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Homework #4
Math 537

1. Spectral approach

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Eigen value of A

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\left|\begin{array}{ccc|cc} 1-\lambda & 1 & 1 & 1-\lambda & 1 \\ 1 & 1-\lambda & 1 & 1 & 1-\lambda \\ 1 & 1 & 1-\lambda & 1 & 1 \end{array}\right| = 0$$

$$\left((1-\lambda)^3 + 1 + 1\right) - \left((1-\lambda) + (1-\lambda) + (1-\lambda)\right) = 0$$

$$(1-\lambda)^3 + 3\lambda - 1 = 0$$

$$-\lambda^3 + 3\lambda^2 = 0 \quad \Rightarrow \lambda = 0, 0, 3$$

Eigen vectors Corresponding to each eigen value

$$Av = \lambda v, \quad \text{let } v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$x + y + z = \lambda x$$

for $\lambda = 0 \Rightarrow x + y + z = 0$

$$z = -x - y$$

$$v_1 = \begin{bmatrix} x \\ y \\ -x-y \end{bmatrix}$$

If $x = -1$, $y = 0 \Rightarrow v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$$(A - \lambda I) v_2 = v_1$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$x + y = -z$$

If $z = 0$, $\Rightarrow x = -y \Rightarrow$ If $y = 1$, $x = -1$

$$v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

for $\lambda = 3$

$$-\left|\begin{array}{l} y + z = 2x \quad -(1) \\ x + z = 2y \quad -(2) \end{array}\right.$$

$$3y = 3x \Rightarrow y = x$$

Putting $y = x$ into (1) $\Rightarrow z = y$

$$V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow V_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$P = Q \Delta Q^T$$

$$Q = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{3} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\Delta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$P = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & \frac{1}{\sqrt{2}} & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & \frac{1}{\sqrt{3}} & 1/\sqrt{3} \end{bmatrix}$$

$$P = 1/9 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

b) Non-Spectral approach.

$$X = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} + Ny$$

$$N = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Compute the minimum norm solution X as Pb

$$X = Pb$$

$$X = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(b) Non-spectral approach

$$X = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} + Ny$$

(i) $$N = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

To Project vector b onto the Column Space of A

Since we obtained linear independent set of vectors that span column space A.

If b project onto ~~the~~ an eigen vector ~~Q~~ in the column space of A corresponding to non zero eigen value, $\lambda = 3$,

$$Q_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\cancel{Q_3^T A x_1 = Q_3^T b}$$

$$\left[\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right] x_1 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$3x_1 = 1 \Rightarrow x_1 = \underline{\underline{1/3}}$$

(ii) $$X = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} + Ny$$

$$N = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$X = \begin{bmatrix} 1/3 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$x = \begin{bmatrix} 1/3 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} -y_1 - y_2 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} + \frac{1}{\sqrt{2}}(-y_1 - y_2) \\ y_1/\sqrt{2} \\ y_2/\sqrt{2} \end{bmatrix}$$

$$\|x\|^2 = \left(\frac{1}{3} + \frac{1}{\sqrt{2}}(-y_1 - y_2)\right)^2 + \left(\frac{y_1}{\sqrt{2}}\right)^2 + \left(\frac{y_2}{\sqrt{2}}\right)^2$$

$$\|x\|^2 = \left(\frac{1}{3} + \frac{1}{\sqrt{2}}(-y_1 - y_2)\right)^2 + y_1^2/2 + y_2^2/2$$

let $f(y_1, y_2) = \|x\|^2$

$$f(y_1, y_2) = \left(\frac{1}{3} + \frac{1}{\sqrt{2}}(-y_1 - y_2)\right)^2 + y_1^2/2 + y_2^2/2$$

at minimum $\frac{\partial f(y_1, y_2)}{\partial y_1} = \frac{\partial f(y_1, y_2)}{\partial y_2} = 0$

$$\frac{\partial f(y_1, y_2)}{\partial y_1} = 2\left(\frac{1}{3} - \frac{1}{\sqrt{2}}(y_1 + y_2)\right)\left(-\frac{1}{\sqrt{2}}\right) + y_1 = 0$$

$$y_1 - 2/3\sqrt{2} + y_1 + y_2 = 0$$

$$2y_1 + y_2 = 2/3\sqrt{2} \quad \text{—— (1)}$$

$$\partial f/\partial y_2 = 2\left(\frac{1}{3} - \frac{1}{\sqrt{2}}(y_1 + y_2)\right)\left(-\frac{1}{\sqrt{2}}\right) + y_2 = 0$$

$$2y_2 + y_1 = \frac{2}{3\sqrt{2}} \quad \text{--- } ②$$

① − ②

$$(2y_1 + y_2) - 2y_2 - y_1 = 0$$

$$y_1 = y_2$$

Into ①

$$3y_1 = \frac{2}{3\sqrt{2}} \Rightarrow y_1 = \frac{2}{9\sqrt{2}} = y_2$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/9 \\ \sqrt{2}/9 \end{bmatrix}$$

Report the minimum norm solution.

So

$$X = \begin{bmatrix} \frac{1}{3} + \frac{1}{\sqrt{2}}(-y_1 - y_2) \\ y_1/\sqrt{2} \\ y_2/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} + \frac{1}{\sqrt{2}}(-2y_1) \\ y_1/\sqrt{2} \\ y_1/\sqrt{2} \end{bmatrix}$$

$$X = \begin{bmatrix} 1/9 \\ 1/9 \\ 1/9 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Find an algebraic expression for the pseudo-inverse

$AX = I,$

$X = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} + Ny, \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- Since I is 3 by 3, then X will also be 3x3 matrix.
- Same are to project the canonical $\overset{(3\times1)}{\text{basis}}$ vectors: $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ onto the Column space of A.

$(1\ 1\ 1)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} x_1 = (1\ 1\ 1)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x_1 = 1/3$

thus x_1 corresponds to $X = 1/9 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as in (b)

$(1\ 1\ 1)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} x_2 = (1\ 1\ 1)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow x_2 = 1/3$

thus x_2 corresponds to $X = 1/9 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as in b)

$(1\ 1\ 1)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} x_3 = (1\ 1\ 1)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow x_3 = 1/3$

Again x_3 corresponds to $x = \frac{1}{9}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as in (b)

thus for $X = PI$

$$X = \begin{bmatrix} x_1 & | & x_2 & | & x_3 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

So

$$P = \begin{bmatrix} P_r \\ 0 \end{bmatrix} + Ny$$

where $y = \operatorname{argmin} \frac{1}{2}\|p\|^2$

$$P = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + Ny$$

2. Both use the Null space of the matrix in question, however the Spectral approach is more straight forward to deal with

- One non zero eigen value with its corresponding eigen vector is enough for us to ~~a~~ to ~~op~~ obtain the vector to project on A.

- For non-spectral method, A must be a rank one matrix.

- But in both the Eigen vectors must be linearly independent.