Brian KYANJO
Homework #7
Math 537.

1. Haar wavelets. The Haar Scaling function can be defined recursively as

$$\phi(x) = \phi(2x) + \phi(2x-1)$$

Initial conditions $\phi(0) = 1$
$\phi(1) = 0$ and $\phi(x) = 0,\ x < 1$ or $x < 1$

for $k_{max} = 2^{m-1} - 1$

for $m = 1,\ k = 0$

$$X = (2k+1)\,2^{-m} = 2^{-1}$$

$$\phi(2^{-1}) = \phi(2\cdot 2^{-1}) + \phi((2\cdot 2^{-1}) - 1) = 1$$

for $m = 2,\ k = 0, 1$

$k = 0$

$$X = (2(0)+1)\,2^{-2} = 1/4$$

$$\phi(1/4) = \phi(1/2) + \phi(-1/2) = \underline{\underline{1}}$$

$k = 1$

$$X = 3/4$$

$$\phi(3/4) = \phi(2\cdot 3/4) + \phi(2\cdot 3/4 - 1) = 1$$

for $m = 3,\ k = 0, 1, 2$

$k = 0$

$$X = 1/8$$

$$\phi(1/8) = \phi(1/4) + \phi(-3/4) = \underline{1}$$

$k=1 \quad , x = 3/8$

$$\phi(3/8) = \phi(2\cdot 3/8) + \phi(2\cdot 3/8 - 1) = 1$$

$k=2 \quad , \quad x = 5/8$

$$\phi(5/8) = \phi(2\cdot 5/8) + \phi(2\cdot 5/8 - 1) = \underline{1}$$

$k=3 \quad , \quad x = 7/8$

$$\phi(7/8) = \phi(7/4) + \phi(3/4) = \underline{\underline{1}}$$

Write a general algorithm for evaluating $\phi(x)$ at dyadic numbers $(2k+1)2^{-m}$, $k=0,1,2,\ldots,2^{m-1}-1$

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X = [0, 1]
Φ = [1 0]

for m=1 to m_max do
    k_max = 2^(m-1) - 1
    for k=0 to k_max do
        x_k = (2k+1) 2^(-m)
        X_q = x_k

        x'_k = 2 x_k
        x''_k = 2x_k - 1

        If x'_k < 0 or x'_k > 1 do

            φ'_k = 0
        else if 0 ≤ x'_k < 1 do
            i'_k = Index of x'_k in X
            φ'_k = Φ(i'_k)
        end if

        If x''_k < 0 or x''_k > 1 do
            φ''_k = 0
        else if 0 ≤ x''_k < 1 do
            i''_k = index of x''_k in X
            φ''_k = Φ(i''_k)
        end if

        φ_k = φ'_k + φ''_k
        φ_q = φ_k
    end for
    X = x_q
    Φ = φ_q
end for.
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Argue that $\phi(x)$ has a closed form solution

$$\phi(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Since from all the above calculation of $\phi(x)$, the value of $\phi(x) = 1$ for any value of $x \in [0, 1)$ and zero else where. Hence $\phi(x)$ has a closed from

2. ~~Haar wavelets~~. Repeat the above for the Haar wavelets defined recursively as.

$$\psi(x) = \psi(2x) - \psi(2x-1)$$

Using.
$\psi(0) = 1$, $\psi(\frac{1}{2}) = -1$, $\psi(x) = 0$ for $x < 0$ and $x \geq 1$

~~for~~ $m \neq 1$ $k_{max} = 2^{m-1} - 1$

for $m = 1$, $k = 0$

$$x = (2k+1)2^{-m} = (2(0)+1)2^{-1} = \frac{1}{2}$$

$$\psi(\tfrac{1}{2}) = \psi(2\cdot\tfrac{1}{2}) - \psi(2\cdot\tfrac{1}{2} - 1) = \underline{\underline{-1}}$$

for $m = 2$, $k = 0, 1$
$k = 0$, $x = \frac{1}{4}$.

$$\psi(\tfrac{1}{4}) = \psi(2\cdot\tfrac{1}{4}) - \psi(2\cdot\tfrac{1}{4} - 1) = \underline{\underline{1}}$$

for $k = 1$, $x = \frac{3}{4}$

$\psi(3/4) = \psi(2 \cdot 3/4) - \psi(2 \cdot 3/4 - 1) = 1$

~~k=~~

for $m=3$, $k=0,1,2$

$k=0$, $x = 1/8$

$\psi(1/8) = \psi(2 \cdot 1/8) - \psi(2 \cdot 1/8 - 1) = 1$

$k=1$, $x = 3/8$

$\psi(3/8) = \psi(2 \cdot 3/8) - \psi(2 \cdot 3/8 - 1) = 1$

$k=2$, $x = 5/8$

$\psi(5/8) = \psi(2 \cdot 5/8) - \psi(2 \cdot 5/8 - 1) = \underline{\underline{-1}}$

Write a general algorithm.

$X = [0 \ \ 1/2]$
$\psi = [1 \ -1]$
for $m = 1$ to M_{max} do
  $k_{max} = 2^{m-1} - 1$
  for $k = 0$ to k_{max} do
    $x_k = (2k+1)\, 2^{-m}$
    $x_a = x_k$
    $x'_k = 2x_k$
    $x''_k = 2x_k - 1$

    If $x'_k < 0$ and $x'_k \geq 1$ do
      $\psi'_k = 0$
    else if $0 \leq x'_k < 1$ do
      i'_k = index of x'_k in X
      $\psi'_k = \psi(i'_k)$
    end if

    If $x''_k < 0$ and $x''_k \geq 1$ do
      $\psi''_k = 0$
    else if $0 \leq x''_k < 1$ do
      i''_k = index of x''_k in X
      $\psi''_k = \psi(i''_k)$
    end if

    $\psi_k = \psi'_k - \psi''_k$
    $\psi_a = \psi_k$
  end for

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$X = x_a$
$\psi = \psi_k$

end for

Argue that $\varphi(x)$ has a closed form sol

According to the previous calculations $\varphi(x) = 1$ or $\varphi(x) = -1$ for all $x \in [0, 1]$ and zero else where. Hence has a closed form.

5. Using integration by parts, show that the solution to the elliptic problem:

$$u''(x) = f$$

on $[0,1]$, subject to $u(0) = a$, $u(1) = b$ is given by

$$u(x) = a(1-x) + b(x) + \int_0^1 G(x,y)\, f(y)\, dy$$

Consider $u''(x) = f$ —— (1)

multiply (1) by $G(x,y)$

$$G(x,y)\, u''(x) = G(x,y)\, f(y)$$

Integrate both sides w.r.t x and y respectively over $[0,1]$

$$\int_0^1 G(x,y)\, u''(x)\, dx = \int_0^1 G(x,y)\, f(y)\, dy$$

from the L.H.S, integrating by parts.

$$U = G(x,y) \Rightarrow \frac{du}{dx} = \frac{d}{dx} G(x,y)$$

$$dv = u''(x)\, dx \Rightarrow v = u'(x) + C$$

$$\int_0^1 u\, dv = uv\Big|_0^1 - \int_0^1 v\, du$$

$$\int_0^1 G(x,y)\, u''(x)\, dx = G(x,y)\, u'(x)\Big|_0^1 - \int_0^1 u'(x) \frac{d}{dx} G(x,y)\, dx$$

$$= G(1,y)\, u'(1) - G(0,y)\, u'(0) - \int_0^1 u'(x) \frac{d}{dx} G(x,y)\, dx$$

Since $G(x,y)$ is a fundamental solution subject to

$$G(0,y) = G(1,y) = 0 \Rightarrow \frac{d^2}{dx^2} G(x,y) = \delta(x,y)$$

then

$$\int_0^1 G(x,y)\, u''(x)\, dx = -\int_0^1 u'(x) \frac{d}{dx} G(x,y)\, dx$$

Applying by parts again.

$$U = \frac{d}{dx} G(x,y) \Rightarrow \frac{dU}{dx} = \frac{d^2}{dx^2} G(x,y)$$

$$dv = u'(x)\, dx \Rightarrow v = u(x) + c$$

$$\int_0^1 G(x,y)\, u''(x)\, dx = -\left[\left(u(x) \frac{d}{dx} G(x,y)\right)\Big|_0^1 - \int_0^1 u(x) \frac{d^2}{dx^2} G(x,y)\, dx \right.$$

$$\int_0^1 G(x,y)\, u''(x)\, dx = -\left[u(1)\, G'(1,y) - u(0)\, G'(0,y) - \int_0^1 u(x) \frac{d^2}{dx^2} G(x,y)\, dx \right]$$

Since $G(x,y) = \begin{cases} (y-1)x & 0 \le x \le y \\ y(x-1) & y \le x \le 1 \end{cases}$

$$\frac{d}{dx} G(x,y) = \begin{cases} (y-1) & 0 \le x \le y \\ y & y \le x \le 1 \end{cases}$$

so $G'(1,y) = y$

$G'(0,y) = y-1$

$u(1) = b, \quad u(0) = a$

$$\int_0^1 G(x,y)\, u''(x)\, dx = -\left[by - a(y-1) - \int_0^1 u(x) \frac{d^2}{dx^2} G(x,y) \right]$$

$$= a(y-1) - by + \int_0^1 u(x) \frac{d^2}{dx^2} G(x,y)\, dx$$

but $\frac{d^2}{dx^2} G(x,y) = \delta(x-y)$

$$\int_0^1 G(x,y)\, u''(x)\, dx = a(y-1) - by + \int_0^1 u(x)\, \delta(x-y)\, dx$$

Since $u(y) = \int_0^1 u(x)\, \delta(x-y)\, dx$

So

$$\int_0^1 G(x,y)\, u''(x)\, dx = a(y-1) - by + u(y)$$

taking $x=y$

$$\int_0^1 G(x,y)\, u''(x)\, dx = a(x-1) - bx + u(x)$$

hence

$$\int_0^1 G(x,y)\, u''(x)\, dx = \int_0^1 G(x,y)\, f(y)\, dy$$

$$a(x-1) - bx + u(x) = \int_0^1 G(x,y)\, f(y)\, dy.$$

$$u(x) = bx + a(1-x) + \int_0^1 G(x,y) \cdot f(y)\, dy$$

therefore:

$$u(x) = a(1-x) + bx + \int_0^1 G(x,y)\, f(y)\, dy.$$

No. 7

$$\theta''(t) + \frac{g}{L}\theta = \frac{1}{mL} f(t)$$

where $t \geq 0$ is time

$$\theta(0) = \theta'(0) = 0$$

a) Using a Laplace Transform, solve

$$\frac{\partial^2 G(t,\tau)}{\partial t^2} + \frac{g}{L} G(t,\tau) = \delta(t-\tau)$$

<u>Solution</u>

$$G''(t,\tau) + \frac{g}{L} G(t,\tau) = \delta(t-\tau)$$

$$L(G''(t,\tau)) + \frac{g}{L} L(G(t,\tau)) = L(\delta(t-\tau)) \quad -①$$

$L(G(t,\tau)) = Y$

$$L(G''(t,\tau)) = s^2 Y - s\,G(0,\tau) - G'(0,\tau)$$

Equation ① becomes

$$s^2 Y + \frac{g}{L} Y = L(\delta(t-\tau))$$

$$\left(s^2 + \frac{g}{L}\right) Y = L(\delta(t-\tau))$$

$$Y = \frac{1}{s^2 + g/L} L(\delta(t-\tau))$$

$$Y = \sqrt{L/g} \cdot \frac{\sqrt{g/L}}{s^2 + \left(\sqrt{g/L}\right)^2} \cdot L(\delta(t-\tau))$$

$$L(\delta(t-\tau)) = e^{-\tau s}$$

$$Y = \sqrt{\frac{l}{g}} \cdot \frac{\sqrt{g/l}}{s^2 + \left(\sqrt{g/l}\right)^2} e^{-\tau s}$$

The inverse of Y, $G(t,\tau)$ is

$$G(t,\tau) = \begin{cases} 0, & \text{if } 0 < t < \tau \\ \sqrt{\frac{l}{g}} \sin\left(\sqrt{\frac{g}{l}}(t-\tau)\right) & \text{if } t > \tau \end{cases}$$

$$G(t,\tau) = \sqrt{\frac{l}{g}} \sin\left(\sqrt{\frac{g}{l}}(t-\tau)\right) H(t-\tau)$$

b) For the Green's function $G(t,\tau)$ and operator L, given by

$$L[G(t,\tau)] = \frac{\partial^2}{\partial t^2} G(t,\tau) + \frac{g}{l} G(t,\tau)$$

(i) $L[G(t,\tau)] = 0, \quad t \neq \tau$

$$G(t,\tau) = \begin{cases} 0, & \text{if } 0 < t < \tau \\ \sqrt{\frac{l}{g}} \sin\left(\sqrt{\frac{g}{l}}(t-\tau)\right) & \text{if } t > \tau \end{cases}$$

$$\frac{\partial G(t,\tau)}{\partial t} = \sqrt{\frac{l}{g}} \sqrt{\frac{g}{l}} \cos\left(\sqrt{\frac{g}{l}}(t-\tau)\right)$$

$$= \cos\left(\sqrt{\frac{g}{l}}(t-\tau)\right), \quad \text{if } t > \tau$$

$$\frac{\partial^2}{\partial t^2} G(t,\tau) = \frac{\partial}{\partial t} \cos\left(\sqrt{g/l}\,(t-\tau)\right)$$

$$= -\sqrt{g/l}\,\sin\left(\sqrt{g/l}\,(t-\tau)\right)$$

So,

$$L[G(t,\tau)] = -\sqrt{g/l}\,\sin\left(\sqrt{g/l}\,(t-\tau)\right) + g/l \cdot \sqrt{l/g}\,\sin\left(\sqrt{g/l}\,(t-\tau)\right)$$

$$= -\sqrt{g/l}\,\sin\left(\sqrt{g/l}\,(t-\tau)\right) + \sqrt{g/l}\,\sin\left(\sqrt{g/l}\,(t-\tau)\right)$$

$$L[G(t,\tau)] = 0$$

hence

$$L[G(t,\tau)] = 0, \quad t > \tau.$$

(ii) for $0 < t < \tau$, $L[G(t,\tau)] = 0$

The $\lim_{t\to\tau} G(t,\tau)$ must exist, hence $\lim_{t\to\tau} G(t,\tau) = G(\tau,\tau)$

$$\lim_{t\to\tau} G(t,\tau) = \begin{cases} \lim_{t\to\tau} 0, & \text{if } 0 < t < \tau \\ \sqrt{l/g}\, \lim_{t\to\tau} \sin\left(\sqrt{g/l}\,(t-\tau)\right), & \text{if } t > \tau \end{cases}$$

$$\lim_{t\to\tau} G(t,\tau) = \begin{cases} 0, & 0 < t < \tau \\ 0, & t > \tau \end{cases} \Rightarrow G(\tau,\tau)$$

hence

$\lim_{t\to\tau} G(t,\tau) = G(\tau,\tau)$ and exists.

(iii) $$\left[\frac{\partial G(t,\tau)}{\partial t}\right]_{t=\tau^-}^{t=\tau^+} = 1$$

$$\frac{\partial}{\partial t} G(t,\tau) = \cos\left(\sqrt{g/l}\,(t-\tau)\right)$$

$$\left[\frac{\partial}{\partial t} G(t,\tau)\right]_{t=\tau^-}^{t=\tau^+} = \lim_{t\to\tau^+}\left(\cos\sqrt{g/l}\,(t-\tau)\right) - \lim_{t\to\tau^-}\left(\cos\sqrt{g/l}\,(t-\tau)\right)$$

Since $G(t,\tau)=0 \quad t<\tau \Rightarrow \frac{\partial}{\partial t} G(t,\tau) = 0$

$$\lim_{t\to\tau^-}\left(\cos\sqrt{g/l}\,(t-\tau)\right) = 0$$

$$\left[\frac{\partial}{\partial t} G(t,\tau)\right]_{t=\tau^-}^{t=\tau^+} = \lim_{t\to\tau^+}\left(\cos\sqrt{g/l}\,(t-\tau)\right) = \underline{\underline{1}}$$

c) Use the Greens' function to solve

$$\theta''(t) + \frac{g}{l}\theta = \frac{1}{ml} f(t), \quad \theta(0) = \theta'(0) = 0$$

$$L[u] = f \Rightarrow u = L^{-1}[f] \quad \text{— ①}$$

$$L[\theta] = \frac{1}{ml}(f(t)) \Rightarrow \theta = \frac{1}{ml} L^{-1}[f(t)]$$

using ①

$$u(x) = a(1-x) + bx + \int_0^1 G(x,y) f(y)\,dy$$

$$u(x) = u(0)(1-x) + u(1)\cancel{b}x + \int_0^1 G(x,y) f(y)\,dy$$

therefore

$$u(t) = u(0)(1-t) + u(1)t + \int_0^1 G(t,y) f(y)\, dy = L^{-1}[f(t)]$$

$$\theta(t) = \int_0^t G(t,y) f(\tau)\, d\tau$$

$$\theta(t) = \int_0^t G(t,y) \frac{1}{ml} f(\tau)\, d\tau$$

$$= \frac{1}{ml} L^{-1}[f(t)]$$

but $G(t,y) = \sqrt{l/g}\, \sin\left(\sqrt{g/l}\,(t-\tau)\right)$

$$\theta(t) = \int_0^t \sqrt{l/g}\, \sin\left[\sqrt{g/l}\,(t-\tau)\right] 1/ml\, f(\tau)\, d\tau$$

$$\theta(t) = \frac{1}{ml}\sqrt{l/g} \int_0^t \sin\left[\sqrt{g/l}\,(t-\tau)\right] f(\tau)\, d\tau.$$