# *Sturm-Lioville Problem*

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## *Introduction*

General form of Sturm-Liouville problem

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + q(x)\phi = -\lambda\omega(x)\phi \quad \text{for } a \le x \le b \tag{1}$$

where
- $p(x), q(x)$ and $\omega(x)$ are continuous.
- $p(x), \omega(x) > 0$ on $[a, b]$
- $\phi(x)$ is an eigen function.

### *problem 1*

consider the following chebyshev's differential equation

$$(1 - x^2)\phi'' - x\phi' = -\lambda\phi \quad \text{for} - 1 < x < 1 \tag{2}$$

with initial conditions $\phi(-1) = 0,\ \phi(1) = 0$.
to rewrite in Sturm-Lioville form first we divide by $\sqrt{1-x^2}$ to get

$$\underbrace{\sqrt{1-x^2}\phi'' - \frac{x}{\sqrt{1-x^2}}\phi'}_{\left(\sqrt{1-x^2}\phi\prime\right)'} = -\frac{\lambda}{\sqrt{1-x^2}}\phi \tag{3}$$

with $p(x) = \sqrt{1-x^2}$, $q(x) = 0$ and $\omega(x) = \frac{1}{\sqrt{1-x^2}}$ so we have

$$\frac{d}{dx}\left[\sqrt{1-x^2}\ \frac{d\phi}{dx}\right] = -\lambda\frac{1}{\sqrt{1-x^2}}\phi \quad \text{for} - 1 < x < 1 \tag{4}$$

### *Solution*

we will use shooting method to solve chebyshev's differential equation. general approach to rewrite second order into first order ODE.

$$u(x) = \phi(x) \tag{5}$$

$$v(x) = \phi(x) \tag{6}$$

$$u'(x) = v(x) \tag{7}$$

$$v'(x) = \frac{p'(x)v + q(x)u + \lambda\omega(x)u}{p(x)} \tag{8}$$

in vector form

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-q(x)+\lambda w(x)}{p(x)} & \frac{p'(x)}{p(x)} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{9}$$

in our case
for first initial value problem

$$u_1{}'(x) = v_1(x) \tag{10}$$

$$\begin{aligned} v_1{}'(x) &= \frac{-\frac{x}{\sqrt{1-x^2}} v_1(x) + \frac{\lambda}{\sqrt{1-x^2}} u_1(x)}{\sqrt{1-x^2}} \\ &= \frac{-x v_1(x) + \lambda u_1(x)}{1-x^2} \end{aligned} \tag{11}$$

$$u_1(-1) = 1, v_1(-1) = 0 \tag{12}$$

for second initial value problem

$$u_2{}'(x) = v_2(x) \tag{13}$$

$$\begin{aligned} v_2{}'(x) &= \frac{-\frac{x}{\sqrt{1-x^2}} v_2(x) + \frac{\lambda}{\sqrt{1-x^2}} u_2(x)}{\sqrt{1-x^2}} \\ &= \frac{-x v_2(x) + \lambda u_2(x)}{1-x^2} \end{aligned} \tag{14}$$

$$u_2(-1) = 0, v_2(-1) = 1 \tag{15}$$

we find appropriate $\alpha_1, \alpha_2, \beta_1, \beta_2$ for Robin boundary conditions

$$\begin{aligned} \alpha_1 \phi(-1) + \beta_1 \phi'(-1) = 0 \\ \alpha_2 \phi(1) + \beta_2 \phi'(1) = 0 \end{aligned} \tag{16}$$

determinant function

$$F(\lambda) = \det \begin{bmatrix} \alpha_1 u_1(-1) + \beta_1 v_1(-1) & \alpha_1 u_2(-1) + \beta_1 v_2(-1) \\ \alpha_2 u_1(1) + \beta_2 v_1(1) & \alpha_2 u_2(1) + \beta_2 v_2(1) \end{bmatrix} \tag{17}$$

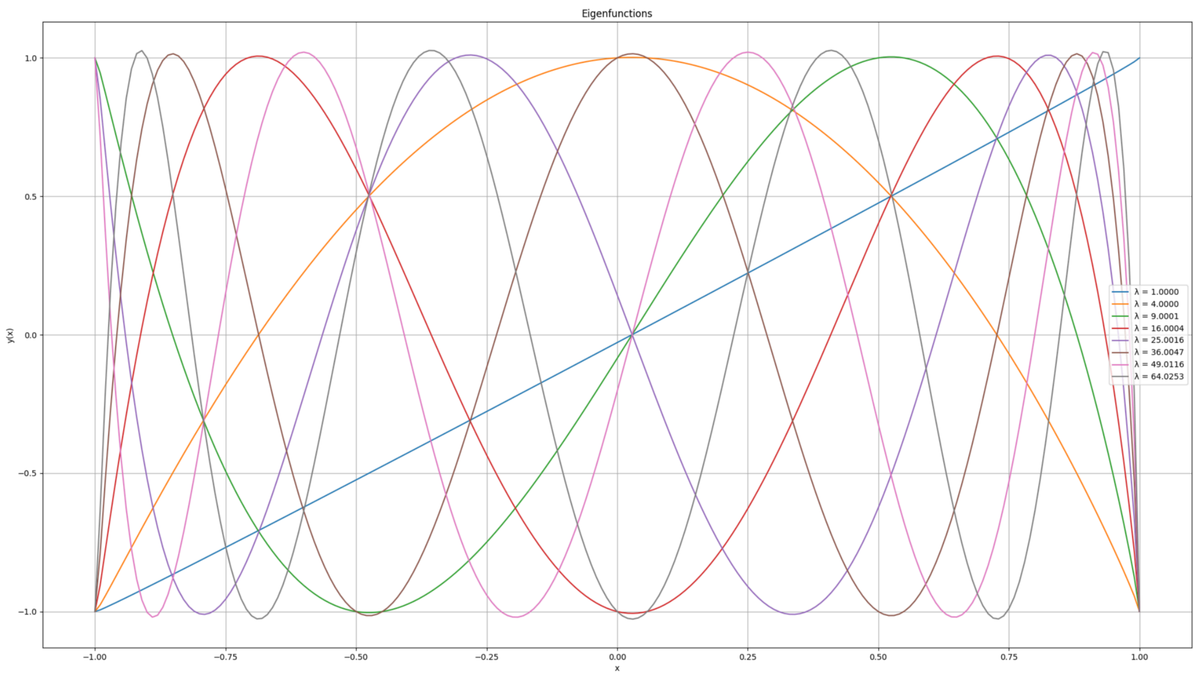
then we calcualte $\lambda_n$ using newton's iteration

$$\lambda_{n+1} = \lambda_n - \frac{F(\lambda)}{F'(\lambda)} \tag{18}$$

and find eigenfunction as linear combination

$$\phi(x) = c_1 u_1(x) + c_2 u_2(x) \tag{19}$$

and since eigenfunction of chebyshev's differential is same as chebyshev's polynomial we get the following result




### *problem 2*

$$-\frac{1}{2}\left(\cos^4(x)\frac{d^2u}{dx^2}+\frac{\cos^3(x)\cos(2x)}{\sin(x)}\frac{du}{dx}\right)+\left(\frac{m^2\cos^2(x)}{2\sin^2(x)}-\frac{\cos(x)}{\sin(x)}\right)u=\lambda u$$
$$u(0)=0,\quad u\left(\frac{\pi}{2}\right)=0 \tag{20}$$

### *Solution*

firstly, we rewrite it in first order form

$$\begin{aligned} v(x) &= \frac{du}{dx} \\ v'(x) &= \frac{d^2u}{dx^2} \end{aligned} \tag{21}$$

thus we have

$$-\frac{1}{2}\left(\cos^4(x)\frac{dv}{dx}+\frac{\cos^3(x)\cos(2x)}{\sin(x)}v\right)+\left(\frac{m^2\cos^2(x)}{2\sin^2(x)}-\frac{\cos(x)}{\sin(x)}\right)u=\lambda u \tag{22}$$

$\Rightarrow$

$$\begin{aligned} \frac{du}{dx} &= v \\ \frac{dv}{dx} &= -\frac{2}{\cos^4(x)}\left(\lambda u+\frac{\cos^3(x)\cos(2x)}{2\sin(x)}v-\left(\frac{m^2\cos^2(x)}{2\sin^2(x)}-\frac{\cos(x)}{\sin(x)}\right)u\right) \end{aligned} \tag{23}$$

then rewrite in matrix form like equation (9)

$$
\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-2}{\cos^4(x)}\left(\lambda - \frac{m^2\cos^2(x)}{2\sin^2(x)} + \frac{\cos(x)}{\sin(x)}\right) & -\frac{\cos(2x)}{\cos(x)\sin(x)} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{24}
$$

replace $\cos(x)$ and $\sin(x)$ by its taylor function approximations to avoid boundary value errors.

$$
\begin{aligned}
\cos(x) &\approx 1 - \frac{x^2}{2} + \frac{x^4}{24} \\
\sin(x) &\approx x - \frac{x^3}{6} + \frac{x^5}{120}
\end{aligned} \tag{25}
$$

then we have

$$
\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-2}{\left(1-\frac{x^2}{2}+\frac{x^4}{24}\right)^4}\left(\lambda - \frac{m^2\left(1-\frac{x^2}{2}+\frac{x^4}{24}\right)^2}{2\left(x-\frac{x^3}{6}+\frac{x^5}{120}\right)^2} + \frac{1-\frac{x^2}{2}+\frac{x^4}{24}}{x-\frac{x^3}{6}+\frac{x^5}{120}}\right) & -\frac{1-2x^2+\frac{4x^4}{6}}{\left(1-\frac{x^2}{2}+\frac{x^4}{24}\right)\left(x-\frac{x^3}{6}+\frac{x^5}{120}\right)} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{26}
$$

we use implicit Runge Kutta formulas with butcher table.

$$
\left[\begin{array}{c|cc} c_1 & a_{11} & a_{12} \\ c_2 & a_{21} & a_{22} \\ \hline & b_1 & b_2 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c|c} \frac{1}{2} & \frac{1}{2} \\ \hline & 1 \end{array}\right] \tag{27}
$$

where for first table coefficients are from 2-stage 4-th order Gauss-Legendre method.

$$
\begin{aligned}
&c_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}, \quad c_2 = \frac{1}{2} + \frac{\sqrt{3}}{6} \\
&a_{11} = \frac{1}{4},\ a_{12} = \frac{1}{4} - \frac{\sqrt{3}}{6},\ a_{21} = \frac{1}{4} + \frac{\sqrt{3}}{6},\ a_{22} = \frac{1}{4} \\
&b_1 = \frac{1}{2}, \quad b_2 = \frac{1}{2}
\end{aligned} \tag{28}
$$

and for second case 1-stage second order Gauss-Legendre method.
some basic notations

$$
Y = \begin{bmatrix} u \\ v \end{bmatrix} \quad A(x) = \begin{bmatrix} 0 & 1 \\ \frac{-2}{\left(1-\frac{x^2}{2}+\frac{x^4}{24}\right)^4}\left(\lambda - \frac{m^2\left(1-\frac{x^2}{2}+\frac{x^4}{24}\right)^2}{2\left(x-\frac{x^3}{6}+\frac{x^5}{120}\right)^2} + \frac{1-\frac{x^2}{2}+\frac{x^4}{24}}{x-\frac{x^3}{6}+\frac{x^5}{120}}\right) & -\frac{1-2x^2+\frac{4x^4}{6}}{\left(1-\frac{x^2}{2}+\frac{x^4}{24}\right)\left(x-\frac{x^3}{6}+\frac{x^5}{120}\right)} \end{bmatrix} \tag{29}
$$

with our problem

$$
Y' = A(x)Y \tag{30}
$$

for each stage $K_1, K_2$ general formula is

$$
K_j = A(x_n + c_j h)\left(Y_n + h\sum_{k=1}^{2} a_{jk}K_k\right) \tag{31}
$$

We solve the resulting nonlinear system for $K = [K_1, K_2]$, minimize the residual

$$R_j(K) = K_j - A\big(x_n + c_j h\big)\left(Y_n + h\sum_{k=1}^{2} a_{jk}K_k\right) = 0 \tag{32}$$

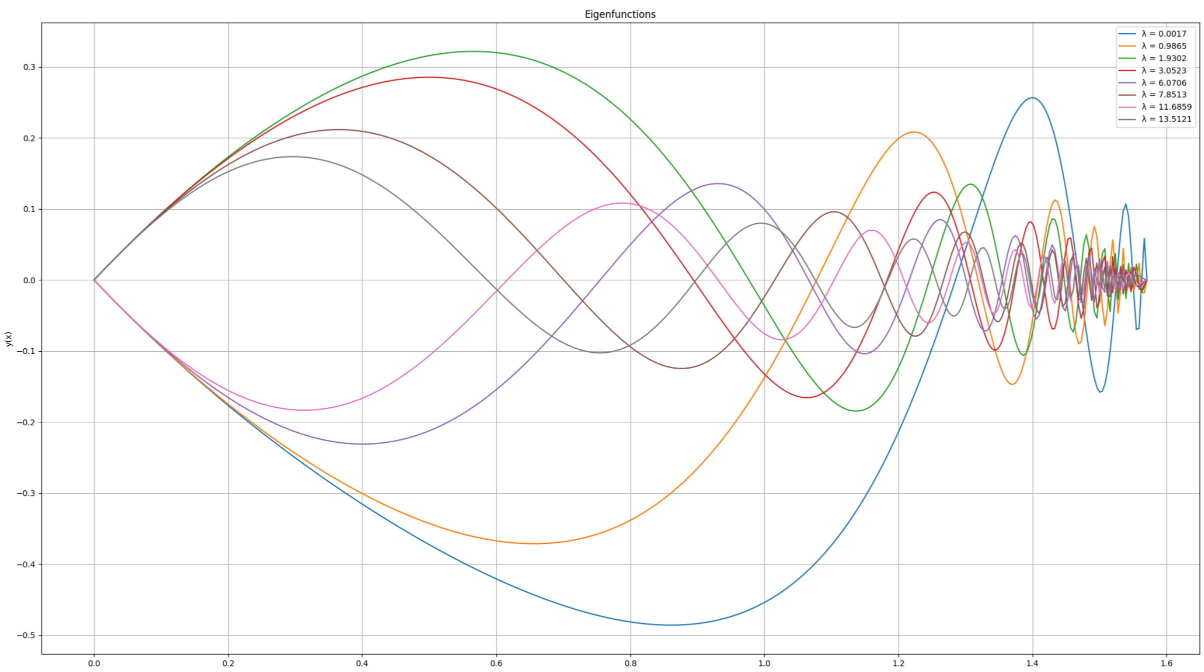
we use Newton's method for that

$$K^{n+1} = K^n - J^{-1}R(K^n) \tag{33}$$

where $J$ is jacobian of $R(k)$ and we update the value

$$Y_{n+1} = Y_n + h(b_1K_1 + b_2K_2) \tag{34}$$

after rk4 and rk2 calculations we follow same as in the first example from equations $(17) - (18)$. ad we get following results.




## ***drawbacks***

- initial guess of lambda being far from the root results in Newton's method not converging