Sturm-Lioville Problem

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Introduction

General form of Sturm-Liouville problem

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + q(x)\phi = -\lambda\omega(x)\phi \quad \text{for } a \leq x \leq b \tag{1}$$

where
- $p(x), q(x)$ and $\omega(x)$ are continuous.
- $\phi(x)$ is an eigen function.

problem 1

consider the following chebyshev's differential equation

$$(1-x^2)\phi'' - x\phi' = -\lambda\phi \quad \text{for} -1 < x < 1 \tag{2}$$

with initial conditions $\phi(-1) = 0,\ \phi(1) = 0$.
to rewrite in Sturm-Lioville form first we divide by $\sqrt{1-x^2}$ to get

$$\underbrace{\sqrt{1-x^2}\phi'' - \frac{x}{\sqrt{1-x^2}}\phi'}_{\left(\sqrt{1-x^2}\phi\prime\right)'} = -\frac{\lambda}{\sqrt{1-x^2}}\phi \tag{3}$$

with $p(x) = \sqrt{1-x^2}$, $q(x) = 0$ and $\omega(x) = \frac{1}{\sqrt{1-x^2}}$ so we have

$$\frac{d}{dx}\left[\sqrt{1-x^2}\ \frac{d\phi}{dx}\right] = -\lambda\frac{1}{\sqrt{1-x^2}}\phi \quad \text{for} -1 < x < 1 \tag{4}$$

Solution

we will use shooting method to solve chebyshev's differential equation. general approach to rewrite second order into first order ODE.

$$u(x) = \phi(x) \tag{5}$$

$$v(x) = \phi(x) \tag{6}$$

$$u'(x) = v(x) \tag{7}$$

$$v'(x) = \frac{p'(x)v + q(x)u + \lambda\omega(x)u}{p(x)} \tag{8}$$

in vector form

$$
\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-q(x)+\lambda w(x)}{p(x)} & \frac{p'(x)}{p(x)} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{9}
$$

in our case
for first initial value problem

$$
{u_1}'(x) = v_1(x) \tag{10}
$$

$$
\begin{aligned} {v_1}'(x) &= \frac{-\frac{x}{\sqrt{1-x^2}} v_1(x) + \frac{\lambda}{\sqrt{1-x^2}} u_1(x)}{\sqrt{1-x^2}} \\ &= \frac{-x v_1(x) + \lambda u_1(x)}{1-x^2} \end{aligned} \tag{11}
$$

$$
u_1(-1) = 1, v_1(-1) = 0 \tag{12}
$$

for second initial value problem

$$
{u_2}'(x) = v_2(x) \tag{13}
$$

$$
\begin{aligned} {v_2}'(x) &= \frac{-\frac{x}{\sqrt{1-x^2}} v_2(x) + \frac{\lambda}{\sqrt{1-x^2}} u_2(x)}{\sqrt{1-x^2}} \\ &= \frac{-x v_2(x) + \lambda u_2(x)}{1-x^2} \end{aligned} \tag{14}
$$

$$
u_2(-1) = 0, v_2(-1) = 1 \tag{15}
$$

we find appropriate $\alpha_1, \alpha_2, \beta_1, \beta_2$ for Robin boundary conditions

$$
\begin{aligned} \alpha_1 \phi(-1) + \beta_1 \phi'(-1) = 0 \\ \alpha_2 \phi(1) + \beta_2 \phi'(1) = 0 \end{aligned} \tag{16}
$$

determinant function

$$
F(\lambda) = \det \begin{bmatrix} \alpha_1 u_1(-1) + \beta_1 v_1(-1) & \alpha_1 u_2(-1) + \beta_1 v_2(-1) \\ \alpha_2 u_1(1) + \beta_2 v_1(1) & \alpha_2 u_2(1) + \beta_2 v_2(1) \end{bmatrix} \tag{17}
$$

then we calcualte λ_n using newton's iteration

$$
\lambda_{n+1} = \lambda_n - \frac{F(\lambda)}{F'(\lambda)} \tag{18}
$$

and find eigenfunction as linear combination

$$
\phi(x) = c_1 u_1(x) + c_2 u_2(x) \tag{19}
$$

and since eigenfunction of chebyshev's differential is same as chebyshev's polynomial we get the following result

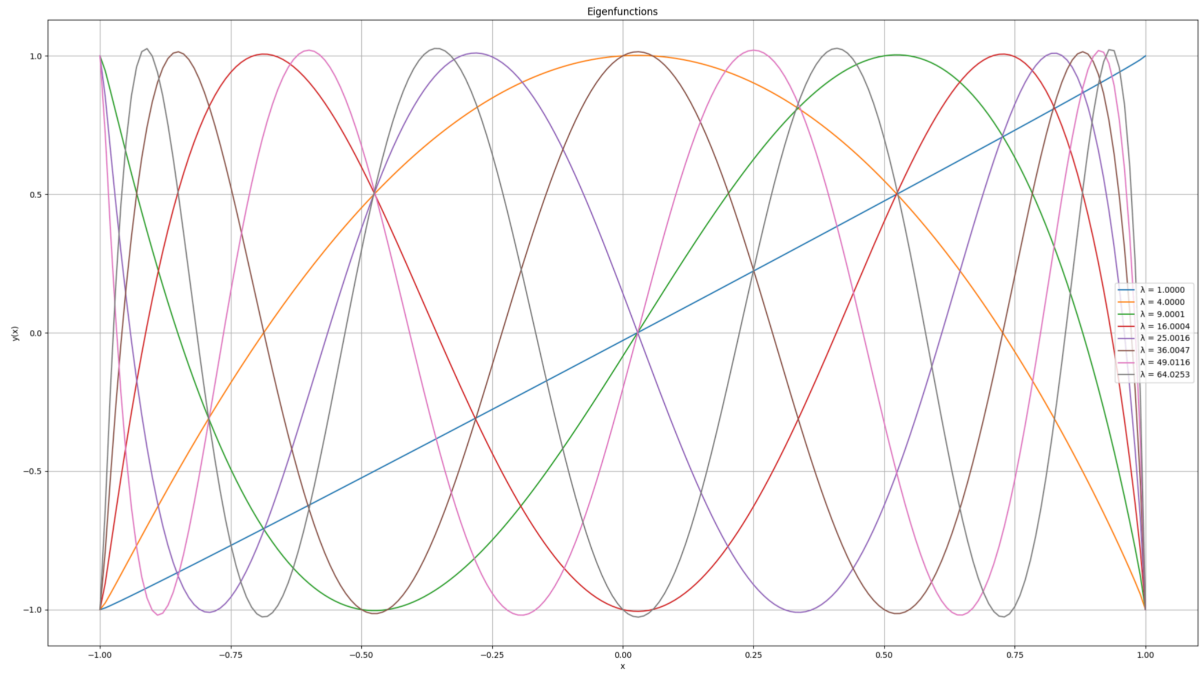

Figure 1: Gauss-Legendre 4th order

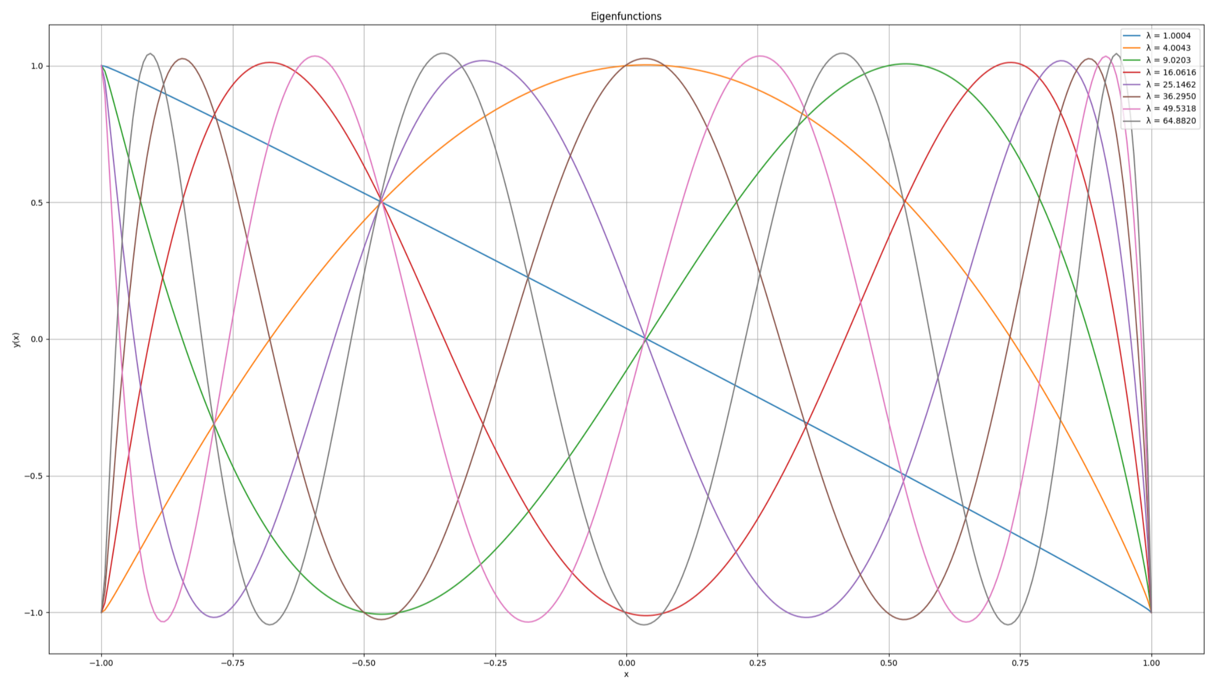

Figure 2: Gauss-Legendre second order

chebyshev differential equation is a popular problem with its lambda values being

$$\lambda = n^2 \quad n \in \mathbb{N} \tag{20}$$

as we see from the plots the lambda values on 4th order have better approximation resulting higher accuracy compared to Gauss-Legendre second order. keeping note that both 4th and second order are ***A*-stable**

problem 2

$$-\frac{1}{2}\left(\cos^4(x)\frac{d^2u}{dx^2}+\frac{\cos^3(x)\cos(2x)}{\sin(x)}\frac{du}{dx}\right)+\left(\frac{m^2\cos^2(x)}{2\sin^2(x)}-\frac{\cos(x)}{\sin(x)}\right)u=\lambda u$$
$$u(0)=0,\quad u\left(\frac{\pi}{2}\right)=0 \tag{21}$$

Solution

firstly, we rewrite it in first order form

$$\begin{aligned} v(x) &= \frac{du}{dx} \\ v'(x) &= \frac{d^2u}{dx^2} \end{aligned} \tag{22}$$

thus we have

$$-\frac{1}{2}\left(\cos^4(x)\frac{dv}{dx}+\frac{\cos^3(x)\cos(2x)}{\sin(x)}v\right)+\left(\frac{m^2\cos^2(x)}{2\sin^2(x)}-\frac{\cos(x)}{\sin(x)}\right)u=\lambda u \tag{23}$$

$\Rightarrow$

$$\begin{aligned} \frac{du}{dx} &= v \\ \frac{dv}{dx} &= -\frac{2}{\cos^4(x)}\left(\lambda u+\frac{\cos^3(x)\cos(2x)}{2\sin(x)}v-\left(\frac{m^2\cos^2(x)}{2\sin^2(x)}-\frac{\cos(x)}{\sin(x)}\right)u\right) \end{aligned} \tag{24}$$

then rewrite in matrix form like equation (9)

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-2}{\cos^4(x)}\left(\lambda-\frac{m^2\cos^2(x)}{2\sin^2(x)}+\frac{\cos(x)}{\sin(x)}\right) & -\frac{\cos(2x)}{\cos(x)\sin(x)} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{25}$$

replace $\cos(x)$ and $\sin(x)$ by its taylor function approximations to avoid boundary value errors.

$$\begin{aligned} \cos(x) &\approx 1-\frac{x^2}{2}+\frac{x^4}{24} \\ \sin(x) &\approx x-\frac{x^3}{6}+\frac{x^5}{120} \end{aligned} \tag{26}$$

then we have

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-2}{\left(1-\frac{x^2}{2}+\frac{x^4}{24}\right)^4}\left(\lambda-\frac{m^2\left(1-\frac{x^2}{2}+\frac{x^4}{24}\right)^2}{2\left(x-\frac{x^3}{6}+\frac{x^5}{120}\right)^2}+\frac{1-\frac{x^2}{2}+\frac{x^4}{24}}{x-\frac{x^3}{6}+\frac{x^5}{120}}\right) & -\frac{1-2x^2+\frac{4x^4}{6}}{\left(1-\frac{x^2}{2}+\frac{x^4}{24}\right)\left(x-\frac{x^3}{6}+\frac{x^5}{120}\right)} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{27}$$

we use implicit Runge Kutta formulas (Gauss-Legendre) both being ***A*-stable** with butcher table.

$$\left[\begin{array}{c|cc} c_1 & a_{11} & a_{12} \\ c_2 & a_{21} & a_{22} \\ \hline & b_1 & b_2 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c|c} \frac{1}{2} & \frac{1}{2} \\ \hline & 1 \end{array}\right] \tag{28}$$

where for first table coefficients are from 2-stage 4-th order Gauss-Legendre method.

$$
\begin{aligned}
&c_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}, \quad c_2 = \frac{1}{2} + \frac{\sqrt{3}}{6} \\
&a_{11} = \frac{1}{4},\ a_{12} = \frac{1}{4} - \frac{\sqrt{3}}{6},\ a_{21} = \frac{1}{4} + \frac{\sqrt{3}}{6},\ a_{22} = \frac{1}{4} \\
&b_1 = \frac{1}{2}, \quad b_2 = \frac{1}{2}
\end{aligned}
\tag{29}
$$

and for second case 1-stage second order Gauss-Legendre method.
some basic notations

$$
Y = \begin{bmatrix} u \\ v \end{bmatrix} \quad A(x) = \begin{bmatrix} 0 & 1 \\ \frac{-2}{\left(1-\frac{x^2}{2}+\frac{x^4}{24}\right)^4}\left(\lambda - \frac{m^2\left(1-\frac{x^2}{2}+\frac{x^4}{24}\right)^2}{2\left(x-\frac{x^3}{6}+\frac{x^5}{120}\right)^2} + \frac{1-\frac{x^2}{2}+\frac{x^4}{24}}{x-\frac{x^3}{6}+\frac{x^5}{120}}\right) & -\frac{1-2x^2+\frac{4x^4}{6}}{\left(1-\frac{x^2}{2}+\frac{x^4}{24}\right)\left(x-\frac{x^3}{6}+\frac{x^5}{120}\right)} \end{bmatrix} \tag{30}
$$

with our problem

$$
Y' = A(x)Y \tag{31}
$$

for each stage K_1, K_2 general formula is

$$
K_j = A(x_n + c_j h)\left(Y_n + h\sum_{k=1}^{2} a_{jk}K_k\right) \tag{32}
$$

We solve the resulting nonlinear system for $K = [K_1, K_2]$, minimize the residual

$$
R_j(K) = K_j - A(x_n + c_j h)\left(Y_n + h\sum_{k=1}^{2} a_{jk}K_k\right) = 0 \tag{33}
$$

we use Newton's method for that

$$
K^{n+1} = K^n - J^{-1}R(K^n) \tag{34}
$$

where J is jacobian of $R(k)$ and we update the value

$$
Y_{n+1} = Y_n + h(b_1K_1 + b_2K_2) \tag{35}
$$

after rk4 and rk2 calculations we follow same as in the first example from equations $(17) - (19)$. ad we get following results.

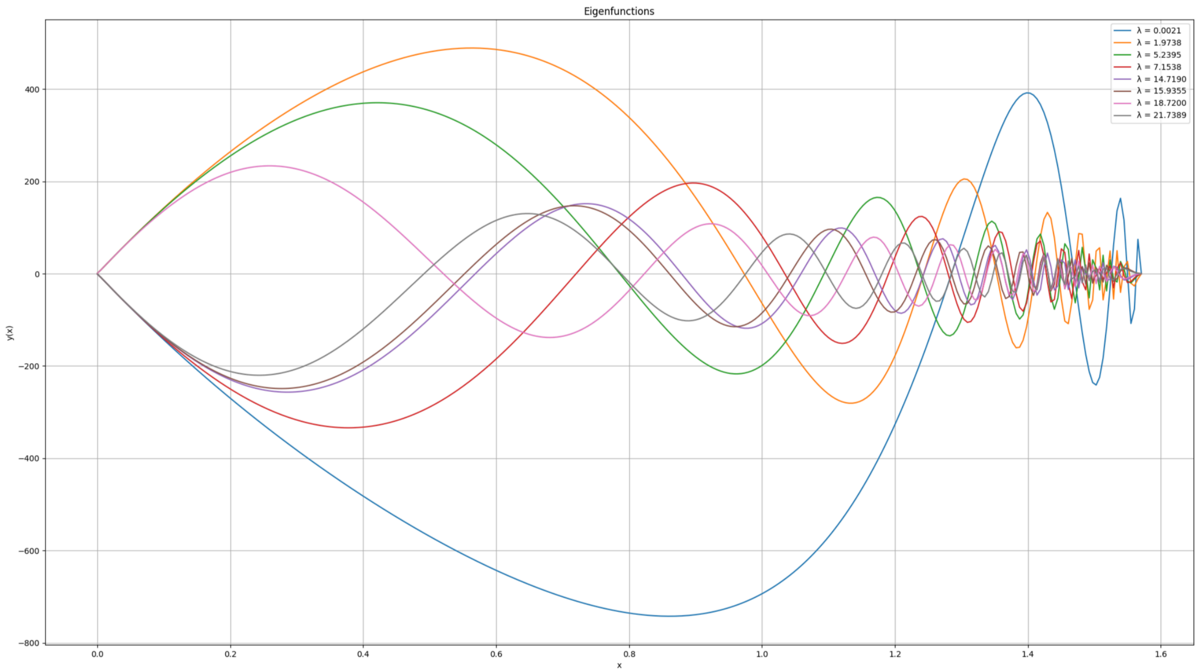

Figure 3: Gauss-Legendre 4th order

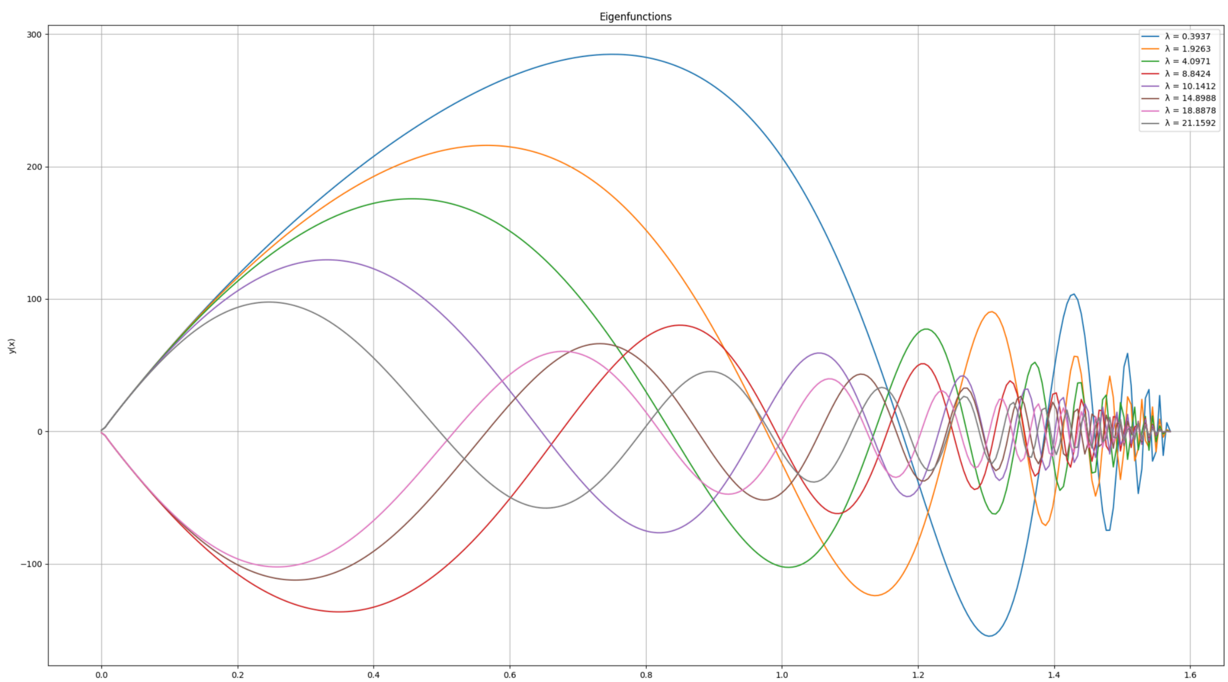

Figure 4: Gauss-Legendre Second order

Note: if $y(x)$ is an eigenfunction is Sturm-Lioville problem $-y(x)$ is also eigenfunction aswell since eigenvalue equation is linear and homogenous. $L[y(x)] = \lambda y(x)$

$$L[-y(x)] = -L[y(x)] = -\lambda y(x) = \lambda(-y(x)) \tag{36}$$

drawbacks

- initial guess of lambda being far from the root results in Newton's method not converging