Brian KYANJO

Homework #3

Math 537

①

1. $x^2 y'' + x\, b(x)\, y' + c(x)\, y = 0$ ---- ①

where: $b(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \dots$

$c(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$

Assume

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m$$

(a) Show that the coefficient of x^r is $p(r) \equiv r^2 + (b_0 - r)b_0 + c_0$ (the indicial equation).

Start by

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m \quad (1)$$

$$y' = r x^{r-1} \sum_{m=0}^{\infty} a_m x^m + x^r \sum_{m=0} m a_m x^{m-1}$$

$$y' = \sum_{m=0}^{\infty} a_m r x^{m+r-1} + \sum_{m=0}^{\infty} m a_m x^{m+r-1}$$

$$y' = \sum_{m=0}^{\infty} (r+m) a_m x^{m+r-1}$$

$$y''(x) = r x^{r-1} \sum_{m=0}^{\infty} (r+m) a_m x^{m-1} + x^r \sum_{m=1}^{\infty} (r+m)(m-1) a_m x^{m-2}$$

$$y''(x) = \sum_{m=0}^{\infty} r(r+m) a_m x^{m+r-2} + \sum_{m=0}^{\infty} (r+m)(m-1) a_m x^{m+r-2}$$

$$y''(x) = \sum_{m=0}^{\infty} \left[r(r+m) + (r+m)(m-1) \right] a_m x^{m+r-2}$$

$$y''(x) = \sum_{m=0}^{\infty} (r+m)(r+m-1)\, a_m x^{m+r-2}$$

Substitute y'', y' and y into equation ①.

$$x^2 y'' + x b(x) y' + c(x) y = 0$$

$$x^2 \sum_{m=0}^{\infty} (m+r-1)(m+r) a_m x^{m+r-2} + x b(x) \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1} +$$
$$c(x) \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

$$\sum_{m=0}^{\infty} (m+r-1)(m+r) a_m x^{m+r} + b(x) \sum_{m=0}^{\infty} (m+r) a_m x^{m+r} + c(x) \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

$$\sum_{m=0}^{\infty} \big((m+r-1)(m+r) a_m + b(x)(m+r) a_m + c(x) a_m \big) x^{m+r} = 0$$

$$\sum_{m=0}^{\infty} \big[(m+r-1)(m+r) + b(x)(m+r) + c(x) \big] a_m x^{m+r} = 0$$

$$\sum_{m=0}^{\infty} \big[(m+r-1)(m+r) + (b_0 + b_1 x + b_2 x^2 + \dots)(m+r) + (c_0 + c_1 x + c_2 x^2 + \dots$$
$$\big] a_m x^{m+r} = 0 \qquad \text{——— ②}$$

Set $m=0$, we obtain the coefficient for x^r

$$\big[(r-1)(r) + br + c_0 \big] a_0 x^r = 0$$

$$p(r) = (r-1)r + b_0 r + c_0 = \underline{r^2 + (b_0 - 1) r + c_0}$$

b) Show that the coefficient of x^{r+m}, $m=1,2,3,\dots$ is given by

$$P(r+m)\,a_m = -\sum_{k=0}^{m-1}\left[(r+k)b_{m-k} + c_{m-k}\right]a_k, \quad m=1,2,3,\dots$$

from Equation (2) above, we have

$$\sum_{m=0}^{\infty}\left[(m+r-1)(m+r) + (b_0+b_1x+b_2x^2+\dots)(m+r) + (c_0+c_1x+c_2x^2+\dots)\right]a_m x^{m+r} = 0$$

$$\left[(r-1)r + (b_0+b_1x+b_2x^2+b_3x^3+\dots)r + (c_0+c_1x+c_2x^2+\dots)\right]a_0x^r + \left[r(r+1) + (b_0+b_1x+b_2x^2+b_3x^3+\dots)(r+1) + (c_0+c_1x+c_2x^2+c_3x^3+\dots)\right]a_1x^{r+1} + \left[(r+1)(r+2) + (b_0+b_1x+b_2x^2+\dots)(r+2) + (c_0+c_1x+c_2x^2+\dots)\right]a_2x^{r+2} + \dots = 0$$

Collecting like terms

$$\left[(r-1)ra_0 + b_0ra_0 + c_0a_0\right]x^r + \left(rb_1a_0 + c_1a_0 + r(r+1)a_1 + b_0(r+1)a_1 + c_0a_1\right)x^{r+1} + \left(b_2ra_0 + c_2a_0 + b_1(r+1)a_1 + c_1a_1 + (r+1)(r+2)a_2 + b_0(r+2)a_2 + c_0a_2\right)x^{r+2} + \dots = 0 \quad \text{---}(3)$$

From equation (3)

$$((r-1)r + b_0r + c_0)a_0x^r = 0$$

$$P(r) = (r-1)r + b_0r + c_0 = r^2 + (b_0-1)r + c_0$$

$$\left(rb_1 a_0 + c_1 a_0 + r(r+1) a_1 + b_0 (r+1) a_1 + c_0 a_1\right) x^{r+1} = 0$$

$$\left(r(r+1) + b_0(r+1) + c_0\right) a_1 + (rb_1 + c_1) a_0 = 0$$

$$\left((r+1)^2 + (b_0-1)(r+1) + c_0\right) a_1 + (rb_1 + c_1) a_0 = 0$$

but $P(r) = r^2 + (b_0 - 1) r + c_0$

$P(r+1) = (r+1)^2 + (b_0-1)(r+1) + c_0$

Therefore

$$P(r+1) a_1 + (rb_1 + c_1) a_0 = 0$$

$$P(r+1) a_1 = -(rb_1 + c_1) a_0 = -((r-0) b_{1-0} + c_{1-0}) a_0$$

$$P(r+1) a_1 = -\sum_{k=0}^{1-1} \left((r+k) b_{1-k} + c_{1-k}\right) a_k$$

Coefficients for x^{r+2}

$$(b_2 r a_0 + c_2 a_0 + b_1 (r+1) a_1 + c_1 a_1 + (r+1)(r+2) a_2 + b_0 (r+2) a_2 + c_0 a_2) x^{r+2} = 0$$

$$\left((r+1)(r+2) + b_0 (r+2) + c_0\right) a_2 + \left(b_1 (r+1) + c_1\right) a_1 + (b_2 r + c_2) a_0 = 0$$

$$\left((r+2)^2 + (b_0 - 1)(r+2) + c_0\right) a_2 + \left(b_1 (r+1) + c_1\right) a_1 + (b_2 r + c_2) a_0 = 0.$$

$$P(r+2) a_2 = -\left[\left(b_1 (r+1) + c_1\right) a_1 + (b_2 r + c_2) a_0\right]$$

$$= -\sum_{k=0}^{2-1} \left[(r+k) b_{2-k} + c_{2-k}\right] a_k$$

$$P(r+2)\,a_2 = -\sum_{k=0}^{2-1}\left[(r+k)\,b_{2-k} + c_{2-k}\right] a_k$$

⋮

$$P(r+m)\,a_m = -\sum_{k=0}^{m-1}\left[(r+k)\,b_{m-k} + c_{m-k}\right] a_k \quad \boxed{}$$

(c) Assume $r_1 > r_2$

Take $r = r_1$, (2) becomes

$$P(r_1+m)\,a_m = -\sum_{k=0}^{m-1}\left[(r_1+k)\,b_{m-k} + c_{m-k}\right] a_k,$$

$m = 1, 2, 3, \ldots$ ~~$m = 1, 2, 3, \ldots$~~

Also $P(r) = r^2 + (b_0 - 1)r + c_0$

$P(r+m) = (r+m)^2 + (b_0 - 1)(r+m) + c_0$

Since r_2 is a root then $p(r) = (r - r_1)(r - r_2)$

$P(r_2) = 0$, $r_2 + m$ ~~can~~ can be equal to r_1

So therefore $P(r_2+m) = P(r_1) = 0$, then $P(r_1+m) \neq 0$, since it's ~~known~~ known that

$$\sum_{k=0}^{m-1}\left[(r_1+k)\,b_{m-k} + c_{m-k}\right] \neq 0,$$ hence $a_m \neq 0$

Since a_m is obtained using a_0, then $a_0 \neq 0$ therefore we can find atleast a solution $y_1(x) = x^{r_1}\sum_{m=0}^{\infty} a_m x^m$, $a_0 \neq 0$.

(i) Case 1:

Consider: $r_1 - r_2 \neq N$, where N is an integer, then this means that $r_2 + m \neq r_1$ or $r_2 + m \neq r_1$

with this we can directly conclude that

$P(r_2 + m) \neq 0$, hence $a_m \neq 0$, therefore we can find a ~~solution~~ second solution

$$y_2(x) = x^{r_2} \sum_{m=0}^{\infty} a_m(r_2) x^m \quad , a_0 \neq 0$$

Since a_m is obtained from a_0. Since $r_1 \neq r_2$, the the ~~solutions~~ are linearly independent.

(ii) Case 2:

Suppose that $r_1 = r_2$. Show that the second solution must be of the form

$$y_2(x) = y_1(x) \ln(x) + x^{r_2} \sum_{m=0}^{\infty} A_m x^m$$

Consider

$$L[y(x;r)] = a_0 P(r) x^r + \sum_{m=1}^{\infty} \left[P(r+m) a_m + \sum_{k=0}^{m-1} \left[(r+m) b_{m-k} \right.\right.$$

$$\left.\left. + C_{m-k} \right] a_k \right] \quad \text{——} (1)$$

Since

$$P(r+m) a_m = -\sum_{k=0}^{m-1} \left[(r+m) b_{m-k} + C_{m-k} \right] a_k$$

Equation 4 becomes.

$$L[y(x;r)] = a_0 p(r) x^r + \sum_{m=1}^{\infty} [\cancel{p(r+m) a_m + -p(r+m) a_m}]^{0}$$

$$L[y(x;r)] = a_0 p(r) x^r + 0 . = a_0 p(r) x^r$$

but $p(r) = (r - r_1)(r - r_2)$

$$L[y(x;r)] = a_0 p(r) x^r = a_0 (r - r_1)(r - r_2) x^r$$

If $r_1 = r_2$ then $(r - r_1)(r - r_2) = (r - r_1)^2$

$$L[y(x;r)] = a_0 (r - r_1)^2 x^r$$

differentiating both sides with respect to r

$$\frac{d}{dr} L[y(x;r)] = \frac{d}{dr}(a_0 (r - r_1)^2 x^r)$$

Since L is a linear operator then

$$\frac{d}{dr} L[y(x;r)] = L\left[\frac{d}{dr} y(x;r)\right]$$

$$L\left[\frac{d}{dr} y(x;r)\right] = 2a_0 (r - r_1) x^r + a_0 (r - r_1)^2 \frac{d}{dr}(x^r)$$

but $x^r = e^{\ln x^r} = e^{r \ln|x|}$

$$\frac{d}{dr}(x^r) = \frac{d}{dr}(e^{r \ln|x|}) = (\ln|x|) e^{r \ln|x|} = x^r \ln|x|$$

$$L\left[\frac{d}{dr} y(x;r)\right] = 2a_0 (r - r_1) x^r + a_0 (r - r_1)^2 x^r \ln|(x)|$$

For $r_1 = r_2 = r$

$$L\left[\frac{d}{dr} y(x; r_1)\right] = 0.$$

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from (1): $x^2 y'' + x b(x) y' + c(x) y = 0.$

$$L[y(x;r)] = x^2 y'' + x b(x) y' + c(x) y$$

$$y_2(x; r_1) = \frac{d}{dr}(y(x; r_1))$$

Since $y(x; r_1) = x^{r_1} \sum_{m=0}^{\infty} a_m(r) x^m.$

$$\frac{d}{dr} y(x; r_1) = \frac{d}{dr}\left[x^{r_1} \sum_{m=0}^{\infty} a_m(r) x^m\right]$$

Using $\frac{d}{dr}(x^{r_1}) = x^{r_1} \ln|x|$

$$\frac{d}{dr} y(x; r_1) = (x^{r_1} \ln|x|) \sum_{m=0}^{\infty} a_m(r) x^m + x^{r_1} \sum_{m=0}^{\infty} a_m'(r) x^m$$

$$y_2(x; r_1) = (\ln|x|)\left(x^{r_1} \sum_{m=0}^{\infty} a_m(r) x^m\right) + x^{r_1} \sum_{m=0}^{\infty} a_m'(r) x^m$$

but $x^{r_1} \sum_{m=0}^{\infty} a_m(r) x^m = y_1(x)$

$$y_2(x; r_1) = y_1(x) \ln|x| + x^{r_1} \sum_{m=0}^{\infty} a_m'(r) x^m$$

for $r_1 = r_2$

$$y_2(x) = y_1(x) \ln|x| + x^{r_2} \sum_{m=0}^{\infty} A_m x^m.$$

(iii) Case 3:

Suppose that $r_1 - r_2 = N$, $\Rightarrow$ $r_1 = r_2 + N$

Using $p(r) = (r-r_1)(r-r_2)$, $p(r_1) = 0$

$$p(r_1) = p(r_2+N) = 0 \quad -(2)$$

Equation (2) would contradicts with

$$p(r_2+N)a_N = -\sum_{k=0}^{N-1}\left((r+N)b_{N-k} + c_{N-k}\right)a_k$$

Since, It would imply that all the coefficients of a_N will be zero, which is impossible since $a_0 \neq 0$ and a_N is obtained from a_0.

With this argument, it will be impossible to obtain a second solution in the form of the frobenius series.

$$L\left[\frac{d}{dr}y(x;r_1)\right] = a_0 p'(r_1)x^{r_1}$$

from above $L[y(x;r)] = a_0 p(r) x^r$

Introduce $(r-r_2)$ on both sides

$$(r-r_2)L[y(x;r)] = a_0(r-r_2)p(r)x^r$$

Since L is a linear operator.

$$L[(r-r_2)y(x;r)] = a_0(r-r_2)p(r)x^r$$

differentiating both sides, we have:

$$L\left[\frac{d}{dr}(r-r_2)\,y(x;r)\right] = \frac{d}{dr}a_0(r-r_2)\,p(r)\,x^r$$

$$L\left[y(x;r) + (r-r_2)\frac{d}{dr}y(x;r)\right] = a_0\,p(r)\,x^r +$$

$$a_0(r-r_2)\,p'(r)\,x^r + a_0(r-r_2)\,p(r)\,x^r\ln|x|$$

$$L[\cancel{y(x;r)}] + (r-r_2)\,L\left[\frac{d}{dr}y(x;r)\right] = a_0\,p(r)\,x^r$$
$$+ a_0(r-r_2)\,p'(r)\,x^r + a_0(r-r_2)\,p(r)\,x^r\ln|x|$$

$$L\left[\frac{d}{dr}y(x;r)\right] = a_0\,p'(r)\,x^r + a_0\,p(r)\,x^r\ln|x| \quad \text{— (4)}$$

for $r = r_1$, $p(r_1) = 0$.

$$L\left[\frac{d}{dr}y(x;r_1)\right] = a_0\,p'(r_1)\,x^{r_1}$$

Show that we can find a series solution to this inhomogeneous equation of the form

$$x^{r_2}\sum_{m=0}^{\infty} c_n x^m$$

from (4)

$$L\left[\frac{d}{dr}y(x;r)\right] = a_0\,p'(r)\,x^r + a_0\,p(r)\,x^r\ln|x|$$

$p(r) = (r-r_1)(r-r_2)$ ~~$\Rightarrow$~~ $p'(r) =$

$p'(r) = (r-r_2) + (r-r_1)$

$p'(r_2) = (r_2 - r_1)$

So put $r = r_2$ in 4, $p(r_2) = 0$

$$L\left[\frac{d}{dr} y(x;r_2)\right] = a_0 p'(r_2) x^{r_2} + a_0 p(r_2) x^r \ln|x|$$

$$L\left[\frac{d}{dr} y(x;r_2)\right] = a_0 p'(r_2) x^{r_2} = a_0 (r_2 - r_1) x^{r_2}$$

$$L\left[\frac{d}{dr} y(x;r_2)\right] = a_0 (r_2 - r_1) x^{r_2}$$

but
$$L[(r-r_2)\, y(x;r)] = a_0 (r-r_2) p(r) x^r$$

if ~~x~~ $= r_1$
$$L[(r_1 - r_2) y(x;r_1)] = a_0 (r_1 - r_2) p(r_1) x^{r_1} = 0$$

$$L[(r_1 - r_2) y(x;r_1)] = 0$$

This means $(r_1 - r_2) y(x;r_1)$ is a solution to
$x^2 y'' + x b(x) y' + c(x) y = 0$.

~~$(r_1 - r_2) y_2(x)$~~ ~~let $f(x) = \frac{d}{dr} y_2(x)$~~

~~$L[f(x)] = a_0 (r_2 - r_1) x^{r_2}$~~

take $f(x) = (r-r_2)\, y(x;r) = (r-r_2) x^r \sum_{m=0}^{\infty} a_m x^m$

$$\frac{d}{dr} f(x) = x^r \sum_{m=0}^{\infty} a_m x^m + \left(x^r \ln|x| \sum_{m=0}^{\infty} a_m x^m\right)(r-r_2)$$

$$+ (r-r_2)\, x^r \sum_{m=0}^{\infty} a_m'(r)\, x^m.$$

at $r=r_2$

$$\frac{d}{dr} f(x) = x^{r_2} \sum_{m=0}^{\infty} a_m x^m + 0$$

$$L\left[\frac{d}{dr} f(x)\right] = x^{r_2} \sum_{m=0}^{\infty} a_m x^m$$