Legendre’s Equation

Power Series Methods - Legendre Polynomials

$$y'' + p(x)y' + q(x)y = r(x)$$

Recall: If $p(x)$, $q(x)$ and $r(x)$ have a power series, than the solution y can be represented as a power series.

Existence of Power Series Solutions

If p, q, and r in (12) are analytic at $x = x_0$, then every solution of (12) is analytic at $x = x_0$ and can thus be represented by a power series in powers of $x - x_0$ with radius of convergence > 0.

- analytic = "infinitely differentiable"
  = has a Taylor series about x_0 and a positive radius of convergence.

* for real valued functions

p, q, r can be written as a power series:

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m$$

and this power series has to converge in an interval

$$|x - x_0| \leq R$$

for some $R > 0$

* for complex valued functions, the definition has to extend to the complex plane.

Homework Problem:

$$y'' - xy = 0$$

→ power series solution will converge on the entire real line

Example: $$y'' - y' + xy = 0$$

$p(x) = -1 \quad q(x) = x, \quad r(x) = 0$

⇒ series converges on $\mathbb{R}$.

more generally, for $p(x)$, $q(x)$, $r(x)$ given by polynomials, the series solution will converge.

- also $\cos(x)$, $\sin(x)$, etc.

$$y'' + \cos(x) y = 0$$

(Gets messy, since we have to multiply infinite series) solve y" + cos(x)*y = 0 for y

Wolfram Alpha

Extended Keyboard   Upload

Input interpretation:

solve	$y''(x) + \cos(x)\, y(x) = 0$	for	$y(x)$

Result:

$$y(x) = c_1 \,\mathrm{Ce}\left(0, -2, \frac{x}{2}\right) + c_2 \,\mathrm{Se}\left(0, -2, \frac{x}{2}\right)$$

$\mathrm{Se}(x)$, $\mathrm{Ce}(x)$

ODE classification:

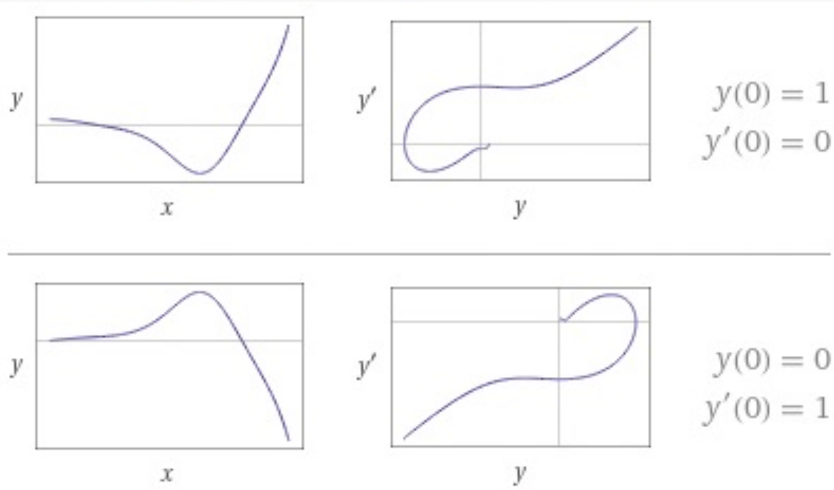
second-order linear ordinary differential equation

Alternate forms:

$$y''(x) = y(x)\,(-\cos(x))$$

$$y''(x) + \frac{1}{2} e^{-i x}\, y(x) + \frac{1}{2} e^{i x}\, y(x) = 0$$

Plots of sample individual solutions:



arise in optics, quantum mechanics, general relativity

$\mathrm{Ce}(x)$, $\mathrm{Se}(x)$ are "mathieu functions" are solutions to the ODE

$$y'' + (a - 2q\cos(2x))y = 0$$

Many important problems don't have such nice properties.

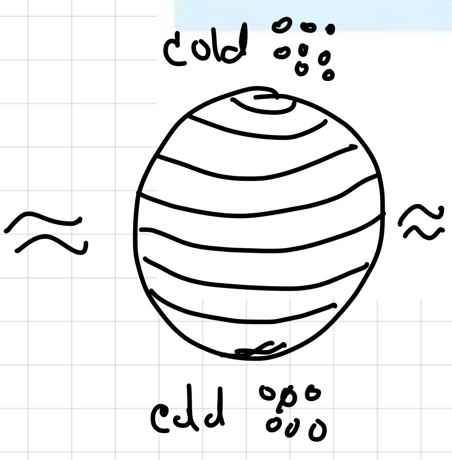
Steady state heat distribution in a Sphere:

Laplacian in spherical coordinates (r, θ, ϕ)

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\cot\phi}{r^2}\frac{\partial u}{\partial \phi} + \cancel{\frac{1}{r^2 \sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}}.$$

If we assume no variation in θ (axis-symmetry), we can eliminate θ: to get

$$\nabla^2 u = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial u}{\partial \phi}\right)\right] = 0.$$



$$u(R, \phi) = f(\phi)$$

$$\lim_{r \to \infty} u(r, \phi) = 0.$$

separation of variables soln : $u = G(r)H(\phi)$

we get ODEs for $G(r)$ and $H(\phi)$:

$$\underbrace{G_n(r) = r^n}, \quad \underbrace{G_n^*(r) = \frac{1}{r^{n+1}}}$$

depending on whether we care about the solution as $r \to 0$ or $r \to \infty$.

The solution $H(\phi)$ satisfies

$$\underbrace{(1-\omega^2)H'' - 2\omega H' + n(n+1)H = 0}_{\text{example of a Legendre equation}}$$

Notice:

$$H'' - \underbrace{\frac{2\omega}{1-\omega^2}}_{p(x)} H' + \underbrace{\frac{n(n+1)}{1-\omega^2}}_{q(x)} H = 0$$

$p(x)$ and $q(x)$ are no longer analytic on $\mathbb{R}$.
Radius of convergence: $|\omega| < 1$.

Legendres Equation

$K = n(n+1)$

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

where n is a parameter that depends on the engineering or physical context.

To solve, we get a power series solution to

$$y = \sum_{m=0}^{\infty} a_m x^m \qquad (2)$$

and its derivatives into (1), and denoting the constant $n(n+1)$ simply by k, we obtain

$$(1-x^2)\sum_{m=2}^{\infty} \overset{y''}{m(m-1)a_m x^{m-2}} - 2x\sum_{m=1}^{\infty} \overset{y'}{m a_m x^{m-1}} + k\sum_{m=0}^{\infty} a_m x^m = 0.$$

By writing the first expression as two separate series we have the equation

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)a_m x^m - \sum_{m=1}^{\infty} 2ma_m x^m + \sum_{m=0}^{\infty} ka_m x^m = 0.$$

It may help you to write out the first few terms of each series explicitly, as in Example 3 of Sec. 5.1; or you may continue as follows. To obtain the same general power x^s in all four series, set $m - 2 = s$ (thus $m = s + 2$) in the first series and simply write s instead of m in the other three series. This gives

shift indices

$K = n(n+1)$

$$\sum_{s=0}^{\infty} (s+2)(s+1)a_{s+2}x^s - \sum_{s=2}^{\infty} s(s-1)a_s x^s - \sum_{s=1}^{\infty} 2sa_s x^s + \sum_{s=0}^{\infty} ka_s x^s = 0.$$

Obtain the recurrence relation

$$a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)} a_s \qquad (s = 0, 1, \cdots).$$

n: parameter from the original problem
s: index on coefficient

This gives us two sets of coefficients:
even $a_0, a_2, a_4, a_6 \ldots$ and odd
$a_1, a_3, a_5, \ldots$ where a_0 and a_1 are
determined using boundary conditions:

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

Idea now: For which values of n
do we get "nice" solutions? Possibly
reducing to polynomials ?

$$a_{s+2} = \frac{(n-s)(n+s-1)}{(s+2)(s+1)} a_s$$

$s=0 \qquad a_2 = \dfrac{n(n+1)}{2\cdot 1} a_0$

even - coefficients

$s=2 \qquad a_4 = \dfrac{(n-2)(n+1)}{4\cdot 3} a_2$

$s=4 \qquad a_6 = \dfrac{(n-4)(n+3)}{6\cdot 5} a_4 \qquad \cdots$

$s=1 \qquad a_3 = \dfrac{(n-1)(n)}{3\cdot 2} a_1$

odd - coefficients

$s=3 \qquad a_5 = \dfrac{(n-3)(n+2)}{5\cdot 4} a_3$

$s=5 \qquad a_7 = \dfrac{(n-5)(n+4)}{7\cdot 6} a_5 \qquad \cdots$

$$y_1(x) = a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6 + \cdots$$

$$y_2(x) = a_1 + a_3 x^3 + a_5 x^5 + a_7 x^7 + \cdots$$

Coefficients depend on choice of n, so we could write

$$a_2(n) = \frac{n(n-1)}{2\cdot 1} a_0$$

$$a_4(n) = \frac{(n-2)(n+1)}{4\cdot 3} a_2(n)$$

$$a_6(n) = \frac{(n-4)(n+3)}{6\cdot 5} a_4(n)$$

$$a_3(n) = \frac{(n-1)(n)}{3\cdot 2} a_1$$

$$a_5(n) = \frac{(n-3)(n+2)}{5\cdot 4} a_3$$

$$a_7(n) = \frac{(n-5)(n+4)}{7\cdot 6} a_5$$

So, in fact, each n generates a pair of lin. ind. solutions:

$$y_1(x;n) = \sum_{m=0}^{\infty} a_{2m}(n)\, x^{2m} \qquad \text{even}$$

$$y_2(x;n) = \sum_{m=0}^{\infty} a_{2m+1}(n)\, x^{2m+1} \qquad \text{odd}$$

In general, we only expect convergence in $|x| < 1$.

But for particular choices of n, we get polynomials for $y_1(x;n)$ and $y_2(x;n)$

$n=0$

$$a_2(0) = \frac{0\cdot(0-1)}{2\cdot 1} = 0$$

$$a_4(0) = \frac{(0-2)(0+1)}{4\cdot 3}\underbrace{a_2(0)}_{=0} = 0$$

$$a_6(0) = \frac{(0-4)(0+3)}{6\cdot 5}\underbrace{a_4(0)}_{=0} = 0$$

$$a_3(0) = \frac{(0-1)(0)}{3\cdot 2}a_1 = 0$$

$$a_5(0) = \frac{(0-3)(0+2)}{5\cdot 4}\underbrace{a_3(0)}_{=0} = 0$$

$$a_7(0) = \frac{(0-5)(0+4)}{7\cdot 6}\underbrace{a_5(0)}_{=0} = 0$$

$$y_1(x;0) = a_0 \qquad y_2(x;0) = a_1 x$$

$n=1$

$$a_2(1) = \frac{1(1-1)}{2\cdot 1}\,a_0$$

$$a_4(1) = \frac{(1-2)(1+1)}{4\cdot 3}\,a_2(1) = 0$$

$$a_5(1) = \dots = 0$$

$$a_3(1) = \frac{(1-1)(1)}{3\cdot 2}\,a_1 = 0$$

$$a_5(1) = \frac{(1-3)(1+2)}{5\cdot 4}\,\underbrace{a_3(1)}_{=0} = 0$$

$$a_7(1) = \dots = 0$$

$$y_1(x;1) = a_0 \qquad y_2(x;1) = a_1 x$$

$n=2$

$$a_2(2) = \frac{2(2-1)}{2\cdot 1}\,a_0 = 1$$

$$a_4(2) = \frac{\cancel{(2-2)}(2+1)}{4\cdot 3}\,a_2(2) = 0$$

$$a_4(2) = a_6(2) = \dots = 0$$

$$a_3(2) = \frac{(2-1)(2)}{3\cdot 2}\,a_1 = \frac{1}{3}$$

$$a_5(2) = \frac{(2-3)(2+2)}{5\cdot 4}\,a_3(2) = -\frac{1}{5}$$

$$a_7(2) = \frac{(2-5)(2+4)}{7\cdot 6}\cdot a_3(2)$$

$$= -\frac{3}{7}\cdot -\frac{1}{5} = \frac{3}{35}$$

$$y_1(x;2) = a_0(1+x^2) \qquad y_2(x;2) = \frac{1}{3}x - \frac{1}{5}x^3 + \frac{3}{35}x^5 + \dots$$

$$y_1(x) = a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6 + \cdots$$

$$y_2(x) = a_1 + a_3 x^3 + a_5 x^5 + a_7 x^7 + \cdots$$

Since $p(x) = \dfrac{-2x}{1-x^2}$, $q(x) = \dfrac{n(n+1)}{1-x^2}$

we only expect convergence of the infinte series $y_1(x)$ and $y_2(x)$ for $|x| < 1$.

But if we choose n carefully, we can reduce $y_1(x)$ and $y_2(x)$ to polynomials!

$$a_{s+2} = \frac{(n-s)(n+s-1)}{(s+2)(s+1)} a_s$$

For $s = n$, $a_{s+2} = a_{s+4} = a_{s+6} = \cdots 0$

So then $y_1(x)$ and $y(x)$ are just polynomials of degree n.

Even powers of s

$$a_{s+2} = \frac{(n-s)(n+s-1)}{(s+2)(s+1)} a_s$$

$$a_2(0) = \frac{(n-0)(n-1)}{2 \cdot 1} a_0$$

$$= \frac{(n-2)(n+1)}{4 \cdot 3} a_2$$

$$a_6 = \frac{(n-4)(n+3)}{6 \cdot 5} a_4$$

$n=0$: $a_2 = a_4 = a_6 = \cdots 0$

$y_1(x) = a_0$

$n=2$: $a_2 = \frac{2(1)}{2 \cdot 1} a_0$ $y_1(x) = a_0(1+x^2)$

$a_4 = a_6 = a_8 = \cdots = 0$

$n=4$: $a_2 = \frac{4 \cdot 3}{2 \cdot 1} a_0 = 6a_0$ $y_1(x) =$

$a_4 = \frac{2 \cdot 5}{4 \cdot 3} a_2 = \cdots$ $a_0(1 + 6x^2 + \boxed{\ } x^4)$

$n=6$ $a_2 \neq 0$ $a_6 = a_8 = a_{10} = \cdots 0$

$a_4 \neq 0$

Legendre Polynomials

$$P_n(x) \quad n = 0, 1, 2, 3, 4, \ldots$$

where a constant is chosen so that

$$P_n(1) = 1$$

<u>Constant</u>

$$a_n = \frac{2n}{2^n (n)^2}$$

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

"Power series solution" to

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$r \geq R$

$$u(r, \phi) = \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} P_n(\cos \phi)$$

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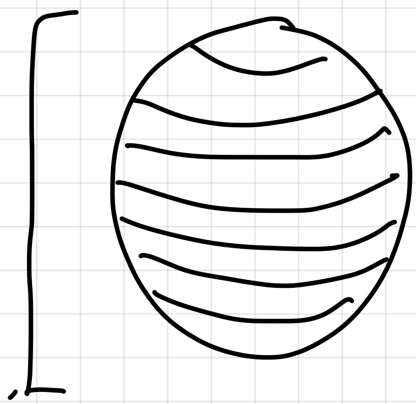

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$u(R, \phi) = f(\phi)$