Power series solutions to ODEs

Power series methods for solving ODEs

- Some of the most important ODEs in applied mathematics have power series solutions.

We will use this method to solve ODEs of the form

$$y'' + p(x)y' + q(x)y = r(x)$$

$y(x)$
$r(x) = 0$
$\Rightarrow$ homogeneous
$r(x) \neq 0$
$\Rightarrow$ inhomogeneous.

- If $p(x)$, $q(x)$, $r(x)$ can be represented as a power series, then the ODE has a power series solution.

- Idea: Assume a solution of the form

$$y = \sum_{m=0}^{\infty} a_m (x - x_0)^m$$

We will take $x_0 = 0$ when possible.

$$y = \sum_{m=0}^{\infty} a_m x^m$$

Example:

$$y' - y = 0$$

$y(x) = y_0 e^{x}$

$y_0 = y(0)$

$x \in [0, \infty)$

Assume

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$$

$$\underbrace{(a_0 + a_1 x + a_2 x^2 + \dots)}_{y} - \underbrace{(a_1 + 2a_2 x + 3a_3 x^2 + \dots)}_{y'} = 0$$

$a_0 - a_1 = 0$ (constant)

$a_1 - 2a_2 = 0$ (linear term)

$a_2 - 3a_3 = 0$ (quadratic)

$a_3 - 4a_4 = 0$ (cubic)

$\vdots$

$a_1 = a_0$

$a_2 = \frac{1}{2} a_1 = \frac{1}{2} a_0$

$a_3 = \frac{1}{3} a_2 = \frac{1}{3} \cdot \frac{1}{2} a_0$

$a_4 = \frac{1}{4} a_3 = \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{2} a_0$

$a_5 = \frac{1}{5} \cdot \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{2} a_0$

$= \frac{1}{5!} a_0$

$$y = a_0 \underbrace{\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots\right)}_{e^x}$$

$$y = a_0 e^x$$

$$y(0) = a_0 = y_0$$

$$y(x) = y_0 e^x$$

Another try:

$$y = \sum_{m=0}^{\infty} a_m x^m$$

$$y' = \sum_{m=0}^{\infty} m a_m x^{m-1}$$

$$\sum_{m=0}^{\infty} a_m x^m - \sum_{m=0}^{\infty} m a_m x^{m-1} = 0$$

$$0 \cdot a_0 \cdot x^{-1} + 1a_1 x^0 + 2a_2 x^1 + \cdots$$

$$= \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m$$

$$\sum_{m=0}^{\infty} a_m x^m - \underbrace{\sum_{m=0}^{\infty} m a_m x^{m-1}} = 0$$

$$0 \cdot a_0 \cdot x^{-1} + 1 a_1 x^0 + 2 a_2 x^1 + \dots$$

$$= \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m$$

$$\sum_{m=0}^{\infty} a_m x^n - \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m = 0$$

$$\sum_{m=0}^{\infty} \big(\underbrace{a_m - (m+1) a_{m+1}}_{=0}\big) x^m = 0$$

Solve: $a_{m+1} = \dfrac{1}{m+1} a_m$

$m=0 \quad a_1 = \dfrac{1}{2} a_0$

$m=1 \quad a_2 = \dfrac{1}{3} \cdot a_1 = \dfrac{1}{3} \cdot \dfrac{1}{2} \cdot a_0$

$\vdots$

$$a_{m+1} = \frac{1}{m!} a_m$$

Example: $y'' + y = 0$

$p(x) = 0$
$q(x) = 1$
$r(x) = 0$

Boundary conditions. $\left\{ \text{s.t.} \quad \begin{array}{l} y(0) = A \\ y'(0) = B \end{array} \right.$

$$y = \sum_{m=0}^{\infty} a_m x^m$$

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1}$$

$$y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

$$y'' + y = \sum_{m=0}^{\infty} a_m x^m + \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

want to start at $m=0$

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m$$

$$\sum_{m=0}^{\infty} \underbrace{\left(a_m + (m+2)(m+1) a_{m+2}\right)}_{=0} x^m = 0$$

$$\sum_{m=0}^{\infty}\Big(\underbrace{a_m+(m+2)(m+1)a_{m+2}}_{=0}\Big)x^m=0$$

$\underline{m=0}$

$$a_0+(2)(1)a_2=0 \Rightarrow a_2=-\frac{1}{2}a_0$$

$\underline{m=1}$

$$a_1+3\cdot 2\,a_3=0 \Rightarrow a_3=\frac{-1}{3\cdot 2}a_1$$

$\underline{m=2}$

$$a_2+4\cdot 3\cdot a_4=0 \Rightarrow a_4=-\frac{1}{4\cdot 3}a_2$$

$$=-\frac{1}{4!}a_0$$

$\underline{m=3}$

$$a_3+5\cdot 4\cdot a_5=0 \Rightarrow a_5=-\frac{1}{5!}a_1$$

$$a_{m+2}=\frac{-1}{(m+2)(m+1)}a_m$$

$$a_{m+2}=\frac{-1}{(m+2)(m+1)}\cdot\frac{1}{(m)(m-1)}a_{m-2}$$

$\cdots$

$$a_{m+2} = \frac{-1}{(m+2)(m+1)} a_m$$

$$= \frac{-1}{(m+2)(m+1)} \cdot \frac{-1}{(m)(m-1)} a_{m-2}$$

Even

$$a_{m+2} = \frac{(-1)^{\frac{m+2}{2}}}{(m+2)!} a_0 \quad , \quad m = 0, 2, 4, 6, 8 \ldots$$

Odd

$$a_{m+2} = \frac{(-1)^{\frac{m+1}{2}}}{(m+2)!} a_1 \quad , \quad m = 1, 3, 5, 7, \ldots$$

Check!

$m = 0 \qquad a_2 = -\frac{1}{2} a_0$ ✓

$m = 1 \qquad a_3 = \frac{-1}{3 \cdot 2} a_1$ ✓

$m = 2 \qquad a_4 = \frac{1}{4!} a_0$ ✓

$m = 3 \qquad a_5 = \frac{-1}{5!} a_1$ ✓

Shifting indices

Even

$$a_m = \frac{(-1)^{m/2}}{m!} a_0 , \quad m = 0, 2, 4, 6 \ldots$$

Odd

$$a_m = \frac{(-1)^{\frac{m-1}{2}}}{m!} a_1 , \quad m = 1, 3, 5, 7$$

Even

$$y = a_0 \left(1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \ldots \right)$$

$$= a_0 \cos(x)$$

odd

$$y = a_1 \left(x^1 - \frac{x^3}{3!} + \frac{x^5}{6!} - \frac{x^7}{8!} + \ldots \right)$$

$$= a_1 \sin(x)$$

Method of superposition:

$$y(x) = a_0 \cos(x) + a_1 \sin(x)$$

Imposing BC: $y(x) = A\cos(x) + B\sin(x)$