Projectile Motion

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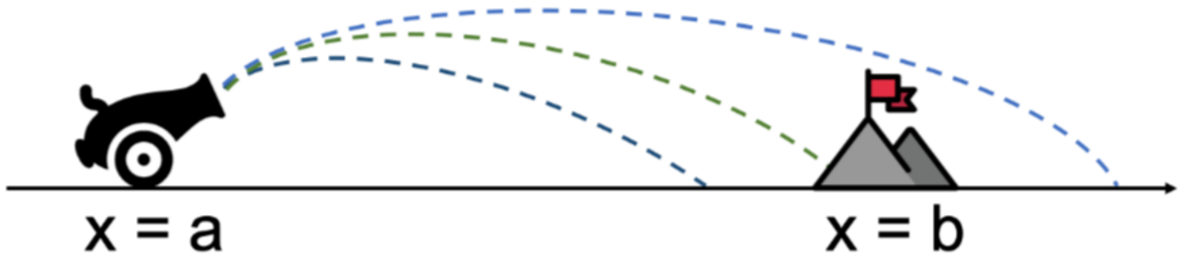
Introduction



Projectile Motion give as Second order ODE

$$\frac{d^2y}{dt^2} = -g \tag{1}$$

where
- $g = 9.81$
- $y(t)$ is the position of the ball

initial conditions $y(a) = a_0, \ y(b) = b_0$

Solution

we solve by shooting method. first transform the ODE into first order

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -g \end{aligned} \tag{2}$$

written in vector form

$$\begin{aligned} S(t) &= \begin{bmatrix} y(t) \\ v(t) \end{bmatrix} \\ \frac{dS(t)}{dt} &= \begin{bmatrix} 0 & -1 \\ 0 & -g/v \end{bmatrix} S(t) \end{aligned} \tag{3}$$

```python
def F(t, s):
  return np.dot(np.array([[0, 1], [0, -9.8 / s[1]]]), s)
```

we use RK4 and Euler to find the optimal velocity in shooting method
- formulation of RK4

$$S_{n+1} = S_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{4}$$

$$
\begin{aligned}
k_1 &= F(t_n, S_n) \\
k_2 &= F\left(t_n + \frac{h}{2}, S_n + \frac{h}{2} k_1\right) \\
k_3 &= F\left(t_n + \frac{h}{2}, S_n + \frac{h}{2} k_2\right) \\
k_4 &= F(t_n + h, S_n + h k_3)
\end{aligned}
\tag{5}
$$

- formulation of Euler

$$
S_{n+1} = S_n + hF(t_n, S_n) \tag{6}
$$

where

- $F(t, S)$ is the system of equations defined as $F(t, s) = \frac{dS}{dt}$
- h is a step size
- t_n is the current time
- S_n is the state vector at time t_n

analysis and comparison of methods

- Rk4 output

```
target coordinates: 618, 375
last trajectory point: 618.0, 375.0000000000109
target coordinates: 229, 387
last trajectory point: 229.0, 387.00000000000307
target coordinates: 487, 377
last trajectory point: 487.0, 377.0000000000409
target coordinates: 356, 379
last trajectory point: 356.0, 379.0000000000023
target coordinates: 91, 379
last trajectory point: 91.0, 378.99999999999454
```

- Euler output

```
target coordinates: 618, 375
last trajectory point: 618.0, 375.0000000000582
target coordinates: 229, 387
last trajectory point: 229.0, 386.99999999998545
target coordinates: 487, 377
last trajectory point: 487.0, 376.9999999998545
target coordinates: 356, 379
last trajectory point: 356.0, 378.9999999997599
target coordinates: 91, 379
last trajectory point: 91.0, 379.0000000000025
```

- Rk4 method achieves extremely high accuracy, with errors on the order of 10^{-10} to 10^{-12}.
  - The errors are consistently small across all target coordinates.
- Euler method also reaches target coordinates with error on the order of 10^{-10} to 10^{-12} aswell.
  - The errors on the other hand result in less consistent form. (i.e 376.9999999998545 vs 377.0000000000409 of Rk4)
  - Euler on the other hand is computationally much cheaper and therefore faster than Rk4

References

heplful article
https://pythonnumericalmethods.studentorg.berkeley.edu/notebooks/chapter23.02-The-Shooting-Method.html