3(a) Fast Poisson Solver with Neumann B.C.

$$\nabla^2 u = f(x,y) \qquad (x,y) \in \Omega = (a,b) \times (a,b)$$

$$n \cdot \nabla u(x,y) = 0 \qquad (x,y) \in \partial\Omega$$

For interior points.

$$\nabla^2 u = f(x,y)$$

$$\nabla^2 u = u_{xx} + u_{yy} = f_{ij}$$

$$u_{xx} = \frac{u_{i-1,j} + u_{i+1,j} - 2u_{ij}}{h^2}$$

$$u_{yy} = \frac{u_{i,j-1} + u_{i,j+1} - 2u_{ij}}{h^2}$$

$$\nabla^2 u = \frac{u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{ij}}{h^2} = f_{ij}$$

$$u_{ij} = \frac{1}{4}\left(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - h^2 f_{ij}\right)$$

For Boundary points.

$$n \cdot \nabla u(x,y) = 0 \Rightarrow \frac{\partial u}{\partial x} = 0, \ \frac{\partial u}{\partial y} = 0$$

Using the Centered Differnce formula and the fitious point method, we have

$$\frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i-1,j}}{2h} = 0 \Rightarrow u_{i+1,j} = u_{i-1,j}$$

at $i=0 \Rightarrow u_{1,j} = u_{-1,j}$

at $i=m+2 \Rightarrow u_{m+2,j} = u_{m+1,j}$

Boundary points along y

$$\frac{\partial u}{\partial y} = \frac{u_{i,j+1} - u_{i,j-1}}{2h} = 0 \Rightarrow u_{i,j+1} = u_{i,j-1}$$

for $j = 0 \Rightarrow u_{i,1} = u_{i,-1}$

for $j = m+2 \Rightarrow u_{i,m+3} = u_{i,m+1}$

therefore the second-order accurate FD method formed is

$$u_{i,j} = \frac{1}{4}\left(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - h^2 f_{i,j}\right)$$

with

$$u_{i,m+3} = u_{i,m+1}$$
$$u_{i,1} = u_{i,-1}$$
$$u_{1,j} = u_{-1,j}$$
$$u_{m+2,j} = u_{m+1,j}$$

NO. 4 Implicit FD method.

a) Using the technique from problem 4 of homework 2, derive the following implicit fourth-order accurate approximation to the 2-D Poisson equation $u_{xx}+u_{yy}=f$:

$$\frac{1}{6h^2}\begin{bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{bmatrix} u = \frac{1}{12}\begin{bmatrix} & 1 & \\ 1 & 8 & 1 \\ & 1 & \end{bmatrix} f + O(h^4)$$

Solution:

from Homework 2, for 1D we have:

$$\frac{1}{h^2}\left[-\frac{1}{12} \quad \frac{4}{3} \quad -\frac{5}{2} \quad \frac{4}{3} \quad -\frac{1}{12}\right] u = f + O(h^4)$$

Now in 2D: $\nabla^2 u(x,y) = f(x,y)$ —— (1), it will become

$$\frac{1}{h^2}\Big[-\frac{1}{12}u_{i-2,j} + \frac{4}{3}u_{i-1,j} - \frac{5}{2}u_{i,j} + \frac{4}{3}u_{i+1,j} - \frac{1}{12}u_{i+2,j} - \frac{1}{12}u_{i,j-2}$$
$$+ \frac{4}{3}u_{i,j-1} - \frac{5}{2}u_{i,j} + \frac{4}{3}u_{i,j+1} - \frac{1}{12}u_{i,j+2}\Big]$$
$$= f_{i,j} + O(h^4) \quad —— (2)$$

Using a technique from Homework (2), differentiate Equation (1) twice with respect to x and y.

$$\nabla^2(\nabla^2 u(x,y)) = \nabla^2 f(x,y)$$

$$\nabla^2 f(x,y) = \frac{1}{h^2}\left[f_{i-1,j} + f_{i,j+1} + f_{i+1,j} + f_{i,j-1} - 4f_{i,j}\right] + O(h^2)$$

$$\nabla^2 u(x,y) = u_{xx} + u_{yy}$$

$$\nabla^2 u(x,y) = \frac{1}{h^2}\left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}\right] + O(h^4)$$

So

$$\nabla^2\left(\nabla^2 u(x,y)\right) = \frac{1}{h^2}\left[\nabla^2 u_{i+1,j} + \nabla^2 u_{i-1,j} + \nabla^2 u_{i,j+1} + \nabla^2 u_{i,j-1} - 4\nabla^2 u_{i,j}\right] + O(h^4)$$

$$\nabla^2 u_{i-1,j} = \frac{1}{h^2}\left[u_{i-2,j} + u_{i,j} + u_{i-1,j-1} - 4u_{i-1,j} + u_{i-1,j+1}\right] + O(h^4) \quad \cdots \quad (3)$$

$$\nabla^2 u_{i,j-1} = \frac{1}{h^2}\left[u_{i,j} + u_{i,j-2} + u_{i+1,j-1} + u_{i-1,j-1} - 4u_{i,j-1}\right] + O(h^4) \quad \text{---} \quad (4)$$

$$\nabla^2 u_{i+1,j} = \frac{1}{h^2}\left[u_{i,j} + u_{i+2,j} + u_{i+1,j-1} + u_{i+1,j+1} - 4u_{i+1,j}\right] + O(h^4) \quad \text{---} \quad (5)$$

$$\nabla^2 u_{i,j+1} = \frac{1}{h^2}\left[u_{i,j} + u_{i,j+2} + u_{i+1,j+1} + u_{i-1,j+1} - 4u_{i,j+1}\right] + O(h^4) \quad \text{---} \quad (6)$$

$$\nabla^2 u_{i,j} = \frac{1}{h^2}\left[u_{i+1,j} + u_{i-1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}\right] + O(h^4)$$

$$-4\nabla^2 u_{i,j} = \frac{1}{h^2}\left[-4u_{i+1,j} - 4u_{i-1,j} - 4u_{i,j-1} - 4u_{i,j+1} + 16u_{i,j}\right] + O(h^4) \quad \text{---} \quad (7)$$

Adding Equations (3), (4), (5), (6), and (7) and then divide by h^2, we obtain:

$$\nabla^2(\nabla^2 U(x,y)) = \frac{1}{h^4}\Big[20U_{i,j} + U_{i-2,j} + 2U_{i-1,j-1} - 8U_{i-1,j} +$$

$$2U_{i-1,j+1} + U_{i,j-2} + 2U_{i+1,j-1} - 8U_{i,j-1} + U_{i+2,j} +$$

$$2U_{i+1,j+1} - 8U_{i+1,j} + U_{i,j+2} - 8U_{i,j+1}\Big] + O(h^4)$$

therefore:

$$\nabla^2(\nabla^2 U(x,y)) = \frac{1}{h^2}\Big[f_{i-1,j} + f_{i,j-1} + f_{i+1,j} + f_{i,j+1} - 4f_{i,j}\Big] \quad \text{--- (8)}$$

Using a technique from Homework (2),

$$(2) + \frac{h^2}{12}(8)$$

$$\frac{1}{h^2}\begin{bmatrix} & & -\frac{1}{12} & & \\ & & \frac{4}{3} & & \\ -\frac{1}{12} & \frac{4}{3} & -5 & \frac{4}{3} & -\frac{1}{12} \\ & & \frac{4}{3} & & \\ & & -\frac{1}{12} & & \end{bmatrix} U + \frac{1}{h^2}\begin{bmatrix} & & \frac{1}{12} & & \\ & \frac{1}{6} & -\frac{2}{3} & \frac{1}{6} & \\ \frac{1}{12} & -\frac{2}{3} & \frac{5}{3} & -\frac{2}{3} & \frac{1}{12} \\ & \frac{1}{6} & -\frac{2}{3} & \frac{1}{6} & \\ & & \frac{1}{12} & & \end{bmatrix} U =$$

$$f_{i,j} + \begin{bmatrix} & \frac{1}{12} & \\ \frac{1}{12} & \frac{2}{3} & \frac{1}{12} \\ & \frac{1}{12} & \end{bmatrix} f + O(h^4)$$

which reduces to:

$$\frac{1}{6h^2}\begin{bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{bmatrix} u = \frac{1}{12}\begin{bmatrix} & 1 & \\ 1 & 8 & 1 \\ & 1 & \end{bmatrix} f + O(h^4)$$