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Homework 1

1. Matrix manipulations

- (a) Consider a 4×4 identity matrix, I , and a 4×4 matrix B to which we may apply the following operations.

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(i) don't apply the following to I

- (i) double Column 1

$$I_1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- (ii) halve row 3

$$I_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- (iii) add row 3 to row 1

$$I_3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(iv) Interchange Columns 1 and 4.

$$I_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

(v) Subtract row 2 from each of the other rows

$$I_5 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

(vi) Replace column 4 by column 3

$$I_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(vii) Delete column 1.

$$I_7 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(g) Write the result as a product of eight matrices.

$$I_5 \times I_3 \times I_2 \times B \times I_1 \times I_4 \times I_6 \times I_7$$

$$I_5 \times I_3 \times I_2 \times B \times I_1 \times I_4 \times I_6 \times I_7 =$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = B$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) Write it again as a product ABC (Same B)
of three matrices.

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1/2 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Finally we get

$$\left[\begin{array}{cccc} 1 & -1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 0 & -1 & 0 & 1 \end{array} \right] B \left[\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array} \right]$$

NO.2 Orthogonal matrices

(a) Show that if a matrix Q is orthogonal and lower triangular, then it is diagonal.

Consider $Q Q^T = I$

Performing forward substitution.

$$Q^T = Q^{-1} \quad \text{--- (1)}$$

Equation (1) is true for a real square matrix which is orthogonal, however Q is also lower triangular, meaning its transpose will be upper triangular matrix.

- But again the inverse of Q , which is lower triangular and orthogonal is lower triangular.

- therefore equation (1) becomes a contradiction to obtain an upper triangular matrix, Hence Q is diagonal.

b) What are the diagonal elements?

Consider $Q Q^T = I$, Since Q is diagonal, then $Q = Q^T$, therefore we have

$$Q \cdot Q = I$$

The diagonal entries of $Q \cdot Q$ will be squared, and since we are equating them to I , whose major diagonal elements are $1, 0, \dots, 0$, then

The diagonal entries will be ± 1

Q3 (Rank one matrices) Show the following

- If A has rank one then $A = ab^T$ for some column vectors a and b .

If $\text{rank}(A) = 1$, this means that its image is one dimensional.

Consider a vector $u \in \mathbb{R}^m$, then, and a scalar α , then with a fixed at \mathbb{R}^n

$$Au = \alpha a \quad \text{--- ①}$$

Equation ①, means that every column of A is a multiple of a , hence

$$A = (b_1 a \ b_2 a \ \dots \ b_m a)$$

$$A = a(b_1 \ b_2 \ b_3 \ \dots \ b_m)$$

$$\underline{A = a b^T}$$

- If $A = ab^T$ for some column vectors a and b then A has rank 1.

Again consider $u \in \mathbb{R}^m$, then

$$Au = ab^T u.$$

$$Au = (u \cdot b) a$$

This implies that A maps every vector in the

\mathbb{R}^m to a scalar multiple of a .

In this case the $\text{rank}(A) = \dim(\text{Column}(A)) = 1$

□

No. 4 Pythagorean theorem

The Pythagorean theorem asserts that for a set of n orthogonal vectors $\{x_i\}$,

$$\left\| \sum_{i=1}^n x_i \right\|^2 = \sum_{i=1}^n \|x_i\|^2$$

- (a) Prove this in the case $n=2$ by an explicit computation $\forall \|x_1 + x_2\|^2$

Consider $\|x\|^2 = x \cdot x$

$$\begin{aligned} \left\| \sum_{i=1}^{n=2} x_i \right\|^2 &= \|x_1 + x_2\|^2 \\ &= (x_1 + x_2) \cdot (x_1 + x_2) \\ &= x_1 \cdot x_1 + x_1 \cdot x_2 + x_2 \cdot x_1 + x_2 \cdot x_2 \end{aligned}$$

For linear orthogonality $x_1 \cdot x_2 = x_2 \cdot x_1 = 0$

$$\begin{aligned} \left\| \sum_{i=1}^{n=2} x_i \right\|^2 &= x_1 \cdot x_1 + 0 + 0 + x_2 \cdot x_2 \\ &= \|x_1\|^2 + \|x_2\|^2 \end{aligned}$$

$$\left\| \sum_{i=1}^{n=2} x_i \right\|^2 = \sum_{i=1}^{n=2} \|x_i\|^2$$

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- b) Show that this computation also establishes the general case, by induction.

Base Case:

$$\text{for } n=2, \left\| \sum_{i=1}^{n=2} x_{ii} \right\|^2 = \sum_{i=1}^{n=2} \|x_{ii}\|^2$$

Induction Case:

for $n=k$

$$\left\| \sum_{i=1}^{n=k} x_{ii} \right\|^2 = \sum_{i=1}^{n=k} \|x_{ii}\|^2$$

Assume it is true for $n=k$, i.e. k mutually orthogonal vectors: x_1, x_2, \dots, x_k

Suppose for $n=k+1$

$$\left\| \sum_{i=1}^{n=k+1} x_{ii} \right\|^2 = \left\| x_1 + x_2 + x_3 + \dots + x_{k+1} \right\|^2$$

assuming x_1, x_2, \dots, x_{k+1} are also mutually orthogonal vectors and that

$$\left\| \sum_{i=1}^{n=k+1} x_{ii} \right\|^2 = \left\| \sum_{i=1}^{n=k} x_{ii} \right\|^2 + \|x_{k+1}\|^2$$

Since x_{k+1} is orthogonal to each x_i for $i=1, \dots, k$, then using linearity of the inner product, $x_i \cdot x_j = 0$, so

$$\left\| \sum_{i=1}^{n=k} x_{ii} \right\|^2 = \|x_1\|^2 + \|x_2\|^2 + \dots + \|x_{k-1}\|^2 + \|x_k\|^2$$

Concluding that

$$\left\| \sum_{i=1}^{n-k+1} a_i e_i \right\|^2 = \|x_1\|^2 + \|x_2\|^2 + \dots + \|x_k\|^2 + \|x_{k+1}\|^2 \\ = \sum_{i=1}^{n-k+1} \|x_i\|^2$$

□

No. 5 (inverse of rank one-perturbation).

If A is non singular, then $A A^{-1} = I$
so

$$(I + uv^*) (I + \alpha uv^*) = I \\ I + \alpha I uv^* + uv^* I + \alpha uv^* u v^* \equiv I$$

$$\alpha uv^* + uv^* + \alpha v^* u v^* = 0$$

$$(\alpha + 1 + \alpha v^* u) uv^* = 0 \quad \text{--- (1)}$$

For (1) to hold $\alpha + 1 + \alpha v^* u = 0$, then
 $\alpha = -1(1+v^* u)^{-1}$, for $v^* u \neq -1$

as long as $v^* u \neq -1$ then A has an inverse
of the form $A^{-1} = I + \alpha uv^*$

For what u and v is A singular?

A is singular iff $\det(A) = 0$, so if α
 $\det(A) = 1 + v^* u = 0$
 $v^* u = -1$

Hence If $v^*u = -1$, then A is singular.

What is $\text{null}(A)$?

Consider $Ax = 0$, for x a m-vector.

$$Ax = 0 \Rightarrow (I + uv^*)x = 0$$

$$uv^*x = -x$$

from above, $v^*u = -1$, then it's possible to obtain $x = u$, hence

$\text{null}(A) = \{ \text{the span spanned by } u \}$

No. 6.

(Matrix Norms) Consider the following Matrix

$$A = \begin{bmatrix} 4 & -1 & 2 \\ 1 & 2 & 3 \\ -1 & 7 & -5 \end{bmatrix}$$

(a) Compute the matrix 1-, 2-, ∞ -, and Frobenius norm of A

1-Norm: $\|A\|_1 = 10$

2-Norm $\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)}$

Let $A^T A = B$

$$B = \begin{bmatrix} 4 & 1 & -1 \\ -1 & 2 & 7 \\ 2 & 3 & -5 \end{bmatrix} \begin{bmatrix} 4 & -1 & 2 \\ 1 & 2 & 3 \\ -1 & 7 & -5 \end{bmatrix}$$

$$B = \begin{bmatrix} 18 & -9 & 16 \\ -9 & 54 & -31 \\ 16 & -31 & 38 \end{bmatrix}$$

Eigen values of B , $|B - \lambda I| = 0$.

$$\begin{vmatrix} 18-\lambda & -9 & 16 \\ -9 & 54-\lambda & -31 \\ 16 & -31 & 38-\lambda \end{vmatrix} = 0$$

$\lambda_1 = 6.846$, $\lambda_2 = 20.652$, $\lambda_3 = 85.5023$.

$$P(B) = \max_{1 \leq j \leq n} |x_j|$$

$$P(B) = \underline{\lambda_3} = 82.502$$

$$\|A\|_2 = \sqrt{P(B)} = \sqrt{82.502} = \underline{\underline{9.083}}$$

∞ -Norm

$$\|A\|_\infty = \max \text{ of row sum } \{7, 6, 13\} = \underline{\underline{13}}$$

Frobenius Norm.

$$\|A\|_F = \sqrt{\sum_{j=1}^m \sum_{k=1}^n |a_{jk}|^2} = \sqrt{\sum_{j=1}^3 \sum_{k=1}^3 |a_{jk}|^2}$$

$$= \sqrt{\sum_{j=1}^3 (|a_{j1}|^2 + |a_{j2}|^2 + |a_{j3}|^2)}$$

$$= \sqrt{|a_{11}|^2 + |a_{12}|^2 + |a_{13}|^2 + |a_{21}|^2 + |a_{22}|^2 + |a_{23}|^2 + |a_{31}|^2 + |a_{32}|^2 + |a_{33}|^2}$$

$$= \sqrt{10} = \underline{\underline{3.16}}$$

(b) For any n -by- n Matrix A the following bounds on hold true.

$$1. \frac{1}{\sqrt{n}} \|A\|_2 \leq \|A\|_1 \leq \sqrt{n} \|A\|_2$$

$$\frac{1}{\sqrt{3}} (9.083) \leq 10 \leq \sqrt{3} (9.083)$$

holds

$$2. \frac{1}{\sqrt{n}} \|A\|_2 \leq \|A\|_\infty \leq \sqrt{n} \|A\|_2$$

$$\frac{1}{\sqrt{3}} (9.083) \leq 13 \leq \sqrt{3} (9.083)$$

holds

$$3. \frac{1}{n} \|A\|_\infty \leq \|A\|_1 \leq n \|A\|_\infty$$

$$\frac{1}{3} (13) \leq 10 \leq 3 (9.083)$$

holds

$$4. \|A\|_1 \leq \|A\|_F \leq \sqrt{n} \|A\|_2$$

$$10 \leq 10 \cdot 4.88 \leq \sqrt{3} (9.083)$$

holds

$$5. \|A\|_p \leq \|A\|_F, \text{ for } p=1, 2, \infty$$

$$\|A\|_p = \max_{1 \leq j \leq n} |\lambda_j|$$

$$|A - \lambda I| = 0 \Rightarrow \lambda_1 = 3.88, \lambda_2 = 4.56, \lambda_3 = 6.916$$

$$P(A) = 6 \cdot 964$$

$$P(A) \leq \|A\|_1 \Rightarrow 6 \cdot 966 \leq 10, \text{ holds}$$

$$P(A) \leq \|A\|_2 \Leftrightarrow 6 \cdot 966 \leq 9.083, \text{ holds.}$$

$$P(A) \leq \|A\|_\infty \Rightarrow 6 \cdot 966 \leq 13, \text{ holds}$$

7. Matrix multiplication

a) $C_{11} = P_1 + P_4 - P_5 + P_7$

$$\begin{aligned} C_{11} = & A_{11}B_{11} + A_{11}B_{22} + A_{12}B_{11} + A_{12}B_{22} + \\ & A_{22}B_{11} - A_{22}B_{22} - A_{11}B_{22} - A_{12}B_{22} + \\ & A_{12}B_{21} + A_{22}B_{22} - A_{22}B_{21} - A_{12}B_{22} \end{aligned}$$

$$\underline{C_{11} = A_{11}B_{11} + A_{12}B_{21}}$$

b) $C_{12} = P_3 + P_5$

$$\begin{aligned} C_{12} = & A_{11}B_{12} - A_{11}B_{22} + A_{12}B_{12} + A_{12}B_{22} \\ & - A_{22}B_{12} - A_{22}B_{22} \end{aligned}$$

$$\underline{C_{12} = A_{11}B_{12} + A_{12}B_{22}}$$

$$C_{21} = P_2 + P_4$$

$$C_{21} = A_{21}B_{11} + A_{22}B_{11} + A_{22}B_{21} - A_{22}B_{11}$$

$$\underline{C_{21} = A_{21}B_{11} + A_{22}B_{21}}$$

$$C_{22} = P_1 - P_2 + P_3 + P_4$$

$$\begin{aligned} C_{22} = & A_1 B_{11} + A_1 B_{22} + A_{21} B_{11} + A_{22} B_{22} - \\ & A_{21} B_{11} - A_{22} B_{11} + A_1 B_{12} - A_1 B_{22} + A_{21} B_{11} \\ & + A_{21} B_{12} - A_1 B_{11} - A_1 B_{12} \end{aligned}$$

$$C_{22} = \underline{\underline{A_{22} B_{22} + A_{21} B_{12}}}$$

b).

- d) i) The algorithm multiplies 2×2 matrix, Using Seven Scalar multiplications, and this is because $N \times N$ matrix can be divided into quadrants.
- Strassen's algorithm does this recursively, so each \sqrt{N} quadrant multiplication is computed recursively and the computational cost of addition and subtraction of quadrants is n^2 .
 - So the linear combination of products are computed in order to obtain the result with a size $n^2 = 4^m$.
 - Thus, the recurrence for the flop count is $f(n) = 7f(n/2) + n^2$ with a base case $f(1) = 1$, yielding $\underline{\underline{f(n) = (n \log_2 7)}}$