Boise State University

BSU

Final Project

INVERSE METHODS FOR SHALLOW WATER EQUATIONS

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Abstract

Contents

1 Introduction

1.1 Background

The shallow water equations (SWE) are a system of hyperbolic partial differential equations (PDEs) describing the flow below a pressure surface in a fluid. They have been frequently used to model several real-life problems i.e. the propagation of tsunamis waves in the ocean (Dias and Dutykh, 2007) and modeling of atmospheric turbulence. Deep knowledge is required to handle such events, therefore first, robust, and computationally efficient methods like inverse methods are required to solve the shallow water equations.

1.2 General Objectives

- To discover a best approach of chosing an initial guess (m_0) for the SWE problem while recovering true estimates (m_{true}).
- To verify if the confidence interval of the recovered estimates captures the true parameter estimates.
- To check the relationship between the recovered estimates.
- To investigate how the order of the roughening matrix impact the χ^2, p-value and L2-norm of the recovered estimates.

1.3 Literature

Inverse methods have been frequently used to handle shallow water problems; for instance **?** proposed technique which reconstructed the initial tsunami waveform using inversion of the remote measurements of the water level data. Monnier et al. (2016), Gessese and Sellier (2012), and **?**.

2 Problem Formulation and Methods

2.1 Formulation of the Forestclaw code

2.1.1 Riemann solvers

Riemann solvers are numerical methods in which time-averaged fluxes of all conserved quantities are calculated to solve fundamental problems in conservation laws named Riemann problems. A Riemann problem can be defined as a specific initial value problem (Cauchy) of a partial differential equation (PDE) that consists of conservation equations (1) combined with piece-wise constant initial data which has a single discontinuity in the domain of interest as shown in equation (2) (George, 2011).

$$q_t + f(q)_x = 0 \tag{1}$$

$$q(x,0) = \begin{cases} q_L, & \text{if } x \leq 0, \\ q_R, & \text{if } x > 0, \end{cases} \tag{2}$$

where $f(q)_x \in \mathbb{R}^m$ is a vector of conserved quantities, q_R and q_L are two piece-wise constant states separated by a discontinuity.

2.1.2 Shallow Water Equations (SWE)

The SWE are a system of hyperbolic PDEs governing the flow below a pressure surface in a fluid. They arise from the Navier-Stokes equations. In one dimension, the SWE can be used to model a fluid in a channel of unit width, taking the vertical velocity negligible, and horizontal velocity roughly constant throughout any cross section of the channel (George, 2008).

Consider small-amplitude waves in a one-dimensional fluid channel that is shallow relative to its wavelength. The conservation of momentum equation is written in terms of pressure, $p(x,t) = \frac{1}{2}gh^2$, and the height field $h(x,t)$ (m), which breaks down into system (3).

$$\begin{aligned} h_t + (hu)_x &= 0 \\ (hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x &= 0 \end{aligned} \tag{3}$$

where hu measures the flow rate of water past a point, ρ (kg/m^3) is the constant density of the in-compressible fluid, and $u(x,t)$ (m/s) is the horizontal velocity.

A very simple set of initial conditions is a single discontinuity at the middle of the channel. In this case, we set h and hu equal to constants on either side of the channel. This problem is a classic Riemann Problem, and for the SWE, has an exact solution. We assume the discontinuity is at $x = 0$. The variation of h and hu on either side of the discontinuity leads the waves in the Riemann problem to move at different speeds creating discontinuities (shocks) or changing regions (rarefactions) (LeVeque et al., 2002). At $x = 0$ and $t = 0$, the discontinuity is located between the left and right state, so the solution at the left $(q_l)$and right(q_r) states are given by:

$$q_l = [h_l,\, u_l,\, hu_l]^T \quad \text{and} \quad q_r = [h_r,\, u_r,\, hu_r]^T \tag{4}$$

As t increases, four distinct regions are created, separated by characteristics. The middle state called the intermediate state$(q_m = [h_m,\, u_m,\, hu_m]^T)$, is generated. The determination of this state characterizes the Riemann problem and how it connects to other states via waves in each respective characteristic family (Bale et al., 2003). This can only hold if the connection wave speeds satisfy the Lax entropy condition. The intermediate state is obtained by solving the Riemann problem using the exact or approximate method, however in this project we considered the exact solver, due to its computational accuracy, robustness, and ability to handle wet and dry states better than the numerical approach.

2.1.3 Exact Riemann Solver

The states can be separated by either *shocks* or *rarefactions*. General left and right states can be connected by a combination of the two (either two shocks, two rarefactions, or one of each). We describe how to determine if two states are connected by a shock. We refer the reader to LeVeque et al. (2002) for other cases.

We can obtain an exact solution to the Riemann Problem for the SWE as follows. The shock speed, $s(t)$, from the shock wave as the solution emerges is determined from the Rankine-Hugoniot jump condition given by equation (5) which must be satisfied across any shock wave. If q_l and q_r are connected by a shock, the Rankine Hugoniot conditions will be satisfied (Mandli et al., 2016).

$$\begin{aligned} s_1(q_m - q_l) &= f(q_m) - f(q_l) \\ s_2(q_r - q_m) &= f(q_r) - f(q_m) \end{aligned} \tag{5}$$

By applying condition (5) to shallow water equations (3) creates a system of four equations (6) that must be satisfied simultaneously.

$$
\begin{aligned}
s_1(h_m - h_l) &= hu_m - hu_l \\
s_1(hu_m - hu_l) &= hu_m^2 - hu_l^2 + \frac{1}{2}g(h_m^2 - h_l^2) \\
s_2(h_r - h_m) &= hu_r - hu_m \\
s_1(hu_r - hu_m) &= hu_r^2 - hu_m^2 + \frac{1}{2}g(h_r^2 - h_m^2)
\end{aligned} \tag{6}
$$

Since the (h_l, u_l) and (h_r, u_r) are fixed, we find all states: (h_m, u_m) and their corresponding speeds: s_1 and s_2 that satisfy system (6). We have four equations and four unknowns, which gives a two parameter family of solutions: one-shock and two-shock. Using h_l and h_r as parameters, corresponding u_l, u_r, s_1,and s_2 are determined for each h_l and h_r.

Consider a general Riemann problem whose known solution consists of two shocks with initial data (4). This problem can be solved by finding the state q_m that can be connected to q_l by a *1-shock* and simultaneously connects to q_r by a *2-shock*. The point q_m lies on the curve (7) of points through point q_r that connects to q_r by a *2-shock* (Berger et al., 2011).

$$
u_m = u_r + (h_m - h_r)\sqrt{\frac{g}{2}\left(\frac{1}{h_m} + \frac{1}{h_r}\right)} \tag{7}
$$

Likewise, the state(q_m), must also lie on the Hugonoit locus (equation (8)) of the 1-shock wave passing through q_l

$$
u_m = u_l - (h_m - h_l)\sqrt{\frac{g}{2}\left(\frac{1}{h_m} + \frac{1}{h_l}\right)} \tag{8}
$$

Equations (7) and (8) form a system of two equations with two unknowns (h_m and u_m) that are equated since the left-hand sides of both equations are equal. A non linear equation that consist of only one unknown h_m is formed and solved using an iterative method such as Newton method to obtain a desired intermediate state in the Riemann solution (LeVeque et al., 2011).

As an example, consider a SWE Riemann problem with $h_l = h_r = 1, u_l = 0.5$, and $u_r = -0.5$. These initial values are used by the Newton solver to solve equations (7) and (8) to produce $h_m = 1.554$. The shock speeds (s_l and s_r) in each region are different due to different wave characteristics, we use this concept to loop through all interfaces and determine h, u, and hu in each region as shown in the code 5.1 . At each interface the function *forestclaw*, is called and respective values of height (h), velocity (u) and momentum (hu) solutions are stored, forming the simulated data $(G(m))$ where $m = [q_l, q_m, q_r]$. According to Lax entropy conditions, a *1-shock* that physically connects q_l to q_m is obtained if $h_m > h_l$, and similarly a *2-shock* wave that physically connects q_m to q_r requires $h_m > h_r$ (LeVeque et al., 2011).

The synthetic data, d_s, is generated at each interface by adding noise (ϵ) to obtained simulated data $(G(m))$ as shown in equation (9).

$$
d_s = G(m) + \epsilon, \qquad \text{for } \epsilon \sim N(0, \sigma^2) \tag{9}
$$

2.2 Occam Inversion

The Occam model was implemented and used to recover the true parameter estimates (m_{true}) for the SWE problem using the following inputs;

- Synthetic data (d_s) given by equation (9)
- A regularization parameter, $\delta = \sigma\sqrt{N}$, where σ is the uncertainity in the simulated data $(G(m))$ and N is the size of the spatial domain.

- A different roughening matrix (L) for each simulation i.e, either zeroth-order Tikhonov regularization (L_0) or a finite difference approximation of a first (L_1) or second (L_2) derivative for higher-order regularization as shown in equation (10)

$$L_0 = \mathbf{I}, \quad L_1 = \begin{pmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & \ddots & \ddots \\ & & & -1 \quad 1 \end{pmatrix}, \quad \text{and} \quad L_2 = \begin{pmatrix} 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & \ddots & \ddots & \ddots \\ & & & 1 \quad -2 \quad 1 \end{pmatrix} \tag{10}$$

  where $\mathbf{I} \in \mathbb{R}^{3\times3}$ is an identity matrix, $L_1 \in \mathbb{R}^{2\times3}$ and $L_2 \in \mathbb{R}^{1\times3}$.

- An initial solution ($m^{(0)}$) given by equation (11)

$$m^{(0)} = \begin{pmatrix} q_l^{(0)} \\ q_m^{(0)} \\ q_r^{(0)} \end{pmatrix}^T = \begin{pmatrix} h_l^{(0)} & u_l^{(0)} & hu_l^{(0)} \\ h_m^{(0)} & u_m^{(0)} & hu_m^{(0)} \\ h_r^{(0)} & u_r^{(0)} & hu_r^{(0)} \end{pmatrix} \tag{11}$$

- A nonlinear forward model, $G(m)$, and its Jacobian, J_{ij}, which is designed based on a centered difference approach as shown in equation (12).

$$J_{ij} \approx \frac{(G(m + h_s e_j)_i - G(m - h_s e_j))_i}{2h_s}, \tag{12}$$

  where h_s is the step size, $e_j \in \mathbb{R}^3$ is a vector of ones.

Since the number of equations are greater than the number of parameter estimates, then at each iterations, the model uses the discrepancy principle to search for a solution that minimizes $||Lm||_2$ subject to the constraint $||G(m) - d_s||_2 \leq \delta$ while updating the Jacobian ($J(m^{(k)})$) and the vector $\hat{d}(m^{(k)})$ given by equation (13) at every k^{th} iteration.

$$\hat{d}(m^{(k)}) = d_s - G(m^{(k)}) + J^{(k)} m^{(k)} \tag{13}$$

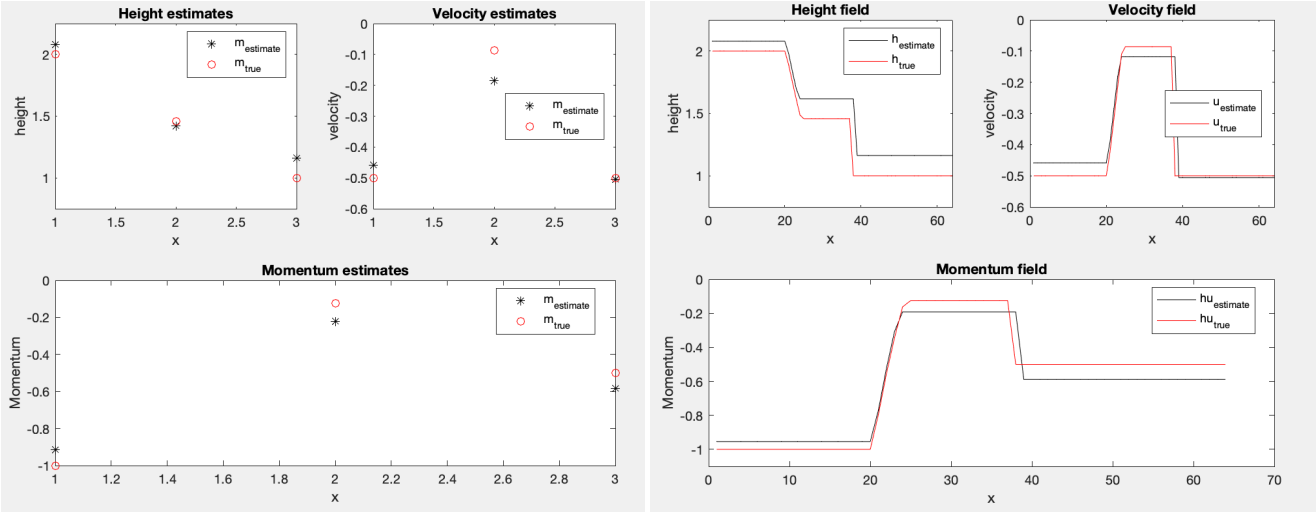
Then the updated modules corresponding to mean of the regularization parameter values are computed using equation (14).

$$m^{k+1} = (J(m^{(k)})^T J(m^{(k)}) + \alpha^2 L^T L)^{-1} J(m^{(k)})^T \hat{d}(m^{(k)}) \tag{14}$$

where α is a regularization parameter. The procedure is repeated several times until when a specific value of m^{k+1} with the maximum value of α such that $\chi^2(m^{(k+1)}) \leq \delta^2$ is found, otherwise instead a value of α that minimizes $\chi^2(m^{(k+1)})$ is used to locate m^{k+1}.

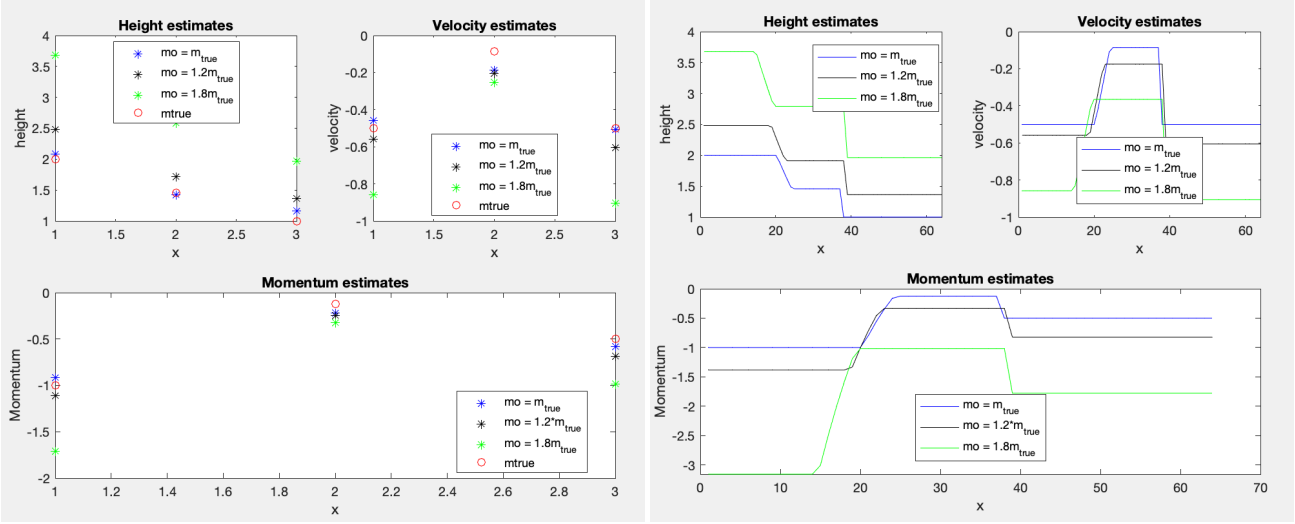
3 Results and Discusion

In this section results obtained from various simulations are presented and discused



(a) (b)

Figure 1: (a) and (b) respectively show the parameter estimates and their corresponding data



(a) (b)

Figure 2: (a) and (b) respectively show the parameter estimates and their corresponding data based on the value of the initial estimates.

Figures 2a and 2b represents recovered estimates depending on the initial estimates (m_0) used,

explains how best we can choose the initial estimates (m_0) while recovering m_{true}. The closer m_0 is to m_{true}

	$m_0 = m_{true}$	$m_0 = 1.2m_{true}$	$m_0 = 1.8m_{true}$
h	0.69264	0.0099329	0
u	0.97701	0.90333	0.062386
hu	0.94026	0.10929	0

Table 1: P-values of different values of m_0

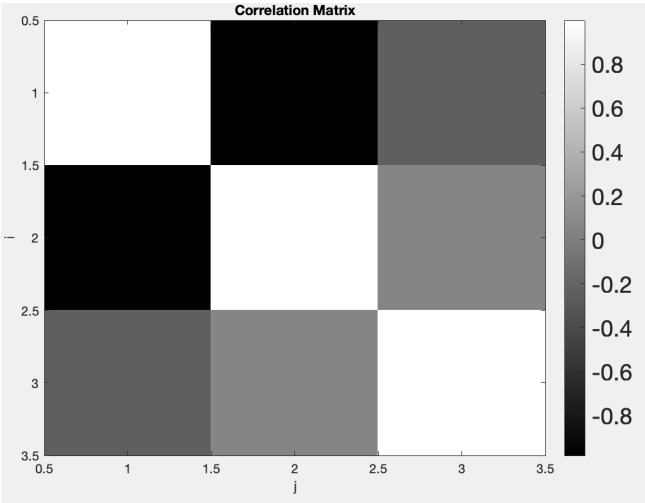


Figure 3: Correlation matrix

Figure. 3, displays the correlations between the recovered parameter estimates.

$$\text{Covariance matrix} = \begin{pmatrix} 4.4814 & -4.4333 & -0.2542 \\ -4.4333 & 4.7621 & -0.0597 \\ -0.2542 & -0.0597 & 0.3228 \end{pmatrix} \tag{15}$$

$$m_{true} = \begin{pmatrix} 2.000 & -0.500 & -1.000 \\ 1.4571 & -0.0858 & -0.1250 \\ 1.000 & -0.500 & -0.500 \end{pmatrix} \tag{16}$$

$$C_h = \begin{pmatrix} -2.0528 & 6.2456 \\ -2.7290 & 5.5694 \\ -2.9935 & 5.3049 \end{pmatrix} \quad C_u = \begin{pmatrix} -4.7359 & 3.8184 \\ -4.4700 & 4.0844 \\ -4.7801 & -3.7742 \end{pmatrix} \quad C_{hu} = \begin{pmatrix} -2.0406 & 0.1867 \\ -1.3347 & 0.8927 \\ -1.6924 & 0.5349 \end{pmatrix} \tag{17}$$

$$\text{Correlation matrix} = \begin{pmatrix} 1.0000 & -0.9597 & -0.2113 \\ -0.9597 & 1.0000 & -0.0481 \\ -0.2113 & -0.0481 & 1.0000 \end{pmatrix} \tag{18}$$

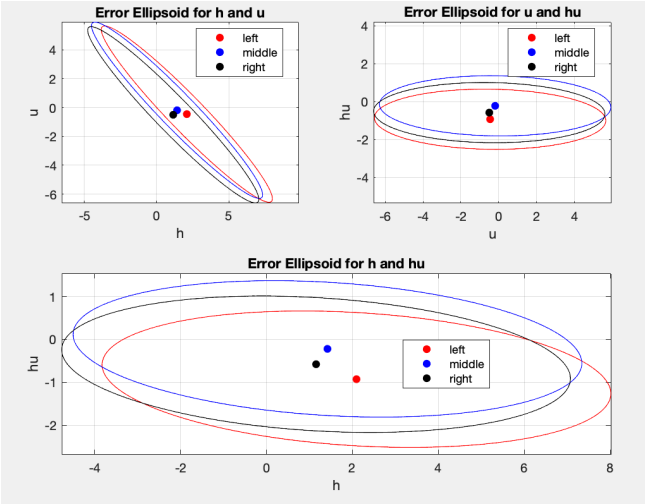


Figure 4: Linear error ellipsiods

Figure. 4, contains three linear error ellipsoids depicting the confidence regions between parameters: h and u, u and hu, and h and hu . The red, blue, and black dots represents the left, middle and right parameter estimates. Since these estimates lie at the center for each of the corresponding colour region, implies that the estimates lie with in the confidence regions.

	L0			L1			L2		
	h	u	hu	h	u	hu	h	u	hu
$\chi 2$	1.4564	0.20341	0.39861	1.4564	0.20356	0.39916	1.4559	0.20311	0.39858
p-value	0.69237	0.97704	0.94053	069237	0.97701	0.94042	0.6925	0.97709	0.94054
L2-norm	0.1888			0.1884			0.1880		

Figure 5: The Chi-square, p-value, and the L2-norm for the three roughening matrices: zeroth, first, and second.

4 Conclusion

I discovered that recovering mtrue, depends on: The initial estimate ,mo, used. The value of the standard deviation used to generate the noise. The step size ,h, The inversion model used which high depends on the range of values of alpha used. The order of the roughening matrix used. P-value is highly affected by input parameters: tfinal, h, roughening matrix, inversion model, N, e.t.c.

References

D. Bale, R. J. LeVeque, S. Mitran, and J. Rossmanith. A Wave Propagation Method for Conservation Laws and Balance Laws with Spatially Varying Flux Functions. 24(3): 955–978, 2003. doi: 10.1137/S106482750139738X. URL `http://dx.doi.org/10.1137/S106482750139738X`.

M. J. Berger, D. L. George, R. J. LeVeque, and K. T. Mandli. The GeoClaw software for depth-averaged flows with adaptive refinement. 34(9):1195–1206, September 2011. doi: 10.1016/j.advwatres.2011.02.016. arXiv:1008.0455v2 [physics.geo-ph].

F. Dias and D. Dutykh. Dynamics of tsunami waves. In *Extreme man-made and natural hazards in dynamics of structures*, pages 201–224. Springer, 2007.

D. L. George. Augmented Riemann solvers for the shallow water equations over variable topography with steady states and inundation. *J Comp Phys*, 227(6):3089 – 3113, 2008. ISSN 0021-9991. doi: https://doi.org/10.1016/j.jcp.2007.10.027. URL `http://www.sciencedirect.com/science/article/pii/S0021999107004767`.

D. L. George. Adaptive finite volume methods with well-balanced Riemann solvers for modeling floods in rugged terrain: Application to the Malpasset dam-break flood (France, 1959). 66 (8):1000–1018, 2011. ISSN 1097-0363. doi: 10.1002/fld.2298. URL `http://dx.doi.org/10.1002/fld.2298`.

A. Gessese and M. Sellier. A direct solution approach to the inverse shallow-water problem. *Mathematical Problems in Engineering*, 2012, 2012.

R. J. LeVeque, D. L. George, and M. J. Berger. Tsunami modelling with adaptively refined finite volume methods. 20:211 – 289, May 2011. doi: 10.1017/S0962492911000043.

R. J. LeVeque et al. *Finite volume methods for hyperbolic problems*, volume 31. Cambridge university press, 2002.

K. T. Mandli, A. J. Ahmadia, M. Berger, D. Calhoun, D. L. George, Y. Hadjimichael, D. I. Ketcheson, G. I. Lemoine, and R. J. LeVeque. Clawpack: Building an open source ecosystem for solving hyperbolic PDEs. 2(68), August 8 2016. doi: https://doi.org/10.7717/peerj-cs.68. URL `https://peerj.com/search/?q=mandli&t=&type=articles&subject=&topic=&uid=&sort=&journal=cs&discipline=&type=articles`.

J. Monnier, F. Couderc, D. Dartus, K. Larnier, R. Madec, and J.-P. Vila. Inverse algorithms for 2d shallow water equations in presence of wet dry fronts: Application to flood plain dynamics. *Advances in Water Resources*, 97:11–24, 2016.

5 Appendix

5.1 Forestclaw solver

```
%%=====================================================================
% Boise State University
% Author: Brian KYANJO
% supervised by: Prof. Jodi Mead
% class: Inverse Methods
% Date: March 17th, 2022
% final project
%
% Description:
% -----------
% Uses the Initial Riemann problem to find an intemediate state (qm) which either
% the intial left or right state connects to it via any combination of shocks and
% rarefactions in the two families.
%
% For Riemann problems with an initial dry state on one side, the exact Riemann
% solution contains only a single rarefaction connecting the wet to dry state.
% The evolving wet dry interface is therefore simply one edge of the rarefaction.
% The propagation speed of this interface can be exactly determined using the
% Riemann invariants of the corresponding characteristics field.
%
% Input:
% -----
% x  - array of spacial points
% t  - array of temporal points
% mq - specifies the output (0 and 1 corresponds to h and hu respectively)
% ql - left initial state
% qr - right initial state
% Returns:
% h  - array of hieght field values
% hu - array of momentum field values
%%=====================================================================

% Exact solver
```

```
function [Q,hs,us] = forestclaw(ql,qr,xi)
global g;
hl = ql(1); hr = qr(1);
ul = ql(2); ur = qr(2);

hs = Newton(hl,hr,ul,ur);  %calling the newton solver
us = ul - phi(hs,hl);

if xi <= us
if hs > hl
s = ul - sqrt(0.5*g*hs/hl*(hl+hs));
if xi <= s
h = hl;
hu = hl*ul;
else
h = hs;
hu = hs*us;
end
else
head = ul -sqrt(g*hl);
tail = us - sqrt(g*hs);
if xi <= head
h = hl;
hu = hl*ul;
elseif xi >= tail
h = hs;
hu = hs*us;
else
h = (((ul + 2*sqrt(g*hl) - xi)/3)^2)/g;
u = xi + sqrt(g*h);
hu = h*u;
end
end
else
if hs > hr
s = ur + sqrt(0.5*g*hs/hr*(hs+hr));
if xi <=s
h = hs;
hu = hs*us;
else
h = hr;
hu = hr*ur;
end
else
head = ur + sqrt(g*hr);
tail = us + sqrt(g*hs);
if xi >= head
h = hr;
hu = hr*ur;
elseif xi <= tail
```

```
h = hr;
hu = hr*ur;
else
h = (((xi-ur+2*sqrt(g*hr))/3)^2)/g;
u = xi - sqrt(g*h);
hu = h*u;
end
end
end
Q = [h;hu./h;hu];
end

% Newton solver
function [hs] = Newton(hl,hr,ul,ur)
global g;
hs = ((sqrt(hl) + sqrt(hr) - (ur-ul)/2/sqrt(g))^2)/4;
tol = 1e-12;
max_iter = 100;

for i=1:max_iter
gk = func(hs,hl,hr,ul,ur);
res = abs(gk);

if (res<tol)
break
else
continue
end

dg = dfun(hs,hl,hr,ul,ur);
dh = -gk/dg;
delta = 1;

for i=1:500
if (abs(func(hs+dh*delta,hl,hr,ul,ur)) >= res)
delta = 0.5*delta;
else
break
end
end
hs = hs + delta*dh;
end
end

% phi function
function [h] = phi(hs,hlr)
global g;
if (hs>hlr)
h = sqrt(0.5*g*(hs + hlr)/(hs*hlr))*(hs - hlr);
else
```

```
h = 2*sqrt(g)*(sqrt(hs) - sqrt(hlr));
end
end

% function f
function [f] = func(hs,hl,hr,ul,ur)
global g;
f = phi(hs,hl) + phi(hs,hr) + ur - ul;
end

% Jacobian of f
function [df] = dfun(hs,hl,hr,ul,ur)
global g;
eps = 1e-7;
df = (func(hs+eps,hl,hr,ul,ur) - func(hs-eps,hl,hr,ul,ur))/(2*eps);
end


function plot_ellipse(DELTA2,C,m,first,second)
n=5000;
%first - first parameter
%second - second parameter
%construct a vector of n equally-spaced angles from (0,2*pi)
theta=linspace(0,2*pi,n)';

%corresponding unit vector
xhat=[cos(theta),sin(theta)];
Cinv=inv(C);

%preallocate output array
r=zeros(n,2);
for i=1:n
%store each (x,y) pair on the confidence ellipse in the corresponding row of r
r(i,:)=sqrt(DELTA2/(xhat(i,:)*Cinv*xhat(i,:)'))*xhat(i,:);
end

% Plot the ellipse and set the axes.
for i = 1:3
a=['r' 'b' 'k'];
plot(m(i,first)+r(:,1), m(i,second)+r(:,2),'color',a(i)); hold on
plot(m(i,first),m(i,second),'.','color',a(i),'MarkerSize',17);grid on
axis equal
end
end
```

5.2 Model

```
%%=====================================================================
% Author: Brian KYANJO
% supervised by: Prof. Jodi Mead
```

```
% class: Inverse Methods
% Date: March 17th, 2022
% final project
%  [Q] = model(m)
%
% INPUT
%   m - a guess at the model
%
% OUTPUT
%   Q   - a model matrix G(ql,qr,x/t) for the forward problem
%%=====================================================================

function [Q] = model(m)
global N; global M; global h;
global t; global x;

for j= 2:M % time loop
for i=1:N % spartial loop
xi = x(i)/t(j);
[Q(:,i),hs,us] = forestclaw(m(1,:),m(3,:),xi);
end
end
Q = Q';
```

5.3 Jacobian

```
%%=====================================================================
% Boise State University
% Author: Brian KYANJO
% supervised by: Prof. Jodi Mead
% class: Inverse Methods
% Date: March 17th, 2022
% final project
%  [J] = Jacobian(m)
%
% INPUT
%   m - a guess at the model
%
% OUTPUT
%   J   - its corresponding Jacobian
%%=====================================================================

function [J] = Jacobian(m)
global N; global M; global h;
global t; global x;

for j= 2:M % time loop
J = [];
for i=1:N % spartial loop
```

```
xi = x(i)/t(j);

% Formulation of Jacobian Matrix
ej = ones(3,1);
[Qmin(:,i),hs,us] = forestclaw(m(1,:)-h*ej,m(3,:)-h*ej,xi);
[Qmax(:,i),hs,us] = forestclaw(m(1,:)+h*ej,m(3,:)+h*ej,xi);
J = [J (Qmax(:,i) - Qmin(:,i))./(2*h)]; % Jacobian
end
end
J = J';
```

5.4 Test script

```
%%=====================================================================
% Boise State University
% Author: Brian KYANJO
% supervised by: Prof. Jodi Mead
% class: Inverse Methods
% Date: March 17th, 2022
% final project
% main_fuction script
%%=====================================================================
function main_function(mtrue,mo,h,L,x,ql,qr,M,t,mq)
global N; global sig;
hr = qr(1); hl = ql(1)
% Initial conditions
Q = zeros(3,N);
for i=1:N
if (x(i)<=0)
Q(:,i) = ql;
else
Q(:,i) = qr;
end
end

% writing a video
v = VideoWriter('dam.avi');
open(v);

for j= 2:M % time loop
J = [];
for i=1:N % spartial loop
xi = x(i)/t(j);
[Q(:,i),hs,us] = forestclaw(ql,qr,xi);
end

noise = sig*randn(size(Q(:,:)));      % genrate noise
d = Q(:,:) + noise;   d = d';         % add noise to the data
```

```
dh = d(:,1); du = d(:,2); dhu = d(:,3);
dnoise = [0.005*dh  0.005*du 0.005*dhu ];
d = Q(:,:) + dnoise';

d = d';                                 % transposing data d
delta = sig*sqrt(N);                    % delta
fun = @(m) model(m);                    % model function handle
jac = @(m) Jacobian(m);                 % Jacobian function handle

m = occam(fun, jac, L, d, mo, delta);    % calling the Occam model to  recover mtrue

% plotting estimates
figure(1)
if mq == 1
plot(m(:,mq),'k*'); hold on
plot(mtrue(:,mq),'ro'); hold off
legend('m_{estimate}','m_{true}',Location='best')
title('Height field');
ylim([hr-0.25,hl+0.25]);
ylabel('height');xlabel('x')
frame = getframe(gcf);
writeVideo(v,frame);
elseif mq == 2
plot(m(:,mq),'k*'); hold on
plot(mtrue(:,mq),'ro'); hold off
legend('m_{estimate}','m_{true}',Location='best')
title('velocity field')
ylabel('velocity');xlabel('x')
frame = getframe(gcf);
writeVideo(v,frame);
elseif mq == 3
plot(m(:,mq),'k*'); hold on
plot(mtrue(:,mq),'ro'); hold off
legend('m_{estimate}','m_{true}',Location='best')
title('momentum field')
ylabel('Momentum');xlabel('x')
frame = getframe(gcf);
writeVideo(v,frame);
else
subplot(2,2,1)
plot(m(:,1),'k*'); hold on
plot(mtrue(:,1),'ro');  hold off
legend('m_{estimate}','m_{true}',Location='best')
title('Height estimates')
ylim([hr-0.25,hl+0.25])
ylabel('height');xlabel('x')
subplot(2,2,2);
plot(m(:,2),'k*'); hold on
plot(mtrue(:,2),'ro'); hold off
legend('m_{estimate}','m_{true}',Location='best')
```

```
title('Velocity estimates')
ylabel('velocity');xlabel('x')
subplot(2,2,[3,4]);
plot(m(:,3),'k*'); hold on
plot(mtrue(:,3),'ro'); hold off
title('Momentum estimates')
ylabel('Momentum');xlabel('x')
legend('m_{estimate}','m_{true}',Location='best')
frame = getframe(gcf);
writeVideo(v,frame);
end

for i=1:N
xi = x(i)/t(j);
[Qest(:,i),hs,us] = forestclaw(m(1,:),m(3,:),xi);
end

% chi-square
chi_s = zeros(3,1); ad = d';
for k = 1:N
chi_s = chi_s + (Qest(:,k) - ad(:,k)).^2;
end

% pvalue
dof = N - 9; %degrees of freedom
p = 1 - chi2cdf(chi_s,3);

% plotting data
figure(2)
if mq == 1
plot(Qest(mq,:),'k'); hold on
plot(Q(mq,:),'r'); hold off
legend('h_{estimate}','h_{true}',Location='best')
title('Height field');
ylim([hr-0.5,hl+0.5]);
ylabel('height');xlabel('x')
frame = getframe(gcf);
writeVideo(v,frame);
elseif mq == 2
plot(Qest(mq,:),'k'); hold on
plot(Q(mq,:),'r'); hold off
legend('u_{estimate}','u_{true}',Location='best')
title('velocity field')
ylabel('velocity');xlabel('x')
frame = getframe(gcf);
writeVideo(v,frame);
elseif mq == 3
plot(Qest(mq,:),'k'); hold on
plot(Q(mq,:),'r'); hold off
legend('hu_{estimate}','hu_{true}',Location='best')
```

```
title('momentum field')
ylabel('Momentum');xlabel('x')
frame = getframe(gcf);
writeVideo(v,frame);
else
subplot(2,2,1)
plot(Qest(1,:),'k'); hold on
plot(Q(1,:),'r'); hold off
legend('h_{estimate}','h_{true}',Location='best')
title('Height field')
ylim([hr-0.25,hl+0.25])
ylabel('height');xlabel('x')
subplot(2,2,2);
plot(Qest(2,:),'k'); hold on
plot(Q(2,:),'r'); hold off
legend('u_{estimate}','u_{true}',Location='best')
title('Velocity field')
ylabel('velocity');xlabel('x')
subplot(2,2,[3,4]);
plot(Qest(3,:),'k'); hold on
plot(Q(3,:),'r'); hold off
ylim([-1.1,0])
title('Momentum field')
ylabel('Momentum');xlabel('x')
legend('hu_{estimate}','hu_{true}',Location='best')
frame = getframe(gcf);
writeVideo(v,frame);

end


%Covariance Matrix
J = Jacobian(m);
C = inv(J'*J);

%confidence interval
za = 1.96; % 95% confidence interval

% first parameter
s1 = sqrt(C(1,1)); % standard deviation
s2 = sqrt(C(2,2));
s3 = sqrt(C(3,3));

% confidence intervals
c1 = m(:,1)- za*s1;
c2 = m(:,1) + za*s1;
c11 = m(:,2)- za*s2;
c22 = m(:,2) + za*s2;
c13 = m(:,3)- za*s3;
c33 = m(:,3) + za*s3;
```

```
%Correlation matrix
rho11 = C(1,1)/sqrt(C(1,1)*C(1,1));
rho22 = C(2,2)/sqrt(C(2,2)*C(2,2));
rho33 = C(3,3)/sqrt(C(3,3)*C(3,3));
rho12 = C(1,2)/sqrt(C(1,1)*C(2,2));
rho13 = C(1,3)/sqrt(C(1,1)*C(3,3));
rho23 = C(2,3)/sqrt(C(2,2)*C(3,3));
Correlation_matrix = [rho11 rho12 rho13;rho12 rho22 rho23;...
rho13 rho23 rho33];


end
close(v);

disp(['chi-square obs = [',num2str(chi_s'),']'])
disp(['pvalue = [',num2str(p'),']'])
L2norm = norm(mtrue - m,2)
mtrue
covariance_matrix = C
cofidence_interval_height = [c1 c2]
cofidence_interval_velocity = [c11 c22]
cofidence_interval_Momentum = [c13 c33]

Correlation_matrix = Correlation_matrix


%Linearised ellipsoid
Delta = chi2inv(0.95,3); %Delta2

figure(4)
subplot(2,2,1)
plot_ellipse(Delta,C(1:2,1:2),m,1,2);
grid on
title('Error Ellipsoid for h and u')
legend('','left','','middle','','right',Location='best')
xlabel('h'); ylabel('u')
subplot(2,2,2)
C1 = [C(2,2) C(2,3); C(3,2) C(3,3)];
plot_ellipse(Delta,C1,m,2,3);
grid on
title('Error Ellipsoid for u and hu')
legend('','left','','middle','','right',Location='best')
xlabel('u'); ylabel('hu')
subplot(2,2,[3,4]);
C2 = [C(1,1) C(1,3); C(3,1) C(3,3)];
plot_ellipse(Delta,C2,m,1,3);
grid on
title('Error Ellipsoid for h and hu')
legend('','left','','middle','','right',Location='best')
```

```
xlabel('h'); ylabel('hu')

Corr = corrcoef(C);

figure(7)
clf
colormap('gray')

imagesc(Corr)
set(colorbar,'Fontsize',18);
xlabel('j')
ylabel('i')
title('Correlation Matrix')
end
```

5.5 Main script

```
%%=====================================================================
% Boise State University
% Author: Brian KYANJO
% supervised by: Prof. Jodi Mead
% class: Inverse Methods
% Date: March 17th, 2022
% final project
% main script
%%=====================================================================

clc
clear
close all

global g; global N;
global sig; global h;
global t; global x;
global M;
warning('off','all')

% problem
hl = 2;                          % left depth
hr = 1;                          % right depth
ul = -0.5;                       % left velocity
ur = -0.5;                       % right velocity

% Spatial domain
ax = -5;
bx = 5;
ay = -2;
by = 4;
```

```
meqn = 2;                       % Number of equations in the system


g = 1;                          % Gravity

to = 0;                         % initial time
Tfinal = 1;                     % final time

ql = [hl; ul; hl*ul];           % left conservation variable
qr = [hr; ur; hr*ur];           % right consrvation variable
qm = [(hl+hr)/2 (ul+ur)/2 ...   % intermediate initial state
(hl*ul+hr*ur)/2];

N = 64;                         % Number of spartial steps

dx = (bx - ax)/N;               % spartial step size

cfl = 0.9;                      % cfl number

a = 1.5;                        % maximum velocity

dt_est = cfl*dx/a;

M = (floor(Tfinal/dt_est) + 1); % number of time steps

dt = Tfinal/(M);                % temporal step size

t = linspace(to,Tfinal,M);      % temporal domain

xe = linspace(ax,bx,N+1);       % edge locations
x = xe(1:end-1) + dx/2;         % Cell-center locations

mq = 4;                         % 1-height,
% 2-velocity,
% 3-momentum,
% 4-all fields
h = 0.9;

sig = 1e-3;                     % standard deviation

L0 = eye(3);          % generate roughening matrices
L1 = get_l_rough(3,1);
L2 = get_l_rough(3,2);


qll = [hl ul hl*ul];            % left conservation variable
qrr = [hr ur hr*ur];            % right conservation variable
xi = x(1)/t(2);                 % initial speed
[Q(:,1),hs,us] = forestclaw(ql,qr,xi); % initial conservation variable
qmm = [hs us hs*us];                   % intermediate state
```

```
%mo = [qll;qm;qrr];
mtrue = [qll; qmm; qrr];                % true parameters

mo = 1.5*mtrue;
main_function(mtrue,mo,h,L0,x,ql,qr,M,t,mq) %calling the main function script
```