Lean course: Project 2

Kasimir Jabben

Main results

My topic was the formalisation of the Laurent series, in particular the convergence. The problem with showing convergence here is the following: one could just show abstract convergence of both of the Taylor series (one in z, one in z^{-1}), but this approach does not necessarily give convergence anywhere. We would get convergence of the first series for $\|z\| < R_2$ and of the second series for $R_1 < \|z\|$, but we would not know $R_1 < R_2$. I thought this is really not interesting so I went for the other option: Showing Laurent's theorem in full, i.e. that the series converges where f is analytic and is equal to $f(z)$. So this is the main statement at the end, which is called **Laurent_theorem**.

Theorem 1 (Laurent)**.** Let f be analytic on an annulus $R_1 < \|z - z_0\| < R_2$ centered at z_0, then we have for all z in the annulus

$$f(z) = \sum_{k \in \mathbb{Z}} c_k (z - z_0)^k$$

with

$$c_k := \frac{1}{2\pi i} \int_{C(z_0,r)} \frac{f(w)}{(w - z_0)^{k+1}} dw$$

where r is some number $R_1 < r < R_2$.

My proof of this theorem generally follows the book "A first course in Complex Analysis" by Matthias Beck, Gerald Marchesi, Dennis Pixton and Lucas Sabalka (which is accessible online). The proof follows roughly these steps:

- Construct the integration path as in the proof (**int_path_real** or **int_path_nonreal**, depending on $z \in \mathbb{R}$ or not)
- Observe by Cauchy's theorem that the integral over the path and the normal Cauchy integrand is $f(z)$ (**Application_Cauchy**)
- Split the path of integration into two circles (**Integral_decomposition**)
- Use the geometric series and Fubini's theorem to convert the integrals into sums over Laurent coefficients (**Outer_integral_to_Sum** and **Inner_integral_to_Sum**)
- Put everything together (**Laurent_theorem**)

Unfinished proofs

There are still a lot of sorry's in my project, a few of which are intentional and a few are due to time reasons (I started a bit late because I had a lot to do with other courses) most of which are analogous to some statements I have proven.

Intentional sorry's

I realised that I cannot use Cauchy's theorem, because Mathlib does not have the tools yet. There is a version for integrals over circles, but no general one. I think this is intended and one is supposed to show homotopy equivalence of the given path to a circle, but changing the path of integration via homotopy equivalence is also not yet in Mathlib. So I left out the proofs for **Application_Cauchy** and **Circle_path_shift** which both depend on this.
Also the definition **analytic_on_annulus** and the corresponding two lemmas below are not given because I do not really need this property for anything except the two statements using shift of the path of integration. The only other thing I need this for is technical stuff like integrability or convergence and for this I assumed **analytic_implies_cont** to get continuity from the function being analytic.

Unfinished due to time reasons

The lemma **Integral_decomposition** still contains a lot of sorry's, mostly some technical conditions like integrability. The first sorry is because both cases on z being real or not are essentially analogous.
The lemma **Outer_integral_to_Sum** still contains two sorry's, the first one is a technical condition for Fubini and the second one is convergence of a series, which is analogous to the convergence of a similar series I have shown above (see Summable_condition).
The lemma **Inner_integral_to_Sum** is completely missing a proof, but it is analogous to **Outer_integral_to_Sum**.

Reference

The book "A first course in Complex Analysis" by Matthias Beck, Gerald Marchesi, Dennis Pixton and Lucas Sabalka.

Usage of AI

I have used chatgpt during the project, but only for researching Mathlib statements or understanding error messages. In particular, there is no strip of code that is written by AI (The most I did was copy a one-line command like a refine).