In [1]:
```
%matplotlib notebook
%pylab
```

```
Using matplotlib backend: nbAgg
Populating the interactive namespace from numpy and matplotlib
```

Infinite dimensional spaces ¶

Definition. A sequence $\{x_n\}$ is called a *Cauchy* sequence if for any $\epsilon > 0$, we can always find an N large enough so that for any $m, n > N$, we have $\|x_n - x_m\| < \epsilon$.

Consider the space of all Cauchy sequences $\{x_n\}$ with the property that

$$\|\mathbf{x}\| = \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2} < \infty$$

This space is a vector space since it is closed under addition, and multiplication by a scalar, contains a zero element.

What is the dimension of this space? We need to find a basis for the space.

Consider sequences $\mathbf{e}_{(k)} = \{\delta_{ik}\}, i = 1, 2, 3, \ldots$.

- The $\mathbf{e}_{(k)}$ are linearly independent,
- Each sequence $\mathbf{x}$ in the sequence space can be expressed uniquely as a linear combination of $\mathbf{e}_{(k)}$.

So the $\mathbf{e}_{(k)}$ form a *basis* for the sequence space. Furthermore, for any finite set of vectors $\mathbf{e}_{(k)}$, we can always find an additional vector that is linearly independent to add to the set. So the basis contains an infinite number of vectors. Therefore, the sequence space is *infinite dimensional*.

Example. Consider the space of all continuous functions defined on an interval $[a, b]$. We denote these space $C[a, b]$. This space is a *vector space*, since it is closed under addition and multiplication by a scalar, and contains a zero element.

We can define a norm on this space in many ways. For example

$$\|f\| = \left(\int_a^b |f(x)|^2 \, dx \right)^{1/2} \equiv \sqrt{\langle f, f \rangle}$$

is the L^2 norm.

Completeness

A *normed vector space* S is complete if every Cauchy sequence in S is convergent in S.

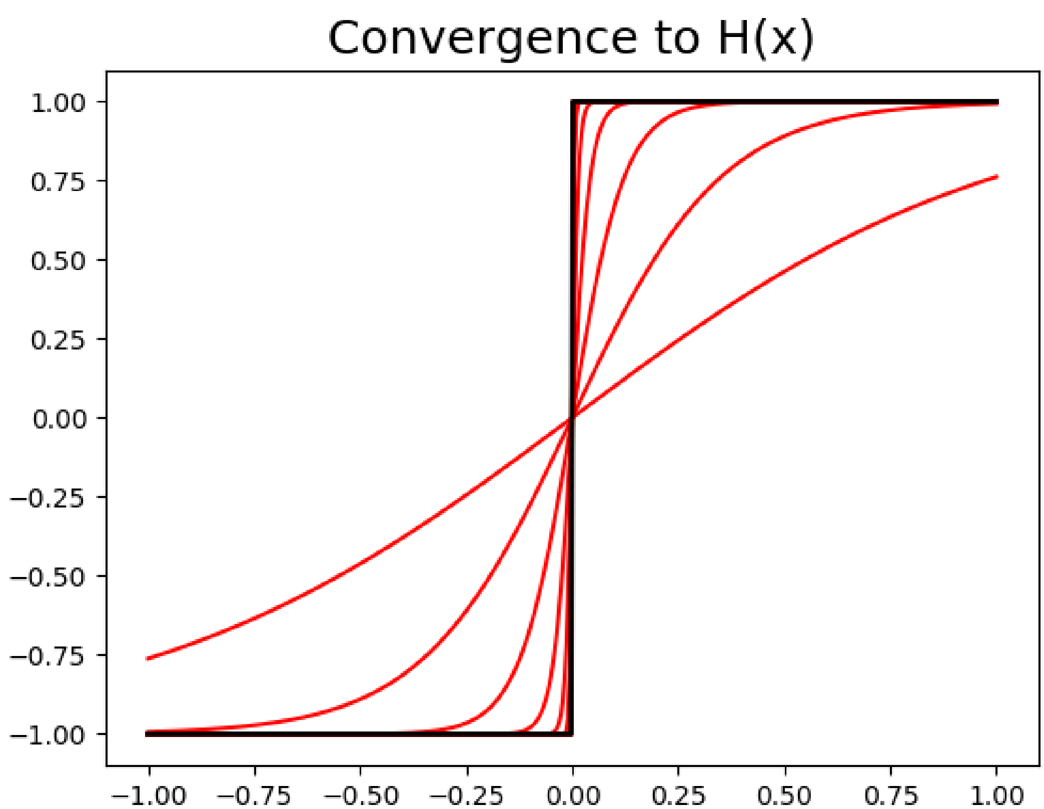
This definition is somewhat problematic, even for our nice function space $C[a, b]$. It is easy to find examples of a sequence of continuous functions which do not converge to a continuous function.

In [3]:
```
figure(1)
clf()

def f(x,e):
    return tanh(x/e)

x = linspace(-1,1,500)
for e in logspace(-5,0,12):
    plot(x,f(x,e),'r',linewidth=1.5)

H = where(x > 0,1,-1)
plot(x,H,'k',linewidth=2)
title('Convergence to H(x)',fontsize=18)
show()
```



Yet, if we want to discuss approximations to function within a set, we will want some notion of completeness.

Solution : We will expand our set of functions to include those which differ from functions in $C[a,b]$ only on a set of measure 0. In the case above, we allow $H(x)$ to be given special membership, because it almost looks like a continuous function. This requires that we extend our definition of integration (used in the definition of the norm) to *Lebesgue integration*. (pronounced le-BEG).

Our continuous function space $C[a,b]$, along with a norm defined in terms of Lebesgue integration, is complete and is called a *Hilbert space*.

This leads to the question of how to approximate functions in Hilbert spaces. The goal will be to approximate a function $f(x)$ as linear combinations of *basis* functions $\phi_i(x)$ so that

$$f(x) = \sum_{i=1}^{\infty} \alpha_i \phi_i$$

The idea is to find α_i so that

$$\|f - \sum_{n=1}^{\infty} \alpha_i \phi_i\|^2 = \|f\|^2 - 2\sum_{i=1}^{\infty} \alpha_i \langle f, \phi_i \rangle + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \alpha_i \alpha_j \langle \phi_i, \phi_j \rangle$$

This norm will be a minimum if we have

$$\Phi \alpha = \beta$$

where $\beta = \langle f, \phi_i \rangle$ and Φ has entries $\phi_{ij} = \langle \phi_i, \phi_j \rangle$

Lebesgue integration

Because our norms will now be defined in terms of integration, it is important that we are clear on what rules of integration we can ssume still hold.

Extending our notion of integration to ***Lebesgue*** integration allows us to preserve many of the rules of integration that we came to take for granted with Riemann integration. Some of these are

1. Suppose $f(x) < g(x)$ on $[a, b]$. Then we have
$$\int f(x)\ dx < \int g(x)\ dx$$
2. If $0 \le f_1(x) \le f_2(x) \le f_3(x) \le \ldots$ and
$$f(x) = \lim_{k\to\infty} f_k(x)$$

then

$$\lim_{k\to\infty} \int f_k(x)\ dx = \int f\ dx$$

1. Linear combinations can be integrated.

$$\int (af(x) + bg(x))\ dx = a\int f(x)\ dx + b\int g(x)\ dx$$

1. Absolute values of integration

$$\left|\int f(x)\ dx\right| \le \int |f(x)|\ \ dx$$

Integration, continued

1. Interchange order of integration (Fubini's theorem)

$$\int_X \int_Y f(x,y)\ dY dX = \int_Y \int_X f(x,y)\ dX dY$$

For this course, we will assume that the above hold for any functions we may consider.

Lebesgue space

Define a norm

$$\|f\|_p = \left(\int |f|^p\ dx \right)^{1/p}, \qquad 0 < p < \infty$$

For $p = \infty$, we have

$$\|f\|_\infty = \max |f(x)|$$

Lebesgue space $\mathcal{L}^p$ is the space of functions for which $\|f\|_p < \infty$.

- $\mathcal{L}^p$ is a vector space

Hilbert spaces

Hilbert spaces is an infinite dimensional normed vector space with an inner product. This inner product gives us a notion of an **angle** and allows us to talk about orthogonality of elements in the space. Every Hilbert space has an orthonomal basis. This makes Hilbert spaces look very much like $\mathcal{R}^n$, with the exception that Hilbert spaces are infinite dimensional.

The **norm** in an Hilbert space is defined in terms of the inner product. Much of our intuition about geometry in $\mathcal{R}^n$ can be extended to Hilbert spaces.

For much of what remains, we will refer to Hilbert spaces $\mathcal{L}^2[a, b]$, the set of all **square integrable** functions on $[a, b]$.

Examples :

- **Cauchy-Schwarz inequality**

$$|\langle f, g\rangle| \leq \|f\| \cdot \|g\|$$

- **Triangle inequality**

$$\|f + g\| \leq \|f\| + \|g\|$$

- **Orthogonal decomposition.**

  Given f and g as elements of an inner product space, we can always decompose f into g and component *orthogonal* to g

$$f = h + \frac{\langle f, g\rangle}{\|g\|^2} g, \qquad \langle g, h\rangle = 0$$

Orthormal sets

This approximation is simplified if our ϕ_i form an *orthnormal set*. Then, we have that

$$\|f - \sum_{i=1}^{n} \alpha_i \phi_i\|^2 = \|f\|^2 - \sum_{i=1}^{n} \langle f, \phi_i \rangle^2 + \sum_{i=1}^{n} (\alpha_i - \langle f, \phi_i \rangle)^2$$

The choice of α that minimizes this expression is then just $\alpha_i = \langle f, \phi_i \rangle$ and we obtain a *generalized Fourier series*

$$f = \sum_{i=1}^{n} \langle f, \phi_i \rangle \phi_i$$

The coefficients α_i are called the Fourier Coefficients.

The error in the approximation is then

$$\|f - \sum_{n=1}^{n} \alpha_i \phi_i\|^2 = \|f\|^2 - \sum_{i=1}^{n} \langle f, \phi_i \rangle^2 \geq 0$$

which leads to *Bessel's inequality*

$$\sum_{i=1}^{n} \langle f, \phi_i \rangle^2 \leq \|f\|^2$$

If we allow $n \to \infty$, we get Parseval's Indentity

$$\sum_{i=1}^{\infty} \langle f, \phi_i \rangle^2 = \|f\|^2$$

Weierstrass Approximation Theorem

For any continuous function $f \in C[a, b]$, and $\epsilon > 0$, there is a polynomial $p(x)$ so that

$$\max_{a \leq x \leq b} |f(x) - p(x)| < \epsilon$$

The polynomials are *dense* in $\mathcal{L}^2[a, b]$.

Trigonometric polynomials : $\sin(nx)$ and $\cos(mx)$. These form an orthogonal, complete set on $\mathcal{L}^2[0, 2\pi]$.