Chapter 1

Model Problems

Multigrid methods were originally applied to simple boundary value problems that arise in many physical applications. For simplicity and for historical reasons, these problems provide a natural introduction to multigrid methods. As an example, consider the two-point boundary value problem that describes the steady-state temperature distribution in a long uniform rod. It is given by the second-order boundary value problem

$$\begin{aligned} -u''(x) + \sigma u(x) &= f(x), \qquad 0 < x < 1, \quad \sigma \geq 0, & (1.1)\\ u(0) = u(1) &= 0. & (1.2) \end{aligned}$$

While this problem can be handled analytically, our present aim is to consider numerical methods. Many such approaches are possible, the simplest of which is a finite difference method (finite element formulations will be considered in Chapter 10). The domain of the problem $\{x : 0 \leq x \leq 1\}$ is partitioned into n subintervals by introducing the grid points $x_j = jh$, where $h = 1/n$ is the constant width of the subintervals. This establishes the grid shown in Fig. 1.1, which we denote Ω^h.

At each of the $n-1$ interior grid points, the original differential equation (1.1) is replaced by a second-order finite difference approximation. In making this replacement, we also introduce v_j as an approximation to the exact solution $u(x_j)$. This approximate solution may now be represented by a vector $\mathbf{v} = (v_1, \ldots, v_{n-1})^T$, whose components satisfy the $n-1$ linear equations

$$\begin{aligned} \frac{-v_{j-1} + 2v_j - v_{j+1}}{h^2} + \sigma v_j &= f(x_j), \qquad 1 \leq j \leq n-1, & (1.3)\\ v_0 = v_n &= 0. \end{aligned}$$

Defining $\mathbf{f} = (f(x_1), \ldots, f(x_{n-1}))^T = (f_1, \ldots, f_{n-1})^T$, the vector of right-side values, we may also represent this system of linear equations in matrix form as

$$\frac{1}{h^2}\begin{bmatrix} 2+\sigma h^2 & -1 & & & & \\ -1 & 2+\sigma h^2 & -1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & -1 \\ & & & & -1 & 2+\sigma h^2 \end{bmatrix} \begin{bmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ v_{n-1} \end{bmatrix} = \begin{bmatrix} f_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ f_{n-1} \end{bmatrix}$$

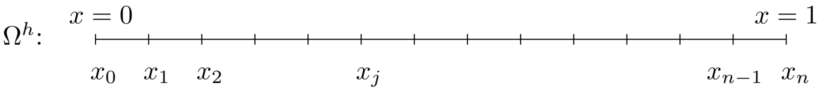

Figure 1.1: *One-dimensional grid on the interval* $0 \le x \le 1$*. The grid spacing is* $h = \frac{1}{n}$ *and the* j*th grid point is* $x_j = jh$ *for* $0 \le j \le n$*.*

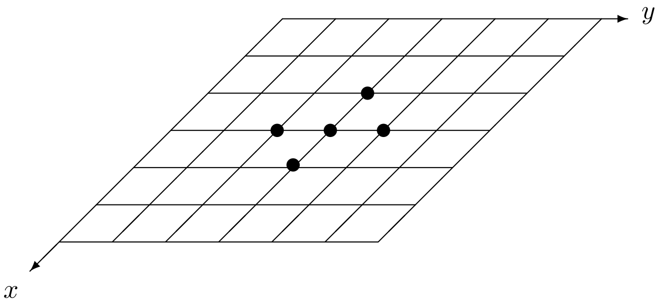

Figure 1.2: *Two-dimensional grid on the unit square. The solid dots indicate the unknowns that are related at a typical grid point by the discrete equations* (1.5)*.*

or even more compactly as $A\mathbf{v} = \mathbf{f}$. The matrix A is $(n-1) \times (n-1)$, tridiagonal, symmetric, and positive definite.

Analogously, it is possible to formulate a two-dimensional version of this problem. Consider the second-order partial differential equation (PDE)

$$-u_{xx} - u_{yy} + \sigma u = f(x, y), \quad 0 < x < 1, \quad 0 < y < 1, \quad \sigma > 0. \tag{1.4}$$

With $\sigma = 0$, this is the Poisson equation; with $\sigma \neq 0$, it is the Helmholtz equation. We consider this equation subject to the condition that $u = 0$ on the boundary of the unit square.

As before, this problem may be cast in a discrete form by defining the grid points $(x_i, y_i) = (ih_x, jh_y)$, where $h_x = \frac{1}{m}$ and $h_y = \frac{1}{n}$. This two-dimensional grid is also denoted Ω^h and is shown in Fig. 1.2. Replacing the derivatives of (1.4) by second-order finite differences leads to the system of linear equations

$$\begin{gathered} \frac{-v_{i-1,j} + 2v_{ij} - v_{i+1,j}}{h_x^2} + \frac{-v_{i,j-1} + 2v_{ij} - v_{i,j+1}}{h_y^2} + \sigma v_{ij} = f_{ij}, \\ v_{i0} = v_{in} = v_{0j} = v_{mj} = 0, \qquad 1 \le i \le m-1, \;\; 1 \le j \le n-1. \end{gathered} \tag{1.5}$$

As before, v_{ij} is an approximation to the exact solution $u(x_i, y_j)$ and $f_{ij} = f(x_i, y_j)$.

There are now $(m-1)(n-1)$ interior grid points and the same number of unknowns in the problem. We can choose from many different orderings of the unknowns. For the moment, consider the *lexicographic* ordering by lines of constant i. The unknowns of the ith row of the grid may be collected in the vector $\mathbf{v}_i =$

$(v_{i1}, \ldots, v_{i,n-1})^T$ for $1 \le i \le m-1$. Similarly, let $\mathbf{f}_i = (f_{i1}, \ldots, f_{i,n-1})^T$. The system of equations (1.5) may then be given in block matrix form as

$$\begin{bmatrix} B & -aI & & & \\ -aI & B & -aI & & \\ \cdot & \cdot & \cdot & & \\ & \cdot & \cdot & -aI & \\ & & -aI & B \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{v}_{m-1} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{f}_{m-1} \end{bmatrix}.$$

This system is symmetric, block tridiagonal, and sparse. It has block dimension $(m-1)\times(m-1)$. Each diagonal block, B, is an $(n-1)\times(n-1)$ tridiagonal matrix that looks much like the matrix for the one-dimensional problem. Each off-diagonal block is a multiple, $a = \frac{1}{h_x^2}$, of the $(n-1)\times(n-1)$ identity matrix I.

Matrix Properties. The matrices produced by the discretization of self-adjoint boundary value problems have some special properties that are desirable for many numerical methods. Let A with elements a_{ij} be such a matrix. It is generally symmetric ($A = A^T$) and sparse (a large percentage of the elements are zero). These matrices are also often *weakly diagonally dominant*, which means that, in magnitude, the diagonal element is at least as large as the sum of the off-diagonal elements in the same row:

$$\sum_{j\neq i}^{n} |a_{ij}| \le |a_{ii}| \quad \text{for} \quad 1 \le i \le n.$$

These matrices are also *positive definite*, which means that, for all vectors $\mathbf{u} \neq \mathbf{0}$, we have $\mathbf{u}^T A\mathbf{u} > 0$. This property is difficult to interpret, but there are several alternate characterizations. For example, a symmetric positive definite matrix has real and positive eigenvalues. It can also be shown that if A is symmetric and diagonally dominant with positive diagonal elements, then A is positive definite. One other matrix property arises in the course of our work: a symmetric positive definite matrix with positive entries on the diagonal and nonpositive off-diagonal entries is called an *M-matrix*.

We occasionally appeal to stencils associated with discrete equations. For the one-dimensional model problem, the stencil representation of the matrix is

$$A = \frac{1}{h^2}(-1 \quad 2+\sigma h^2 \quad -1).$$

The two-dimensional stencil for $h_x = h_y = h$ is

$$A = \frac{1}{h^2}\begin{pmatrix} & -1 & \\ -1 & 4+\sigma h^2 & -1 \\ & -1 & \end{pmatrix}.$$

Stencils are useful for representing operators that interact locally on a grid. However, they must be used with care near boundaries.

The two model linear systems (1.3) and (1.5) provide the testing ground for many of the methods discussed in the following chapters. Before we proceed, however, it is useful to give a brief summary of existing methods for solving such systems.

During the past 50 years, a tremendous amount of work was devoted to the numerical solution of sparse systems of linear equations. Much of this attention was given to structured systems such as (1.3) and (1.5) that arise from boundary value problems. Existing methods of solution fall into two large categories: *direct methods* and *iterative* (or *relaxation*) methods. This tutorial is devoted to the latter category.

Direct methods, of which Gaussian elimination is the prototype, determine a solution exactly (up to machine precision) in a finite number of arithmetic steps. For systems such as (1.5) that arise from a two-dimensional elliptic equation, very efficient direct methods have been developed. They are usually based on the fast Fourier transform or the method of cyclic reduction. When applied to problems on an $n \times n$ grid, these methods require $O(n^2 \log n)$ arithmetic operations. Because they approach the minimum operation count of $O(n^2)$ operations, these methods are nearly optimal. However, they are also rather specialized and restricted primarily to systems that arise from separable self-adjoint boundary value problems.

Relaxation methods, as represented by the Jacobi and Gauss–Seidel iterations, begin with an initial guess at a solution. Their goal is to improve the current approximation through a succession of simple updating steps or iterations. The sequence of approximations that is generated (ideally) converges to the exact solution of the linear system. Classical relaxation methods are easy to implement and may be successfully applied to more general linear systems than most direct methods [23, 24, 26].

As we see in the next chapter, relaxation schemes suffer from some disabling limitations. Multigrid methods evolved from attempts to overcome these limitations. These attempts have been largely successful: used in a multigrid setting, relaxation methods are competitive with the fast direct methods when applied to the model problems, and they have more generality and a wider range of application.

In Chapters 1–5 of this tutorial, we focus on the two model problems. In Chapters 6–10, we extend the basic multigrid methods to treat more general boundary conditions, operators, and geometries. The basic methods can be applied to many elliptic and other types of problems without significant modification. Still more problems can be treated with more sophisticated multigrid methods.

Finally, the original multigrid ideas have been extended to what are more appropriately called *multilevel methods*. Purely algebraic problems (for example, network and structural problems) have led to the development of *algebraic multigrid* or *AMG*, which is the subject of Chapter 8. Beyond the boundaries of this book, multilevel methods have been applied to time-dependent problems and problems in image processing, control theory, combinatorial optimization (the traveling salesman problem), statistical mechanics (the Ising model), and quantum electrodynamics. The list of problems amenable to multilevel methods is long and growing. But first we must begin with the basics.

Exercises

1. **Derivative (Neumann) boundary conditions.** Consider model problem (1.1) subject to the *Neumann boundary conditions* $u'(0) = u'(1) = 0$. Find the system of linear equations that results when second-order finite differences are used to discretize this problem at the grid points $x_0, \ldots, x_n$. At the end

points, x_0 and x_n, one of many ways to incorporate the boundary conditions is to let $v_1 = v_0$ and $v_{n-1} = v_n$. (We return to this problem in Chapter 7.) How many equations and how many unknowns are there in this problem? Give the matrix that corresponds to this boundary value problem.

2. **Ordering unknowns.** Suppose the unknowns of system (1.5) are ordered by lines of constant j (or y). Give the block structure of the resulting matrix and specify the dimensions of the blocks.

3. **Periodic boundary conditions.** Consider model problem (1.1) subject to the *periodic boundary conditions* $u(0) = u(1)$ and $u'(0) = u'(1)$. Find the system of linear equations that results when second-order finite differences are used to discretize this problem at the grid points $x_0, \ldots, x_{n-1}$. How many equations and unknowns are there in this problem?

4. **Convection terms in two dimensions.** A convection term can be added to the two-dimensional model problem in the form

$$-\epsilon(u_{xx} + u_{yy}) + au_x = f(x).$$

Using the grid described in the text and second-order central finite difference approximations, find the system of linear equations associated with this problem. What conditions must be met by a and ϵ for the associated matrix to be diagonally dominant?

5. **Three-dimensional problem.** Consider the three-dimensional Poisson equation

$$-u_{xx} - u_{yy} - u_{zz} = f(x, y, z).$$

Write out the discrete equation obtained by using second-order central finite difference approximations at the grid point (x_i, y_j, z_k). Assuming that the unknowns are ordered first by lines of constant x, then lines of constant y, describe the block structure of the resulting matrix.