Bessel's equation

General Bessel Equation

$$x^2 y'' + xy' + (x^2 - \nu^2) y = 0 \quad , \quad \nu \geq 0$$

$$\sum_{m=0}^{\infty} (m+r)^2 a_m x^{m+r} + (x^2 - \nu^2) \sum_{m=0} a_m x^{m+r} = 0$$

$$= \sum_{m=0}^{\infty} \left((m+r)^2 - \nu^2\right) a_m x^{m+r} + \sum_{m=2}^{\infty} a_{m-2} x^{m+r} = 0$$

$$\Rightarrow x^r \Big((r^2 - \nu^2) a_0 + \left((r+1)^2 - \nu^2\right) a_1$$

$$+ \sum_{m=2}^{\infty} \left[\left((m+r)^2 - \nu^2\right) a_m + a_{m-2} \right] x^m \Big) = 0$$

$\Rightarrow$ Indicial equation : $r^2 = \nu^2 \Rightarrow r = \pm \nu$

choose $r^1 = \nu$, $r^2 = -\nu$, $r^1 > r^2$

$\Rightarrow \left[(\nu+1)^2 - \nu^2 \right] a_1 \Rightarrow \nu = \frac{-1}{2}$ or $a_1 = 0$

Go for the series solution using $r^1 = \nu \geq 0$

$$\left[\left((m+\nu)^2-\nu^2\right)a_m + a_{m-2}\right] = 0$$

Since $a_1 = 0$ all odd indices will be 0.

Consider even indices m:

$$\Rightarrow a_m = \frac{-1}{(m+\nu)^2-\nu^2}\, a_{m-2} = \frac{-1}{m^2+2m\nu}\, a_{m-2}$$

Set $m = 2j$:

$$a_{2j} = \frac{-1}{(2j)^2+2(2j)\nu}\, a_{2j-2} = \overbrace{\frac{-1}{2^2 j(j+\nu)}}^{b_j}\, a_{2(j-1)}$$

$$= \frac{1}{2^2 j(j+\nu)} \cdot \frac{1}{2^2(j-1)(j+\nu-1)}\, a_{2(j-2)}$$

$$= \underbrace{\frac{-1}{2^2 j(j+\nu)}}_{b_j} \cdot \underbrace{\frac{-1}{2^2(j-1)(j+\nu-1)}}_{b_{j-1}} \cdot \underbrace{\frac{-1}{2^2(j-2)(j+\nu-2)}}_{b_{j-2}}\, a_{2(j-3)}$$

$a_{2j} =$

$$= \underbrace{\frac{-1}{2^2 j(j+\nu)}}_{b_j} \cdot \underbrace{\frac{-1}{2^2(j-1)(j+\nu-1)}}_{b_{j-1}} \cdot \underbrace{\frac{-1}{2^2(j-2)(j+\nu-2)}}_{b_{j-2}} a_{2(j-3)}$$

$$= \underbrace{b_j}_{b_j} \cdot \underbrace{b_{j-1}}_{b_{j-1}} \cdots \underbrace{b_{j-(j-1)}}_{b_1} \underbrace{a_{2(j-j)}}_{a_0}$$

$j+\nu-(j-1)$
$= \nu+1$

$$= \frac{(-1)^j a_0}{2^{2j}[\underbrace{(j)(j-1)(j-2)\cdots(1)}_{j!}][(j+\nu)(j+\nu-1)(\cdots)(\nu+1)]}$$

$$= \frac{(-1)^j a_0}{2^{2j} j! (\nu+1)(\nu+2)\cdots(\nu+j)}$$

$$y_1(x) = x^{\nu} \sum_{j=0}^{\infty} a_{2j} x^{2j} \qquad r' = \nu \geq 0$$

where

$$a_{2j} = \frac{(-1)^j a_0}{2^{2j} j! (\nu+1)(\nu+2) \cdots (\nu+j)}$$

Assume $\nu = n$ is an integer, $n \geq 0$

$$a_{2j} = \frac{(-1)^j}{2^{2j} j! (n+1)(n+2) \cdots (n+j)} a_0$$

Since a_0 is arbitrary, we can define it

$$a_0 = \frac{1}{2^n n!}$$

So that

$$a_{2j} = \frac{(-1)^j}{2^{2j+n} j! (n+j)!}$$

Bessel function of the First Kind of order n

$$x^2 y'' + x y' + (x^2 - n^2) y = 0$$

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m! (m+n)!}$$

Solution for any $\nu \geq 0$

$$a_{2j} = \frac{(-1)^j a_0}{2^{2j} j! \underbrace{(\nu+1)(\nu+2)\cdots(\nu+j)}}$$

Choose

$$a_0 = \frac{1}{2^\nu \Gamma(\nu+1)}$$

integer ν.

$a_0 = \frac{1}{2^n n!}$

Recall: for integer n, $\Gamma(n+1) = n!$.

So now, we have

$$a_{2j} = \frac{(-1)^j a_0}{2^{2j} j! (\nu+1)(\nu+2)\cdots(\nu+j)}$$

$$= \frac{(-1)^j}{2^{2j+\nu} j! (\nu+1)(\nu+2)\cdots(\nu+j)\,\Gamma(\nu+1)}$$

Use $x\,\Gamma(x) = \Gamma(x+1)$ to get

$$(\nu+1)\,\Gamma(\nu+1) = \Gamma(\nu+2)$$
$$(\nu+2)\,\Gamma(\nu+2) = \Gamma(\nu+3)$$
$$(\nu+3)\,\Gamma(\nu+3) = \Gamma(\nu+4)$$

So that

$$(\nu+1)(\nu+2)(\nu+3)\cdots(\nu+j)\,\Gamma(\nu+1)$$
$$= (\nu+j)\,\Gamma(\nu+j) = \Gamma(\nu+j+1)$$

For any $\nu \geq 0$, we hae

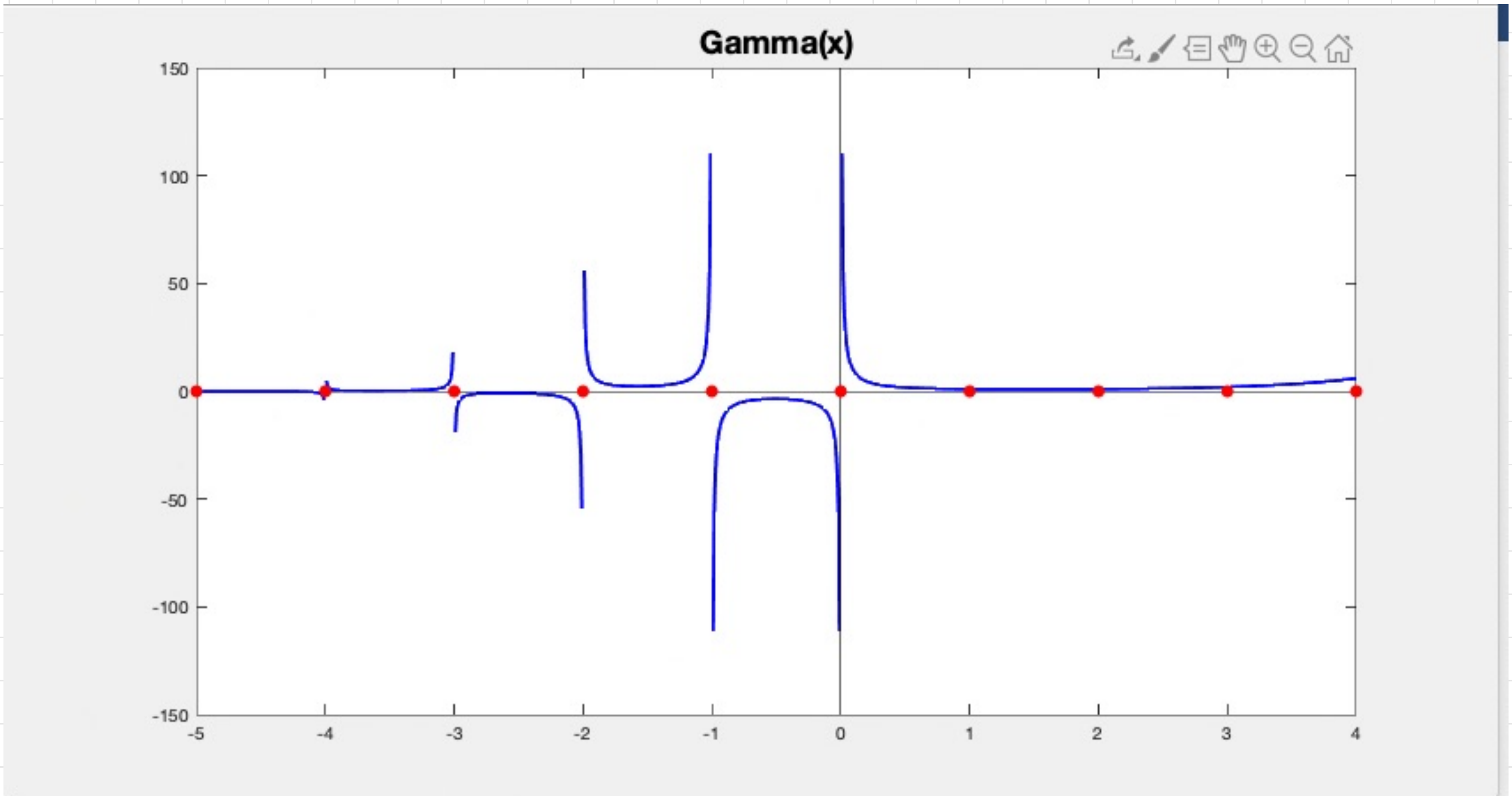
$$J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu}\, m!\, \underbrace{\Gamma(m+\nu+1)}_{=(m+\nu)!,\ \nu \text{ integer.}}}$$

Bessel Function of the first kind of order ν.

Recall that we had two roots of the indicial equation: $r^1 = \nu$ and $r^2 = -\nu$

What is the solution corresponding to $r^2 = -\nu$?

We need to consider the behavior of the $\Gamma(m+\nu+1)$ for integer and non integer values



- For non-positive integer values,
  $\Gamma(-n) = \pm\infty \quad -n = -1, -2, -3, \ldots$
- For positive integer values, we have $\Gamma(n+1) = n!$, $n = 0, 1, 2, 3, \ldots$

If ν is not an integer, then $\Gamma(m+n-1)$ is finite and we can construct a second linearly independent solution

If ν is not an integer, then

$$J_{-\nu}(x) = x^{-\nu} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-\nu}\, m!\, \Gamma(m-\nu+1)}$$

The solution to Bessel's equation is then

$$y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x)$$

ν not an integer.

If ν is an integer, the term

term $\Gamma(m+n-1)$ in the series coefficients for $J_{-\nu}(x)$ is infinite, for $m+n-1 < 0$ or $m < -n+1$. thus

$$B_m = 0, \quad m < -n+1$$

and the series coefficients are shifted.

With some algebra, it is possible to show that

$$J_{-n}(x) = (-1)^n J_n(x)$$

and so a second linearly independent solution must involve a $\ln(x)$ term.

A general second solution is defined as

$$Y_\nu(x) = \frac{1}{\sin(\nu\pi)}\left[J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)\right]$$

$$Y_n(x) = \lim_{\nu \to n} Y_\nu(x)$$

$\Rightarrow$ Bessel function of the second kind.

log term

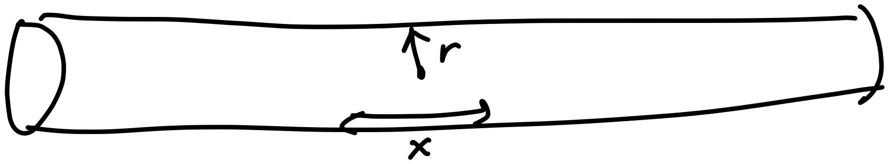
$$Y_n(x) = \frac{2}{\pi} J_n(x)\left(\ln\frac{x}{2} + \gamma\right) + \frac{x^n}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m-1}(h_m + h_{m+n})}{2^{2m+n} m!\,(m+n)!}x^{2m}$$

$$-\frac{x^{-n}}{\pi}\sum_{m=0}^{n-1}\frac{(n-m-1)!}{2^{2m-n} m!}x^{2m}$$

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

$$y(x) = c_1 J_\nu(x) + c_2 Y_\nu(x)$$

$$\nu \geq 0$$



The classic text "Conduction of Heat in Solids", by H. S. Carslaw and J. C. Jaeger (Oxford University Press, 1959) proposes a model for heat flow in a wire. The model geometry is a cylinder of radius a and the distribution of temperature $T(r)$ in the wire is a function of radius r only. Assuming constant thermal resistivity $R = 1/K$, a simple model of heat flow in the wire is given by

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{A_0}{K} = 0, \qquad 0 \le r \le a \tag{1}$$

where K is the thermal conductivity of the wire and A_0 is constant rate of heat production due to Joule heating.

1. A more realistic model allows the thermal resistivity R to vary linearly with temperature as $R = R_0(1 + \alpha(T - T_0))$, where R_0 is the resistance at a reference temperature T_0 and α is the temperature coefficient of resistivity. Show that the model in (1) becomes

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \beta^2 T = -\frac{A_0}{K_0}\left(1 - \alpha T_0\right), \qquad \beta^2 = \alpha A_0/K_0, \tag{2}$$

   where $K_0 = 1/R_0$.

2. Find the general solution to this model. **Hint 1** : First solve the homogeneous problem, then find a very simple solution to the non-homogeneous problem. Use the principle of superposition to find the general solution. **Hint 2** : Be sure your solution is physical over the domain $0 \le r \le a$.

3. Suppose the temperature on the surface of the wire is held fixed at $T(a) = T_0$. Find the particular solution to model equation.

4. Find physical values for resistivity R_0, rate of heat production, temperature coefficient α at a reference temperature T_0, diameter, and a surface temperature T_0 for copper wiring. Verify that the units you use are consistent. Plot your solution $T(r)$ using these values. Cite the sources you used.

   Hint : For the units part of this question, convince yourself that the equation makes sense when K_0 is a thermal conductivity. Then to convert between thermal conductivity and electrical conductivity of a metal, use the Wiedemann-Franz Law, which states that

$$K_0 = LT_0\sigma$$

   where σ is the electical conductivity ($\Omega^{-1}\,\mathrm{m}^{-1}$) and $L = 2.44 \times 10^{-8}$ is the Lorenz number ($\mathrm{W}\,\Omega\,\mathrm{K}^{-2}$).

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constant resistivity! $R_0 = \frac{1}{K}$

linear dependence on Temp.

$$R(T) = R_0(1 + \alpha(T - T_0))$$

$$\frac{1}{r}\left(r\frac{d^2T}{dr^2} + \frac{dT}{dr}\right) + A_0\left(R_0(1+\alpha(T-T_0))\right) = 0$$

$$T''(r) + \frac{1}{r}T' + \underbrace{\frac{A_0\alpha}{K_0}}_{\beta^2}T = -\frac{A_0}{K_0}(1-\alpha T_0)$$

$$T''(r) + \frac{1}{r}T' + \beta^2 T = -\frac{A_0}{K_0}(1-\alpha T_0)$$

$$\Rightarrow \quad r^2T''(r) + rT'(r) + \beta^2 r^2 T = -r^2\frac{A_0}{K_0}(1-\alpha T_0)$$

$$r^2 T''(r) + rT'(r) + \beta^2 r^2 T = -r^2 \frac{A_0}{K_0}(1-\alpha T_0)$$

$$\rho^2 = r^2\beta^2 \Rightarrow r = \frac{\rho}{\beta} \quad dr = \frac{1}{\beta} d\rho$$

$$\frac{d}{dr} = \beta \cdot \frac{1}{d\rho} ; \quad \frac{d^2}{dr^2} = \beta^2 \frac{1}{d\rho^2}$$

$$\frac{\rho^2}{\beta^2}\left(\beta^2 T''(\rho)\right) + \left(\frac{\rho}{\beta}\right)\left(\beta T'(\rho)\right) + \left(\frac{\rho^2}{\beta^2}\beta^2 T\right) =$$

$$-\left(\frac{\rho^2}{\beta^2}\right)\frac{A_0}{K_0}(1-\alpha T_0)$$

$$\beta^2 = \frac{A_0 \alpha}{K_0} \Rightarrow \quad -\frac{\rho^2}{\beta^2}\frac{A_0}{K_0} = -\rho^2 \cdot \frac{1}{\alpha}$$

$$\rho^2 T'' + \rho T' + \rho^2 T = -\rho^2\left(\frac{1}{\alpha} - T_0\right)$$

Homogeneous Solution

$$\rho^2 T'' + \rho T' + (\rho^2 - 0)T = 0 \qquad T(a) = T_0 \qquad T(0) < \infty$$

$$T(\rho) = c_1 J_0(\rho) + B Y_0(\rho)$$

$$T(r) = c_1 J_0(\rho\beta) + \cancel{B Y_0(\rho B)}$$

since $Y_0(0) = \infty$

$$\rho^2 T'' + \rho T' + \rho^2 T = -\rho^2 \cdot \left(\frac{1}{\alpha} - T_0\right)$$

$$\rho = r\beta = \frac{A_0 \alpha}{K} r, \quad \beta^2 = \frac{A_0 \alpha}{K}$$

$$K_0 = \frac{1}{R_0}$$

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non-homogeneous solution

Try a constant solution:

$$T = \text{constant.}$$

In this case, we can kick out $T''(\rho)$ and $T'(\rho)$, and are left with

$$\rho^2 T = -\rho^2 \left(\frac{1}{\alpha} - T_0\right)$$

or

$$T = T_0 - \frac{1}{\alpha}$$

inhomogeneous solution

General solution: $T(r) = c_1 J_0(r\beta) + T_0 - \frac{1}{\alpha}$

3. Suppose the temperature on the surface of the wire is held fixed at $T(a) = T_0$. Find the particular solution to model equation.

General solution

$$T(r) = C_1 J_0(r\beta) + T_0 - \frac{1}{\alpha}$$

at $r = a$, we have $T(a) = T_0$.

So

$$T(a) = C_1 J_0(a\beta) + T_0 - \frac{1}{\alpha} = T_0$$

$$\Rightarrow C_1 = \frac{1}{\alpha J_0(a\beta)}$$

and the final solution is:

$$T(r) = \frac{1}{\alpha}\left(\frac{J_0(r\beta)}{J_0(a\beta)} - 1\right) + T_0$$

Check: $T(a) = T_0$

$$T(0) = \frac{1}{\alpha}\left(\frac{1}{J_0(a\beta)} - 1\right) + T_0 < \infty$$

as long as $a\beta = aA_0\alpha/K \neq$ zero of $J_0(x)$.