The TV splitting for a class of hyperbolic systems

Eleuterio F. TORO

Laboratory of Applied Mathematics
University of Trento, Italy
Email: eleuterio.toro@unitn.it
Webpage: eleuteriotoro.com

Short Course on:
Advanced numerical methods for hyperbolic equations From 1 February to 5 February 2021.
University of Trento, Italy

February 4, 2021

Aims of this talk:

1. **To review the TV flux (TV)**

    - The TV flux vector splitting for the Euler equations

    - The TV associated numerical method and results

2. **To briefly mention some extensions of the TV flux:**

    - General equation of state

    - Multiple space dimensions

    - High-order of accuracy

    - Explicit/implicit approaches: all-Mach number flows

(TV) E F Toro and M E Vázquez-Cendón. Flux splitting schemes for the Euler equations. Computers and Fluids. Vol. 70, Pages 1-12, 2012).

Context: The Numerical Flux Saga

$$\partial_t \mathbf{Q} + \partial_x \mathbf{F}(\mathbf{Q}) = \mathbf{0} \tag{1}$$

$$\mathbf{Q}_i^{n+1} = \mathbf{Q}_i^n - \frac{\Delta t}{\Delta x}[\mathbf{F}_{i+\frac{1}{2}} - \mathbf{F}_{i-\frac{1}{2}}] \tag{2}$$

- $\mathbf{F}_{i+\frac{1}{2}}$ is the numerical flux, to be determined
- There are several approaches to find $\mathbf{F}_{i+\frac{1}{2}}$
  - The Godunov approach. The Riemann problem. Upwinding
  - The flux vector splitting approach. Upwinding
  - The centred (or central) approach. No upwinding, e.g. the Lax-Friedrichs flux, the FORCE flux
- The search for the ideal flux seems to be endless!

The Euler equations in 1D

$$\partial_t \mathbf{Q} + \partial_x \mathbf{F}(\mathbf{Q}) = \mathbf{0} \tag{3}$$

$$\mathbf{Q} = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \quad \mathbf{F}(\mathbf{Q}) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \end{bmatrix} \tag{4}$$

- ρ is density
- u is particle velocity
- p is pressure
- E is total energy

$$E = \rho(\frac{1}{2}u^2 + e) \tag{5}$$

Equation of state. The specific internal energy is

$$e = e(\rho, p) \quad \text{or} \quad p = p(\rho, e) \tag{6}$$

For ideal gases

$$e(\rho, p) = \frac{p}{\rho(\gamma - 1)} \tag{7}$$

Conservative method and flux splitting

For solving (3)-(4) numerically we adopt the conservative method

$$\mathbf{Q}_i^{n+1} = \mathbf{Q}_i^n - \frac{\Delta t}{\Delta x}[\mathbf{F}_{i+\frac{1}{2}} - \mathbf{F}_{i-\frac{1}{2}}] \tag{8}$$

- $\mathbf{F}_{i+\frac{1}{2}}$ is the numerical flux, to be determined
- Here we adopt the flux vector splitting (FVS) approach

$$\mathbf{F}(\mathbf{Q}) = \mathbf{A}(\mathbf{Q}) + \mathbf{P}(\mathbf{Q}) \tag{9}$$

Numerical flux:

$$\mathbf{F}_{i+\frac{1}{2}} = \mathbf{A}_{i+\frac{1}{2}} + \mathbf{P}_{i+\frac{1}{2}} \tag{10}$$

Existing FVS schemes

- Steger and Warming (JCP, Vol. 4, pp 263-293, 1981)
- van Leer (ICASE Repor 82-30, NASA Langley, 1982)
- Zha and Bilgen (IJNMF Vol. 17, pp 115-144, 1993)
- Liou and Steffen (JCP Vol. 107, pp 23-39, 1993)
- Toro and Vázquez (2012)
- Chalons, Girardin and Kokh (2014)

Remarks on FVS schemes

- Upwinding is simpler, compared to the Godunov approach
- Resolution of contact waves is an issue

For background on FVS methods see:

E F Toro. Riemann solvers and numerical methods for fluid dynamics, 3rd edition, Springer, 2009. Chap. 8

TV Splitting for Euler in 1D

Recalling

$$\mathbf{F}(\mathbf{Q}) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \end{bmatrix} \quad , \qquad E = \rho[\frac{1}{2}u^2 + e(\rho, p)] \tag{11}$$

The flux may be split thus (TV)

$$\mathbf{F}(\mathbf{Q}) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \end{bmatrix} = \begin{bmatrix} \rho u \\ \rho u^2 \\ \frac{1}{2}\rho u^3 \end{bmatrix} + \begin{bmatrix} 0 \\ p \\ u(\rho e + p) \end{bmatrix} \tag{12}$$

Advection and pressure fluxes for **general equation of state**

$$\mathbf{A}(\mathbf{Q}) = u \begin{bmatrix} \rho \\ \rho u \\ \frac{1}{2}\rho u^2 \end{bmatrix} \quad , \quad \mathbf{P}(\mathbf{Q}) = \begin{bmatrix} 0 \\ p \\ u(\rho e + p) \end{bmatrix} \tag{13}$$

(TV) E F Toro and M E Vázquez-Cendón. Flux splitting schemes for the Euler equations. Computers and Fluids. Vol. 70, Pages 1-12, 2012).
See also:
Park JH, Munz CD. Multiple pressure variables methods for fluid flow at all mach numbers. Int. J. Numer. Methods Fluids. 2005;49:905-931

The TV flux in 1D for ideal gases

For ideal gases

$$e(\rho, p) = \frac{p}{\rho(\gamma - 1)} \tag{14}$$

$$\mathbf{A}(\mathbf{Q}) = u \begin{bmatrix} \rho \\ \rho u \\ \frac{1}{2}\rho u^2 \end{bmatrix}, \quad \mathbf{P}(\mathbf{Q}) = \begin{bmatrix} 0 \\ p \\ \frac{\gamma}{\gamma-1} up \end{bmatrix} \tag{15}$$

The **Numerical flux** is proposed as

$$\mathbf{F}_{i+\frac{1}{2}} = \mathbf{A}_{i+\frac{1}{2}} + \mathbf{P}_{i+\frac{1}{2}} \tag{16}$$

with

$$\left.\begin{array}{ll} \mathbf{A}_{i+\frac{1}{2}} \leftarrow \text{Riemann problem for advection:} & \partial_t \mathbf{Q} + \partial_x \mathbf{A}(\mathbf{Q}) = \mathbf{0} \\ \mathbf{P}_{i+\frac{1}{2}} \leftarrow \text{Riemann problem for pressure:} & \partial_t \mathbf{Q} + \partial_x \mathbf{P}(\mathbf{Q}) = \mathbf{0} \end{array}\right\} \tag{17}$$

IVP for analogue problem

Consider IVP for linear advection

$$\left.\begin{array}{l}\partial_t q(x,t) + \lambda \partial_x q(x,t) = 0 \ , -\infty < x < \infty \ , t > 0 \\ \\ q(x,0) = h(x)\end{array}\right\} \qquad (18)$$

Split the characteristic speed as

$$\lambda = \beta\lambda + (1-\beta)\lambda = \lambda_a + \lambda_p \ , \quad 0 \le \beta \le 1 \qquad (19)$$

Then solve two Riemann problems in succession:

$$\left.\begin{array}{l}a_{i+\frac{1}{2}} \leftarrow \text{RP for advection:} \quad \partial_t q + \lambda_a \partial_x q = 0 \\ \\ p_{i+\frac{1}{2}} \leftarrow \text{RP for pressure:} \quad \partial_t q + \lambda_p \partial_x q = 0\end{array}\right\} \qquad (20)$$

Claim: The exact solution of the IVP (20) for a time step Δt can be obtained by solving the IVPs (20) in succession.

The advection system

$$\partial_t \mathbf{Q} + \partial_x \mathbf{A}(\mathbf{Q}) = \mathbf{0} \tag{21}$$

$$\mathbf{Q} = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix} \;, \;\; \mathbf{A}(\mathbf{Q}) = \begin{bmatrix} \rho u \\ \rho u^2 \\ \frac{1}{2}\rho u^3 \end{bmatrix} \tag{22}$$

Real eigenvalues

$$\lambda_1 = 0 \;, \;\; \lambda_2 = \lambda_3 = u \tag{23}$$

Eigenvectors

$$\mathbf{R}_1 = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \;, \;\; \mathbf{R}_2 = \alpha_2 \begin{bmatrix} 1 \\ u \\ \frac{1}{2}u^2 \end{bmatrix} \tag{24}$$

- System is weakly hyperbolic
- It turns out that the pressure system furnishes all needed information

The pressure system

In terms of physical variables the pressure system reads

$$\partial_t \mathbf{V} + \mathbf{B}(\mathbf{V})\partial_x \mathbf{V} = \mathbf{0} \tag{25}$$

with

$$\mathbf{V} = \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/\rho \\ 0 & \gamma p & u \end{bmatrix} \tag{26}$$

The real and distinct eigenvalues are

$$\lambda_1 = \frac{1}{2}(u - A), \quad \lambda_2 = 0, \quad \lambda_3 = \frac{1}{2}(u + A) \tag{27}$$

$$A = \sqrt{u^2 + 4a^2} \tag{28}$$

The pressure system

- The pressure system (25) is **hyperbolic**

- The pressure system (25) is **subsonic**

$$\lambda_1 = \frac{1}{2}(u - A) < \lambda_2 = 0 < \lambda_3 = \frac{1}{2}(u + A) \tag{29}$$

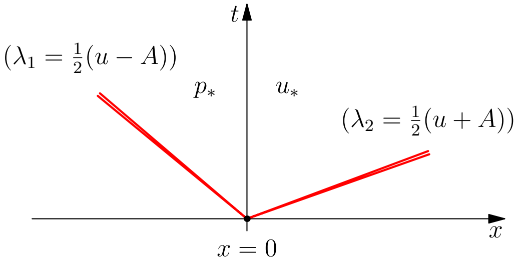
- Solution of the Riemann problem always **subsonic**

- **No transonic** states

- **No sampling** of the solution is necessary

Riemann problem for the pressure system

$$\left.\begin{array}{l} \partial_t \mathbf{V} + \mathbf{B}(\mathbf{V})\partial_x \mathbf{V} = \mathbf{0}\,, \\ \\ \mathbf{V}(x,0) = \left\{ \begin{array}{lllll} \mathbf{V}_L & \equiv & \mathbf{V}_i^n & \text{if} & x < 0 \\ \\ \mathbf{V}_R & \equiv & \mathbf{V}_{i+1}^n & \text{if} & x > 0 \end{array} \right. \end{array} \right\} \tag{30}$$



Structure of the solution of the Riemann problem for the pressure system.

Very simple solution of Riemann problem

Linearised Riemann invariants yield

$$\left.\begin{aligned}
u^*_{i+\frac{1}{2}} &= \frac{C_R u_R - C_L u_L}{C_R - C_L} - \frac{2}{C_R - C_L}(p_R - p_L) \\
p^*_{i+\frac{1}{2}} &= \frac{C_R p_L - C_L p_R}{C_R - C_L} + \frac{1}{2}\frac{C_R C_L}{C_R - C_L}(u_R - u_L) \\
\rho_K &= \begin{cases} \rho_i^n & \text{if} \quad u^*_{i+\frac{1}{2}} \geq 0 \\ \rho_{i+1}^n & \text{if} \quad u^*_{i+\frac{1}{2}} < 0 \end{cases}
\end{aligned}\right\} \tag{31}$$

with

$$C_L = \rho_L(u_L - A_L)\,, \quad C_R = \rho_R(u_R + A_R) \tag{32}$$

$$A = \begin{cases} \sqrt{u^2 + 4a^2} & \text{for ideal gases} \\ \sqrt{u^2 + 4h/(\rho e_p)} & \text{for real gases} \end{cases} \tag{33}$$

Remarks on solution of Riemann problem

- Solution (31) is obtained by applying the simplest of approximations, namely linearised Riemann invariants

- Such solver is useless when used in conventional formulations:
  - entropy violation
  - spurious vacuum
  - it crashes easily in the presence of strong shocks

- It is highly surprising that with the proposed flux vector splitting approach, the scheme works so well

- For general EOS, the numerical flux requires three EOS-function evaluations

The TV Numerical Flux

$$\mathbf{F}_{i+\frac{1}{2}} = \mathbf{A}_{i+\frac{1}{2}} + \mathbf{P}_{i+\frac{1}{2}} \ , \tag{34}$$

Advection flux $$\mathbf{A}_{i+\frac{1}{2}} = \begin{cases} u^*_{i+\frac{1}{2}} \begin{bmatrix} \rho \\ \rho u \\ \frac{1}{2}\rho u^2 \end{bmatrix}^n_i & \text{if} \quad u^*_{i+\frac{1}{2}} \geq 0 \\ u^*_{i+\frac{1}{2}} \begin{bmatrix} \rho \\ \rho u \\ \frac{1}{2}\rho u^2 \end{bmatrix}^n_{i+1} & \text{if} \quad u^*_{i+\frac{1}{2}} < 0 \end{cases} \tag{35}$$

Pressure flux $$\mathbf{P}_{i+\frac{1}{2}} = \begin{bmatrix} 0 \\ p^*_{i+\frac{1}{2}} \\ u^*_{i+\frac{1}{2}} \left[\rho_K e(\rho_K, p^*_{i+\frac{1}{2}}) + p^*_{i+\frac{1}{2}} \right] \end{bmatrix} \tag{36}$$

Properties of the scheme

1. **Positivity of density.** The update formula for density becomes

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x}[u^*_{i+\frac{1}{2}}\rho_{K,i+\frac{1}{2}} - u^*_{i-\frac{1}{2}}\rho_{K,i-\frac{1}{2}}] \ , \tag{37}$$

where $u^*_{i-\frac{1}{2}}$, $u^*_{i+\frac{1}{2}}$, $\rho_{K,i-\frac{1}{2}}$ and $\rho_{K,i+\frac{1}{2}}$ are given by (31). Introducing Courant numbers

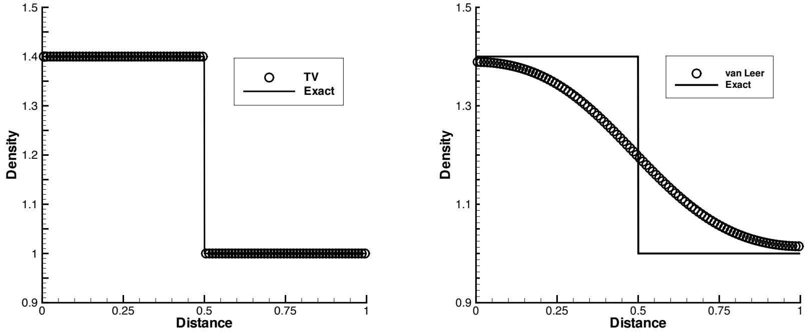
$$c_{i-\frac{1}{2}} = \frac{\Delta t}{\Delta x}u^*_{i-\frac{1}{2}} \ , \quad c_{i+\frac{1}{2}} = \frac{\Delta t}{\Delta x}u^*_{i+\frac{1}{2}} \tag{38}$$

the condition for positivity of density is ρ_i^{n+1} is

$$max_i \left\{ c_{i+\frac{1}{2}} \right\} \leq \frac{1}{2} \tag{39}$$

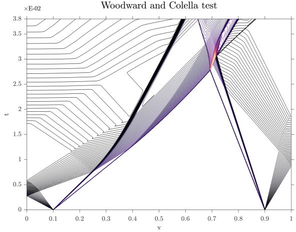
2. **Recognition of contacts/shear waves**. The scheme **recognises exactly**, isolated stationary contact discontinuities/shear waves for the Euler equations. For proofs see Toro and Vázquez (2012).

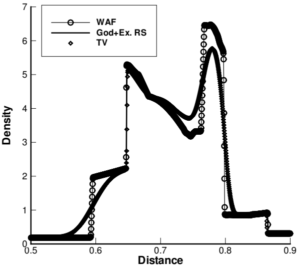
Tests for ideal gases: isolated stationary contact



Isolated stationary contact discontinuity.
Left: Toro-Vázquez flux (symbol) and exact solution (line)
Right: van Leer flux splitting.

Woodward & Collela blast wave problem



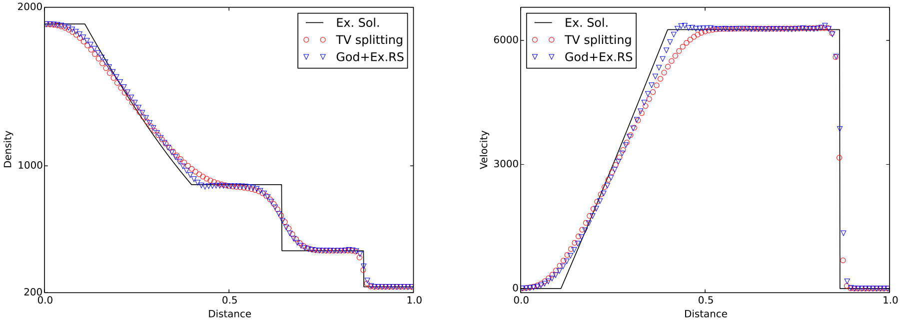


EXTENSIONS

- Eleuterio F. Toro, Cristóbal E. Castro, Bok Jik Lee. **A novel numerical flux for the 3D Euler equations with general equation of state**. Journal of Computational Physics. Vol. 303, pages 80-94, 2015.
- Dinshaw S. Balsara, Gino Montecinos, and Eleuterio F. Toro. **Exploring Various Flux Vector Splittings for the Magnetohydrodynamic System.** Journal of Computational Physics. Vol. 311, Pages 1-21, 2016.
- Svetlana Tokareva and Eleuterio F. Toro. **A flux vector splitting method for the Baer-Nunziato equations of compressible two-phase flow.** Journal of Computational Physics. Submitted, 2016.
- M. Dumbser, D. S. Balsara, M. Tavelli and F. Fambri. **A divergence-free semi-implicit finite volume scheme for ideal, viscous, and resistive magnetohydrodynamics.** Int. J. Numer. Meth. Fluids. Vol. 89, pp 1642 (2019)

SAMPLE NUMERICAL RESULTS

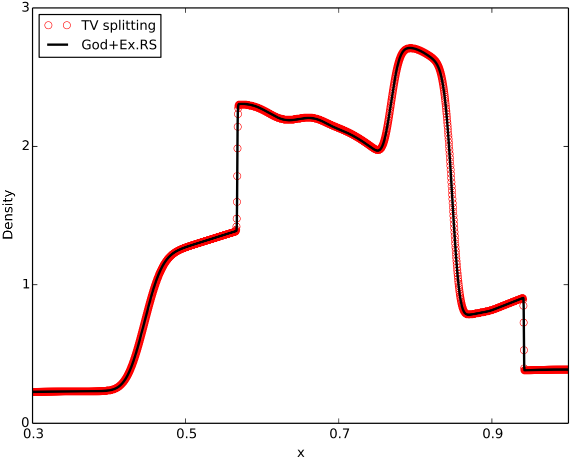
1D Euler, Jones-Wilkins-Lee EOS, TV flux



Accuracy: shock-tube problem for the Euler equations with the Jones-Wilkins-Lee equation of state. Comparison between the exact (line) and first-order numerical solutions: TV flux (circles) denotes the present scheme and God+Ex.RS (triangles) denotes the Godunov method in conjunction with the exact solution of the Riemann problem.

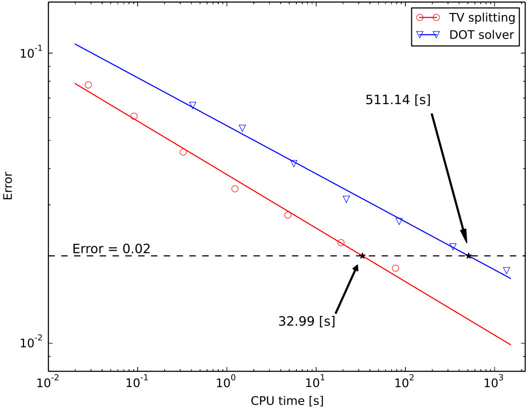
Eleuterio F. Toro, Cristóbal E. Castro, Bok Jik Lee. A novel numerical flux for the 3D Euler equations with general equation of state. Journal of Computational Physics. Vol. 303, pages 80-94, 2015

1D Euler, Jones-Wilkins-Lee EOS, TV flux



Robustness: the Woodward and Colella test problem for the Euler equations with the Jones-Wilkins-Lee equation of state. Comparison between the present TV flux (circles) and the first-order Godunov method in conjunction with the exact Riemann solver (full line) at time $t_{out} = 0.038$.

Efficiency

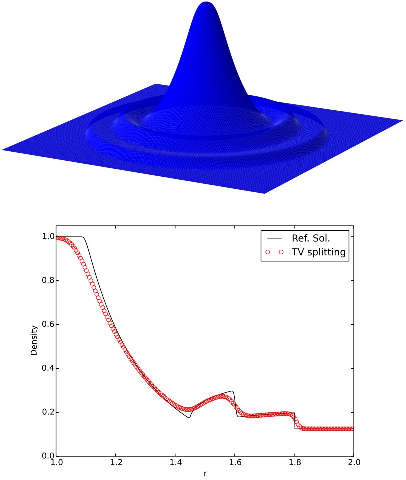


Efficiency. Efficiency plot $Error$ versus $CPUtime$ for a sequence of 7 meshes, for a shock-tube problem for the Euler equations with the Jones-Wilkins-Lee equation of state. Two methods are compared: the present TV splitting scheme and the DOT scheme. It is seen that for the same $Error = 0.02$ the present TV scheme is 15 times more efficient than the DOT scheme.

Reference for DOT solver:

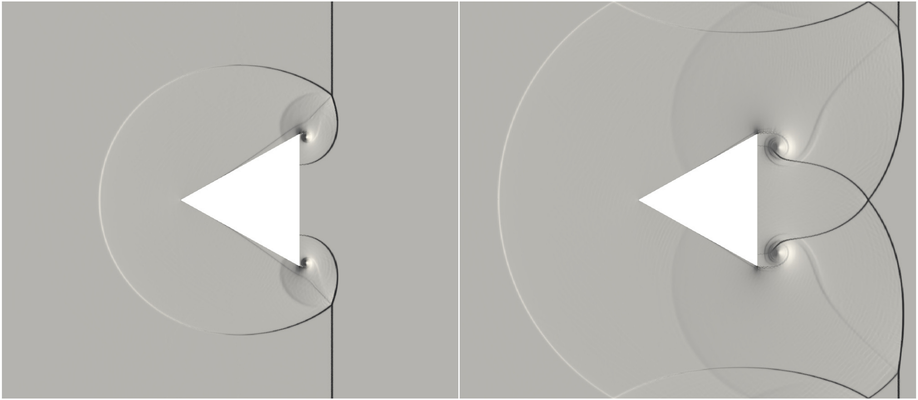
Michael Dumbser and Eleuterio Toro. A simple extension of the Osher Riemann solver to general non-conservative hyperbolic systems. Journal of Scientific Computing. Volume 48, Pages 70-88, 2011

2D Euler, van der Waals EOS, TV flux



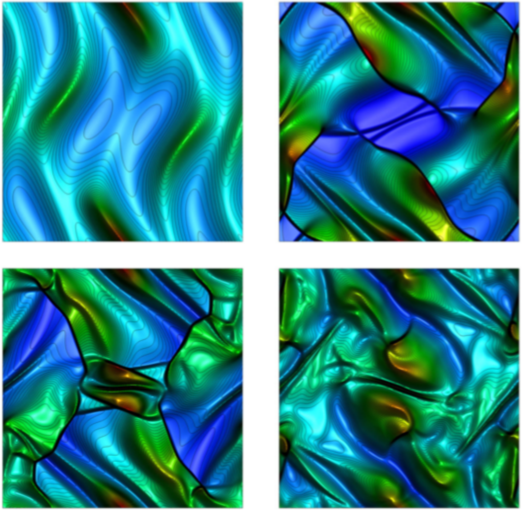
Spherical explosion test problem for the Euler equations with the van der Waals equation of state on a Cartesian mesh. Top frame: first-order TV solution for density at $z = 0$. Bottom frame: First-order TV solution (symbol) along radial direction compared to reference numerical solution (line) along the radial direction.

High-order ADER with TV flux



Shock wave impinging on stationary triangular solid body. Incident shock Mach number is $= 1.3$. The 2D Euler equations with van der Waals equation of state are solved on unstructured mesh with $\sim 1 \times 10^6$ triangles. Numerical method: ADER third order with TV flux. Density contours shown at times $t = 1.6 \times 10^{-3}$ (top) and $t = 2.2 \times 10^{-3}$ (bottom).

Explicit/Implicit implementation (Dumbser et al. 2019)



M. Dumbser, D.S. Balsara, M Tavelli and F. Fambri. A divergence-free semi-implicit finite volume scheme for ideal, viscous, and resistive magnetohydrodynamics. Int. J. Numer. Meth. Fluids. Vol. 89, pp 1642 (2019)

Concluding Remarks

- We have reviewed:
  - the TV splitting for Euler and
  - the TV flux for Euler
- We have refered to extensions of the TV flux:
  - General equation of state for compressible media
  - Multiple space dimensions
  - High order of accuracy in space and time on unstructured meshes (ADER)
  - Explicit/implicit schemes
- Performance is satisfactory
- **Future:** The TV flux can be used, for example
  - Semi-discrete schemes, eg WENO-TVDRK
  - Fully discrete one-step Discontinuous Galerkin; arbitrary accuracy in space and time (ADER)
  - Hyperbolic balance laws with stiff source terms
  - Advection-diffusion-reaction PDEs via hyperbolization

Acknowledgements

- Elena Vázquez, Universidad de Santiago de Compostela, Spain.
- Cristóbal E Castro, Universidad de Tarapacá, Chile.
- Bok Jik Lee, King Abdullah University of Science and Technology, Saudi Arabia.
- Dinshaw Balsara, University of Notre Dame, USA.
- Gino Montecinos. Universidad de Aysén, Chile
- Svetlana Tokareva, Los Alamos National Laboratory, USA

THANK YOU