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Description of a Global Shallow Water Model Based on the Spectral Transform Method

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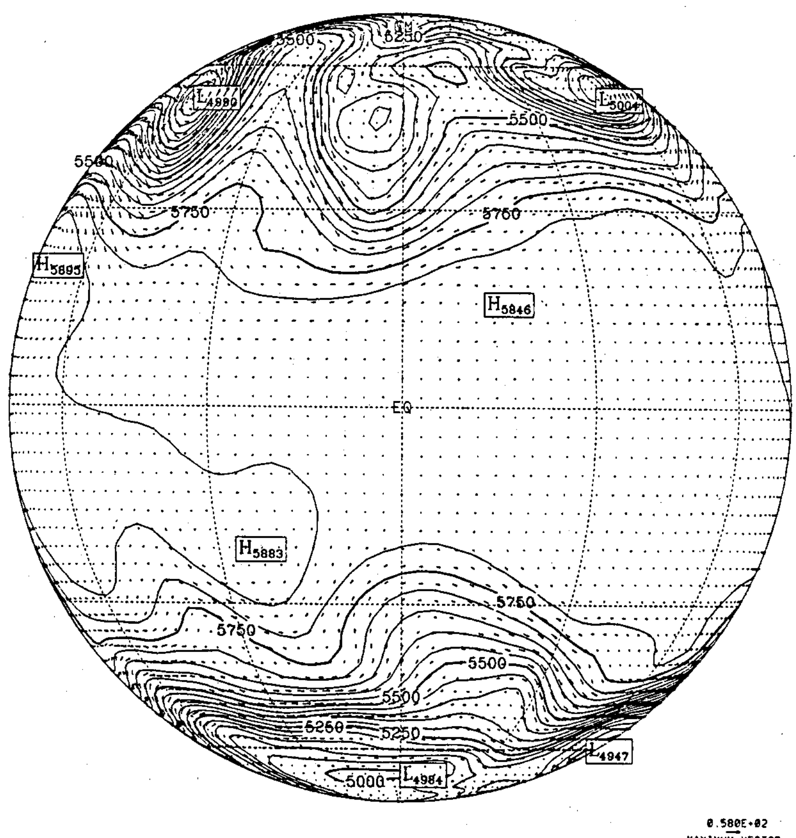

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CONTENTS

1. Introduction

The most common formulation of the equations of motion for simulating large-scale atmospheric flow, otherwise known as the meteorological primitive equations, includes the vertical dependence of the atmospheric state variables. Many of the mathematical and computational properties of these equations, however, are embodied in a simpler 2-dimensional system of equations that govern the behavior of a rotating, homogeneous, incompressible and hydrostatic fluid with a finite free surface height. This system of equations is equivalently referred to as either the one-level primitive equations, the divergent barotropic system of equations, or the shallow water equations.

The shallow water equations provide a useful framework for the analysis of the dynamics of large-scale atmospheric flow (e.g., the geostrophic adjustment process), as well as the analysis of innovative numerical methods that might be applied to the solution of baroclinic formulations of the primitive equations. The purpose of this technical note is to document a global shallow water equation model based on the spectral transform method. This technical note includes a derivation of the governing system along with a detailed discussion of the spectral transform algorithm as applied to the shallow water system in spherical geometry, including the use of semi-implicit time differencing. The algorithmic formalism closely follows the presentation on the NCAR Community Climate Model made in Williamson et al. (1987). Finally, a complete application code, including graphics and post analysis packages is briefly described. This code, which also includes the standard numerical methods test set proposed by Williamson et al. (1992), is available via anonymous FTP as described in Section 6. It is the authors' expectation that the availability of these modeling tools, coupled with the description provided in this technical report, will be of benefit to both the educational and research community.

2. Formulation of the Governing Equations

a. *The Spectral Transform Method*

The earliest of global atmospheric modeling initatives modified existing numerical methods to accommodate the solution of the meteorological equations in spherical coordinates. Most of these efforts focused on finite difference solution techniques, many of which are reviewed by Williamson (1979). The modeling community soon discovered that the adaptation and development of computational methods for solving partial differential equations in spherical geometry is complicated by the unique characteristics of the coordinate system itself. Since longitude is multivalued at the pole, non-zero vector functions will have multivalued or discontinuous components, even though the same functions have smooth properties in Cartesian coordinates. As another example, the use of a uniformly distributed latitude-longitude finite-difference grid requires an unnecessarily small time step (to satisfy the local linear stability criteria) or some form of empirical filtering of longitudinal waves near the poles because of the convergence of longitude lines. Such difficulties, which are uniquely associated with the spherical coordinate system, are collectively referred to as the "pole problem".

The spectral method presents a more natural solution to the problems introduced by spherical geometry. Spectral techniques were first used for the solution of meteorological equations by Silberman (1954) who solved the non-divergent barotropic vorticity equation using an interaction coefficient method to calculate the nonlinear advection terms. In practice, this procedure proved to be computationally tractable for only a small number of waves. Modest increases in spectral resolution gave rise to a rapid increase in the number of interaction coefficients, resulting in prohibitively large storage and computational requirements. Another major drawback to the approach was its inability to include "local" diabatic physical processes. Although the numerical properties of spectral methods continued to be explored by a number of investigators over the next 15 years (e.g., Platzman, 1960; Baer and Platzman, 1961; Ellsaesser, 1966) the computational problems associated with the interaction coefficient procedure could not be overcome. Consequently, spectral methods remained a curious but impractical alternative to traditional finite difference methods until the simultaneous, but independent introduction of the spectral transform method by Eliassen et al. (1970) and Orzag (1970).

The basic idea behind the spectral transform method is to locally evaluate all nonlinear terms (including diabatic physical processes) in physical space on an associated finite-difference-like grid, most often referred to as the transform grid. These terms are then transformed back into wavenumber space to calculate linear terms and derivatives, and to obtain tendencies for the time-dependent state variables. Because of the efficiency of the Fast Fourier Transform (FFT) (see Cooley and Tukey, 1965), the procedure enjoys a computational cost comparable to the most efficient finite difference procedures for comparable accuracy. The technique also offers a number of other computational advantages for global atmospheric models, among which are the ease with which semi-implicit time differencing can be incorporated (due to the simple form of the ∇^2 operator in wavenumber space using a spherical harmonic basis), and the absence of a "pole problem" when formulating the fluid problem in terms of vorticity and divergence (i.e., by raising the order of the system). Bourke (1972) and Machenhauer and Rasmussen (1972) were the first to test the full two dimensional transform procedure in a shallow water model, later followed by more complex implementations in multi-level global spectral models (e.g., Bourke, 1974; Hoskins and Simmons, 1975; Daley et al., 1976). Within a decade, the procedure had become a widely utilized numerical method for global modeling investigations. A more complete discussion of the history and numerical characteristics of the spectral transform procedure can be found in Bourke et al. (1977) and Machenhauer (1979).

b. *Vorticity/Divergence Form of the Shallow-Water System*

In vector form, the horizontal momentum and mass continuity equations governing the behavior of a rotating, homogeneous, incompressible, inviscid and hydrostatic fluid are written as

$$\frac{d\mathbf{V}}{dt} = -f\mathbf{k} \times \mathbf{V} - \nabla\Phi, \tag{2.1}$$

and

$$\frac{d\Phi}{dt} = -\Phi\nabla \cdot \mathbf{V}, \tag{2.2}$$

where $\mathbf{V} \equiv \mathbf{i}u + \mathbf{j}v$ is the horizontal vector velocity, $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors in the eastward and northward directions respectively, $\Phi \equiv gh$ is the free surface geopotential, g is the gravitational acceleration, $f \equiv 2\Omega \sin\phi$ is the Coriolis parameter, Ω is the angular velocity of the earth, ϕ denotes latitude, λ denotes longitude, the substantial derivative is given by

$$\frac{d}{dt}(\ ) = \frac{\partial}{\partial t}(\ ) + (\mathbf{V} \cdot \nabla)(\ ), \tag{2.3}$$

and the ∇ operator is defined in spherical coordinates as

$$\nabla(\ ) \equiv \frac{\mathbf{i}}{a\cos\phi}\frac{\partial}{\partial\lambda}(\ ) + \frac{\mathbf{j}}{a}\frac{\partial}{\partial\phi}(\ ), \tag{2.4}$$

where a is the radius of the earth.

As noted by Bourke (1972), the horizontal velocity field may also be equivalently specified in terms of the vertical component of the relative vorticity

$$\zeta \equiv \mathbf{k} \cdot (\nabla \times \mathbf{V}), \tag{2.5}$$

and horizontal divergence

$$\delta \equiv \nabla \cdot \mathbf{V}. \tag{2.6}$$

where $\mathbf{k}$ is the vertical unit vector. Using the vector identity

$$(\mathbf{V} \cdot \nabla)\mathbf{V} \equiv \nabla\left(\frac{\mathbf{V} \cdot \mathbf{V}}{2}\right) + \zeta\mathbf{k} \times \mathbf{V}, \tag{2.7}$$

the horizontal momentum equation can be expanded to the form

$$\frac{\partial\mathbf{V}}{\partial t} = -(\zeta + f)\mathbf{k} \times \mathbf{V} - \nabla\left(\Phi + \frac{\mathbf{V} \cdot \mathbf{V}}{2}\right). \tag{2.8}$$

By applying the curl ($\mathbf{k} \cdot \nabla \times [\ ]$) and divergence ($\nabla \cdot [\ ]$) operators to (2.8), we arrive at prognostic equations for the scalar quantities ζ and δ,

$$\frac{\partial \zeta}{\partial t} = -\nabla \cdot (\zeta + f)\mathbf{V}, \tag{2.9}$$

$$\frac{\partial \delta}{\partial t} = \mathbf{k} \cdot \nabla \times (\zeta + f)\mathbf{V} - \nabla^2 \left(\Phi + \frac{\mathbf{V} \cdot \mathbf{V}}{2} \right). \tag{2.10}$$

The mass continuity equation (2.2) can also be manipulated to give a form amenable to semi-implicit time integration

$$\frac{\partial \Phi'}{\partial t} = -\nabla \cdot (\Phi' \mathbf{V}) - \overline{\Phi}\delta, \tag{2.11}$$

where the geopotential has been divided into the time invariant spatial mean $\overline{\Phi} \equiv g\overline{h}$ and the time-dependent deviation from this mean $\Phi' = \Phi - \overline{\Phi}$.

After dropping primes (i.e., hereafter Φ is understood to be equivalent to Φ') and setting $\mu = \sin\phi$, further expansion of (2.9)–(2.11) yields the system

$$\frac{\partial \eta}{\partial t} = -\frac{1}{a(1-\mu^2)} \frac{\partial}{\partial \lambda}(U\eta) - \frac{1}{a}\frac{\partial}{\partial \mu}(V\eta), \tag{2.12}$$

$$\frac{\partial \delta}{\partial t} = +\frac{1}{a(1-\mu^2)} \frac{\partial}{\partial \lambda}(V\eta) - \frac{1}{a}\frac{\partial}{\partial \mu}(U\eta) - \nabla^2 \left(\Phi + \frac{U^2 + V^2}{2(1-\mu^2)} \right), \tag{2.13}$$

$$\frac{\partial \Phi}{\partial t} = -\frac{1}{a(1-\mu^2)} \frac{\partial}{\partial \lambda}(U\Phi) - \frac{1}{a}\frac{\partial}{\partial \mu}(V\Phi) - \overline{\Phi}\delta, \tag{2.14}$$

where

$$\eta \equiv \zeta + f = \frac{1}{a(1-\mu^2)} \frac{\partial V}{\partial \lambda} - \frac{1}{a}\frac{\partial U}{\partial \mu} + f, \tag{2.15}$$

$$\delta = \frac{1}{a(1-\mu^2)} \frac{\partial U}{\partial \lambda} + \frac{1}{a}\frac{\partial V}{\partial \mu}, \tag{2.16}$$

and

$$\left. \begin{aligned} U &\equiv u\cos\phi \\ V &\equiv v\cos\phi \end{aligned} \right\} \tag{2.17}$$

The definitions in (2.17) follow Robert (1966) who observed that the zonal (u) and meridional (v) components of the horizontal vector velocity are not well suited to representation in terms of scalar spectral expansions. The variables U and V can be diagnostically determined by making use of the Helmholtz theorem which separates the horizontal velocity vector $\mathbf{V}$ into two terms, a scalar stream function ψ and a scalar velocity potential χ,

$$\mathbf{V} = \mathbf{k} \times \nabla\psi + \nabla\chi. \tag{2.18}$$

Expanding (2.18) gives

$$U = \frac{1}{a}\frac{\partial \chi}{\partial \lambda} - \frac{(1-\mu^2)}{a}\frac{\partial \psi}{\partial \mu}, \tag{2.19}$$

and

$$V = \frac{1}{a}\frac{\partial \psi}{\partial \lambda} + \frac{(1-\mu^2)}{a}\frac{\partial \chi}{\partial \mu}. \tag{2.20}$$

Application of the curl and divergence operators to (2.18) gives relationships for the prognostic variables η and δ in terms of the stream function and velocity potential

$$\eta = \nabla^2\psi + f, \tag{2.21}$$

and

$$\delta = \nabla^2\chi. \tag{2.22}$$

Relationships (2.21) and (2.22) have a particularly simple form in wavenumber space when spherical harmonic functions are utilized as the orthogonal basis for the spectral expansion. The variables U and V can then be readily determined from η and δ, as we will show in Section 5.

The shallow water system can be easily generalized to accommodate viscous flow and irregular lower boundaries (i.e. surface topography). A description of the necessary changes to the equations and the numerical algorithms is contained in Jakob et al. (1992).

3. Horizontal Approximation to the Continuous Governing Equations

The horizontal representation of an arbitrary scalar quantity ξ consists of a truncated series of spherical harmonic functions,

$$\xi(\lambda, \mu) = \sum_{m=-M}^{M} \sum_{n=|m|}^{N(m)} \xi_n^m P_n^m(\mu) e^{im\lambda}, \tag{3.1}$$

where M is the highest Fourier wavenumber included in the east-west representation and $N(m)$, which can be a function of the Fourier wavenumber (m), is the highest degree of the associated Legendre functions included in the north-south representation. The spherical harmonic functions, $P_n^m(\mu)e^{im\lambda}$, used in the spectral expansion are the eigensolutions of the Laplacian operator in spherical coordinates, and constitute a complete and orthogonal expansion basis. Additional discussion of the properties of these functions can be found in Machenhauer (1979).

The shallow water model code discussed in Section 6 provides for the general pentagonal spectral truncation framework illustrated in Figure 1 and defined by three parameters: M the largest Fourier wavenumber, K the highest degree of the associated Legendre functions, and N the highest degree of the Legendre functions for $m = 0$. The most commonly used spectral truncations are subsets of this general framework:

$$\begin{aligned} \text{Triangular}: M = N = K, \\ \text{Rhomboidal}: K = N + M. \end{aligned} \tag{3.2}$$

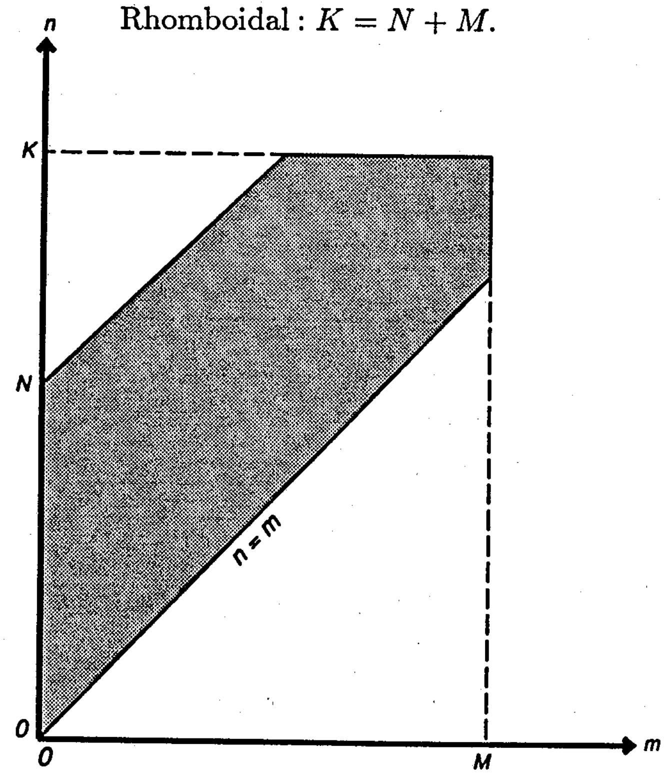

Fig. 1 General pentagonal truncation framework

Note that, as drawn, the complex components of the expansion have been limited to points in the half plane $m \geq 0$ with integer values of m and n at and above the line $n = m$. This is possible due to the symmetry of the Fourier basis functions where

$$e^{-im\lambda} = \left(e^{im\lambda}\right)^*, \tag{3.3}$$

and

$$P_n^{-m}(\mu) = (-1)^m P_n^m(\mu), \tag{3.4}$$

where $(\ )^*$ denotes the complex conjugate. Thus, for real valued functions, $\xi(\lambda, \mu)$, the spectral coefficients must have the symmetry property

$$\xi_n^{-m} = (-1)^m (\xi_n^m)^*. \tag{3.5}$$

The associated Legendre functions are normalized such that

$$\int_{-1}^{+1} \left[P_n^m(\mu)\right]^2 d\mu = 1. \tag{3.6}$$

Examples of the normalized associated Legendre functions are given in Table 1, which can be derived using (B.5-B.7), (B.36-B.38) and (B.44) in Washington and Parkinson (1986).

$m =$	0	1	2
$n = 0$	$\frac{1}{2}\sqrt{2}$		
$n = 1$	$\frac{1}{2}\sqrt{6}\mu$	$\frac{1}{2}\sqrt{3}\sqrt{1-\mu^2}$	
$n = 2$	$\frac{1}{4}\sqrt{10}(3\mu^2 - 1)$	$\frac{1}{2}\sqrt{15}\mu\sqrt{1-\mu^2}$	$\frac{1}{4}\sqrt{15}(1-\mu^2)$

Table 1: Associated Legendre Functions, $P_n^m(\mu)$

The complex coefficients of the spectral expansion (3.1) can then be determined by projecting the scalar field $\xi(\lambda, \mu)$ onto the normalized orthogonal basis

$$\xi_n^m = \int_{-1}^{+1} \frac{1}{2\pi} \int_0^{2\pi} \xi(\lambda, \mu) e^{-im\lambda} d\lambda P_n^m(\mu) d\mu. \tag{3.7}$$

The inner integral in (3.7) represents a Fourier transform,

$$\xi^m(\mu) = \frac{1}{2\pi}\int_0^{2\pi} \xi(\lambda,\mu)e^{-im\lambda}d\lambda \;=\; \frac{1}{I}\sum_{\mathrm{i}=1}^{I}\xi(\lambda_\mathrm{i},\mu)e^{-im\lambda_\mathrm{i}}, \tag{3.8}$$

where

$$\lambda_\mathrm{i} = \frac{2\pi\mathrm{i}}{I}, \tag{3.9}$$

which can be evaluated using a Fast Fourier Transform (FFT) procedure. In order to allow an exact, unaliased Fourier transform of quadratic terms (i.e., $\xi(\lambda_\mathrm{i},\mu) = A(\lambda_\mathrm{i},\mu)\cdot B(\lambda_\mathrm{i},\mu)$ where A and B are scalars representable by a discrete spherical harmonic expansion), the number of gridpoints I in the east-west direction must satisfy

$$I \geq 3M + 1. \tag{3.10}$$

The outer integral in (3.7) is performed via Gaussian quadrature,

$$\xi_n^m = \int_{-1}^{+1} \xi^m(\mu)P_n^m(\mu)d\mu \;=\; \sum_{\mathrm{j}=1}^{J}\xi^m(\mu_\mathrm{j})P_n^m(\mu_\mathrm{j})w_\mathrm{j}, \tag{3.11}$$

where μ_j denotes the Gaussian latitudes, w_j the Gaussian weight at latitude μ_j, and J the number of Gaussian latitudes from pole to pole. The Gaussian latitudes (μ_j) are determined from the roots of the Legendre polynomial $P_J(\mu)$, where the corresponding weights are given by

$$w_\mathrm{j} = \frac{2(1-\mu_\mathrm{j}^2)}{\left[J\, P_{J-1}(\mu_\mathrm{j})\right]^2}, \tag{3.12}$$

which also satisfy the relation

$$\sum_{\mathrm{j}=1}^{J} w_\mathrm{j} = 2. \tag{3.13}$$

As in the case of the east-west Fourier transformation, the Gaussian grid used for the north-south transformation is generally chosen to allow exact, unaliased computations of quadratic terms. In this instance, the number of Gaussian latitudes J must satisfy

$$J \geq (2N + K + M + 1)/2 \quad \text{for } \mathrm{M} \leq 2(\mathrm{K}-\mathrm{N}), \tag{3.14}$$

$$J \geq (3K + 1)/2 \qquad\qquad \text{for } \mathrm{M} \geq 2(\mathrm{K}-\mathrm{N}). \tag{3.15}$$

For the commonly used spectral truncations, these become

$$J \geq (3K+1)/2 \qquad \text{for triangular,} \tag{3.16}$$

$$J \geq (3N+2M+1)/2 \qquad \text{for rhomboidal.} \tag{3.17}$$

The actual values of J and I are frequently set slightly higher than the required lower bound in order to allow use of the most efficient Fast Fourier Transform algorithm.

4. Temporal Approximation to the Continuous Governing Equations

The model uses the spectral transform method for the evaluation of all nonlinear terms and local physical processes, and is discussed in some detail in the next section. Qualitatively, the numerical integration procedure as implemented steps through time in gridpoint space by an evaluation of all nonlinear terms on the Gaussian grid, forward transformation via an FFT and Gaussian quadrature of the nonlinear products from gridpoint space to spectral space where spatial derivatives are evaluated, computation of spectral coefficients of the prognostic variables at time $t + \Delta t$ and an inverse transform of the spectral quantities to physical (gridpoint) space. The time integration procedure in this model can be either explicit (forward or centered), or semi-implicit (centered differencing) in which the terms responsible for the fast moving gravitational mode are treated implicitly, and the remaining terms explicitly. We leave the details of the semi-implicit procedure for discussion in Section 5 and conceptually illustrate the basic time differencing procedures by applying them to the equation

$$\frac{\partial \xi(t)}{\partial t} = F(\xi). \tag{4.1}$$

The forward and centered approximations to (4.1) can be written respectively as

$$\xi^{(\tau+1)} = \xi^{(\tau)} + \Delta t F(\xi^{(\tau)}), \tag{4.2}$$

and

$$\xi^{(\tau+1)} = \tilde{\xi}^{(\tau-1)} + 2\Delta t F(\xi^{(\tau)}). \tag{4.3}$$

When using centered time differencing (4.3), a linear time filter (see Asselin, 1972) is applied at the completion of each time step. This procedure filters values of the prognostic variables at time τ after the values at time $\tau+1$ have been computed, and helps to prevent modal splitting. The filter has the form

$$\tilde{\xi}^{(\tau)} = \xi^{(\tau)} + \alpha\Big[\tilde{\xi}^{(\tau-1)} - 2\xi^{(\tau)} + \xi^{(\tau+1)}\Big], \tag{4.4}$$

where α is the filter coefficient, which is typically quite small ($\alpha \leq 0.01$).

5. Numerical Algorithms

a. *Spectral Form of the Governing Equations*

Formally, (2.12)–(2.14) are transformed to spectral space by applying the relationship given in (3.7) to each term. Inspection of the equations reveals that they contain undifferentiated terms, longitudinally differentiated terms, meridionally differentiated terms and a Laplacian operator.

Transformation of undifferentiated terms are obtained by straightforward application of (3.8) and (3.11),

$$\xi^m(\mu_j) = \frac{1}{I}\sum_{i=1}^{I} \xi(\lambda_i, \mu_j) e^{-im\lambda_i}, \tag{5.1}$$

and

$$\xi_n^m = \sum_{j=1}^{J} \xi^m(\mu_j) P_n^m(\mu_j) w_j, \tag{5.2}$$

where $\xi^m(\mu_j)$ is the complex Fourier coefficient of ξ with wavenumber m at the Gaussian latitude μ_j and is evaluated using an FFT procedure. Note that application of (3.8) and (3.11) to the Coriolis parameter f results in the one term expansion

$$f_n^m = \begin{cases} \frac{\Omega}{\sqrt{0.375}} & \text{for } n = 1 \text{ and } m = 0 \\ 0 & \text{otherwise} \end{cases}, \tag{5.3}$$

which will be utilized later in the calculation of absolute vorticity η (see eqs. 2.15 and 2.21). For a derivation of this term see Appendix B.2.

The longitudinally differentiated terms are determined by an integration by parts using cyclic boundary conditions,

$$
\begin{aligned}
\left\{\frac{\partial \xi}{\partial \lambda}\right\}^m &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial \xi}{\partial \lambda} e^{-im\lambda} d\lambda \\
&= im\left[\frac{1}{2\pi}\int_0^{2\pi} \xi e^{-im\lambda} d\lambda\right] \\
&= im\xi^m,
\end{aligned}
\tag{5.4}
$$

so that the Fourier transform of ξ is performed first, followed by differentiation (multiplication by im) in wavenumber space. The final step in the transformation follows eq. (5.2)

$$
\left\{\frac{1}{a(1-\mu^2)}\frac{\partial \xi}{\partial \lambda}\right\}_n^m = \sum_{\mathrm{j}=1}^{J} im\xi^m(\mu_\mathrm{j})\frac{P_n^m(\mu_\mathrm{j})}{a(1-\mu_\mathrm{j}^2)} w_\mathrm{j}. \tag{5.5}
$$

The latitudinally differentiated terms in (2.12)–(2.14) can be determined by an integration by parts using zero boundary conditions at the poles, (since $\xi^m(\pm 1) = 0$),

$$
\begin{aligned}
\left\{\frac{1}{a}\frac{\partial \xi}{\partial \mu}\right\}_n^m &= \int_{-1}^{+1} \frac{1}{a}\frac{\partial \xi^m}{\partial \mu} P_n^m(\mu) d\mu \\
&= -\int_{-1}^{+1} \frac{1}{a}\xi^m \frac{dP_n^m(\mu)}{d\mu} d\mu.
\end{aligned}
\tag{5.6}
$$

Defining the derivatives of the associated Legendre functions by

$$
H_n^m(\mu) \equiv (1-\mu^2)\frac{dP_n^m(\mu)}{d\mu}, \tag{5.7}
$$

(5.6) can be written

$$
\left\{\frac{1}{a}\frac{\partial \xi}{\partial \mu}\right\}_n^m = -\sum_{\mathrm{j}=1}^{J} \xi^m(\mu_\mathrm{j})\frac{H_n^m(\mu_\mathrm{j})}{a(1-\mu_\mathrm{j}^2)} w_\mathrm{j}. \tag{5.8}
$$

Similarly, the ∇^2 operator (e.g., in the divergence equation) can be converted to spectral space by sequential integration by parts and application of the relationship,

$$
\nabla^2 P_n^m(\mu) e^{im\lambda} = \frac{-n(n+1)}{a^2} P_n^m(\mu) e^{im\lambda}, \tag{5.9}
$$

to each spherical harmonic function individually so that

$$\{\nabla^2\xi\}_n^m = \frac{-n(n+1)}{a^2}\sum_{j=1}^{J}\xi^m(\mu_j)P_n^m(\mu_j)w_j, \tag{5.10}$$

where, as before, $\xi^m(\mu_j)$ is the complex Fourier coefficient of the original grid variable $\xi(\lambda_i, \mu_j)$.

Before transforming the governing equations, let us redefine the nonlinear products in terms of the intermediate variables

$$\left.\begin{aligned} A &\equiv U\eta, \\ B &\equiv V\eta, \\ C &\equiv U\Phi, \\ D &\equiv V\Phi, \\ E &\equiv \frac{U^2+V^2}{2(1-\mu^2)}, \end{aligned}\right\} \tag{5.11}$$

so that (2.12)–(2.14) can be rewritten as

$$\frac{\partial\eta}{\partial t} = -\frac{1}{a(1-\mu^2)}\frac{\partial A}{\partial\lambda} - \frac{1}{a}\frac{\partial B}{\partial\mu}, \tag{5.12}$$

$$\frac{\partial\delta}{\partial t} = +\frac{1}{a(1-\mu^2)}\frac{\partial B}{\partial\lambda} - \frac{1}{a}\frac{\partial A}{\partial\mu} - \nabla^2 E - \nabla^2\Phi, \tag{5.13}$$

$$\frac{\partial\Phi}{\partial t} = -\frac{1}{a(1-\mu^2)}\frac{\partial C}{\partial\lambda} - \frac{1}{a}\frac{\partial D}{\partial\mu} - \overline{\Phi}\delta. \tag{5.14}$$

Transforming (5.12)–(5.14) yields the spectral form of the governing equations

$$\frac{\partial\eta_n^m}{\partial t} = -\sum_{j=1}^{J}\{imA^m(\mu_j)P_n^m(\mu_j) - B^m(\mu_j)H_n^m(\mu_j)\}\frac{w_j}{a(1-\mu_j^2)}, \tag{5.15}$$

$$\frac{\partial \delta_n^m}{\partial t} = -\sum_{\mathrm{j}=1}^{J} \{-imB^m(\mu_\mathrm{j})P_n^m(\mu_\mathrm{j}) - A^m(\mu_\mathrm{j})H_n^m(\mu_\mathrm{j})\} \frac{w_\mathrm{j}}{a(1-\mu_\mathrm{j}^2)}$$

$$+ \frac{n(n+1)}{a^2} \sum_{\mathrm{j}=1}^{J} [E^m(\mu_\mathrm{j}) + \Phi^m(\mu_\mathrm{j})] \, P_n^m(\mu_\mathrm{j}) w_\mathrm{j} \ , \tag{5.16}$$

$$\frac{\partial \Phi_n^m}{\partial t} = -\sum_{\mathrm{j}=1}^{J} \{imC^m(\mu_\mathrm{j})P_n^m(\mu_\mathrm{j}) - D^m(\mu_\mathrm{j})H_n^m(\mu_\mathrm{j})\} \frac{w_\mathrm{j}}{a(1-\mu_\mathrm{j}^2)} - \overline{\Phi}\delta_n^m, \tag{5.17}$$

where $A^m(\mu_\mathrm{j})$, $B^m(\mu_\mathrm{j})$, $C^m(\mu_\mathrm{j})$, $D^m(\mu_\mathrm{j})$, $E^m(\mu_\mathrm{j})$, and $\Phi^m(\mu_\mathrm{j})$ are the complex Fourier coefficients of the nonlinear products defined in (5.11), and the free surface geopotential perturbation $\Phi(\lambda_\mathrm{i}, \mu_\mathrm{j})$.

As was discussed in section 2, it is necessary to provide diagnostic relationships between the prognostic variables η and δ and the horizontal velocity components U and V to close this system of equations. This will be done with the aid of the stream function ψ and velocity potential χ which were defined in (2.19) and (2.20). Let

$$\psi(\lambda, \mu) = \sum_{m=-M}^{M} \sum_{n=|m|}^{N(m)} \psi_n^m P_n^m(\mu) e^{im\lambda}, \tag{5.18}$$

and

$$\chi(\lambda, \mu) = \sum_{m=-M}^{M} \sum_{n=|m|}^{N(m)} \chi_n^m P_n^m(\mu) e^{im\lambda}, \tag{5.19}$$

where ψ_0^0 and χ_0^0 are undefined since the global mean of ψ and χ can be arbitrarily chosen. Then

$$\frac{\partial \psi}{\partial \mu} = \sum_{m=-M}^{M} \sum_{n=|m|}^{N(m)} \psi_n^m \frac{dP_n^m(\mu)}{d\mu} e^{im\lambda}, \tag{5.20}$$

$$\frac{\partial \chi}{\partial \mu} = \sum_{m=-M}^{M} \sum_{n=|m|}^{N(m)} \chi_n^m \frac{dP_n^m(\mu)}{d\mu} e^{im\lambda}, \tag{5.21}$$

and

$$\frac{\partial \psi}{\partial \lambda} = \sum_{m=-M}^{M} \sum_{n=|m|}^{N(m)} im\psi_n^m P_n^m(\mu) e^{im\lambda}, \tag{5.22}$$

$$\frac{\partial \chi}{\partial \lambda} = \sum_{m=-M}^{M} \sum_{n=|m|}^{N(m)} im\chi_n^m P_n^m(\mu) e^{im\lambda}. \tag{5.23}$$

By substituting (5.20)–(5.23) into (2.19) and (2.20), and using (2.21), (2.22), (5.7) and (5.9) we arrive at diagnostic relationships for U and V in terms of η_n^m and δ_n^m

$$U(\lambda_{\mathrm{i}}, \mu_{\mathrm{j}}) = -\sum_{m=-M}^{M} \sum_{\substack{n=|m| \\ n\neq 0}}^{N(m)} \frac{a}{n(n+1)} \left[im\ \delta_n^m P_n^m(\mu_{\mathrm{j}}) - (\eta_n^m - f_n^m) H_n^m(\mu_{\mathrm{j}}) \right] e^{im\lambda_{\mathrm{i}}}, \tag{5.24}$$

and

$$V(\lambda_{\mathrm{i}}, \mu_{\mathrm{j}}) = -\sum_{m=-M}^{M} \sum_{\substack{n=|m| \\ n\neq 0}}^{N(m)} \frac{a}{n(n+1)} \left[im(\eta_n^m - f_n^m) P_n^m(\mu_{\mathrm{j}}) + \delta_n^m H_n^m(\mu_{\mathrm{j}}) \right] e^{im\lambda_{\mathrm{i}}}. \tag{5.25}$$

The inverse procedure is obtained by transforming (2.15) and (2.16) and using the expansion for the Coriolis parameter given in (5.3)

$$\eta_n^m = \sum_{\mathrm{j}=1}^{J} \{imV^m(\mu_{\mathrm{j}}) P_n^m(\mu_{\mathrm{j}}) + U^m(\mu_{\mathrm{j}}) H_n^m(\mu_{\mathrm{j}})\} \frac{w_{\mathrm{j}}}{a(1-\mu_{\mathrm{j}}^2)} + f_n^m, \tag{5.26}$$

and

$$\delta_n^m = \sum_{\mathrm{j}=1}^{J} \{imU^m(\mu_{\mathrm{j}}) P_n^m(\mu_{\mathrm{j}}) - V^m(\mu_{\mathrm{j}}) H_n^m(\mu_{\mathrm{j}})\} \frac{w_{\mathrm{j}}}{a(1-\mu_{\mathrm{j}}^2)}, \tag{5.27}$$

where $U^m(\mu_{\mathrm{j}})$ and $V^m(\mu_{\mathrm{j}})$ are the complex Fourier coefficients of $U(\lambda_{\mathrm{i}}, \mu_{\mathrm{j}})$ and $V(\lambda_{\mathrm{i}}, \mu_{\mathrm{j}})$. With these last two expression, the system of equations is complete and in a form suitable for numerical integration in time.

b. *Explicit Time Differencing Procedure*

The model is designed to be primarily used with second order accurate centered time differencing (with either an explicit or semi-implicit procedure) but can be easily modified to incorporate a first order accurate explicit forward time difference (4.2). A forward time step of $\frac{\Delta t}{2}$ is invoked to start the model integration (*i.e.*, by setting variables at $t = -\frac{\Delta t}{4}$ equal to those at $t = 0$ and using a centered time step of $2\Delta t/4$), followed by a centered time step of Δt. All subsequent time steps are $2\Delta t$ with centered time differencing.

Using (4.3), (5.2) and (5.15)–(5.17), the explicit centered equations for vorticity, divergence and geopotential are simply written as

$$\{\eta_n^m\}^{(\tau+1)} = \sum_{\mathrm{j}=1}^{J} \Big\{ \{\eta^m(\mu_\mathrm{j})\}^{(\tau-1)} P_n^m(\mu_\mathrm{j}) - \frac{2\Delta t}{a(1-\mu_\mathrm{j}^2)} \Big[imA^m(\mu_\mathrm{j}) P_n^m(\mu_\mathrm{j}) - B^m(\mu_\mathrm{j}) H_n^m(\mu_\mathrm{j}) \Big] \Big\} w_\mathrm{j} \tag{5.28}$$

$$\{\delta_n^m\}^{(\tau+1)} = \sum_{\mathrm{j}=1}^{J} \Big\{ \{\delta^m(\mu_\mathrm{j})\}^{(\tau-1)} P_n^m(\mu_\mathrm{j}) + \frac{2\Delta t}{a(1-\mu_\mathrm{j}^2)} \Big[imB^m(\mu_\mathrm{j}) P_n^m(\mu_\mathrm{j}) + A^m(\mu_\mathrm{j}) H_n^m(\mu_\mathrm{j}) \Big] + 2\Delta t \frac{n(n+1)}{a^2} \Big[\{\Phi^m(\mu_\mathrm{j})\}^{(\tau)} + E^m(\mu_\mathrm{j}) \Big] P_n^m(\mu_\mathrm{j}) \Big\} w_\mathrm{j} \tag{5.29}$$

$$\{\Phi_n^m\}^{(\tau+1)} = \sum_{\mathrm{j}=1}^{J} \Big\{ \{\Phi^m(\mu_\mathrm{j})\}^{(\tau-1)} P_n^m(\mu_\mathrm{j}) - \frac{2\Delta t}{a(1-\mu_\mathrm{j}^2)} \Big[imC^m(\mu_\mathrm{j}) P_n^m(\mu_\mathrm{j}) - D^m(\mu_\mathrm{j}) H_n^m(\mu_\mathrm{j}) \Big] - 2\Delta t \overline{\Phi} \{\delta^m(\mu_\mathrm{j})\}^{(\tau)} P_n^m(\mu_\mathrm{j}) \Big\} w_\mathrm{j} \tag{5.30}$$

c. *Semi-implicit Time Differencing Procedure*

As in the case of the meteorological primitive equations, the shallow water equations presented in Section 2 admit both high frequency gravitational solutions as well as lower frequency meteorological (i.e., Rossby wave) solutions. In order to satisfy the CFL linear computational stability criterion (e.g., see Haltiner and Williams, 1980), the time step utilized by explicit time differencing methods is thus limited by the speed of fastest gravitational wave ($\sqrt{g\overline{h}}$ in the case of the shallow water system where $\overline{h}$ is the mean depth of the fluid). The semi-implicit time integration procedure, first proposed by Robert (1969), removes this constraint by treating terms that give rise to gravitational oscillations implicitly, and the remaining terms explicitly. Therefore the time step is limited by the highest frequency meteorological modes, rather than by the highest frequency gravitational modes, allowing a several-fold increase in the size of the time step associated with a fully explicit procedure.

We begin our discussion of the semi-implicit procedure by rewriting the coupled system (5.16)–(5.17) in the form

$$\frac{\partial \delta_n^m}{\partial t} = \mathcal{D}_n^m + \frac{n(n+1)}{a^2}\Phi_n^m, \tag{5.31}$$

$$\frac{\partial \Phi_n^m}{\partial t} = \mathcal{P}_n^m - \overline{\Phi}\delta_n^m, \tag{5.32}$$

where

$$\mathcal{D}_n^m = -\sum_{\mathrm{j}=1}^{J}\{-imB^m(\mu_\mathrm{j})P_n^m(\mu_\mathrm{j}) - A^m(\mu_\mathrm{j})H_n^m(\mu_\mathrm{j})\}\frac{w_\mathrm{j}}{a(1-\mu_\mathrm{j}^2)}$$

$$+\frac{n(n+1)}{a^2}\sum_{\mathrm{j}=1}^{J}E^m(\mu_\mathrm{j})P_n^m(\mu_\mathrm{j})w_\mathrm{j}, \tag{5.33}$$

$$\mathcal{P}_n^m = -\sum_{\mathrm{j}=1}^{J}\{imC^m(\mu_\mathrm{j})P_n^m(\mu_\mathrm{j}) - D^m(\mu_\mathrm{j})H_n^m(\mu_\mathrm{j})\}\frac{w_\mathrm{j}}{a(1-\mu_\mathrm{j}^2)}. \tag{5.34}$$

Discretizing (5.33)–(5.34) in time using a centered difference (4.3) in which the linear terms $\frac{n(n+1)}{a^2}\Phi_n^m$ and $\overline{\Phi}\delta_n^m$ are also averaged in time results in

$$\{\delta_n^m\}^{(\tau+1)} = \{\delta_n^m\}^{(\tau-1)} + 2\Delta t\{\mathcal{D}_n^m\}^{(\tau)}$$
$$+ 2\Delta t\frac{n(n+1)}{a^2}\,\frac{\{\Phi_n^m\}^{(\tau-1)} + \{\Phi_n^m\}^{(\tau+1)}}{2}, \tag{5.35}$$
$$\{\Phi_n^m\}^{(\tau+1)} = \{\Phi_n^m\}^{(\tau-1)} + 2\Delta t\{\mathcal{P}_n^m\}^{(\tau)}$$
$$- 2\Delta t\overline{\Phi}\,\frac{\{\delta_n^m\}^{(\tau-1)} + \{\delta_n^m\}^{(\tau+1)}}{2}. \tag{5.36}$$

Equations (5.35) and (5.36) are coupled through the time-averaged terms, and thus constitute a 2×2 simultaneous linear system of equations. In matrix form, they may be written

$$\begin{bmatrix} 1 & -\frac{n(n+1)}{2a^2}(2\Delta t) \\ \frac{\overline{\Phi}}{2}(2\Delta t) & 1 \end{bmatrix} \begin{bmatrix} \{\delta_n^m\}^{(\tau+1)} \\ \{\Phi_n^m\}^{(\tau+1)} \end{bmatrix} = \begin{bmatrix} \mathcal{R} \\ \mathcal{Q} \end{bmatrix} \tag{5.37}$$

where

$$\mathcal{R} \equiv \{\delta_n^m\}^{(\tau-1)} + 2\Delta t\{\mathcal{D}_n^m\}^{(\tau)} + 2\Delta t\frac{n(n+1)}{2a^2}\{\Phi_n^m\}^{(\tau-1)}, \tag{5.38}$$

$$\mathcal{Q} \equiv \{\Phi_n^m\}^{(\tau-1)} + 2\Delta t\{\mathcal{P}_n^m\}^{(\tau)} - 2\Delta t\frac{\overline{\Phi}}{2}\{\delta_n^m\}^{(\tau-1)}. \tag{5.39}$$

Using Cramer's rule, the solutions to (5.37) can be written

$$\{\delta_n^m\}^{(\tau+1)} = \frac{\begin{vmatrix} \mathcal{R} & -\frac{n(n+1)}{2a^2}(2\Delta t) \\ \mathcal{Q} & 1 \end{vmatrix}}{1 + \overline{\Phi}\frac{n(n+1)}{4a^2}(2\Delta t)^2} = \frac{\mathcal{R} + \mathcal{Q}\frac{n(n+1)}{2a^2}(2\Delta t)}{1 + \overline{\Phi}\frac{n(n+1)}{4a^2}(2\Delta t)^2}$$
$$= \left[1 + \overline{\Phi}\frac{n(n+1)}{4a^2}(2\Delta t)^2\right]^{-1} \times$$
$$\left\{ \{\delta_n^m\}^{(\tau-1)} + 2\Delta t\left[\{\mathcal{D}_n^m\}^{(\tau)} + \frac{n(n+1)}{a^2}\{\Phi_n^m\}^{(\tau-1)}\right]\right.$$
$$\left. + (2\Delta t)^2\frac{n(n+1)}{2a^2}\left[\{\mathcal{P}_n^m\}^{(\tau)} - \frac{\overline{\Phi}}{2}\{\delta_n^m\}^{(\tau-1)}\right]\right\}, \tag{5.40}$$

and

$$\{\Phi_n^m\}^{(\tau+1)} = \frac{\begin{vmatrix} 1 & \mathcal{R} \\ \frac{\overline{\Phi}}{2}(2\Delta t) & \mathcal{Q} \end{vmatrix}}{1+\overline{\Phi}\frac{n(n+1)}{4a^2}(2\Delta t)^2} = \frac{\mathcal{Q}-\mathcal{R}\frac{\overline{\Phi}}{2}(2\Delta t)}{1+\overline{\Phi}\frac{n(n+1)}{4a^2}(2\Delta t)^2}$$

$$= \left[1+\overline{\Phi}\frac{n(n+1)}{4a^2}\,(2\Delta t)^2\right]^{-1} \times$$

$$\left\{\{\Phi_n^m\}^{(\tau-1)} + 2\Delta t\left[\{\mathcal{P}_n^m\}^{(\tau)} - \overline{\Phi}\,\{\delta_n^m\}^{(\tau-1)}\right]\right.$$

$$\left. -\,(2\Delta t)^2\frac{\overline{\Phi}}{2}\,\left[\{\mathcal{D}_n^m\}^{(\tau)} + \frac{n(n+1)}{2a^2}\,\{\Phi_n^m\}^{(\tau-1)}\right]\right\}. \tag{5.41}$$

Using (5.33)–(5.34), the semi-implicit difference equations can also be written as

$$\{\delta_n^m\}^{(\tau+1)} = \left[1+\frac{\overline{\Phi}}{4}\,\frac{n(n+1)}{a^2}\,(2\Delta t)^2\right]^{-1} \times \sum_{\mathrm{j}=1}^{J} w_{\mathrm{j}}$$

$$\left\{\left[1-\frac{\overline{\Phi}}{4}\,\frac{n(n+1)}{a^2}\,(2\Delta t)^2\right]\,\{\delta^m(\mu_{\mathrm{j}})\}^{(\tau-1)}\;P_n^m(\mu_{\mathrm{j}})\right.$$

$$+\,\frac{n(n+1)}{a^2}\,2\Delta t\left[\{\Phi^m(\mu_{\mathrm{j}})\}^{(\tau-1)} + E^m(\mu_{\mathrm{j}})\right]\;P_n^m(\mu_{\mathrm{j}})$$

$$+\,\frac{2\Delta t}{a(1-\mu_{\mathrm{j}}^2)}\;[imB^m(\mu_{\mathrm{j}})\,P_n^m(\mu_{\mathrm{j}}) + A^m(\mu_{\mathrm{j}})\,H_n^m(\mu_{\mathrm{j}})]$$

$$+\,\frac{(2\Delta t)^2}{2a(1-\mu_{\mathrm{j}}^2)}\;\frac{n(n+1)}{a^2}\times$$

$$\left.[-imC^m(\mu_{\mathrm{j}})\,P_n^m(\mu_{\mathrm{j}}) + D^m(\mu_{\mathrm{j}})\,H_n^m(\mu_{\mathrm{j}})]\right\}, \tag{5.42}$$

and

$$
\begin{aligned}
\{\Phi_n^m\}^{(\tau+1)} = &\left[1 + \frac{\overline{\Phi}}{4}\,\frac{n(n+1)}{a^2}\,(2\Delta t)^2\right]^{-1} \times \sum_{\mathrm{j}=1}^{J} w_{\mathrm{j}} \\
&\left\{\left[1 - \frac{\overline{\Phi}}{4}\,\frac{n(n+1)}{a^2}\,(2\Delta t)^2\right]\,\{\Phi^m(\mu_{\mathrm{j}})\}^{(\tau-1)}\,P_n^m(\mu_{\mathrm{j}})\right. \\
&- 2\Delta t\overline{\Phi}\,\{\delta^m(\mu_{\mathrm{j}})\}^{(\tau-1)}\,P_n^m(\mu_{\mathrm{j}}) \\
&+ \frac{2\Delta t}{a(1-\mu_{\mathrm{j}}^2)}\,\left[-imC^m(\mu_{\mathrm{j}})\,P_n^m(\mu_{\mathrm{j}}) + D^m(\mu_{\mathrm{j}})\,H_n^m(\mu_{\mathrm{j}})\right] \\
&- \frac{\overline{\Phi}}{2}\,\frac{n(n+1)}{a^2}\,(2\Delta t)^2\,E^m(\mu_{\mathrm{j}})\,P_n^m(\mu_{\mathrm{j}}) \\
&\left.- \frac{\overline{\Phi}}{2}\,\frac{(2\Delta t)^2}{a(1-\mu_{\mathrm{j}}^2)}\,\left[imB^m(\mu_{\mathrm{j}})\,P_n^m(\mu_{\mathrm{j}}) + A^m(\mu_{\mathrm{j}})\,H_n^m(\mu_{\mathrm{j}})\right]\right\}. \qquad (5.43)
\end{aligned}
$$

d. Meridional Symmetry of Associated Legendre Basis

Although we have consistently denoted the Gaussian quadrature as a sum from pole to pole, for an even number of Gaussian latitudes one can take advantage of the meridional symmetry of the associated Legendre functions,

$$
P_n^m(-\mu) = sgn_n^m\;P_n^m(\mu) \qquad (5.44)
$$

where

$$
sgn_n^m = \begin{cases} 1 & \text{for } n - |m| \text{ even}, \\ -1 & \text{for } n - |m| \text{ odd}. \end{cases} \qquad (5.45)
$$

In addition, the spectral transform algorithm uses the derivatives of the associated Legendre functions

$$
H_n^m(\mu) \equiv (1-\mu^2)\,\frac{dP_n^m(\mu)}{d\mu} = (2n+1)\varepsilon_n^m\,P_{n-1}^m(\mu) - n\mu P_n^m(\mu), \qquad (5.46)
$$

where $\varepsilon_n^m = \left(\frac{n^2-m^2}{4n^2-1}\right)^{\frac{1}{2}}$ as in (B.45-B.46) in Washington and Parkinson (1986).

Their symmetry properties are given by

$$
H_n^m(-\mu) = -sgn_n^m\;H_n^m(\mu) \qquad (5.47)
$$

The Gaussian latitudes and weights also exhibit hemispherical symmetry where

$$\mu_{\mathrm{j}} = -\mu_{J+1-\mathrm{j}}, \tag{5.48}$$

$$w_{\mathrm{j}} = w_{J+1-\mathrm{j}}. \tag{5.49}$$

For a given hemisphere, let us define a new index which goes from 1 at the point next to the pole to $\hat{J} = J/2$ at the point next to the equator. Then, the Gaussian sums can be rewritten as:

$$\begin{aligned}
\xi_n^m &= \sum_{\mathrm{j}=1}^{J} \Big\{ \alpha^m(\mu_{\mathrm{j}}) P_n^m(\mu_{\mathrm{j}}) + \beta^m(\mu_{\mathrm{j}}) H_n^m(\mu_{\mathrm{j}}) \Big\} w_{\mathrm{j}} \\
&= \sum_{\mathrm{j}=1}^{J/2} \Big\{ \alpha^m(\mu_{\mathrm{j}}) P_n^m(\mu_{\mathrm{j}}) + \beta^m(\mu_{\mathrm{j}}) H_n^m(\mu_{\mathrm{j}}) \Big\} w_{\mathrm{j}} \\
&\quad + \sum_{\mathrm{j}=1}^{J/2} \Big\{ \alpha^m(\mu_{J+1-\mathrm{j}}) P_n^m(-\mu_{\mathrm{j}}) + \beta^m(\mu_{J+1-\mathrm{j}}) H_n^m(-\mu_{\mathrm{j}}) \Big\} w_{J+1-\mathrm{j}} \\
&= \sum_{\mathrm{j}=1}^{\hat{J}} \Big\{ \left[\alpha^m(\mu_{\mathrm{j}}) + sgn_n^m \alpha^m(\mu_{J+1-\mathrm{j}})\right] P_n^m(\mu_{\mathrm{j}}) \\
&\quad + \left[\beta^m(\mu_{\mathrm{j}}) - sgn_n^m \beta^m(\mu_{J+1-\mathrm{j}})\right] H_n^m(\mu_{\mathrm{j}}) \Big\} w_{\mathrm{j}}
\end{aligned} \tag{5.50}$$

Using a similar symmetry argument, the Fourier coefficients are computed by

$$\begin{aligned}
\xi^m(\mu_{\mathrm{j}}) &= \sum_{n=|m|}^{N(m)} \alpha_n^m P_n^m(\mu_{\mathrm{j}}) + \beta_n^m H_n^m(\mu_{\mathrm{j}}) \\
&= \begin{cases} \sum_{n=|m|}^{N(m)} \alpha_n^m P_n^m(\mu_{\mathrm{j}}) + \beta_n^m H_n^m(\mu_{\mathrm{j}}) & \text{for } \mu_{\mathrm{j}} \geq 0 \\ \sum_{n=|m|}^{N(m)} sgn_n^m \left[\alpha_n^m P_n^m(-\mu_{\mathrm{j}}) - \beta_n^m H_n^m(-\mu_{\mathrm{j}})\right] & \text{for } \mu_{\mathrm{j}} < 0 \end{cases}
\end{aligned} \tag{5.51}$$

6. Software Description and Availability

The spectral transform shallow water model (STSWM) has been implemented as a Fortran program and is available via anonymous FTP from the authors. The software includes initialization routines for steady and unsteady fields, including real data cases which have been initialized using a Nonlinear Normal Mode Initialization procedure, error analysis routines based on known analytic or computed reference solutions, analysis routines (conservation properties, etc.) and graphics code for 2D plots of computed and analytic solutions and their differences.

The spherical basis functions P_n^m are computed using the highly accurate and stable recurrence relationship of Belousov (1962). Explicit and semi-implicit time differencing options have been implemented. Vorticity/divergence or momentum forcing terms can be added to the model as described in Appendix A of this Technical Note. The software is furthermore implemented to handle rotated coordinate systems as required by several of the test cases presented in Williamson et al. (1992). The rotational transformation equations are described in detail in Appendix B. The code has been used to generate reference solutions for the shallow water test cases proposed in Williamson et al. (1992) which are documented in Jakob et al. (1992).

The code is written in standard Fortran 77 with INCLUDE-files and NAMELIST input parameters. It has been compiled and executed on Sun workstations as well as Cray supercomputers. The time integration routines use a Fast Fourier Transform Library from Temperton (1983). A Fortran version of these routines is included with the source to facilitate portability. Access to the reference solutions requires the netCDF library (Unidata Program Center, 1991). Graphical output is based on the NCAR Graphics library (Clare et al., 1987; Clare and Kennison, 1989). These are the only libraries used by the model source code.

A more detailed description of the software is available in electronic form via anonymous FTP from the machine

ftp.ucar.edu (IP address: 128.117.64.4)

in the plain text file

/chammp/shallow/docu/description.txt

Detailed instructions on how to obtain a copy of the software are contained in the file

/chammp/shallow/README

Difficulties in accessing these files should be reported to the NCAR computer consulting office at 303-497-1278 (email: consult1@ncar.ucar.edu). Software bugs, along with suggested fixes, should be reported electronically to stswm@ncar.ucar.edu.

The spectral transform shallow water model (STSWM) is being made available for scientific research and educational purposes only. The source code is available AS IS without warranty, expressed or implied, as to its suitability for any purpose or application. The copyright in and to STSWM is held by UCAR. UCAR will not indemnify any user of the model for any copyright, patent, or other proprietary interest held by a third party. NCAR/CGD Division cannot provide support for the software, and therefore will only provide access to copies of the source code and the accompanying technical note.

Users are requested to acknowledge NCAR as the source of the software in any resulting research or publications, and are encouraged to send reprints of their work with this model when available to the NCAR CGD Division Office.

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APPENDIX A: Shallow Water Model with Forcing

Test case (4) in Williamson et al. (1992) involves the use of a forcing term for each of the three state variables. The vorticity/divergence form of the shallow water system can make use of either a simple momentum forcing or a more complicated vorticity/divergence forcing. These two approaches are illustrated in this Appendix.

A.1 Vorticity/Divergence Form with Forcing

The forced nonlinear shallow water equations add forcing terms F_η, F_δ and F_Φ to the system of equations (2.12)–(2.14) (e.g., as shown in Browning et al., 1989):

$$\frac{\partial \eta}{\partial t} + \frac{1}{a(1-\mu^2)} \frac{\partial}{\partial \lambda}(U\eta) + \frac{1}{a}\frac{\partial}{\partial \mu}(V\eta) = F_\eta\,, \tag{A.1}$$

$$\frac{\partial \delta}{\partial t} - \frac{1}{a(1-\mu^2)} \frac{\partial}{\partial \lambda}(V\eta) + \frac{1}{a}\frac{\partial}{\partial \mu}(U\eta) + \nabla^2\left(\Phi + \frac{U^2+V^2}{2(1-\mu^2)}\right) = F_\delta\,, \tag{A.2}$$

and

$$\frac{\partial \Phi}{\partial t} + \frac{1}{a(1-\mu^2)} \frac{\partial}{\partial \lambda}(U\Phi) + \frac{1}{a}\frac{\partial}{\partial \mu}(V\Phi) + \overline{\Phi}\delta = F_\Phi\,. \tag{A.3}$$

The vorticity and divergence forcing F_η and F_δ in (A.1) and (A.2) can also be expressed in terms of an equivalent momentum forcing F_u and F_v:

$$\frac{\partial \eta}{\partial t} + \frac{1}{a(1-\mu^2)} \frac{\partial}{\partial \lambda}(U\eta - F_v\cos\phi) + \frac{1}{a}\frac{\partial}{\partial \mu}(V\eta + F_u\cos\phi) = 0, \tag{A.4}$$

$$\frac{\partial \delta}{\partial t} - \frac{1}{a(1-\mu^2)} \frac{\partial}{\partial \lambda}(V\eta + F_u\cos\phi) + \frac{1}{a}\frac{\partial}{\partial \mu}(U\eta - F_v\cos\phi) + \nabla^2\left(\Phi + \frac{U^2+V^2}{2(1-\mu^2)}\right) = 0. \tag{A.5}$$

A.2 Derivation of Forcing Terms

For the convenience of the reader we derive the forcing terms for test case (4) in Williamson et al. (1992).

Assume a stream function $\widetilde{\psi}$ that resembles a circular low $\overline{\psi}$ which is moving in the center of a mid-latitude jet stream $\overline{U} = u_0 \sin^{14}(2\phi)$ of the form

$$\widetilde{\psi}(\lambda, \mu, t) = \overline{\psi}\left(\lambda - \frac{u_0}{a}t, \mu, 0\right) - a \int_{-1}^{\mu} \frac{\overline{U}(\mu')}{1-\mu'^2}\, d\mu', \tag{A.6}$$

so that

$$\widetilde{U} = -\frac{1-\mu^2}{a}\frac{\partial \widetilde{\psi}}{\partial \mu} = \overline{U}(\mu) - \frac{1-\mu^2}{a}\frac{\partial \overline{\psi}}{\partial \mu}, \tag{A.7}$$

$$\widetilde{V} = \frac{1}{a}\frac{\partial \widetilde{\psi}}{\partial \lambda}, \tag{A.8}$$

$$\widetilde{\Phi} = \Phi_0(\mu) + f\overline{\psi}, \tag{A.9}$$

$$\widetilde{\eta} = \frac{1}{a^2(1-\mu^2)}\frac{\partial^2 \overline{\psi}}{\partial \lambda^2} + \frac{1}{a^2}\left[(1-\mu^2)\frac{\partial^2 \overline{\psi}}{\partial \mu^2} - 2\mu \frac{\partial \overline{\psi}}{\partial \mu}\right] + f - \frac{1}{a}\frac{\partial}{\partial \mu}(\overline{U}(\mu)), \tag{A.10}$$

and

$$\widetilde{\delta} = 0. \tag{A.11}$$

For a steady state solution the geopotential must satisfy the differential equality

$$\frac{\partial \Phi_0}{\partial \mu} = -\frac{a}{1-\mu^2} f\overline{U} - \frac{\mu \overline{U}^2}{(1-\mu^2)^2}.$$

Relations (A.6)–(A.11) are a solution of (A.1)–(A.3) for

$$F_\eta = \frac{\partial \widetilde{\eta}}{\partial t} + \frac{1}{a(1-\mu^2)}\frac{\partial}{\partial \lambda}(\widetilde{U}\widetilde{\eta}) + \frac{1}{a}\frac{\partial}{\partial \mu}(\widetilde{V}\widetilde{\eta}), \tag{A.12}$$

$$F_\delta = -\frac{1}{a(1-\mu^2)}\frac{\partial}{\partial \lambda}(\widetilde{V}\widetilde{\eta}) + \frac{1}{a}\frac{\partial}{\partial \mu}(\widetilde{U}\widetilde{\eta}) + \nabla^2\left(\widetilde{\Phi} + \frac{\widetilde{U}^2 + \widetilde{V}^2}{2(1-\mu^2)}\right), \tag{A.13}$$

and

$$F_\Phi = \frac{\partial \widetilde{\Phi}}{\partial t} + \frac{1}{a(1-\mu^2)}\frac{\partial}{\partial \lambda}(\widetilde{U}\widetilde{\Phi}) + \frac{1}{a}\frac{\partial}{\partial \mu}(\widetilde{V}\widetilde{\Phi}). \tag{A.14}$$

When using momentum forcing, relations (A.6)–(A.11) are a solution of (A.3)–(A.5) for

$$F_u \cos\phi = \frac{\partial \widetilde{U}}{\partial t} + \frac{\widetilde{U}}{a(1-\mu^2)}\frac{\partial \widetilde{U}}{\partial \lambda} + \frac{\widetilde{V}}{a}\frac{\partial \widetilde{U}}{\partial \mu} + \frac{1}{a}\frac{\partial \widetilde{\Phi}}{\partial \lambda} - f\widetilde{V} \tag{A.15}$$

$$\begin{aligned} F_v \cos\phi = & \frac{\partial \widetilde{V}}{\partial t} + \frac{\widetilde{U}}{a(1-\mu^2)}\frac{\partial \widetilde{V}}{\partial \lambda} + \frac{\widetilde{V}}{a}\frac{\partial \widetilde{V}}{\partial \mu} + \\ & \frac{\mu}{a(1-\mu^2)}\left(\widetilde{U}\widetilde{U} + \widetilde{V}\widetilde{V}\right) + \frac{1-\mu^2}{a}\frac{\partial \widetilde{\Phi}}{\partial \mu} + f\widetilde{U} \end{aligned} \tag{A.16}$$

Expansion of (A.12)–(A.14) yields

$$F_\eta = \frac{\partial \widetilde{\eta}}{\partial t} + \frac{\widetilde{U}}{a(1-\mu^2)}\frac{\partial \widetilde{\eta}}{\partial \lambda} + \frac{\widetilde{V}}{a}\frac{\partial \widetilde{\eta}}{\partial \mu}, \tag{A.17}$$

$$\begin{aligned} F_\delta = & \nabla^2 \widetilde{\Phi} - f(\widetilde{\eta} - f) + \frac{\widetilde{U}}{a}\frac{\partial f}{\partial \mu} \\ & + \frac{2\mu}{a(1-\mu^2)}\left\{\frac{\widetilde{U}}{a(1-\mu^2)}\frac{\partial \widetilde{V}}{\partial \lambda} - \frac{\widetilde{V}}{a(1-\mu^2)}\frac{\partial \widetilde{U}}{\partial \lambda} + \frac{\widetilde{U}}{a}\frac{\partial \widetilde{U}}{\partial \mu} + \frac{\widetilde{V}}{a}\frac{\partial \widetilde{V}}{\partial \mu}\right\} \\ & + 2\left\{\left(\frac{1}{a(1-\mu^2)}\frac{\partial \widetilde{V}}{\partial \lambda}\right)\left(\frac{1}{a}\frac{\partial \widetilde{U}}{\partial \mu}\right) - \left(\frac{1}{a(1-\mu^2)}\frac{\partial \widetilde{U}}{\partial \lambda}\right)\left(\frac{1}{a}\frac{\partial \widetilde{V}}{\partial \mu}\right)\right\} \\ & + (\widetilde{U}^2 + \widetilde{V}^2)\left[\frac{2\mu^2}{a^2(1-\mu^2)^2} + \frac{1}{a^2(1-\mu^2)}\right], \end{aligned} \tag{A.18}$$

$$F_\Phi = \frac{\partial \tilde{\Phi}}{\partial t} + \frac{\widetilde{U}}{a(1-\mu^2)}\frac{\partial \tilde{\Phi}}{\partial \lambda} + \frac{\widetilde{V}}{a}\frac{\partial \tilde{\Phi}}{\partial \mu}, \tag{A.19}$$

where

$$\tilde{\Phi}\left(\frac{1}{a(1-\mu^2)}\frac{\partial \tilde{U}}{\partial \lambda} + \frac{1}{a}\frac{\partial \tilde{V}}{\partial \mu}\right) = \tilde{\Phi}\,\tilde{\delta} = 0.$$

The terms required for calculation of F_η, F_δ, (or F_u, F_v) and F_Φ are given by (A.7)–(A.10) and

$$\frac{\partial\tilde{\eta}}{\partial t} = -\left(\frac{u_0}{a}\right)\left[\frac{1}{a^2(1-\mu^2)}\,\frac{\partial^3\overline{\psi}}{\partial\lambda^3} + \frac{1}{a^2}\left[(1-\mu^2)\,\frac{\partial}{\partial\lambda}\left(\frac{\partial^2\overline{\psi}}{\partial\mu^2}\right) - 2\mu\,\frac{\partial}{\partial\lambda}\left(\frac{\partial\overline{\psi}}{\partial\mu}\right)\right]\right], \quad (A.20)$$

$$\frac{\partial\tilde{\eta}}{\partial\lambda} = \frac{1}{a^2(1-\mu^2)}\,\frac{\partial^3\overline{\psi}}{\partial\lambda^3} + \frac{1}{a^2}\left[(1-\mu^2)\,\frac{\partial}{\partial\lambda}\left(\frac{\partial^2\overline{\psi}}{\partial\mu^2}\right) - 2\mu\,\frac{\partial}{\partial\lambda}\left(\frac{\partial\overline{\psi}}{\partial\mu}\right)\right], \quad (A.21)$$

$$\frac{\partial\tilde{\eta}}{\partial\mu} = \frac{1}{a^2}\left[\frac{1}{(1-\mu^2)}\,\frac{\partial}{\partial\mu}\left(\frac{\partial^2\overline{\psi}}{\partial\lambda^2}\right) + \frac{2\mu}{(1-\mu^2)^2}\,\frac{\partial^2\overline{\psi}}{\partial\lambda^2} + (1-\mu^2)\,\frac{\partial^3\overline{\psi}}{\partial\mu^3} - 4\mu\frac{\partial^2\overline{\psi}}{\partial\mu^2} - 2\frac{\partial\overline{\psi}}{\partial\mu}\right]$$
$$+\frac{\partial f}{\partial\mu} - \frac{1}{a}\,\frac{\partial^2}{\partial\mu^2}\left(\overline{U}(\mu)\right), \quad (A.22)$$

$$\nabla^2\tilde{\Phi} = \frac{f}{a^2(1-\mu^2)}\,\frac{\partial^2\overline{\psi}}{\partial\lambda^2} + \frac{1}{a^2}\left[(1-\mu^2)\left\{f\frac{\partial^2\overline{\psi}}{\partial\mu^2} + 2\frac{\partial f}{\partial\mu}\frac{\partial\overline{\psi}}{\partial\mu}\right\} - 2\mu\left\{f\frac{\partial\overline{\psi}}{\partial\mu} + \overline{\psi}\frac{\partial f}{\partial\mu}\right\}\right]$$
$$-\frac{f}{a}\,\frac{\partial\overline{U}}{\partial\mu} - \frac{\overline{U}}{a}\,\frac{\partial f}{\partial\mu} - \frac{\overline{U}}{a^2(1-\mu^2)^2}\left[\left(1+\mu^2\right)\overline{U} + 2\mu\left(1-\mu^2\right)\,\frac{\partial\overline{U}}{\partial\mu}\right], \quad (A.23)$$

$$\frac{\partial\tilde{V}}{\partial\lambda} = \frac{1}{a}\,\frac{\partial^2\overline{\psi}}{\partial\lambda^2}, \quad (A.24)$$

$$\frac{\partial\tilde{V}}{\partial\mu} = \frac{1}{a}\,\frac{\partial}{\partial\mu}\left(\frac{\partial\overline{\psi}}{\partial\lambda}\right), \quad (A.25)$$

$$\frac{\partial\tilde{U}}{\partial\mu} = \frac{\partial\overline{U}}{\partial\mu} - \left[\frac{1-\mu^2}{a}\,\frac{\partial^2\overline{\psi}}{\partial\mu^2} - \frac{2\mu}{a}\,\frac{\partial\overline{\psi}}{\partial\mu}\right], \quad (A.26)$$

$$\frac{\partial\tilde{U}}{\partial\lambda} = -\,\frac{1-\mu^2}{a}\,\frac{\partial}{\partial\lambda}\left(\frac{\partial\overline{\psi}}{\partial\mu}\right), \quad (A.27)$$

$$\frac{\partial\tilde{\Phi}}{\partial t} = -f\left(\frac{u_0}{a}\right)\,\frac{\partial\overline{\psi}}{\partial\lambda}, \quad (A.28)$$

$$\frac{\partial\tilde{\Phi}}{\partial\lambda} = f\frac{\partial\overline{\psi}}{\partial\lambda}, \quad (A.29)$$

$$\frac{\partial\tilde{\Phi}}{\partial\mu} = f\frac{\partial\overline{\psi}}{\partial\mu} + \overline{\psi}\frac{\partial f}{\partial\mu} - \frac{a}{1-\mu^2}\,f\overline{U} - \frac{\mu\overline{U}^2}{(1-\mu^2)^2}, \quad (A.30)$$

$$\frac{\partial\tilde{U}}{\partial t} = \frac{u_0}{a}\frac{1-\mu^2}{a}\,\frac{\partial^2\overline{\psi}}{\partial\lambda\partial\mu} \quad (A.31)$$

and

$$\frac{\partial \widetilde{V}}{\partial t} = -\frac{u_0}{a^2}\frac{\partial^2 \overline{\psi}}{\partial \lambda^2} \tag{A.32}$$

We assume that $\overline{\psi}$ takes the form

$$\overline{\psi}(\lambda, \mu, t) = \psi_0 e^{-\sigma(1-c)/(1+c)} \tag{A.33}$$

where

$$c(\lambda, \mu, t) = \sin\phi_0\,\mu + \cos\phi_0(1-\mu^2)^{1/2}\cos(\lambda - \frac{u_0}{a}t - \lambda_0). \tag{A.34}$$

Equations (A.7)–(A.10) and (A.20)–(A.32) require the determination of the following expressions

$$\frac{\partial \overline{\psi}}{\partial \mu} = \frac{2\sigma\overline{\psi}}{(1+c)^2}\frac{\partial c}{\partial \mu} \tag{A.35}$$

$$\frac{\partial \overline{\psi}}{\partial \lambda} = \frac{2\sigma\overline{\psi}}{(1+c)^2}\frac{\partial c}{\partial \lambda} \tag{A.36}$$

$$\frac{\partial^2 \overline{\psi}}{\partial \mu^2} = \frac{2\sigma\overline{\psi}}{(1+c)^2}\left\{\frac{\partial^2 c}{\partial \mu^2} + 2\left(\frac{\partial c}{\partial \mu}\right)^2\left[\frac{\sigma-(1+c)}{(1+c)^2}\right]\right\} \tag{A.37}$$

$$\frac{\partial^2 \overline{\psi}}{\partial \lambda^2} = \frac{2\sigma\overline{\psi}}{(1+c)^2}\left\{\frac{\partial^2 c}{\partial \lambda^2} + 2\left(\frac{\partial c}{\partial \lambda}\right)^2\left[\frac{\sigma-(1+c)}{(1+c)^2}\right]\right\} \tag{A.38}$$

$$\begin{aligned}\frac{\partial^3 \overline{\psi}}{\partial \mu^3} = \frac{2\sigma\overline{\psi}}{(1+c)^2}&\left\{\frac{\partial^3 c}{\partial \mu^3} + 2\left(\frac{\partial c}{\partial \mu}\right)^3\left[\frac{(1+c)^2 - 2\sigma(1+c)}{(1+c)^4}\right]\right.\\ &\left. + 2\frac{\partial c}{\partial \mu}\left[\frac{\sigma-(1+c)}{(1+c)^2}\right]\left[3\frac{\partial^2 c}{\partial \mu^2} + 2\left(\frac{\partial c}{\partial \mu}\right)^2\left[\frac{\sigma-(1+c)}{(1+c)^2}\right]\right]\right\}\end{aligned} \tag{A.39}$$

$$\begin{aligned}\frac{\partial^3 \overline{\psi}}{\partial \lambda^3} = \frac{2\sigma\overline{\psi}}{(1+c)^2}&\left\{\frac{\partial^3 c}{\partial \mu^3} + 2\left(\frac{\partial c}{\partial \lambda}\right)^3\left[\frac{(1+c)^2 - 2\sigma(1+c)}{(1+c)^4}\right]\right.\\ &\left. + 2\frac{\partial c}{\partial \lambda}\left[\frac{\sigma-(1+c)}{(1+c)^2}\right]\left[3\frac{\partial^2 c}{\partial \lambda^2} + 2\left(\frac{\partial c}{\partial \lambda}\right)^2\left[\frac{\sigma-(1+c)}{(1+c)^2}\right]\right]\right\}\end{aligned} \tag{A.40}$$

$$\frac{\partial}{\partial\lambda}\left(\frac{\partial\overline{\psi}}{\partial\mu}\right)=\frac{2\sigma\overline{\psi}}{(1+c)^2}\left\{\frac{\partial}{\partial\mu}\left(\frac{\partial c}{\partial\lambda}\right)+2\frac{\partial c}{\partial\lambda}\frac{\partial c}{\partial\mu}\left[\frac{\sigma-(1+c)}{(1+c)^2}\right]\right\} \tag{A.41}$$

$$\begin{aligned}\frac{\partial}{\partial\lambda}\left(\frac{\partial^2\overline{\psi}}{\partial\mu^2}\right)=\frac{2\sigma\overline{\psi}}{(1+c)^2}\Bigg\{&\frac{\partial}{\partial\lambda}\left(\frac{\partial^2 c}{\partial\mu^2}\right)+2\left(\frac{\partial c}{\partial\mu}\right)^2\frac{\partial c}{\partial\lambda}\left[\frac{(1+c)^2-2\sigma(1+c)}{(1+c)^4}\right]\\&+2\frac{\partial c}{\partial\lambda}\left[\frac{\sigma-(1+c)}{(1+c)^2}\right]\left[\frac{\partial^2 c}{\partial\mu^2}+2\left(\frac{\partial c}{\partial\mu}\right)^2\left[\frac{\sigma-(1+c)}{(1+c)^2}\right]\right]\\&+4\frac{\partial c}{\partial\mu}\frac{\partial}{\partial\lambda}\left(\frac{\partial c}{\partial\mu}\right)\left[\frac{\sigma-(1+c)}{(1+c)^2}\right]\Bigg\}\end{aligned} \tag{A.42}$$

$$\begin{aligned}\frac{\partial}{\partial\mu}\left(\frac{\partial^2\overline{\psi}}{\partial\lambda^2}\right)=\frac{2\sigma\overline{\psi}}{(1+c)^2}\Bigg\{&\frac{\partial}{\partial\mu}\left(\frac{\partial^2 c}{\partial\lambda^2}\right)+2\left(\frac{\partial c}{\partial\lambda}\right)^2\frac{\partial c}{\partial\mu}\left[\frac{(1+c)^2-2\sigma(1+c)}{(1+c)^4}\right]\\&+2\frac{\partial c}{\partial\mu}\left[\frac{\sigma-(1+c)}{(1+c)^2}\right]\left[\frac{\partial^2 c}{\partial\lambda^2}+2\left(\frac{\partial c}{\partial\lambda}\right)^2\left[\frac{\sigma-(1+c)}{(1+c)^2}\right]\right]\\&+4\frac{\partial c}{\partial\lambda}\frac{\partial}{\partial\mu}\left(\frac{\partial c}{\partial\lambda}\right)\left[\frac{\sigma-(1+c)}{(1+c)^2}\right]\Bigg\}\end{aligned} \tag{A.43}$$

where

$$\frac{\partial c}{\partial\mu}=\sin\phi_0-\frac{\mu}{(1-\mu^2)^{1/2}}\cos\phi_0\cos(\lambda-\frac{u_0}{a}t-\lambda_0) \tag{A.44}$$

$$\frac{\partial c}{\partial\lambda}=-\cos\phi_0(1-\mu^2)^{1/2}\sin(\lambda-\frac{u_0}{a}t-\lambda_0) \tag{A.45}$$

$$\frac{\partial^2 c}{\partial\mu^2}=-\frac{\cos\phi_0\cos(\lambda-\frac{u_0}{a}t-\lambda_0)}{(1-\mu^2)^{3/2}} \tag{A.46}$$

$$\frac{\partial^2 c}{\partial\lambda^2}=-\cos\phi_0(1-\mu^2)^{1/2}\cos(\lambda-\frac{u_0}{a}t-\lambda_0) \tag{A.47}$$

$$\frac{\partial^3 c}{\partial\mu^3}=-\cos\phi_0\cos(\lambda-\frac{u_0}{a}t-\lambda_0)\frac{3\mu}{(1-\mu^2)^{5/2}} \tag{A.48}$$

$$\frac{\partial^3 c}{\partial\lambda^3}=\cos\phi_0\sin(\lambda-\frac{u_0}{a}t-\lambda_0)(1-\mu^2)^{1/2} \tag{A.49}$$

$$\frac{\partial}{\partial\mu}\left(\frac{\partial c}{\partial\lambda}\right)=\cos\phi_0\sin(\lambda-\frac{u_0}{a}t-\lambda_0)\frac{\mu}{(1-\mu^2)^{1/2}} \tag{A.50}$$

$$\frac{\partial}{\partial\lambda}\left(\frac{\partial^2 c}{\partial\mu^2}\right) = \frac{\cos\phi_0 \sin(\lambda - \frac{u_0}{a}t - \lambda_0)}{(1-\mu^2)^{3/2}} \tag{A.51}$$

and

$$\frac{\partial}{\partial\mu}\left(\frac{\partial^2 c}{\partial\lambda^2}\right) = \frac{\mu\cos\phi_0 \cos(\lambda - \frac{u_0}{a}t - \lambda_0)}{(1-\mu^2)^{1/2}} \tag{A.52}$$

A.3 Inclusion of Forcing in Numerical Algorithm

In the explicit time differencing procedure, (5.28)–(5.30) must be expanded to include the forcing terms:

$$\{\eta_n^m\}_{\text{forced}}^{(\tau+1)} = \{\eta_n^m\}^{(\tau+1)} + \sum_{j=1}^{J} 2\Delta t\, \{(F_\eta)^m(\mu_j)\}^{(\tau)}\, P_n^m(\mu_j) w_j \tag{A.53}$$

$$\{\delta_n^m\}_{\text{forced}}^{(\tau+1)} = \{\delta_n^m\}^{(\tau+1)} + \sum_{j=1}^{J} 2\Delta t\, \{(F_\delta)^m(\mu_j)\}^{(\tau)}\, P_n^m(\mu_j) w_j \tag{A.54}$$

$$\{\Phi_n^m\}_{\text{forced}}^{(\tau+1)} = \{\Phi_n^m\}^{(\tau+1)} + \sum_{j=1}^{J} 2\Delta t\, \{(F_\Phi)^m(\mu_j)\}^{(\tau)}\, P_n^m(\mu_j) w_j \tag{A.55}$$

In the semi-implicit time differencing procedure, (5.33) and (5.34) must be expanded to include the forcing terms:

$$\{\mathcal{D}_n^m\}_{\text{forced}} = \mathcal{D}_n^m + \sum_{j=1}^{J} (F_\delta)^m(\mu_j)\, P_n^m(\mu_j) w_j \tag{A.56}$$

$$\{\mathcal{P}_n^m\}_{\text{forced}} = \mathcal{P}_n^m + \sum_{j=1}^{J} (F_\Phi)^m(\mu_j)\, P_n^m(\mu_j) w_j \tag{A.57}$$

The momentum forcing terms are included in (5.11) by setting

$$A_{\text{forced}} = A - F_v cos\phi \tag{A.58}$$

$$B_{\text{forced}} = B + F_u cos\phi \tag{A.59}$$

APPENDIX B: Rotational Transformations in a Spherical Coordinate System

In this appendix we derive the transformation equations for the Coriolis parameter and horizontal velocities on a sphere for the case of a rotation around one axis. The equations can be used to test numerical methods for geophysical flows, since most physical laws are invariant under a rotational transformation of the coordinate system. Let u, u' be the eastward velocities and v, v' the northward velocities in the original and rotated coordinate system, respectively. The rotation angle between the two polar axis is α, and the plane of rotation is defined by longitude $\lambda = 0$ for the coordinate systems. As a consistency test, the transformations must reduce to identities for $\alpha = 0$, and they must be symmetric for the inverse rotation $u' \to u$, $v' \to v$ and $\alpha \to -\alpha$.

B.1 Coordinate Transformation

For the purpose of this derivation we introduce a Cartesian coordinate system with origin in the center of the sphere, z-axis toward the North Pole, x-axis toward longitude 0 in the original spherical system. The y-axis is the axis of rotation. The Cartesian coordinates are thus given by

$$x = \cos\lambda\cos\phi \tag{B.1}$$

$$y = \sin\lambda\cos\phi \tag{B.2}$$

$$z = \sin\phi \tag{B.3}$$

In the Cartesian coordinate system, the rotation transformation can be described by a matrix multiplication

$$\begin{pmatrix} x' \\ y' \\ x' \end{pmatrix} = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{B.4}$$

The Cartesian coordinates in the rotated system are then given by

$$x' = \cos\alpha\cos\lambda\cos\phi + \sin\alpha\sin\phi \tag{B.5}$$

$$y' = \sin\lambda\cos\phi \tag{B.6}$$

$$z' = -\sin\alpha\cos\lambda\cos\phi + \cos\alpha\sin\phi \tag{B.7}$$

We can return to spherical coordinates using the coordinate transformation (B.1)–(B.3)

$$x' = \cos\lambda' \cos\phi' \tag{B.8}$$

$$y' = \sin\lambda' \cos\phi' \tag{B.9}$$

$$z' = \sin\phi' \tag{B.10}$$

Eliminating x', y', z' from (B.5)–(B.10) yields the spherical coordinates in the original system as a function of the rotated coordinates

$$\phi = \arcsin(\sin\alpha\cos\lambda'\cos\phi' + \cos\alpha\sin\phi') \tag{B.11}$$

$$\lambda^* = \arcsin(\sin\lambda'\cos\phi'/\cos\phi) \tag{B.12}$$

Since the inverse sine function arcsin has multiple branches, (B.12) must be corrected so that

$$\begin{aligned} \text{if} \quad & x' = \cos\alpha\cos\lambda\cos\phi + \sin\alpha\sin\phi < 0 \\ & \text{then } \lambda = \pi - \lambda^* \\ & \text{else } \lambda = \lambda^* \end{aligned} \tag{B.13}$$

B.2 Scalar Transformations

Using the coordinate transformations (B.11)–(B.13), the values of a scalar field f' in rotated coordinates λ', ϕ' are given by

$$f'(\lambda', \phi') = f(\lambda, \phi). \tag{B.14}$$

As an example, we rotate the Coriolis parameter $f' = 2\Omega\sin\phi'$ where

$$\sin\phi' = -\sin\alpha\cos\lambda\cos\phi + \cos\alpha\sin\phi. \tag{B.15}$$

which yields

$$f = 2\Omega\ (\mu\cos\alpha - \sqrt{1-\mu^2}\sin\alpha\sin\phi). \tag{B.16}$$

Making use of the associated Legendre functions contained in Table 1 and their symmetry properties, the Coriolis parameter can be written as

$$
\begin{aligned}
f &= 2\Omega\left[\frac{\sqrt{6}}{3}\cos\alpha\ P_1^0(\mu) - \frac{2\sqrt{3}}{3}\sin\alpha\ P_1^1(\mu)\cos\lambda\right] \\
&= \frac{2}{3}\sqrt{6}\Omega\cos\alpha\ P_1^0(\mu) \\
&- \frac{2}{3}\sqrt{3}\Omega\sin\alpha\ P_1^1(\mu)e^{i\lambda} \\
&+ \frac{2}{3}\sqrt{3}\Omega\sin\alpha\ P_1^{-1}(\mu)e^{-i\lambda}
\end{aligned}
\tag{B.17}
$$

The general rotated spectral form of the Coriolis parameter defined in (5.3) is thus the two term expansion

$$
\begin{aligned}
f_1^0 &= \frac{2}{3}\sqrt{6}\Omega\cos\alpha \\
f_1^1 &= -\frac{2}{3}\sqrt{3}\Omega\sin\alpha \\
f_n^m &= 0 \text{ for all other } n \text{ and } m \geq 0.
\end{aligned}
\tag{B.18}
$$

B.3 Field Transformations

The mapping of vector fields between Cartesian and spherical coordinates is described by the following two transform matrices:

$$
\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \begin{pmatrix} -\sin\lambda & -\sin\phi\cos\lambda & \cos\phi\cos\lambda \\ \cos\lambda & -\sin\phi\sin\lambda & \cos\phi\sin\lambda \\ 0 & \cos\phi & \sin\phi \end{pmatrix} \begin{pmatrix} f_\lambda \\ f_\phi \\ f_r \end{pmatrix}
\tag{B.19}
$$

and its inverse transform

$$
\begin{pmatrix} f_\lambda \\ f_\phi \\ f_r \end{pmatrix} = \begin{pmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\sin\phi\cos\lambda & -\sin\phi\sin\lambda & \cos\phi \\ \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \end{pmatrix} \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}
\tag{B.20}
$$

For a surface velocity field, $f_\lambda \equiv u$, $f_\phi \equiv v$ and $f_r \equiv 0$. We translate this field into Cartesian coordinates (B.14), and change to the rotated coordinate system using (B.4):

$$f'_x = -u\cos\alpha\sin\lambda - v\cos\alpha\sin\phi\cos\lambda + v\sin\alpha\cos\phi \tag{B.21}$$

$$f'_y = u\cos\lambda - v\sin\phi\sin\lambda \tag{B.22}$$

$$f'_z = u\sin\alpha\sin\lambda + v\sin\alpha\sin\phi\cos\lambda + v\cos\alpha\cos\phi \tag{B.23}$$

Using the inverse transform (B.20), we obtain the rotated surface velocities:

$$\begin{aligned} u' = {} & u(\cos\alpha\sin\lambda\sin\lambda' + \cos\lambda'\cos\lambda) \\ & + v(-\sin\alpha\cos\phi\sin\lambda' + \cos\alpha\sin\phi\cos\lambda\sin\lambda' - \sin\phi\sin\lambda\cos\lambda') \end{aligned} \tag{B.24}$$

$$\begin{aligned} v' = {} & v\big(\sin\phi'\cos\lambda'(\cos\alpha\sin\phi\cos\lambda - \sin\alpha\cos\phi) \\ & + \sin\phi'\sin\lambda'\sin\phi\sin\lambda \\ & + \cos\phi'(\cos\alpha\cos\phi + \sin\alpha\sin\phi\cos\lambda)\big) \\ & + u\big(\sin\phi'\cos\lambda'\cos\alpha\sin\lambda \\ & - \sin\phi'\sin\lambda'\cos\lambda + \cos\phi'\sin\alpha\sin\lambda\big) \end{aligned} \tag{B.25}$$

which is equivalent to (5.16) and (5.17) in Williamson and Rasch (1989) for $\lambda_A = 0$.