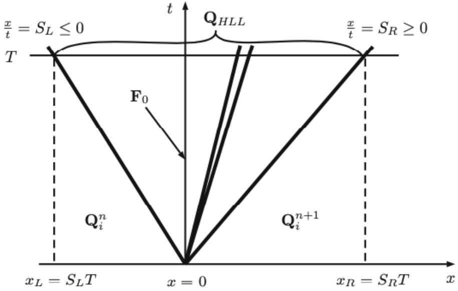


Fig. 3. Derivation of the HLL flux for a subcritical, or subsonic, wave pattern, $S_L \leq 0$ and $S_R \geq 0$.

Useful integral relations I

- For an arbitrary time $T > 0$ we define the distances

$$x_L = TS_L \ , \quad x_R = TS_R \ . \tag{8}$$

- Consider the control volume $[x_L, 0] \times [0, T]$ in x-t space, see Fig. 3, and apply the integral form of the conservation laws

$$\partial_t \mathbf{Q} + \partial_x \mathbf{F}(\mathbf{Q}) = \mathbf{0} \tag{9}$$

in $[x_L, 0] \times [0, T]$.

- We obtain

$$\int_{x_L}^{0} \mathbf{Q}(x,T)dx = \int_{x_L}^{0} \mathbf{Q}(x,0)dx + \int_{0}^{T} \mathbf{F}(\mathbf{Q}(x_L,t))dt - \int_{0}^{T} \mathbf{F}(\mathbf{Q}(0,t))dt \tag{10}$$

- Evaluation of the first and second terms on the right hand side gives

$$\int_{x_L}^{0} \mathbf{Q}(x,0)dx = -S_L T \mathbf{Q}_L \ ; \quad \int_{0}^{T} \mathbf{F}(\mathbf{Q}(x_L,t))dt = T\mathbf{F}(\mathbf{Q}_L) \ . \tag{11}$$

Useful integral relations II

- Inserting these into (10) and dividing through by T gives

$$\mathbf{F}_0 = \frac{1}{T}\int_0^T \mathbf{F}(\mathbf{Q}(0,t))dt = -S_L\mathbf{Q}_L + \mathbf{F}(\mathbf{Q}_L) - \frac{1}{T}\int_{x_L}^0 \mathbf{Q}(x,T))dx \; . \tag{12}$$

- To approximate $\mathbf{F}_0$ find an approximation to the integral on the right hand side of (12).
- This is accomplished by finding an approximate state $\mathbf{Q}(x,T)$.
- Applying the integral form of the conservation laws (9) in the control volume $[x_L, x_R] \times [0,T]$, see Fig. 3, we obtain

$$\int_{x_L}^{x_R} \mathbf{Q}(x,T)dx = \int_{x_L}^{x_R} \mathbf{Q}(x,0)dx + \int_0^T \mathbf{F}(\mathbf{Q}(x_L,t))dt - \int_0^T \mathbf{F}(\mathbf{Q}(x_R,t))dt \; . \tag{13}$$

- The first term on the right hand side gives

$$\int_{x_L}^{x_R} \mathbf{Q}(x,0)dx = -S_L T\mathbf{Q}_L + S_R T\mathbf{Q}_R \; . \tag{14}$$

Harten-Lax-van Leer flux I

- Substitution of this expression into (13) and evaluating the remaining integrals yield

$$\int_{x_L}^{x_R} \mathbf{Q}(x,T)dx = T[S_R\mathbf{Q}_R - S_L\mathbf{Q}_L + \mathbf{F}(\mathbf{Q}_L) - \mathbf{F}(\mathbf{Q}_R)] \qquad (15)$$

- On division through by $x_R - x_L = T(S_R - S_L)$ we obtain the averaged state

$$\left.\begin{array}{rcl} \mathbf{Q}^{HLL} & = & \dfrac{1}{(x_R - x_L)} \displaystyle\int_{x_L}^{x_R} \mathbf{Q}(x,T)dx \\ & = & \dfrac{S_R\mathbf{Q}_R - S_L\mathbf{Q}_L + \mathbf{F}(\mathbf{Q}_L) - \mathbf{F}(\mathbf{Q}_R)}{S_R - S_L} \; . \end{array}\right\} \qquad (16)$$

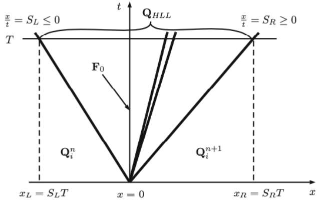
- We now use the state $\mathbf{Q}^{HLL}$ to evaluate the integral on the RHS of (12). The resulting intercell flux is

$$\mathbf{F}_0 = \frac{S_R\mathbf{F}(\mathbf{Q}_L) - S_L\mathbf{F}(\mathbf{Q}_R) + S_L S_R(\mathbf{Q}_R - \mathbf{Q}_L)}{S_R - S_L} \; . \qquad (17)$$

Harten-Lax-van Leer flux II

Wave pattern samplig gives:

$$\mathbf{F}_{i+\frac{1}{2}}^{HLL} = \begin{cases} \mathbf{F}_L & 0 \le S_L \ , \\ \dfrac{S_R \mathbf{F}(\mathbf{Q}_L) - S_L \mathbf{F}(\mathbf{Q}_R) + S_L S_R (\mathbf{Q}_R - \mathbf{Q}_L)}{S_R - S_L} \ , & S_L \le 0 \le S_R \ , \\ \mathbf{F}_R & 0 \ge S_R \ . \end{cases} \quad (18)$$



- **Pending: wave speed estimates for** S_L and S_R.
- **Missing intermediate waves.**

HLL and other simple methods

- **The Rusanov flux from HLL**
  - Define:

$$\left.\begin{array}{rcl} S^+ & = & max\{|S_L|,|S_R|\} \\ S_R & = & S^+ \ , \quad S_L = -S^+ \end{array}\right\} \tag{19}$$

  - Substitute these speeds into the HLL flux, to obtain the Rusanov flux

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{2}\left[\mathbf{F}(\mathbf{Q}_L) + \mathbf{F}(\mathbf{Q}_R)\right] - \frac{1}{2}S^+(\mathbf{Q}_R - \mathbf{Q}_L) \tag{20}$$

  - The Rusanov scheme adopts a **one-wave model**. It is the simplest upwind, non-linear method, but incomplete for any system.
  - Sometimes the Rusanov flux is called the **Local Lax-Friedrichs Flux**, incorrectly in my view

- The (true) **Lax-Friedrichs Flux** can be obtained from Rusanov with $S^+ = \frac{\Delta x}{\Delta t}$

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{2}\left[\mathbf{F}(\mathbf{Q}_L) + \mathbf{F}(\mathbf{Q}_R)\right] - \frac{1}{2}\frac{\Delta x}{\Delta t}(\mathbf{Q}_R - \mathbf{Q}_L) \tag{21}$$

  This is the most diffusive of all stable schemes.

The HLLC Riemann Solver

- The HLLC approximate Riemann solver (Toro et al. 1992/1994) is a modification of the basic HLL scheme
- HLLC accounts for intermediate waves in the structure of the solution of the Riemann problem. See Fig. 4.
- Originally, the intermediate waves were contacts and shear waves. Extensions are possible.

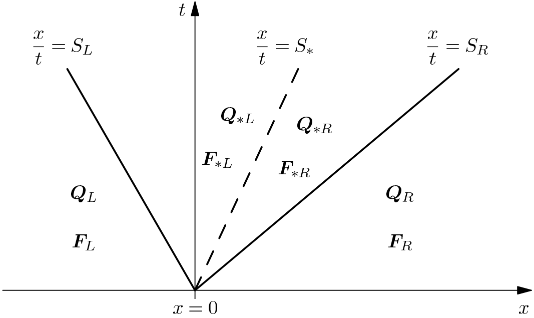


Fig. 4. Assumed wave pattern for the HLLC Riemann solver.

Integral relations revisited

$$\frac{1}{T(S_R - S_L)} \int_{TS_L}^{TS_R} \mathbf{Q}(x,T)dx = \mathbf{Q}_{*L} + \mathbf{Q}_{*R} \ , \tag{22}$$

$$\left.\begin{aligned} \mathbf{Q}_{*L} &= \frac{1}{T(S_* - S_L)} \int_{TS_L}^{TS_*} \mathbf{Q}(x,T)dx \ , \\ \mathbf{Q}_{*R} &= \frac{1}{T(S_R - S_*)} \int_{TS_*}^{TS_R} \mathbf{Q}(x,T)dx \ . \end{aligned}\right\} \tag{23}$$

$$\left(\frac{S* - S_L}{S_R - S_L}\right)\mathbf{Q}_{*L} + \left(\frac{S_R - S_*}{S_R - S_L}\right)\mathbf{Q}_{*R} = \mathbf{Q}^{hll} \ , \tag{24}$$

The HLLC approximate Riemann solution is:

$$\tilde{\mathbf{Q}}(x,t) = \begin{cases} \mathbf{Q}_L & , \text{ if } \quad \frac{x}{t} \leq S_L \ , \\ \mathbf{Q}_{*L} & , \text{ if } \quad S_L \leq \frac{x}{t} \leq S_* \ , \\ \mathbf{Q}_{*R} & , \text{ if } \quad S_* \leq \frac{x}{t} \leq S_R \ , \\ \mathbf{Q}_R & , \text{ if } \quad \frac{x}{t} \geq S_R \ . \end{cases} \tag{25}$$

The HLLC flux: general

- We seek a corresponding HLLC numerical flux of the form

$$\mathbf{F}^{hllc}_{i+\frac{1}{2}} = \begin{cases} \mathbf{F}_L & , \text{ if } \quad 0 \leq S_L \ , \\ \mathbf{F}_{*L} & , \text{ if } \quad S_L \leq 0 \leq S_* \ , \\ \mathbf{F}_{*R} & , \text{ if } \quad S_* \leq 0 \leq S_R \ , \\ \mathbf{F}_R & , \text{ if } \quad 0 \geq S_R \ , \end{cases} \tag{26}$$

  with $\mathbf{F}_{*L}$ and $\mathbf{F}_{*R}$ still to be determined, see Fig. 4.
- By integrating over appropriate control volumes we obtain

$$\mathbf{F}_{*L} = \mathbf{F}_L + S_L(\mathbf{Q}_{*L} - \mathbf{Q}_L) \ , \tag{27}$$

$$\mathbf{F}_{*R} = \mathbf{F}_{*L} + S_*(\mathbf{Q}_{*R} - \mathbf{Q}_{*L}) \ , \tag{28}$$

$$\mathbf{F}_{*R} = \mathbf{F}_R + S_R(\mathbf{Q}_{*R} - \mathbf{Q}_R) \ . \tag{29}$$

- These are three equations for the four unknowns vectors $\mathbf{Q}_{*L}$, $\mathbf{F}_{*L}$, $\mathbf{Q}_{*R}$, $\mathbf{F}_{*R}$.

HLLC for Multidimensional Euler equations

The three-dimensional Euler equations in the x-direction:

$$\partial_t \mathbf{Q} + \partial_x \mathbf{F}(\mathbf{Q}) = \mathbf{0}$$

$$\mathbf{Q} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u w \\ u(E+p) \end{bmatrix}.$$

The HLLC flux for the Euler equations I

- Seek intermediate fluxes $\mathbf{F}_{*L}$ and $\mathbf{F}_{*R}$.
- More unknowns than equations.
- Extra conditions imposed to solve algebraic problem.
- 1D. We impose

$$\left.\begin{array}{ccccc} p_{*L} & = & p_{*R} & = & p_* \ , \\ u_{*L} & = & u_{*R} & = & u_* \ . \end{array}\right\} \text{for pressure and normal velocity} \tag{30}$$

- 3D. We impose

$$\left.\begin{array}{ccccccc} v_{*L} & = & v_L & , & v_{*R} & = & v_R \ , \\ w_{*L} & = & w_L & , & w_{*R} & = & w_R \ . \end{array}\right\} \text{for tangential velocities} \tag{31}$$

- Conditions (30), (31) are identically satisfied by the exact solution.
- In addition we impose

$$S_* = u_* \ . \tag{32}$$

- Thus if an estimate for S_* is known, the normal velocity component u_* in the *Star Region* is known.

The HLLC flux for the Euler equations II

- Equations (27) and (29) can be re–arranged as

$$S_L \mathbf{Q}_{*L} - \mathbf{F}_{*L} = S_L \mathbf{Q}_L - \mathbf{F}_L \ , \tag{33}$$

$$S_R \mathbf{Q}_{*R} - \mathbf{F}_{*R} = S_R \mathbf{Q}_R - \mathbf{F}_R \ , \tag{34}$$

where the right–hand sides of (33) and (34) are known constant vectors (data).

- We also note the useful relation

$$\mathbf{F}(\mathbf{Q}) = u\mathbf{Q} + p\mathbf{D} \ , \quad \mathbf{D} = [0, 1, 0, 0, u]^T \ . \tag{35}$$

- Assuming S_L and S_R to be known and performing algebraic manipulations of the first and second components of equations (33)–(34) one obtains

$$p_{*L} = p_L + \rho_L (S_L - u_L)(S_* - u_L) \ , \quad p_{*R} = p_R + \rho_R (S_R - u_R)(S_* - u_R) \ . \tag{36}$$

The HLLC flux for the Euler equations II

- From (30) $p_{*L} = p_{*R}$, which from (36) gives

$$S_* = \frac{p_R - p_L + \rho_L u_L (S_L - u_L) - \rho_R u_R (S_R - u_R)}{\rho_L (S_L - u_L) - \rho_R (S_R - u_R)} \; . \tag{37}$$

- Manipulation of (33) and (34) and using p_{*L} and p_{*R} from (36) gives

$$\mathbf{F}_{*K} = \mathbf{F}_K + S_K (\mathbf{Q}_{*K} - \mathbf{Q}_K) \; , \tag{38}$$

for K=L and K=R, with the intermediate states given as

$$\mathbf{Q}_{*K} = \rho_K \left(\frac{S_K - u_K}{S_K - S_*} \right) \begin{bmatrix} 1 \\ S_* \\ v_K \\ w_K \\ \frac{E_K}{\rho_K} + (S_* - u_K) \left[S_* + \frac{p_K}{\rho_K (S_K - u_K)} \right] \end{bmatrix} . \tag{39}$$

- The final choice of the HLLC flux is made according to (26).

Variation 1 of HLLC

- From equations (33) and (34) we may write the following solutions for the state vectors $\mathbf{Q}_{*L}$ and $\mathbf{Q}_{*R}$

$$\mathbf{Q}_{*K} = \frac{S_K \mathbf{Q}_K - \mathbf{F}_K + p_{*K} \mathbf{D}_*}{S_L - S_*} \ , \quad \mathbf{D}_* = [0, 1, 0, 0, S_*] \ , \tag{40}$$

  with p_{*L} and p_{*R} as given by (36).

- Substitution of p_{*K} from (36) into (40) followed by use of (27) and (29) gives direct expressions for the intermediate fluxes as

$$\mathbf{F}_{*K} = \frac{S_*(S_K \mathbf{Q}_K - \mathbf{F}_K) + S_K(p_K + \rho_L(S_K - u_K)(S_* - u_K))D_*}{S_K - S_*} \ , \tag{41}$$

  with the final choice of the HLLC flux made again according to (26).

Variation 2 of HLLC

- A different HLLC flux is obtained by assuming a single mean pressure value in the *Star Region*, and given by the arithmetic average of the pressures in (36), namely

$$P_{LR} = \frac{1}{2}[p_L + p_R + \rho_L(S_L - u_L)(S_* - u_L) + \rho_R(S_R - u_R)(S_* - u_R)] \; . \tag{42}$$

- Then the intermediate state vectors are given by

$$\mathbf{Q}_{*K} = \frac{S_K \mathbf{Q}_K - \mathbf{F}_K + P_{LR}\mathbf{D}_*}{S_K - S_*} \; . \tag{43}$$

- Substitution of these into (27) and (29) gives the fluxes $\mathbf{F}_{*L}$ and $\mathbf{F}_{*R}$ as

$$\mathbf{F}_{*K} = \frac{S_*(S_K \mathbf{Q}_K - \mathbf{F}_K) + S_K P_{LR}\mathbf{D}_*}{S_K - S_*} \; . \tag{44}$$

- Again the final choice of HLLC flux is made according to (26).

Remarks

- The original HLLC formulation (38)-(39) enforces the condition $p_{*L} = p_{*R}$, which is satisfied by the exact solution.

- In the alternative HLLC formulation (41) we relax such condition, being more consistent with the pressure approximations (36).

- There is limited practical experience with the alternative HLLC variations (41) and (44).

- Full details in: EF Toro *Riemann solvers and numerical methods for fluid dynamics. Springer, 3rd edition, 2009.*

- **General equation of state.** All manipulations, assuming that wave speed estimates for S_L and S_R are available, are valid for any equation of state; this only enters when prescribing estimates for S_L and S_R.

HLLC for Multidimensional multicomponent flow I

- Consider the advection of m *chemical species* of concentrations q_l by the normal flow speed u. Then we can write the following advection equations

$$\partial_t q_l + u \partial_x q_l = 0 \;,\; \text{for } l = 1, \ldots, m \;.$$

- Note that these equations are written in non–conservative form. However, by combining these with the continuity equation we obtain a conservative form of these equations, namely

$$\partial_t (\rho q_l) + \partial_x (\rho u q_l) = 0 \;,\; \text{for } l = 1, \ldots, m \;.$$

- The eigenvalues of the enlarged system are as before, with the exception of $\lambda_2 = u$, which now, in three space dimensions, has multiplicity $m + 3$.

HLLC for Multidimensional multicomponent flow II

These conservation equations can then be added as new components to the conservation equations in (1) or (4), with the enlarged vectors of conserved variables and fluxes given as

$$\mathbf{Q} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \\ \rho q_1 \\ \dots \\ \rho q_l \\ \dots \\ \rho q_m \end{bmatrix} , \quad \mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u w \\ u(E+p) \\ \rho u q_1 \\ \dots \\ \rho u q_l \\ \dots \\ \rho u q_m \end{bmatrix} . \tag{45}$$

HLLC for Multidimensional multicomponent flow III

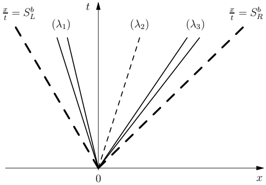
The HLLC flux accommodates these new equations in a very natural way, and nothing special needs to be done. If the HLLC flux (38) is used, with $\mathbf{F}$ as in (45), then the intermediate state vectors are given by

$$\mathbf{Q}_{*K} = \rho_K \left(\frac{S_K - u_K}{S_K - S_*} \right) \left[\begin{array}{c} 1 \\ S_* \\ v_K \\ w_K \\ \dfrac{E_K}{\rho_K} + (S_* - u_K) \left[S_* + \dfrac{p_K}{\rho_K (S_K - u_K)} \right] \\ (q_1)_K \\ \dots \\ (q_l)_K \\ \dots \\ (q_m)_K \end{array} \right] . \quad (46)$$

for $K = L$ and $K = R$.

Wave speed estimates

- We need estimates S_L, S_* and S_R.
- In fact S_L and S_R are sufficient.
- Such estimates must **BOUND** the true wave speeds.
- Also relevant for the Courant condition to choose stable time step.



1. Davis (1988): fail to bound true wave speeds
2. Einfeldt (1988): fail to bound true wave speeds
3. Toro et al. (1994): **bound true wave speeds but larger than necessary**
4. Batten et al. (1997): fail to bound true wave speeds
5. Toro et al. (2020): **bound true wave speeds, more accurate than in 3.**

Bounding wave speed estimates

- Toro et al. (1994) suggested

$$S_L = u_L - a_L q_L \ , \quad S_R = u_R + a_R q_R \ , \tag{47}$$

$$q_K = \begin{cases} 1 & \text{if} \quad p_* \leq p_K \\ \left[1 + \frac{\gamma+1}{2\gamma}(p_*/p_K - 1)\right]^{1/2} & \text{if} \quad p_* > p_K \ , \end{cases} \tag{48}$$

- with the Two-Rarefaction approximation for pressure

$$p_* = p_{tr} = \left[\frac{a_L + a_R - \frac{\gamma-1}{2}(u_R - u_L)}{a_L/p_L^z + a_R/p_R^z}\right]^{1/z} \ , \tag{49}$$

$$P_{LR} = \left(\frac{p_L}{p_R}\right)^z \ ; \quad z = \frac{\gamma-1}{2\gamma} \ . \tag{50}$$

- Estimates (47)-(50) are BOUNDS, first conjectured by Toro (1994) and then proved theoretically by Guermond and Popov (2016).
- Theoretically proved sharper bounds given in Toro et al. (2020).

Summary of HLLC Fluxes I

- **Step I: pressure estimate p_* and wave speed estimates:**

$$S_L = u_L - a_L q_L \; , \quad S_R = u_R + a_R q_R \; , \tag{51}$$

with

$$q_K = \begin{cases} 1 & \text{if} \quad p_* \le p_K \\ \left[1 + \dfrac{\gamma+1}{2\gamma}(p_*/p_K - 1)\right]^{1/2} & \text{if} \quad p_* > p_K \; . \end{cases} \tag{52}$$

and

$$S_* = \frac{p_R - p_L + \rho_L u_L (S_L - u_L) - \rho_R u_R (S_R - u_R)}{\rho_L (S_L - u_L) - \rho_R (S_R - u_R)} \; . \tag{53}$$

- **Step II: HLLC flux.** Compute the HLLC flux, according to

$$\mathbf{F}^{hllc}_{i+\frac{1}{2}} = \begin{cases} \mathbf{F}_L & \text{if} \quad 0 \le S_L \; , \\ \mathbf{F}_{*L} & \text{if} \quad S_L \le 0 \le S_* \; , \\ \mathbf{F}_{*R} & \text{if} \quad S_* \le 0 \le S_R \; , \\ \mathbf{F}_R & \text{if} \quad 0 \ge S_R \; , \end{cases} \tag{54}$$

Summary of HLLC Fluxes II

with

$$\mathbf{F}_{*K} = \mathbf{F}_K + S_K(\mathbf{Q}_{*K} - \mathbf{Q}_K) \tag{55}$$

and

$$\mathbf{Q}_{*K} = \rho_K \left(\frac{S_K - u_K}{S_K - S_*} \right) \begin{bmatrix} 1 \\ S_* \\ v_K \\ w_K \\ \frac{E_K}{\rho_K} + (S_* - u_K) \left[S_* + \frac{p_K}{\rho_K (S_K - u_K)} \right] \end{bmatrix} . \tag{56}$$

HLLC variations

There are two variants of the HLLC flux in the second step, as seen below.

- **Step II: HLLC flux, Variant 1.** Compute the numerical fluxes as

$$\mathbf{F}_{*K} = \frac{S_*(S_K\mathbf{Q}_K - \mathbf{F}_K) + S_K(p_K + \rho_L(S_K - u_K)(S_* - u_K))\mathbf{D}_*}{S_K - S_*} ,$$

$$\mathbf{D}_* = [0, 1, 0, 0, S_*]^T , \tag{57}$$

  and the final HLLC flux chosen according to (54).

- **Step II: HLLC flux, Variant 2.** Compute the numerical fluxes as

$$\mathbf{F}_{*K} = \frac{S_*(S_K\mathbf{Q}_K - \mathbf{F}_K) + S_K P_{LR}\mathbf{D}_*}{S_K - S_*} , \tag{58}$$

  with $\mathbf{D}_*$ as in (57) and

$$P_{LR} = \frac{1}{2}[p_L + p_R + \rho_L(S_L - u_L)(S_* - u_L) + \rho_R(S_R - u_R)(S_* - u_R)] . \tag{59}$$

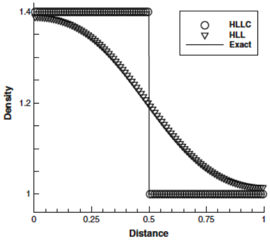
  The final HLLC flux is chosen according to (54).

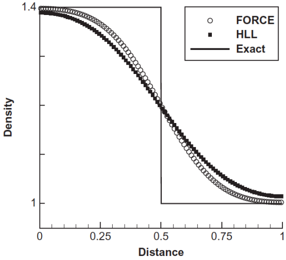
Some numerical results

Test	ρ_{L}	u_{L}	p_{L}	ρ_{R}	u_{R}	p_{R}
1	1.0	0.75	1.0	0.125	0.0	0.1
2	1.0	-2.0	0.4	1.0	2.0	0.4
3	1.0	0.0	1000.0	1.0	0.0	0.01
4	5.99924	19.5975	460.894	5.99242	-6.19633	46.0950
5	1.0	-19.59745	1000.0	1.0	-19.59745	0.01
6	1.4	0.0	1.0	1.0	0.0	1.0
7	1.4	0.1	1.0	1.0	0.1	1.0

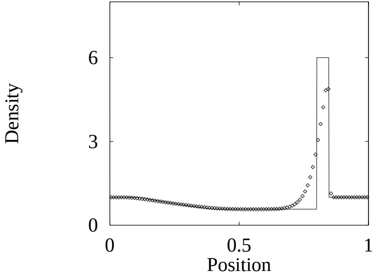
Table 1. Data for seven test problems with exact solution

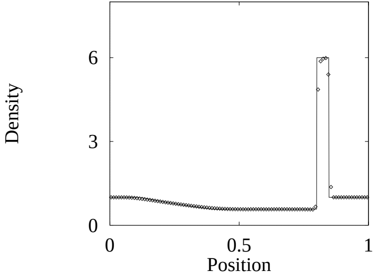
Numerical results: isolated contact





Numerical results: non-isolated contact





Numerous extensions and applications

- We have presented HLLC for the Euler equations (1D, 3D).
- There are other variations in the literature: HLLE, HLLEM, HLLI, HLLX, not studied here.
- 2D shallow water equations (*Toro E F. Shock capturing methods for free-surface shallow flows. Wiley and Sons, 2001*).
- Turbulent flow applications (implicit version of HLLC), see Batten, Leschziner and Goldberg (1997).
- MHD equations, see Gurski (2004), Li (2005), Mignone et al. (2006++).
- Two-phase compressible flow, see Tokareva and Toro, JCP (2010).
- Extensions: Takahiro (2005), Bouchut (2007), Mignone (2005).
- Aerospace, industrial and environmental problems.
- Astrophysics and cosmology. Example: Yohei Miki, Masao Mori and Toshihiro Kawaguchi. **Destruction of the central black hole gas reservoir through head-on galaxy collisions. Nature Astronomy, 2021.**

THANK YOU