Math-6316 & CSE-7366, homework 2

Due Feb. 28, 2008

Problem 1. Let $W \in \mathbb{C}^{n\times k}$ with $k \leq n$. Show that $\left|\left|WW^*\right|\right|_2 \leq trace(WW^*)$.

Proof: Let $M = WW^*$, we show $\left|\left|M\right|\right|_2 \leq trace(M)$. Note that M is hermitian, therefore it is unitarily diagonalizable. Let $M = QDQ*$ be its eigen-decomposition, with Q unitary. Note also that M is positive semidefinite, therefore all the diagonal elements d_i in D are nonnegative. Hence,

$$\left|\left|M\right|\right|_2 = \left|\left|D\right|\right|_2 = \max_{1\leq i\leq n} d_i \leq trace(D) = trace(QDQ^*) = trace(M) \ .$$

Proof 2: Let the SVD of W be $W = U\Sigma V^*$. Then

$$\left|\left|WW^*\right|\right|_2 = \left|\left|U\Sigma^2 U^*\right|\right|_2 = \left|\left|W\right|\right|_2^2 \leq \left|\left|W\right|\right|_F^2 = trace(WW^*) \ .$$

Proof 3: Let the SVD of WW^* be $WW^* = U\Sigma V^*$, since WW^* is (hermitian) positive semidefinite, we have $U = V$. Therefore

$$\left|\left|WW^*\right|\right|_2 = \sigma_1 \leq trace(\Sigma) = trace(U\Sigma V^*) = trace(WW^*) \ .$$

Problem 2. Show that for any matrices A, B, C such that the matrix product ABC is defined,

$$\left|\left|ABC\right|\right|_F \leq \left|\left|A\right|\right|_2 \left|\left|B\right|\right|_F \left|\left|C\right|\right|_2 \ .$$

Proof: We first show that $\left|\left|AB\right|\right|_F \leq \left|\left|A\right|\right|_2 \left|\left|B\right|\right|_F$: Let the SVD of A be $A = U\Sigma V^*$, where U and V are unitary, then

$$\begin{aligned} \left|\left|AB\right|\right|_F &= \left|\left|U\Sigma V^* B\right|\right|_F = \left|\left|\Sigma V^* B\right|\right|_F \\ &\leq \sigma_1(A)\left|\left|V^* B\right|\right|_F = \sigma_1(A)\left|\left|B\right|\right|_F = \left|\left|A\right|\right|_2 \left|\left|B\right|\right|_F \end{aligned}$$

Similarly we can show $\left|\left|BC\right|\right|_F \leq \left|\left|B\right|\right|_F \left|\left|C\right|\right|_2$, therefore

$$\left|\left|ABC\right|\right|_F \leq \left|\left|A\right|\right|_2 \left|\left|BC\right|\right|_F \leq \left|\left|A\right|\right|_2 \left|\left|B\right|\right|_F \left|\left|C\right|\right|_2 \ .$$

Problem 3. Let an SVD of $A \in \mathbb{C}^{m\times m}$ be $A = U\Sigma V^*$. Find an eigenvalue decomposition of the $2m \times 2m$ matrix

$$\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} .$$

(You may use Matlab to figure out the relation, then prove the relation you find.)

Answer: Note that $AV = U\Sigma$, and $A^*U = V\Sigma$, we have

$$\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} \begin{bmatrix} U & -U \\ V & V \end{bmatrix} = \begin{bmatrix} U\Sigma & U\Sigma \\ V\Sigma & -V\Sigma \end{bmatrix} = \begin{bmatrix} U & -U \\ V & V \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix}$$

which means an eigendecomposition is

$$\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} = \begin{bmatrix} U & -U \\ V & V \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix} \begin{bmatrix} U & -U \\ V & V \end{bmatrix}^{-1} .$$

Let $X = \begin{bmatrix} U & -U \\ V & V \end{bmatrix}$, note that $XX^* = 2I$, we see $\frac{1}{\sqrt{2}}X$ is unitary, so a unitary eigendecomposition can be written as

$$\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} = \left(\frac{1}{\sqrt{2}}X\right) \begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix} \left(\frac{1}{\sqrt{2}}X\right)^* .$$

Problem 4. Write a single M-file to perform the *Thin QR decomposition* using Classical Gram-Schmidt (CGS) and modified Gram-Schmidt (MGS). (The choice of CGS or MGS is by an input variable).

Here is a sample GS code, but you are **strongly** recommended to write your own code first before trying the sample code.

Problem 5. Write a M-file to perform the *QR decomposition* using Householder reflectors.

Here is a sample Householder code. Again, please try to code on your own first!

Problem 6. Test correctness of your code. Here is a sample test code .

Then use the provided example (you can construct your own examples), to test the orthogonality of the Q factor obtained from your code for Problems 4 & 5. I.e., plot the $||I - Q^*Q||_2$.

Also, plot the factorization error $||A - QR||_2$. What do you observe? Does lost of orthogonality in the Q factor imply a large $||A - QR||_2$ error ?

Problem 7. Modify your CGS and MGS code so that it performs a *Thin QL decomposition*, where L is a lower triangular.

(You need to first modify the CGS and MGS algorithms, then code them up.)

Problem 8. Modify the generation of Householder reflectors so that they can be used to perform QL decomposition. Modify your M-file for Householder QR for the Householder QL decomposition.

Answer: A sample QL by Householder code is available here.

Problem 9. Use the same example as in Problem 6 to test the orthogonality of the Q factor obtained from your code for Problems 7 & 8. I.e., plot the $||I - Q^*Q||_2$.

Also, plot the factorization error $||A - QL||_2$.

Problem 10. Modify your Householder QR code in Problem 5 so that it can solve a least square problem $\min_x ||Ax - b||$. In this code, do not form Q explicitly. (Assume A is full rank. Compare your solution with the one obtained by $x = A\backslash b$.)

Problem 11. (optional) Let $S \in \mathbb{R}^{n\times n}$ be a skew-symmetric matrix (i.e., $S^T = -S$), then from homework 1 we know the Cayley-transform of S defined as

$$C(S) = (I + S)(I - S)^{-1}$$

is orthogonal. Construct a rank-2 matrix S such that if $x \in \mathbb{R}^n$ then $C(S)x$ is zero except in the first component.

Answer: It is proved in homework one that $C(S) = (I + S)(I - S)^{-1}$ is an orthogonal matrix, therefore it preserves the 2-norm. So

$$||C(S)x||_2 = ||\alpha e_1||_2 = |\alpha| = ||x||_2.$$

We can choose $\alpha = ||x||_2$.

$$\begin{aligned} C(S)x = \alpha e_1, \quad C^{-1}(S) = C^T(S) \\ \Longrightarrow \quad x = C^T(S)\alpha e_1 = (I - S)^{-T}(I + S)^T \alpha e_1 \\ \Longrightarrow \quad (I - S)^{-T} x = (I + S)^T \alpha e_1 \\ \Longrightarrow \quad (I + S)x = (I - S)\alpha e_1 \end{aligned}$$

Note also that a rank-2 skew symmetric matrix can be written as

$$S = uv^T - vu^T$$

where $u, v \in \mathbb{R}^n$ Plug this S into $(I + S)x = (I - S)\alpha e_1$ we get

$$uv^T(x + \alpha e_1) - vu^T(x + \alpha e_1) = \alpha e_1 - x \tag{1}$$

The RHS is in a 2-d subspace $span\{u, v\}$, the RHS is in a 2-d subspace $span\{e_1, x\}$. We can simply set $u = \beta_1 e_1, v = \beta_2 x$ and plug them into (1), and compare coefficients for the x and e_1 terms on both sides. This leads to (denote $x_1 = e_1^T x$)

$$\beta_1\beta_2(||x||_2 + x_1) = 1$$

We can choose, for example, $\beta_1 = 1, \beta_2 = \dfrac{1}{||x||_2 + x_1}$, and construct a desired rank-2 matrix

$$S = e_1 \frac{x^T}{||x||_2 + x_1} - \frac{x}{||x||_2 + x_1} e_1^T \ .$$

A test code is available here.