In [1]:
```
%matplotlib notebook
%pylab
```

```
Using matplotlib backend: nbAgg
Populating the interactive namespace from numpy and matplotlib
```

Haar Wavelets

Advantages of Fourier transforms (Discrete transform)

$$\hat{f}_n = \sum_{k=0}^{N-1} f_k e^{2\pi i n x_k}$$

- Frequency content
- Derivatives are easy to compute

But one major disadvange : We don't get any information about where the high frequencies occur. For truly periodic signals, this isn't a problem. But it makes Fourier transforms ill-suited for doing analysis of images or voice recordings, for example.

The *wavelet* transforms were deveoped to solve this problem.

Reference : ["Wavelets Made Easy", Yves Nievergelt, (Birkhäuser,, 1999). (https://www.amazon.com/Wavelets-Made-Easy-Yves-Nievergelt/dp/0817640614)](https://www.amazon.com/Wavelets-Made-Easy-Yves-Nievergelt/dp/0817640614)

Haar transform

To start, we define a *step* function (also referred to as a *scaling function*) as

$$\phi_{[0,1)}(x) = \begin{cases} 1 & \text{if} \quad 0 \le x < 1 \\ 0 & \quad \text{otherwise} \end{cases}$$

A related function is the *Haar wavelet*, given by

$$\psi_{[0,1)}(x) = \begin{cases} 1 & \text{if} \quad 0 \le x < \frac{1}{2} \\ -1 & \text{if} \quad \frac{1}{2} \le x < 1 \\ 0 & \quad \text{otherwise} \end{cases}$$

- The subscripts $[0, 1)$ on each function name indicates the support of the function. In general, the domains will be *dyadic* intervals $[k2^j, (k+1)2^j)$.
- The Haar wavelet function can be written as a difference of the step functions as

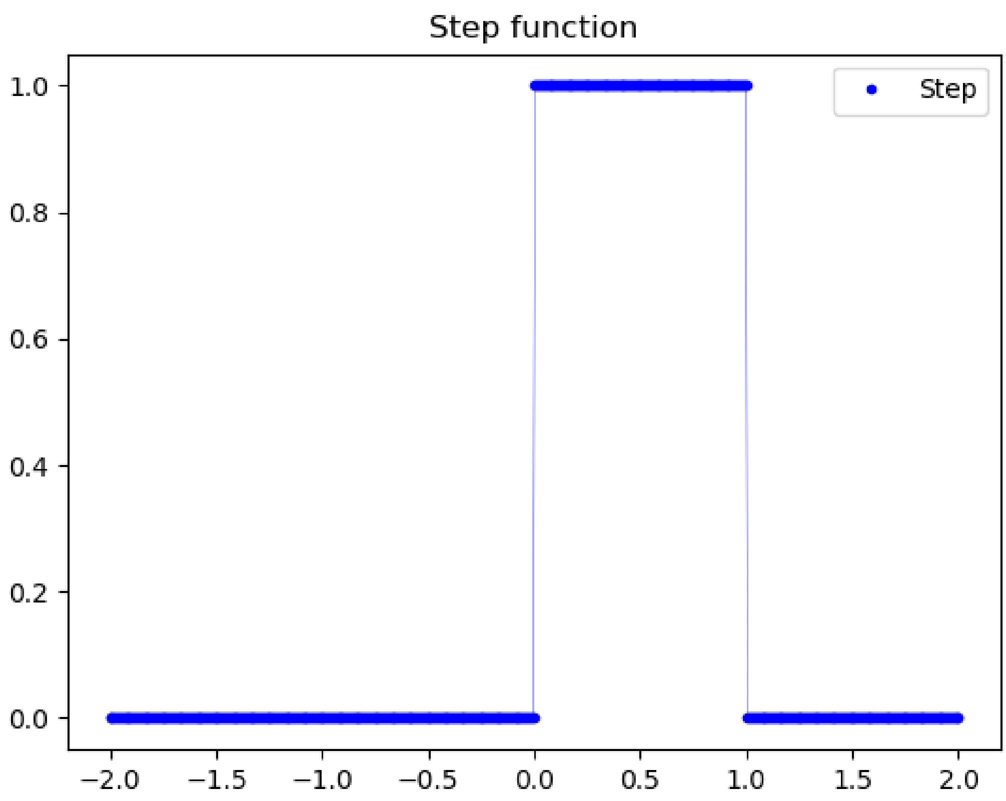
$$\psi_{[0,1)} = \phi_{[0,1/2)} - \phi_{[1/2,1)}$$

We can easily code and plot the step function $\phi_{[u,w)}(x)$ and $\psi_{[u,w)}(x)$.

In [2]:
```
def step(x,u=0,w=1):
    return where(logical_and(x >= u, x < w),1,0)

def haar(x,u=0,w=1):
    v = (u+w)/2
    b1 = logical_and(u <= x,x < v)
    b2 = logical_and(v <= x,x < w)
    return where(b1,1,0) + where(b2,-1,0)
```

In [3]:
```
figure(1)
clf()
x = linspace(-2,2,500)
plot(x,step(x),'b.',label="Step")
plot(x,step(x),'b-',linewidth=0.25)
title('Step function')
legend()
```



Out[3]: <matplotlib.legend.Legend at 0x11bd58390>

In [ ]:
```
figure(2)
clf()

plot(x,haar(x),'r.',label="Haar")
plot(x,haar(x),'r-',linewidth=0.25)
title('Haar wavelet')
legend()
```

Example - 2 point signal

We can use the step function to define a function on $C^0[0,1]$. For a signal $\mathbf{s} = (s_0, s_1)$ with two points, our reconstructed function is given by

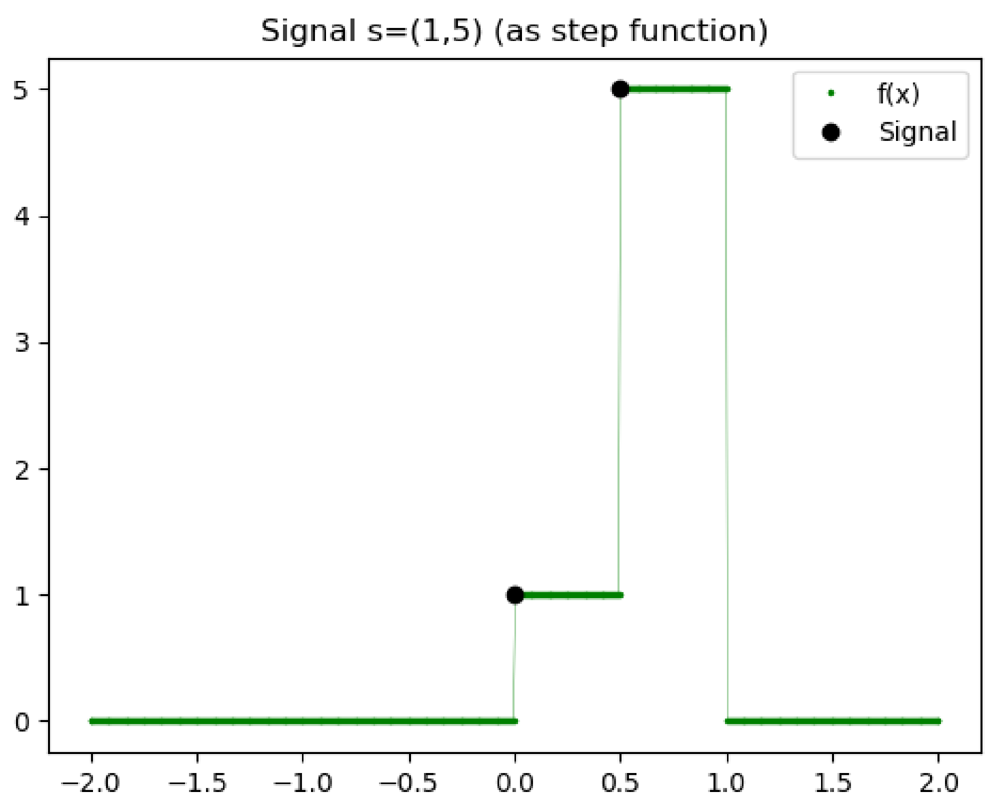
$$f(x) = s_0 \phi_{[0,1/2)}(x) + s_1 \phi_{[1/2,1)}(x)$$

In [4]:
```
figure(3)
clf()

s = [1,5]

f = s[0]*step(x,0,0.5) + s[1]*step(x,0.5,1)
plot(x,f,'g.',label='f(x)',markersize=3)
plot(x,f,'g',linewidth=0.25)
plot([0,0.5],s,'k.',markersize=12,label='Signal')

title('Signal s=(1,5) (as step function)')
legend()
```



Out[4]: `<matplotlib.legend.Legend at 0x11c21ced0>`

Haar Transform

Suppose we have a signal consisting of two data values (r_0, s_0) and (r_1, s_1). For simplicity, we will assume that $r_0 = 0$ and $r_1 = 1/2$, and we denote the signal of length 2^1 as $\mathbf{s}^1 = (s_0, s_1)$.

Writing the signal as a function on $C^0[0, 1]$ as

$$f(x) = s_0 \phi_{[0,1/2)}(x) + s_1 \phi_{[1/2,1)}(x)$$

we can then decompose $f(x)$ into an *average* plus a *difference* as

$$f(x) = \left(\frac{s_1 + s_2}{2}\right) \phi_{[0,1)}(x) + \left(\frac{s_1 - s_2}{2}\right) \psi_{[0,1)}(x)$$

This is the *wavelet* decomposition in terms of Haar bases. We write the *ordered* Haar transform on 2^1 points as

$$\mathbf{s}^0 = \left(\frac{s_1 + s_2}{2}; \frac{s_1 - s_2}{2}\right) = \left(a_0^0; c_0^0\right)$$

If we have four points in our signal $\mathbf{s}^2 = (s_0, s_1, s_2, s_3)$, we need two sweeps to produce a transform of four points.

Sweep 1

$$\mathbf{s}^1 = \left(\frac{s_0 + s_1}{2}, \frac{s_2 + s_3}{2}; \frac{s_0 - s_1}{2}, \frac{s_2 - s_3}{2}\right) = \left(a_0^1, a_1^1; c_0^1, c_1^1\right)$$

Sweep 2 (transform)

$$\mathbf{s}^0 = \left(\frac{s_0 + s_1 + s_2 + s_3}{4}; \frac{s_0 + s_1 - s_2 - s_3}{2}, \frac{s_0 - s_1}{2}, \frac{s_2 - s_3}{2}\right) = \left(a_0^0; c_0^0, c_0^1, c_1^1\right)$$

- This is an example of the *ordered* transform. Notice that the average of the data shows up as the first entry.
- The Haar transform does not depend on the abscissa $\{r_j\}$.

Inverse Haar Transform

Given the Haar Transform of our signal of length 2^1,

$$\mathbf{s}^0 = \left(\frac{s_0 + s_1}{2}; \frac{s_0 - s_1}{2}\right) = \left(a_0^0; c_0^0\right)$$

we can recover the original signal (s_0, s_1) exactly as follows.

$$\begin{aligned} s_0 &= a_0^0 + c_0^0 \\ s_1 &= a_0^0 - c_0^0 \end{aligned}$$

Example :

Suppose our signal is given by $\mathbf{s}^2 = (5, 1, 2, 8)$. The the two sweeps produce

Sweep 1

$$\mathbf{s}^1 = (3, 5; 2, -3)$$

Sweep 2 (transform)

$$\mathbf{s}^0 = (4; -1, 2, -3)$$

The "in-place" transform

A form that is easier for computation and analysis is the "in-place" transform. This produces the same coefficients as above, but in a different order.

Using the example from above - $\mathbf{s}^2 = (5, 1, 2, 8)$ - the two sweeps produce

Sweep 1

$$\mathbf{s}^1 = (3, 2, 5, -3)$$

Sweep 2 (transform)

$$\mathbf{s}^0 = (4, 2, -1, -3)$$

We can try this on a larger sample $\mathbf{s} = (3, 1, 0, 4, 8, 6, 9, 9)$ to get

Sweep 1

$$\mathbf{s}^2 = (2, 1, 2, -2, 7, 1, 9, 0)$$

Sweep 2

$$\mathbf{s}^1 = (2, 1, 0, -2, 8, 1, -1, 0)$$

Sweep 3 (transform)

$$\mathbf{s}^0 = (5, 1, 0, -2, -3, 1, -1, 0)$$

In [5]:
```
# Coding the In-Place Haar Transform

def haar_transform(s,prt=False):
    N = len(s)
    p = int(log2(N))
    inc = 1
    for k in range(p):
        n = 2**(p-k)
        s1 = s[range(0,N,inc)]
        r1 = range(0,n,2)
        r2 = range(1,n,2)
        a = (s1[r1] + s1[r2])/2.0
        c = (s1[r1] - s1[r2])/2.0
        s[range(0,N,2*inc)] = a
        s[range(inc,N,2*inc)] = c
        inc *= 2
        if prt:
            print("Sweep {:d}    ".format(p-k),s)

    return array(s).astype(float)

def inverse_Haar_transform(s,prt=False):
    N = len(s)
    p = int(log2(N))
    inc = N//2
    for k in range(p-1,-1,-1):
        n = 2**(p-k)
        s1 = s[range(0,N,inc)]
        r1 = range(0,n,2)
        r2 = range(1,n,2)
        a = (s1[r1] + s1[r2])
        c = (s1[r1] - s1[r2])
        s[range(0,N,2*inc)] = a
        s[range(inc,N,2*inc)] = c
        inc //= 2
        if prt:
            print("Sweep {:d}    ".format(p-k),s)

    return s
```

Example

We can decompose a signal of 8 points using the Haar wavelets.

In [6]:
```
s0 = array([3,1,0,4,8,6,9,9])
# s0 = array([9,1])
# s0 = array([5,1,2,8])

print("Signal s3")
print(s0)
print("")

print("Transform")
a = haar_transform(s0,prt=True)
# print(a)
print("")

print("Inverse Transform")
s = inverse_Haar_transform(a,prt=True)
# print(s)
```

```
Signal s3
[3 1 0 4 8 6 9 9]

Transform
Sweep 3      [ 2  1  2 -2  7  1  9  0]
Sweep 2      [ 2  1  0 -2  8  1 -1  0]
Sweep 1      [ 5  1  0 -2 -3  1 -1  0]

Inverse Transform
Sweep 1      [ 2.  1.  0. -2.  8.  1. -1.  0.]
Sweep 2      [ 2.  1.  2. -2.  7.  1.  9.  0.]
Sweep 3      [3. 1. 0. 4. 8. 6. 9. 9.]
```

(see notes)

Example : Temperature data

We can decompose temperature weekly temperature data, and determine where the largest variations occur.

In [7]:
```
T = array([32,10,20,38,37,28,38,34,18,24,18,9,23,24,28,34]).astype(float
);

print("T")
print(T)
print("")

print("Transform")
a = haar_transform(T)
print(a)
```

```
T
[32. 10. 20. 38. 37. 28. 38. 34. 18. 24. 18.  9. 23. 24. 28. 34.]

Transform
[25.9375 11.      -4.      -9.      -4.625    4.5     -1.75     2.       3.687
5
 -3.       3.75     4.5     -5.      -0.5     -3.75    -3.     ]
```

(see notes)

Properties of the Haar wavelets

The Haar wavelets can be expressed a shift and dilation of a base wavelet $\psi(x) \in L^2[R]$ as

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \qquad j, k = 0, \pm 1, \pm 2, \pm 3, \ldots$$

- The integral of $\psi_{n,k}(x)$ on R is 0, so $\psi_{n,k}(x) \in L^2[R]$
- $\|\psi_{n,k}(x)\| = 1$.
- The Haar wavelets are pair-wise orthogonal, i.e.

$$\int_R \psi_{n,k}(x)\psi_{n',k'}(x)\, dx = \delta_{n,n'}\delta_{k,k'}$$

- The Haar system $\psi_{n,k}(x)$ is *complete and forms an orthonormal system* in $L^2[R]$.
- Extensions to 2d are done via tensor products.
- A related transform, the *Walsh* transform, is described in Keener, Section 2.2.5 (page 79).

A disadvantage of the Haar functions is that they are discontinuous. Ingrid Daubechies solved this problem with the Daubechie wavelets.

In [ ]: