Brian KYANJO
Homework #7
Math 537.

1. Haar wavelets. The Haar Scaling function can be defined recursively as

$$\phi(x) = \phi(2x) + \phi(2x-1)$$

Initial conditions $\phi(0) = 1$
$\phi(1) = 0$ and $\phi(x) = 0$, $x<1$ or $x<1$

for $k_{max} = 2^{m-1}-1$
for $m=1$, $k=0$

$$X = (2k+1)2^{-m} = 2^{-1}$$

$$\phi(2^{-1}) = \phi(2\cdot 2^{-1}) + \phi((2\cdot 2^{-1})-1) = 1$$

for $m=2$, $k=0,1$
$k=0$

$$X = (2(0)+1)2^{-2} = \tfrac{3}{4}$$

$$\phi(\tfrac{1}{4}) = \phi(\tfrac{1}{2}) + \phi(-\tfrac{1}{2}) = \underline{\underline{1}}$$

$k=1$

$$X = \tfrac{3}{4}$$

$$\phi(\tfrac{3}{4}) = \phi(2\cdot\tfrac{3}{4}) + \phi(2\cdot\tfrac{3}{4}-1) = 1$$

for $m=3$, $k=0,1,2$
$k=0$

$$X = \tfrac{1}{8}$$

$$\phi(\tfrac{1}{8}) = \phi(\tfrac{1}{4}) + \phi(-\tfrac{3}{4}) = \underline{1}$$

$k=1$, $x=3/8$

$\Phi(3/8) = \Phi(2\cdot 3/8) + \Phi(2\cdot 3/8 - 1) = 1$

$k=2$, $x = 5/8$

$\Phi(5/8) = \Phi(2\cdot 5/8) + \Phi(2\cdot 5/8 - 1) = \underline{1}$

$k=3$, $x = 7/8$

$\Phi(7/8) = \Phi(7/4) + \Phi(3/4) = \underline{\underline{1}}$

Write a general algorithm for evaluating $\Phi(x)$ at dyadic numbers $(2k+1)2^{-m}$, $k=0,1,2,\ldots,2^{m-1}-1$

$X = [0, 1]$
$\phi = [1\ 0]$

for $m = 1$ to m_{max} do
  $k_{max} = 2^{m-1} - 1$
  for $k = 0$ to k_{max} do
    $x_k = (2k+1)2^{-m}$
    $x_q = x_k$

    $x'_k = 2x_k$
    $x''_k = 2x_k - 1$

    If $x'_k < 0$ or $x'_k > 1$ do

      $\phi'_k = 0$
    else if $0 \le x'_k < 1$ do
      i'_k = Index of x'_k in X
      $\phi'_k = \phi(i'_k)$
    end if

    If $x''_k < 0$ or $x''_k > 1$ do
      $\phi''_k = 0$
    else if $0 \le x''_k < 1$ do
      i''_k = index of x''_k in X
      $\phi''_k = \phi(i''_k)$
    end if

    $\phi_k = \phi'_k + \phi''_k$
    $\phi_q = \phi_k$
  end for
  $X = x_q$
  $\phi = \phi_q$
end for.

Argue that $\Phi(x)$ has a closed form solution

$$\Phi(x) = \begin{cases} 1 & \text{if } 0 \le x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Since from all the above calculation of $\Phi(x)$, the value of $\Phi(x) = 1$ for any value of $x \in [0, 1)$ and zero else where. Hence $\Phi(x)$ has a closed from

2. Haar wavelets. Repeat the above for the Haar wave lets defined recursively as.

$$\psi(x) = \psi(2x) - \psi(2x-1)$$

Using. $\psi(0) = 1$, $\psi(\tfrac{1}{2}) = -1$, $\psi(x) = 0$ for $x < 0$ and $x \geq 1$

for/ $m \neq 1$ $k_{max} = 2^{m-1} - 1$

for $m = 1$, $k = 0$

$$x = (2k+1)2^{-m} = (2(0)+1)2^{-1} = \tfrac{1}{2}$$

$$\psi(\tfrac{1}{2}) = \psi(2 \cdot \tfrac{1}{2}) - \psi(2 \cdot \tfrac{1}{2} - 1) = \underline{\underline{-1}}$$

for $m = 2$, $k = 0, 1$

$k = 0$, $x = \tfrac{1}{4}$.

$$\psi(\tfrac{1}{4}) = \psi(2 \cdot \tfrac{1}{4}) - \psi(2 \cdot \tfrac{1}{4} - 1) = \underline{\underline{1}}$$

for $k = 1$, $x = \tfrac{3}{4}$

$$\psi(3/4) = \psi(2 \cdot 3/4) - \psi(2 \cdot 3/4 - 1) = 1$$

~~k=~~

for $m=3$, $k=0,1,2$

$k=0$, $x = 1/8$

$$\psi(1/8) = \psi(2 \cdot 1/8) - \psi(2 \cdot 1/8 - 1) = 1$$

$k=1$, $x = 3/8$

$$\psi(3/8) = \psi(2 \cdot 3/8) - \psi(2 \cdot 3/8 - 1) = 1$$

$k=2$, $x = 5/8$

$$\psi(5/8) = \psi(2 \cdot 5/8) - \psi(2 \cdot 5/8 - 1) = \underline{\underline{-1}}$$

Write a general algorithm.

$X = [0 \ \ 1/2]$
$\psi = [1 \ -1]$
for $m = 1$ to m_{max} do
  $k_{max} = 2^{m-1} - 1$
  for $k = 0$ to k_{max} do
    $x_k = (2k+1)\,2^{-m}$
    $x_a = x_k$
    $x'_k = 2x_k$
    $x''_k = 2x_k - 1$

    If $x'_k < 0$ and $x'_k \geq 1$ do
      $\psi'_k = 0$
    else if $0 \leq x'_k < 1$ do
      i'_k = index of x'_k in X
      $\psi'_k = \psi(i'_k)$
    end if

    If $x''_k < 0$ and $x''_k \geq 1$ do
      $\psi''_k = 0$
    else if $0 \leq x''_k < 1$ do
      i''_k = index of x''_k in X
      $\psi''_k = \psi(i''_k)$
    end if

    $\psi_k = \psi'_k - \psi''_k$
    $\psi_a = \psi_k$
  end for

*
$X = x_a$
$\psi = \psi_k$

end for

Argue that $\varphi(x)$ has a closed form solut

According to the previous calculations $\varphi(x)=1$ or $\varphi(x)=-1$ for all $x \in [0,1)$ and zero else where. hence has a closed form.

5. Using integration by parts, show that the solution to the elliptic problem:

$$u''(x) = f$$

on $[0,1]$, subject to $u(0) = a$, $u(1) = b$ is given by

$$u(x) = a(1-x) + b(x) + \int_0^1 G(x,y)\, f(y)\, dy$$

Consider $u''(x) = f$ ——— ①

multiply ① by $G(x,y)$

$$G(x,y)\, u''(x) = G(x,y)\, f(y)$$

Integrate both sides w.r.t x and y respectively over $[0,1]$

$$\int_0^1 G(x,y)\, u''(x)\, dx = \int_0^1 G(x,y)\, f(y)\, dy$$

from the L.H.S, integrating by parts.

$$U = G(x,y) \Rightarrow \frac{du}{dx} = \frac{d}{dx} G(x,y)$$

$$dv = u''(x)\, dx \Rightarrow v = u'(x) + C$$

$$\int_0^1 u\, dv = uv\Big|_0^1 - \int_0^1 v\, du$$

$$\int_0^1 G(x,y)\, u''(x)\, dx = G(x,y)\, u'(x)\Big|_0^1 - \int_0^1 u'(x) \frac{d}{dx} G(x,y)\, dx$$

$$= G(1,y)\, u'(1) - G(0,y)\, u'(0) - \int_0^1 u'(x) \frac{d}{dx} G(x,y)\, dx$$

Since $G(x,y)$ is a fundamental solution subject to

$$G(0,y) = G(1,y) = 0 \Rightarrow \frac{d^2}{dx^2} G(x,y) = \delta(x,y)$$

then

$$\int_0^1 G(x,y)\, u''(x)\, dx = -\int_0^1 u'(x) \frac{d}{dx} G(x,y)\, dx$$

Applying by parts again.

$$U = \frac{d}{dx} G(x,y) \Rightarrow \frac{dU}{dx} = \frac{d^2}{dx^2} G(x,y)$$

$$dv = u'(x)\, dx \Rightarrow v = u(x) + c$$

$$\int_0^1 G(x,y)\, u''(x)\, dx = -\left[\left(u(x) \frac{d}{dx} G(x,y) \right) \Big|_0^1 - \int_0^1 u(x) \frac{d^2}{dx^2} G(x,y)\, dx \right.$$

$$\int_0^1 G(x,y)\, u''(x)\, dx = -\left[u(1)\, G'(1,y) - u(0)\, G'(0,y) - \int_0^1 u(x) \frac{d^2}{dx^2} G(x,y)\, dx \right]$$

Since $G(x,y) = \begin{cases} (y-1)x & 0 \leq x \leq y \\ y(x-1) & y \leq x \leq 1 \end{cases}$

$$\frac{d}{dx} G(x,y) = \begin{cases} (y-1) & 0 \leq x \leq y \\ y & y \leq x \leq 1 \end{cases}$$

so $G'(1,y) = y$

$G'(0,y) = y-1$

$u(1) = b, \quad u(0) = a$

$$\int_0^1 G(x,y)\, u''(x)\, dx = -\left[by - a(y-1) - \int_0^1 u(x) \frac{d^2}{dx^2} G(x,y) \right]$$

$$= a(y-1) - by + \int_0^1 u(x) \frac{d^2}{dx^2} G(x,y)\, dx$$

but $\frac{d^2}{dx^2} G(x,y) = \delta(x-y)$

$$\int_0^1 G(x,y)\, u''(x)\, dx = a(y-1) - by + \int_0^1 u(x)\, \delta(x-y)\, dx$$

Since $u(y) = \int_0^1 u(x)\, \delta(x-y)\, dx$

So

$$\int_0^1 G(x,y)\, u''(x)\, dx = a(y-1) - by + u(y)$$

taking $x = y$

$$\int_0^1 G(x,y)\, u''(x)\, dx = a(x-1) - bx + u(x)$$

hence

$$\int_0^1 G(x,y)\, u''(x)\, dx = \int_0^1 G(x,y)\, f(y)\, dy$$

$$a(x-1) - bx + u(x) = \int_0^1 G(x,y)\, f(y)\, dy.$$

$$u(x) = bx + a(1-x) + \int_0^1 G(x,y) \cdot f(y)\, dy$$

therefore:

$$u(x) = a(1-x) + bx + \int_0^1 G(x,y)\, f(y)\, dy.$$

No. 7

$$\theta''(t) + \frac{g}{L}\theta = \frac{1}{mL} f(t)$$

where $t \geq 0$ is time

$$\theta(0) = \theta'(0) = 0$$

a) Using a Laplace Transform, solve

$$\frac{\partial^2 G(t,\tau)}{\partial t^2} + \frac{g}{L} G(t,\tau) = \delta(t-\tau)$$

Solution

$$G''(t,\tau) + \frac{g}{L} G(t,\tau) = \delta(t-\tau)$$

$$L(G''(t,\tau)) + \frac{g}{L} L(G(t,\tau)) = L(\delta(t-\tau)) \quad \text{--- (1)}$$

$L(G(t,\tau)) = Y$

$$L(G''(t,\tau)) = s^2 Y - s\,G(0,\tau) - G'(0,\tau)$$

Equation (1) becomes

$$s^2 Y + \frac{g}{L} Y = L(\delta(t-\tau))$$

$$\left(s^2 + \frac{g}{L}\right) Y = L(\delta(t-\tau))$$

$$Y = \frac{1}{s^2 + \frac{g}{L}} L(\delta(t-\tau))$$

$$Y = \sqrt{\frac{L}{g}} \cdot \frac{\sqrt{\frac{g}{L}}}{s^2 + \left(\sqrt{\frac{g}{L}}\right)^2} \cdot L(\delta(t-\tau))$$

$$L(\delta(t-\tau)) = e^{-\tau s}$$

$$Y = \sqrt{\frac{l}{g}} \cdot \frac{\sqrt{g/l}}{s^2 + \left(\sqrt{g/l}\right)^2} e^{-\tau s}$$

The inverse of Y, $G(t,\tau)$ is

$$G(t,\tau) = \begin{cases} 0, & \text{if } 0 < t < \tau \\ \sqrt{\frac{l}{g}} \sin\left(\sqrt{\frac{g}{l}}(t-\tau)\right) & \text{if } t > \tau \end{cases}$$

$$G(t,\tau) = \sqrt{\frac{l}{g}} \sin\left(\sqrt{\frac{g}{l}}(t-\tau)\right) H(t-\tau)$$

b) For the Green's function $G(t,\tau)$ and operator L, given by

$$L[G(t,\tau)] = \frac{\partial^2}{\partial t^2} G(t,\tau) + \frac{g}{l} G(t,\tau)$$

(1) $L[G(t,\tau)] = 0$, $t \neq \tau$

$$G(t,\tau) = \begin{cases} 0, & \text{if } 0 < t < \tau \\ \sqrt{\frac{l}{g}} \sin\left(\sqrt{\frac{g}{l}}(t-\tau)\right) & \text{if } t > \tau \end{cases}$$

$$\frac{\partial G(t,\tau)}{\partial t} = \sqrt{\frac{l}{g}} \sqrt{\frac{g}{l}} \cos\left(\sqrt{\frac{g}{l}}(t-\tau)\right)$$

$$= \cos\left(\sqrt{\frac{g}{l}}(t-\tau)\right), \quad \text{if } t > \tau$$

$$\frac{\partial^2}{\partial t^2} G(t,\tau) = \frac{\partial}{\partial t} \cos\left(\sqrt{g/_l}\,(t-\tau)\right)$$

$$= -\sqrt{g/_l}\,\sin\left(\sqrt{g/_l}\,(t-\tau)\right)$$

So,

$$L[G(t,\tau)] = -\sqrt{g/_l}\,\sin\left(\sqrt{g/_l}\,(t-\tau)\right) + g/_l \cdot \sqrt{l/_g}\,\sin\left(\sqrt{g/_l}\,(t-\tau)\right)$$

$$= -\sqrt{g/_l}\,\sin\left(\sqrt{g/_l}\,(t-\tau)\right) + \sqrt{g/_l}\,\sin\left(\sqrt{g/_l}\,(t-\tau)\right)$$

$$L[G(t,\tau)] = 0$$

hence

$$L[G(t,\tau)] = 0, \quad t > \tau.$$

(ii) for $0 < t < \tau$, $L[G(t,\tau)] = 0$

The $\lim_{t \to \tau} G(t,\tau)$ must exist, hence $\lim_{t \to \tau} G(t,\tau) = G(\tau,\tau)$

$$\lim_{t \to \tau} G(t,\tau) = \begin{cases} \lim_{t \to \tau} 0, & \text{if } 0 < t < \tau \\ \sqrt{l/_g}\,\lim_{t \to \tau} \sin\left(\sqrt{g/_l}\,(t-\tau)\right), & \text{if } t > \tau \end{cases}$$

$$\lim_{t \to \tau} G(t,\tau) = \begin{cases} 0, & 0 < t < \tau \\ 0, & t > \tau \end{cases} \Rightarrow G(\tau,\tau)$$

hence

$$\lim_{t \to \tau} G(t,\tau) = G(\tau,\tau) \quad \text{and exists.}$$

(iii) $$\left[\frac{\partial G(t,\tau)}{\partial t}\right]_{t=\tau^-}^{t=\tau^+} = 1$$

$$\frac{\partial}{\partial t} G(t,\tau) = \cos\left(\sqrt{g/l}\,(t-\tau)\right)$$

$$\left[\frac{\partial}{\partial t} G(t,\tau)\right]_{t=\tau^-}^{t=\tau^+} = \lim_{t\to\tau^+}\left(\cos\sqrt{g/l}\,(t-\tau)\right) - \lim_{t\to\tau^-}\left(\cos\sqrt{g/l}\,(t-\tau)\right)$$

Since $G(t,\tau)=0 \quad t<\tau \Rightarrow \frac{\partial}{\partial t} G(t,\tau) = 0$

$$\lim_{t\to\tau^-}\left(\cos\sqrt{g/l}\,(t-\tau)\right) = 0$$

$$\left[\frac{\partial}{\partial t} G(t,\tau)\right]_{t=\tau^-}^{t=\tau^+} = \lim_{t\to\tau^+}\left(\cos\sqrt{g/l}\,(t-\tau)\right) = \underline{\underline{1}}$$

c) Use the Green's function to solve

$$\theta''(t) + \frac{g}{l}\theta = \frac{1}{ml} f(t), \quad \theta(0) = \theta'(0) = 0$$

$$L[u] = f \Rightarrow u = L^{-1}[f] \quad \text{—— (1)}$$

$$L[\theta] = \frac{1}{ml}(f(t)) \Rightarrow \theta = \frac{1}{ml} L^{-1}[f(t)]$$

Using (1)

$$u(x) = a(1-x) + bx + \int_0^1 G(x,y) f(y)\,dy$$

$$u(x) = u(0)(1-x) + u(1)\,x + \int_0^1 G(x,y) f(y)\,dy$$

therefore

$$u(t) = u(0)(1-t) + u(1)t + \int_0^1 G(t,y) f(y)\, dy = L^{-1}[f(t)]$$

$$\theta(t) = \int_0^t G(t,y) f(\tau)\, d\tau$$

$$\theta(t) = \int_0^t G(t,y) \frac{1}{ml} f(\tau)\, d\tau$$

$$= \frac{1}{ml} L^{-1}[f(t)]$$

but $G(t,y) = \sqrt{l/g}\, \sin\left(\sqrt{g/l}\,(t-\tau)\right)$

$$\theta(t) = \int_0^t \sqrt{l/g}\, \sin\left[\sqrt{g/l}\,(t-\tau)\right] \frac{1}{ml} f(\tau)\, d\tau$$

$$\theta(t) = \frac{1}{ml} \sqrt{l/g} \int_0^t \sin\left[\sqrt{g/l}\,(t-\tau)\right] f(\tau)\, d\tau.$$

In [1]:
```
%matplotlib notebook
%pylab
from scipy.integrate import quad
from scipy.special import erf
```

```
Using matplotlib backend: nbAgg
Populating the interactive namespace from numpy and matplotlib
```

No6

Solve

$$u''(x) = e^{-100(x-0.5)^2} \tag{1}$$

Use boundary conditions $u(0) = u(1) = 0$. Evaluate your solution at $xj = jh, j = 0, 1, \ldots, N$, for $h = 1/N$ and $N = 16$.

In [2]:
```
N = 16
a = 0; b = 1
h = 1/N

x = zeros(N+1)
I = zeros(N+1)

#rhs
def f(x):
    return exp(-100*(x-0.5)**2)

#Green's function
def G(x,y):
    if 0 <= x <= y:
        return (y-1)*x
    elif y <= x <= 1:
        return y*(x-1)

#Qudrature Rule
for j in range(N+1):
    x[j] = j*h
    xj = x[j]
    Gf = lambda y: G(xj,y)*f(y)
    I[j] =  quad(Gf,a,b)[0]
```

Compare your solution to the exact solution, which you can obtain by using WolframAlpha to evaluate the integral form of the solution in (5)

In [3]:
```
def wolfram(x):

    w1 = (x-1)*(-(1/40)*sqrt(pi)*erf(5-10*x)-(1/200)*exp(-25*(1-2*x)**2)) - \
    (x-1)*(-(1/40)*sqrt(pi)*erf(5)-(1/200)*exp(-25))

    w2 = x*((1/40)*sqrt(pi)*erf(-5)-(1/200)*exp(-25))- \
    (x*((1/40)*sqrt(pi)*erf(5-10*x)-(1/200)*exp(-25*(1-2*x)**2)))

    return w1, w2

w1,w2 = wolfram(x)
wolf_exact = w1 + w2
```

In [4]:
```
error = abs(wolf_exact[1:-1]-I[1:-1])
print('Error between Wolfram solution and the intergral solution is:',max(error))
print('\n The error is too small, hence the solution is well approximated.')
```

```
Error between Wolfram solution and the intergral solution is: 3.0852900477496004e-09

 The error is too small, hence the solution is well approximated.
```

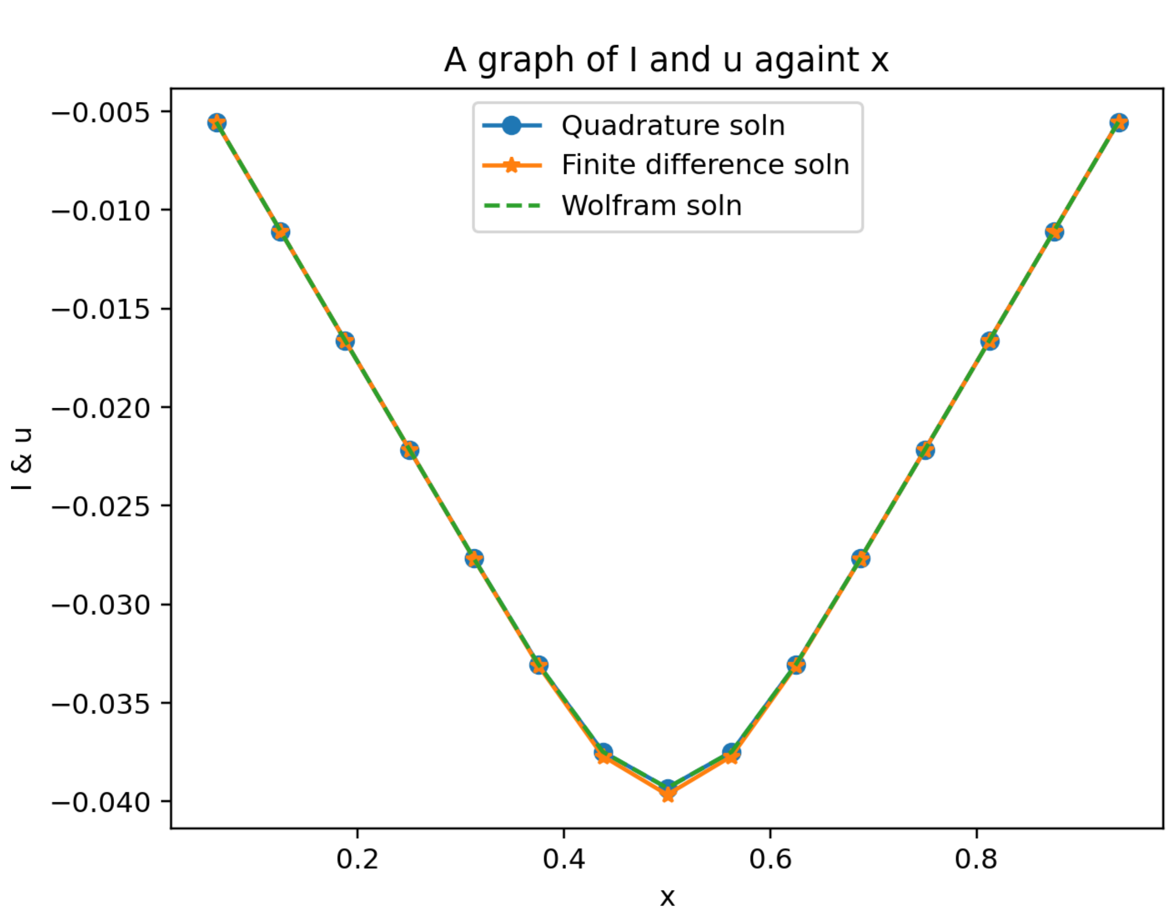
Compare the solution to the finite difference solution obtained in class.

In [5]:
```
#Using Finite Difference
f = f(x[1:-1])

# Construct A and compute inv(A)
z = ones(N-1)
A = (diag(z[:-1],-1) -2*diag(z) + diag(z[:-1],1))/h**2
Ainv = inv(A)
u = solve(A,f)
```

In [6]:
```
error = abs(u-I[1:-1])
print('Error between Quadrature solution and the intergral solution is:',max(error))
print('\n The error is too small, hence the solution is well approximated, but not better than the wolfram solution.')
```

```
Error between Quadrature solution and the intergral solution is: 0.0003396181193379699

 The error is too small, hence the solution is well approximated, but not better than the wolfram solution.
```

In [7]:
```
#ploting
figure(1)
clf()
plot(x[1:-1],I[1:-1],'-o',label='Quadrature soln')
plot(x[1:-1],u,'-*',label='Finite difference soln')
plot(x[1:-1],wolf_exact[1:-1],'--',label='Wolfram soln')
xlabel('x'); ylabel('I & u')
title('A graph of I and u againt x')
legend()
show()
```



What are some potential advantages/disadvantages of evaluating the integral from of the solution rather than solving the linear system arising from the finite difference approach or the exact formulation obtained from Wolfram Alpha

Advantages

- We obtain the exact solution.
- computationally cheap.
- Doesn't depend on the number of iterations,N, for any value of N choosen we still get the actual solution.

Disadvantages

- Take long to get to close to the exact solution.
- They can't obtain solution to every problem i.e., only work for simple models.

In [ ]: