

```
In [1]: %matplotlib notebook
%pylab
from scipy.integrate import quad
from scipy.special import erf

Using matplotlib backend: nbAgg
Populating the interactive namespace from numpy and matplotlib
```

1.e) Verify that your formulation holds for the function

$$u(x) = e^x,$$

subject to boundary conditions $u(0) = 1$, $u(1) = e$.

The function to be plotted is

$$u(x) = \frac{1}{2} \int_0^1 (|x-y| - x + 2xy - y)e^y dy + 1 - x + xe$$

In [12]:

```
N = 16
a = 0; b = 1
h = 1/N

x = zeros(N+1)
I = zeros(N+1)

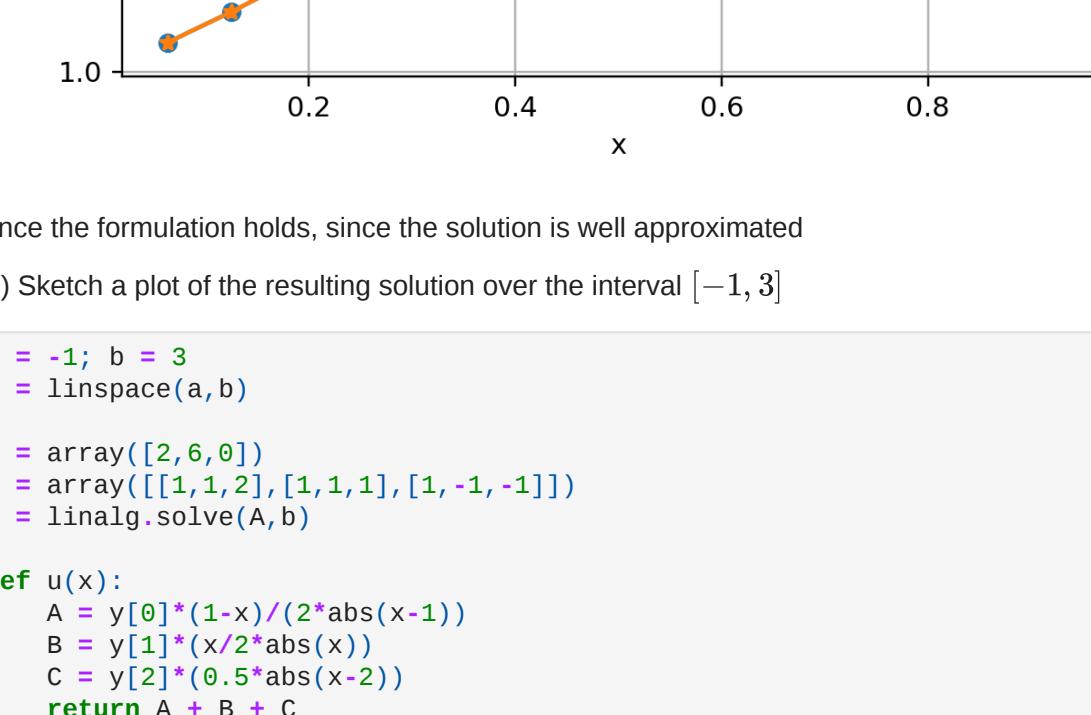
#rhs
def f(x):
    return exp(x)

#Green's function
def G(x,y):
    return 0.5*(abs(x-y)-x+2*x*y-y)

#Quadrature Rule
for j in range(N+1):
    x[j] = j*h
    xj = x[j]
    Gf = lambda y: G(xj,y)*f(y)
    I[j] = quad(Gf,a,b)[0]
    I = I + 1 - x + x*exp(1)
```

In [13]:

```
#plotting
figure(1)
clf()
plot(x[1:-1],I[1:-1],'-o',label='u(x)')
plot(x[1:-1],exp(x[1:-1]),'*',label='exp(x)')
xlabel('x'); ylabel('u and $e^{x}$')
title('A graph of u and $e^{x}$ against x')
grid()
legend()
show()
```



Hence the formulation holds, since the solution is well approximated

2.d) Sketch a plot of the resulting solution over the interval $[-1, 3]$

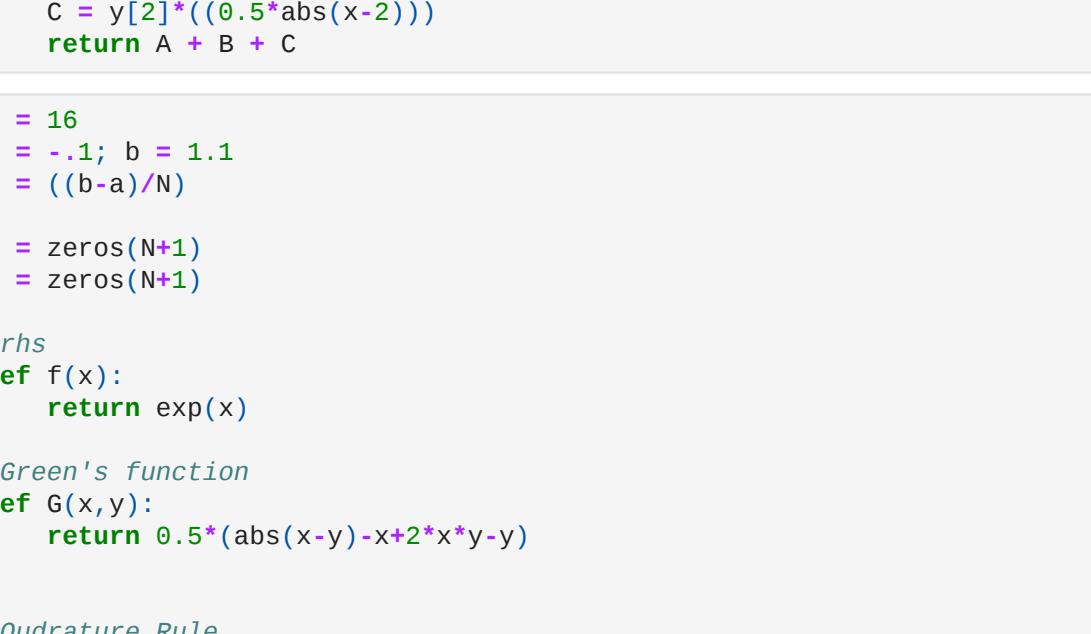
In [14]:

```
a = -1; b = 3
x = linspace(a,b)

b = array([2,6,0])
A = array([[1,1,2],[1,1,1],[1,-1,-1]])
y = linalg.solve(A,b)

def u(x):
    A = y[0]*(1-x)/(2*abs(x-1))
    B = y[1]*x/(2*abs(x))
    C = y[2]*((0.5*abs(x-2)))
    return A + B + C

figure(2)
plot(x,u(x),'-*')
xlabel("x")
ylabel("u")
title('A plot of u against x')
grid()
show()
```



There is a jump, before and after $x = 1$

2.e) Solve the problem

$$u''(x) = e^x$$

$\forall x \in [0, 1]$ subject to $u(0) = 1$ and $u(1) = e$, using a volume integral plus a distribution of sources and dipoles.

In [15]:

```
def u(x):
    b = array([2,2*exp(1),0])
    A = array([[1,1,2],[1,1,1],[1,-1,-1]])
    y = linalg.solve(A,b)
    A = y[0] * ((1-x)/(2*abs(x-1)))
    B = y[1] * (x/(2*abs(x)))
    C = y[2] * ((0.5*abs(x-2)))
    return A + B + C
```

In [16]:

```
N = 16
a = -.1; b = 1.1
h = ((b-a)/N)

x = zeros(N+1)
I = zeros(N+1)

#rhs
def f(x):
    return exp(x)

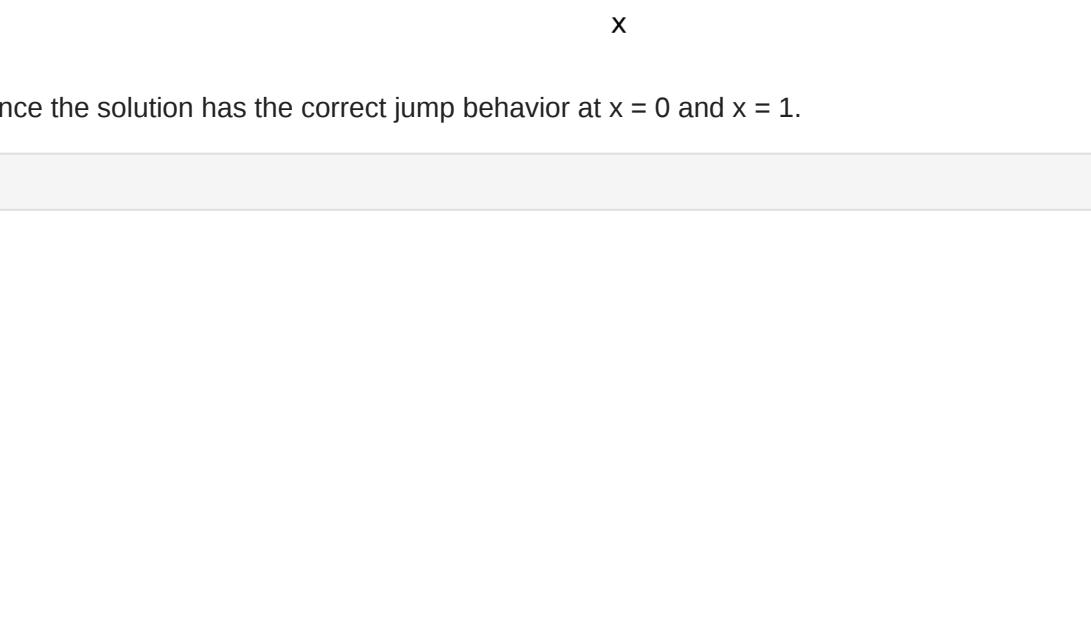
#Green's function
def G(x,y):
    return 0.5*(abs(x-y)-x+2*x*y-y)

#Quadrature Rule
for j in range(N+1):
    x[j] = j*h
    xj = x[j]
    Gf = lambda y: G(xj,y)*f(y)
    I[j] = quad(Gf,a,b)[0]
    I = I + u(x)

cipython-input-15-8ea98c7eb7bb>: RuntimeWarning: invalid value encountered in true_divide
B = y[1] * (x/(2*abs(x)))
```

In [17]:

```
#plotting
figure(3)
clf()
plot(x[1:-1],I[1:-1],'-o',label='u(x)')
plot(x[1:-1],exp(x[1:-1]),'*',label='exp(x)')
xlabel('x'); ylabel('u and $e^{x}$')
title('A graph of u and $e^{x}$ against x')
xlim(-0.1,1.1)
grid()
legend()
show()
```



Hence the solution has the correct jump behavior at $x = 0$ and $x = 1$.

In []:

Brian KYANJO

Homework 8

MATH 573

1. Harmonic function in 1d

(a) Show that

$$G_0(x, y) = \frac{1}{2} |x-y|$$

is a fundamental solution for the operator $L[U] = \frac{\partial^2 U}{\partial x^2}$

Integrate $U''(x) = f(x)$ twice

$$U'(x) = H(x) + C_0$$

$$U(x) = xH(x) + C_0x + C_1$$

but $R(x) = xH(x)$

$$U(x) = R(x) + C_0x + C_1$$

but

$$G_0(x, y) = U(x-y)$$

$$G_0(x, y) = R(x-y) + C_0(x-y) + C_1 \quad \text{--- (1)}$$

Using $G_0(x, y) = G_0(y, x)$

$$R(x-y) + C_0(x-y) + C_1 = R(y-x) + C_0(y-x) + C_1$$

but $R(x-y) = \begin{cases} x-y & x>y \\ 0 & x \leq y \end{cases}$

$$R(y-x) = \begin{cases} y-x & y>x \\ 0 & y \leq x \end{cases}$$

Assume $x>y$, $R(x-y) = x-y$ and $R(y-x) = 0$
 $x-y = 2C_0(y-x) \Rightarrow C_0 = -\frac{1}{2}$.

So Equation ① becomes

$$G_0(x,y) = R(x-y) - \frac{1}{2}(x-y)$$

but $R(x-y) = \begin{cases} x-y & x>y \\ 0 & x\leq y \end{cases}$

So

$$\begin{aligned} G_0(x,y) &= \begin{cases} x-y - \frac{1}{2}(x-y) & x>y \\ 0 - \frac{1}{2}(x-y) & x\leq y \end{cases} \\ &= \begin{cases} \frac{1}{2}(x-y) & x>y \\ -\frac{1}{2}(x-y) & x\leq y \end{cases} \end{aligned}$$

$$G_0(x,y) = \frac{1}{2}|x-y|$$

b) Using the method of Images to find an "Image" source $\tilde{G}_0(x,y) = f(G_0(x,y))$ so that

$$G(x,y) = G_0(x,y) + \tilde{G}_0(x,y)$$

$$U(0)=U(1)=0 \Rightarrow G(0,y)=G(1,y)=0$$

To verify that $\tilde{G}_0(x,y)$ is harmonic in $[0,1]$ then

$$U(0)=ax+b$$

$$U(0)=U(1)=0$$

From max

$$G_0(x,y) \approx \begin{cases} \frac{1}{2}(x-y) & x>y \\ -\frac{1}{2}(x-y) & x\leq y \end{cases}$$

$$G_0(0, y) = -\frac{1}{2}y, \quad y < 0$$

$$G_0(1, y) = -\frac{1}{2}(1-y), \quad y \geq 1$$

$$\text{So } u = av + b$$

$$\text{for } v=0, u = -\frac{1}{2}y$$

$$-\frac{1}{2}y = a(0) + b \Rightarrow b = -\frac{1}{2}y$$

$$\text{for } v=1, u = -\frac{1}{2}(1-y)$$

$$-\frac{1}{2}(1-y) = a + b \Rightarrow a = -\frac{1}{2}(1-y) + \frac{1}{2}y$$

$$\underline{a = y - \frac{1}{2}}$$

$$u = (y - \frac{1}{2})v + -\frac{1}{2}y$$

$$\text{Therefore } \bar{G}_0(x, y) = (y - \frac{1}{2})x - \frac{1}{2}y$$

$$\bar{G}_0(x, y) = -\frac{1}{2}x(1-2y) - \frac{1}{2}y$$

$$\bar{G}_0(x, y) = -\frac{1}{2}(x + y(1-2x)) = -\frac{1}{2}(x - y(2x-1))$$

$$\bar{G}_0(x, y) = -\frac{1}{2}|x - \bar{y}|, \text{ where } \bar{y} = (2x-1)y$$

$$G(x, y) = G_0(x, y) + \bar{G}_0(x, y)$$

$$G(x, y) = \frac{1}{2}|x - y| - \frac{1}{2}(x(1-2y) + y)$$

for $x > y$

$$\begin{aligned} G(x, y) &= \frac{1}{2}(x - y) - \frac{1}{2}(x(1-2y) + y) \\ &= \underline{\underline{y(x-1)}} \end{aligned}$$

for $x < y$

$$G(x,y) = -\frac{1}{2}(x-y) - \frac{1}{2}x + xy - \frac{1}{2}y \\ = \underline{x(y-1)}$$

therefore

$$G(x,y) = \begin{cases} y(x-1) & x < y \\ x(y-1) & x > y \end{cases}$$

As obtained in class

- c) Verify Green's Second Identity for a harmonic function $U(x)$ satisfying (1) and the Green's function $V(y) = G(x,y)$

$$\int_0^1 (U(y)V''(y) - V(y)U''(y)) dy = (U(y)V'(y) - V(y)U'(y)) \Big|_{y=0}^{y=1}$$

$$\int_0^1 (UV'' - VU'') dy = U(1)V'(1) - U(0)V'(0)$$

$$V'(y) = \frac{y-x}{2|x-y|} + x - \frac{1}{2} \Leftrightarrow \begin{cases} V'(0) = x-1 \\ V'(1) = x \end{cases}$$

$$\int_0^1 (UV'' - VU'') dy = U(1)x - U(0)(x-1)$$

$$\text{but } U(1) = b, \quad U(0) = a$$

$$\int_0^1 (UV'' - VU'') dy = bx - a(x-1) = a(1-x) + bx$$

$$\text{but } U'''(y) = 0$$

$$\int_0^1 u(y) v''(y) dy = q(1-x) + bx$$

$$v''(y) = f(x-y) \Rightarrow \int_0^1 u(y) f(x-y) dy = u(x)$$

$$u(x) = q(1-x) + bx \quad \text{hence proved.}$$

d) Verify Green's Second Identity using free space Green's function $G_0(x,y) = \frac{1}{4\pi} \frac{1}{|x-y|}$ and a harmonic function $u(x)$ satisfying 2 boundary conditions given in (1).

Using

$$\int_0^1 (u(y) v_0''(y) - v_0(y) u''(y)) dy = (u(y) v_0'(y) - v_0(y) u'(y)) \Big|_0^1$$

$$\text{but } u''(y) = 0$$

$$\int_0^1 (u(y) v_0''(y)) dy = u(1) v_0'(1) - v_0(1) u'(1) - u(0) v_0'(0) + v_0(0) u'(0)$$

$$\text{but } u'(0) = v_0'(1)$$

$$\int_0^1 (u(y) v_0''(y)) dy = u(1) v_0'(1) - u(0) v_0'(0) + (v_0(0) - v_0(1)) u(1)$$

$$u(1) = b, \quad u(0) = q$$

$$V(y) = \frac{y-x}{2|x-y|} = \begin{cases} V(1) = \frac{1}{2} \\ V(0) = -\frac{1}{2} \end{cases}$$

$$\int_0^1 (u(y) v_0''(y)) dy = \frac{1}{2} b + \frac{1}{2} q + (v_0(0) - v_0(1)) u(1)$$

$$\text{but } u(1) = b - q$$

$$\int_0^1 (U(y) V_0''(y)) dy = \frac{1}{2} (b + a) + \left(\frac{1}{2}x - \frac{1}{2}(1-x) \right) (b-a)$$

$$\int_0^1 U(y) V_0''(y) dy = bx + (1-x)a$$

but

$$\int_0^1 (U(y) V_0''(y)) dy = \int_0^1 U(y) S(x-y) dy = U(x)$$

$$U(x) = a(1-x) + bx$$

2) For a general problem $U''(x) = f(x)$ on $[0,1]$, use Green

$$U(x) = e^x, \quad U(0) = 1, \quad U(1) = e$$

$$U(x) = \int_0^1 G(x,y) f(y) dy + bx + a(1-x)$$

$$U(x) = \int_0^1 \frac{1}{2} (1|x-y| - x + 2xy - y) e^y dy + 1 - x + xe$$

Q. Potentials due to sources and dipole distribution

a) Verify (b) using the representation of the delta function as

$$\delta(x) \approx \frac{1}{2\pi\alpha} e^{-x^2/4\alpha}$$

$$\delta'(x) = \frac{1}{2\pi\alpha} \left(-\frac{2x}{4\alpha} \right) e^{-x^2/4\alpha}$$

$$-\delta'(x) = \frac{x}{4\sqrt{\pi}\alpha} e^{-x^2/4\alpha} \quad \text{--- (1)}$$

but

$$-\delta'(x) = \lim_{\epsilon \rightarrow 0} \frac{\delta(x - \epsilon_2) - \delta(x + \epsilon_2)}{\epsilon}$$

$$-\delta'(x) = \lim_{\epsilon \rightarrow 0} \frac{1}{2\sqrt{\pi}\alpha} \frac{e^{-(x - \epsilon_2)^2/4\alpha} - e^{-(x + \epsilon_2)^2/4\alpha}}{\epsilon}$$

Using L'Hopital's rule

$$-\delta'(x) = \frac{1}{2\sqrt{\pi}\alpha} \left(\lim_{\epsilon \rightarrow 0} \left(\frac{-2(x - \epsilon_2)(-\frac{1}{2}) e^{-(x - \epsilon_2)^2/4\alpha} - (-2(x + \epsilon_2)(\frac{1}{2}) e^{-(x + \epsilon_2)^2/4\alpha}}{4\alpha} \right) \right)$$

$$-\delta'(x) = \frac{x}{4\sqrt{\pi}\alpha} e^{-x^2/4\alpha} \quad \text{--- (2)}$$

Hence verified $\lim_{\epsilon \rightarrow 0} \underline{(2)} = \underline{(1)}$

- b) Use the 1d free-space Green's function $G_0(x, y) = \frac{1}{2} |x - y|$ to obtain a dipole potential by evaluating

$$\lim_{\epsilon \rightarrow 0} \frac{\frac{1}{2} |x - \epsilon_2| - \frac{1}{2} |x + \epsilon_2|}{\epsilon}$$

Rationalizing

$$\lim_{\varepsilon \rightarrow 0} \left(\frac{1}{2} |x - \xi_2| - \frac{1}{2} |x + \xi_2| \right) \left(\frac{1}{2} |x - \xi_2| + \frac{1}{2} |x + \xi_2| \right)$$

$$\varepsilon \left(\frac{1}{2} |x - \xi_2| + \frac{1}{2} |x + \xi_2| \right)$$

$$= \lim_{\varepsilon \rightarrow 0} \frac{-x}{|x - \xi_2| + |x + \xi_2|} = \frac{-x}{2|x|}$$

$$\text{but } G_0(x, y) = \frac{1}{2} |x - y|$$

$$\frac{\partial}{\partial y} G_0(x, y) = \frac{1}{2} \frac{y - x}{|x - y|} \Rightarrow \frac{\partial}{\partial y} G_0(x, 0) = \frac{-x}{2|x|}$$

Hence verified.

- c) Show that if we try to represent the solution distribution of dipoles on the boundary of the interval $[0, 1]$, we can only capture constant solutions to (1).

$$u(x) = \mu(1) \left(-\frac{\partial G_0(x, 1)}{\partial y} \right) - \mu(0) \left(-\frac{\partial G_0(x, 0)}{\partial y} \right)$$

At $x = 0$

$$u(0) = \mu(1) \lim_{x \rightarrow 0} \left(-\frac{\partial G_0(x, 1)}{\partial y} \right) - \mu(0) \lim_{x \rightarrow 0} \left(-\frac{\partial G_0(x, 0)}{\partial y} \right)$$

$$\frac{\partial G_0}{\partial y} = \frac{y - x}{2|x - y|}$$

$$u(0) = \mu(1) \lim_{x \rightarrow 0^+} -\left(\frac{1-x}{2|x-1|} \right) - \mu(0) \lim_{x \rightarrow 0^+} -\left(\frac{-x}{2|x|} \right)$$

$$u(0) = \mu(1) \left(\frac{1}{2}\right) + \mu(0) \left(\frac{1}{2}\right)$$

$$\text{but } u(0) = a$$

$$\underline{\mu(1) + \mu(0) = 2a}$$

At $x=1$

$$u(1) = \mu(1) \lim_{x \rightarrow 1^-} -\left(\frac{1-x}{2|x-1|}\right) - \mu(0) \lim_{x \rightarrow 1^-} -\left(\frac{-x}{2|x-1|}\right)$$

$$\underline{\mu(1) + \mu(0) = 2b}$$

Hence the resulting 2×2 system is

$$\mu(1) + \mu(0) = 2a \quad \text{--- (1)}$$

$$\mu(1) + \mu(0) = 2b \quad \text{--- (2)}$$

$$(2) - (1) \Rightarrow a - b = 0$$

Hence it's only solvable iff $a=b$, otherwise if $a \neq b$, we can't solve the system.

d) If we add an additional source term at a location outside the domain

$$u(x) = \mu(1) \left(-\frac{\partial g_0(x_1)}{\partial y}\right) - \mu(0) \left(-\frac{\partial g_0(x_1)}{\partial y}\right) + \\ \mu(2) G_0(x_1, 2)$$

At $x=0$

$$u(0) = \frac{1}{2} \mu(1) + \frac{1}{2} \mu(0) + \mu(2) \lim_{x \rightarrow 0^-} G_0(x_1, 2)$$

$$u(x) = \frac{1}{2} \mu(1) + \frac{1}{2} \mu(0) + \mu(2) = 9$$

$$\mu(1) + \mu(0) + 2\mu(2) = 29$$

At $x=1$

$$u(x) = \frac{1}{2} \mu(1) + \frac{1}{2} \mu(0) + \mu(2) \lim_{x \rightarrow 1^+} g_0(x, 2)$$

$$\lim_{x \rightarrow 1^+} g_0(x, 2) = \frac{1}{2}, \text{ outside the domain}$$

$$\mu(1) + \mu(0) + \mu(2) = 2b$$

and constraint

$$\mu(2) = \mu(1) - \mu(0)$$

3x3

the system is

$$\begin{cases} \mu(1) + \mu(0) + 2\mu(2) = 2 \\ \mu(1) + \mu(0) + \mu(2) = b \\ \mu(1) + \mu(0) - \mu(2) = 0 \end{cases}$$

Solving the 3 equations we obtain

$$\mu(1) = 3$$

$$\mu(0) = 7$$

$$\mu(2) = -4$$

Show that at the boundary points $x=0$ and $x=1$ the resulting solution has a jumps equal to $\mu(0)$ and $\mu(1)$

$$\mu(0+) = \frac{1}{2} \mu(1) + \frac{1}{2} \mu(0) + \mu(2) \quad \text{--- ①}$$

$$\mu(0-) = \frac{1}{2} \mu(1) - \frac{1}{2} \mu(0) + \mu(2) \quad \text{--- ②}$$

$$\textcircled{1} - \textcircled{2} \\ \underline{\mu(0^+) - \mu(0^-) = \mu(0)}$$

$$\mu(1^+) = \frac{1}{2}\mu(1) + \frac{1}{2}\mu(0) + \frac{1}{2}\mu(2) \quad - \textcircled{3}$$

$$\mu(1^-) = -\frac{1}{2}\mu(1) + \frac{1}{2}\mu(0) + \frac{1}{2}\mu(2) \quad - \textcircled{4}$$

$$\textcircled{3} - \textcircled{4} \Rightarrow \underline{\mu(1^+) - \mu(1^-) = \mu(1)}$$

but that the derivative is continuous across the boundary.

$$u(x) = \mu(1) \left[-\frac{\partial G_0(x, 1)}{\partial y} \right] - \mu(0) \left[-\frac{\partial G_0(x, 0)}{\partial y} \right] + \mu(2) G_0(x, 2)$$

Substituting in for $\mu(1)$, $\mu(0)$ and $\mu(2)$

$$u(x) = 3 \left(\frac{1-x}{2|x-1|} \right) + 7 \left(\frac{x}{2|x|} \right) - 4 \left(\frac{1}{2}|x-2| \right)$$

$$u(x) = -\frac{3}{2} + \frac{7}{2} + 2(x-2) = 2 + 2(x-2)$$

$u'(x) = 2, \forall x \in [0, 1], u'(x)$, exists hence it's continuous across the boundary.

- What is the behaviour of the solution at the same at $x=2$?

The behaviour of the solution at the same ~~at~~ $x=2$ is continuous since

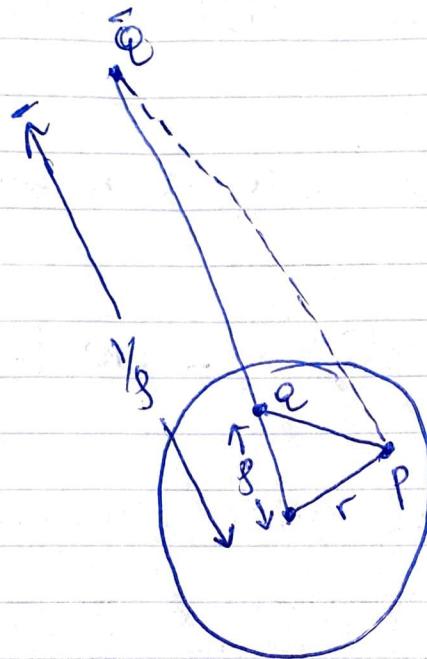
$$U'(x) = \begin{cases} -2(x-2) \\ \frac{-2(x-2)}{|x-2|}, & x \neq 0 \text{ or } x \neq 1 \end{cases}$$

Indeterminate, Removable.

$$\lim_{x \rightarrow 2^-} U'(x) = \lim_{x \rightarrow 2^+} U'(x) = 0$$

3' Green's function for the disk

$P(r, \theta)$ and $Q = (p, \theta')$, interior,
 $\bar{Q} = (1/p, \theta')$ outside the disk.



$$G(P, Q) = \frac{1}{2\pi} \log \left(\frac{r_{PQ}}{pr_{PQ}} \right) = \frac{1}{2\pi} \log(r_{PQ}) - \left(\frac{1}{2\pi} \log(r_{PQ}) + \frac{1}{2\pi} \log(p) \right)$$

Using

$$G(P, Q) = G_0(P, Q) + \bar{G}(P, Q)$$

Using Cosine rule.

$$r_{PQ} = \sqrt{(r^2 + p^2 - 2rp \cos(\theta - \theta'))^{1/2}}$$

$$r_{pq} = \left(r^2 + \frac{1}{r^2} - 2r \cdot \frac{1}{r} \cos(\theta - \theta') \right)^{\frac{1}{2}}$$

$$pr_{pq} = \left(p^2 r^2 + 1 - 2rp \cos(\theta - \theta') \right)^{\frac{1}{2}}$$

Suppose $\bar{G}(P, Q) = -G(P, Q)$, then

but $\bar{G}(P, \bar{Q}) = \frac{1}{2\pi} \log(pr_{pq})$

then

$$\bar{G}(P, \bar{Q}) = -\frac{1}{2\pi} \log(pr_{pq})$$

$$\text{So } G(P, Q) = G_0(P, Q) + \bar{G}(P, Q)$$

but $G_0(P, Q) = \frac{1}{2\pi} \log(r_{pq})$

$$G(P, Q) = \frac{1}{2\pi} \log(r_{pq}) + -\frac{1}{2\pi} \log(pr_{pq})$$

$$G(P, Q) = \frac{1}{2\pi} \log \left(\frac{r_{pq}}{pr_{pq}} \right)$$

Show that $G(P, Q) = 0$, on the boundary
at the disk at the boundary $r=1$

$$r_{pq} = \left(1 + p^2 - 2p \cos(\theta - \theta') \right)^{\frac{1}{2}} \quad \text{--- ①}$$

$$pr_{pq} = \left(1 + p^2 - 2p \cos(\theta - \theta') \right)^{\frac{1}{2}} \quad \text{--- ②}$$

$$\textcircled{4} / \textcircled{2} \Rightarrow \frac{r_{pq}}{pr_{pq}} = 1$$

hence

$$G(p_1, q) = \frac{1}{2\pi} \log C(1) = 0$$

$$\underline{\underline{G(p_1, q) = 0}}$$