1. Prove that $(GM_{\alpha,L}^{[k]})_k - d_k = \dfrac{(GM_{\alpha,L})_k - d_k}{1-(GG^{\#})_{k,k}}$ in the following steps

a) Use the definition $\bar{d}_i$ to explain why $(GG^{\#}\bar{d})_k - (GG^{\#}d)_k = (GG^{\#})_{k,k}(\bar{d}_k - d_k)$.

$$\hat{d}_i = \begin{cases} (GM_{\alpha,L}^{[k]})_k & \text{if } i = k \\ d_i & \text{if } i \neq k \end{cases}$$

if $i = k$, $\hat{d}_k = (GM_{\alpha,L}^{[k]})_k$, but $M_{\alpha,L}^{[k]} = G^{\#}\hat{d}$

$$\hat{d}_k = (GG^{\#}\hat{d})_k$$

Since $\hat{d}_k = (GM_{\alpha,L}^{[k]})_k = (GG^{\#}\hat{d})_k$, then

$$\begin{cases} (GG^{\#}\bar{d})_k \approx (GG^{\#})_{k,k}\bar{d}_k & \text{—} ① \\ (GG^{\#}d)_k \approx (GG^{\#})_{k,k} d_k & \text{—} ② \end{cases}$$

Eqn ① − ② $\Rightarrow$ $(GG^{\#}\bar{d})_k - (GG^{\#}d)_k = (GG^{\#})_{k,k}\bar{d}_k - (GG^{\#})_{k,k}d_k$

$$= (GG^{\#})_{k,k}(\bar{d}_k - d_k)$$

b) Show that $\dfrac{\bar{d}_k - d_k - (GG^{\#}\bar{d})_k + (GG^{\#}d)_k}{\bar{d}_k - d_k} = 1 - (GG^{\#})_{k,k}$

from L.H.S

$$\frac{\bar{d}_k - d_k - (GG^{\#}\bar{d})_k + (GG^{\#}d)_k}{\bar{d}_k - d_k} = \frac{\bar{d}_k - d_k - \left((GG^{\#}\bar{d})_k - (GG^{\#}d)_k\right)}{\bar{d}_k - d_k}$$

but $(GG^{\#}\bar{d})_k - (GG^{\#}d)_k = (GG^{\#})_{k,k}(\bar{d}_k - d_k)$

then

$$\frac{\bar{d}_k - d_k - (GG^\# \bar{d})_k + (GG^\# d)_k}{\bar{d}_k - d_k} = \frac{\bar{d}_k - d_k - (GG^\#)_{k,k}(\bar{d}_k - d_k)}{\bar{d}_k - d_k}$$

$$= \frac{(\bar{d}_k - d_k)(1 - (GG^\#)_{k,k})}{(\bar{d}_k - d_k)}$$

$$= 1 - (GG^\#)_{k,k} \qquad \square$$

c) Use $\tilde{d}_k = (GM^{[k]}_{\alpha,L})_k = (GG^\# \tilde{d})_k$ to simplify 1b.

from 1b, we have;

$$\frac{\bar{d}_k - d_k - (GG^\# \bar{d})_k + (GG^\# d)_k}{\bar{d}_k - d_k} = 1 - (GG^\#)_{k,k}$$

Substituting in $\tilde{d}_k$

$$\frac{(GG^\# \tilde{d})_k - d_k - (GG^\# \bar{d})_k + (GG^\# d)_k}{\bar{d}_k - d_k} = 1 - (GG^\#)_{k,k}$$

$$-d_k + (GG^\# d)_k = (\bar{d}_k - d_k)(1 - (GG^\#)_{k,k})$$

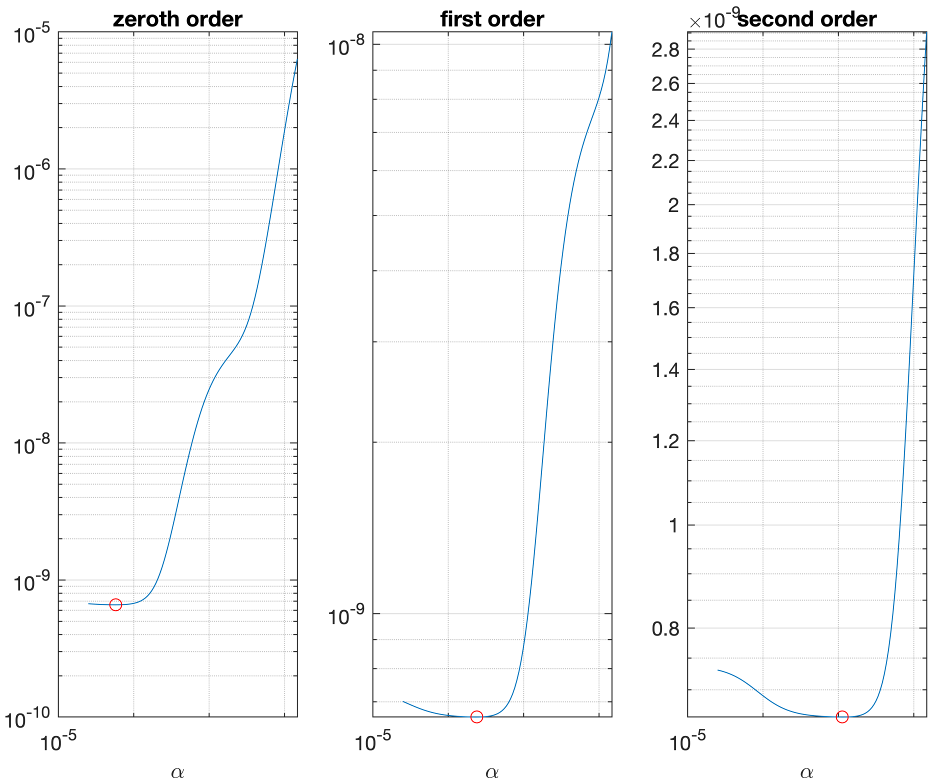
$$\bar{d}_k - d_k = \frac{(GG^\# d)_k - d_k}{(1 - (GG^\#)_{k,k})}$$

but $\bar{d}_k = (GM^{[k]}_{\alpha,L})_k$

therefore

$$(GM^{[k]}_{\alpha,L})_k - d_k = \frac{(GG^\# d)_k - d_k}{(1 - (GG^\#)_{k,k})} \qquad \square$$

## No.2 Plot the predictive errors.




## Discuss how you chose αmin and αmax and how you would chose an optimal value for α by looking at the graphs.

I used the minmum and maximum values of the alpha obtained from zeroth, first, and second order Tikhonov regularization of both the regularized and the non regularized discrepancy principles to yeild $\alpha_{min}$ and $\alpha_{max}$ respectively. The optimal value of $\alpha$ will be the value of $\alpha$ at which $g(\alpha)$ is minimum, and those are the red small circles in every plot.

## Appendix

```
clear
clc

load ifk.mat

[m,n] = size(d);
a = 0; b = 1;

n = m;

sig  = 5e-5;

dx = (b-a)/m;
g = @(x,y) x*exp(-x*y);

x = [];
for j = 1:n
    x = [x  a + (dx/2) + (j-1)*dx]; %form x
end
y = x;

G = zeros(m,n);

for i = 1:m
    for j = 1:n
        G(i,j) = g(x(j),y(i)).*dx; %form G
    end
end


%form L1 and L2 matrices
L1 = zeros(n-2,n-1);
for i=2:n-1
    for j=1:n
        L1(1,1) = -1;
        L1(1,2) = 1;
        if (i==j)
            L1(i,j) = -1;
        elseif (j==i+1)
            L1(i,j) =1;
        end
    end
end

L2 = zeros(n-3,n-1);
for i=2:n-2
    for j=1:n
        L2(1,1) = 1;
        L2(1,2) = -2;
        L2(1,3) = 1;
```

```
        if (i==j)
            L2(i,j) = 1;
        elseif (j==i+1)
            L2(i,j) = -2;
        elseif (j==i+2)
            L2(i,j) = 1;
        end
    end
end
```

No.2 Plot the predictive errors.

```
alpha_max = 0.0148;
alpha_min = 2.5058e-5;

nbar = 100;
alp_i = [];
g0 = []; g1 =[]; g2 = [];
for i = 1:nbar
    apha = alpha_min*(alpha_max/alpha_min)^((nbar-i)/(nbar-1));
    alp_i = [alp_i apha];
    g0 = [g0 galpha(m,G,eye(m),d,apha)];
    g1 = [g1 galpha(m,G,L1,d,apha)];
    g2 = [g2 galpha(m,G,L2,d,apha)];
end

[op0 i0] = min(g0);
[op1 i1] = min(g1);
[op2 i2] = min(g2);

figure(1)
subplot(1,3,1)
loglog(alp_i,g0); hold on
loglog(alp_i(i0),op0,'or'); grid on
title('zeroth order');
xlabel('\alpha'); %ylabel('g(\alpha)')

subplot(1,3,2)
loglog(alp_i,g1); hold on
loglog(alp_i(i1),op1,'or'); grid on
title('first order');
xlabel('\alpha'); %ylabel('g(\alpha)')

subplot(1,3,3)
loglog(alp_i,g2); hold on
loglog(alp_i(i2),op2,'or'); grid on

title('second order');
xlabel('\alpha'); %ylabel('g(\alpha)')

function g = galpha(m,G,L,d,alpha)
    Gpound = (G'*G + (alpha^2).*L'*L)\G';
    m_alpha = Gpound*d;
```

```
    g = m*(norm(G*m_alpha - d)^2)/trace(eye(m) - G*Gpound)^2;

end
```