Brian KYANJO Homework #2

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5a)

$$(a+b)^n = \frac{a^n b^0}{0!} + \frac{n a^{n-1} b^1}{1!} + \frac{n(n-1) a^{n-2} b^2}{2!} + \frac{n(n-1)(n-2) a^{n-3} b^3}{3!}$$

$$+ \frac{n(n-1)(n-2)(n-3) a^{n-4} b^4}{4!} + \cdots$$

$$(1-v)^{-1/2} = 1 - \frac{1}{2}(-v) - \frac{\frac{1}{2}\left(-\frac{1}{2}-1\right)(-v)^2}{2!} - \frac{\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)(-v)^3}{3!}$$

$$- \frac{\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\left(-\frac{1}{2}-3\right)(-v)^4}{4!} + \cdots$$

$$(1-v)^{-1/2} = 1 + \frac{1}{2}v + \frac{3}{8}v^2 + \frac{5}{16}v^3 + \frac{35}{128}v^4 + \cdots$$ 4

Set $v = 2xu - u^2$

$$(1-2xu+u^2)^{-1/2} = 1 + \frac{1}{2}(2xu-u^2) + \frac{3}{8}(2xu-u^2)^2$$

$$+ \frac{5}{16}(2xu-u^2)^3 + \frac{35}{128}(2xu-u^2)^4 + \cdots$$

$$= 1 + xu - \frac{1}{2}u^2 + \frac{3}{8}u^4 - \frac{3}{2}u^3x + \frac{3}{2}u^2x^2 - \frac{5}{6}u^6 +$$

$$\frac{15}{8}u^5x - \frac{15}{4}u^4x^2 + \frac{5}{2}u^3x^3 + \frac{35}{128}u^8 - \frac{35}{16}u^7x + \frac{105u^6x^2}{16}$$

$$- \frac{35}{4}u^5x^3 + \frac{35}{8}u^4x^4 + \cdots$$

$$(1-2xu+u^2)^{-1/2} = 1u^0 + xu^1 + \left(-\tfrac{1}{2} + \tfrac{3}{2}x^2\right)u^2 +$$

$$\left(-\tfrac{3}{2}x + \tfrac{5}{2}x^3\right)u^3 + \left(-\tfrac{15}{4}x^2 + \tfrac{35}{8}x^4 + \tfrac{3}{8}\right)u^4 + \cdots$$

Looking at the coefficients of u

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \left(-\tfrac{1}{2} + \tfrac{3}{2}x^2\right) = \tfrac{1}{2}(3x^2 - 1)$

$P_3(x) = \tfrac{1}{2}(5x^3 - 3x)$

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$P_4(x) = -\tfrac{15}{4}x^2 + \tfrac{35}{8}x^4 + \tfrac{3}{8} = \tfrac{1}{8}(35x^4 - 30x^3 + 3)$

$\vdots \qquad \vdots$

<u>To get higher order terms</u>

$$G(u,x) = \frac{1}{\sqrt{1-2xu+u^2}} = \sum_{n=0}^{\infty} P_n(x)\,u^n$$

Derive $P_{n+1}(x)$ in terms of $P_n(x)$ and $P_{n-1}(x)$

$$\frac{d}{du}G(u,x) = \frac{d}{du}(1-2xu+u^2)^{-1/2} = \sum_{n=1}^{\infty} n\,P_n(x)\,u^{n-1}$$

$$G'(u,x) = \frac{(x-u)}{(1-2xu+u^2)^{3/2}} = \sum_{n=1}^{\infty} n\,P_n(x)\,u^{n-1}$$

$$\frac{x-u}{(1-2xu+u^2)^{1/2}} = (1-2xu+u^2)\sum_{n=1}^{\infty} n\,P_n(x)\,u^{n-1}$$

$$\frac{(x-u)}{\sqrt{1-2xu+u^2}} = \sum_{n=1}^{\infty} n P_n(x) u^{n-1} - 2\sum_{n=0}^{\infty} x n P_n(x) u^n + \sum_{n=0}^{\infty} n P_n(x) u^{n+1}$$

but $$\frac{1}{\sqrt{1-2xu+u^2}} = \sum_{n=0}^{\infty} P_n(x) u^n$$

$$(x-u)\sum_{n=0}^{\infty} P_n(x) u^n = \sum_{n=1}^{\infty} n P_n(x) u^{n-1} - \sum_{n=0}^{\infty} 2xn P_n(x) u^n + \sum_{n=0}^{\infty} n P_n(x) u^{n+1}$$

$$\sum_{n=0}^{\infty}(2n+1) x P_n(x) u^n - \sum_{n=0}^{\infty} P_n(x) u^{n+1} = \sum_{n=1}^{\infty} n P_n(x) u^{n-1}$$

$$+ \sum_{n=0}^{\infty} n P_n(x) u^{n+1}$$

$$\sum_{n=0}^{\infty}(2n+1) x P_n(x) u^n - \sum_{n=1}^{\infty} n P_{n-1}(x) u^n = \sum_{n=0}^{\infty} n P_n(x) u^{n-1}$$

$$\sum_{n=0}^{\infty}(2n+1) x P_n(x) u^n - \sum_{n=0}^{\infty} n P_{n-1}(x) u^n = \sum_{n=0}^{\infty} (n+1) P_{n+1}(x) u^n$$

$$\underline{\underline{(2n+1) x P_n(x) - n P_{n-1}(x) = (n+1) P_{n+1}}} \quad \square$$

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Using $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$
verify up to $n=6$

Set $P_0(x) = 1$, $P_1(x) = x$

$n=1$: $2P_2(x) = 3xP_1(x) - P_0(x) =$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$n=2$: $3P_3(x) = 5xP_2(x) - 2P_1(x)$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$n=3$: $4P_4(x) = 7xP_3(x) - 3P_2(x)$

$$P_4(x) = \frac{1}{4}\left(7x\left(\frac{1}{2}(5x^3-3x)\right) - 3\left(\frac{1}{2}(3x^2-1)\right)\right)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$n=4$: $5P_5(x) = 9xP_4(x) - 4P_3(x)$

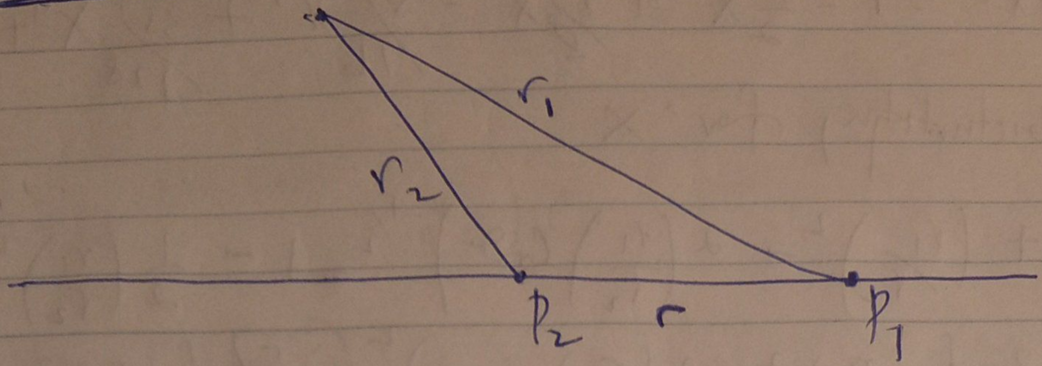
$$P_5(x) = \frac{1}{5}\left(9x\left(\frac{1}{8}(35x^4-30x^2+3)\right) - 4\left(\frac{1}{2}(5x^3-3x)\right)\right)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

$n=5$: $6P_6(x) = 11xP_5(x) - 5P_4(x)$

$$P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$$

Q10.5(b).



~~$r^2 = r_1^2$~~ using Cosine rule

$$r^2 = r_1^2 + r_2^2 - 2r_1r_2\cos\theta$$

$$\frac{1}{r} = \frac{1}{\sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\theta}}$$

$$\frac{1}{r} = \frac{1}{r_2}\left(1 + \left(\frac{r_1}{r_2}\right)^2 - 2\left(\frac{r_1}{r_2}\right)\cos\theta\right)^{-1/2}$$

Let $x = \left(\frac{r_1}{r_2}\right)^2 - 2\left(\frac{r_1}{r_2}\right)\cos\theta$, $n = -1/2$

then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \cdots$$

$$(1+x)^{-1/2} = 1 - \frac{1}{2}x - \frac{\frac{1}{2}\left(-\frac{1}{2}-1\right)}{2!}x^2 - \frac{\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}x^3$$

$$+ \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\left(-\frac{1}{2}-3\right)}{4!}x^4 + \cdots$$

$$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4 + \cdots$$

Substituting for x

$$\left(1 + \left(\frac{r_1}{r_2}\right)^2 - 2\left(\frac{r_1}{r_2}\right)\cos\theta\right)^{-1/2} = 1 - \frac{1}{2}\left(\frac{r_1}{r_2}\right)^2 + \left(\frac{r_1}{r_2}\right)\cos\theta$$

$$+ \frac{3}{8}\left(\left(\frac{r_1}{r_2}\right)^2 - 2\left(\frac{r_1}{r_2}\right)\cos\theta\right)^2 - \frac{5}{16}\left(\left(\frac{r_1}{r_2}\right)^2 - 2\left(\frac{r_1}{r_2}\right)\cos\theta\right)^3$$

$$+ \cdots$$

$$\left(1+\left(\frac{r_1}{r_2}\right)^2 - 2\left(\frac{r_1}{r_2}\right)\cos\theta\right)^{-1/2} = 1 - \frac{1}{2}\left(\frac{r_1}{r_2}\right)^2 + \left(\frac{r_1}{r_2}\right)\cos\theta$$

$$+ \frac{3}{8}\left(\left(\frac{r_1}{r_2}\right)^4 + 4\left(\frac{r_1}{r_2}\right)^2\cos^2\theta - 4\left(\frac{r_1}{r_2}\right)^3\cos\theta\right) -$$

$$\frac{5}{16}\left(\left(\frac{r_1}{r_2}\right)^6 - 8\left(\frac{r_1}{r_2}\right)^3\cos^3\theta + 3\left(\frac{r_1}{r_2}\right)^4\left(-2\left(\frac{r_1}{r_2}\right)\cos\theta\right)\right.$$

$$\left. + 3\left(\frac{r_1}{r_2}\right)^2\left(-2\left(\frac{r_1}{r_2}\right)\cos\theta\right)^2\right) + \cdots$$

$$= 1 - \frac{1}{2}\left(\frac{r_1}{r_2}\right)^2 + \left(\frac{r_1}{r_2}\right)\cos\theta + \frac{3}{8}\left(\frac{r_1}{r_2}\right)^4 + \frac{3}{2}\left(\frac{r_1}{r_2}\right)^2\cos^2\theta$$

$$- \frac{3}{2}\left(\frac{r_1}{r_2}\right)^3\cos\theta - \frac{5}{16}\left(\frac{r_1}{r_2}\right)^6 + \frac{5}{2}\left(\frac{r_1}{r_2}\right)^3\cos^3\theta +$$

$$\frac{15}{8}\left(\frac{r_1}{r_2}\right)^5\cos\theta + \frac{15}{4}\left(\frac{r_1}{r_2}\right)^4\cos\theta + \cdots$$

$$= \frac{r}{r_2}\left(1 + \left(\frac{r_1}{r_2}\right)^2 - 2\left(\frac{r_1}{r_2}\right)\cos\theta\right)^{-1/2} = \frac{1}{r_2}\left(1 + \left(\frac{r_1}{r_2}\right)\cos\theta + \right.$$

$$\frac{1}{2}(3\cos^2\theta - 1)\left(\frac{r_1}{r_2}\right)^2 + \frac{1}{2}(5\cos^3\theta - 3\cos\theta)\left(\frac{r_1}{r_2}\right)^3 +$$

$$+\frac{1}{8}\left(35\cos^4\theta-30\cos^2\theta+3\right)\left(\frac{r_1}{r_2}\right)^4+\cdots\Big)$$

$$=\frac{1}{r_2}\Big(P_0(\cos\theta)\left(\frac{r_1}{r_2}\right)^0+P_1(\cos\theta)\left(\frac{r_1}{r_2}\right)+P_2(\cos\theta)\left(\frac{r_1}{r_2}\right)^2$$

$$+P_3(\cos\theta)\left(\frac{r_1}{r_2}\right)^3+P_4(\cos\theta)\left(\frac{r_1}{r_2}\right)^4+\cdots\Big)$$

$$=\frac{1}{r_2}\left(\sum_{n=0}^{\infty}P_n(\cos\theta)\left(\frac{r_1}{r_2}\right)^n\right)$$

$$\frac{1}{r}=\frac{1}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos\theta}}=\frac{1}{r_2}\sum_{n=0}^{\infty}P_n(\cos\theta)\left(\frac{r_1}{r_2}\right)^n$$

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c). A point charge in space creates an electric field given by $\vec{E}=\frac{k\vec{q}}{r^2}$, where $k=9\times10^9$, and q is the charge ~~at that~~ at that point. $\vec{E}$ is a vector field around it, so considering a Gaussian or any surface about this point charge, the generating function above helps us to acess charge ~~at~~ in special values e.g $x=\pm1$. This function can enable us to obtain charge even between limits $-1\leq x\leq1$ e.g $|P_n(\cos\theta)|\leq1$, therefore the solution for the generating function is convergent for $|r|<1$ and $|r|=1$ except for $r=\pm1$, when $|P_n(\pm1)|=1$. With the equation above, ~~where~~ potential at 2 that point can be evaluated explicitly at

Special values.