Average-Cost Mean-Field Games: Forward and Inverse Perspectives

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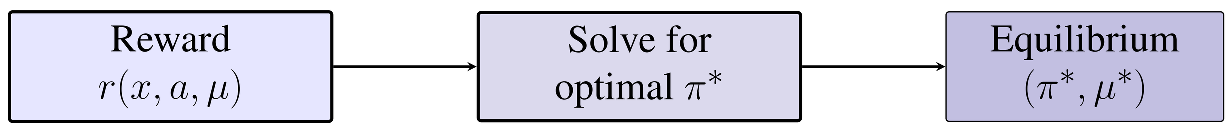
1 Forward Reinforcement Learning



Mean-field Term

Let $(x_{1,t},\dots,x_{N,t})$ be exchangeable. Then

$$\mu_t^N := \frac{1}{N}\sum_{j=1}^{N}\delta_{x_{j,t}} \;\to\; \mu_t \quad \text{(a.s.) as } N\to\infty,$$

under Assumption (1). And conditioned on μ_t, agents are independent.

Representative-Agent Problem (Average Cost)

Fix a stationary population law μ (or a stationary flow $\{\mu_t\}$ that has converged). The representative agent chooses a stationary policy π to minimise the long-run average cost

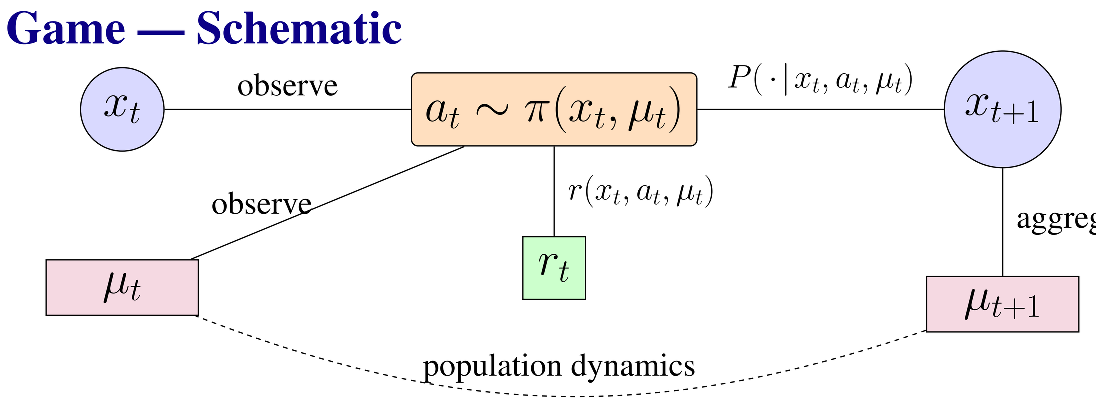
$$V_\mu(\pi,\mu_0) = \limsup_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}^{\pi}\Big[r(x_t,\ a_t,\ \mu)\Big] \quad \textit{where} \quad x_0\sim\mu_0.$$

Here, dynamics follows $x_{t+1}\sim P(\cdot\mid x_t,a_t,\mu),\ \ a_t\sim\pi(\cdot\mid x_t)$. Now define

$$\rho(\mu) := \max_{\pi} V_\mu(\pi,\mu_0).$$

This criterion will serve as our reference for optimality.

Mean-Field Game — Schematic



State $X_t\in S$ (discrete)
Population $\mu_t\in\mathcal{P}(S)$ empirical distribution of states
Action $A_t\in\mathcal{A}(X_t)$ chosen by policy $\pi(\,\cdot\,|X_t,\mu_t)$
Reward $R_t = r(X_t,A_t,\mu_t)$
Dynamics $X_{t+1}\sim P(\cdot\mid X_t,A_t,\mu_t)$
Coupling: each agent uses (X_t,μ_t)*, while* μ_t *aggregates everyone's states.*

2 Mean-Field Average-Cost Soft Bellman Equation

Mean-Field Soft Average-Cost Bellman

For a fixed population law μ and parameter $\delta>0$:

$$\rho(\mu)+h(x;\mu) \;=\; \max_{\pi(\cdot|x)}\Big\{\sum_a \pi(a\mid x)\Big[r(x,a;\mu)+\sum_y P(y\mid x,a;\mu)\,h(y;\mu)\Big]+\delta\,H(\pi(\cdot\mid x))\Big\},$$

where $H(\pi) := -\sum_a \pi(a)\log\pi(a)$.

Softmax Closed Form

$$\rho(\mu)+h(x;\mu) \;=\; \delta\,\log\sum_a \exp\Big(\tfrac{1}{\delta}\Big[r(x,a;\mu)+\sum_y P(y\mid x,a;\mu)\,h(y;\mu)\Big]\Big).$$

Optimal Stochastic Policy

$$\pi^*(a\mid x;\mu) \;=\; \frac{\exp\Big(\frac{1}{\delta}\Big[r(x,a;\mu)+\sum_y P(y\mid x,a;\mu)\,h(y;\mu)\Big]\Big)}{\sum_{a'}\exp\Big(\frac{1}{\delta}\Big[r(x,a';\mu)+\sum_y P(y\mid x,a';\mu)\,h(y;\mu)\Big]\Big)}.$$

3 Fixed-Point Iteration for Mean-Field Equilibrium

(1) Best-Response / Softmax Policy For a population measure μ and entropy weight $\delta>0$, define

$$\begin{aligned}\pi_\mu(\cdot\mid x) &= \operatorname{softmax}_{\frac{1}{\delta}}\big(Q_\mu(x,\cdot)\big)\\ &= \arg\max_{\pi(\cdot|x)\in\Delta(\mathcal{A})}\Big[\sum_{a\in\mathcal{A}}\pi(a\mid x)\big(r(x,a;\mu)+\sum_{y\in\mathcal{X}}P(y\mid x,a;\mu)\,h_\mu(y)\big)+\delta\,H\big(\pi(\cdot\mid x)\big)\Big]\end{aligned}$$

with h_μ solving the average-cost Bellman equation under μ.

(2) Population-Update Map Using π_μ, propagate the population forward:

$$\mu'(y) = \sum_{x\in\mathcal{X}}\sum_{a\in\mathcal{A}}P\big(y\mid x,a,\mu\big)\,\pi_\mu(a\mid x)\,\mu(x), \qquad y\in\mathcal{X}.$$

A *mean-field equilibrium* is a fixed point $\mu^*=\mu'$ of this two-step operator.

Corollary (Policy Lipschitz)

For the soft-max policy

$$\pi_\mu(a\mid x)\ \propto\ \exp\!\big(Q^*_\mu(x,a)/\tau\big),$$

there exists a constant $C_\pi>0$ such that

$$\|\pi_\mu-\pi_{\mu'}\|_{\ell^1}\ \le\ C_\pi\,\|\mu-\mu'\|_{\ell^1}.$$

Algorithm 1 Contraction Fixed-Point Iteration for Mean-Field Equilibrium

Require: Initial measure $\mu_0\in\mathcal{P}(\mathcal{X})$, temperature $\delta>0$, tolerance $\varepsilon>0$

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μ ← μ0
repeat
    h ← BellmanSolve(μ)                      // span-contraction step
    π ← Softmax_{1/δ}(h)
    μ' ← PopUpdate(μ, π)
    Δ ← ‖μ' − μ‖_1;   μ ← μ'
until Δ ≤ ε
return Equilibrium measure μ and policy π
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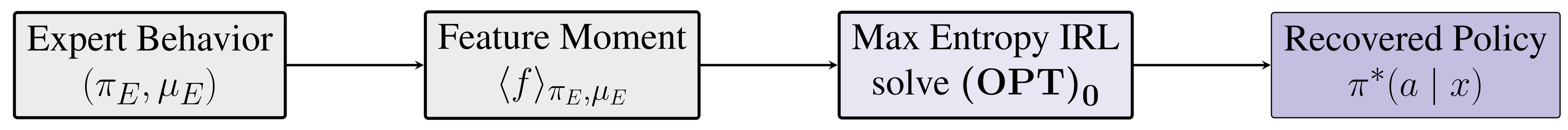
1 Inverse Reinforcement Learning



A widely used assumption in IRL for MFGs is that the reward function can be expressed as a linear combination of feature vectors, which depend on the state, action, and mean-field term:

$$\mathcal{R} := \Big\{r(x,a,\mu) = \langle\theta, f(x,a,\mu)\rangle\ \mid\ \theta\in\mathbb{R}^k,\ f:\mathcal{X}\times\mathcal{A}\times\mathcal{P}(\mathcal{X})\to\mathbb{R}^k\Big\}$$

where $f(x,a,\mu)\in\mathbb{R}^k$ is the feature vector that includes information about the state-action pair (x,a) and the mean-field term μ.

Assumption 1 *We assume that for any policy* π *and mean-field term* μ*, the state process is positive Harris recurrent and aperiodic, guaranteeing the convergence of empirical averages [1, Thm. 13.3.3]. In other words, the state process exhibits ergodic behavior under any policy and mean-field term, ensuring long-run statistical stability. Assuming the expert population distribution* μ_E *is known is standard, as it can be consistently estimated from long-run trajectories. [1, Thm. 13.3.3]*

Definition 1 *average-reward expected feature vector under the mean-field equilibrium* (π_E,μ_E) *for* $x_0\sim\mu_E$ *is defined as,*

$$\langle f\rangle_{\pi_E,\mu_E} := \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}^{\pi_E,\mu_E}\left[f(x_t,a_t,\mu_E)\right]. \tag{1}$$

2 Entropy-Regularized Inverse Learning

Definition 2 *Average-reward causal entropy* $H(\pi)$ *of the policy* $\pi\in\Pi$ *is defined as follows*

$$H(\pi) := \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}^{\pi,\mu_E}\left[-\log\pi(a_t\,|\,x_t)\right].$$

This principle selects the least biased policy among those consistent with expert demonstrations [2] . We define the average-cost maximum causal entropy IRL problem:

$$\begin{aligned}(\mathbf{OPT})_0\ \ \text{maximize}_\pi\ \ & H(\pi)\\ \text{subject to}\ \ & \pi(a\mid x)\ge 0\ \forall(x,a)\in\mathcal{X}\times\mathcal{A}\\ & \textstyle\sum_{a\in\mathcal{A}}\pi(a\mid x)=1\ \forall x\in\mathcal{X}\\ & \mu_E(x)=\textstyle\sum_{(a,y)\in\mathcal{A}\times\mathcal{X}}p(x\mid y,a,\mu_E)\,\pi(a\mid y)\,\mu_E(y)\ \forall x\in\mathcal{X}\\ & \lim_{T\to\infty}\tfrac{1}{T}\textstyle\sum_{t=0}^{T-1}\ \mathbb{E}^{\pi,\mu_E}[f(x_t,a_t,\mu_E)]=\langle f\rangle_{\pi_E,\mu_E}\end{aligned}$$

3 Convex Reformulation via Occupation Measures

Definition 3 *We define the state-action occupation measure* ν_π *for any policy* $\pi\in\Pi$ *and corresponding state occupation measure as*

$$\nu_\pi(x,a) := \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}^{\pi,\mu_E}\left[\mathbf{1}_{\{(x_t,a_t)=(x,a)\}}\right] \qquad \nu_\pi^{\mathcal{X}}(x) := \sum_{a\in\mathcal{A}}\nu_\pi(x,a).$$

Under ergodicity, ν_π *defines a valid probability measure over* $\mathcal{X}\times\mathcal{A}$*.*

Lemma 1 *Suppose* $\pi\in\Pi$ *is a feasible point for the optimization problem* (OPT)*. Then:*

1. ***State Marginals:*** $\nu_\pi^{\mathcal{X}}(x)=\mu_E(x)=\sum_{(y,a)}p(x\mid y,a,\mu_E)\,\nu_\pi(y,a)$ *for all* $x\in\mathcal{X}$*.*
2. ***Entropy:*** $H(\pi)=\sum_{(x,a)}-\log\left(\frac{\nu_\pi(x,a)}{\mu_E(x)}\right)\nu_\pi(x,a)$.
3. ***Feature Expectation:*** $\langle f\rangle_{\pi,\mu_E}=\sum_{(x,a)}f(x,a,\mu_E)\,\nu_\pi(x,a)$.

In view of above results, let us now reformulate $(\mathbf{OPT})_0$ in terms of state-action occupation measure [3] . Although $(\mathbf{OPT})_0$ is nonconvex, it can be reformulated as an equivalent convex program over occupation measures. This new formulation is denoted by $(\widehat{\mathbf{OPT}})$, which is convex and yield the same solution as $(\mathbf{OPT})_0$:

$$\begin{aligned}(\widehat{\mathbf{OPT}})\ \ \text{maximize}_\nu\ \ & \textstyle\sum_{(x,a)\in\mathcal{X}\times\mathcal{A}}-\log\left(\frac{\nu(x,a)}{\mu_E(x)}\right)\ \nu(x,a)\\ \text{subject to}\ \ & \textstyle\sum_{(x,a)\in\mathcal{X}\times\mathcal{A}}f(x,a,\mu_E)\,\nu(x,a)=\langle f\rangle_{\pi_E,\mu_E}\\ & \mu_E(x)=\textstyle\sum_{(y,a)\in\mathcal{X}\times\mathcal{A}}p(x\mid y,a,\mu_E)\,\nu(y,a)\ \forall x\in\mathcal{X}\\ & \nu^{\mathcal{X}}(x)=\mu_E(x)\ \forall x\in\mathcal{X}\\ & \nu(x,a)\ge 0\ \forall(x,a)\in\mathcal{X}\times\mathcal{A}.\end{aligned}$$

The optimization problem $(\widehat{\mathbf{OPT}})$ is convex: its objective combines a strongly concave entropy term with a linear component, and all constraints are linear in the occupation measure ν. Using convex duality [4] and Sion's minimax theorem [5], the problem $(\widehat{\mathbf{OPT}})$ admits the dual formulation:

$$\min_{\boldsymbol{\alpha}\in\mathbb{R}^k\boldsymbol{\beta},\boldsymbol{\theta}\in\mathbb{R}^{\mathcal{X}}}\Big\{\log\sum_{(x,a)\in\mathcal{X}\times\mathcal{A}}e^{k_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}(x,a)}-\langle\boldsymbol{\alpha},\ \langle f\rangle_{\pi_E,\mu_E}\rangle-\sum_{x\in\mathcal{X}}\boldsymbol{\theta}_x\,\mu_E(x)\Big\}$$

Here, $k_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}(x,a) := \langle\boldsymbol{\alpha},\ f(x,a,\mu_E)\rangle+\boldsymbol{\theta}_x+\sum_{z\in\mathcal{X}}\boldsymbol{\beta}_z\,(p(z\mid x,a,\mu_E)-\mu_E(z))$. For fixed dual variables, the inner maximization over ν is solved by the Boltzmann distribution, and the corresponding recovered policy is given by:

$$\nu^*(x,a) := \frac{e^{k_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}(x,a)}}{Z_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}}, \qquad \pi^*(a\mid x) := \frac{\nu^*(x,a)}{\mu_E(x)}$$

which uniquely maximizes the entropy-regularized objective. This yields a fully tractable dual with no duality gap.

Theorem 1 *The dual objective* h *is* L*-smooth and* $\rho(D)$*-strongly convex over any compact subset* $D\subset\mathbb{R}^m$*, where* $m=k+2|\mathcal{X}|$*. Specifically,*

$$L := 2M^2\sqrt{|\mathcal{X}|\cdot|\mathcal{A}|},$$

with M *a uniform bound on the gradients of the function* $k_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}}(x,a)$*. Moreover, if the set*

$$\{(f(x,a,\mu_E),\ p(\cdot\mid x,a,\mu_E),\ e(\cdot\mid x,a)) : (x,a)\in\mathcal{X}\times\mathcal{A}\}$$

spans $\mathbb{R}^k\times\mathbb{R}^{\mathcal{X}}\times\mathbb{R}^{\mathcal{X}}$*, then* h *is uniformly strongly convex on* D*.*

Now we can introduce the gradient descent algorithm for finding the minimizer of h as follows.

Algorithm 1 Gradient Descent for Parameter Optimization

Require: $(\boldsymbol{\alpha}_0,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0)\in\mathbb{R}^m$ and $\delta\in\left(0,\ \frac{1}{L}\right]$ and $K\in\mathbb{N}$

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Set (α, β, θ) ← (α0, β0, θ0)
for k = 0, 1, ..., K − 1 do
    (α, β, θ) ← (α, β, θ) − δ ∇h(α, β, θ)
end for
Compute ν*_{α,β,θ} from (α, β, θ)
return (α, β, θ) and ν*_{α,β,θ}
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References

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