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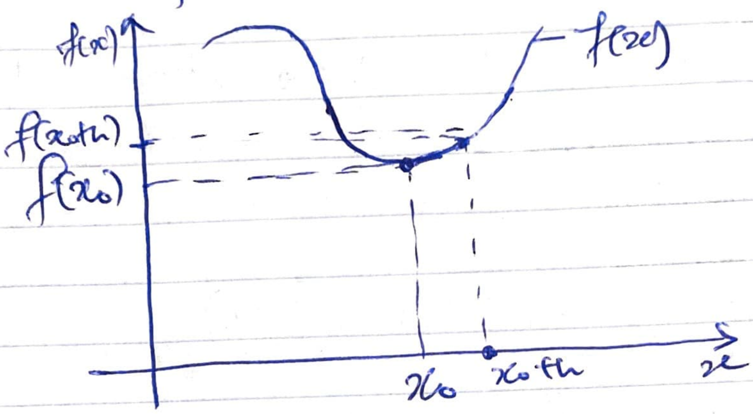
MATH 537
Final Project

1. Show that If a Continuous, differential function $f(x)$ $\mathbb{R}\to\mathbb{R}$ has a local minimum at a point x_0, then $f'(x_0) = 0$.

Consider the function $f(x)$



the gradient of the curve at x_0 = gradient of tangent at x_0.

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{(x_0+h) - x_0} = \frac{f(x_0+h) - f(x_0)}{h}$$

$\lim_{h\to 0^+} f'(x_0) \geq 0$ and $\lim_{h\to 0^-} f'(x_0) \leq 0$, then

at the local minimum point $f'(x_0) = 0$

2. Counter example:

Consider a function $f(x) = 3x + 2x^2$ $\forall x \in [0,1]$

at $x=0 \Rightarrow f(0)=0$

$f'(x)=3+4x \Rightarrow f'(0)=3\neq 0$, hence it is not sufficient.

2. Show that, if a differential function $J(x)=f(x)+\lambda g(x)$ has a local minimum at a point x_0, then $g(x_0)=0$.

$$J(x)=f(x)+\lambda g(x) \quad -(1)$$

differentiating (1) w.r.t λ

$$\frac{\partial J(x)}{\partial \lambda}=g(x)$$

$$\left.\frac{\partial J(x)}{\partial \lambda}\right|_{x=x_0}=g(x_0)=0.$$

3. Duhamel's principle

$$u_t+cu_x=f(x,t)$$

The resulting solution is

$$u(x,t)=u(x-ct,0)+\int_0^t f(x-c(t-\tau),\tau)\,d\tau$$

Use this solution to obtain the fundamental

Solution $G_0(x,t,\xi,\tau)$ for the equation

Since $u_t + cu_x = \delta(t-\tau)\,\delta(x-\xi)$,
$(x,t) \in (-\infty,\infty) \times [0,T]$

$$u(x,t) = u(x-ct,0) + \int_0^t f(x-c(t-\tau),\tau)\,d\tau$$

Can be written as,

$$G(x,t,\xi,\tau) = G(x-ct,0,\xi,\tau) + \int_0^t f(x-c(t-\tau),\tau)\,d\tau \quad \text{---} (1)$$

Since $u_t + cu_x = f(x,t) = \delta(t-\tau)\,\delta(x-\xi)$
$f(x,t) = \delta(t-\tau, x-\xi)$

So
$$f(x-c(t-\tau),\tau) = \delta(x-c(t-\tau)-\xi)\,\delta(\tau-\tau)$$
$$= \delta(x-c(t-\tau)-\xi, 0)$$

then equation (1) becomes

$$G(x,t,\xi,\tau) = G(x-ct,0,\xi,\tau) + \int_0^{t-\tau} \delta(x-\xi-c(t-\tau-\eta),\eta)\,d\eta$$

$$G(x,t,\xi,\tau) = G(x-ct,0,\xi,\tau) + \int_0^{t-\tau} \delta(x-\xi-c(t-\tau)+c\eta)\,\delta(\eta)\,d\eta$$

$$G(x,t,\xi,\tau) = G(x-ct,0,\xi,\tau) + \int_0^{t-\tau} \delta\left(c\left(\frac{1}{c}(x-\xi)-(t-\tau)+\eta\right)\right)\delta(\eta)\,d\eta$$

Using a property $\delta(c\eta) = \frac{1}{|c|}\delta(\eta)$

$$G(x,t,\xi,\tau) = G(x-ct,0,\xi,\tau) + \frac{1}{c}\int_0^{t-\tau} \delta\left(\eta+\frac{1}{c}(x-\xi)-(t-\tau)\right)\delta(\eta)\,d\eta$$

$$G(x,t,\xi,\tau) = G(x-ct,0,\xi,\tau) + \frac{1}{c}\int_0^{t-\tau} \delta(\eta - (t-\tau) - \frac{1}{c}(x-\xi))\,\delta(\eta)\,d\eta$$

$$G(x,t,\xi,\tau) = G(x-ct,0,\xi,\tau) + \frac{1}{c}\delta((t-\tau) - \frac{1}{c}(x-\xi))\,H(t-\tau)$$

Since $\int_{-\infty}^{\infty} \delta(\eta-a)\,\delta(\eta-b)\,d\eta = \delta(a-b)$

So

$$G(x,t,\xi,\tau) = G(x-ct,0,\xi,\tau) + \delta(c(t-\tau) - (x-\xi))\,H(t-\tau)$$

Since $\delta(x)$ is an even function i.e. $\delta(-x) = \delta(x)$ then

$$G(x,t,\xi,\tau) = G(x-ct,0,\xi,\tau) + \delta(x-\xi - c(t-\tau))\,H(t-\tau)$$

where $H(t-\tau) = \begin{cases} 0, & t<\tau \\ 1, & t>\tau \end{cases}$

4) (Euler Lagrange equations) Consider the model problem.

$$u_t + c u_x = 0, \quad c>0$$
$$u(x,0) = 0$$
$$u(0,t) = 0$$

a) Show that $\beta_m = (\hat{u}(x_m,t_m) - d_m)$

$$\hat{u}(x,t) = \sum_{m=1}^{M} \beta_m \gamma_m(x,t) \quad \text{—— (1)}$$

differentiate (1) w.r.t t

$$\frac{\partial}{\partial t}\hat{u}(x,t) = \sum_{m=1}^{M} \beta_m \frac{\partial \Gamma_m(x,t)}{\partial t} \quad \text{--- (2)}$$

differentiate (2) w.r.t x

$$\frac{\partial}{\partial x}\hat{u}(x,t) = \sum_{m=1}^{M} \beta_m \frac{\partial \Gamma_m(x,t)}{\partial x} \quad \text{--- (3)}$$

Equation (2) + c(equation (3))

$$\hat{u}_t(x,t) + c\hat{u}(x,t) = \sum_{m=1}^{M} \beta_m \left(\frac{\partial \Gamma_m(x,t)}{\partial t} + c\frac{\partial \Gamma_m(x,t)}{\partial x}\right)$$

$$\frac{\partial \Gamma_m(x,t)}{\partial t} + c\frac{\partial \Gamma_m(x,t)}{\partial x} = \alpha_m$$

$$\hat{u}_t(x,t) + c\hat{u}_x(x,t) = \sum_{m=1}^{M} \beta_m \alpha_m$$

but $\hat{u}_t + c\hat{u}_x = \lambda(x,t)$

$$\lambda(x,t) = \sum_{m=1}^{M} \beta_m \alpha_m \quad \text{--- (4)}$$

$$\frac{\partial \lambda}{\partial x} = \sum_{m=1}^{M} \beta_m \frac{\partial \alpha_m}{\partial x} \quad \text{--- (5)}$$

differentiate (4) w.r.t t

$$\frac{\partial \lambda}{\partial t} = \sum_{m=1}^{M} \beta_m \frac{\partial \alpha_m}{\partial t} \quad \text{--- (6)}$$

c(Equation (4)) + Equation (6)

$$c\frac{\partial \lambda}{\partial x} + \frac{\partial \lambda}{\partial t} = \sum_{m=1}^{M} \beta_m \left(\frac{\partial \alpha_m}{\partial t} + c\frac{\partial \alpha_m}{\partial x}\right)$$

but

$$\frac{\partial \alpha_m}{\partial t} + c\frac{\partial \alpha_m}{\partial x} = -\delta(x - x_m)\,\delta(t - t_m)$$

and

$$c\lambda_x + \lambda_t = \sum_{m=1}^{M} (\hat{u}(x,t) - d_m)\,\delta(x - x_m)\,\delta(t - t_m)$$

$$\sum_{m=1}^{M} (\hat{u}(x,t) - d_m)\,\delta(x - x_m)\,\delta(t - t_m) = \sum_{m=1}^{M} \beta_m(-\delta(x - x_m)\delta(t - t_m))$$

$$\hat{u}(x_m, t_m) - d_m = -\beta_m$$

c) Using (4), show that solutions for $\alpha_m(x,t)$ and $r_m(x,t)$ are given by

$$\alpha_m(x,t) = \delta(x - x_m - c(t - t_m))\,H(t_m - t) \quad , m = 1, 2, 3, \dots M$$

$$G(x, \xi, t, \tau) = G(x - ct, 0, \xi, \tau) + \delta(x - \xi - c(t - \tau))\,H(t - \tau)$$

but $G(x - ct, 0, \xi, \tau) = 0$ at the boundary.
replace ξ with x_m, and τ with t_m

$$G(x, x_m, t, t_m) = \delta(x - x_m - c(t - t_m))\,H(t - t_m) = \alpha_m(x,t)$$

so

$$-\alpha_m(x,t) = \delta((x - x_m) - c(t - t_m))\,H(t - t_m)$$

dividing through by negative.

$$\alpha_m(x,t) = \delta(x - x_m - c(t - t_m))\, H(t_m - t)$$

$$r_m(x,t) = \delta(x - x_m - c(t - t_m))\,(t + (t_m - t)\,H(t - t_m))$$

$m = 1, 2, \dots, M$

$$\frac{\partial r_m}{\partial t} + c\,\frac{\partial r_m}{\partial x} = \alpha_m$$

$$r_m(x,t) = \int_0^t \alpha_m(x_0 - c(t-\eta), \eta)\, d\eta$$

$$r_m(x,t) = \int_0^t \delta(x - c(t-\eta) - x_m - c(\eta - t_m))\, H(t_m - \eta)\, d\eta$$

$$= \int_0^t \delta(x - ct + \cancel{c\eta} - x_m - \cancel{c\eta} + ct_m)\, H(t_m - \eta)\, d\eta$$

$$r_m(x,t) = \int_0^t \delta(x - x_m - c(t - t_m))\, H(t_m - \eta)\, d\eta$$

$$r_m(x,t) = \delta(x - x_m - c(t - t_m)) \int_{t_m - t}^{t} H(t_m - \eta)\, d\eta$$

$$r_m(x,t) = \delta(x - x_m - c(t - t_m))\,(-R(t_m - \eta))\Big|_{t_m - t}^{t}$$

$$r_m(x,t) = \delta(x - x_m - c(t - t_m))\,(-R(t_m - t) + R(t_m - (t_m - t)))$$

$$r_m(x,t) = \delta(x - x_m - c(t - t_m))\,(R(t) - R(t_m - t))$$

but $R(x) = x\,H(x)$

$$r_m(x,t) = \delta(x - x_m - c(t - t_m))\,(t\,H(t) - (t_m - t)\,H(t_m - t))$$

$$r_m(x,t) = \delta\left(x - x_m - c(t - t_m)\right) \cdot \left(t + (t_m - t)\, H(t - t_m)\right)$$

$$m = 1, 2, \dots, M$$

4(b) Using (6) formulate a linear system

$$(R+I)\beta = d$$

using $\hat{u}(x_m, t_m) - d_m = -\beta_m$ — ① from 4(a)
and that

$$\hat{u}(x,t) = \sum_{m=1}^{M} \beta_m r_m(x,t) \quad — ②$$

Since $r_{ij} = r_i(x_j, t_j)$ then ② can be written as

$$\hat{u}(x_j, t_j) = \sum_{m=1}^{M} \beta_m r_m(x_j, t_j), \quad j = 1, 2, \ldots M$$

from Equation ①, if $\hat{u}(x_m, t_m) = d_m - \beta_m$
then

$$\hat{u}(x_j, t_j) = \sum_{m=1}^{M} \beta_m r_m(x_j, t_j) = d_j - \beta_j, \quad j = 1, 2, \ldots M$$

$$\sum_{m=1}^{M} \beta_m r_m(x_j, t_j) + \beta_j = d_j, \quad j = 1, 2, \ldots, M$$

$$\beta_1 r_{1j} + \beta_2 r_{2j} + \cdots + \beta_M r_{Mj} + \beta_j = d_j; \quad j = 1, 2, \ldots, M$$

Since R is symmetric then $r_{ij} = r_i(x_j, t_j) = r_j(x_i, t_i)$

$$\underbrace{\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1M} \\ r_{21} & r_{22} & \cdots & r_{2M} \\ \vdots & & \ddots & \vdots \\ r_{M1} & \cdots & \cdots & r_{MM} \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_M \end{bmatrix}}_{\beta} + \underbrace{\begin{bmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots \\ 0 & & & 1 \end{bmatrix}}_{I} \underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_M \end{bmatrix}}_{\beta} = \underbrace{\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_M \end{bmatrix}}_{d}$$

$$(R\beta + I\beta) = d$$

$$(R+I)\beta = d$$