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Homework 8
MATH 573

1. Harmonic function in 1d

(a) Show that

$$G_0(x,y) = \frac{1}{2}|x-y|$$

is a fundamental solution for the operator $L[u] = \frac{\partial^2 u}{\partial x^2}$

Integrate $u''(x) = \delta(x)$ twice

$$u'(x) = H(x) + C_0$$
$$u(x) = xH(x) + C_0 x + C_1$$

but $R(x) = xH(x)$

$$u(x) = R(x) + C_0 x + C_1$$

but

$$G_0(x,y) = u(x-y)$$

$$G_0(x,y) = R(x-y) + C_0(x-y) + C_1 \quad \text{—} (1)$$

Using $G_0(x,y) = G_0(y,x)$

$$R(x-y) + C_0(x-y) + C_1 = R(y-x) + C_0(y-x) + C_1$$

but $R(x-y) = \begin{cases} x-y & x>y \\ 0 & x<y \end{cases}$

$$R(y-x) = \begin{cases} y-x & y>x \\ 0 & y<x \end{cases}$$

Assume $x>y$, $R(x-y) = x-y$ and $R(y-x) = 0$

$$x-y = 2C_0(y-x) \Rightarrow C_0 = -\tfrac{1}{2}.$$

So Equation ① becomes

$$G_0(x,y) = R(x-y) - \frac{1}{2}(x-y)$$

but $R(x-y) = \begin{cases} x-y & x>y \\ 0 & x<y \end{cases}$

So

$$G_0(x,y) = \begin{cases} x-y-\frac{1}{2}(x-y) & x>y \\ 0-\frac{1}{2}(x-y) & x<y \end{cases}$$

$$= \begin{cases} \frac{1}{2}(x-y) & x>y \\ -\frac{1}{2}(x-y) & x<y \end{cases}$$

$$G_0(x,y) = \frac{1}{2}|x-y|$$

b) Using the method of Images to find an "Image" source $\bar{G}_0(x,y) = \beta(G_0(x,\bar{y}))$ so that

$$G(x,y) = G_0(x,y) + \bar{G}_0(x,y)$$

$$u(0) = u(1) = 0 \Rightarrow G(0,y) = G(1,y) = 0$$

To verify that $\bar{G}_0(x,y)$ is harmonic in $[0,1]$ then

$$u(x) = ax+b$$

$$u(0) = u(1) = 0$$

~~For~~ $$G_0(x,y) = \begin{cases} \frac{1}{2}(x-y) & x>y \\ -\frac{1}{2}(x-y) & x<y \end{cases}$$

$$G_0(0,y) = -\tfrac{1}{2} \quad , \; y<0$$
$$G_0(1,y) = -\tfrac{1}{2}(1-y) \quad , \; y>1$$

So $u = av + b$

for $v=0$, $u = -\tfrac{1}{2}y$

$$-\tfrac{1}{2}y = a(0) + b \Rightarrow b = -\tfrac{1}{2}y$$

for $v=1$, $u = -\tfrac{1}{2}(1-y)$

$$-\frac{1}{2}(1-y) = a + b. \Rightarrow a = -\tfrac{1}{2}(1-y) + \tfrac{1}{2}y$$
$$\underline{\underline{a = y - \tfrac{1}{2}}}$$

$$u = (y - \tfrac{1}{2})v + -\tfrac{1}{2}y$$

Therefore $\bar{G}_0(x,y) = (y - \tfrac{1}{2})x - \tfrac{1}{2}y$

$$\bar{G}_0(x,y) = -\tfrac{1}{2}x(1-2y) - \tfrac{1}{2}y$$

$$\bar{G}_0(x,y) = -\tfrac{1}{2}(x + y(1-2x)) = -\tfrac{1}{2}(x - y(2x-1))$$

$$\bar{G}_0(x,y) = -\tfrac{1}{2}|x - \bar{y}| \quad , \text{ where } \bar{y} = (2x-1)y$$

$$G(x,y) = G_0(x,y) + \bar{G}_0(x,y)$$

$$G(x,y) = \frac{1}{2}|x-y| - \frac{1}{2}(x(1-2y) + y)$$

for $x > y$

$$G(x,y) = \frac{1}{2}(x-y) - \frac{1}{2}(x(1-2y) + y)$$
$$\underline{\underline{= y(x-1)}}$$

for $x < y$

$$G(x,y) = -\frac{1}{2}(x-y) - \frac{1}{2}x + xy - \frac{1}{2}y$$

$$= \underline{x(y-1)}$$

therefore

$$G(x,y) = \begin{cases} y(x-1) & x < y \\ x(y-1) & x > y \end{cases}$$

As obtained in class

c) Verify Green's Second Identity for a harmonic function $u(x)$ satisfying (1) and the Green's function $v(y) = G(x,y)$

$$\int_0^1 (u(y)v''(y) - v(y)u''(y))dy = (u(y)v'(y) - v(y)u'(y))\Big|_{y=0}^{y=1}$$

$$\int_0^1 (uv'' - vu'')dy = u(1)v'(1) - u(0)v'(0)$$

$$v'(y) = \frac{y-x}{2|x-y|} + x - \frac{1}{2} \Rightarrow \begin{cases} v'(0) = x-1 \\ v'(1) = x \end{cases}$$

$$\int_0^1 (uv'' - vu'')dy = u(1)x - u(0)(x-1)$$

but $u(1) = b$, $u(0) = a$

$$\int_0^1 (uv'' - vu'')dy = bx - a(x-1) = a(1-x) + bx$$

but $u''(y) = 0$

$$\int_0^1 u(y)\, v''(y)\, dy = a(1-x) + bx$$

$$v''(y) = \delta(x-y) \Rightarrow \int_0^1 u(y)\, \delta(x-y)\, dy = u(x)$$

$u(x) = a(1-x) + bx$ hence proved.

d) Verify Green's second identity using free space Green's function $G_0(x,y) = \frac{1}{2}|x-y|$ and a harmonic function $u(x)$ satisfying 2 boundary conditions given in (1).

Using

$$\int_0^1 (u(y)\, v_0''(y) - v_0(y)\, u''(y))\, dy = (u(y)\, v_0'(y) - v_0(y)\, u'(y)) \Big|_0^1$$

but $u''(y) = 0$

$$\int_0^1 (u(y)\, v_0''(y))\, dy = u(1)\, v_0'(1) - v_0(1)\, u'(1) - u(0)\, v_0'(0) + v_0(0)\, u'(0)$$

but $u'(0) = u'(1)$

$$\int_0^1 (u(y)\, v_0''(y))\, dy = u(1)\, v_0'(1) - u(0)\, v_0'(0) + (v_0(0) - v_0(1))\, u'$$

$u(1) = b, \quad u(0) = a$

$$v'(y) = \frac{y-x}{2|x-y|} = \begin{cases} v'(1) = \frac{1}{2} \\ v'(0) = -\frac{1}{2} \end{cases}$$

$$\int_0^1 u(y)\, v_0''(y)\, dy = \frac{1}{2} b + \frac{1}{2} a + (v_0(0) - v_0(1))\, u'(0)$$

but $u'(0) = b - a$

$$\int_0^1 \left(u(y)\, v_0''(y)\right) dy = \frac{1}{2}(b+a) + \left(\frac{1}{2}x - \frac{1}{2}(1-x)\right)(b-a)$$

$$\int_0^1 u(y)\, v_0''(y)\, dy = bx + (1-x)\, a$$

but $\int_0^1 u(y)\, v_0''(y)\, dy = \int_0^1 u(y)\, \delta(x-y)\, dy = u(x)$

$$u(x) = a(1-x) + bx$$

2) For a general poisson problem $u''(x) = f(x)$ on $[0,1]$, use Green

$$u(x) = e^x, \quad u(0) = 1, \quad u(1) = e$$

$$u(x) = \int_0^1 G(x,y)\, f(y)\, dy + bx + a(1-x)$$

$$u(x) = \int_0^1 \frac{1}{2}\left(|x-y| - x + 2xy - y\right) e^y\, dy + 1 - x + xe$$

2. Potentials due to sources and dipoles distribution

a) Verify (6) using the representation of the delta function as

at $$\delta(x) \approx \frac{1}{2\sqrt{\pi\alpha}}\, e^{-x^2/4\alpha}$$

$$\delta'(x) = \frac{1}{2\sqrt{\pi\alpha}}\left(-\frac{2x}{4\alpha}\right) e^{-x^2/4\alpha}$$

$$-\delta'(x) = \frac{x}{4\alpha\sqrt{\pi\alpha}} e^{-x^2/4\alpha} \quad \text{--- (1)}$$

but

$$-\delta'(x) = \lim_{\varepsilon \to 0} \frac{\delta(x - \varepsilon/2) - \delta(x + \varepsilon/2)}{\varepsilon}$$

$$-\delta'(x) = \lim_{\varepsilon \to 0} \frac{\frac{1}{2\sqrt{\pi\alpha}} e^{-(x-\varepsilon/2)^2/4\alpha} - \frac{1}{2\sqrt{\pi\alpha}} e^{-(x+\varepsilon/2)^2/4\alpha}}{\varepsilon}$$

Using lahopitals rule

$$-\delta'(x) = \frac{1}{2\sqrt{\pi\alpha}} \left| \lim_{\varepsilon \to 0} \left(\frac{-2(x-\varepsilon/2)(-1/2)}{4\alpha} e^{-(x-\varepsilon/2)^2/4\alpha} - \left(\frac{-2(x+\varepsilon/2)}{4\alpha}\left(\frac{1}{2}\right) \right) e^{-(x+\varepsilon/2)^2/4\alpha} \right) \right.$$

$$-\delta'(x) = \frac{x}{4\alpha\sqrt{\pi\alpha}} e^{-x^2/4\alpha} \quad \text{--- (2)}$$

Hence verified since (2) = (1)

b) Use the 1d free-space Green's function G_0 $G_0(x,y) = \frac{1}{2}|x-y|$ to obtain a dipole potential by evaluating

$$\lim_{\varepsilon \to 0} \frac{\frac{1}{2}|x - \varepsilon/2| - \frac{1}{2}|x + \varepsilon/2|}{\varepsilon}$$

Rationalizing

$$\lim_{\varepsilon\to 0}\frac{\left(\frac{1}{2}\left|x-\frac{\varepsilon}{2}\right|-\frac{1}{2}\left|x+\frac{\varepsilon}{2}\right|\right)\left(\frac{1}{2}\left|x-\frac{\varepsilon}{2}\right|+\frac{1}{2}\left|x+\frac{\varepsilon}{2}\right|\right)}{\varepsilon\left(\frac{1}{2}\left|x-\frac{\varepsilon}{2}\right|+\frac{1}{2}\left|x+\frac{\varepsilon}{2}\right|\right)}$$

$$=\lim_{\varepsilon\to 0}\frac{-x}{\left|x-\frac{\varepsilon}{2}\right|+\left|x+\frac{\varepsilon}{2}\right|}=\underline{\underline{\frac{-x}{2|x|}}}$$

but $G_0(x,y)=\frac{1}{2}|x-y|$

$$\frac{\partial}{\partial y}G_0(x,y)=\frac{1}{2}\frac{y-x}{|x-y|}\Rightarrow\frac{\partial}{\partial y}G_0(x,0)=\underline{\underline{\frac{-x}{2|x|}}}$$

Hence verified.

c) Show that if we try to represent the solution distribution of dipoles on the boundary of the interval $[0,1]$, we can only capture constant solutions to (1).

$$u(x)=\mu(1)\left(-\frac{\partial G_0(x,1)}{\partial y}\right)-\mu(0)\left(-\frac{\partial G_0(x,0)}{\partial y}\right)$$

At $x=0$

$$u(0)=\mu(1)\lim_{x\to 0}\left(-\frac{\partial G_0(x,1)}{\partial y}\right)-\mu(0)\lim_{x\to 0}\left(-\frac{\partial G_0(x,0)}{\partial y}\right)$$

$$\frac{\partial G_0}{\partial y}=\frac{y-x}{2|x-y|}$$

$$u(0)=\mu(1)\lim_{x\to 0^+}-\left(\frac{1-x}{2|x-1|}\right)-\mu(0)\lim_{x\to 0^+}-\left(\frac{-x}{2|x|}\right)$$

$$u(0) = \mu(1)\left(\tfrac{1}{2}\right) + \mu(0)\left(\tfrac{1}{2}\right)$$

but $u(0) = a$

$$\underline{\mu(1) + \mu(0) = 2a}$$

At $x = 1$

$$u(1) = \mu(1) \lim_{x \to 1^-} -\left(\frac{1-x}{2|x-1|}\right) - \mu(0) \lim_{x \to 1^-} -\left(\frac{-x}{2|x|}\right)$$

$$\underline{\mu(1) + \mu(0) = 2b}$$

Hence the resulting 2×2 system is

$$\mu(1) + \mu(0) = 2a \quad — ①$$
$$\mu(1) + \mu(0) = 2b \quad — ②$$

① $-$ ② $\Rightarrow$ $a - b = 0$

Hence its only solvable iff $a = b$, otherwise if $a \neq b$, we can't ~~solve the~~ solve the system

d) If we add an additional source term at a location outside the domain

$$u(x) = \mu(1)\left(-\frac{\partial G_0(x,1)}{\partial y}\right) - \mu(0)\left(-\frac{\partial G_0}{\partial y}(x,0)\right) + \mu(2)\, G_0(x,2)$$

At $x = 0$

$$u(0) = \frac{1}{2}\mu(1) + \frac{1}{2}\mu(0) + \mu(2) \lim_{x \to 0^-} G_0(x,2)$$

$$u(0) = \frac{1}{2}\mu(1) + \frac{1}{2}\mu(0) + \mu(2) = 9$$

$$\mu(1) + \mu(0) + 2\mu(2) = 29$$

At $x=1$

$$u(1) = \frac{1}{2}\mu(1) + \frac{1}{2}\mu(0) + \mu(2)\lim_{x\to 1^+} G_0(x,2)$$

$$\lim_{x\to 1^+} G_0(x,2) = \frac{1}{2}\text{ , outside the domain}$$

$$\mu(1) + \mu(0) + \mu(2) = 26$$

and Constraint

$$\mu(2) = \mu(1) - \mu(0)$$

The 3×3 system is

$$\begin{cases} \mu(1) + \mu(0) + 2\mu(2) = 2 \\ \mu(1) + \mu(0) + \mu(2) = 6 \\ \mu(1) + \mu(0) - \mu(2) = 0 \end{cases}$$

Solving the 3 equations we obtain

$$\mu(1) = 3$$
$$\mu(0) = 7$$
$$\mu(2) = -4$$

Show that at the boundary points $x=0$ and $x=1$ the resulting solution has a jumps equal to $\mu(0)$ and $\mu(1)$

$$\mu(0^+) = \frac{1}{2}\mu(1) + \frac{1}{2}\mu(0) + \mu(2) \quad \text{—① }$$

$$\mu(0^-) = \frac{1}{2}\mu(1) \cancel{+} - \frac{1}{2}\mu(0) + \mu(2) \quad \text{—②}$$

① − ②

$$\mu(0^+) - \mu(0^-) = \mu(0)$$

$$\mu(1^+) = \tfrac{1}{2}\mu(1) + \tfrac{1}{2}\mu(0) + \tfrac{1}{2}\mu(2) \quad — ③$$

$$\mu(1^-) = -\tfrac{1}{2}\mu(1) + \tfrac{1}{2}\mu(0) + \tfrac{1}{2}\mu(2) \quad — ④$$

③ − ④ $\Rightarrow$ $\mu(1^+) - \mu(1^-) = \mu(1)$

But that ~~the~~ the derivative is continuous across the boundary.

$$u(x) = \mu(1)\left(-\frac{\partial G_0(x,1)}{\partial y}\right) - \mu(0)\left(-\frac{\partial G_0(x,0)}{\partial y}\right) + \mu(2)G_0(x,2)$$

Substituting in for $\mu(1)$, $\mu(0)$ and $\mu(2)$

$$u(x) = 3\left(\frac{1-x}{2|x-1|}\right) + 7\left(\frac{x}{2|x|}\right) - 4\left(\frac{1}{2}|x-2|\right)$$

$$u(x) = -\frac{3}{2} + \frac{7}{2} + 2(x-2) = 2 + 2(x-2)$$

$u'(x) = 2$; $\forall x \in [0,1]$., $u'(x)$, exists hence its continuous across the bondary.

- What is the behaviour of the solution at the same at $x=2$?

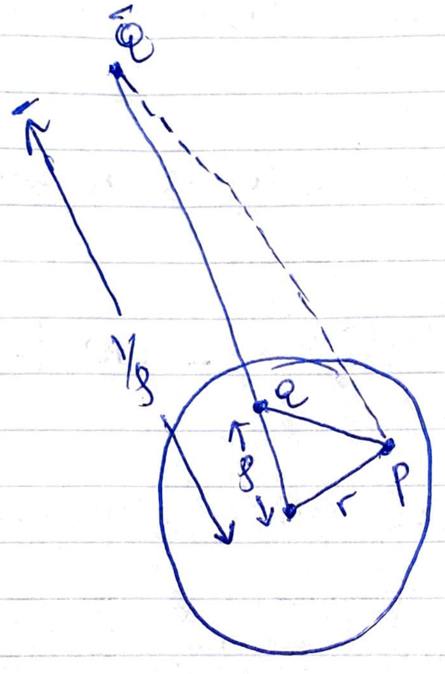
The behaviour of the solution at the same ~~at~~ at $x=2$; is continuous since

$$U'(x) = \begin{cases} \frac{-2(x-2)}{|x-2|}, & x \neq 0 \cap x \neq 1 \\ \text{indeterminate}, & \text{otherwise.} \end{cases}$$

$$\lim_{x \to 2^-} u'(x) = \lim_{x \to 2^+} U'(x) = 0$$

3. Green's function for the disk

$P(r, \theta)$ and $Q = (\rho, \theta')$, interior,
$\bar{Q} = (1/\rho, \theta')$ outside the disk.



$$G(P, Q) = \frac{1}{2\pi} \log\left(\frac{r_{PQ}}{\rho\, r_{P\bar{Q}}}\right) = \frac{1}{2\pi}\log(r_{PQ}) - \left(\frac{1}{2\pi}\log(r_{P\bar{Q}}) + \frac{1}{2\pi}\log(\rho)\right)$$

Using
$G(P, Q) = G_0(P, Q) + \bar{G}(P, Q)$

Using Cosine rule.
$r_{PQ} = \left(r^2 + \rho^2 - 2r\rho\cos(\theta - \theta')\right)^{1/2}$

$$r_{p\bar{q}} = \left(r^2 + 1/\rho^2 - 2r \cdot 1/\rho \cos(\theta - \theta')\right)^{1/2}$$

$$\rho r_{p\bar{q}} = \left(\rho^2 r^2 + 1 - 2r\rho \cos(\theta - \theta')\right)^{1/2}$$

Suppose $\bar{G}(P, Q) = -G(P, \bar{Q})$, ~~then~~

but $G(P, \bar{Q}) = \frac{1}{2\pi} \log(\rho r_{p\bar{q}})$

then

$$\bar{G}(P, Q) = -\frac{1}{2\pi} \log(\rho r_{p\bar{q}})$$

So $G(P, Q) = G_0(P, Q) + \bar{G}(P, Q)$

but $G_0(P, Q) = \frac{1}{2\pi} \log(r_{pq})$

$$G(P, Q) = \frac{1}{2\pi} \log(r_{pq}) + -\frac{1}{2\pi} \log(\rho r_{p\bar{q}})$$

$$G(P, Q) = \frac{1}{2\pi} \log\left(\frac{r_{pq}}{r_{p\bar{q}}}\right)$$

Show that $G(P, Q) = 0$, on the boundary of the disk at the boundary $r = 1$

$$r_{pq} = (1 + \rho^2 - 2\rho \cos(\theta - \theta'))^{1/2} \quad — ①$$
$$\rho r_{p\bar{q}} = (1 + \rho^2 - 2\rho \cos(\theta - \theta'))^{1/2} \quad - ②$$

$$① / ② \Rightarrow r_{pq} / \rho r_{p\bar{q}} = 1$$

Hence

$$G(P,Q) = \frac{1}{2\pi} \log(1) = 0$$

$$\underline{G(P,Q) = 0}$$