In [1]:
```
%matplotlib notebook
%pylab
```

```
Using matplotlib backend: nbAgg
Populating the interactive namespace from numpy and matplotlib
```

Sturm Liouville Problems

We can extend the idea of a spectral solution we used in solving a symmetric matrix problem to that of solving differential operators whose solutions are in infinite dimensional space.

Consider the second order differential operator

$$\left[p(x)y'\right]' + \left[q(x) + \lambda r(x)\right] y = 0$$

on some interval $[a, b]$, with either *separated* boundary conditions

$$\begin{aligned} k_1 y(x) + k_2 y'(x) = 0, &\quad \text{at} \quad x = a \\ l_1 y(x) + l_2 y'(x) = 0, &\quad \text{at} \quad x = b \end{aligned}$$

or periodic boundary conditions $y(a) = y(b)$ and $y'(a) = y'(b)$, provided $p(a) = p(b)$.

This is a *boundary value problem* (since conditions are imposed at both a and b) and is called a "Sturm-Liouville* problem.

Eigenvectors and eigenvalue solutions

If we can write our Sturm-Liouville problem as

$$L[y] = \frac{1}{r(x)}\left(\left[p(x)y'\right]' + q(x)y\right) = -\lambda y$$

this problem can be viewed as an eigenvalue problems for eigenvalues λ and eigenfunctions $y(x)$.

Example.

$$y'' + \lambda y = 0$$

on $[0, \pi]$ with boundary conditions $y(0) = y(\pi) = 0$. The eigenvalues are $\lambda = \nu^2$ and the eigenfunctions are $\sin(\nu x)$, $\nu = \pm 1, \pm 2, \pm 3, \ldots$.

A key property of the solutions to this system is that they eigenfunctions corresponding to distinct eigenvalues are orthogonal.

Orthogonality of the solutions to Sturm-Liouville problems

Suppose that the functions $p(x)$, $q(x)$, $r(x)$ and $p'(x)$ are continuous and $r(x) > 0$ on an interval $[a, b]$. Let $y_n(x)$ correspond to eigenfunctions associated with eigenvalue λ_n of the Sturm-Liouville problem. Then the $y_n(x)$ form an orthogonal set with respect to the weight function $r(x)$. That is,

$$\langle y_n, y_m \rangle = \int_a^b y_m(x) y_n(x) r(x)\, dx = 0, \qquad m \neq n.$$

Furthermore, in analogy to the symmetric matrix case, the Sturm-Liouville operator is *self-adjoint*,

$$\langle L[u], v \rangle = \langle u, L[v] \rangle$$

where $\langle \cdot, \cdot \rangle$ is an inner product

$$\langle u, v \rangle = \int_a^b u(x) v(x) r(x)\, dx$$

Example : Fourier series

Trigonometric functions. Solutions to

$$y'' + \lambda y = 0$$

will be functions $\{\cos(nx), \sin(mx)\}$, $m, n = 0, 1, 2, \ldots$ on $[0, 2\pi]$. This form an orthogonal set relative to the weight function $r(x) = 1$.

Example : Legendre-Fourier series

Legendre polynomials. These are eigenfunctions of the Sturm-Liouville problem

$$\left[(1 - x^2) y'\right]' + \lambda y = 0$$

on $[-1, 1]$ where $\lambda = n(n + 1)$. The eigenfunctions are the Legendre polynomials $P_n(x)$. These are orthogonal with respect to the weight function $r(x) = 1$.

Note. The Legendre polynomials do not form an orthornomal set, since

$$\|P_n\|^2 = \langle P_n, P_n \rangle = \frac{2}{2n + 1}$$

These functions can be used in *Legendre-Fourier* series to write

$$f = \sum_{m=0}^{\infty} a_m P_m(x)$$

where

$$a_m = \frac{2m + 1}{2} \int_{-1}^{1} f(x) P_m(x)\, dx$$

Example : Bessel-Fourier series

Bessel functions $J_n(x)$ are solutions to

$$\left[xy'\right]' + \left(-\frac{n^2}{x} + \lambda x\right) y = 0,$$

The eigenfunctions are solutions $y(x) = J_n(kx)$, with associated eigenvalue $\lambda = k^2$. The $J_n(kx)$ are orthogonal and satisfy

$$\int_0^R xJ_n(k_{n,m}x)J_n(k_{n,j}x)\, dx = 0 \qquad (j \neq m).$$

where $k_{n.m}R$ is a zero of $J_n(x)$.

Chebyshev series

The Chebyshev polynomials are the eigenfunctions of the Chebyshev differential equation

$$\left[\sqrt{1-x^2}\,y'\right]' + \frac{n^2}{\sqrt{1-x^2}} y = 0, \qquad x \in [-1, 1]$$

The eigenfunctions are the *Chebyshev* polynomials $T_n(x)$, which form a complete orthogonal set with respect to the weight $1/\sqrt{1-x^2}$.

What next?

Using the above sets of orthogonal functions, we can developed generalized Fourier series.