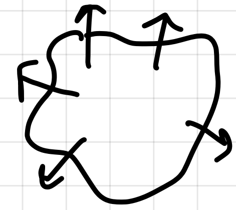
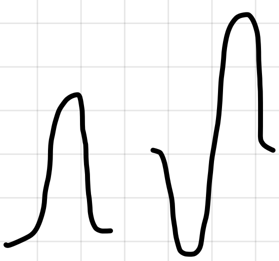
Green's Second Identity



$$\int_C \underbrace{(u\nabla^2 v - v\nabla^2 u)}_{\text{volume}}\, dV = \int_{\partial C} \underbrace{\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)}_{\text{surface}} dA$$

Suppose u is a harmonic function, and v is a free space Green's function



$$u \Rightarrow \nabla^2 u = 0 \qquad x, y \in \mathbb{R}^d$$

$$v \Rightarrow \nabla^2 G = \delta(x-y)$$

$$u(x) = \int_{\partial C} \underbrace{u(y)\frac{\partial G}{\partial n}\, dy}_{\text{dipoles}} - \int_{\partial C} \underbrace{\frac{\partial u}{\partial n}(y)\, G(x,y)\, dy}_{\text{sources.}}$$

This means u can be seen as a result of distribution of sources of strength $\frac{\partial u}{\partial n}$ and dipoles of strength $u(y)$

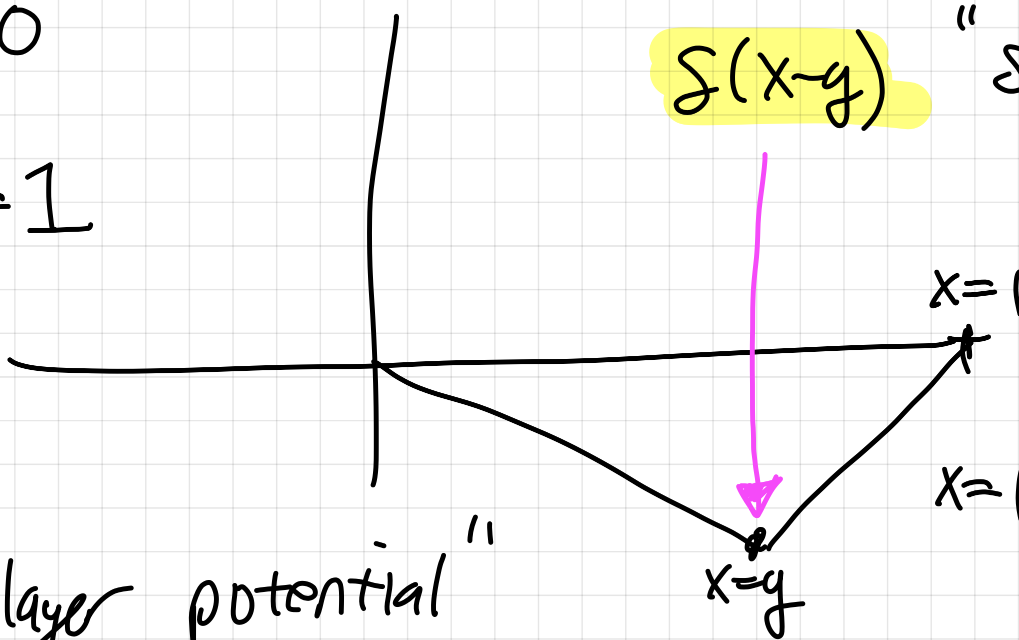
In 1d

$$u''(x) = \delta(x-y)$$

$[G] = 0$

$\left[\frac{\partial G}{\partial n}\right] = 1$

$\delta(x-y)$ "source"



"Single layer potential"

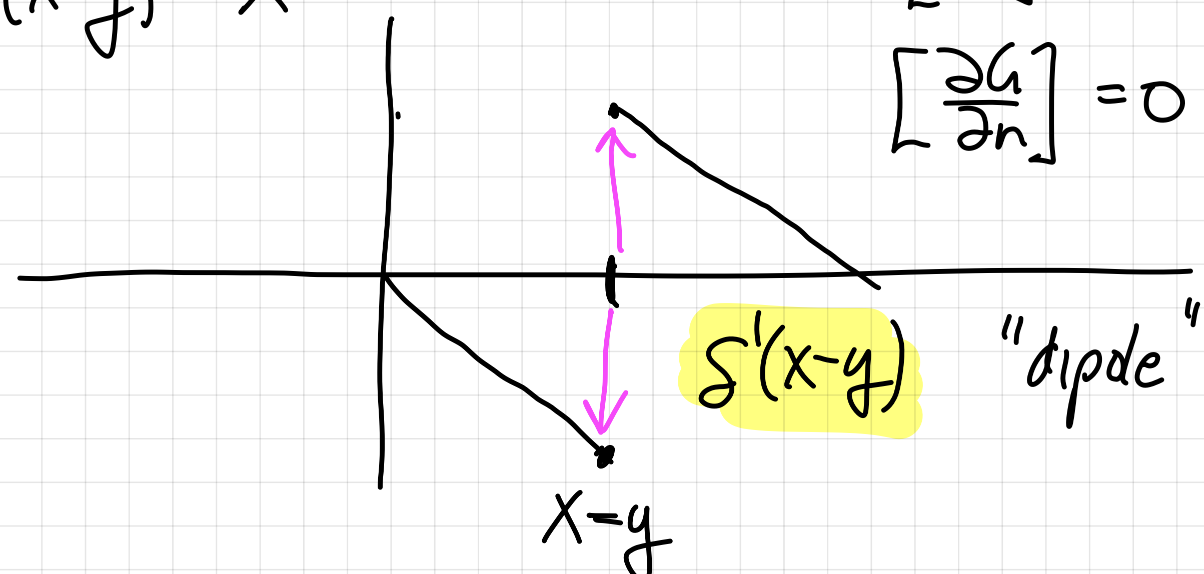
$$u''(x) = \delta'(x-y)$$

$[G] = 1$

$\left[\frac{\partial G}{\partial n}\right] = 0$

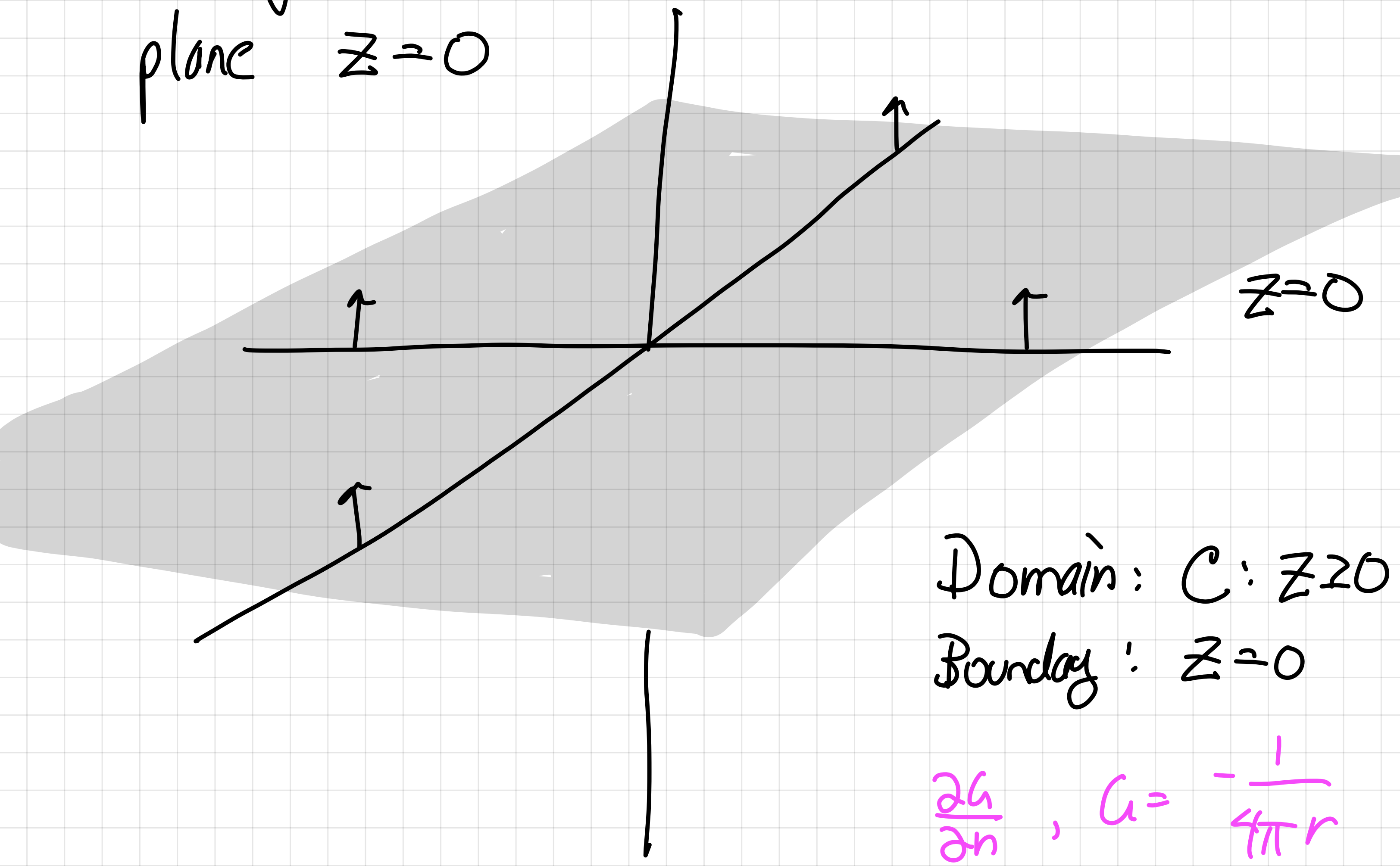
$$u(x) = H(x-y) - x$$

$\delta'(x-y)$ "dipole"



"double layer potential"

"**Double layer potential**" of strength ρ_0" distributed on the plane $z=0$



Domain: $C: z \geq 0$

Boundary: $z=0$

$\frac{\partial G}{\partial n}$, $G = -\frac{1}{4\pi r}$

Potential $w(x,y,z)$:

$$w(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho_0 \frac{\partial}{\partial n}\left(-\frac{1}{4\pi r}\right) d\xi\, d\eta$$

distribution of dipoles on the x-y plane.

$\vec{n} = (0,0,1)$

$\frac{\partial}{\partial n} \equiv \underset{\sim}{\nabla} \cdot \vec{n}$

Double layer Potential.

$$w(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_0 \frac{\partial}{\partial n}\left(-\frac{1}{4\pi r}\right) d\xi\, d\eta$$

$$r = \left((x-\xi)^2 + (y-\eta)^2 + z^2\right)^{1/2}, \quad \xi, \eta \in x\text{-}y \text{ plane}$$

What is the value of w at $z=0$?

We will evaluate the potential at z close to zero.

$$w(x,y,z) = -\frac{P_0}{4\pi} \int_{-\infty}^{\infty}\int \frac{\partial}{\partial n}\left(\frac{1}{r}\right) d\xi\, d\eta$$

$$\frac{\partial}{\partial n}\left(\frac{1}{r}\right) = -\frac{1}{r^2} \nabla r \cdot \vec{n}$$

$$= -\frac{P_0 z}{4\pi} \int_{-\infty}^{\infty}\int \frac{d\xi\, d\eta}{\left((x-\xi)^2 + (y-\eta)^2 + z^2\right)^{3/2}}$$

$$w(x,y,z) = -\frac{p_0 z}{4\pi} \int_{-\infty}^{\infty}\int \frac{d\xi\, d\eta}{\left((x-\xi)^2+(y-\eta)^2+z^2\right)^{3/2}}$$

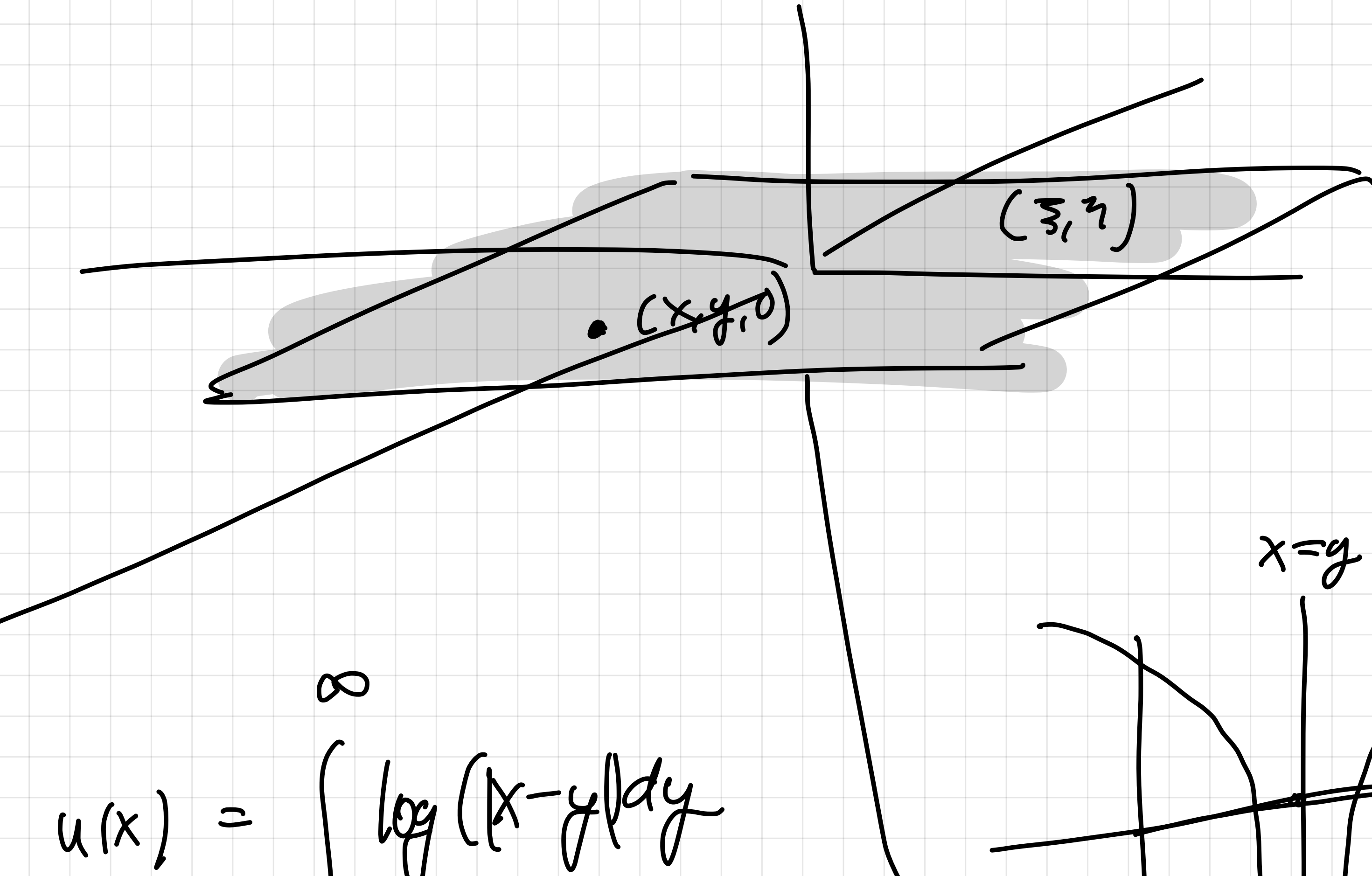
(x,y,z) : in upper half space $z \geq 0$

$(\xi,\eta,0)$: on x-y plane (boundary)

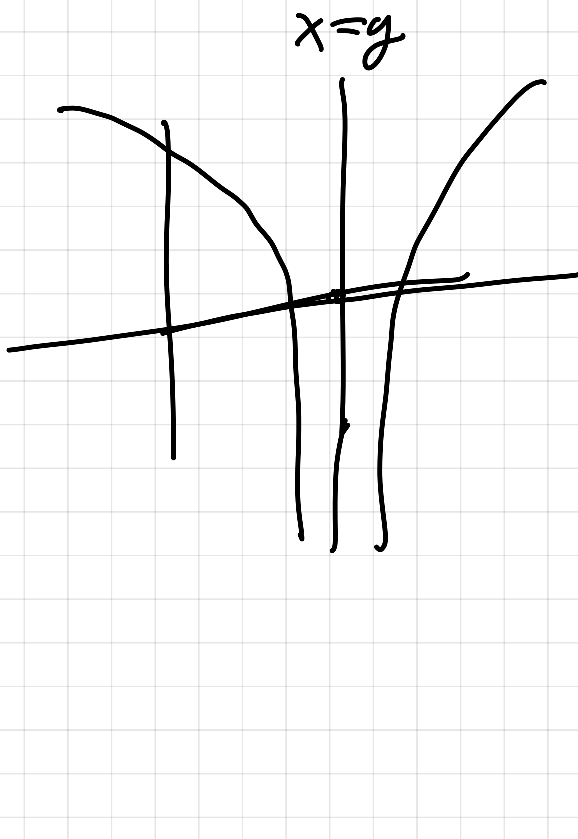
$w(x,y,z)$ is a "double layer potential"

How do we evaluate $w(x,y,z)$ on the boundary $z=0$?

Idea: Evaluate close to $z=0$ and take a limits



$$u(x) = \int_0^\infty \log(|x-y|)\,dy$$



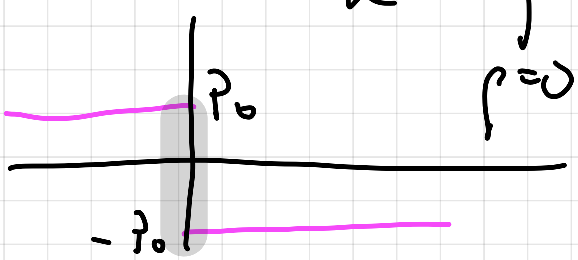
Change of variables:

$$w(x,y,z) = -\frac{P_0 z}{4\pi}\iint_{-\infty}^{\infty}\frac{d\xi\, d\eta}{\underbrace{\xi^2+\eta^2}_{\rho^2}+z^2} \qquad z\neq 0$$

Introduce polar coordinates: (ρ, ϕ)

$$w(x,y,z) = \frac{-P_0 z}{4\pi}\int_{\rho=0}^{\infty}\int_{\phi=0}^{2\pi}\frac{\rho\, d\phi\, d\rho}{(\rho^2+z^2)^{3/2}} \qquad z\neq 0$$

$$\underset{z=0}{=} -\frac{P_0 z}{2}\int_{\rho=0}^{\infty}\frac{\rho\, d\rho}{(\rho^2+z^2)^{3/2}} = \frac{P_0 z}{2}\cdot\frac{1}{(\rho^2+z^2)^{1/2}}\Bigg|_{\rho=0}^{\infty}$$

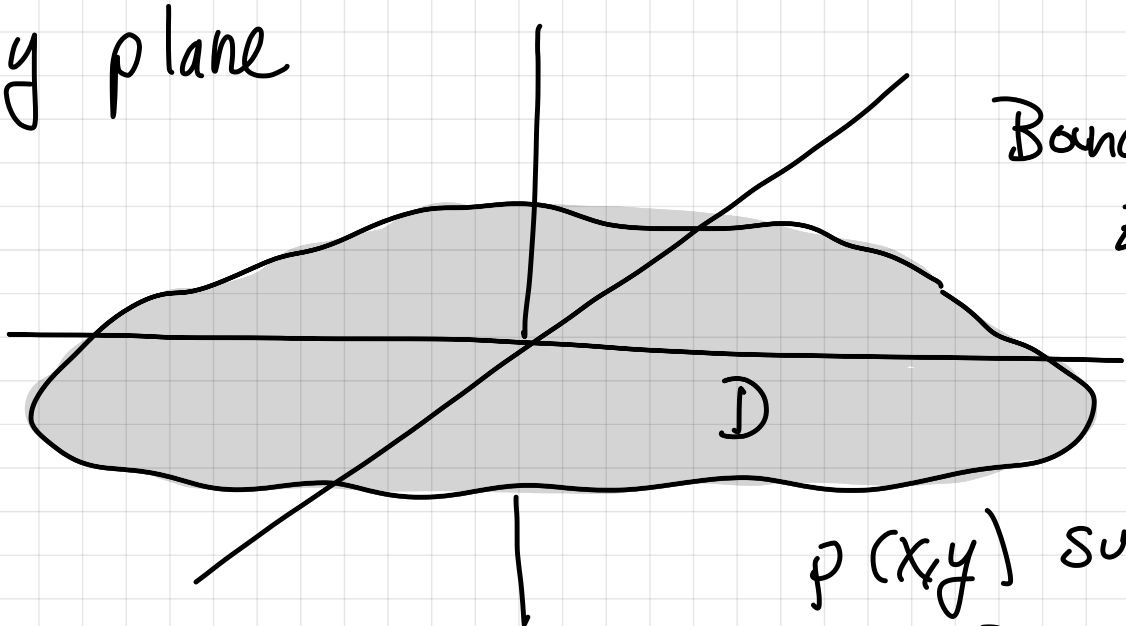


$$= -\frac{P_0 z}{2}\cdot\frac{1}{|z|} = \begin{cases} -P_0/2, & z>0 \\ P_0/2, & z<0 \end{cases}$$

w is constant in each half space.

Note: $[w]_{z=0^-}^{z=0^+} = -P_0$. The dipole induces a jump P_0 $\blacksquare$

More generally: Suppose we had a dipole distribution of strength $p(x,y)$ defined in some domain D in the x-y plane



Boundary: $z=0$

$p(x,y)$ supported on D.

$$w(x,y,z) = -\frac{z}{4\pi}\iint_D \frac{p(\xi,\eta)\,d\xi\,d\eta}{\left[(x-\xi)^2+(y-\eta)^2+z^2\right]^{3/2}}$$

$$\left.\begin{aligned} w(x,y,0^+) &= -\frac{p(x,y)}{2} \\ w(x,y,0^-) &= \frac{p(x,y)}{2} \end{aligned}\right\} \text{"Double layer potential"}$$

Dirichlet Problem

Suppose we want to solve a Dirichlet problem on a region Ω subject to boundary conditions:

$$\nabla^2 u = 0 \quad \text{on } \Omega$$

$$u = g \quad \text{on } \partial\Omega$$

If we had a Green's function $G(x,y)$ that satisfied

$$\nabla^2 G = \delta(x-y) \quad \text{on } \Omega$$

$$G(x,y) = 0 \quad \text{on } \partial\Omega,$$

we would have

$$u(x) = \int_{\partial\Omega} g(y) \frac{\partial G}{\partial n}(x,y)\, dy$$

But we don't have G. for general domains

Use the "double layer potential" idea to write

unknown

$$u(P) = \iint_{\partial\Omega} \mu(y) \frac{\partial}{\partial n} G(x,y)\, dy$$

$P = (x, y, z)$

free space

where we use the free space Green's function instead. The function $\mu(x)$ is called a "density" and is unknown.

On the boundary, we want to satisfy

$$g(x) = \iint_{\partial\Omega} \mu(y) \frac{\partial}{\partial n} G(x,y)\, dy$$

$$\frac{\partial}{\partial n}\left(-\frac{1}{4\pi r}\right)$$

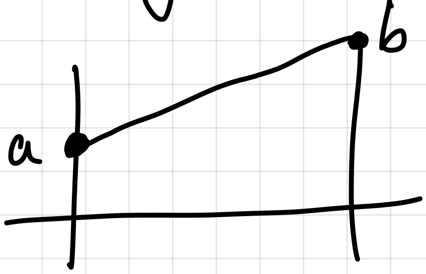
$x \in \partial\Omega$

$$g(x) = \iint_{\partial\Omega} \mu(y) \frac{\partial}{\partial n} G(x,y)\, dy$$

solve for $\mu(x)$

We can use this integral equation to "solve" for $\mu(x)$. To evaluate the integral, however, we need to be careful around the singularity:

$$g(x) = \frac{\mu(x)}{2} + \iint_{\underbrace{\partial\Omega \setminus x}_{\text{avoid singularity.}}} \mu(y) \frac{\partial G}{\partial n}(y)\, dy$$



Discretizing the quadrature leads to a well-conditioned linear system

Fast multiple method

$$\boxed{\left(\frac{1}{2} I \pm A\right)\mu = g}$$