Computational Mathematics Assignment V of Math577

Based on HWs of Yuhan Ding

20.2 Suppose $A \in \mathbb{C}^{m\times m}$ satisfies the upper-left $k \times k$ block $A_{1:k,1:k}$ is nonsingular and is banded with bandwidth $2p+1$, i.e., $a_{ij} = 0$ for $|i-j| > p$. What can you say about the sparsity patterns of the factors L and U of A?

Solution:
L is lower triangular matrix with $l_{ij} = 0$ for $i - j > p$.
U is upper triangular matrix with $u_{ij} = 0$ for $j - i > p$.
As A is a banded matrix,

$$A = \begin{pmatrix} a_{11} & \dots & a_{1,p+1} & & & \\ \vdots & \ddots & \vdots & \ddots & & \\ a_{p+1,1} & \dots & \dots & \dots & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & a_{m-p,m} \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & a_{m,m-p} & \dots & a_{mm} \end{pmatrix}.$$

A has a LU factorization, where L is a lower triangular matrix and U is a upper triangular matrix.

$$L = \begin{pmatrix} 1 & & \\ \vdots & \ddots & \\ l_{m1} & \dots & 1 \end{pmatrix}, U = \begin{pmatrix} u_{11} & \dots & u_{1m} \\ & \ddots & \vdots \\ & & u_{mm} \end{pmatrix}.$$

Then when $i - j > p$, due to the property of the upper triangular matrix,

$$a_{i,j} = \sum_{k=1}^{j} l_{i,k}u_{k,j} = 0, j = 1, \dots, m-p-1.$$

We can obtain $l_{i,j} = 0, \text{ when } i - j > p$.
Similarly $j - i > p$, due to the property of the lower triangular matrix,

$$a_{i,j} = \sum_{k=1}^{i} l_{i,k}u_{k,j} = 0, i = 1, \dots, m-p-1.$$

We can obtain $u_{i,j} = 0, \text{ when } j - i > p$.

20.3 Suppose an $m \times m$ matrix A is written in the block form $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where A_{11} is $n \times n$ and A_{22} is $(m-n) \times (m-n)$. Assume that the upper-left $k \times k$ block $A_{1:k,1:k}$ is nonsingular,
(a) Verify the formula

$$\begin{pmatrix} I & \\ -A_{21}A_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix}$$

for "elimination" of the block A_{21}. The matrix $A_{22} - A_{21}A_{11}^{-1}A_{12}$ is known as the *Schurcomplement* of A_{11} in A.
(b) Suppose A_{21} is eliminated row by row by means of n steps of Gaussian elimination. Show that the bottom-right $(m-n) \times (m-n)$ block of the result is again $A_{22} - A_{21}A_{11}^{-1}A_{12}$.

Proof:
(a)

$$\begin{pmatrix} I & \\ -A_{21}A_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$
$$= \begin{pmatrix} A_{11} & A_{12} \\ -A_{21}A_{11}^{-1}A_{11} + A_{21} & -A_{21}A_{11}^{-1}A_{12} + A_{22} \end{pmatrix}$$
$$= \begin{pmatrix} A_{11} & A_{12} \\ & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix}$$

(b)The n steps of Gaussian elimination is equivalent to multiply a lower triangular matrix to A such that A_{11} become an upper triangular matrix and the block A_{21} become 0. Then consider the LU factorization of A_{11}. Suppose $A_{11} = L_{11}U_{11}$, then the lower triangular matrix for A can be represented as

$$L = \begin{pmatrix} L_{11}^{-1} & 0 \\ X & I \end{pmatrix}$$

where L_{11} is $n \times n$, I is $m-n$ Identity matrix, X is $(m-n) * n$ unknown matrix. Then

$$LA = \begin{pmatrix} L_{11}^{-1} & 0 \\ X & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} U_{11} & L_{11}^{-1}A_{12} \\ XA_{11} + A_{21} & XA_{12} + A_{22} \end{pmatrix},$$

where $XA_{11} + A_{21} = 0$. A_{11} is nonsingular, then $X = -A_{21}A_{11}^{-1}$. Hence, the bottom-right block is

$$XA_{12} + A_{22} = -A_{21}A_{11}^{-1}A_{12} + A_{22} = A_{22} - A_{21}A_{11}^{-1}A_{12},$$

The same result of (a).

21.2 Suppose $A \in \mathbb{C}^{m\times m}$ is banded with bandwidth $2p+1$, as in Exercise 20.2, and a factorization $PA = LU$ is computed by Gaussian elimination with partial pivoting. What can you say about the sparsity patterns of L and U?

Solution:
L is a lower triangular matrix.
U is a upper triangular matrix with $u_{ij} = 0$ for $j - i > 2p$.

As A is a banded matrix,

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1,p+1} & & & \\ \vdots & \ddots & \vdots & \ddots & & \\ a_{p+1,1} & \cdots & \cdots & \cdots & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & a_{m-p,m} \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & a_{m,m-p} & \cdots & a_{mm} \end{pmatrix}.$$

The extreme situiation of PA is that

$$PA = \begin{pmatrix} a_{p+1,1} & a_{p+2,1} & \cdots & a_{2p+1,1} & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & a_{m-p,m-2p} & \cdots & \cdots & a_{m-p,m} \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & a_{m,m-p} & \cdots & a_{mm} \\ a_{11} & \cdots & a_{1,p+1} & & & \\ \vdots & \ddots & \vdots & \ddots & & \\ a_{p1} & \cdots & \cdots & \cdots & a_{p,2p} & \end{pmatrix}.$$

Due to the property of LU factorization, we can find that L is still a lower triangular matrix and U is a upper triangular matrix with $u_{ij} = 0$ for $j - i > 2p$.
When $j - i > 2p$, we can get $j = 4, i = 1$ and $u_{ij} = 0$.

4. Perform the Gaussian Elimination with Partial Pivoting (on paper) on the matrix

$$A = \begin{pmatrix} 2 & 4 & 8 & 6 \\ 1 & 3 & 7 & 7 \\ 1 & 3 & 9 & 9 \\ 0 & 1 & 5 & 8 \end{pmatrix}.$$

Find the matrices P, L and U.

Solution:
As the diagonal entry is bigger,

$$P_1 = I, P_1 A = \begin{pmatrix} 2 & 4 & 8 & 6 \\ 1 & 3 & 7 & 7 \\ 1 & 3 & 9 & 9 \\ 0 & 1 & 5 & 8 \end{pmatrix} = A^{(1)}.$$

Then

$$L_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} L_1 A^{(1)} = \begin{pmatrix} 2 & 4 & 8 & 6 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 5 & 6 \\ 0 & 1 & 5 & 8 \end{pmatrix} = A^{(2)}$$

Also because the diagonal entry is bigger, $P_2 = I$. So $P_2A^{(2)} = A^{(2)}$. Let

$$L_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \Rightarrow L_2A^{(2)} = \begin{pmatrix} 2 & 4 & 8 & 6 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 4 \end{pmatrix} = A^{(3)}$$

As the diagonal entry is bigger, $P_3 = I$. And $P_3A^{(3)} = A^{(3)}$. Let

$$L_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \Rightarrow L_3A^{(3)} = \begin{pmatrix} 2 & 4 & 8 & 6 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix} = U.$$

Hence,

$$P = P_3P_2P_1 = I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, L = L_1^{-1}L_2^{-1}L_3^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{2} & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}, U = \begin{pmatrix} 2 & 4 & 8 & 6 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

21.6 Suppose $A \in \mathbb{C}^{m\times m}$ is *strictly column diagonally dominant*, which means that for each k,

$$|a_{kk}| > \sum_{j\neq k} |a_{jk}|.$$

Show that if Gaussian elimination with partial pivoting is applied to A, no row interchanges take place.

Proof:

We want to show that a strictly column diagonally dominant matrix $A \in \mathbb{C}^{m\times m}$ after the first step of Gaussian elimination with partial pivoting, $A^{(1)}_{2:m,2:m}$ is still a strictly column diagonally dominant matrix.

Because $|a_{11}| > \sum_{j\neq 1} |a_{j1}|$, there is no need to interchange row for the first step. Then we do the Gaussian elimination. We denote the $A^{(1)}$ is the matrix A after the first step Gaussian elimination. Then we obtain for $j > 1$, $a^{(1)}_{ij} = a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j}$. Hence, $\forall k > 1$,

$$\begin{aligned} |a^{(1)}_{kk}| &= |a_{kk} - \frac{a_{k1}}{a_{11}} a_{1k}| \\ &\geq |a_{kk}| - |\frac{a_{k1}}{a_{11}}||a_{1k}| \\ &> \sum_{i\neq k} |a_{ik}| - |\frac{a_{k1}}{a_{11}}||a_{1k}| \\ &= \sum_{i>1, i\neq k} a^{(1)}_{ik}. \end{aligned}$$

Then $A^{(1)}_{2:m,2:m}$ is also a strictly column diagonally dominant matrix.

We repeat the Gaussian elimination with pivoting to $A^{(1)}_{2:m,2:m}$, then we can obtain $A^{(2)}_{3:m,3:m}$ is also a strictly column diagonally dominant matrix. We repeat the process again and again. We can obtain that, during $m-1$ steps Gaussian elimination with pivoting, there is no need to do row interchanges.

22.1 Show that for Gaussian elimination with partial pivoting applied to any matrix $A \in \mathbb{C}^{m\times m}$, the growth factor $\rho = \frac{\max\limits_{i,j}|u_{ij}|}{\max\limits_{i,j}|a_{ij}|} \leq 2^{m-1}$.

Proof:

For Gaussian elimination with partial pivoting, the permutation matrix P will not change $\max\limits_{i,j}|a_{ij}|$. Hence, we can denote $PA = (a_{ij}^{(0)})_{m\times m}$, where $\max\limits_{i,j}|a_{ij}^{(0)}| = \max\limits_{i,j}|a_{ij}|$. Apply 1 step of Gaussian elimination to PA :

$$\begin{pmatrix} a_{11}^{(0)} & a_{12}^{(0)} & \dots & a_{1m}^{(0)} \\ a_{21}^{(0)} & a_{22}^{(0)} & \dots & a_{2m}^{(0)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}^{(0)} & a_{m2}^{(0)} & \dots & a_{mm}^{(0)} \end{pmatrix} \to \begin{pmatrix} a_{11}^{(0)} & a_{12}^{(0)} & \dots & a_{1m}^{(0)} \\ 0 & a_{22}^{(1)} & \dots & a_{2m}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2}^{(1)} & \dots & a_{mm}^{(1)} \end{pmatrix},$$

where the entries $a_{ij}^{(1)} = a_{ij}^{(0)} - \frac{a_{i1}^{(0)}}{a_{11}^{(0)}} a_{1j}^{(0)}$. And due to the Gaussian elimination with partial pivoting, $|\frac{a_{i1}^{(0)}}{a_{11}^{(0)}}| \leq 1$. Hence, we can obtain,

$$|a_{ij}^{(1)}| \leq |a_{ij}^{(0)}| + |a_{1j}^{(0)}| \leq 2\max_{i,j}|a_{ij}^{(0)}| = 2\max_{i,j}|a_{ij}|.$$

Repeat the above process, we can obtain after k steps of Gaussian elimination,

$$|a_{ij}^{(k)}| \leq |a_{ij}^{(k-1)}| + |a_{kj}^{(k-1)}| \leq 2\max_{i,j}|a_{ij}^{(k-1)}|.$$

We need to do $m-1$ steps to form U. Hence,

$$\begin{aligned} |u_{ij}| = |a_{ij}^{(m-1)}| &\leq |a_{ij}^{(m-2)}| + |a_{kj}^{(m-2)}| \leq 2\max_{i,j}|a_{ij}^{(m-2)}| \\ &\leq 2^2 \max_{i,j}|a_{ij}^{(m-3)}| \leq \dots \\ &\leq 2^{m-1}\max|a_{ij}^{(0)}| = 2^{m-1}\max|a_{ij}|. \end{aligned}$$

Hence, the growth factor

$$\rho = \frac{\max\limits_{i,j}|u_{ij}|}{\max\limits_{i,j}|a_{ij}|} \leq 2^{m-1}.$$

22.2 Experiment with solving 60×60 systems of equations $Ax = b$ by Gaussian elimination with partial pivoting, with A having the form (22.4). Do you observe that the results are useless because of the growth factor of order 2^{60}? At your first attempt you may not observe this, because the integer entries of A may prevent any rounding errors from occurring. If so, find a way to modify your problem slightly so that the growth factor is the same or nearly so and catastrophic rounding errors really do take place.

Solution:

We consider the true solution x equals to the right singular vector of A corresponding to σ_m. In the following figure, x is the true solution of $Ax = b$, x_1 is the solution of using LU factorization of A and x_2 is the solution of using LU factorization of A after diagonal entry perturbation. The rounding error between x_2 and x is about 40. And we can find the growth factor ρ_1 and ρ_2 is nearly same.

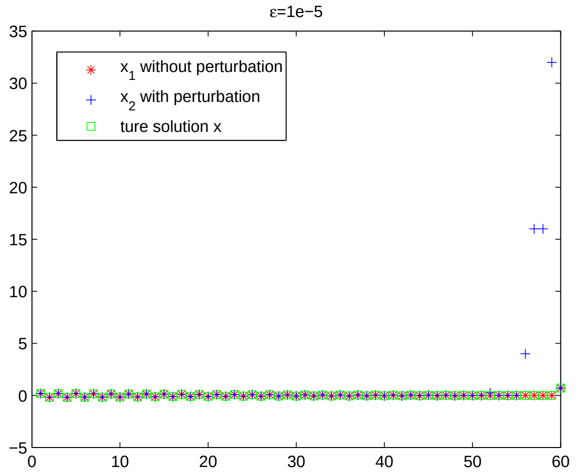

FIGURE 1. different solution of x