INVERSE METHODS FOR SOLVING SHALLOW WATER EQUATIONS

Project Proposal

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Motivation

The shallow water equations (SWE) are a system of hyperbolic partial differential equations (PDEs) describing the flow below a pressure surface in a fluid. They have been frequently used to model several real-life problems i.e. the propagation of tsunamis waves in the ocean (Dias and Dutykh, 2007) and modeling of atmospheric turbulence. Deep knowledge is required to handle such events, therefore first, robust, and computationally efficient methods like inverse methods are required to solve the shallow water equations.

Problem description and methods used

Inverse methods have been frequently used to handle shallow water problems; see for instance Monnier et al. (2016), Gessese and Sellier (2012), and **?**. Consider small-amplitude waves in a one-dimensional fluid channel that is shallow relative to its wavelength. The conservation of mass and momentum equations are written in terms of height $h(x,t)$ (m) and momentum $h(x,t)u(x,t)$ (m^2/s) as shown in system (1)

$$\begin{aligned} h_t + (hu)_x &= 0, \\ (hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x &= 0, \end{aligned} \tag{1}$$

where g (m/s^2) is acceleration due to gravity, hu measures the flow rate of water past a point and $u(x,t)$ (m/s) is the horizontal velocity (LeVeque et al., 2002; Toro, 2001). The conservation laws in system (1) can be solved by developing numerical methods based on an eigendecomposition of the Jacobian matrix. And this can be achieved by expressing system (1) in quasi linaer form as shown in equation (2)

$$m_t + f'(m)m_x = 0, \tag{2}$$

where $m(x,t) = (h(x,t), hu(x,t))$ and $f'(m) \in \mathbb{R}^{m\times m}$ is a flux Jacobian matrix given by;

$$f'(m) = \begin{bmatrix} 0 & 1 \\ -u^2 + gh & 2u \end{bmatrix}. \tag{3}$$

Consider a simple set of initial conditions to equation (2) with a single discontinuity at the middle of the channel. Setting h and hu equal to constants on either side of the interface, and assuming the discontinuity is at $x = 0$ yields a Riemann problem with initial conditions given by equation (4).

$$m(x,0) \quad = \quad \begin{cases} (h_l, h_l u_l), & \text{if} \quad x \leq 0, \\ (h_l, h_l u_l), & \text{if} \quad x > 0, \end{cases} \tag{4}$$

where m_l and m_r are two piece-wise constant states separated by a discontinuity. Suppose that we know the nature of the solution (i.e. consits of two shocks or a single shock) to equation (2), then we can solve the Riemann problem (equation (4)) by finding the intermediate state $m_m(x,t) = (h_m(x,t), hu_m(x,t))$ that can be connected to m_l by a left going shock or to m_r by a right going shock and vise varsa. Through the points m_l and m_r there is a curve of points that connects m_l to m_m or m_m to m_r via a left going shock or right going shock. For shallow water equations, these points must satisfy the non linear equation (5) (LeVeque et al., 2002).

$$F(h_m) = u_r - u_l + (h_m - h_r)\sqrt{\frac{g}{2}\left(\frac{1}{h_m} + \frac{1}{h_r}\right)} + (h_m - h_l)\sqrt{\frac{g}{2}\left(\frac{1}{h_m} + \frac{1}{h_l}\right)} \tag{5}$$

The Methods that will be used in this study are the forward and inverse methods for solving a Riemann problem. The idea behind the forward approach (based on Forestclaw) is to find Roe-averaged parameters: height field (h), velocity field (u), and root mean square speed (c) for a Riemann problem at every interface. These are used to obtain the corresponding eigenvalues and eigenvectors for the Jacobian matrix (3) that yield the waves and speeds used in obtaining fluctuations at each interface. These fluctuations as shown in equation (6) are used to obtain the forestclaw output M.

$$M_i^{n+1} = M_i^n - \frac{\Delta t}{\Delta x}(\mathcal{A}^+\Delta M_{i-\frac{1}{2}}^n + \mathcal{A}^- M_{i+\frac{1}{2}}^n) \tag{6}$$

where the fluctuations $\mathcal{A}^-\Delta M_{i+\frac{1}{2}}^n$ and $\mathcal{A}^+\Delta M_{i-\frac{1}{2}}^n$ are the net updating contributions from the leftward and rightward moving waves into the grid cell from the right and left interfaces respectively, Δt and Δx are the temporal and spatial step size respectively. Then the simulated data d_s is generated at each interface by adding noise to the forestclaw output as shown in equation (7).

$$d_s = G(m) + \epsilon, \qquad \text{for } \epsilon \sim N(0, \sigma^2) \tag{7}$$

where $G(m) = M$ represents the forestclaw output, with $m = (h, hu)$ such that h and hu are chosen values. Then Jacobian matrix is obtained by;

$$J(m^{k+1}) \approx \frac{(G(m^{k+1} + h) - (G(m^{k+1})))}{h}, \tag{8}$$

And then iterate the estimate m^k that satisfy

$$\min_{m^{k+1}} \left\{ ||J(m^k)m^{k+1} - d_s||_2^2 + \alpha^2 ||m^{k+1} - m^k||_2^2 \right\} \tag{9}$$

where α is obtained using a regularized discrepancy principle. The value of the parameter estimate for which the norm of the change in the estimates is less than 10^{-6}, will be the solution that will be validated with results in the next method. To check the nature of the simulated data, the χ^2_{obs} and p-value will be calculated.

Time Frame

All the stages of this project are planned for the remaining time of the semester and that is the four weeks in the month of April and the first five days of May 2022. During this time the following will be accomplished;

- First week: Accomplishment of the coding part of the project.
- Second week: simulation of the problem, visualization, and analysis of results.
- Third Week: Start and finish the write up of the project
- Fourth Week: Project presentation, implementation, and finalizing remarks.
- First five days of May: Submission of the project report.

The accomplishment of all these tasks in the anticipated time frame will prove a high level of success.

Take home

At the end of the project, I will have learned how to: write up a project proposal, formulate and implement an inverse shallow-water problem from a forward one, use the Gauss-Newton method to solve nonlinear problems, and think as an inverse method expert.

References

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