# Method of Frobenius

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# Ordinary points, regular singular points and irregular singular points.

Consider ODE:

(1) $$y''(x) + p(x)y' + q(x)y = 0$$

A point $x_0$ is an "ordinary point" if $p(x)$ and $q(x)$ are analytic at $x_0$

Fuch's (1866): All linearly independent solutions of (1) are analytic in the neighborhood of an ordinary point.

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A point $x_0$ is a "regular singular point" if $p(x)$ or $q(x)$ are not both analytic at $x_0$ but

$$(x-x_0)^2 p(x) \quad \text{and} \quad (x-x_0)q(x)$$

are both analytic at $x_0$

**Example:** $x=1$ and $x=-1$ "regular singular points" of Legendre's equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$$y'' - \frac{2x}{1-x^2}y' + \frac{n(n+1)}{1-x^2}y = 0$$

since

$$(1-x)^2\left(\frac{-2x}{1-x^2}\right) = (1-x)\left(\frac{-2x}{1+x}\right)$$

and

$$(1-x)\left(\frac{n(n+1)}{1-x^2}\right) = \frac{n(n+1)}{1+x}$$

are both analytic at $x=1$.

Similarly,

$$(1+x)^2\left(\frac{-2x}{1-x^2}\right) = (1+x)\left(\frac{-2x}{1-x}\right)$$

and

$$(1+x)\left(\frac{n(n+1)}{1-x^2}\right) = \frac{n(n+1)}{1-x}$$

are both analytic at $x=-1$. ∎

We assumed a power series solution to Legendre's equation about $x = 0$

$$y(x) = \sum_{m=0}^{\infty} a_m x^m$$

All $x$, $|x| < 1$ are "ordinary points" and so the solution is analytic for $|x| < 1$.

But we also found solutions in the form of polynomials which were analytic everywhere. In general, the solution may be analytic at a regular singular point, or it will be a pole, or an algebraic or logarithmic branch point.

# Method of Frobenius

Consider the ODE

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0$$

$x = 0$ is a "regular singular point"

## Example:

$$y'' + \frac{1}{4x^2} y = 0$$

Write this as:

$$4x^2 y'' + y = 0$$

and assume a solution:

$$y(x) = \sum_{m=0}^{\infty} a_m x^m$$

**Example** $4x^2y'' + y = 0$

Assume

$$y = \sum_{m=0}^{\infty} a_m x^m$$

$\Rightarrow$

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1} =$$

$$y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

$$4x^2y'' + y = \sum_{m=2}^{\infty} 4(m)(m-1) a_m x^m + \sum_{m=0}^{\infty} a_m x^m$$

$$= a_0 + a_1 x + \sum_{m=2}^{\infty} \left(4m(m-1) + 1\right) a_m x^m = 0$$

$$= a_0 + a_1 x + \sum_{m=2}^{\infty} 4\left(m - \tfrac{1}{2}\right)^2 a_m x^m = 0$$

$$\Rightarrow a_0 = a_1 = a_2 = \dots = 0$$

Power series solution is not general enough.

# Method of Frobenius

Assume a more general solution of the form

$$y = x^r \sum_{m=0}^{\infty} a_m x^m \qquad (2)$$

where $r$ is a real or complex number.

- $r = \frac{1}{2} \Rightarrow$ introduces $\sqrt{x}$

- At least one solution will be of this form.
- The exponent $r$ is called the "indicial" exponent.

$\Longrightarrow$

Example: Using the solution (2) for the ODE

$$y'' + \frac{1}{4x^2} y = 0$$

Assume:

$$y = x^r \sum_{m=0}^{\infty} a_m x^m$$

Get:

$$= \sum_{m=0}^{\infty} a_m x^{m+r} \Rightarrow 4x^2 y'' + y = 0$$

$$\left[ (m+r)(m+r-1) + \frac{1}{4} \right] a_m = 0, \quad m = 0, 1, 2, \cdots$$

Consider the case $m=0$, and assume $a_0 \neq 0$. The we get:

indicial equation.

$$r(r-1) + \frac{1}{4} = 0 \Rightarrow r = \frac{1}{2}$$

then $a_1 = a_2 = a_3 = \cdots = 0$. The

$m^2 a_m = 0$

Frobenius solution is then

$$y(x) = \sqrt{x}\, a_0$$

Check! $y' = \frac{1}{2} x^{-1/2}$ $\quad y'' = -\frac{1}{4} x^{-3/2}$

$$\left[ 4x^2 \left( -\frac{1}{4} x^{-3/2} \right) + \sqrt{x} \right] = 0 \checkmark$$

Applying the method of Frobenius to the more general equation

$$x^2 y'' + x b(x) y' + c(x) y = 0$$

where

$$b(x) = b_0 + b_1 x + b_2 x^2 + \cdots$$

$$c(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$

where we have assumed $b(x)$ and $c(x)$ are analytic at 0.

... some algebra ...

we get that the coefficient of $x^r$ is

$$[r(r-1) + b_0 r + c_0] a_0 = 0$$

Assume $a_0$ is not 0. The indicial equation is then

$$r(r-1) + b_0 r + c_0 = 0$$

Solving the indicial equation for $r$ gives us at least one solution of the form

$$y(x) = x^{r^1} \sum_{m=0}^{\infty} a_m x^m \quad , \; r^1 > r^2$$

The second solution will take various forms, depending on the roots of the indicial equation.

Case 1: Distinct roots not differing by an integer $r^1 - r^2 \neq n$ integer

Case 2: A double root $r^1 = r^2$

Case 3: Roots differing by an integer

Why? Homework exercise!

**Case 1**: Distinct roots not differing by an integer. Roots $r_1$ and $r_2$

$r_1 - r_2 \notin \mathbb{I}$

Solutions:

$$y_1(x) = x^{r_1} \sum_{m=0}^{\infty} a_m x^m$$

$$y_2(x) = x^{r_2} \sum_{m=0}^{\infty} A_m x^m$$

different values for the coefficient.

**Case 2** Double root:

$$r_1 = r_2 = r$$

$$y_1(x) = x^r \sum_{m=0}^{\infty} a_m x^m$$

$$y_2(x) = y_1(x)\ln(x) + x^{r_2} \sum_{m=0}^{\infty} A_m x^m$$

**Case 3** Roots differ by an integer

$$y_1(x) = x^{r_1} \sum_{m=0}^{\infty} a_m x^m$$

$$y_2(x) = k\ln(x)\, y_1(x) + x^{r_2} \sum_{m=0}^{\infty} A_m x^m \quad (k \text{ might be } 0)$$

$$r_1 > r_2$$

## Example

$$xy'' + y' - xy = 0$$

"Standard form":

$$x^2y'' + xy' - x^2y = 0$$

$$b(x) = 1, \quad c(x) = -x^2$$

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$$-x^2 y = \sum_{m=0}^{\infty} a_m x^{m+r-2} = \sum_{m=2}^{\infty} a_{m-2} x^{m+r}$$

$\Longrightarrow$

$$xy' = \sum_{m=0}^{\infty} (m+r) a_m x^{m+r}$$

$$x^2 y'' = \sum_{m=0}^{\infty} (m+r-1)(m+r) a_m x^{m+r}$$

Note: $$x^2y'' + xy' = \sum_{m=0}^{\infty} (m+r)^2 a_m x^{m+r}$$

$$x^2 y'' + xy' - x^2 y = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} (m+r)^2 a_m x^{m+r} - \sum_{m=2}^{\infty} a_{m-2} x^{m+r} = 0$$

$$\Rightarrow x^r \left( r^2 a_0 + (r+1)^2 a_1 x - \sum_{m=2}^{\infty} \left[ (m+r)^2 a_m - a_{m-2} \right] \right) x^m = 0$$

$\Rightarrow$ Indicial equation: $r^2 a_0 = 0 \Rightarrow r = 0$

double root.

Setting $r=0$, we have $(r+1)^2 a_1 = a_1 = 0$, and more generally

$$a_m = \frac{1}{m^2} a_{m-2}, \quad m = 2, 3, 4, 5, \ldots$$

From this, and the fact that $a_1 = 0$, we get that for odd values of $m$,

$a_m = 0$.

$\Rightarrow$

For even indices, we have

$$a_m = \frac{1}{m^2} a_{m-2} = \frac{1}{m^2 (m-2)^2} a_{m-4}$$

$$= \frac{1}{m^2 (m-2)^2 (m-4)^2 \cdots 2^2} a_0$$

We have $m = 2j$ for some $j$, (since $m$ even)

$$a_{2j} = \frac{1}{(2j)^2 (2(j-1))^2 (2(j-2))^2 \cdots 2^2} a_0$$

$$= \frac{1}{2^{2j} (j!)^2} a_0 \quad , j = 0, 1, 2, \cdots$$

$$y_1(x) = \sum_{m=0}^{\infty} \frac{1}{2^{2m} (m!)^2} x^{2m}$$

Modified Bessel function of the First Kind of order 0.

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