In [1]:
```
%matplotlib notebook
%pylab
```

```
Using matplotlib backend: nbAgg
Populating the interactive namespace from numpy and matplotlib
```

# Daubechies wavelets

The idea behind the Daubechies wavelet was to develop a basic building block that was continuous, but has the nice properties of the Haar wavelet.

The Daubechies scaling function (analogous to the Haar "step" function) is defined *recursively* in terms of coefficients $h_0, h_1, h_2, h_3$ as

$$\phi(x) = h_0\phi(2x) + h_1\phi(2x-1) + h_2\phi(2x-2) + h_3\phi(2x-3)$$

where the coefficients are given by

$$h_0 = \frac{1+\sqrt{3}}{4}, \qquad h_1 = \frac{3+\sqrt{3}}{4}, \qquad h_2 = \frac{3-\sqrt{3}}{4}, \qquad h_3 = \frac{1-\sqrt{3}}{4}$$

The scaling function is supported on the interval $[0,3]$ and is zero for $r <= 0$ or $r >= 3$. Daubechie proved that $\phi(x)$ has no simple closed from expression in terms of elementary functions, but it does satisfy some convenient properties.

Reference :

["Wavelets Made Easy", Yves Nievergelt, (Birkhäuser,, 1999). (https://www.amazon.com/Wavelets-Made-Easy-Yves-Nievergelt/dp/0817640614)](https://www.amazon.com/Wavelets-Made-Easy-Yves-Nievergelt/dp/0817640614)

[Ten Lectures in Wavelets (https://epubs.siam.org/doi/book/10.1137/1.9781611970104)](https://epubs.siam.org/doi/book/10.1137/1.9781611970104), Ingrid Daubechies, (SIAM Publishing).

# Daubechies scaling function

A (slow) way to plot an approximation to $\phi(x)$ is to define a linaer operator on a function $g(x)$

$$T[g](x) = h_0 g(2x) + h_1 g(2x-1) + h_2 g(2x-2) + h_3 g(2x-3)$$

and then look for "fixed points" of this operator, e.g. functions $\phi$ that satisfy

$$\phi = T[\phi]$$

Just as we do with a "fixed point" iteration used to solve $x = g(x)$, we can get an idea as to what $\phi(x)$ might look like by plotting successive iterates

$$g_{k+1}(x) = T[g_k](x) \equiv T^k[g_0]$$

where $T^0[g] = g, T^1[g] = T[g], T^2[g] = T[T[g]]]$ and so on. The Daubechie scaling function is then

$$\phi(x) = \lim_{n\to\infty} T^n[g]$$

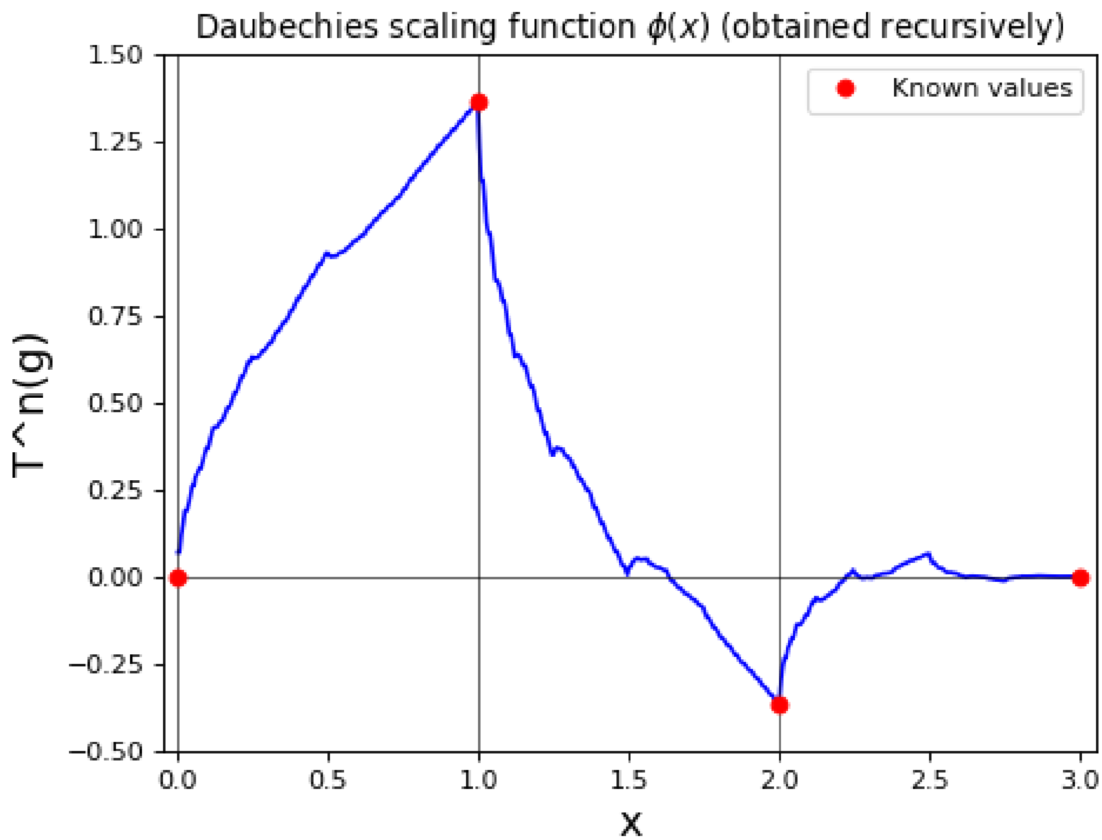
For the Daubechie scaling function, the initial function $g_0(x)$ is set to the Haar "box" function

$$g_0(x) = \begin{cases} 1 & \text{if} \quad 0 \le x < 1 \\ 0 & \quad \text{otherwise} \end{cases}$$

In [91]:

```
def box(x,u=0,w=1):
    return where(logical_and(x >= u, x < w),1,0)

h0 = (1 + sqrt(3))/4
h1 = (3 + sqrt(3))/4
h2 = (3 - sqrt(3))/4
h3 = (1 - sqrt(3))/4

def T(x,n):
    if n == 0:
        return box(x)
    else:
        n -= 1
        return h0*T(2*x,n) + h1*T(2*x-1,n) + h2*T(2*x - 2,n) + h3*T(2*x-
3,n)
```

In [90]:
```python
fig = figure(1)
clf()

N = 512
x = linspace(0,3,N+1)

hdl, = plot(x,box(x,0),'b-')

# Reference lines
yl = [-0.5,1.5]
plot([0,3],[0,0],'k-',linewidth=0.5)
plot([0,0],yl,'k-',linewidth=0.5)
plot([1,1],yl,'k-',linewidth=0.5)
plot([2,2],yl,'k-',linewidth=0.5)

xp = [0,1,2,3]
yp = [0,(1 + sqrt(3))/2, (1 - sqrt(3))/2,0]
plot(xp,yp,'r.',markersize=12,label='Known values')


xlim([-0.05,3.05])
ylim([-0.5,1.5])

title('Daubechies scaling function $\phi(x)$ (obtained recursively)')
xlabel('x',fontsize=16)
ylabel('T^n(g)',fontsize=16)

legend()

for k in range(8):
    hdl.set_ydata(T(x,k))

    fig.canvas.draw()

    pause(1)
```


Daubechies scaling function $\phi(x)$ (obtained recursively)
Known values
1.50
1.25
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
T^n(g)
0.0
0.5
1.0
1.5
2.0
2.5
3.0
x

# Efficient evalution of the scaling function

The scaling function has known values given by

$$\phi(0) = 0, \qquad \phi(1) = \frac{1+\sqrt{3}}{2}, \qquad \phi(2) = \frac{1-\sqrt{3}}{2}, \qquad \phi(3) = 0$$

We can use these as initial values to obtain other values of $\phi(x)$ at *dyadic* points $k2^{-j}$, where $k, j$ are integers.

For example, we can compute $\phi(1/2)$ as

$$\begin{aligned}\phi\left(\frac{1}{2}\right) &= h_0\phi(1) + h_1\phi(1-1) + h_2\phi(1-2) + h_3\phi(1-3) \\ &= h_0\phi(1) + h_1 \cdot 0 + h_2 \cdot 0 + h_3 \cdot 0 \\ &= \frac{1+\sqrt{3}}{4} \cdot \frac{1+\sqrt{3}}{2} = \frac{2+\sqrt{3}}{4}\end{aligned}$$

and $\phi(3/2)$

$$\begin{aligned}\phi\left(\frac{3}{2}\right) &= h_0\phi(3) + h_1\phi(3-1) + h_2\phi(3-2) + h_3\phi(3-3) \\ &= h_0 \cdot 0 + h_1\phi(2) + h_2\phi(1) + h_3 \cdot 0 \\ &= \frac{3+\sqrt{3}}{4} \cdot \frac{1-\sqrt{3}}{2} + \frac{3-\sqrt{3}}{4} \cdot \frac{1+\sqrt{3}}{2} \\ &= 0\end{aligned}$$

and $\phi(5/2)$

$$\begin{aligned}\phi\left(\frac{5}{2}\right) &= h_0\phi(5) + h_1\phi(5-1) + h_2\phi(5-2) + h_3\phi(5-3) \\ &= h_0 \cdot 0 + h_1 \cdot 0 + h_2 \cdot 0 + h_3\phi(2) \\ &= \frac{1-\sqrt{3}}{4} \cdot \frac{1-\sqrt{3}}{2} \\ &= \frac{2-\sqrt{3}}{4}\end{aligned}$$

Continuing in this manner, we can compute odd multiples of $1/4$ since these values will rely on odd multiples of $1/2$ and so on.

# Daubechies wavelet

Just as in the case of the Haar wavelet, the Daubechies wavelet is defined in terms of the scaling function as

$$\psi(x) = h_3\phi(2x+2) - h_2\phi(2x+1) + h_1\phi(2x) - h_0\phi(2x-1)$$

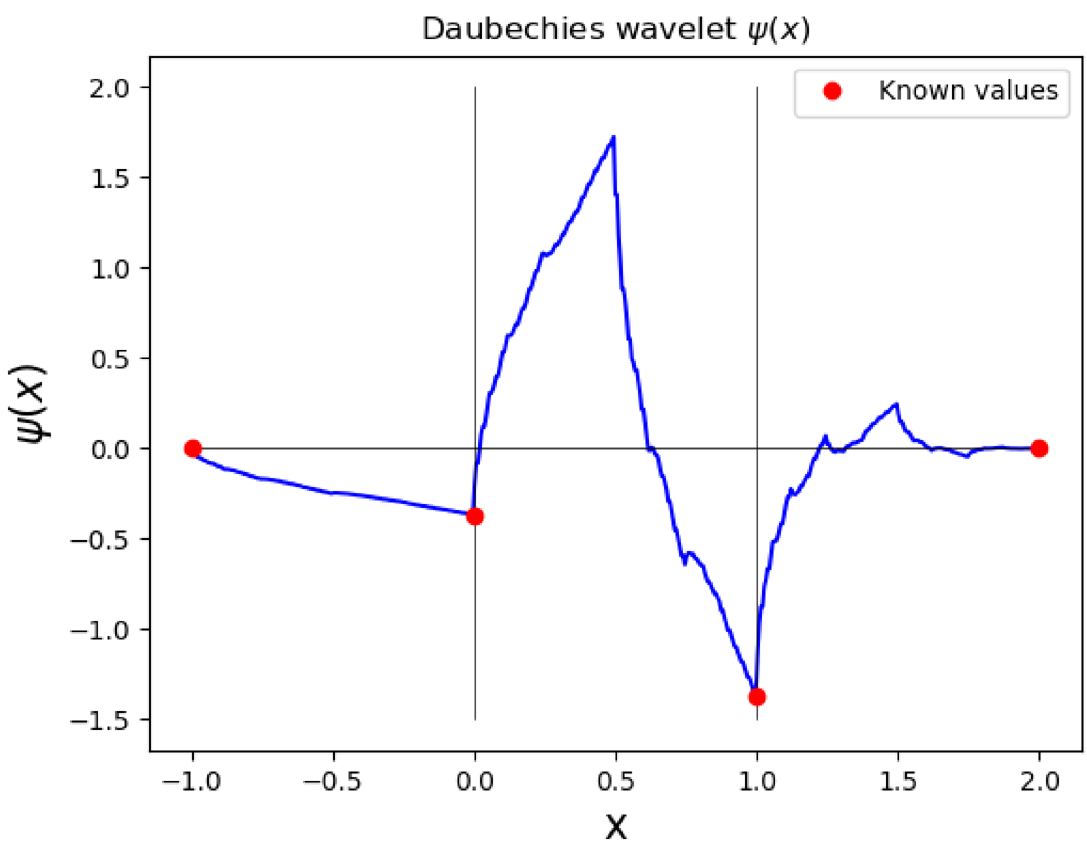
We can plot this wavelet in the same way we plotted the scaling function.

Note that the wavelet is non-zero on the interval $[-1, 2]$.

Because of values can be obtained recursively (as for $\phi(x)$), fast algorithms can be developed based on the Daubechies wavelet.

In [73]:
```
def phi(x,n=6):
    return T(x,n)

def wavelet(x):
    return h3*phi(2*x+2) - h2*phi(2*x+1) + h1*phi(2*x) - h0*phi(2*x-1)
```

In [76]:
```
fig = figure(2)
clf()

N = 512
x = linspace(-1,2,N+1)

plot(x,wavelet(x),'b-')

# Reference lines
yl = [-1.5,2]
plot([-1,2],[0,0],'k-',linewidth=0.5)
plot([0,0],yl,'k-',linewidth=0.5)
plot([1,1],yl,'k-',linewidth=0.5)

xp = [-1,0,1,2]
yp = [0,(1 - sqrt(3))/2, -(1 + sqrt(3))/2,0]
plot(xp,yp,'r.',markersize=12,label='Known values')

title('Daubechies wavelet $\psi(x)$')
xlabel('x',fontsize=16)
ylabel('$\psi(x)$',fontsize=16)


legend()
```




Out[76]: `<matplotlib.legend.Legend at 0x121b83d90>`

# Connection to Haar wavelets

The Haar scaling functions also satisfy a recursion relation, given by

$$\phi(x) = h_0\phi(2x) + h_1\phi(2x-1)$$

with $h_0 = h_1 = 1$. In this case, the "box" function satisifies this recursion exactly, so there is no need for a recursive definition as with the Daubechies scaling function.

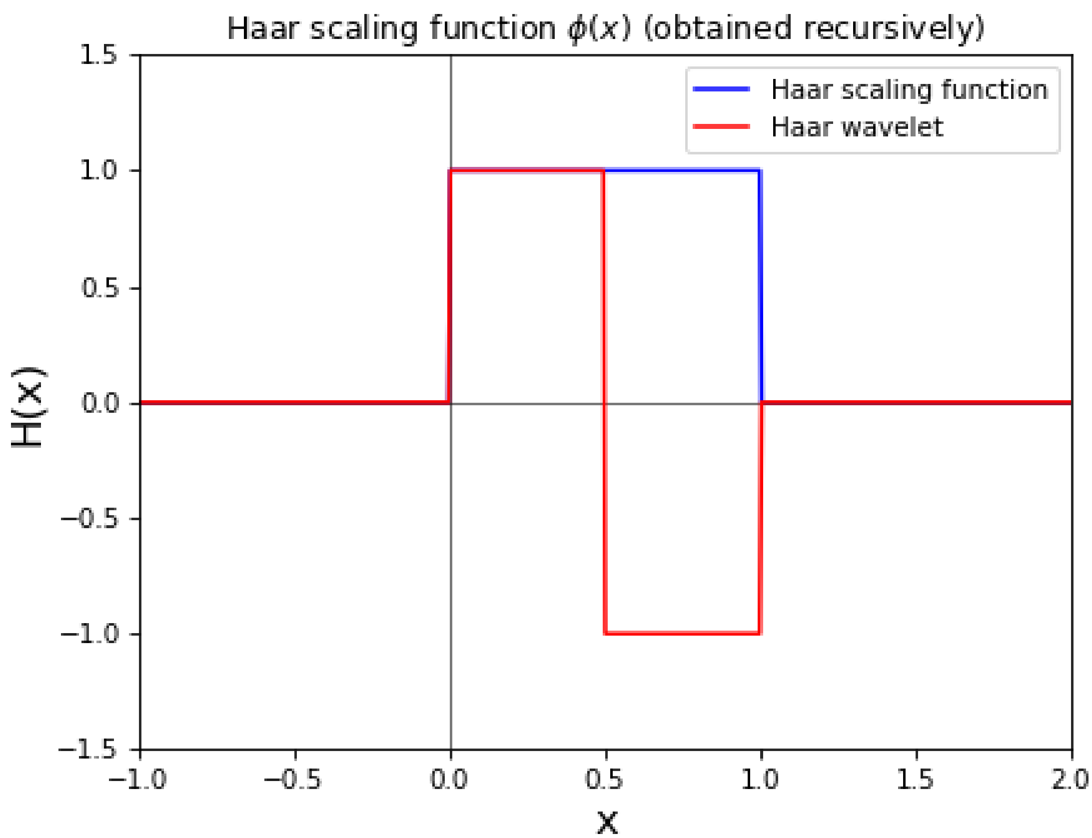
The corresponding wavelet is defined analogously as

$$\psi(x) = h_0\phi(2x) - h_1\phi(2x-1)$$

which is also satisfied exactly by the Haar wavelet.

In [92]:
```
def H(x,n):
    h0 = 1
    h1 = 1
    if n == 0:
        return box(x)
    else:
        return h0*H(2*x,n-1) + h1*H(2*x - 1,n-1)

def haar_wavelet(x):
    h0 = 1
    h1 = 1
    return h0*box(2*x) - h1*box(2*x - 1)
```

In [94]:
```
fig = figure(3)
clf()

N = 512
x = linspace(-1,2,N+1)

hdl, = plot(x,H(x,0),'b-',label='Haar scaling function')

# Reference lines
yl = [-1.5,1.5]
plot([-1,2],[0,0],'k-',linewidth=0.5)
plot([0,0],yl,'k-',linewidth=0.5)

xlim([-1,2])
ylim(yl)

title('Haar scaling function $\phi(x)$ (obtained recursively)')
xlabel('x',fontsize=16)
ylabel('H(x)',fontsize=16)

for k in range(4):
    hdl.set_ydata(H(x,k))

    fig.canvas.draw()

    pause(0.1)

plot(x,haar_wavelet(x),'r',label='Haar wavelet')

legend()
```




```
Out[94]: <matplotlib.legend.Legend at 0x1210f53d0>
```

## Design of Daubechies wavelets

1. The idea is to seek functions which satisfy $T[g] = g$. This is translated to an eigenvalue problem in which which an eigenfunction of $T$ is sought that has associated eigenvalue equal to 1.

1. Does the corresponding eigenspace $\mathrm{Ker}(T - I)$ have dimension 1 so that the solution is unique?

1. Does the "fixed point iteration" $g = T[g]$ actually converge?

Through Fourier analysis, the above questions are answered in the affirmative. In fact, there is exactly one function satisfying the recursion relation 1 (above) with $||\phi|| = 1$.

# Orthogonality

Define

$$\phi_k^{(m)}(x) = \phi(2^m x - k)$$

Then for all indices $k, \ell, m$,

$$\langle \phi_k^{(m)}, \phi_\ell^{(m)} \rangle = \begin{cases} 2^{-m} & \text{if} \quad k = \ell, \\ 0 & \text{if} \quad k \neq \ell, \end{cases}$$

Defining a family of wavelet functions analogously, we have

$$\langle \psi_k^{(m)}, \psi_\ell^{(m)} \rangle = \begin{cases} 2^{-m} & \text{if} \quad k = \ell, \\ 0 & \text{if} \quad k \neq \ell, \end{cases}$$

Furthermore, we have $\langle \phi_\ell^{(m)}, \psi_h^{(n)} \rangle$ for $(m, \ell) \neq (n, h)$.

- These can be use to interpolate functions *in an approximate sense*
- Availability of fast transforms makes these wavelets a practical choice for many applications

```
In [ ]:
```