1

Brian KYANJO

Homework 2

1. Question 5-3

Consider the matrix

$$A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}$$

a) Determine, on a paper, a real SVD of A in the form $A = U \Sigma V^*$.

The non zero singular values of A = $\sqrt{\text{eigen values of } A^*A}$

$$A^*A = \begin{bmatrix} -2 & -10 \\ 11 & 5 \end{bmatrix} \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix} = \begin{bmatrix} 104 & -72 \\ -72 & 146 \end{bmatrix}$$

let $A^*A = B$

Eigen values of B

$$|B - \lambda I| = 0 \quad \Rightarrow \lambda_1 = 200, \ \lambda_2 = 50$$

the singular values of A = $\sqrt{\text{eigen value of } B}$.

$$\sigma_1 = \sqrt{200} = 10\sqrt{2}$$

$$\sigma_2 = \sqrt{50} = 5\sqrt{2}$$

U and V will be the eigen vectors of AA^* and A^*A respectively.

$Bv = \lambda v$, let $v = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{bmatrix} 104 & -72 \\ -72 & 146 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

$104x - 72y = \lambda x$

for $\lambda = 200 \Rightarrow x = -3, y = 4$.

$$V_1 = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

for $\lambda = 50 \Rightarrow x = 4, y = 3$

$$V_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

therefore the Right Singular vectors.

$$V_1 = \frac{1}{\sqrt{3^2+4^2}} \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$V_2 = \frac{1}{\sqrt{4^2+3^2}} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$\Rightarrow V = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$$

let $C = AA^T = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix} \begin{bmatrix} -2 & -10 \\ 11 & 5 \end{bmatrix} = \begin{bmatrix} 125 & 75 \\ 75 & 125 \end{bmatrix}$

Eigen values of C

$|C - \lambda I| = 0 \Rightarrow \lambda_1 = 200, \lambda_2 = 50$.

Eigen vector of C

$Cu = \lambda u$, let $u = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{bmatrix} 125 & 75 \\ 75 & 125 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

$125x + 75y = \lambda x$

for $\lambda_1 = 200$, $x = 1$, $y = 1$ $\Rightarrow u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

for $\lambda_2 = 50$, $x = -1$, $y = 1$ $\Rightarrow u_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

left singular vectors

$u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$U = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

therefor the SVD is

$$A = U \Sigma V^T = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix} \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$$

b) List the singular values,

$$\sigma_1 = 10\sqrt{2}, \quad \sigma_2 = 5\sqrt{2}$$

Left singular vectors

$$u_1 = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

Right singular vectors

$$v_1 = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$$

c) What are the 1-, 2-, ∞-, and frobenius norm of A?

$$\|A\|_1 = \max\{12, 16\} = 16$$

$$\|A\|_2 = \sqrt{\rho(A^*A)} = 10\sqrt{2}$$

$$\|A\|_\infty = \max\{13, 15\} = 15$$

$$\|A\|_F = \sqrt{2^2 + 11^2 + 10^2 + 5^2} = \sqrt{250} = \underline{\underline{5\sqrt{10}}}$$

d) Find A^{-1} not directly, but via the SVD.

Consider

$$A = (U\Sigma V^*)$$

$$A^{-1} = (U\Sigma V^*)^{-1} = (V^*)^{-1}\Sigma^{-1}U^{-1}$$

but V and U are Unitary, $V^* = V^{-1}$, $U^* = U^{-1}$

$$A^{-1} = V\Sigma^{-1}U^* = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix} \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix}^{-1} \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0.05 & -0.11 \\ 0.1 & -0.02 \end{bmatrix}$$

e) Find the eigen values λ_1, λ_2 of A.

$$\lambda^2 - \text{trace}(A)\lambda + \det(A) = 0$$

$$\det(A) = 100, \quad tr(A) = 3$$

$$\lambda^2 + 3\lambda + 100 = 0$$

$$\lambda_1 = \frac{3 + \sqrt{391}\,i}{2}, \quad \lambda_2 = \frac{3 - \sqrt{391}\,i}{2}$$

f) Verify that $\det A = \lambda_1\lambda_2$ and $|\det(A)| = \sigma_1\sigma_2$

$$\lambda_1 \cdot \lambda_2 = \left(\frac{3 + \sqrt{391}\,i}{2}\right)\left(\frac{3 - \sqrt{391}\,i}{2}\right)$$

$$\lambda_1 \cdot \lambda_2 = \frac{1}{4}\left(3^2 - (\sqrt{391}\,i)^2\right) = 100.$$

have $\det(A) = \lambda_1 \lambda_2$

$|\det A| = \sigma_1 \sigma_2$

Since U is Unitary.

$$\det(UU^*) = \det(U) . \det(U^*) = 1$$

but $\det(U) = \det(U^*) = \pm 1$

$\det(A) = \det(U \Sigma V^*) \Rightarrow (\det(A))^2 = \det(A^* A)$

~~$(\det(A))^2 = (\det(U\Sigma V^*))^2 = \det(U\Sigma V^* U \Sigma V^*)$~~

$$A^* = V \Sigma^* U^*$$

$(\det(A))^2 = \det(V \Sigma^* U^* U \Sigma V^*)$

$(\det(A))^2 = \det(V \Sigma^* \Sigma V^*)$

$(\det(A))^2 = \det(V) . \det(\Sigma^* \Sigma) . \det(V^*)$

$$\text{but } \det(V) = \det(V^*) = -1$$

$(\det(A))^2 = \det(\Sigma^* \Sigma)$

$|\det(A)| = \sqrt{\det(\Sigma^* \Sigma)}$

$|\det(A)| = \sqrt{\sigma_1^2 \sigma_2^2}$

$|\det(A)| = \sigma_1 \sigma_2$

(g) What is the area of the ellipsoid onto which A maps the unit ball of $\mathbb{R}^2$?

$$A = \pi r_1 r_2$$

$$A = \pi(\sigma_1 \sigma_2) = \pi(10\sqrt{2})(5\sqrt{2})$$

$$\underline{A = 100\pi}$$

2. Question 5.4

Consider $A = U\Sigma V^* \Rightarrow A^* = V\Sigma^* U^*$

Let $B = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}$

$$B = \begin{bmatrix} 0 & V\Sigma^* U^* \\ U\Sigma V^* & 0 \end{bmatrix}$$

B can be flipped, Let I be an m×m Identity Matrix

$$B = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} U\Sigma V^* & 0 \\ 0 & V\Sigma^* U^* \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma^* \end{bmatrix} \begin{bmatrix} V^* & 0 \\ 0 & U^* \end{bmatrix} \quad \text{---} \; (1)$$

Since Σ is diagonal matrix then $\Sigma^* = \Sigma$

from ①, let $\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}$ be C, $C^{-1} = \begin{bmatrix} V^* & 0 \\ 0 & U^* \end{bmatrix}$,

Since

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \text{ then}$$

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} = \begin{bmatrix} V^* & 0 \\ 0 & U^* \end{bmatrix}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

therefore,

$$B = \left(\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}\right)\left(\begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma \end{bmatrix}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\right)\left(\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}\right)^{-1}$$

$$B = \left(\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}\right)\left(\begin{bmatrix} 0 & \Sigma \\ \Sigma & 0 \end{bmatrix}\right)\left(\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}\right)^{-1} \quad \ldots ②$$

Since Σ has non negative ~~entri~~ elements then

$$\begin{bmatrix} 0 & \Sigma \\ \Sigma & 0 \end{bmatrix}\begin{bmatrix} I & I \\ I & -I \end{bmatrix} = \begin{bmatrix} I & I \\ I & -I \end{bmatrix}\begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix} \quad — ③$$

but $\begin{bmatrix} I & I \\ I & -I \end{bmatrix}\begin{bmatrix} I & I \\ I & -I \end{bmatrix} = 2I_2$, where I_2 is a $2n \times 2n$ identity matrix.

So equation ③ becomes.

$$\begin{bmatrix} 0 & \Sigma \\ \Sigma & 0 \end{bmatrix} = \begin{bmatrix} I & I \\ I & -I \end{bmatrix}\begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix}\begin{bmatrix} I & I \\ I & -I \end{bmatrix}^{-1}$$

$$\begin{bmatrix} 0 & \Sigma \\ \Sigma & 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ I & -I \end{bmatrix}\begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ I & -I \end{bmatrix}$$

Subsituting $\begin{bmatrix} 0 & \Sigma \\ \Sigma & 0 \end{bmatrix}$ back into Equation ②

$$B = \left(\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}\right)\left(\frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ I & -I \end{bmatrix}\begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ I & -I \end{bmatrix}\right)\left(\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}\right)^{-1} \quad \text{--- ④}$$

If we let

$$Q = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ I & -I \end{bmatrix}$$

Then Equation ④ becomes

$$B = Q\begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix}Q^{-1}$$

3. Question 6.2

Given $Ex = \frac{(x + fx)}{2}$

To know the nature of E we need to know the nature of F.

F flips $(x_1, x_2 ..., x_m)^*$ to $(x_m, ... x_2, x_1)^*$

$$F\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & & & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} x_m \\ x_{m-1} \\ \vdots \\ x_1 \end{bmatrix}$$

Since F just flips x, then it has the form above however it's Unitary and Symmetrically, therefore $F^2 = I$.

So there for we can take advantage of the square

$$Ex = \frac{(x + Fx)}{2} = \frac{(I + F)}{2} x$$

$$E = \frac{I + F}{2} \Rightarrow E^2 = \frac{1}{4}(I + F)^2$$

$$E^2 = \frac{1}{4}(I^2 + F^2 + 2IF),$$

Since $F^2 = I$

$$E^2 = \frac{1}{4}(I^2 + I^2 + 2IF) = \frac{1}{4}(2I^2 + 2IF)$$

$$E^2 = \frac{1}{4} 2(F + I) = \frac{(F+I)}{2}$$

$$E^2 = \frac{(F+I)}{2}$$

$$\underline{\underline{E^2 = E}}$$

Since both E and F are Unitary and Symmetrical

$E = E^*$, $F = F^*$, this means E is orthogonal projector.

What are the entries of E?

$$E = \frac{1}{2}(F + I)$$

$$E = \frac{1}{2}\left(\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & & \iddots & & \vdots \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & 0 \\ 0 & & & & 1 \end{bmatrix} \right)$$

$$E = \frac{1}{2}\begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 1 & 0 \\ & & \ddots \; 2 \; \ddots & & \\ 0 & 1 & \cdots & 1 & 0 \\ 1 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

5. (Householder reflections)

a) verify directly that if we let $v = \frac{y-x}{\|y-x\|}$ and $H = I - 2vv^T$ then $Hx = (I-2vv^T)x = y$

from $H = I - 2vv^T$

$$Hx = (I - 2vv^T)x$$
$$Hx = x - 2vv^Tx$$

$$Hx = x - 2\left(\frac{y-x}{\|y-x\|}\right)\left(\frac{y-x}{\|y-x\|}\right)^T x$$

$$Hx = x - \frac{2(y-x)(y-x)^T}{\|y-x\|^2} x$$

Introducing a useful zero $-y+y$

$$Hx = x - y + y - \frac{2(y-x)(y-x)^T}{\|y-x\|^2} x + y$$

$$Hx = \frac{-(y-x)}{\|y-x\|^2}\left[\|y-x\|^2 + 2(y-x)^T x\right] + y$$

Since $x^Tx = \|x\|^2 \Rightarrow \|y-x\|^2 = (y-x)^T(y-x)$

$$Hx = \frac{-(y-x)}{\|y-x\|^2}\left[(y-x)^T(y-x) + 2(y-x)^T x\right] + y$$

$$Hx = \frac{-(y-x)}{\|y-x\|^2}\left[(y^T-x^T)(y-x) + 2(y^T-x^T)x\right] + y$$

$$Hx = \frac{-(y-x)}{\|y-x\|^2}\left[y^Ty - y^Tx - x^Ty + x^Tx + 2y^Tx - 2x^Tx\right] + y$$

Since $y^Ty = x^Tx$, $y^Tx = x^Ty$

$$Hx = \frac{-(y-x)}{\|y-x\|^2}\left[-2y^Tx + 2y^Tx\right] + y = y$$

$$\underline{\underline{Hx = y}}$$

b) H be a House holder matrix of size m. , $x^Tx = y^Ty$
H is both symmetrical and orthogonal, eigen values
$(H) = I1$

(i) $Tr(H)$
Consider $H = I - 2vv^T$

$$Tr(H) = Tr(I - 2vv^T) = Tr(I) - Tr(2vv^T)$$

$$= Tr(I) - 2Tr(vv^T)$$

but $v = \frac{y-x}{\|y-x\|}$, $\Rightarrow vv^T = \frac{(y-x)(y-x)^T}{\|(y-x)\|^2} = 1$

$$Tr(H) = Tr(I) - 2Tr(1)$$

$$Tr(H) = Tr(I) - 2$$

Since I is $m \times m$ then $Tr(I) = m$

$$\underline{\underline{Tr(H) = m-2}}$$

Eigen values of H,

Let the eigen values of H be λ.

$\det(H-\lambda I)=0$, $H=I-2vv^T$

$\det(I-2vv^T-\lambda I)=0$

$\det((1-\lambda)I-2vv^T)=0$

$(-1)^m \det((\lambda-1)I+2vv^T)=0$

$\det((\lambda-1)I+2vv^T)=0$

$\det((\lambda-1)I(I+2((\lambda-1)I)^{-1}vv^T)=0$

$(\lambda-1)^m \det(I+2((\lambda-1)I)^{-1}vv^T)=0$

Since $vv^T=1$

$(\lambda-1)^m \det[I+2((\lambda-1)I)^{-1}]=0$

$(\lambda-1)^m (1+2(\lambda-1)^{-1}) \det(I)=0$

$(\lambda-1)^m (1+2(\lambda-1)^{-1})=0$

Either

$(\lambda-1)^m=0$ or $\lambda=-1$

(ii) Show that $Hv=-v$ and that $Hu=u$ for any $u\in\mathbb{R}^m$ that is orthogonal to v

$H=I-2vv^T$

$$Hv = (I - 2vv^T)v$$

$$Hv = v - 2(vv^T)v \quad , \text{but } v^Tv = 1$$

$$Hv = v - 2v = -v$$

$$\underline{Hv = -v} \quad \text{——————— (1)}$$

$$Hu = (I - 2vv^T)u$$

$$Hu = (u - 2(vv^T)u)$$

$$Hu = u - 2(v^Tu)v \quad ; \text{but } v^Tu = 0$$

$$\underline{Hu = u} \quad \text{——————— (2)}$$

Using the definition $Hv = \lambda v$, where λ are the eigen values and v are the eigen vectors

$$Hv = -v \Rightarrow \lambda = -1$$
$$Hu = u, \Rightarrow \lambda = 1$$

therefore the Eigen values of H are $\lambda = \pm 1$, and this is true since H is both symmetric and orthogonal matrix.

(iii) Using the properties of eigen values, determine $\det(H)$

$$\det(H) = \text{Product of all eigen values of } H$$

$$\det(H) = \prod_{i=1}^{m} \lambda_i = 1^{m-1} \cdot (-1) = -1$$

$\det(H) = -1$