Homework #8

Math 537

This assignment is designed to give you a feel for some of the issues in potential theory, and using free-space Green's functions to solve elliptic problems.

In the following, $H(x)$ is the Heaviside function, and $R(x) = xH(x)$ is the "ramp" function.

1. **Harmonic functions in 1d.** The follow problems are all aimed at solving

$$u''(x) = 0, \qquad 0 \le x \le 1 \tag{1}$$

subject to $u(0) = a$, $u(1) = b$.

We start by solving the free-space Green's function, and then add an "image" source to obtain Green's function satisfying homogeneous boundary conditions.

(a) Show that

$$G_0(x, y) = \frac{1}{2}|x - y| \tag{2}$$

is a fundamental solution (or "free-space" Green's function) for the operator $L[u] = \frac{d^2u}{dx^2}$. Use the following steps.

- Integrate $u''(x) = \delta(x)$ twice to get a function $u(x)$. Then define $G_0(x, y) \equiv u(x - y)$.
- Determine constants of integration C_0 and C_1 so that the resulting function $G_0(x, y)$ satisfies $G_0(x, y) = G_0(y, x)$.

You may use the distributional derivatives $H'(x) = \delta(x)$ and $R'(x) = H(x)$. See above for definitions of $H(x)$ and $R(x)$.

(b) Use the method of images to find an "image" source $\overline{G}_0(x, y) \equiv \rho G_0(x, \overline{y})$ so that

$$G(x, y) = G_0(x, y) + \overline{G}_0(x, y) \tag{3}$$

satisfies $u''(x) = \delta(x - y)$ and boundary conditions $u(0) = u(1) = 0$. The value $\overline{y}$, which depends on y, is located outside of the domain $[0, 1]$.

- Verify that the function $\overline{G}_0(x, y)$ is harmonic in $[0, 1]$. **Hint:** Show that $\overline{y}$ always remains outside the domain $[0, 1]$.
- Show that $G(x, y)$ as expressed as in (3) is the same 1d Green's function we have seen in class and previous homework.

(c) Verify Green's Second Identity for a harmonic function $u(x)$ satisfying (1) and the Green's function $v(y) = G(x, y)$ you found using the method of images.

Green's Second Identity in 1d :

$$\int_0^1 (u(y)v''(y) - v(y)u''(y))dy = (u(y)v'(y) - v(y)u'(y))\Big|_{y=0}^{y=1} \tag{4}$$

Hint: Show that you obtain the harmonic solution $u(x) = a(1 - x) + bx$.

(d) Verify Green's Second Identity using the free-space Green's function $G_0(x, y) = \frac{1}{2}|x - y|$ and a harmonic function $u(x)$ satisfying boundary conditions given in (1). **Hint:** Use the fact that you know what $u'(0) = u'(1)$ are for the 1d harmonic function. You should get that $u(x) = a(1-x)+bx$.

(e) For a general Poisson problem $u''(x) = f(x)$ on $[0, 1]$, use Green's identity to write the solution as a harmonic function plus a volume integral.

- Verify that your formulation holds for the function $u(x) = e^x$, subject to boundary conditions $u(0) = 1$, $u(1) = e$.

2. **Potentials due to sources and dipoles distributions.** In this set of problems, we will solve the Dirichlet problem by expressing the solution as a potential resulting from a distribution of sources and dipoles.

   A dipole is the potential that results from solving the free-space problem

$$u''(x) = -\delta'(x) \tag{5}$$

   where

$$-\delta'(x) = \lim_{\varepsilon \to 0} \frac{\delta(x - \frac{\varepsilon}{2}) - \delta(x + \frac{\varepsilon}{2})}{\varepsilon} \tag{6}$$

   (a) Verify (6) using the representation of the delta function as

$$\delta(x) \approx \frac{1}{2\sqrt{\pi\alpha}} e^{-x^2/4\alpha} \tag{7}$$

   for small α.

   (b) Use the 1d free-space Green's function $G_0(x, y) = \frac{1}{2}|x-y|$ to obtain a dipole potential by evaluating

$$\lim_{\varepsilon \to 0} \frac{\frac{1}{2}|x - \frac{\varepsilon}{2}| - \frac{1}{2}|x + \frac{\varepsilon}{2}|}{\varepsilon} \tag{8}$$

   Show that the limiting configuration you obtain is equal to $-\frac{\partial G_0(x,0)}{\partial y}$.

   (c) Show that if we try to represent the solution as a distribution of dipoles on the boundary of the interval $[0, 1]$, we can only capture constant solutions to (1).

   - Represent the solution to (1) as a linear combination of two dipoles, i.e.

$$u(x) = \mu(1) \left(-\frac{\partial G_0(x, 1)}{\partial y} \right) - \mu(0) \left(-\frac{\partial G_0(x, 0)}{\partial y} \right) \tag{9}$$

   - Impose boundary conditions at $x = 0$ and $x = 1$ by taking limits as you approach the boundary from within the interval $[0, 1]$ and show that the resulting 2×2 system for coefficients $\mu(0)$ and $\mu(1)$ is solvable only if $a = b$.

   (d) If we add an additional source term at a location outside the domain, we can construct a non-singular system for the unknown dipole strengths. Suppose we add a source term at $y = 2$. Then the proposed solution to (1) is represented as

$$u(x) = \mu(1) \left(-\frac{\partial G_0(x, 1)}{\partial y} \right) - \mu(0) \left(-\frac{\partial G_0(x, 0)}{\partial y} \right) + \mu(2) G_0(x, 2). \tag{10}$$

   - Show that the above representation leads to a 3×3 non-singular system.
   - Solve for $\mu(0)$, $\mu(1)$ and $\mu(2)$ by imposing the boundary conditions $u(0) = 1$ and $u(1) = 3$, and using the additional constraint $\mu(2) = \mu(1) - \mu(0)$.
   - Show that at the boundary points $x = 0$ and $x = 1$, the resulting solution has a jumps equal to $\mu(0)$ and $\mu(1)$, respectively, but that the derivative is continuous across the boundary.
   - What is the behavior of the solution at the source at $y = 2$?
   - Sketch a plot of the resulting solution over the interval $[-1, 3]$.

Hint: By "jump" in a function, we mean the difference between the function as we approach a boundary point from within the domain and the value as we approach the same boundary point from outside the domain. The jump in a function $u(x)$ at $x = 0$ can be computed as $u(0^+) - u(0^-)$, where the "+" means take the limiting value from within the domain $[0, 1]$, and "-" means take the limit from outside the domain. For example, for $u(x) = H(x)$, we have $u(0^+) = 1$, and $u(0^-) = 0$. For $u(x) = H(x - 1)$ though, we have $u(1^+) = 0$, and $u(1^-) = 1$

The jump in the derivative $u'(x)$ can be computed in an analogous fashion.

(e) Solve the problem

$$u''(x) = e^x, \qquad x \in [0, 1] \tag{11}$$

subject to $u(0) = 1$, $u(1) = e$ using a volume integral plus a distribution of sources and dipoles. Show that the solution has the correct jump behavior at $x = 0$ and $x = 1$.

Note: The fact that a pure dipole distribution is singular for this problem is a pecularity of the 1d case. In two and three dimensions, a dipole distribution in a simply connected domain would lead to a very well-conditioned system. In the multiply-connected domains in higher dimensions, however, additional sources outside the domain are also required to construct non-singular systems.

3. **Green's function for the disk.** Consider two points $P = (r, \theta)$ and $Q = (\rho, \theta')$ in the interior of a unit disk, and a point $\bar{Q} = (1/\rho, \theta')$ outside the disk. Let the distance PQ be denoted r_{PQ} and the distance $P\bar{Q}$ be denoted $r_{P\bar{Q}}$. Using the "method of images", we can show that the Green's function $G(P, Q)$ for the disk is given as the sum of two source potentials,

$$G(P, Q) = \frac{1}{2\pi} \log\left(\frac{r_{PQ}}{\rho r_{P\bar{Q}}}\right) = \frac{1}{2\pi} \log\left(r_{PQ}\right) - \left(\frac{1}{2\pi} \log\left(r_{P\bar{Q}}\right) + \frac{1}{2\pi} \log\left(\rho\right)\right) \tag{12}$$

- Derive the above expression for $G(P, Q)$.
- Show that $G(P, Q) = 0$ on the boundary of the disk.

Hint : Use the Law of Cosines.