

```
In [1]: %matplotlib notebook
        %pylab
```

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Using matplotlib backend: nbAgg
Populating the interactive namespace from numpy and matplotlib
```

Where are we now

Overview of material so far ...

Solution methods for second order ODEs.

- Power series methods
- **Method of Frobenius**
- Special functions : Airy functions, Legendre polynomials, Bessel functions

This material largely came from Kreyszig, Chapter 5.

Spectral theory for matrices

- Solve $A\mathbf{x} = \mathbf{b}$ using eigenvalue/eigenvector decomposition.
- Spectral Theorem for matrices (Theorem 1.5, Keener Page 16)
- Fredholm Alternative (Theorem 1.9, 1.10, Keener, Page 26).
- Key idea : If we can approximate vectors in an complete, orthogonal basis, life is easy.
- Orthogonal basis vectors and associated eigenvalues came from "operator" A . Then, to solve an actual problem, we expand \mathbf{b} in this orthogonal basis, and obtain a solution \mathbf{x} in this basis.

$$\mathbf{b} = \sum_{i=1}^n \langle \mathbf{b}, \mathbf{q}_i \rangle \mathbf{q}_i = \sum_{i=1}^n (\mathbf{b}^T \mathbf{q}_i) \mathbf{q}_i$$

- **Parseval's Identity** : $\|Q^T \mathbf{u}\|^2 = \|\mathbf{u}\|^2$. (generalization of Pythagorean theorem).
- Pseudo-inverse. What happens if we don't have a complete set of eigenvectors? Can we still obtain a solution?
- Projection onto $\text{Col}(A)$. Orthogonal spaces : $\text{Col}(A) \perp \text{Null}(A^T)$.

This material is largely covered in Keener, Section 1.2 - 1.5

Finite dimensional inner product spaces (\mathcal{R}^n)

- Inner product : $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T \mathbf{v}$.
- Norm : $\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{\mathbf{u}^T \mathbf{u}}$.
- Approximation in \mathcal{R}^n is about solving linear systems of equations. "Given a vector \mathbf{b} , how do we find a suitable \mathbf{x} ?" In many cases, we can find \mathbf{x} exactly. In other cases, we find the "best" \mathbf{x} in some sense.

Function spaces : Approximation in Hilbert spaces

- Infinite dimensional vector spaces. "Sequence spaces".
- Inner product spaces (definition of an inner product)
- Complete normed linear spaces (every Cauchy sequence converges in the space).
- $L^2[a, b]$: Function space containing all functions f for which L^2 norm is finite.
- **Hilbert space** : Complete inner product space with an induced norm (or "natural norm"). $L^2[a, b]$ is a motivating example.
- A notion of an **angle** in Hilbert space allows us to talk about "orthogonality". What is the angle between two functions in a Hilbert space?

$$\cos \theta = \frac{\langle \mathbf{f}, \mathbf{g} \rangle}{\|\mathbf{f}\| \cdot \|\mathbf{g}\|}$$

- **Approximation in Hilbert spaces** : Generalized Fourier series in terms of orthonormal basis functions.
- **Parseval's Identity** : Sum of squares of Fourier coefficients are equal to norm of the function (squared).
- **Weierstrass Approximation Theorem**. We can approximate any continuous function on $C[a, b]$ with a polynomial. Polynomials are dense in $L^2[a, b]$.
- Classical orthogonal sets of functions which we can use to approximate $f \in L^2[a, b]$.
- Ultimately, we want to solve problems for which we cannot obtain the exact solution analytically, so we approximate the solution by taking a finite number of terms in a series expansion.

This material is largely in Keener, Chapter 2.

Sturm-Liouville Theory

- Ties together solutions to ODEs done earlier in the course with orthogonal sets of functions needed for approximating in Hilbert space.
- Connects Spectral Theory in \mathcal{R}^n (symmetric matrices lead to real eigenvalues and orthogonal eigenvectors) to a spectral theory in infinite dimensional space (self-adjoint operators lead to real eigenvalues and orthogonal sets of eigenfunctions).
- We can find orthogonal sets of functions by solving Sturm-Liouville problems.
- One example of problems we can actually solve using function approximation in Hilbert space.

This material is in Kreyszig, Section 11.5 - 11.6. and in Keener Section 4.3 (Differential operators).

Approximation in Hilbert spaces

Suppose we want to approximate a given function f (assume in $L^2[a, b]$) by a sum of other functions which are easily obtained. What do we need?

- We need a complete set of orthonormal (or at least orthogonal) set of functions.
- Trigonometric series (ideal for periodic functions, but can also be use for non-periodic functions).
- Discrete Fourier Transforms
- Walsh functions (wavelet basis)
- Finite element spaces (use in numerical methods).

Material in Keener, 2.2.3

Trigonometric series

Use orthonormal set $\{\cos nx, \sin nx\}$, $n = 0, 1, \dots, \infty$. We will approximate functions on $[0, 2\pi]$.

- What is our goal here? Approximate $f(x)$ as infinite sum of sines and cosines

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

- Two functions are two linearly independent solutions to $y'' + n^2 y = 0$.

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx) \, dx, \quad n > 0$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) \, dx, \quad n > 0$$

$b_0 = 0$, and

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) \, dx$$

Best fit "least squares" approximation.

- Use a finite number of "basis functions" and project f onto these basis functions. Analogous to solving a least squares problem in \mathcal{R}^n .
- What is the general form that the series will take?

Fourier Convergence Theorem

Suppose $f(x)$ is piecewise continuous in $C^1[0, 2\pi]$. The Fourier series converges to

$$\frac{1}{2}[f(x^+) + f(x^-)].$$

for every x in $[0, 2\pi]$ (assuming we identify endpoints with their periodic counterpart).

Complex basis functions

General solution $y'' + n^2 y = 0$ can also be written in terms of complex basis functions as

$$\phi_n(x) = e^{inx}, \quad n = 0, \pm 1, \pm 2, \dots, \quad x \in [0, 2\pi]$$

with inner product given by

$$\langle \phi_n, \phi_m \rangle = \frac{1}{2\pi} \int_0^{2\pi} \bar{\phi}_n(x) \phi_m(x) dx$$

Note the "conjugate" $\bar{\phi}_n$ needed, since ϕ_n is complex.

We can show that

$$\langle \phi_n, \phi_m \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{-inx} e^{imx} dx = \delta_{mn}$$

By scaling x , we can obtain basis functions for a general interval $[0, L]$.

$$\phi_n(x) = e^{in(2\pi x/L)}, \quad x \in [0, L]$$

with inner product

$$\langle \phi_n, \phi_m \rangle = \frac{1}{L} \int_0^L \bar{\phi}_n(x) \phi_m(x) dx$$

A general $f(x)$ on $[0, L]$ can then be written in this basis as

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{in(2\pi x/L)}$$

with

$$a_n = \langle f, \phi_n \rangle = \frac{1}{L} \int_0^L f(x) e^{-in(2\pi x/L)} dx$$

To get the L -periodic extension of f , we evaluate $f(x)$ on the domain $-\infty < x < \infty$.

Example. What does the L -periodic extension of the function $f(x) = x$ look like in the trigonometric basis?

Fourier Transforms

From here, we can go in two different directions

1. Can we express f as a set of basis functions even if we only know f at a discrete set of points?
2. Can we extend this idea to functions which are not periodic? We already saw that we can get a L -periodic extension of any function $f(x)$ defined on $[0, L]$. But what if we want to recover f exactly?

Discrete Fourier Transform

If we only know $f(x)$ at a discrete set of points $f(x_j)$, we can still obtain representation of $f(x)$ in terms of trigonometric functions. In this case, the procedure for finding the coefficients resembles what we did in \mathcal{R}^n to solve a linear system.

Suppose we have n values $f(x_j)$. We wish to find an "interpolant"

$$p(x) = \sum_{k=0}^{n-1} a_k e^{ikx},$$

that agrees with our function $f(x)$ at n points x_j . To this end, we seek a_k such that

$$f_j \equiv f(x_j) = \sum_{k=0}^{\infty} a_k e^{ikx_j}, \quad x_j = \frac{2\pi j}{n}, \quad j = 0, 1, 2, \dots, n-1$$

It will be convenient to think of \mathbf{f} as a vector whose components are f_j , and the $\phi_{(k)}$ as a vector whose components are $\phi_{(k),j} = e^{ikx_j}$. We then have

$$\mathbf{f} = \sum_{k=0}^{n-1} a_k \phi_{(k)}$$

The $\phi_{(k)}$ then satisfy an orthogonality condition which we can write in terms of the usual inner product in \mathcal{R}^n .

$$\langle \phi_{(k)}, \phi_{(\ell)} \rangle = \frac{1}{n} \phi_{(k)}^* \phi_{(\ell)} = \delta_{k\ell}$$

where $\phi_{(m)}^*$ is the Hermitian transpose.

We can then compute the a_k as

$$a_k = \langle \mathbf{f}, \phi_{(k)} \rangle = \frac{1}{n} \sum_{j=0}^{n-1} f_j \bar{\phi}_{(k),j} = \frac{1}{n} \sum_{j=0}^{n-1} f_j e^{-ikx_j}$$

If we think of the $\phi_{(k)}$ as columns of an orthogonal matrix Q , we obtain the a_k by multiplying \mathbf{f} by a matrix Q^* .

This transform is what is known as the "Discrete Fourier Transform" (DFT). Variants of this include the "Discrete Sine Transform" (DST) and the "Discrete Cosine Transform" (DCT).

The Fast Fourier Transform (FFT)

This might seem like the end of the story, but for large n , a matrix-vector multiply requiring $\mathcal{O}(n^2)$ calculations can be prohibitively expensive. Using a key observation about the nature of the basis functions, we can reduce the operation count of this matrix-vector multiply to $n \log_2(n)$ from $\mathcal{O}(n^2)$.

The algorithm for obtaining this reduction is the "Fast Fourier Transform". The most popular variant of this algorithm, and the most widely used is that by Cooley and Tukey, "[An algorithm for the machine calculation of complex Fourier series](https://www.ams.org/journals/mcom/1965-19-090/S0025-5718-1965-0178586-1/home.html)" (<https://www.ams.org/journals/mcom/1965-19-090/S0025-5718-1965-0178586-1/home.html>) (1965). This paper is barely five pages!

The FFT is cited by the ACM (Associated of Computing Machinery) as one of the top 10 Algorithms of the 20th Century.

See Krevszia. Chapter 11.

Fourier Transforms - to ∞ and beyond

Suppose f is absolutely integrable over \mathcal{R} . Then we can consider a limit $L \rightarrow \infty$. When we do this, infinite sums become integrals, and we pass from a discrete spectrum with a discrete set of eigenvalue/eigenfunction to a *continuous spectrum*. In this case, we write $f(x)$ as

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} dw \quad (\text{infinite sum replaced by an integral})$$

where now, the continuous variable w replaces the integer variable n in our earlier sum. Our coefficients a_n are replaced with a single function of w to get

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx \quad (\text{sequence } \{a_n\} \text{ replaced by continuous function } \hat{f}(w))$$

The expression for $\hat{f}(w)$ still looks like an inner product, but we no longer think of "projecting onto a set of orthogonal functions".

- The term "Fourier Transform" $\mathcal{F}[f]$ is used to refer to the functional that computes $\hat{f}(w)$ from $f(x)$. The expression "transform to Fourier space" or transform to the "frequency domain" is often used.
- The inverse Fourier Transform $\mathcal{F}^{-1}[f]$ is the functional that computes $f(x)$ from $\hat{f}(w)$.

The function $\hat{f}(w)$ is viewed in some sense as equivalent to $f(x)$, because both are functions of a continuous variable. Tables of Fourier Transforms give us transforms of many well know functions.

Example. The Fourier Transform of the function $f(x) = 1, |x| < 1$ and zero otherwise, is given by

$$\mathcal{F}[f(x)] = \sqrt{\frac{\pi}{2}} \frac{\sin w}{w}$$

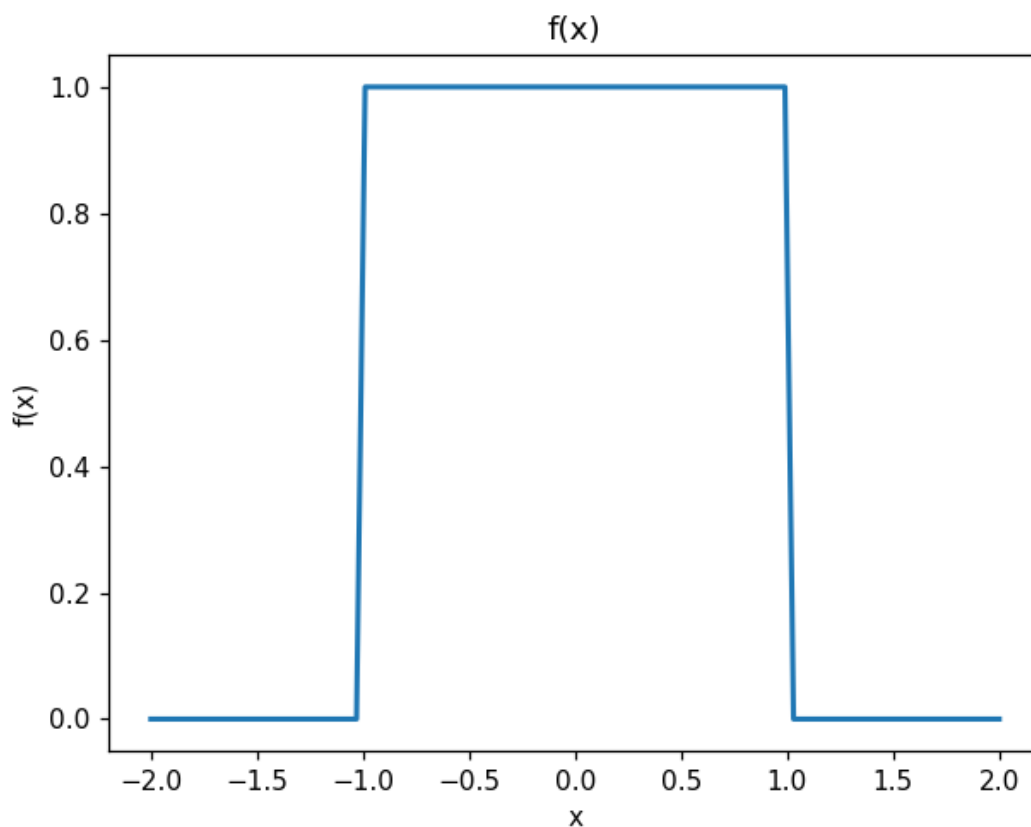
```
In [12]: figure(1)
         clf()

         x = linspace(-2,2,100)

         f = where(abs(x) <= 1, 1, 0)

         plot(x,f,linewidth=2)

         title('f(x)')
         xlabel('x')
         ylabel('f(x)')
```



```
Out[12]: Text(0, 0.5, 'f(x)')
```



```

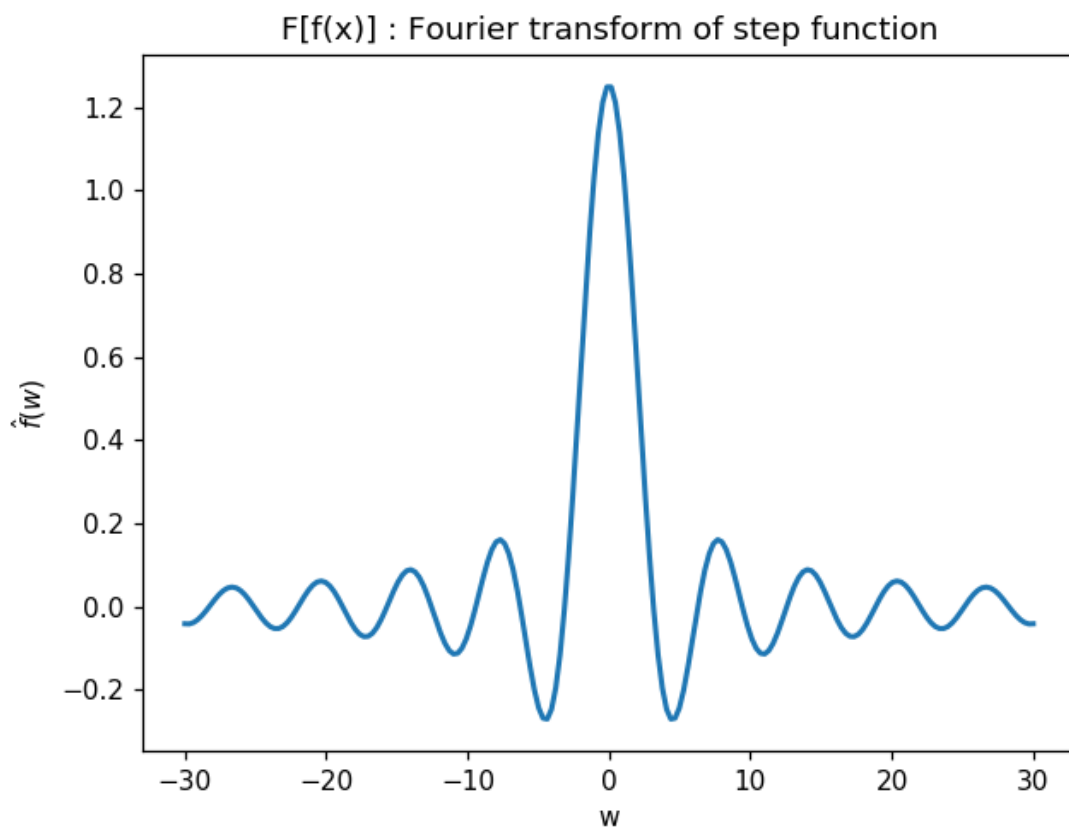
In [14]: figure(2)
         clf()
         w = linspace(-30,30,200)

         fhat = lambda w : sqrt(pi/2)*sin(w)/w

         plot(w,fhat(w),linewidth=2)

         title('F[f(x)] : Fourier transform of step function')
         xlabel('w')
         ylabel('$\hat{f}(w)$')

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Out[14]: Text(0, 0.5, '$\hat{f}(w)$')

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In [ ]:

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