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Homework #5
Math 537

1(a) Given vectors x and y in V, verify that the choice

$$\alpha = \frac{\langle x, y\rangle}{\|y\|^2}$$

minimizes $\|x - \alpha y\|^2$

Using $\|x\| = \sqrt{\langle x, x\rangle}$

$$\|x - \alpha y\| = \sqrt{\langle x - \alpha y, x - \alpha y\rangle}$$

$$\|x - \alpha y\|^2 = \langle x - \alpha y, x - \alpha y\rangle$$

$$= \langle x, x - \alpha y\rangle - \langle \alpha y, x - \alpha y\rangle$$

$$= \langle x, x\rangle - \langle x, \alpha y\rangle - \langle \alpha y, x\rangle + \langle \alpha y, \alpha y\rangle$$

$$= \langle x, x\rangle - \langle \alpha y, x\rangle - \langle \alpha y, x\rangle + \langle \alpha y, \alpha y\rangle$$

$$= \langle x, x\rangle - \alpha\langle y, x\rangle - \alpha\langle y, x\rangle + \alpha^2\langle y, y\rangle$$

but $\langle x, x\rangle = \|x\|^2$, $\langle y, y\rangle = \|y\|^2$

then

$$\|x - \alpha y\|^2 = \|x\|^2 - 2\alpha\langle x, y\rangle + \alpha^2\|y\|^2$$

at minimum $\frac{d}{d\alpha}\|x - \alpha y\|^2 = 0$,

$$\frac{d}{d\alpha}\|x - \alpha y\|^2 = -2\langle x, y\rangle + 2\alpha\|y\|^2 = 0$$

$$\langle x, y\rangle = \alpha\|y\|^2$$

$$\alpha = \frac{\langle x, y\rangle}{\|y\|^2}$$

Show that this leads directly to the Cauchy-Schwarz inequality.

$$|\langle x, y\rangle|^2 \leq \|x\|^2 \|y\|^2$$

Assume $\|x\| = \|y\| = 1$, then $\|x - \alpha y\|^2 \geq 0$

$$\|x - \alpha y\|^2 = \|x\|^2 - 2\alpha\langle x, y\rangle + \alpha^2\|y\|^2 \geq 0$$

but since $\alpha = \dfrac{\langle x, y\rangle}{\|y\|^2}$

$$\|x - \alpha y\|^2 = \|x\|^2 - 2\frac{\langle x, y\rangle}{\|y\|^2}\langle x, y\rangle + \frac{|\langle x, y\rangle|^2}{\|y\|^4}\|y\|^2 \geq 0$$

$$\|x - \alpha y\|^2 = \|x\|^2 - 2\frac{|\langle x, y\rangle|^2}{\|y\|^2} + \frac{|\langle x, y\rangle|^2}{\|y\|^2} \geq 0$$

$$\|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2} \geq 0$$

$$-|\langle x, y\rangle|^2 \geq -\|x\|^2 \cdot \|y\|^2$$

$$|\langle x, y\rangle|^2 \leq \|x\|^2 \cdot \|y\|^2$$

b) Given vectors x and y in V, show that $|\langle x,y\rangle|^2 = \|x\|^2 \cdot \|y\|^2$ iff x and y are linearly dependent.

Given $x, y \in V$, x and y are said to be linearly dependent if x can be written as a linear combination of y. ie. $x = ay$, where a is constant.

Consider

$$\|x - \alpha y\|^2 = \|x\|^2 - 2\alpha\langle x,y\rangle + \alpha^2\|y\|^2$$

$$\|x - \alpha y\|^2 = \|x\|^2 - \frac{|\langle x,y\rangle|^2}{\|y\|^2}, \quad \text{given } \alpha = \frac{\langle x,y\rangle}{\|y\|^2}$$

$$\|x - \alpha y\|^2 = \frac{\|x\|^2 \cdot \|y\|^2 - |\langle x,y\rangle|^2}{\|y\|^2}$$

If

$$|\langle x,y\rangle|^2 = \|x\|^2 \cdot \|y\|^2$$

then

$$\|x - \alpha y\|^2 = \frac{\|x\|^2 \cdot \|y\|^2 - \|x\|^2 \cdot \|y\|^2}{\|y\|^2} = 0$$

$$\|x - \alpha y\|^2 = 0 \Rightarrow \underline{x = \alpha y}$$

Hence $x = \alpha y$, which is linearly dependence iff

$$|\langle x,y\rangle|^2 = \|x\|^2 \cdot \|y\|^2$$

2- Starting with the set $\{1, x, x^2, x^3, \dots\}$, use the Gram-Schmidt process and the inner product.

$$\langle f, g\rangle = \int_{-1}^{1} f(x)\, g(x)\, w(x)\, dx, \quad w(x) > 0$$

to find the first four orthogonal polynomials for:

Since $\{1, x, x^2, x^3, x^4, \dots\}$ is a basis for P.

let $P_0(x) = x$

$$P_1(x) = x - \frac{\langle x, P_0(x)\rangle}{\langle P_0(x), P_0(x)\rangle} P_0(x)$$

$$P_2(x) = x^2 - \frac{\langle x^2, P_0(x)\rangle}{\langle P_0(x), P_0(x)\rangle} P_1(x) - \frac{\langle x^2, P_1(x)\rangle}{\langle P_1(x), P_1(x)\rangle} P_1(x)$$

$$P_3(x) = x^3 - \frac{\langle x^3, P_0(x)\rangle}{\langle P_0(x), P_0(x)\rangle} P_0(x) - \frac{\langle x^3, P_1(x)\rangle}{\langle P_1(x), P_1(x)\rangle} P_1(x)$$

$$- \frac{\langle x^3, P_2(x)\rangle}{\langle P_2(x), P_2(x)\rangle} P_2(x)$$

for (a) $w(x) = 1$.

$$\langle f, g\rangle = \int_{-1}^{1} f(x)\, g(x)\, dx$$

$P_0(x) = 1$

$P_1(x) = x - \frac{\langle x,1\rangle}{\langle 1,1\rangle} 1 = x - \frac{\langle x,1\rangle}{\langle 1,1\rangle}$

but $\langle x,1\rangle = \int_{-1}^{1} x\,dx = \frac{1}{2}x^2\Big|_{-1}^{1} = 0$

$\langle 1,1\rangle = \int_{-1}^{1} dx = 2$

$\underline{\underline{P_1(x) = x}}$

$P_2(x) = x^2 - \frac{\langle x^2,1\rangle}{\langle 1,1\rangle} - \frac{\langle x^2,x\rangle}{\langle x,x\rangle}\cdot x$

$\langle x^2,1\rangle = \int_{-1}^{1} x^2\,dx = 2/3$

$\langle x^2,x\rangle = \int_{-1}^{1} x^2\cdot x\,dx = 0$

$\langle x,x\rangle = \int_{-1}^{1} x\cdot x\,dx = 2/3$

$P_2(x) = x^2 - \frac{2/3}{2} - 0 = x^2 - \frac{1}{3} = \underline{\underline{\frac{1}{3}(3x^2-1)}}$

$P_3(x) = x^3 - \frac{\langle x^3,1\rangle}{\langle 1,1\rangle} - \frac{\langle x^3,x\rangle}{\langle x,x\rangle}x - \frac{\langle x^3, x^2-1/3\rangle}{\langle x^2-1/3, x^2-1/3\rangle}(x^2-\frac{1}{3})$

$$\langle x^3, 1 \rangle = \int_{-1}^{1} x^3 \, dx = 0$$

$$\langle x^3, x \rangle = \int_{-1}^{1} x^2 \, dx = 2/5$$

$$\langle x^3, x^2 - 1/3 \rangle = \int_{-1}^{1} x^3 (x^2 - 1/3) \, dx = 0$$

$$P_3(x) = x^3 - \frac{2/5}{2/3} x - 0 = x^3 - 3/5 x$$

$$\underline{\underline{P_3(x) = x^3 - 3/5 x}}$$

So the first four polynomials will be, before standardization are:

$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{3}(3x^2 - 1)$, $P_3(x) = \frac{1}{5}(5x^3 - 3x)$

<u>standardizing</u>.

$P_n(1) = 1$

$P_0(1) = 1$, $P_1(1) = 1$, $P_2(1) = 2/3$, $P_3(1) = 2/5$

Then the Legendre polynomials.

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{2}{3} \cdot \frac{1}{3}(3x^2-1) = \frac{1}{2}(3x^2-1)$

$P_3(x) = \frac{5}{2} \cdot \frac{1}{5}(5x^3-3x) = \frac{1}{2}(5x^3-3x)$

b) $w(x) = (1-x^2)^{-1/2}$ (chebyshev polynomials).

$\langle f, g\rangle = \int_{-1}^{1} f(x)\, g(x)\, (1-x^2)^{-1/2}\, dx$

$T_0(x) = 1$

$T_1(x) = x - \frac{\langle x, 1\rangle}{\langle 1, 1\rangle}$

$\langle x, 1\rangle = \int_{-1}^{1} x(1-x^2)^{-1/2} dx = 0$

$\langle 1, 1\rangle = \int_{-1}^{1} (1-x^2)^{-1/2} dx = \pi$

$T_1(x) = \underline{\underline{x}}$

$T_2(x) = x^2 - \frac{\langle x^2, 1\rangle}{\langle 1, 1\rangle} - \frac{\langle x^2, x\rangle}{\langle x, x\rangle} x$

$\langle x^2, 1\rangle = \int_{-1}^{1} x^2 (1-x^2)^{-1/2} dx = \pi/2$

$\langle x^2, x\rangle = \int_{-1}^{1} x^2 \cdot x (1-x^2)^{-1/2} dx = 0$

$\langle x, x\rangle = \int_{-1}^{1} x \cdot x (1-x^2)^{-1/2} dx = \pi/2$

$$T_2(x) = x^2 - \frac{\pi/2}{\pi} - 0 = x^2 - 1/2$$

$$T_3(x) = x^3 - \frac{\langle x^3, 1\rangle}{\langle 1,1\rangle} - \frac{\langle x^3, x\rangle}{\langle x, x\rangle} x - \cancel{\langle x^3, x^2 - 1/2\rangle}$$

$$\frac{\langle x^3, x^2 - 1/2\rangle}{\langle x^2 - 1/2, x^2 - 1/2\rangle}(x^2 - 1/2)$$

$$\langle x^3, 1\rangle = \int_{-1}^{1} x^3 (1-x)^{-1/2} dx = 0$$

$$\langle x^3, x\rangle = \int_{-1}^{1} x^3 \cdot x (1-x)^{-1/2} dx = 3/8\,\pi$$

$$\langle x^3, x^2 - 1/2\rangle = \int_{-1}^{1} x^3 \cdot (x^2 - 1/2)(1-x)^{-1/2} dx = 0$$

$$T_3(x) = x^3 - 0 - \frac{3/8\,\pi}{\pi/2} x - 0 = x^3 - 3/4\, x$$

Chebyshev polynomials before standardizing.

$T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = x^2 - 1/2$, $T_3(x) = x^3 - 3/4\, x$

Standardizing.

$T_0(1) = 1$, $T_1(1) = 1$, $T_2(1) = \frac{1}{2}$, $T_3(1) = 1 - 3/4 = 1/4$.

After standardizing.

$T_0(x) = 1$

$T_1(x) = x$

$T_2(x) = 2\left(\frac{1}{2}(2x^2-1)\right) = 2x^2-1$

$T_3(x) = 4\left(\frac{1}{4}(4x^3-3x)\right) = 4x^3-3x$

3. Find the best quadratic polynomial fit to $f(x) = |x|$ on the interval $[-1,1]$ relative to the inner product $\langle f,g \rangle$ used in problem (2) relative to weights $w(x)=1$ and $w(x) = (1-x^2)^{-1/2}$.

$$\langle f,g \rangle = \int_{-1}^{1} f(x)\, g(x)\, w(x)\, dx \quad , \quad w(x) > 0$$

Finding the best quadratic polynomial fit to $f(x)$ is equivalent to finding a least square approximation that minimizes the integral norm:

$\cancel{\|f-P\|}$ Let the polynomial be P.

$$\|f-P\|_2 = \left(\int_{-1}^{1} |f(x) - P(x)|^2 dx\right)^{1/2}$$

where $\|f-P\|_2$ is induced by the inner product $\langle f,g \rangle$.

- If P is the orthogonal projection of f on

The subspace of quadratic polynomials, then $\|f - P\|_2$ is minimal $\therefore$ Therefore the legendre polynomials $P_0, P_1, P_2, \ldots P_n$ form an orthogonal basis for P_n.

$$P(x) = \frac{\langle f, P_0 \rangle}{\langle P_0, P_0 \rangle} P_0(x) + \frac{\langle f, P_1 \rangle}{\langle P_1, P_1 \rangle} P_1(x) + \frac{\langle f, P_2 \rangle}{\langle P_2, P_2 \rangle} P_2(x)$$

$$+ \frac{\langle f, P_3 \rangle}{\langle P_3, P_3 \rangle} P_3(x) + \frac{\langle f, P_4 \rangle}{\langle P_4, P_4 \rangle} P_4(x) + \frac{\langle f, P_5 \rangle}{\langle P_5, P_5 \rangle} P_5(x)$$

$$+ \frac{\langle f, P_6 \rangle}{\langle P_6, P_6 \rangle} P_6(x).$$

for $w(x) = 1$

$$\langle f, P_0 \rangle = \int_{-1}^{1} |x|\, dx = 1$$

but $\langle P_n, P_n \rangle = \frac{2}{2n+1} \Rightarrow \langle P_0, P_0 \rangle = 2, \langle P_1, P_1 \rangle = \frac{2}{3}$

$\langle P_2, P_2 \rangle = \frac{2}{5}, \langle P_3, P_3 \rangle = \frac{2}{7}, \langle P_4, P_4 \rangle = \frac{2}{9}$

$\langle P_5, P_5 \rangle = \frac{2}{11}, \langle P_6, P_6 \rangle = \frac{2}{13}.$

$$\langle f, P_1 \rangle = \int_{-1}^{1} |x|\, x\, dx = 0$$

$$\langle f, P_2 \rangle = \int_{-1}^{1} |x| \left(\frac{3x-1}{2}\right) dx = \frac{1}{4}$$

$$\langle f, P_3 \rangle = \int_{-1}^{1} |x| \left(\frac{1}{2}(5x^3 - 3x)\right) dx = 0$$

$$\langle f, P_4 \rangle = \int_{-1}^{1} |x| \left(\frac{1}{8}(35x^4 - 30x^2 + 3)\right) dx = \frac{-1}{24}$$

$$\langle f, P_5 \rangle = \int_{-1}^{1} |x| \left(\frac{1}{8}(63x^5 - 70x^3 + 15x\right) dx = 0$$

$$\langle f, P_6 \rangle = \int_{-1}^{1} |x| \left(\frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5\right) dx = \frac{1}{64}$$

The polynomial will be:

$$p(x) = \frac{1}{2} + \frac{1/4}{2/5} \cdot \frac{1}{2}(3x^2 - 1) - \frac{\frac{1}{24}}{2/9}\left(\frac{1}{8}(35x^4 - 30x^2 + 3)\right)$$

$$+ \frac{1/64}{2/13}\left(\frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)\right)$$

$$p(x) = \frac{1}{2} + \frac{5}{8} \cdot \frac{1}{2}(3x^2 - 1) - \frac{9}{48} \cdot \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$+ \frac{13}{128} \cdot \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$$

$$p(x) = \frac{1}{2} + \frac{15}{16}x^2 - \frac{5}{16} - \frac{3}{128}(35x^4 - 30x^2 + 3)$$

$$+ \frac{13}{2048}(231x^6 - 315x^4 + 105x^2 - 5)$$

$$\cancel{p(x)} \quad p(x) = \frac{1}{2} + \frac{5}{8}P_2(x) - \frac{9}{48}P_4(x) + \frac{13}{128}P_6(x).$$

for $w(x) = (1-x^2)^{-1/2}$

$$\langle f, g\rangle = \int_{-1}^{1} f(x)\, g(x)\, (1-x^2)^{-1/2}\, dx$$

$$T(x) = \frac{\langle f, T_0\rangle}{\langle T_0, T_0\rangle} T_0(x) + \frac{\langle f, T_1\rangle}{\langle T_1, T_1\rangle} T_1(x) + \cancel{\frac{\langle f, T_2\rangle}{\langle T_2, T_2\rangle}}$$

$$\frac{\langle f, T_2\rangle}{\langle T_2, T_2\rangle} T_2(x) + \frac{\langle f, T_3\rangle}{\langle T_3, T_3\rangle} T_3(x) + \frac{\langle f, T_4\rangle}{\langle T_4, T_4\rangle} T_4(x) +$$

$$\frac{\langle f, T_5\rangle}{\langle T_5, T_5\rangle} T_5(x) + \frac{\langle f, T_6\rangle}{\langle T_6, T_6\rangle} T_6(x).$$

$$\langle f, T_0\rangle = \int_{-1}^{1} |x|\, dx = 1$$

$$\langle f, T_1\rangle = \int_{-1}^{1} |x|\, x\, dx = 0$$

$$\langle f, T_2\rangle = \int_{-1}^{1} |x|\,(2x^2-1)\, dx = 0$$

$$\langle f, T_3\rangle = \int_{-1}^{1} |x|\,(4x^3-3x)\, dx = 0$$

$$\langle f, T_4\rangle = \int_{1}^{1} |x|\,(8x^4-8x^2+1)\, dx = -1/3$$

$\langle f, T_5\rangle = 0$, $\langle f, T_6\rangle = 0$, $\langle f, T_7\rangle = 0$

$$\langle f, T_8\rangle = \int_{-1}^{1} |x|\,(128x^8 - 256x^6 + 160x^4 - 32x^2 + 1)\, dx = -\frac{1}{15}$$

$\langle f, T_9\rangle = 0$

$$\langle T_0, T_0\rangle = \int_{-1}^{1} (1-x^2)^{-1/2}\,dx = \pi$$

$$\langle T_4, T_4\rangle = \int_{-1}^{1} (8x^4-8x^2+1)^2(1-x^2)^{-1/2}\,dx = \pi/2$$

$$\langle T_8, T_8\rangle = \int_{-1}^{1} (128x^8-256x^6+160x^4-32x^2+1)^2(1-x^2)^{-1/2}\,dx$$

$$= \pi/2$$

$$T(x) = 1/\pi + \frac{-\frac{1}{3}}{\frac{\pi}{2}}(8x^4-8x^2+1) - \frac{1/15}{\pi/2}(128x^8 -$$

$$256x^6 + 160x^4 - 32x^2 + 1)$$

$$T(x) = 1/\pi - \frac{2}{3\pi}(8x^4-8x^2+1) - \frac{2}{15\pi}(128x^8 - 256x^6$$

$$+ 160x^4 - 32x^2 + 1)$$