Homework #4

Math 537

This assignment is a review of linear algebra as well as spectral approaches to solving problems in finite dimensional vector spaces.

1. The "pseudo-inverse" solution to a square, singular matrix problem $A\mathbf{x} = \mathbf{b}$ is similar to a least-squares problem, with the additional complication that the solution may not be unique. For this problem, you will solve the following linear system to get the solution returned when using the "pseudo-inverse" operator P. There are two basic approaches. One approach is a "spectral" approach, and the other is a non-spectral approach.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \tag{1}$$

   (a) **Spectral approach.** This is the approach described in the class notes. For this approach, compute the pseudo-inverse P, and compute the minimum norm solution $\mathbf{x}$ as $P\mathbf{b}$. To get eigenvalues and eigenvectors, you may use something like WolframAlpha, but be aware that for the spectral approach, you will need a set of orthonormal eigenvectors.

   (b) **Non-spectral approach.** This is approach is closely related to a linear least squares solution and is solved in part by solving the normal equations. In this approach, we express the solution $\mathbf{x}$ as the sum of components in the column space of A and components in the null space of A, as

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{0} \end{bmatrix} + N\mathbf{y} \tag{2}$$

   The columns of the matrix N form a basis for the null space of A. The goal in this approach is to find $\mathbf{x}_1$ and $\mathbf{y}$ that minimize the norm of $\mathbf{x}$. The steps for obtaining the solution are as follows.

      i. Project the vector $\mathbf{b}$ onto the column space of A to obtain a vector $\mathbf{x}_1 \in \mathcal{R}^{r\times 1}$, where $r = \text{rank}(A)$.

      ii. Find $\mathbf{y}$ such that the norm of $\mathbf{y}$ is a minimum. It suffices to find $\mathbf{y}$ that satisfies

$$\mathbf{y} = \text{argmin}\, \frac{1}{2} \left\| \mathbf{x} \right\|^2 . \tag{3}$$

      where $\mathbf{x}$ is defined as in (2).

   - Report the minimum norm solution you find. This should be the same solution you found in the spectral approach.
   - Find an algebraic expression for the pseudo-inverse P. To find P using this approach, solve $A\mathbf{x} = I$, where I is the 3×3 identity matrix. Express P in terms of the projection of I onto the column space and additional components from the null space. You don't have to work out the actual entries of P (the will be the same as for the spectral approach).

   This approach doesn't require that you find the eigenvalues and eigenvectors. But you do need to find a basis for the null space of A. You can use the eigenvectors found in the spectral approach, or you can row-reduce A to get the basis.

2. The two approaches above are closely related. What kinds of conclusions can you draw from working through the details of the above?