Chapter 3: Generalized inverse,

Consider solving the inverse problem.

$$Gm = d \quad , \quad G \in \mathbb{R}^{m \times n}, \quad m \in \mathbb{R}^n, \quad d \in \mathbb{R}^m$$

1. Assume that G is full rank.

a) form $G^T G$

Using SVD of G, $G = USV^T$

$$G^T G = (USV^T)^T (USV^T)$$

$$= (V S^T U^T (USV^T) = VS^T U^T U S V^T$$

but $U^T U = I$

$$G^T G = VS^T I S V^T$$

but $S^T I S = S^T S = S^2$

$$\underline{G^T G = V(S^T S) V^T} = \underline{VS^2 V^T}$$

b) <u>Case 1:</u> $m > n$

$$G^T G = V(S^T S) V^T$$

Since the eigenvalues of $G^T G$ are the diagonal elements of $S^T S$, then if $m > n$

$$S^T S = \begin{bmatrix} s_1^2 & & & 0 \\ & s_2^2 & & \\ & & \ddots & \\ 0 & & & s_n^2 \end{bmatrix}_{n \times n}$$

Hence the eigen values of $S^T S$ are square of the singular values of $G = USV^T$.

Case 2: $m < n$

Similarly $S^TS = \begin{bmatrix} s_1^2 & & & & & \\ & \ddots & & & 0 & \\ & & s_m^2 & & & \\ & & & 0 & & \\ & 0 & & & \ddots & \\ & & & & & 0 \end{bmatrix}_{n\times n}$

hence the eigen values of S^TS are square of the singular values of $G = USV^T$.

c) Find the generalized inverse G^+ from the compact SVD form of G, $G = U_p S_p V_p^T$

$$G V_p = U_p S_p \Rightarrow G V_p S_p^{-1} = U_p$$

$$U_p^T = (G V_p S_p^{-1})^T \Rightarrow U_p^T = S_p^{-1} V_p^T G^T$$

from $G^+ = V_p S_p^{-1} U_p^T$

substitute for U_p^T

$$G^+ = V_p S_p^{-1} S_p^{-1} V_p^T G^T$$

$$G^+ = V_p S_p^{-2} V_p^T G^T$$

but $G^TG = V_p S_p^2 V_p^T \Rightarrow (G^TG)^{-1} = V_p S_p^{-2} V_p^T$

hence $\underline{G^+ = (G^TG)^{-1} G^T}$, which is the least square matrix

2. Show that G^+ satisfies these four properties.

let $G = U_p S_p V_p^T$

a) $G G^+ G = G$

from L.H.S

$$G G^+ G = U_p S_p \underbrace{V_p^T V_p}_{I} S_p^{-1} \underbrace{U_p^T U_p}_{I} S_p V_p^T$$

$$= U_p \underbrace{S_p^{-1} S_p}_{I} V_p^T = U_p S_p V_p^T = \underline{G}$$

b) $G^\dagger G G^\dagger = G^\dagger$

from L.H.S

$$G^\dagger G G^\dagger = V_p S_p^{-1} \underbrace{U_p^T U_p}_{I} S_p \underbrace{V_p^T V_p}_{I} S_p^{-1} U_p^T$$

$$= V_p \underbrace{S_p^{-1} S_p}_{I} S_p^{-1} U_p^T = V_p S_p^{-1} U_p^T = \underline{\underline{G^\dagger}}$$

c) $(G G^\dagger)^T = G G^\dagger$

from L.H.S

$$(G G^\dagger)^T = \left((U_p S_p V_p^T)(V_p S_p^{-1} U_p^T)\right)^T = (U_p S_p \underbrace{V_p^T V_p}_{I} S_p^{-1} U_p^T)^T$$

$$= (U_p S_p S_p^{-1} U_p^T)^T$$

$$= U_p S_p^{-1} S_p U_p^T = U_p U_p^T = U_p S_p S_p^{-1} U_p^T$$

$$= U_p S_p V_p^T V_p S_p^{-1} U_p^T = \underline{\underline{G G^\dagger}}$$

d) $(G^\dagger G)^T = G^\dagger G$

from L.H.S

$$(G^\dagger G)^T = \left((V_p S_p^{-1} U_p^T)(U_p S_p V_p^T)\right)^T$$

$$= (V_p S_p^{-1} S_p V_p^T)^T = (V_p V_p^T)^T = V_p V_p^T$$

$$= V_p S_p^{-1} S_p V_p^T = V_p S_p^{-1} U_p^T U_p S_p V_p^T$$

$$= \underline{\underline{G^\dagger G}}$$

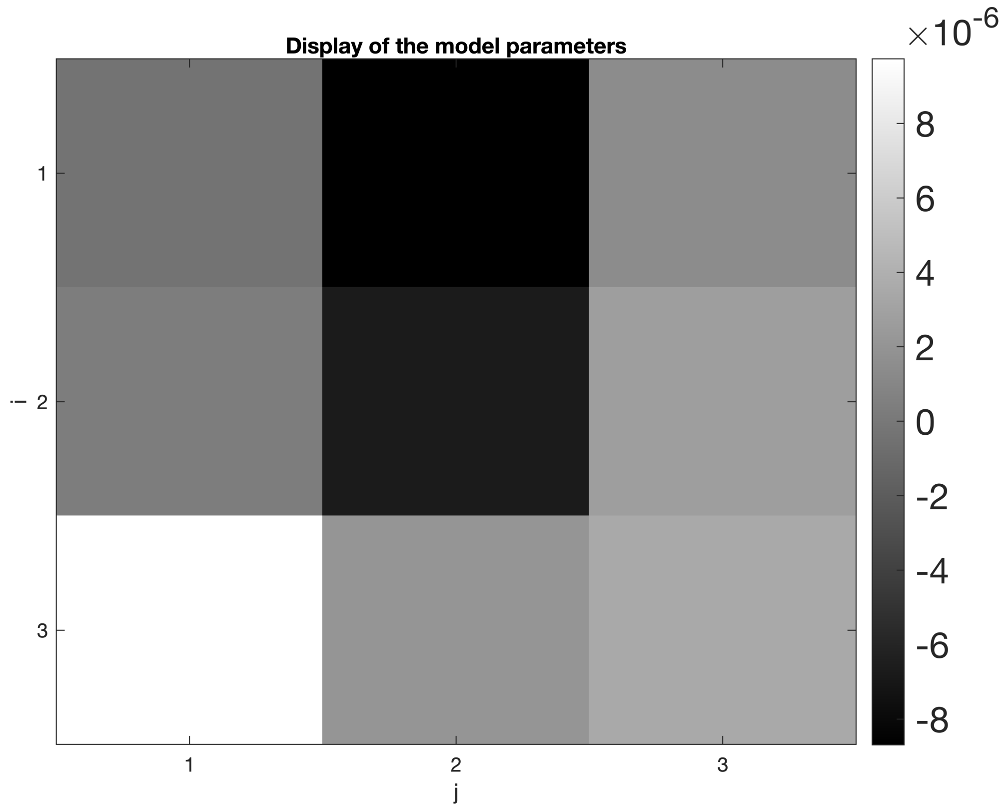
3.a) i) Use the generalized inverse of G, with the compact SVD decomposition.

```
model_parameters_1 = 9×1
10^-5 ×
   -0.0369
   -0.8697
    0.1399
    0.0303
   -0.6702
    0.2732
    0.9732
    0.2066
    0.3536
```

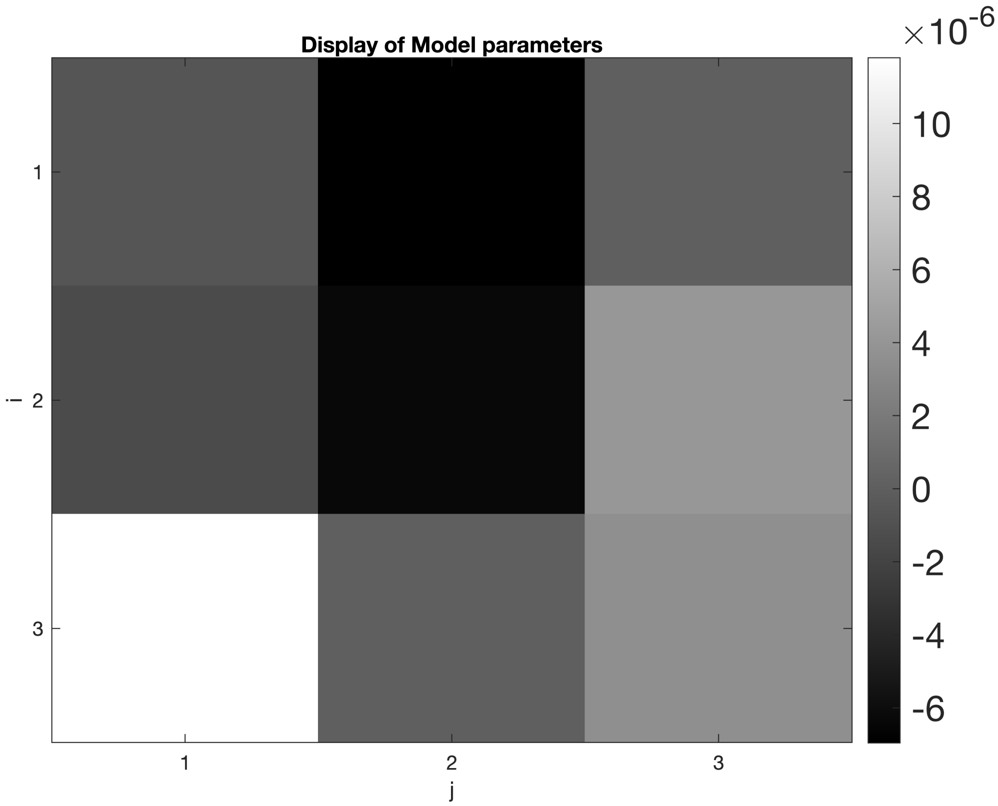
ii) Use available software, e.g. the backslash operator in MATLAB.

```
Warning: Rank deficient, rank = 7, tol =  4.895064e-15.
model_parameters_2 = 9×1
10^-4 ×
   -0.0070
   -0.0696
         0
   -0.0143
   -0.0637
    0.0413
    0.1180
         0
    0.0354
```

Plot the model parameters from 3(a)i



Plot the model parameters from 3(a)ii



Discuss the difference between estimates in 3(a)i and 3(a)ii

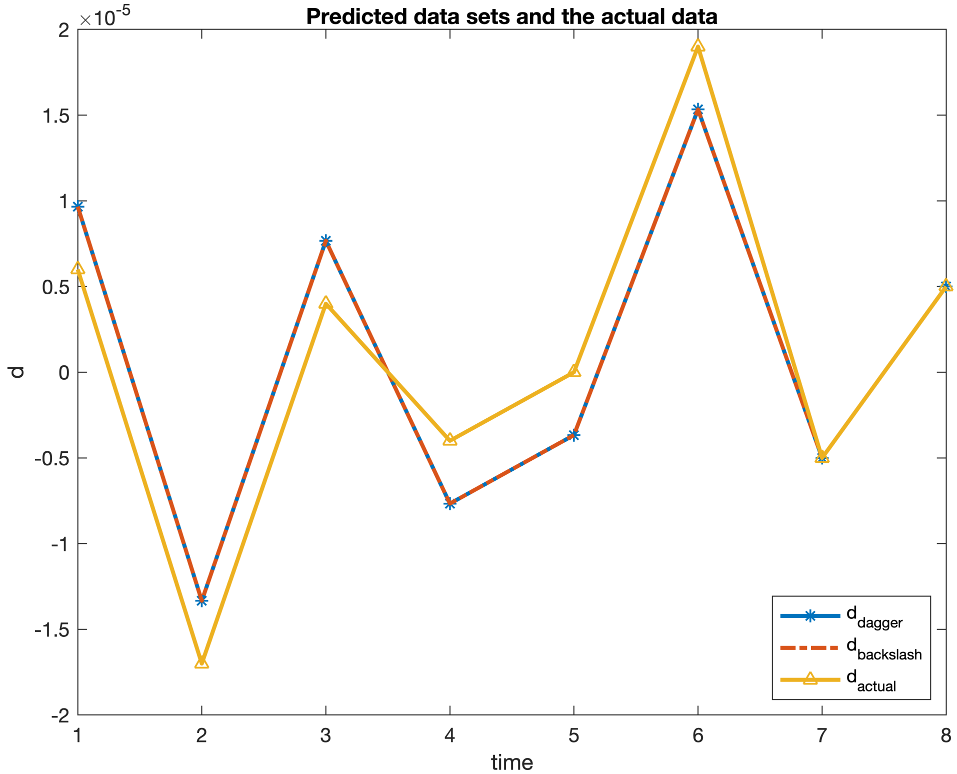
The estimates in 3(a) i, are to the order of 10^{-5} while those in 3(a)ii, are to the order of 10^{-4}.

Use each set of model parameter estimates to predict data.

```
d_dagger = 8×1
10^-4 ×
    0.0967
   -0.1333
    0.0767
   -0.0767
   -0.0367
    0.1533
   -0.0500
    0.0500
```

```
d_back_slash = 8×1
10^-4 ×
    0.0967
   -0.1333
    0.0767
   -0.0767
   -0.0367
    0.1533
   -0.0500
    0.0500
```

Compare both sets of predicted data to each other, and to the actual data.

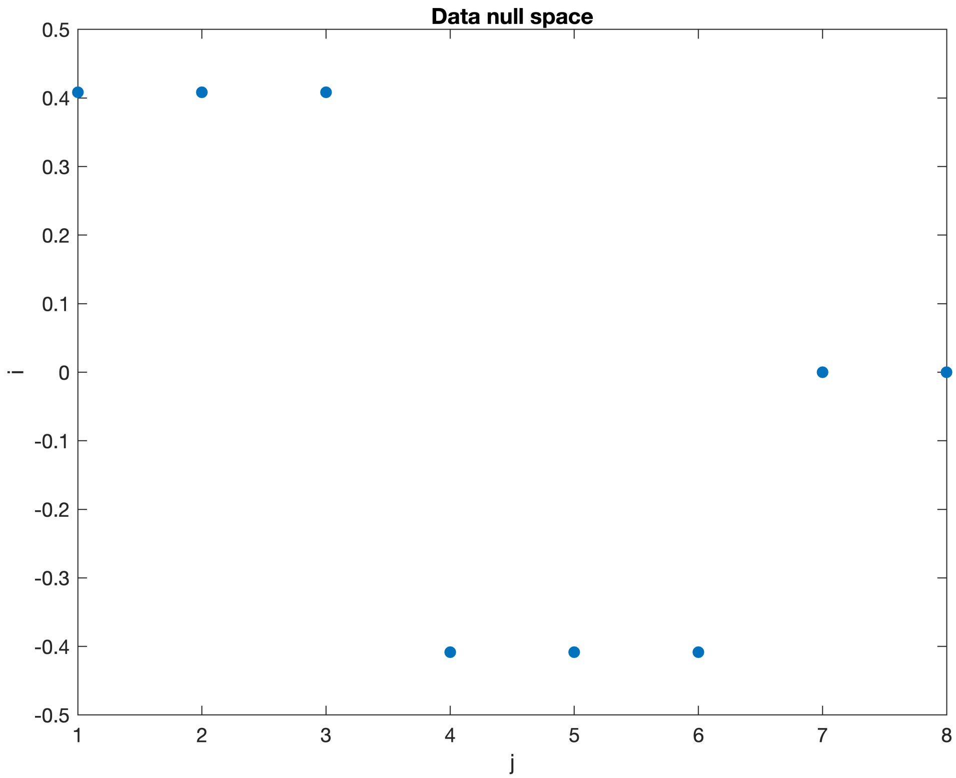


Both sets of predicted data are the same basing to the graph compared tot the actual data.

b) Determine the dimension of the data null space.

```
dim_data_Null_space = 1
```

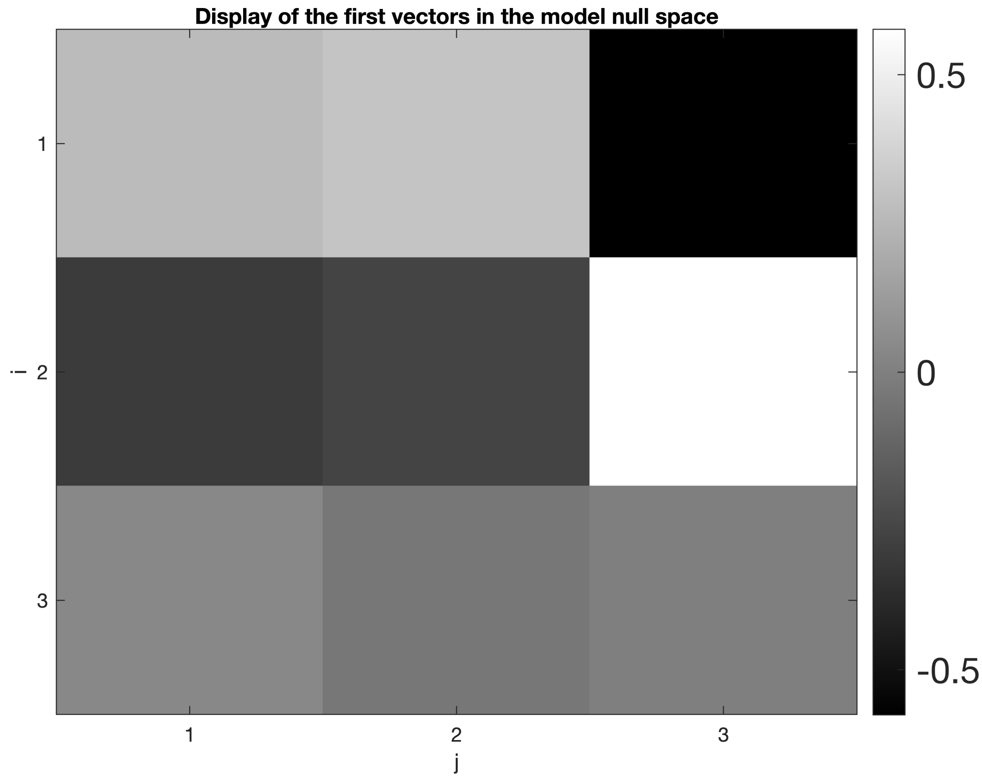
Plot the vectors in the data null space.

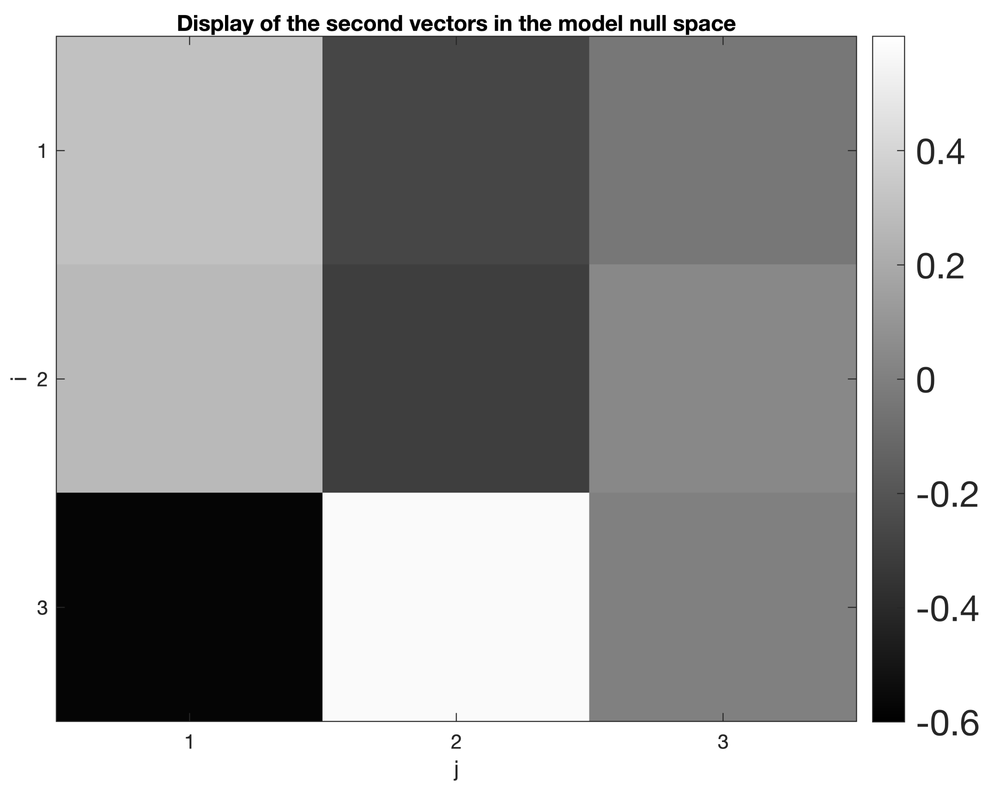


c) Determine the dimension of the model null space.

```
dim_model_Null_space = 2
```

Plot the vectors in the model null space on 3×3 grids, as they are illustrated in Figure 3.2.





(d) Is it possible to have two sets of parameters that produce the same data? Explain why or why not, and give an example if possible.

Yes, since all the model parameters are in the model null space, hence any model parameter obtained as aresult of the linear combination of the elements in the model null space set plus any model parameter will always yield the same fitted data. see for instance, example 3.a) i and 3.a) ii. and also in the calculation below, the value of the fitted data is the same for different linear combinations of model null space vectors.

```
d1 = 8×1
10^-4 ×
    0.0967
   -0.1333
    0.0767
   -0.0767
   -0.0367
    0.1533
   -0.0500
    0.0500

d2 = 8×1
10^-4 ×
    0.0967
   -0.1333
    0.0767
   -0.0767
   -0.0367
    0.1533
   -0.0500
    0.0500
```

(e) Is it possible to have two sets of data that produce the same model parameters? Explain why or why not, and give an example if possible.

Yes, Since our data exists in the the data null space, hence any two sets of data produced as aresult of a linear combination of elements in the data null space plus any data will yeild the same model parameters. see for instance, the example below in the calculation gives the same model parameters for two different linear combinations.

```
Warning: Rank deficient, rank = 7, tol =  4.895064e-15.
M1 = 9×1
10^-4 ×
   -0.0070
   -0.0696
         0
   -0.0143
   -0.0637
    0.0413
    0.1180
         0
    0.0354

Warning: Rank deficient, rank = 7, tol =  4.895064e-15.
M2 = 9×1
10^-4 ×
   -0.0070
   -0.0696
         0
   -0.0143
   -0.0637
    0.0413
    0.1180
         0
    0.0354
```

Appendix

```
d =  [6e-06 -1.7e-05 4e-06 -4e-06 0 1.9e-05 -5e-06 5e-06]';

% Construct system matrix for the ray path models
s2=sqrt(2);
G = [1,0,0,1,0,0,1,0,0;
    0,1,0,0,1,0,0,1,0;
    0,0,1,0,0,1,0,0,1;
    1,1,1,0,0,0,0,0,0;
    0,0,0,1,1,1,0,0,0;
    0,0,0,0,0,0,1,1,1;
    s2,0,0,0,s2,0,0,0,s2;
    0,0,0,0,0,0,0,0,s2];

% Get the singular values for the system matrix
[U,S,V] = svd(G);
```

3.a) i) Use the generalized inverse of G, with the compact SVD decomposition.

```
% Find dimensions of G
[m,n]=size(G);

%disp('System rank:')
p=rank(G);

Gdagger = V(:,1:p)*inv(S(1:p,1:p))*U(:,1:p)';

%Estimates
model_parameters_1 = Gdagger*d
model_parameters_11 = pinv(G, 4.8e-15)*d;
```

ii) Use available software, e.g. the backslash operator in MATLAB.

```
model_parameters_2 = G\d
```

Plot the model parameters from 3(a)i

```
figure(7)
clf
colormap('gray')
imagesc(reshape(model_parameters_1,3,3)');
```

```
%caxis([-0.9e-5 1e-5])
set(colorbar,'Fontsize',18);
set(gca,'xtick',[1,2,3]);
set(gca,'ytick',[1,2,3]);
xlabel('j')
ylabel('i')
title('Display of the model parameters')
```

Plot the model parameters from 3(a)ii

```
figure(8)
clf
colormap('gray')
imagesc(reshape(model_parameters_2,3,3)');
%caxis([-0.1e-4 0.2e-4])
set(colorbar,'Fontsize',18);
set(gca,'xtick',[1,2,3]);
set(gca,'ytick',[1,2,3]);
xlabel('j')
ylabel('i')
title('Display of Model parameters')
```

Use each set of model parameter estimates to predict data.

```
d_dagger = G*model_parameters_1 %using model parameter_1
d_back_slash = G*model_parameters_2 %using model parameter_2
```

Compare both sets of predicted data to each other, and to the actual data.

```
figure(10)
plot(d_dagger,'-*','LineWidth', 2)
hold on
plot(d_back_slash,'-.','LineWidth', 2)
hold on
plot(d,'-^','LineWidth', 2)
legend('d_{dagger}','d_{backslash}','d_{actual}','Location','southeast')
title('Predicted data sets and the actual data')
xlabel('time'); ylabel('d');
```

b) Determine the dimension of the data null space.

```
dim_data_Null_space = m-p
```

Plot the vectors in the data null space.

```
figure(9)
plot(U(:,8),'.','MarkerSize',20)
title('Data null space')
xlabel('j')
ylabel('i')
```

c) Determine the dimension of the model null space.

```
dim_model_Null_space = 2
```

Plot the vectors in the model null space on 3×3 grids, as they are illustrated in Figure 3.2.

```
m01=reshape(V(:,p+1),3,3)';
m02=reshape(V(:,p+2),3,3)';

figure(1)
clf
colormap('gray')
imagesc(m01)
%caxis([-0.6 0.6]);
set(colorbar,'Fontsize',18);
set(gca,'xtick',[1,2,3]);
set(gca,'ytick',[1,2,3]);
xlabel('j')
ylabel('i')
title('Display of the first vectors in the model null space');

figure(2)
clf
colormap('gray')
imagesc(m02)
caxis([-0.6 0.6]);
set(colorbar,'Fontsize',18);
set(gca,'xtick',[1,2,3]);
set(gca,'ytick',[1,2,3]);
xlabel('j')
ylabel('i')
title('Display of the second vectors in the model null space');
```

(d) Is it possible to have two sets of parameters that produce the same data?

```
M1 = model_parameters_1 + 6*V(:,8) + 5*V(:,9);
d1 = G*M1

M2 = model_parameters_2 + 0.6*V(:,8) + 0.9*V(:,9);
d2 = G*M2
```

(e) Is it possible to have two sets of data that produce the same model parameters? Explain why or why not, and give an example if possible.

```
d1 = d_dagger + 6*U(:,8);
M1 = G\d1

d2 = d_back_slash + 0.9*U(:,8);
M2 = G\d2
```