Homework #2

Math 537

The first few problems are to encourage to think more deeply about Legendre polynomials. You only need to turn in work for the last problem.

1. (**Review only**) Legendre polynomials play a central role in many physics problems posed in spherical coordinates. For example, electrostatics, Coulomb potential problems and the gravitational potential can all be expressed as solutions to Laplace's equation $\nabla^2 u = 0$.

   Read the section "Boundary Value Problem in Spherical Coordinates" (Kreyszig, Section 12.11, page 594) to fill in details on how the Legendre equation arise in an electrostatics problem. Work through the details on the separation of variables techniques used.

2. (**Review only**) Check that you follow how the "Legendre polynomials" arise as solutions to the Legendre equation

   $$(1 - x^2)y'' - 2xy' + n(n+1)y = 0 \tag{1}$$

   From Problem 1, you should see why we use "$n(n+1)$" in the equation. Also be sure you understand why in some cases, we get polynomials and not infinite series as solutions.

3. (**Review only**) Check your favorite mathematical platform (Matlab, SciPy, Wolfram Alpha and so on) to see what the Legendre polynomial function is. For example, in Matlab, you can call `legendre` to get the 'associated Legendre function", which is a more general form of the Legendre functions we discussed in class. Create plots of the first few Legendre polynomials and see that they agree with the analytic form of each polynomial.

4. (**Review only.**) Check that the Legendre polynomials do in fact satisfy the Legendre equation for appropriate values of n.

5. (**Turn in**) Given a sequence $f_n(x)$, we can define a *generating function* $G(u, x)$ as

   $$G(u, x) = \sum_{n=0}^{\infty} = f_n(x)u^n$$

   The idea is that the series expansion for $G(u, x)$ "generates" the sequence $f_n(x)$.

   A remarkable property of the Legendre polynomials is that they show up in the generating function for problems in potential theory, a field of applied mathematics dedicated to understanding solutions to Laplace's equation.

   (a) Show that

   $$G(u, x) = \frac{1}{\sqrt{1 - 2xu + u^2}} = \sum_{n=0}^{\infty} P_n(x)u^n \tag{2}$$

   is a generating function of the Legendre polynomials $P_n(x)$. **Hint:** Start by using the "generalized binomial formula" to expand $(1 - v)^{\frac{1}{2}}$. Then set $v = 2xu - u^2$, multiply the powers of $2ux - u^2$ out, collect all terms involving u^n and verify that the coefficients of u^n is $P_n(x)$. Obtain the first few Legendre polynomials in this manner.

   To get higher degree Legendre polynomials, use (2) to derive a recursion relation for $P_{n+1}(x)$ in terms of $P_n(x)$ and $P_{n-1}(x)$. **Hint:** Differentiate (2) with respect to u. Verify that your recursion formula works for degrees up to 6.

   Historical note: The Legendre polynomials were first derived using this approach.

(b) Suppose a point charge at point $\mathbf{p}_1$ is located a distance r_1 from the origin in a Cartesian coordinate system. Show that the charge that is felt at a second point $\mathbf{p}_2$ at a distance r_2 from the origin in 3d space is given by

$$\frac{1}{r} = \frac{1}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta}} = \frac{1}{r_2} \sum_{n=0}^{\infty} P_n(\cos\theta) \left(\frac{r_1}{r_2}\right)^n$$

where r is the distance between $\mathbf{p}_1$ and $\mathbf{p}_2$.

(c) You can read an excellent discussion of on Legendre polynomials and their use in physics here. From this resource (or from many other available online), describe how the above formula might be useful in evaluating the effects of a point charge in space.