Homework #6

Math 537

Show enough work on the Review problems to convincingly show you know how to carry out the steps. The Review problems will only be graded for completeness.

1. **Review.** A *self-adjoint* operator is one for which

$$\langle Lu, v\rangle = \langle u, Lv\rangle \tag{1}$$

for functions u, v in a Hilbert space. Show that the Sturm-Liouville operator

$$L[u] = -\frac{1}{w(x)}\Big((p(x)y')' + q(x)u\Big) \tag{2}$$

is self-adjoint if separated boundary conditions are used. **Hint:** Use integration by parts twice. See Keener, Section 4.3.2 (page 155).

2. **Review.** Show that by using an integrating factor, a second order ODE

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{3}$$

can be written as a Sturm-Liouville operator.

3. **Laguerre polynomials.** Do Problem 14 (b), from Problem set 11.5 in Kreyszig (page 504).

4. **Cauchy sequence.** Show that the sequence $\{x_n\}$,

$$x_n = \sum_{k=1}^{n} \frac{1}{k!} \tag{4}$$

is a Cauchy sequence using the measure of distance $d(x, y) = |x - y|$.

You may use this definition of a Cauchy sequence from Keener (Section 2.1, page 60):

> A sequence $\{x_n\}$ is called a Cauchy sequence if for any $\epsilon > 0$, there is an integer N (depending on ϵ) so that for every $m, n \geq N$, $d(x_m, x_n) < \epsilon$.

5. **Fourier series.** Suppose that $f(t)$ and $g(t)$ are 2π periodic functions with Fourier series representations

$$f(t) = \sum_{k=-\infty}^{\infty} f_k e^{ikt}, \qquad g(t) = \sum_{k=-\infty}^{\infty} g_k e^{ikt} \tag{5}$$

Find the Fourier series of the convolution

$$h(t) = \int_0^{2\pi} f(t-x)g(x)\, dx \tag{6}$$

6. **Discrete Fourier Transform.** Show that the Discrete Fourier Transform $\{f_n\} \to \{g_n\}$ can be viewed as a change of coordinate system in $\mathcal{R}^n$. What is the matrix T corresponding to this change of basis?

7. **DFT to FFT.** Describe how the you can optimize the computations needed for the Discrete Fourier Transform to obtain a Fast Fourier Transform. Refer to the original paper by Cooley and Tukey, available on the course Slack site. Provide enough details in your own words so that you understand how the algorithm works. I will ask a few of your to make a short presentation on the algorithm in class on Friday.