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Homework 6
MATH 537

1. A "self-adjoint" operator is one for which

$$<Lu, v> = <u, Lv>$$

for functions u, v in a Hilbert space. Show that the Sturm - Liouville operator.

$$L[u] = \frac{-1}{w(x)}\left((p(x)\,u')' + q(x)\,u\right)$$

from definition of a linear operator

$$<u,v> = \int_a^b u(x)\,v(x)\,w(x)\,dx, \quad w(x) > 0$$

we have to show that $<Lu, v> = <u, Lv>$

from the L.H.S

$$<Lu, v> = \int_a^b Lu\, v(x)\, w(x)\, dx$$

$$<Lu, v> = \int_a^b \frac{-1}{w(x)}\left((p(x)u')' + q(x)\,u\right) v(x)\, w(x)\, dx$$

$$= -\int_a^b \left((p(x)\,u')' + q(x)\,u(x)\right) v(x)\, dx$$

$$<Lu, v> = -\int_a^b (p(x)\,u')'\,v(x)\,dx - \int_a^b q(x)\,u(x)\,v(x)\,dx \quad -①$$

Integrate $\int_a^b (p(x)\,u')'\,v(x)\,dx$ by parts.

$$\int_a^b (P(x)u')' v(x)\, dx$$

Let $u_1 = v(x) \Rightarrow u_1' = v'$

$dv_1 = (P(x)u')' \Rightarrow \int dv_1 = \int (P(x)u')' dx$

$$v_1 = \int \frac{d}{dx}(P(x)u')\, dx = P(x)u'$$

$$\int_a^b u_1\, dv_1 = u_1 v_1 \Big|_a^b - \int_a^b v_1\, du_1$$

$$\int_a^b (P(x)u')' v(x)\, dx = v(x)P(x)u' \Big|_a^b - \int_a^b P(x)u'v'\, dx$$

Integrate $\int_a^b P(x)u'v'\, dx$ again by parts.

Let $u_2 = P(x)v' \Rightarrow du_2 = (P(x)v')' dx$

$dv_2 = du \Rightarrow v_2 = u$

$$\int_a^b u_2\, dv_2 = u_2 v_2 \Big|_a^b - \int_a^b v_2\, du_2$$

$$\int_a^b P(x)u'v'\, dx = (P(x)v')u(x) \Big|_a^b - \int_a^b u(x)(P(x)v')' dx$$

$$\int_a^b (P(x)u')' v(x)\, dx = v(x)P(x)u' \Big|_a^b - \int_a^b P(x)u'v'\, dx$$

$$= v(x)P(x)u' \Big|_a^b - (P(x)v')u(x) \Big|_a^b + \int_a^b u(x)(P(x)v')' dx$$

$$\int_a^b (p(x)\,u')'\,v(x)\,dx = v(x)\,p(x)\,u'\Big|_a^b - \int_a^b p(x)\,u'v'\,dx$$

$$= v(x)\,p(x)\,u'\Big|_a^b - p(x)\,v'u(x)\Big|_a^b + \int_a^b u(x)\,(p(x)v')'\,dx$$

$$\int_a^b (p(x)\,u')'\,v(x)\,dx = v(b)\,p(b)\,u'(b) - v(a)\,p(a)\,u'(a)$$
$$- p(b)\,v'(b)\,u(b) + p(a)\,v'(a)\,u(a) + \int_a^b u(x)\,(p(x)v')'\,dx$$

$$\int_a^b (p(x)\,u')'\,v(x)\,dx = \Big(v(b)\,p(b)\,u'(b) - p(b)\,v'(b)\,u(b)\Big) -$$

$$\Big(v(a)\,p(a)\,u'(a) - p(a)\,v'(a)\,u(a)\Big) + \int_a^b u(x)\,(p(x)v')'\,dx \quad \text{—— ②}$$

Using Separated boundary Conditions.
This term;

$$p(b)\Big[v(b)\,u'(b) - v'(b)\,u(b)\Big] - p(a)\Big[v(a)\,u'(a) - v'(a)\,u(a)\Big]$$

If we have;

$$\begin{array}{r|l} v(b) & \alpha_1 u(b) + \beta_1 u'(b) = 0 \\ u(b) & \alpha_1 v(b) + \beta_1 v'(b) = 0 \end{array}$$

$$v(b)\,u'(b) - v'(b)\,u(b) = 0$$

also

$$- \begin{array}{r|l} v(a) & \alpha_1 u(a) + \beta_1 u'(a) = 0 \\ u(a) & \alpha_1 v(a) + \beta_1 v'(a) = 0 \end{array}$$

$$v(a)\,u'(a) - v'(a)\,u(a) = 0$$

So Equation ② reduces.

$$\int_a^b (p(x)u')'v(x)\,dx = \int_a^b u(x)\,(p(x)v')'\,dx$$

Then Equation ① becomes.

$$\langle Lu, v\rangle = -\int_a^b u(x)\,(p(x)v')'\,dx - \int_a^b q(x)\,u(x)v(x)\,dx$$

from the R.H.S

$$\langle u, Lv\rangle = \int_a^b u(x)\,Lv\,w(x)\,dx$$

$$\langle u, Lv\rangle = -\int_a^b u(x)\frac{1}{w(x)}\left((p(x)v')' + q(x)\,v(x)\right)w(x)\,dx$$

$$\langle u, Lv\rangle = -\int_a^b (u(x)\,(p(x)v')' + q(x)\,u(x)v(x)\,dx$$

Hence

L.H.S = R.H.S, therefore

$$\underline{\underline{\langle Lu, v\rangle = \langle u, Lu\rangle}}$$

(2) $P(x)\, y'' + q(x)\, y' + R(x)\, y = 0$

Make the coefficients of y'' one.

$$y'' + \frac{q(x)}{P(x)} y' + \frac{R(x)}{P(x)} y = 0 \quad -(1)$$

the integrating factor $I(x) = e^{\int \frac{q(x)}{P(x)} dx}$

multiply Equation (1) through by $I(x)$

$$I(x)\, y'' + I(x) \frac{q(x)}{P(x)} y' + I(x) \frac{R(x)}{P(x)} y = 0 \quad -(2)$$

Since ~~the~~ $I(x)$ is obtained from.

$$\frac{dI}{dx} = I(x) \frac{q(x)}{P(x)}$$

then

$$I(x)\, y'' + I(x) \frac{q(x)}{P(x)} y' = (I y')'$$

Then Equation (2) becomes

$$(I y')' + I(x) \frac{R(x)}{P(x)} y = 0$$

take $p(x) = I(x)$

$$q(x) = I(x) \frac{R(x)}{P(x)}$$

then we have

$$\frac{d}{dx}\left(p(x) \frac{dy}{dx}\right) + q(x)\, y = 0$$

Hence: $L[y] = (py')' + qy$, for Sturm-Liouville form.

3. Laguerre Polynomials

14 (b). Laguerre Polynomials are defined by $L_0 = 1$, and

$$L_n(x) = \frac{e^x}{n!}\frac{d^n(x^n e^{-x})}{dx^n}, \quad n = 1, 2, \ldots$$

Show that

$$L_1(x) = 1 - x,$$

for $n = 1$

$$L_1(x) = \frac{e^x}{1}\frac{d}{dx}(xe^{-x}) = e^x(-xe^{-x} + e^{-x})$$

$$\underline{\underline{L_1(x) = 1 - x}}$$

for $n = 2$

$$L_2(x) = \frac{e^x}{2}\frac{d^2}{dx^2}(x^2e^{-x}) = \frac{e^x}{2}\frac{d}{dx}(2xe^{-x} - x^2e^{-x})$$

$$L_2(x) = \frac{e^x}{2}(2e^{-x} - 2xe^{-x} - 2xe^{-x} + x^2e^{-x})$$

$$\underline{\underline{L_2(x) = 1 - 2x + \tfrac{1}{2}x^2}}$$

$n = 3$

$$L_3(x) = \frac{e^x}{3!}\frac{d^3(x^3e^{-x})}{dx^3}$$

$$L_3(x) = \frac{e^x}{6} \frac{d^2}{dx^2} \left(3x^2 e^{-x} - x^3 e^{-x}\right)$$

$$= \frac{e^x}{6} \frac{d}{dx} \left(6x e^{-x} - 3x^2 e^{-x} - 3x^2 e^{-x} + x^3 e^{-x}\right)$$

$$L_3(x) = \frac{e^x}{6} \left(6 e^{-x} - 6x e^{-x} - 12x e^{-x} + 6x^2 e^{-x} + 3x^2 e^{-x} - x^3 e^{-x}\right)$$

$$L_3(x) = \frac{1}{6}\left(6 - 18x + 9x^2 - x^3\right)$$

$$L_3(x) = 1 - 3x + \frac{9}{6}x^2 - \frac{1}{6}x^3$$

$$L_3(x) = 1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3$$

Prove that the Laguerre polynomials, are orthogonal on the positive axis $0 \le x < \infty$ w.r.t weight function $r(x) = e^{-x}$, Show that

$$\int_0^\infty e^{-x} x^k L_n \, dx = 0$$

$$\int_0^\infty e^{-x} x^k L_n \, dx = \int_0^\infty e^{-x} x^k \left[\frac{e^x}{n!} \frac{d^n}{dx^n} (x^n e^{-x})\right] dx$$

$$= \frac{1}{n!} \int_0^\infty x^k \frac{d^n}{dx^n} (x^n e^{-x}) \, dx$$

Integration by Parts.

let $u = x^k \Rightarrow du = k x^{k-1} dx$

$$dv = \frac{d^n}{dx^n}(x^n e^{-x})\,dx \Rightarrow v = \frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})$$

$$\int_0^\infty u\,dv = uv\Big|_0^\infty - \int_0^\infty v\,du$$

$$\frac{1}{n!}\int_0^\infty x^k\left[\frac{d^n}{dx^n}(x^n e^{-x})\right]dx = \frac{1}{n!}\left(x^k\frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})\right)\Big|_0^\infty - \int_0^\infty kx^{k-1}\frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})$$

$$\lim_{x\to\infty} x^k\frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x}) = 0$$

$$\frac{1}{n!}\int_0^\infty x^k\left[\frac{d^n}{dx^n}(x^n e^{-x})\right]dx = -\int_0^\infty \frac{kx^{k-1}}{n!}\frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})\,dx$$

Integrating again by parts.

Let $u = x^{k-1} \Rightarrow du = (k-1)x^{k-2}dx$

$$dv = \frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})\,dx \Rightarrow v = \frac{d^{n-2}}{dx^{n-2}}(x^n e^{-x})$$

$$\frac{k}{n!}\int_0^\infty x^{k-1}\frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})\,dx = \frac{k}{n!}\left(x^{k-1}\frac{d^{n-2}}{dx^{n-2}}(x^n e^{-x})\Big|_0^\infty - \int_0^\infty (k-1)x^{k-2}\frac{d^{n-2}}{dx^{n-2}}(x^n e^{-x})\,dx\right)$$

Again $\lim_{x\to\infty} x^{k-1}\frac{d^{n-2}}{dx^{n-2}}(x^n e^{-x}) = 0$

$$\frac{k}{n!}\int_0^\infty x^{k-1}\frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})\,dx = \frac{-k}{n!}\int_0^\infty (k-1)x^{k-2}\frac{d^{n-2}}{dx^{n-2}}(x^n e^{-x})\,dx$$

So

$$\frac{1}{n!}\int_0^\infty x^k \frac{d^n}{dx^n}(x^n e^{-x})\,dx = \frac{K}{n!}\int_0^\infty (k-1)\,x^{k-2}\frac{d^{n-2}}{dx^{n-2}}(x^n e^{-x})\,dx$$

for the first Integration, we obtained, when $j=1$

$$\int_0^\infty x^k e^{-x} L_n(x)\,dx = \frac{-k}{n!}\int_0^\infty x^{k-1}\frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})\,dx$$

for the second one, $j=2$

$$\int_0^\infty x^k e^{-x} L_n(x)\,dx = \frac{k}{n!}\int_0^\infty (k-1)\,x^{k-2}\frac{d^{n-2}}{dx^{n-2}}(x^n e^{-x})\,dx$$

for the third one, $j=3$

$$\int_0^\infty x^k e^{-x} L_n(x)\,dx = \frac{-k}{n!}\int_0^\infty (k-1)(k-2)\,x^{k-3}\frac{d^{n-3}}{dx^{n-3}}(x^n e^{-x})\,dx$$

$\vdots$

for the j^{th} one

$$\int_0^\infty x^k e^{-x} L_n(x)\,dx = \frac{(-1)^j}{n!}\int_0^\infty k(k-1)\cdots(k-j+1)\,x^{k-j}\frac{d^{n-j}}{dx^{n-j}}(x^n e^{-x})\,dx$$

for the $j=k$

$$\int_0^\infty x^k e^{-x} L_n(x)\,dx = \frac{(-1)^k}{n!}k(k-1)(k-2)\cdots(k-k+1)\int_0^\infty x^{k-k}\frac{d^{n-k}}{dx^{n-k}}(x^n e^{-x})\,dx$$

$$= \frac{(-1)^k}{n!}k!\int_0^\infty \frac{d^{n-k}}{dx^{n-k}}(x^n e^{-x})\,dx$$

$$= \frac{(-1)^k}{n!}k!\left[\frac{d^{n-k-1}}{dx^{n-k-1}}(x^n e^{-x})\right]_0^\infty$$

$$\int_0^\infty x^k e^{-x} \ln(x)\, dx = \frac{(-1)^k k!}{n!}\left[\lim_{x\to\infty}\left(\frac{d^{n-k-1}}{dx^{n-k-1}}(x^n e^{-x})\right) - \lim_{x\to 0}\frac{d^{n-k-1}}{dx^{n-k-1}}(x^n e^{-x})\right]$$

$$\int_0^\infty x^k e^{-x} \ln(x)\, dx = 0$$

Hence

$\int_0^\infty x^k e^{-x} \ln(x)\, dx = 0$, at the k^{th} Integration.

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4) <u>Cauchy Sequence.</u>

Show that the Sequence $\{x_n\}$.

$$x_n = \sum_{k=1}^{n} \frac{1}{k!}$$

Is a Cauchy Sequence Using the measure of distance.

$$d(x, y) = |x - y|$$

let $\varepsilon > 0$, $\exists\, N$, Such that $n > N$ implies

$$|x_n - x| < \varepsilon/2$$

Consider $n, m \geq N$

$$d(x_n, x_m) = |x_n - x_m| \leq |x_n - x| + |x_m - x|$$

$$d(x_n, x_m) = \left|\sum_{k=1}^{n}\frac{1}{k!} - \sum_{k=1}^{m}\frac{1}{k!}\right|$$

$$= \left|\frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} - \left(1 + \frac{1}{2!} + \cdots \frac{1}{m!}\right)\right|$$

$$d(x_n, x_m) \leq \frac{1}{n!} \leq \left| x - \sum_{k=1}^{n} \frac{1}{k!} \right| + \left| x - \sum_{k=1}^{m} \frac{1}{k!} \right|$$

$$d(x_n, x_m) \leq \frac{1}{n!} + \frac{1}{m!}$$

Since $|x_n - x| = \frac{1}{n!} < \varepsilon/2$, $|x_m - x| = 1/m! < \varepsilon/2$

$$d(x_n, x_m) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$$\underline{\underline{d(x_n, x_m) < \varepsilon}}$$

5. <u>Fourier Series.</u>

$$f(t) = \sum_{k=-\infty}^{\infty} f_k e^{ikt} \quad , \quad g(t) = \sum_{k=-\infty}^{\infty} g_k e^{ikt}$$

Find the Fourier Series of the Convolution

$$h(t) = \int_0^{2\pi} f(t-x)\, g(x)\, dx$$

$$f(t-x) = \sum_{k=-\infty}^{\infty} f_k (t-x)\, e^{ik(t-x)}$$

$$g(x) = \sum_{k=-\infty}^{\infty} g_k (x)\, e^{ikx}$$

$$h(t) = \int_0^{2\pi} \sum_{k=-\infty}^{\infty} f_k (t-x)\, e^{ikt} e^{-ikx} \sum_{k=-\infty}^{\infty} g_k(x)\, e^{ikx}\, dx.$$

$$h(t) = \int_0^{2\pi} \sum_{k=-\infty}^{\infty} f_k(t-x) \sum_{k=-\infty}^{\infty} g_k(x)\, e^{ikt}\, dx$$

$$h(t) = \int_0^{2\pi} \sum_{k=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} f_k(t-x)\, g_k(x)\, e^{ikt}\, dx$$

Let $w(t) = \sum_{k=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} f_k(t-x)\, g_k(x) = \sum_{k=-\infty}^{\infty} f_k g_k$

$$w(t) = g(t) * f(t)$$

$$h(t) = \int_0^{2\pi} w(t)\, e^{ikt}\, dx$$

$$h(t) = w(t)\, e^{ikt} \int_0^{2\pi} dx = w(t)\, e^{ikt} (2\pi)$$

$$h(t) = 2\pi\, w(t)\, e^{ikt}$$

6) Discrete Fourier Transform.

$$\{f_n\} \to \{g_n\}$$

Let g_n be a vector in $\mathbb{R}^n$, its coordinates relative $f_n = (f_0, f_1, \dots, f_{N-1})$ relative to the Fourier basis.

$$\{f_n\} \to \{g_n\} \Rightarrow f_n = T g_n$$

where T is a N×N matrix called N-point fourier matrix.

If f_n is the DFT of g_n

$$f = (f_n)_{n=0}^{N-1}$$

The change of coordinates performed by the DFT is

$$g = f_0 \phi_0 + f_1 \phi_1 + \cdots + f_{N-1} \phi_{N-1}$$

$$g = (\phi_0 \quad \phi_1 \quad \cdots \quad \phi_{N-1}) f = T_N^{-1} f$$

where ϕ_n are columns of T_N^{-1}, which is the change of coordinates matrix from Fourier basis to the standard basis

Since columns of T_N^{-1} are orthonormal, then

$$(T_N)_{nk} = \frac{1}{\sqrt{N}} e^{-2\pi i n k/N}, \quad 0 \le n, k \le N-1$$

<u>T_N is a N-Point Fourier Matrix.</u>

7. DFT to FFT

If we consider a periodic function f,

$$f_j = \sum_{k=0}^{N-1} f_k W_N^{jk}, \quad j = 0, 1, \ldots N-1$$

given $W_N = e^{-2\pi i/N}$ and $W_N^{jk} = 1$

then if we take $N = 2^p$, then

$$f_j = \sum_{k=0}^{M-1} \left(f_{2k} W_N^{2jk} + f_{2k+1} W_N^{(2k+1)j} \right),$$

Assume $M = \frac{N}{2}$ then $W_N^{2jk} = W_M^{jk}$, $\hat{f}_j$ becomes

$$f_j = \sum_{k=0}^{M-1} f_{2k} W_M^{jk} + W_N^{j} \sum_{k=0}^{M-1} f_{2k+1} W_M^{jk}$$

In this case our computation reduces to two DFTs of length $M = \frac{N}{2}$, therefore we can split f_j into even and odd parts.

$$f_j^{even} = \sum_{k=0}^{M-1} f_{2k} W_M^{jk}, \quad \cancel{f}$$

$$f_j^{odd} = \sum_{k=0}^{M-1} f_{2k+1} W_M^{jk}$$

So

$$f_j = f_j^{even} + W_N^{j} f_j^{odd} \qquad \text{for } j=0, \dots M-1$$

and

$$f_{j+M} = f_{j+M}^{even} + W_N^{j+M} f_{j+M}^{odd}$$

Repeating the same steps, f_{j+M} becomes

$$f_{j+M} = f_j^{even} - W_N^{j} f_j^{odd}$$

So the two function f_j and f_{j+M} given what is called butterfly relationship.

Since we assume $N = 2^P \Rightarrow M = 2^{P-1}$ then we can conclude that repeating the butterfly relationship $(P-1)$ times will lead us to $\frac{N}{2}$ problems of dimension 2.

Hence the number of stages, P will become

$$\underline{\underline{P = \log_2 N}}$$

As seen from the Butterfly relationship, the total multiplication time is

$$T = \frac{N}{2} \log_2 N,$$

($O(N^2)$ operations becomes $O(N \log_2 N)$ using $\underline{\underline{N = 2^P}}$)