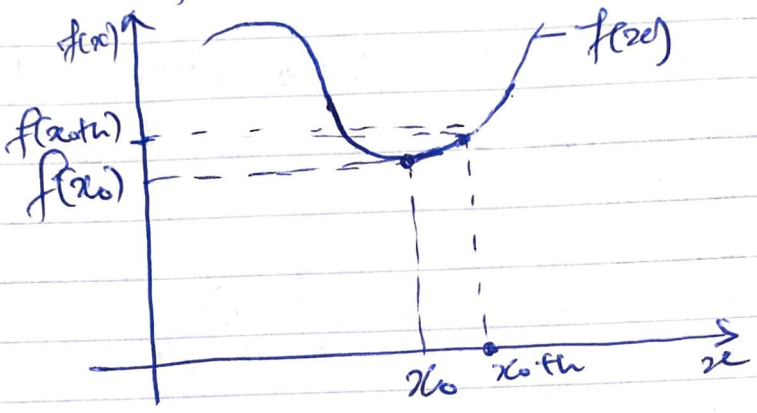
Brian KYANJO
MATH 537
Final Project

1. Show that If a Continuous, differentiable function $f(x)$ $\mathbb{R} \to \mathbb{R}$ has a local minimum at a point x_0, then $f'(x_0) = 0$.

Consider the function $f(x)$



the gradient of the curve at x_0 = gradient of tangent at x_0.

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{(x_0+h) - x_0} = \frac{f(x_0+h) - f(x_0)}{h}$$

$\lim_{h \to 0^+} f'(x_0) \geq 0$ and $\lim_{h \to 0^-} f'(x_0) \leq 0$, then at the local minimum point $f'(x_0) = 0$

2. Counter example:

Consider a function $f(x) = 3x + 2x^2$ $\forall x \in [0,1]$

at $x=0 \Rightarrow f(0) = 0$

$f'(x) = 3+4x \Rightarrow f'(0) = 3 \neq 0$, hence it is not sufficient.

2. Show that, If a differentiable function $J(x) = f(x) + \lambda g(x)$ has a local minimum at a point x_0, then $g(x_0) = 0$.

$$J(x) = f(x) + \lambda g(x) \quad \text{— (1)}$$

differentiating (1) w.r.t λ

$$\frac{\partial J(x)}{\partial \lambda} = g(x)$$

$$\left.\frac{\partial J(x)}{\partial \lambda}\right|_{x=x_0} = g(x_0) = 0.$$

□

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3. Duhamel's principle

$$u_t + cu_x = f(x,t)$$

The resulting solution is

$$u(x,t) = u(x-ct, 0) + \int_0^t f(x-c(t-\tau), \tau)\, d\tau$$

Use this solution to obtain the fundamental

Solution $G_0(x,t,\xi,\tau)$ for the equation

Same $u_t + cu_x = \delta(t-\tau)\,\delta(x-\xi)$,

$(x,t) \in (-\infty,\infty) \times [0,T]$

$$u(x,t) = u(x-ct,0) + \int_0^t f(t-c(t-\tau),\tau)\,d\tau$$

Can be written as,

$$G(x,t,\xi,\tau) = G(x-ct,0,\xi,\tau) + \int_0^t f(x-c(t-\tau),\tau)\,d\tau \quad \text{---(2)}$$

Same $u_t + cu_x = f(x,t) = \delta(t-\tau)\,\delta(x-\xi)$

$$f(x,t) = \delta(t-\tau,\, x-\xi)$$

So

$$f(x-c(t-\tau),\tau) = \delta(x-c(t-\tau)-\xi)\,\delta(\tau-\tau)$$
$$= \delta(x-c(t-\tau)-\xi,\,0)$$

then equation (1) becomes

$$G(x,t,\xi,\tau) = G(x-ct,0,\xi,\tau) + \int_0^{t-\tau} \delta(x-\xi-c(t-\tau-\eta),\eta)\,d\eta$$

$$G(x,t,\xi,\tau) = G(x-ct,0,\xi,\tau) + \int_0^{t-\tau} \delta(x-\xi-c(t-\tau)+c\eta)\,\delta(\eta)\,d\eta$$

$$G(x,t,\xi,\tau) = G(x-ct,0,\xi,\tau) + \int_0^{t-\tau} \delta\left(c\left(\frac{1}{c}(x-\xi)-(t-\tau)+\eta\right)\right)\delta(\eta)\,d\eta$$

Using a property $\delta(c\eta) = \frac{1}{|c|}\delta(\eta)$

$$\frac{\partial}{\partial t} G(x,t,\xi,\tau) = G(x-ct,0,\xi,\tau) + \frac{1}{c}\int_0^{t-\tau} \delta\left(\eta+\frac{1}{c}(x-\xi)-(t-\tau)\right)\delta(\eta)\,d\eta$$

$$G(x,t,\xi,\tau) = G(x-ct,0,\xi,\tau) + \frac{1}{c}\int_0^{t-\tau} \delta(\eta - (t-\tau) - \frac{1}{c}(x-\xi))\, \delta(\eta)\, d\eta$$

$$G(x,t,\xi,\tau) = G(x-ct,0,\xi,\tau) + \frac{1}{c}\delta\left(\left(t-\tau\right) - \frac{1}{c}(x-\xi)\right) H(t-\tau)$$

Since $\int_{-\infty}^{\infty} \delta(\eta - a)\,\delta(\eta - b)\, d\eta = \delta(a-b)$

So

$$G(x,t,\xi,\tau) = G(x-ct,0,\xi,\tau) + \delta\left(c(t-\tau) - (x-\xi)\right) H(t-\tau)$$

Since $\delta(x)$ is an even function i.e. $\delta(-x) = \delta(x)$ then

$$G(x,t,\xi,\tau) = G(x-ct,0,\xi,\tau) + \delta(x-\xi-c(t-\tau))\, H(t-\tau)$$

where $H(t-\tau) = \begin{cases} 0, & t<\tau \\ 1, & t>\tau \end{cases}$

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4) (Euler Lagrange equations) Consider the model problem.

$$u_t + c u_x = 0, \quad c>0$$
$$u(x,0) = 0$$
$$u(0,t) = 0$$

a) Show that $\beta_m = (\hat{u}(x_m,t_m) - d_m)$

$$\hat{u}(x,t) = \sum_{m=1}^{M} \beta_m r_m(x,t) \quad \text{——} (1)$$

differentiate (1) w.r.t t

$$\frac{\partial}{\partial t}\hat{u}(x,t) = \sum_{m=1}^{M} \beta_m \frac{\partial r_m(x,t)}{\partial t} \quad \text{— (2)}$$

differentiate (2) w.r.t x

$$\frac{\partial}{\partial x}\hat{u}(x,t) = \sum_{m=1}^{M} \beta_m \frac{\partial}{\partial x} r_m(x,t) \quad \text{— (3)}$$

Equation (2) + c(Equation (3))

$$\hat{u}_t(x,t) + c\hat{u}(x,t) = \sum_{m=1}^{M} \beta_m \left(\frac{\partial r_m(x,t)}{\partial t} + c\frac{\partial r_m(x,t)}{\partial x} \right)$$

$$\frac{\partial r_m(x,t)}{\partial t} + c\frac{\partial r_m(x,t)}{\partial x} = \alpha_m$$

$$\hat{u}_t(x,t) + c\hat{u}_x(x,t) = \sum_{m=1}^{M} \beta_m \alpha_m$$

but $\hat{u}_t + c\hat{u}_x = \lambda(x,t)$

$$\lambda(x,t) = \sum_{m=1}^{M} \beta_m \alpha_m \quad \text{— (4)}$$

$$\frac{\partial \lambda}{\partial x} = \sum_{m=1}^{M} \beta_m \frac{\partial \alpha_m}{\partial x} \quad \text{— (5)}$$

differentiate (4) w.r.t t

$$\frac{\partial \lambda}{\partial t} = \sum_{m=1}^{M} \beta_m \frac{\partial \alpha_m}{\partial t} \quad \text{— (6)}$$

c(Equation (4)) + Equation (6)

$$c\frac{\partial \lambda}{\partial x} + \frac{\partial \lambda}{\partial t} = \sum_{m=1}^{M} \beta_m \left(\frac{\partial \alpha_m}{\partial t} + c\frac{\partial \alpha_m}{\partial x} \right)$$

but

$$\frac{\partial \alpha_m}{\partial t} + c\frac{\partial \alpha_m}{\partial x} = -\delta(x - x_m)\,\delta(t - t_m)$$

and

$$c\lambda_x + \lambda_t = \sum_{m=1}^{M} (\hat{u}(x,t) - d_m)\,\delta(x - x_m)\,\delta(t - t_m)$$

$$\sum_{m=1}^{M} (\hat{u}(x,t) - d_m)\,\delta(x - x_m)\,\delta(t - t_m) = \sum_{m=1}^{M} f_m(-\delta(x - x_m)\delta(t - t_m)$$

$$\hat{u}(x_m, t_m) - d_m = -f_m$$

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c) Using (4), Show that Solutions for $\alpha_m(x,t)$ and $r_m(x,t)$ are given by

$$\alpha_m(x,t) = \delta(x - x_m - c(t - t_m))\,H(t - t_m) \quad , m = 1, 2, 3, \ldots M$$

$$G(x, \xi, t, \tau) = G(x - ct, 0, \xi, \tau) + \delta(x - \xi - c(t - \tau))\,H(t - \tau)$$

but $G(x - ct, 0, \xi, \tau) = 0$ at the boundary. replace ξ with x_m, and τ with t_m

$$G(x, x_m, t, t_m) = \delta(x - x_m - c(t - t_m))\,H(t - t_m) = \alpha_m(x,t)$$

so

$$-\alpha_m(x,t) = \delta((x - x_m) - c(t - t_m))\,H(t - t_m)$$

dividing through by negative.

$$\alpha_m(x,t) = \delta(x - x_m - c(t - t_m))\, H(t_m - t)$$

$$r_m(x,t) = \delta(x - x_m - c(t - t_m))\,[t + (t_m - t)\,H(t - t_m)]$$

$$m = 1, 2, \ldots, M$$

$$\frac{\partial r_m}{\partial t} + c\frac{\partial r_m}{\partial x} = \alpha_m$$

$$r_m(x,t) = \int_0^t \alpha_m(x_0 - c(t-\eta), \eta)\, d\eta$$

$$r_m(x,t) = \int_0^t \delta(x - c(t-\eta) - x_m - c(\eta - t_m))\, H(t_m - \eta)\, d\eta$$

$$= \int_0^t \delta(x - ct + c\eta - x_m - c\eta + ct_m)\, H(t_m - \eta)\, d\eta$$

$$r_m(x,t) = \int_0^t \delta(x - x_m - c(t - t_m))\, H(t_m - \eta)\, d\eta$$

$$r_m(x,t) = \delta(x - x_m - c(t - t_m)) \int_{t_m - t}^{t} H(t_m - \eta)\, d\eta$$

$$r_m(x,t) = \delta(x - x_m - c(t - t_m))\,(-R(t_m - \eta))\Big|_{t_m - t}^{t}$$

$$r_m(x,t) = \delta(x - x_m - c(t - t_m))\,(-R(t_m - t) + R(t_m - (t_m - t)))$$

$$r_m(x,t) = \delta(x - x_m - c(t - t_m))\,(R(t) - R(t_m - t))$$

but $R(x) = x\,H(x)$

$$r_m(x,t) = \delta(x - x_m - c(t - t_m))\,(t\,H(t) - (t_m - t)\,H(t_m - t))$$

$$r_m(x,t) = \delta(x - x_m - c(t - t_m)) \cdot (t + (t_m - t)\, H(t - t_m))$$

$m = 1, 2, \cdots, M$

4(b) Using (6) formulate a linear system

$$(R+I)\beta = d$$

using $\hat{u}(x_m, t_m) - d_m = -\beta_m$ — ① from 4(a) and that

$$\hat{u}(x,t) = \sum_{m=1}^{M} \beta_m r_m(x,t) \quad — ②$$

Since $r_{ij} = r_i(x_j, t_j)$ then ② can be written as

$$\hat{u}(x_j, t_j) = \sum_{m=1}^{M} \beta_m r_m(x_j, t_j), \quad j = 1, 2, \ldots M$$

from Equation ①, if $\hat{u}(x_m, t_m) = d_m - \beta_m$

then

$$\hat{u}(x_j, t_j) = \sum_{m=1}^{M} \beta_m r_m(x_j, t_j) = d_j - \beta_j, \quad j = 1, 2, \ldots M$$

$$\sum_{m=1}^{M} \beta_m r_m(x_j, t_j) + \beta_j = d_j, \quad j = 1, 2, \ldots, M$$

$$\beta_1 r_{ij} + \beta_2 r_{2j} + \cdots + \beta_M r_{Mj} + \beta_j = d_j; \quad j = 1, 2, \ldots, M$$

Since R is Symmetric then $r_{ij} = r_i(x_j, t_j) = r_j(x_i, t_i)$

$$\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1M} \\ r_{21} & r_{22} & \cdots & r_{2M} \\ \vdots & \vdots & & \vdots \\ r_{M1} & \cdots & \cdots & r_{MM} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_M \end{bmatrix} + \begin{bmatrix} 1 & & 0 \\ & 1 & \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_M \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_M \end{bmatrix}$$

R β I β d

$$(R\beta + I\beta) = d$$

$$(R + I)\beta = d$$

∎

N0.5

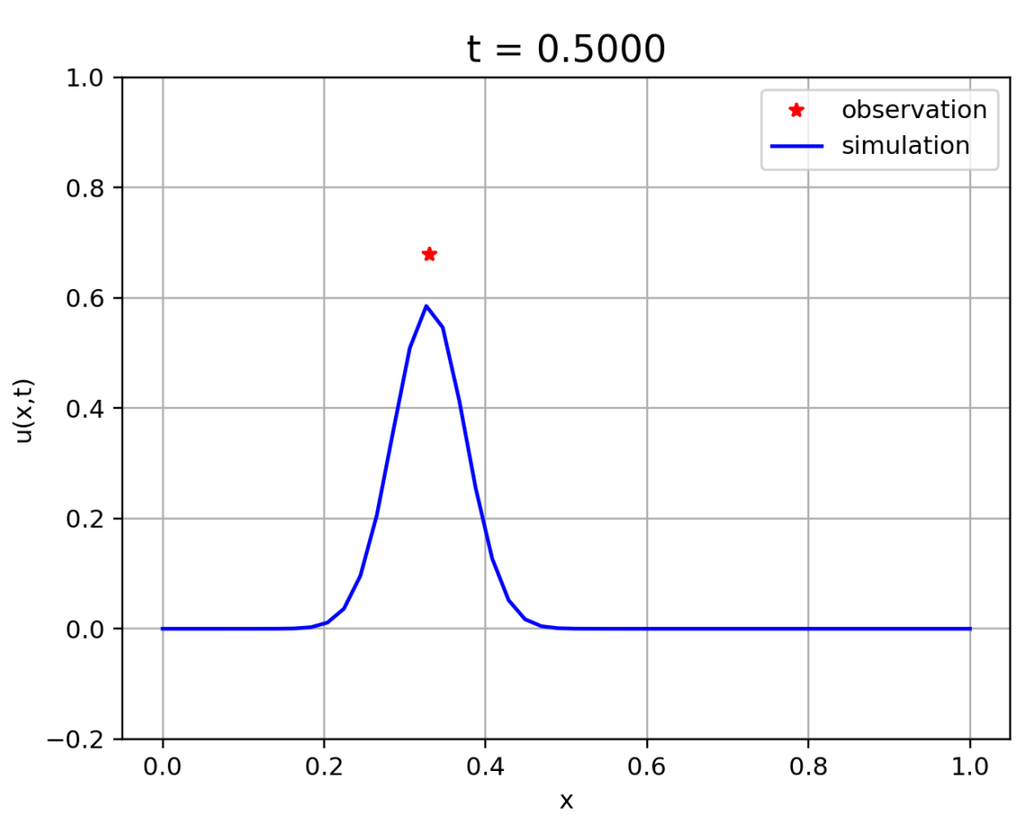
In [36]:
```
%matplotlib notebook
%pylab
```

```
Using matplotlib backend: nbAgg
Populating the interactive namespace from numpy and matplotlib
```

In [37]:
```
xo = 0.33
to = 0.5
do = 0.68
ep = 1e-3
c  = 0.5

x = linspace(0,1)

ro = (1/sqrt(2*pi*ep))*to

#beta_o of the system
bo = do/(ro+1)
print("beta_o of the system =",bo)

#delta functin
def delta(x,xo,ep,t,to):
    A = 1/sqrt(2*pi*ep)
    B = ((x-xo-(c*(t-to)))**2)/(4*ep)
    return A*exp(-B)

def H(t,to):
    if t-to >= 0:
        return 1
    else:
        return 0

def ucap(x,xo,t,to,do,ep,bo):
    B = t +(t-to)*H(t,to)
    return bo*delta(x,xo,ep,t,to)*B
```

```
beta_o of the system = 0.09305086168724995
```

In [39]:
```
uint = ucap(x,xo,0,to, do,ep,bo) #when t=0

fig = figure(1)

plot(xo,do,'r*',label='observation')

hdl,= plot(x,uint,'b-',label='simulation')


xlabel('x')
ylabel('u(x,t)')
ylim(-0.2,1.0)
legend()
tstr = 't = {:.4f}'
htitle = title(tstr.format(0),fontsize=15)
grid()

t = linspace(0,0.5)

for i in range(len(t)):

    u = ucap(x,xo,t[i],to,do,ep,bo)

    hdl.set_ydata(u)
    htitle.set_text(tstr.format(t[i]))

    pause(0.1)

    fig.canvas.draw()
```



In [ ]: