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## ***Sturm-Liouville Problem***

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### ***Introduction***

General form of Sturm-Liouville problem

$$\frac{d}{dx} \left[ p(x) \frac{d\phi}{dx} \right] + q(x)\phi = -\lambda\omega(x)\phi \quad \text{for } a \leq x \leq b \quad (1)$$

where

- $p(x), q(x)$  and  $\omega(x)$  are continuous.
- $p(x), \omega(x) > 0$  on  $[a, b]$
- $\phi(x)$  is an eigen function.

#### ***problem 1***

consider the following chebyshev's differential equation

$$(1-x^2)\phi'' - x\phi' = -\lambda\phi \quad \text{for } -1 < x < 1 \quad (2)$$

with initial conditions  $\phi(-1) = 0, \phi(1) = 0$ .

to rewrite in Sturm-Liouville form first we divide by  $\sqrt{1-x^2}$  to get

$$\underbrace{\sqrt{1-x^2}\phi'' - \frac{x}{\sqrt{1-x^2}}\phi'}_{(\sqrt{1-x^2}\phi')'} = -\frac{\lambda}{\sqrt{1-x^2}}\phi \quad (3)$$

with  $p(x) = \sqrt{1-x^2}$ ,  $q(x) = 0$  and  $\omega(x) = \frac{1}{\sqrt{1-x^2}}$  so we have

$$\frac{d}{dx} \left[ \sqrt{1-x^2} \frac{d\phi}{dx} \right] = -\lambda \frac{1}{\sqrt{1-x^2}} \phi \quad \text{for } -1 < x < 1 \quad (4)$$

### ***Solution***

we will use shooting method to solve chebyshev's differential equation. general approach to rewrite second order into first order ODE.

$$u(x) = \phi(x) \quad (5)$$

$$v(x) = \phi'(x) \quad (6)$$

$$u'(x) = v(x) \quad (7)$$

$$v'(x) = \frac{p'(x)v + q(x)u + \lambda\omega(x)u}{p(x)} \quad (8)$$

in vector form

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-q(x)+\lambda w(x)}{p(x)} & \frac{p'(x)}{p(x)} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \quad (9)$$

in our case  
for first initial value problem

$$u_1'(x) = v_1(x) \quad (10)$$

$$\begin{aligned} v_1'(x) &= \frac{-\frac{x}{\sqrt{1-x^2}}v_1(x) + \frac{\lambda}{\sqrt{1-x^2}}u_1(x)}{\sqrt{1-x^2}} \\ &= \frac{-xv_1(x) + \lambda u_1(x)}{1-x^2} \end{aligned} \quad (11)$$

$$u_1(-1) = 1, v_1(-1) = 0 \quad (12)$$

for second initial value problem

$$u_2'(x) = v_2(x) \quad (13)$$

$$\begin{aligned} v_2'(x) &= \frac{-\frac{x}{\sqrt{1-x^2}}v_2(x) + \frac{\lambda}{\sqrt{1-x^2}}u_2(x)}{\sqrt{1-x^2}} \\ &= \frac{-xv_2(x) + \lambda u_2(x)}{1-x^2} \end{aligned} \quad (14)$$

$$u_2(-1) = 0, v_2(-1) = 1 \quad (15)$$

we find appropriate  $\alpha_1, \alpha_2, \beta_1, \beta_2$  for Robin boundary conditions

$$\begin{aligned} \alpha_1\phi(-1) + \beta_1\phi'(-1) &= 0 \\ \alpha_2\phi(1) + \beta_2\phi'(1) &= 0 \end{aligned} \quad (16)$$

determinant function

$$F(\lambda) = \det \begin{bmatrix} \alpha_1 u_1(-1) + \beta_1 v_1(-1) & \alpha_1 u_2(-1) + \beta_1 v_2(-1) \\ \alpha_2 u_1(1) + \beta_2 v_1(1) & \alpha_2 u_2(1) + \beta_2 v_2(1) \end{bmatrix} \quad (17)$$

then we calculate  $\lambda_n$  using newton's iteration

$$\lambda_{n+1} = \lambda_n - \frac{F(\lambda)}{F'(\lambda)} \quad (18)$$

and find eigenfunction as linear combination

$$\phi(x) = c_1 u_1(x) + c_2 u_2(x) \quad (19)$$

and since eigenfunction of chebyshev's differential is same as chebyshev's polynomial we get the following result



### problem 2

$$-\frac{1}{2} \left( \cos^4(x) \frac{d^2 u}{dx^2} + \frac{\cos^3(x) \cos(2x)}{\sin(x)} \frac{du}{dx} \right) + \left( \frac{m^2 \cos^2(x)}{2 \sin^2(x)} - \frac{\cos(x)}{\sin(x)} \right) u = \lambda u \quad (20)$$

$$u(0) = 0, \quad u\left(\frac{\pi}{2}\right) = 0$$

### Solution

firstly, we rewrite it in first order form

$$v(x) = \frac{du}{dx} \quad (21)$$

$$v'(x) = \frac{d^2 u}{dx^2}$$

thus we have

$$-\frac{1}{2} \left( \cos^4(x) \frac{dv}{dx} + \frac{\cos^3(x) \cos(2x)}{\sin(x)} v \right) + \left( \frac{m^2 \cos^2(x)}{2 \sin^2(x)} - \frac{\cos(x)}{\sin(x)} \right) u = \lambda u \quad (22)$$

$\Rightarrow$

$$\frac{du}{dx} = v \quad (23)$$

$$\frac{dv}{dx} = -\frac{2}{\cos^4(x)} \left( \lambda u + \frac{\cos^3(x) \cos(2x)}{2 \sin(x)} v - \left( \frac{m^2 \cos^2(x)}{2 \sin^2(x)} - \frac{\cos(x)}{\sin(x)} \right) u \right)$$

then rewrite in matrix form like equation (9)

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-2}{\cos^4(x)} \left( \lambda - \frac{m^2 \cos^2(x)}{2 \sin^2(x)} + \frac{\cos(x)}{\sin(x)} \right) & -\frac{\cos(2x)}{\cos(x) \sin(x)} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \quad (24)$$

replace  $\cos(x)$  and  $\sin(x)$  by its taylor function approximations to avoid boundary value errors.

$$\begin{aligned} \cos(x) &\approx 1 - \frac{x^2}{2} + \frac{x^4}{24} \\ \sin(x) &\approx x - \frac{x^3}{6} + \frac{x^5}{120} \end{aligned} \quad (25)$$

then we have

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-2}{\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)^4} \left( \lambda - \frac{m^2 \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)^2}{2 \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)^2} + \frac{1 - \frac{x^2}{2} + \frac{x^4}{24}}{x - \frac{x^3}{6} + \frac{x^5}{120}} \right) & -\frac{1 - 2x^2 + \frac{4x^4}{6}}{\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \quad (26)$$

we use implicit Runge Kutta formulas with butcher table.

$$\left[ \begin{array}{c|cc} c_1 & a_{11} & a_{12} \\ c_2 & a_{21} & a_{22} \\ \hline & b_1 & b_2 \end{array} \right] \quad \text{and} \quad \left[ \begin{array}{c|c} \frac{1}{2} & \frac{1}{2} \\ \hline & 1 \end{array} \right] \quad (27)$$

where for first table coefficients are from 2-stage 4-th order Gauss-Legendre method.

$$\begin{aligned} c_1 &= \frac{1}{2} - \frac{\sqrt{3}}{6}, \quad c_2 = \frac{1}{2} + \frac{\sqrt{3}}{6} \\ a_{11} &= \frac{1}{4}, \quad a_{12} = \frac{1}{4} - \frac{\sqrt{3}}{6}, \quad a_{21} = \frac{1}{4} + \frac{\sqrt{3}}{6}, \quad a_{22} = \frac{1}{4} \\ b_1 &= \frac{1}{2}, \quad b_2 = \frac{1}{2} \end{aligned} \quad (28)$$

and for second case 1-stage second order Gauss-Legendre method.  
some basic notations

$$Y = \begin{bmatrix} u \\ v \end{bmatrix} \quad A(x) = \begin{bmatrix} 0 & 1 \\ \frac{-2}{\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)^4} \left( \lambda - \frac{m^2 \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)^2}{2 \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)^2} + \frac{1 - \frac{x^2}{2} + \frac{x^4}{24}}{x - \frac{x^3}{6} + \frac{x^5}{120}} \right) & -\frac{1 - 2x^2 + \frac{4x^4}{6}}{\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)} \end{bmatrix} \quad (29)$$

with our problem

$$Y' = A(x)Y \quad (30)$$

for each stage  $K_1, K_2$  general formula is

$$K_j = A(x_n + c_j h) \left( Y_n + h \sum_{k=1}^2 a_{jk} K_k \right) \quad (31)$$

We solve the resulting nonlinear system for  $K = [K_1, K_2]$ , minimize the residual

$$R_j(K) = K_j - A(x_n + c_j h) \left( Y_n + h \sum_{k=1}^2 a_{jk} K_k \right) = 0 \quad (32)$$

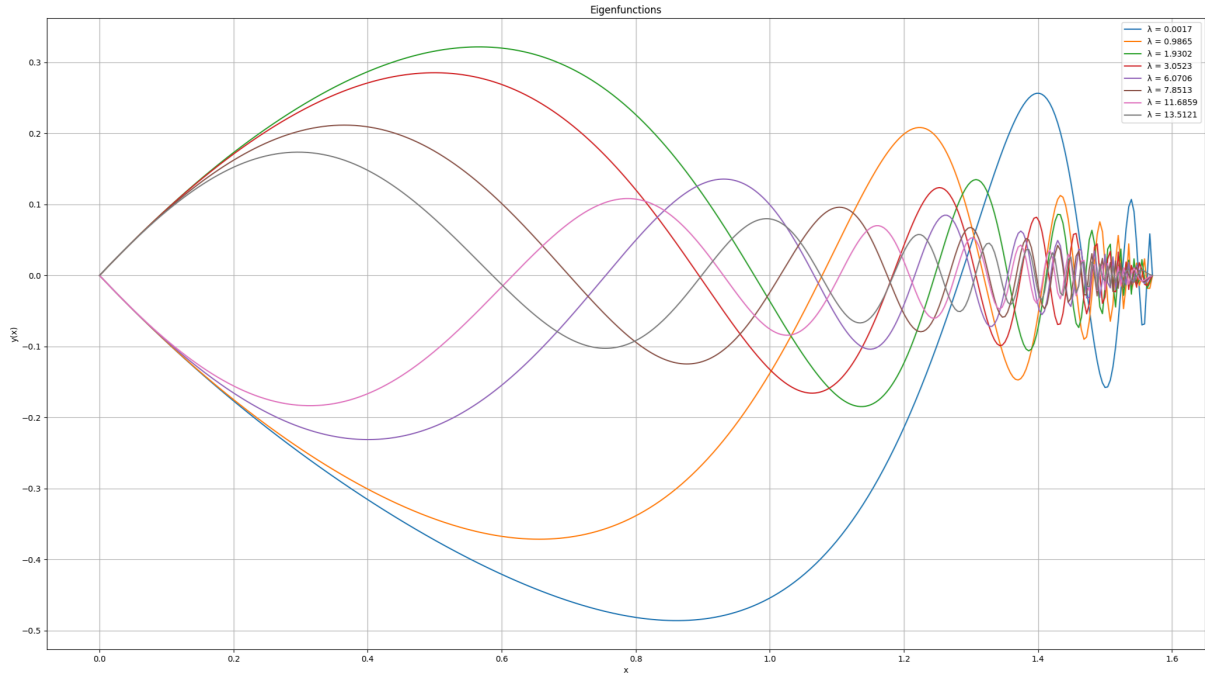
we use Newton's method for that

$$K^{n+1} = K^n - J^{-1} R(K^n) \quad (33)$$

where  $J$  is jacobian of  $R(k)$  and we update the value

$$Y_{n+1} = Y_n + h(b_1 K_1 + b_2 K_2) \quad (34)$$

after rk4 and rk2 calculations we follow same as in the first example from equations (17) – (18). and we get following results.



### **drawbacks**

- initial guess of lambda being far from the root results in Newton's method not converging