Homework #3

Math 537

1. Work through the details needed to show why the Method of Frobenius, applied to

$$x^2y'' + xb(x)y' + c(x)y = 0 \tag{1}$$

leads to three cases, depending on the roots of the indicial equation.

Assume that $b(x)$ and $c(x)$ are both analytic at $x = 0$ and have power series solutions of the form

$$\begin{aligned} b(x) &= b_0 + b_1x + b_2x^2 + b_3x^3 + \dots \\ c(x) &= c_0 + c_1x + c_2x^2 + c_3x^3 + \dots \end{aligned}$$

Assume a Frobenius series solution of the form

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m$$

(a) Show that the coefficient of x^r is $P(r) \equiv r^2 + (b_0 - 1)r + q_0$ (the indicial equation).

(b) Show that the coefficient of x^{r+m}, $m = 1, 2, 3, \dots$ is given by

$$P(r+m)a_m = -\sum_{k=0}^{m-1} [(r+k)b_{m-k} + c_{m-k}]a_k, \qquad m = 1, 2, 3, \dots \tag{2}$$

This recursion relation is key to determining the cases we need to consider. Let r_1 and r_2 be the two roots of the indicial equation.

(c) **A solution Frobenius series solution:** Show that you can always get at least one solution $y_1(x)$ in the form of a Frobenius series. **Hint:** Assume that $r_1 > r_2$ and show that you can compute all of the necessary coefficients using $r = r_1$ in (2).

To find the second solution, we have to consider three cases.

i. **Case 1:** Show that if r_1 and r_2 do not differ by an integer, than we can find a second linearly independent solution in the form of a Frobenius series using the root r_2. Why are the solutions you get linearly independent? **Hint:** Show that if $r = r_2$ in (2), then you can compute all of the coefficients.

ii. **Case 2:** Suppose that $r_1 = r_2$. Show that the second solution must be of the form

$$y_2(x) = y_1(x)\ln(x) + x^{r_2} \sum_{m=0}^{\infty} A_m x^m \tag{3}$$

Hint: Argue that for any choice of r, we have

$$L[y(x;r)] \equiv x^2y'' + xb(x)y' + c(x)y = (r - r_1)^2 a_0 x^r \tag{4}$$

Then show that

$$\frac{d}{dr}L[y(x;r)] = L\left[\frac{d}{dr}y(x;r)\right] = 0 \tag{5}$$

and so that $y_2(x;r_1) = \frac{d}{dr}y(x;r_1)$ is a solution to the ODE. What are the coefficients A_m in the second solution?

iii. **Case 3:** Suppose that $r_1 - r_2 = N$ (a positive integer). Show that in this case, we can't find a second solution in the form of a Frobenius series. Try the trick used in Problem 1(c)ii and show that we would need a solution to the inhomogeneous equation

$$L\left[\frac{d}{dr}y(x;r_1)\right] = a_0 P'(r_1)x^{r_1} \tag{6}$$

Show that we can find a series solution to this inhomogeneous equation of the form

$$x^{r_2}\sum_{m=0}^{\infty} c_n x^m \tag{7}$$

Show that subtracting $\frac{d}{dr}y(x;r_1)$ from this series solution is then a solution to (1).