Brian KYANJO
Homework 1

1. Matrix manipulations

(a) Consider a 4×4 identity Matrix, I, and a 4×4 matrix B to which we may apply the following operations.

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

* ~~Idem~~ Apply the following to I

(i) double column 1

$$I_1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(ii) Halve row 3

$$I_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(iii) add row 3 to row 1

$$I_3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(iv) Interchange Columns 1 and 4.

$$I_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

(v) Subtract row 2 from each of the other rows

$$I_5 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

(vi) Replace column 4 by column 3

$$I_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(vii) Delete column 1.

$$I_7 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) Write the result as a product of eight matrices.

$$I_5 \times I_3 \times I_2 \times B \times I_1 \times I_4 \times I_6 \times I_7$$

$$I_5 \times I_3 \times I_2 \times B \times I_1 \times I_4 \times I_6 \times I_7 =$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} B$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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b) Write it again as a product ABC (same B) of three matrices.

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

finally we get

$$\begin{bmatrix} 1 & -1 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1/2 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} B \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

2.

NO.2 Orthogonal matrices

(a) Show that if a matrix Q is orthogonal and lower triangular, then it is diagonal.

Consider $QQ^T = I$

Performing forward substitution.

$$Q^T = Q^{-1} \quad — (1)$$

Equation (1) is true for a real square matrix which is orthogonal, however Q is also lower triangular, meaning ~~the~~ its transpose will be upper triangular matrix.

- But again the inverse of Q, which is lower triangular and orthogonal is lower triangular.

- therefore equation (1) becomes a contradiction to obtain an upper triangular matrix, Hence Q is diagonal.

b) What are its diagonal elements?

Consider $QQ^T = I$, Since Q is diagonal, then $Q = Q^T$, therefore we have

$$Q \cdot Q = I$$

The diagonal entries of $Q \cdot Q$ will be squared and since we are equating them to I, whose major diagonal elements are ~~1~~, ones, then

The diagonal entries will be ± 1

No-3 (Rank one matrices) Show the following

- If A has rank one then $A = ab^T$ for some Column vectors a and b.

if ~~A~~ rank$(A) = 1$, this means that its Image is one dimensional.

Consider a vector $u \in \mathbb{R}^m$, then, with a fixed $a \in \mathbb{R}^n$ and a scalar α, then

$$Au = \alpha a \quad \text{——} ①$$

Equation ①, means that every Column of A is a multiple of a, hence

$$A = (b_1 a \quad b_2 a \quad \cdots \quad b_m a)$$
$$A = a(b_1 \quad b_2 \quad b_3 \quad \cdots \quad b_m)$$

$$A = ab^T$$

- If $A = ab^T$ for some Column vectors a and b then A has rank 1.

Again consider $u \in \mathbb{R}^m$, then

$$Au = ab^T u.$$
$$Au = (u \cdot b)\, a$$

This implies that A maps every vector in the

$\mathbb{R}^m$ to a Scalar multiple of a.

In this case the $\text{rank}(A) = \dim(\text{Colum}(A)) = 1$

$\boxed{}$

XLO-4 Pythagorean theorem

The Pythagorean theorem asserts that for a set of n orthogonal vectors $\{x_i\}$,

$$\left\| \sum_{i=1}^{n} x_i \right\|^2 = \sum_{i=1}^{n} \| x_i \|^2$$

(a) Prove this in the case $n=2$ by an explicit computation of $\|x_1 + x_2\|^2$

Consider $\|x\|^2 = x \cdot x$

$$\left\| \sum_{i=1}^{n=2} x_i \right\|^2 = \| x_1 + x_2 \|^2$$

$$= (x_1 + x_2)(x_1 + x_2)$$

$$= x_1 \cdot x_1 + x_1 \cdot x_2 + x_2 \cdot x_1 + x_2 \cdot x_2$$

for linear orthogonality $x_1 \cdot x_2 = x_2 \cdot x_1 = 0$

$$\left\| \sum_{i=1}^{n=2} x_i \right\|^2 = x_1 \cdot x_1 + 0 + 0 + x_2 \cdot x_2$$

$$= \|x_1\|^2 + \|x_2\|^2$$

$$\left\| \sum_{i=1}^{n=2} x_i \right\|^2 = \sum_{i=1}^{n=2} \| x_i \|^2$$

b) Show that this computation also establishes the general case, by induction.

Base Case:

for $n=2$, $\left\| \sum_{i=1}^{n=2} x_i \right\|^2 = \sum_{i=1}^{n=2} \|x_i\|^2$

Induction Case:

for $n=k$

$$\left\| \sum_{i=1}^{n=k} x_i \right\|^2 = \sum_{i=1}^{n=k} \|x_i\|^2$$

Assume It is true for $n=k$, i.e k mutually orthogonal vectors: $x_1, x_2, \dots x_k$

Suppose for $n=k+1$

$$\left\| \sum_{i=1}^{n=k+1} x_i \right\|^2 = \left\| x_1 + x_2 + x_3 + \dots + x_{k+1} \right\|^2$$

assuming $x_1, x_2, \dots, x_{k+1}$ are also mutually orthogonal vectors and that

$$\left\| \sum_{i=1}^{n=k+1} x_i \right\|^2 = \left\| \sum_{i=1}^{n=k} x_i \right\|^2 + \|x_{k+1}\|^2$$

Since x_{k+1} is orthogonal to each x_i for $i=1,\dots,k$, then using linearity of the inner product, $x_i \cdot x_j = 0$, so

$$\left\| \sum_{i=1}^{n=k} x_i \right\|^2 = \|x_1\|^2 + \|x_2\|^2 + \dots + \|x_{k-1}\|^2 + \|x_k\|^2$$

Concluding that

$$\left\| \sum_{i=1}^{n=k+1} x_i \right\|^2 = \|x_1\|^2 + \|x_2\|^2 + \dots + \|x_k\|^2 + \|x_{k+1}\|^2$$

$$= \sum_{i=1}^{n=k+1} \|x_i\|^2$$

$\boxed{\cdot}$

<u>2.6.5</u> (Inverse of rank one-perturbation).

If A is non singular, then $AA^{-1} = I$
so

$$(I + uv^*)(I + \alpha uv^*) = I$$

$$I + \alpha I uv^* + uv^* I + \alpha uv^* u v^* = I$$

$$\alpha uv^* + uv^* + \alpha v^* u uv^* = 0$$

$$(\alpha + 1 + \alpha v^* u)\, uv^* = 0 \quad -①$$

for ① to hold $\alpha + 1 + \alpha v^* u = 0$, therefore
$\alpha = -1(1 + v^*u)^{-1}$, for $v^* u \neq -1$

as long as $v^* u \neq -1$ then A has an Inverse of the form $A^{-1} = I + \alpha uv^*$

For what u and v is A singular?

A is Singular iff $\det(A) = 0$, so if
$\det(A) = 1 + v^* u = 0$
$v^* u = -1$

Hence If $v^*u = -1$, then A is Singular.

What is null(A)?

Consider $Ax = 0$, for x a m-vector.

$$Ax = 0 \Rightarrow (I + uv^*)x = 0$$

$$uv^*x = -x$$

from above $v^*u = -1$, then its possible to obtain $x = u$, hence

$$\text{null}(A) = \{\text{the space spanned by } u\}$$

No. 6.

(Matrix Norms) Consider the following Matrix

$$A = \begin{bmatrix} 4 & -1 & 2 \\ 1 & 2 & 3 \\ -1 & 7 & -5 \end{bmatrix}$$

(a) Compute the matrix 1-, 2-, ∞-, and Frobenius norm of A

1-Norm. $\|A\|_1 = \underline{\underline{10}}$

2-Norm $\|A\|_2 = \sqrt{\rho(A^*A)}$

Let $A^*A = B$

$$B = \begin{bmatrix} 4 & 1 & -1 \\ -1 & 2 & 7 \\ 2 & 3 & -5 \end{bmatrix} \begin{bmatrix} 4 & -1 & 2 \\ 1 & 2 & 3 \\ -1 & 7 & -5 \end{bmatrix}$$

$$B = \begin{bmatrix} 18 & -9 & 16 \\ -9 & 54 & -31 \\ 16 & -31 & 38 \end{bmatrix}$$

Eigen value of B, $|B - \lambda I| = 0$.

$$\begin{vmatrix} 18-\lambda & -9 & 16 \\ -9 & 54-\lambda & -31 \\ 16 & -31 & 38-\lambda \end{vmatrix} = 0$$

$\lambda_1 = 6.846$, $\lambda_2 = 20.652$, $\lambda_3 = 85.5023$.

$$\rho(B) = \max_{1 \le j \le n} |\lambda_j|$$

$$\rho(B) = \lambda_3 = 82.502$$

$$\|A\|_2 = \sqrt{\rho(B)} = \sqrt{82.502} = 9.083$$

<u>∞ - Norm</u>

$$\|A\|_\infty = \text{Max}\{\text{row sum}\} = \{7, 6, 13\}$$
$$= 13$$

<u>Frobenius Norm.</u>

$$\|A\|_F = \sqrt{\sum_{j=1}^{m} \sum_{k=1}^{n} |a_{jk}|^2} = \sqrt{\sum_{j=1}^{3} \sum_{k=1}^{3} |a_{jk}|^2}$$

$$= \sqrt{\sum_{j=1}^{3} \left(|a_{j1}|^2 + |a_{j2}|^2 + |a_{j3}|^2 \right)}$$

$$= \sqrt{|a_{11}|^2 + |a_{12}|^2 + |a_{13}|^2 + |a_{21}|^2 + |a_{22}|^2 + |a_{23}|^2 + |a_{31}|^2 + |a_{32}|^2 + |a_{33}|^2}$$

$$= \sqrt{110} = 10.488$$

(b) For any n-by-n Matrix A the following bounds on hold true.

1. $$\frac{1}{\sqrt{n}}\|A\|_2 \leq \|A\|_1 \leq \sqrt{n}\,\|A\|_2$$

$$\frac{1}{\sqrt{3}}(9.083) \leq 10 \leq \sqrt{3}\,(9.083)$$

holds

2. $$\frac{1}{\sqrt{n}}\|A\|_2 \leq \|A\|_\infty \leq \sqrt{n}\,\|A\|_2$$

$$\frac{1}{\sqrt{3}}(9.083) \leq 13 \leq \sqrt{3}\,(9.083)$$

holds

3. $$\frac{1}{n}\|A\|_\infty \leq \|A\|_1 \leq n\,\|A\|_\infty$$

$$\frac{1}{3}(13) \leq 10 \leq 3(13)$$

holds

4. $$\|A\|_1 \leq \|A\|_F \leq \sqrt{n}\,\|A\|_2$$

$$10 \leq 10.488 \leq \sqrt{3}\,(9.083)$$

holds.

5. $\rho(A) \leq \|A\|_p$, for $p = 1, 2, \infty$

$$\rho(A) = \max_{1 \leq j \leq n} |\lambda_j|$$

$$|A - \lambda I| = 0 \Rightarrow \lambda_1 = 3.58,\ \lambda_2 = 4.56,\ \lambda_3 = 6.96$$

$\rho(A) = 6.966$

$\rho(A) \leq \|A\|_1 \Rightarrow 6.966 \leq 10$, holds

$\rho(A) \leq \|A\|_2 \Rightarrow 6.966 \leq 9.083$, holds.

$\rho(A) \leq \|A\|_\infty \Rightarrow 6.966 \leq 13$, holds

7. (Matrix multiplication)

a) $C_{11} = P_1 + P_4 - P_5 + P_7$

$$C_{11} = A_{11}B_{11} + A_{11}\cancel{B_{22}} + A_{22}\cancel{B_{11}} + \cancel{A_{22}}B_{22} + A_{22}\cancel{B_{21}} - \cancel{A_{22}}B_{11} - A_{11}\cancel{B_{22}} - A_{12}\cancel{B_{22}} + A_{12}B_{21} + A_{12}\cancel{B_{22}} - \cancel{A_{22}}B_{21} - \cancel{A_{22}}B_{22}$$

$$\underline{C_{11} = A_{11}B_{11} + A_{12}B_{21}}$$

b) $C_{12} = P_3 + P_5$

$$C_{12} = A_{11}B_{12} - \cancel{A_{11}}B_{22} + \cancel{A_{11}B_{22}} + A_{12}B_{22}$$

$$\underline{C_{12} = A_{11}B_{12} + A_{12}B_{22}}$$

$C_{21} = P_2 + P_4$

$$C_{21} = A_{21}B_{11} + \cancel{A_{22}B_{11}} + A_{22}B_{21} - \cancel{A_{22}B_{11}}$$

$$\underline{C_{21} = A_{21}B_{11} + A_{22}B_{21}}$$

$$C_{22} = P_1 - P_2 + P_3 + P_4$$

$$C_{22} = \cancel{A_{11}B_{11}} + \cancel{A_{11}B_{22}} + \cancel{A_{22}B_{11}} + A_{22}B_{22} -$$

$$\cancel{A_{21}B_{11}} - \cancel{A_{22}B_{11}} + \cancel{A_{11}B_{12}} - \cancel{A_{11}B_{22}} + \cancel{A_{21}B_{11}}$$

$$+ A_{21}B_{12} - \cancel{A_{11}B_{11}} - \cancel{A_{11}B_{12}}$$

$$\underline{\underline{C_{22} = A_{22}B_{22} + A_{21}B_{12}}}$$

b).

d): The algorithm multiplies 2x2 matrix, using seven scalar multiplies, and there is because nxn matrix can be divided into quadrants.

- Strassen's algorithm does this recursively, so each of the seven quadrants multiplications is computed recursively, and the computational cost of addition and subtraction of quadrants is n^2

- So the linear combination of products are computed in order to obtain the result with a size $n^2 = 4^m$.

- Thus, the recursion for the flop count is

$F(n) = 7F(n/2) + n^2$ with a base case

$F(1) = 1$, yielding $F(n) = (n^{\log_2 7})$

```
clear all
%close all

clc

for m =4:7
    n = 2^m;
    a = rand(n,n); b = rand(n,n);
    c_s = strass(a,b);
    C_re = a*b;
    diff =  norm(c_s) - norm(C_re)

end
fprintf('There is almost no difference in the two methods, due to very small error')

function c = strass(a,b)
nmin = 16;
[n,n] = size(a);
if n <= nmin;
c = a*b;
else
m = n/2; u = 1:m; v = m+1:n;
p1 = strass(a(u,u)+a(v,v),b(u,u)+b(v,v));
p2 = strass(a(v,u)+a(v,v),b(u,u));
p3 = strass(a(u,u),b(u,v)-b(v,v));
p4 = strass(a(v,v),b(v,u)-b(u,u));
p5 = strass(a(u,u)+a(u,v),b(v,v));
p6 = strass(a(v,u)-a(u,u),b(u,u)+b(u,v));
p7 = strass(a(u,v)-a(v,v),b(v,u)+b(v,v));
c = [p1+p4-p5+p7,p3+p5; p2+p4, p1-p2+p3+p6];
end
end
```

```
diff =

     0


diff =

     0


diff =

   4.5475e-13


diff =

   1.8190e-12

There is almost no difference in the two methods, due to very small error
```

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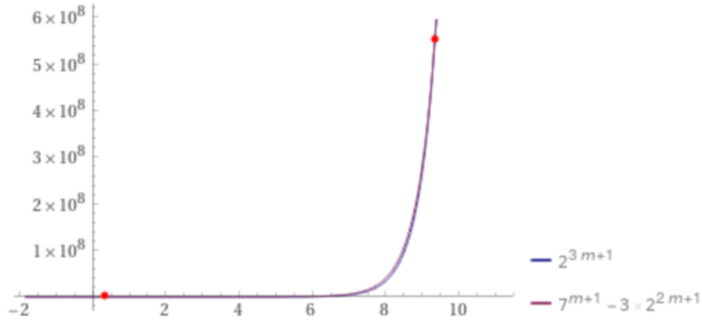
No. 7c

$$2^{3m+1} = 7^{m+1} - 6 \times 4^m$$

Plot:



Alternate forms:

$$2^{2m+1}\left(2^m + 3\right) = 7^{m+1}$$

$$7^{m+1} = 3 \times 2^{2m+1} + 2^{3m+1}$$

Number line:

Solutions: Exact forms More digits

$m \approx 0.361502$

$m \approx 9.34737$

Numerical solution: More digits

$m \approx 0.361501983573749\ldots$

- From above since m = 9, then n = 2^m = **512**.

How practical does Strassen's algorithm seem?

The algorithm seems practical for large values of m, which makes n very big.