Potential Theory in Multiply Connected Domains

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Outline

Introduction

Potential Theory

This the basically the study of harmonic function and its properties as a function according to mathematical physics (Garabedian and Schiffer, 1950).

Dirichlet problem

This is the problem of finding a function which solves a specified PDE in the interior of a given region that takes prescribed values on the boundary of the region.

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Multiple Connected Domains

Consider the Dirichlet problem in a finite open region D in the plane which is $(M+1)$-ply connected.

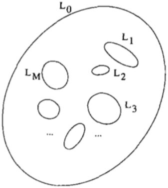

Figure: A bounded multiply connected domain D in the plane. (Greenbaum et al., 1993)

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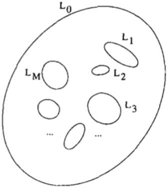

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In multiply connected domains, it is not so simple to reduce the cost of each iteration to $O(NlogN)$ or $O(N)$. Since the net cost of solving a linear system is $O(N^2)$.

Second Kind integral equation methods for the Dirichlet problem develop a nullspace of dimensions equal to the number of bondary components M.

The solution is obtained by:

- Projecting the Dirichlet data onto the range of operators (Subjecting data to certain capability constraints).
- Using a modified integral equation.

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Interior and Exterior Problems in 2D

Consider a finite open region D in the plane with boundary L, which assumes to be smooth and to have continuous curvature.

The Interior Dirichlet problem involves the determination of a function $U(P)$ that satisfies;

$$\Delta U(P) = 0 \qquad \text{for} \qquad P \in D$$

with boundary condition

$$\lim_{\substack{P \to Q \\ P \in D}} U(P) = f(Q) \qquad \text{for} \qquad Q \in L$$

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Denote E to be the open region in the plane exterior to L.
The exterior Dirichlet problem is defined by:

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The double layer potential solution $(U(P))$:

$$U(P) = \frac{1}{2\pi} \int_L \mu(Q) \frac{\partial}{\partial V_Q} \ln |Q - P| dQ$$

where P is a point inside D, $\mu(Q)$ is the value value of the unknown dipole distribution at boundary point Q. $\frac{\partial}{\partial V_Q}$ is the outward normal derivative at the point Q.

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For any point Q_o on the boundary $U(P)$ satisfies the jump relation:

$$\lim_{\substack{P\to Q_o\\ P\in D}} U(P) = \frac{1}{2}\mu(Q_o) + \frac{1}{2\pi}\int_L \mu(Q)\frac{\partial}{\partial V_Q}\ln|Q-Q_o|dQ \tag{1}$$

$$\lim_{\substack{P\to Q_o\\ P\in E}} U(P) = -\frac{1}{2}\mu(Q_o) + \frac{1}{2\pi}\int_L \mu(Q)\frac{\partial}{\partial V_Q}\ln|Q-Q_o|dQ \tag{2}$$

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Difference.

For interior problem, $\frac{\partial}{\partial V_Q}$ refers to the normal derivative in the direction outward from the domain D. Thus, weather the boundary point Q_o lies on the outer boundary or one of the interior curves, the relevant jump condition is equation (1).

For exterior problem, $\frac{\partial}{\partial V_Q}$ refers to the inward normal derivative and the relevant jump condition is equation (2).

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Proposed Solution

The author proposed and presented a new integral equation method for the solution of the Dirichlet problem in multiply connected domains.

Fast multipole methods were combined with the new formulation to create algorithm capable of solving Laplace equations in domains of hundreds of distinct boundary components in minutes.

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Thank you!

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