Homework #7

Math 537

1. **Haar wavelets.** The Haar scaling function can be defined recursively as

$$\phi(x) = \phi(2x) + \phi(2x-1). \tag{1}$$

Use initial conditions $\phi(0) = 1$, $\phi(1) = 0$, and $\phi(x) = 0$, $x < 0$ or $x > 1$ to compute the value of the Haar scaling function at odd multiples of the 2^{-m} for $m = 1, 2, 3$. Write a general algorithm (or pseudo-code) for evaluating $\phi(x)$ at dyadic numbers $(2k+1)2^{-m}$, $k = 0, 1, 2, ..., m/2 - 1$. Argue that $\phi(x)$ has a closed form solution

$$\phi(x) = \begin{cases} 1 & \text{if} \quad 0 \le x < 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

2. **Haar wavelets.** Repeat the above for the Haar wavelet, defined recursively as

$$\psi(x) = \phi(2x) - \phi(2x-1) \tag{3}$$

using initial conditions $\psi(0) = 1$, $\psi\left(\frac{1}{2}\right) = -1$, $\psi(x) = 0$ for $x < 0$ and $x \ge 1$.

3. **Haar wavelets.** TBA

4. **Daubechies wavelets.** TBA

5. Using integration by parts, show that the solution to the elliptic problem

$$u''(x) = f \tag{4}$$

on $[0, 1]$, subject to $u(0) = a$, $u(1) = b$ is given by

$$u(x) = a(1-x) + bx + \int_0^1 G(x,y)f(y)\,dy \tag{5}$$

where $G(x, y)$ is the Green's function solution for this problem.

Hint: Solve $L[u] = f$ using the inverse operator to get $u = L^{-1}[f]$.

6. Solve

$$u''(x) = e^{-100(x-0.5)^2} \tag{6}$$

numerically on $[0, 1]$ by using a quadrature rule to evaluate (5). Use boundary conditions $u(0) = u(1) = 0$. Evaluate your solution at $x_j = jh, j = 0, 1, \ldots, N$, for $h = 1/N$ and $N = 16$.

   - Since $G(x, y)$ is discontinuous, you will need to break up the integral.
   - You can use the Scipy function `quad` to evaluate a definite integral. For example, to evaluate

$$I = \int_a^b f(x)\,dx \tag{7}$$

   use the Scipy function `quad` :

```
from scipy.integrate import quad
I =  quad(f,a,b)
```

where `f` is a function defined as a `lambda` function in Python, or using `def`. The value of the approximated definite integral will be in `I[0]`, and the approximate error is reported in `I[1]`.

- Compare your solution to the exact solution, which you can obtain by using WolframAlpha to evaluate the integral form of the solution in (5). **Hint :** Ask WolframAlpha to evaluate the definite integrals needed to evaluate the Green's function solution. To get the exact solution, use the indefinite form returned by WolframAlpha and evaluate the indefinite form at the endpoints 0 and 1. (For some reason, asking Wolfram to evaluate the indefinite integral returns only a numerical solution.)
- Compare the solution to the finite difference solution obtained in class (see class notes).
- What are some potential advantages/disadvantages of evaluating the integral from of the solution rather than solving the linear system arising from the finite difference approach or the exact formulation obtained from Wolfram Alpha?

7. Consider the initial value problem describing the motion of a forced pendulum. Suppose the pendulum has mass m suspended from a weightless rod of length ℓ and makes an angle θ with a vertical line. Balancing forces, and assuming a small angle θ, the equation of motion for angle θ is given by

$$\theta''(t) + \frac{g}{\ell}\theta = \frac{1}{m\ell}f(t) \tag{8}$$

where $t \geq 0$ is time and g is the acceleration due to gravity. The initial conditions are given by $\theta(0) = \theta'(0) = 0$.

(a) Using a Laplace Transform, solve

$$\frac{\partial^2 G(t,\tau)}{\partial t^2} + \frac{g}{\ell}G(t,\tau) = \delta(t-\tau) \tag{9}$$

to get

$$G(t,\tau) = \sqrt{\frac{\ell}{g}}\sin\left(\sqrt{\frac{g}{\ell}}(t-\tau)\right)H(t-\tau) \tag{10}$$

where $H(t)$ is the Heaviside function and $\delta(t)$ is the delta distribution. **Hint:** See Kreysig, Example 2, page 227 for an example of how to use the Laplace Transform to solve this problem.

(b) For the Green's function $G(t,\tau)$ and operator L, given by

$$L[G(t,\tau)] = \frac{\partial^2 G(t,\tau)}{\partial t^2} + \frac{g}{\ell}G(t,\tau), \tag{11}$$

verify that the fundamental solution $G(t,\tau)$ you obtained in Problem 7a satisfies

i. $L[G(t,\tau)] = 0, t \neq \tau$

ii. $G(t,\tau)$ is continuous at $t = \tau$, and

iii. $\left[\frac{\partial G(t,\tau)}{\partial t}\right]_{t=\tau^-}^{t=\tau^+} = 1$

When these three conditions are satisfied, the solution obtained in Problem 7a is a *Green's function* for the second order operator L.

(c) Use the Green's function to solve (8). **Hint:** Obtain $\theta(t) = L^{-1}[f(t)/m\ell]$.