Homework #5

Math 537

1. In the following problems, assume that $\mathbf{x}$ and $\mathbf{y}$ are vectors in a normed vector space V with inner product $\langle \mathbf{x}, \mathbf{y} \rangle$ and an induced norm $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$.

   (a) Given vectors $\mathbf{x}$ and $\mathbf{y}$ in V, verify that the choice

   $$\alpha = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{y}\|^2} \tag{1}$$

   minimizes $\|\mathbf{x} - \alpha \mathbf{y}\|^2$. Show that this leads directly to the *Cauchy-Schwarz Inequality*

   $$|\langle \mathbf{x}, \mathbf{y} \rangle|^2 \leq \|\mathbf{x}\|^2 \cdot \|\mathbf{y}\|^2. \tag{2}$$

   (b) Given vectors $\mathbf{x}$ and $\mathbf{y}$ in V, show that $|\langle \mathbf{x}, \mathbf{y} \rangle|^2 = \|\mathbf{x}\|^2 \cdot \|\mathbf{y}\|^2$ if and only if $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent.

2. Starting with the set $\left\{1, \mathbf{x}, \mathbf{x}^2, \mathbf{x}^3, \ldots, \right\}$, use the Gram-Schmidt process and the inner product

   $$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) \omega(x) dx, \qquad \omega(x) > 0 \tag{3}$$

   to find the first four orthogonal polynomials for

   (a) $\omega(x) = 1$ (Legendre polynomials)

   (b) $\omega(x) = (1 - x^2)^{-1/2}$ (Chebyshev polynomials).

   Hint: Review the Gram-Schmidt procedure for generating a set of orthonomal vectors in $\mathcal{R}^n$ from a linearly independent set. The resulting orthogonal set will span the same space as the original set. A good reference is the Wikipedia page on "Gram-Schmidt process".

3. Find the best quadratic polynomial fit to $f(x) = |x|$ on the interval $[-1, 1]$ relative to the inner product $\langle f, g \rangle$ used in Problem 2 relative to weights $\omega(x) = 1$ and $\omega(x) = (1 - x^2)^{-1/2}$. Which approximation does a better job?

   For this problem, Wolfram Alpha makes is very easy to compute the integrals needed to go beyond a quadratic approximation, so don't hesitate to try finding a degree 4, 6 or 8 polynomial approximations to $f(x) = |x|$. Be careful to normalize the series terms properly. For example, $\langle P_n(x), P_n(x) \rangle \neq 1$.