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MATH 568
Linear Algebra review

$\frac{25}{25}$

1. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \\ 2 & 3 & -2 \end{pmatrix}$$

a) $$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ ✓

b) Find the null space of A

$$N(A) = \{x \in \mathbb{R}^n : Ax = 0,\ A \in \mathbb{R}^{m \times n}\}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 + 0x_2 - x_3 = 0 \\ 0x_1 + x_2 + 0x_3 = 0 \end{array}$$

$$\downarrow$$

$$x_1 = x_3$$
$$x_2 = 0$$

So x_3 is a free variable

so $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ Choose $x_3 = 1$, since its a free variable

$$N(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$ ✓

$\dim(N(A)) = 1$ ✓

Is $N(A) \in \mathbb{R}^3$: Yes ✓

c) Find the Column Space of A

$R(A) = \{ b \in \mathbb{R}^m : Ax = b \text{ for some } x \in \mathbb{R}^n \}$

$$\underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \\ 2 & 3 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}}_{b}$$

So

$$b = x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ -2 \end{bmatrix}$$

$$R(A) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\} \quad — ①$$

The vectors in equation ① must be linearly independent, i.e $Ax = 0$. Since x_3 is a free parameter, ~~so~~ I make it zero

then

$$b = x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\therefore \quad R(A) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right\} \text{ forms a basis}$$

$\dim(R(A)) = 2$

$R(A) \in \mathbb{R}^4$

d) Find the rank of A

Consider a RREF matrix: $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, Since rank(A) is the number of non zero rows in the reduced matrix, then

$$\text{rank}(A) = 2$$

Discuss if its possible for there to be a solution of $Ax=b$

- from A, $m > n$, which makes it an over determined problem.
- Since $\text{Rank}(A) < \min(m,n)$, then A is rank defficient hence it is possible to get a solution to $Ax=b$, but its not unique. This is because x_3 is a free variable, that means they are infinitely many values that can be assigned to it, making many solutions.

2. Let $A \in \mathbb{R}^{m \times n}$, $Ax = b \Rightarrow A^T A x = A^T b$

a) Show that $A^T A$ is symmetric.

$A^T A$ is symmetric if $A^T A = (A^T A)^T$

$$A^T A = (A^T A)^T$$
$$= A^T (A^T)^T$$
$$A^T A = A^T A \quad \boxed{\cdot}$$

b) If $y \in \mathbb{R}^m$, identify the dimensions of $y^T y$

$$y \in \mathbb{R}^m \Rightarrow \dim(y) = m \times 1$$
$$\dim(y^T) = 1 \times m$$

$$\dim(y^T y) = \dim(y^T) \cdot \dim(y)$$
$$= 1 \times m \cdot m \times 1$$
$$= 1 \times 1$$
$$\underline{\dim(y^T y) = 1}$$

Re-write the product of vectors using sums.

Let $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$, $y^T = [y_1 \ y_2 \ y_3 \ \cdots \ y_m]$

$$y^T \cdot y = [y_1 \ y_2 \ y_3 \cdots y_m] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$y^T \cdot y = y_1^2 + y_2^2 + y_3^2 + \cdots y_m^2 = \underline{\sum_{i=1}^{m} y_i^2}$$

c) Use matrix algebra to find a y so that you can express

$$X^T (A^T A) X \text{ as } y^T y.$$

$$X^T (A^T A) X = X^T A^T A X$$

Since $X^T A^T = (AX)^T$ then,

$$X^T (A^T A) X = (AX)^T AX$$

Let $AX = y$

then $\underline{X^T (A^T A) X = y^T y}$, so $y = \underline{AX}$

d) Show that $x^T(A^TA)x \geq 0$

$$x^T A^T A x = (Ax)^T Ax$$

let $Ax = y$

$$x^T A^T Ax = y^T y$$

but $y^Ty = \sum_{i=1}^{m} y_i^2$

$\forall\ y_i \in y,\ y_i^2 \geq 0 \Rightarrow y^Ty \geq 0 \Rightarrow x^T(A^TA)x \geq 0$

e) let rank $(A) = n$

(i) Find dim $R(A)$ and dim $N(A)$

dim $R(A) = n$ (from Definition A.16)

Using dim $N(A)$ + dim $R(A) = n \Rightarrow$ dim $N(A) = 0$

(ii) Can you find a non zero vector x such that $Ax = 0$?

No, since $N(A) = 0$, so we cannot find a non zero vector x such that $Ax = 0$.

(iii) Use 2(e)(ii)

$$\sum_{i=1}^{m} (Ax)_i^2 \neq 0 \text{ if } x \neq 0$$

from 2(e)(ii) if $x \neq 0$, then $Ax \neq \vec{0}$

$\Rightarrow (Ax)_i^2 \neq 0\ \forall\ i$

hence $\sum_{i=1}^{m} (Ax)_i^2 \neq 0$

(iv) Use 2d and 2(e) (iii) to show that $x^T(A^TA)x>0$ if $x \neq 0$

from 2d) $x^T(A^TA)x \geq 0 \Rightarrow A^TA$ is positive semi-definite

and from 2(e) (iii) $\sum_{i=1}^{m} (Ax)_i^2 \neq 0$ if $x \neq 0$

So since rank(A) = n, A is full rank matrix, therefore

if $x \neq 0 \Rightarrow Ax \neq \vec{0}$ and $(Ax)^T \neq \vec{0}$

then $\underline{x^T(A^TA)x = (Ax)^T Ax > 0}$