Appendix

```
clc
clear all

data = load('gdata.mat');
dn = data.dn;
x = data.x;
rhox = data.rhox;

a = 0; b = 1000; n = 100; m = 500;

R = 6.67428e-11; % Newton's gravitational constant
A = 1;
h = 25 - sqrt(A/pi);

dxc = (b-a)/n;

g = @(xc,x) (R*h./((xc - x).^2 + h^2)^(3/2));
xc = [];
for j = 1:n
   xc = [xc a + (dxc/2) + (j-1)*dxc]; %form xc
end
xc;

G = zeros(m,n);
for i = 1:m
   for j = 1:n
      G(i,j) = g(xc(j), x(i)).*dxc;
   end
end

G;
```

a. Invert for density perturbations along the pipe transect in kg/m3 using least squares.

```
ML2 = inv(G'*G)*G'*dn;
min(ML2);
max(ML2);
```

Analyze the resolution in your model parameter estimates.

```
% Get the singular values for the system matrix
[U,S,V] = svd(G);

[m,n] = size(G);

%rank
p=rank(G);

% model resolution matrix
Vp=V(:,1:p);
Rm=Vp*Vp';

figure(1)
```

```
clf
colormap('gray')

imagesc(Rm)
set(colorbar,'Fontsize',18);
xlabel('j')
ylabel('i')
title('Model resolution matrix R_{m}')
```

b. Invert for density perturbations along the pipe transect in kg/m3 using second order Tikhonov regularization and TGSVD.

```
%generate roughening matrices
L1 = get_l_rough(n,1);
L2 = get_l_rough(n,2);

% TGSVD
[U,V,X,S,M] = gsvd(G,eye(n)); Y = (inv(X)');
[U1,V1,X1,S1,M1] = gsvd(G,L1); Y1 = (inv(X1)');
[U2,V2,X2,S2,M2] = gsvd(G,L2); Y2 = (inv(X2)');

% generalized singular values of G and L
lam = @(S) sqrt(diag(S'*S));
mu = @(M) sqrt(diag(M'*M));

k = 0; %m>n
q2 = 100;

Lam = lam(S); Lam1 = lam(S1); Lam2 = lam(S2);

M0 = model_parameters(q2,U,Lam,Y,k,dn,n);
M_1 = model_parameters(q2,U1,Lam1,Y1,k,dn,n);
M_2 = model_parameters(q2,U2,Lam2,Y2,k,dn,n);

figure(2)
clf
plot(ML2,'o','LineWidth',2,"MarkerSize",5); hold on
plot(M0,'^','LineWidth',2,"MarkerSize",5);
plot(M_1,'*','LineWidth',2,"MarkerSize",5);
plot(M_2,'x','LineWidth',2,"MarkerSize",5); hold off
legend('least squares','m_{zeroth}','m_{first}','m_{second}')
xlabel('i'); ylabel('Parameter Estimates')
title('Model parameter estimates')
```

c. Solve the problem using second-order Tikhonov regularization combined with BVLS and a TGSVD analysis.

```
%zeroth order
L = eye(n);

% get the points and solutions for the first and second order TGSVD L-curve
[rho,eta,reg_param,ms]=l_curve_tgsvd(U,dn,X,S,G,L);
[rho1,eta1,reg_param1,m1s]=l_curve_tgsvd(U1,dn,X1,S1,G,L1);
```

```
[rho2,eta2,reg_param2,m2s]=l_curve_tgsvd(U2,dn,X2,S2,G,L2);

ireg_corner = 90;
alpha0=rho(ireg_corner);
alpha1=rho1(ireg_corner);
alpha2=rho2(ireg_corner);

alpha = [alpha0 alpha1 alpha2];

%generate lower and upper bounds
l= -150.*ones(n,1);
u= 150.*ones(n,1);

%stack the dn matrices
d0 = [dn; zeros(n,1)];
d1 = [dn; zeros(n-1,1)];
d2 = [dn; zeros(n-2,1)];

%stack the matrices
A = [G; alpha(1)*L];
A1 = [G; alpha(2)*L1];
A2 = [G; alpha(3)*L2];

%find positions of 91 and 899 in m
pos = find(xc < 91); pos_91 = pos(end);
pos = find(xc > 899); pos_899 = pos(1);

%setting bounds for xc
l(1:pos_91) = 0; l(pos_899:end) = 0;
u(1:pos_91) = 0; u(pos_899:end) = 0;

ML2 = bvls(G, dn,l, u);  %least squares
M0 = bvls(A, d0,l, u);
M1 = bvls(A1, d1,l, u);
M2 = bvls(A2, d2,l, u);

figure(3)
clf
plot(ML2,'o','LineWidth',2,"MarkerSize",5); hold on
plot(M0,'^','LineWidth',2,"MarkerSize",5);
plot(M1,'*','LineWidth',2,"MarkerSize",5);
plot(M2,'x','LineWidth',2,"MarkerSize",5); hold on
legend('least squares','m_{zeroth}','m_{first}','m_{second}')
xlabel('i'); ylabel('Parameter Estimates')
title('Model parameter estimates')
```

Where are the prominent maxima and minima located?

```
%minima
find(ML2==min(ML2))';
minima_locations =find(M0==min(M0))'
find(M1==min(M1))' ;
find(M2==min(M2))';
```

```
%maxima
maxima_locations = find(ML2==max(ML2))'
maxima_locations_zeroth =find(M0==max(M0))'
find(M1==max(M1))' ;
find(M2==max(M2))';
```

Analyze the resolution in your model parameter estimates.

```
figure(4)
clf
colormap('gray')
subplot(2,2,1)
[U,S,V] = svd(G); p=rank(G);
Vp=V(:,1:p);
Rm=Vp*Vp';
imagesc(Rm)
set(colorbar,'Fontsize',18);
xlabel('j')
ylabel('i')
title('Model resolution matrix R_{m} least squares')

subplot(2,2,2)
[U,S,V] = svd(A); p=rank(A);
Vp=V(:,1:p);
Rm=Vp*Vp';
imagesc(Rm)
set(colorbar,'Fontsize',18);
xlabel('j')
ylabel('i')
title('Model resolution matrix R_{m} zeroth order')

subplot(2,2,3)
[U,S,V] = svd(A1); p=rank(A1);
Vp=V(:,1:p);
Rm=Vp*Vp';
imagesc(Rm)
set(colorbar,'Fontsize',18);
xlabel('j')
ylabel('i')
title('Model resolution matrix R_{m} first order')

subplot(2,2,4)
[U,S,V] = svd(A2); p=rank(A2);
Vp=V(:,1:p);
Rm=Vp*Vp';
imagesc(Rm)
set(colorbar,'Fontsize',18);
xlabel('j')
ylabel('i')
title('Model resolution matrix R_{m} second order')

function [M] = model_parameters(q,U,Lam,Y,k,d,n)
    M = 0;
```

```
    for i = n-q+1:n
        M = M + (U(:,i-k)'*d)/(Lam(i))*Y(:,i);
    end
end
```