Final Project

Math 537

Here are a few topics to consider for a final project. Some are just extended homework problems and others are more open ended.

The first few questions are on the Euler Lagrange Equations.

1. (**Review**) Show that if a continuous, differentiable function $f(x) : \mathcal{R} \to \mathcal{R}$ has a local minimum at a point x_0, then $f'(x_0) = 0$. This condition is a *necessary* condition. Provide a counter example to show that it is not sufficient.

2. (**Review**) Show that the if a differential function $J(x) = f(x) + \lambda g(x)$ has a local minimum at a point x_0, then $g(x_0) = 0$. This is the idea behind the Lagrange Multiplier used to solve constrained minimization problems on functionals, given by

$$\mathbf{J}[u] = \mathbf{F}[u] + \lambda G[u]. \tag{1}$$

for functions $u(\mathbf{x})$in an admissible function space.

3. (**Duhamel's Principle**) We can use Duhamel's Principle to solve the scalar advection equation

$$u_t + cu_x = f(x,t) \tag{2}$$

for constant velocity field c. The resulting solution is

$$u(x,t) = u(x-ct,0) + \int_0^t f(x - c(t-\tau), \tau)\, d\tau. \tag{3}$$

Use this solution to obtain the fundamental solution $G_0(x,t,\xi,\tau)$ for the equation

$$u_t + cu_x = \delta(t-\tau)\delta(x-\xi), \qquad (x,t) \in (-\infty, \infty) \times [0,T] \tag{4}$$

where $\delta(t)$ is the Dirac Delta function.

4. (**Euler Lagrange equations**) Consider the model problem

$$\begin{aligned} &u_t + cu_x = 0, \qquad c > 0 \\ &u(x,0) = 0 \\ &u(0,t) = 0 \end{aligned}$$

on the domain $(x,t) \in [0,L] \times [0,T]$. The solution is $u(x,t) = 0$. We also have M observations d_m at space-time points (x_m, t_m) and so would like to find a solution which minimizes the functional

$$\mathbf{J}[u] = W \int_0^T \int_0^L (u_t + cu_x)^2 \; dx\, dt + w \sum_{m=1}^{M} (u(x_m, t_m) - d_m)^2 \tag{5}$$

In class, we derived the Euler-Lagrange equations for this system and saw that the boundary value problem on $[0,L] \times [0,T]$ for the adjoint variable $\lambda(x,t)$ could be re-formulated as two initial values problems. Setting $W = w = 1$ these two problems become

Backward problem

$$
\begin{aligned}
-\lambda_t - c\lambda_x &= -\sum_{m=1}^{M} (\widehat{u}(x,t) - d_m)\delta(x - x_m)\,\delta(t - t_m)\\
\lambda(L,t) &= 0\\
\lambda(x,T) &= 0
\end{aligned}
$$

Forward problem

$$
\begin{aligned}
\widehat{u}_t + c\widehat{u}_x &= \lambda(x,t)\\
\widehat{u}(x,0) &= 0\\
\widehat{u}(0,t) &= 0
\end{aligned}
$$

where $\widehat{u}(x,t)$ is the function that minimizes $\mathbf{J}[u]$.

While it may seem that these are two separate initial value problems, the backward problem depends on values of $\widehat{u}(x,t)$, while the forward problem depends on $\lambda(x,t)$. To decouple these problems, we can use linearity and superposition. We replace the backward problem with one which does not involve the source strengths $(\widehat{u}(x,t) - d_m)$ and instead just "back propagate" and source of unit strength located in space and time at (x_m, t_m) by solving

$$
\begin{aligned}
-\frac{\partial \alpha_m}{\partial t} - c\frac{\partial \alpha_m}{\partial x} &= \delta(x - x_m)\,\delta(t - t_m)\\
\alpha_m(L,t) &= 0\\
\alpha_m(x,T) &= 0
\end{aligned}
$$

for adjoint variables $\alpha_m(x,t)$.

We forward propagate this source to get

$$
\begin{aligned}
\frac{\partial r_m}{\partial t} + c\frac{\partial r_m}{\partial x} &= \alpha_m\\
r_m(L,t) &= 0\\
r_m(x,T) &= 0
\end{aligned}
$$

We can then write our least-squares solution as

$$
\widehat{u}(x,t) = \sum_{m=1}^{M} \beta_m r_m(x,t) \tag{6}
$$

for unknown coefficients.

(a) Show that $\beta_m = (\widehat{u}(x_m, t_m) - d_m)$.

(b) Using (6), formulate a linear system

$$
(R + I)\beta = \mathbf{d} \tag{7}
$$

for the vector $\beta = (\beta_1, \beta_2, \ldots, \beta_M)$, where R is a symmetric $M \times M$ matrix whose entries are given by $r_{ij} = r_i(x_j, t_j)$ and $\mathbf{d} = (d_1, d_2, \ldots, d_M)$ is the vector of observations.

(c) Using (4), show that the solutions for $\alpha_m(x,t)$ and $r_m(x,t)$ are given by

$$
\alpha_m(x,t) = \delta(x - x_m - c(t - t_m))\,H(t_m - t), \qquad m = 1, 2, \ldots, M \tag{8}
$$

$$
r_m(x,t) = \delta(x - x_m - c(t - t_m))\,(t + (t_m - t)H(t - t_m)), \qquad m = 1, 2, \ldots, M \tag{9}
$$

5. (**Euler Lagrange**) Consider the model problem presented in Problem 4 on the domain $[0,1]$. You have an observation $d_0 = 0.68$ at location $x_0 = 0.33$, at time $t_0 = 0.5$. What is the solution to your Euler-Lagrange system? **Hint:** What is β_0 for this system? Sketch or plot your solution for several times leading up to and including time $t = 0.5$. Express your solution in terms of the Dirac delta function approximation

$$\delta(x) \approx \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{-x^2}{4\varepsilon}} \tag{10}$$

for $\varepsilon \ll 1$. For example, set $\varepsilon \approx 10^{-3}$.

Projects

Please come to class on Wednesday prepared to talk about one of the following topics. You may present slides (no more than 10), or use some other means to give a 15 minute lecture on one of the topics below.

If you have another topic you would like to present, that is fine too. Just let me know what you decide.

1. (**Euler-Lagrange equations**) Suppose you are given two or more observations for the model problem in Problem 4. How do you formulate the solution? **Hint:** What is the linear system for β? Plot the solution for two or three observations.

2. (**Heat kernel**) Consider the 1d heat equation in the infinite domain given by

$$u_t - u_{xx} = \delta(t)\ f(x), \qquad x \in (-\infty, \infty) \tag{11}$$

where $u(x, 0^+) = f(x)$ is the initial condition. The solution is given in terms of a "heat kernel" as

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(\xi) e^{-(x-\xi)^2/4t}\, d\xi \tag{12}$$

   (a) Verify that (12) satisfies (11). In particular, show that the initial condition is satisfied.

   (b) Use the method of images to find a Green's function solution $G(x, \xi, t)$ to (11) that satisfies $G(0, \xi, t) = 0$. **Hint:** Use the free-space Green's function given by the heat kernel

$$G_0(x,\xi,t) = \frac{1}{\sqrt{4\pi t}} e^{-(x-\xi)^2/4t} \tag{13}$$

   (c) Use $G(x, \xi, t)$ to solve (11) on $[0, \infty)$, subject to $u(0,t) = 0$ for the special case of $f(x) = c$, $x > 0$. Write the solution in terms of the error function $\mathrm{erf}(x)$, given by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2}\, d\xi \tag{14}$$

3. (**Potential theory in multiply connected domains**) Read the sections 1,2 and 3 of the article "Laplace's Equation and the Dirichlet-Neumann Map in Multiply connected domains" (A. Greenbaum, L. Greengard and G. B. McFadden).

   (a) What is the difference between the interior and exterior Dirichlet problems in 2d?

   (b) What is the problem with the multiply connected domain?

   (c) What is the solution that the authors propose?

4. (**Multipole expansions and the Fast Multipole Method**) See "Tutorial notes" on class Blackboard site. For this project, you should discuss

   (a) For what purpose do multipole expansions are useful?

(b) For what types of problems do we use multipole expansions?

(c) Describe the multipole expansion in 3d. What is the connection to generating functions for Legendre polynomials?

(d) Provide an intuitive overview of the idea behind the Fast Multipole Method (FMM).