Grover's Search Algorithm

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Problem Description

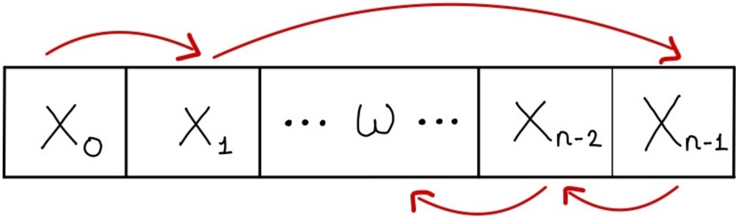
1. **Input:** We have a function $f : \{0, 1, \ldots, N-1\} \to \mathbb{B}$, where

$$f(x) = \begin{cases} 1 & \text{if } x = \omega \\ 0 & \text{if } x \neq \omega \end{cases}$$

2. **The goal** is to determine ω.

Description of Classical case

For a database of size **N**, the linear search algorithm sequentially checks up to **N** elements to find the target. Algorithm checks each element one-by-one. In the worst case, **N-1** operations occur, which makes the complexity is **O(N)**



Description of Grover's algorithm

Grover's Algorithm is designed to search for a marked element in an unsorted database of **N** items where $N = 2^n$

Null Entries

Note that If N is not a power of 2, we add database "dummy enteries" to reach the closest power of 2. The algorithm operates on this extended space. Since this dummy entries are not desired solution, they will not effect the outcome.

Example

We can label the database $L = \{a, b, c\}$ of $|L| = 2^2 - 1$ elements using 2-bit binary form as follows:

$$a \rightarrow |00\rangle \qquad b \rightarrow |01\rangle$$
$$c \rightarrow |10\rangle \qquad n_0 \rightarrow |11\rangle$$

Presenting $\omega_\perp$

To search for the state $|\omega\rangle$, we introduce the orthogonal state:

$$|\omega_\perp\rangle := \frac{1}{\sqrt{N-1}} \sum_{x \neq \omega} |x\rangle$$

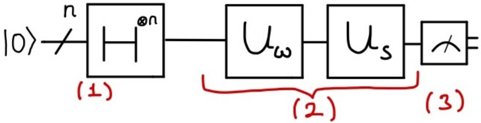
Notice that,

$$\langle \omega | \omega_\perp \rangle = 0$$

Database is contains an orthogonal basis.

Algorithm description

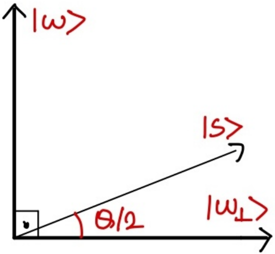
- **(1) Initialize:** Set n qubits to the state $H^{\otimes n}|0^n\rangle$
- **(2) Iterate:** Apply the Grover operation t times.
- **(3) Measure:** A basis measurement yields a solution.



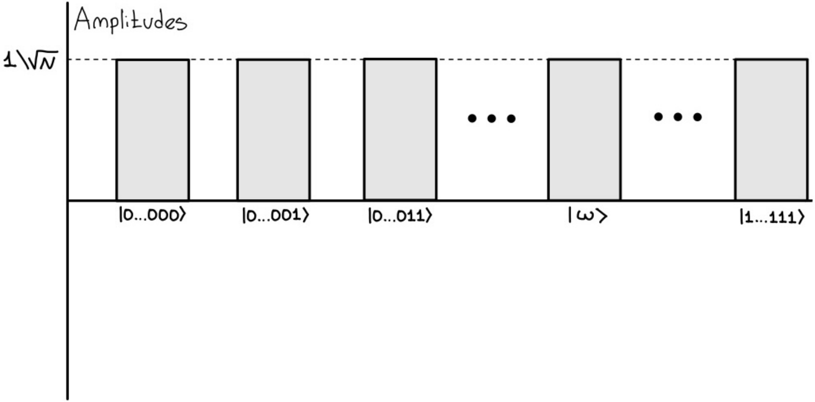
(1) Initialize.

The register is first initialized in the state $|0\rangle^{\otimes n} = |00\ldots0\rangle$, Then, a Hadamard transform is applied to each qubit, which creates an equal superposition of all possible states. This results in a state where each of the possible states has equal amplitude. The amplitudes are normalized so that the total probability is 1.

$$H^{\otimes n}|0^n\rangle = \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{B}^n} |x\rangle =: |s\rangle$$



Frame Title



(2) Iterate.

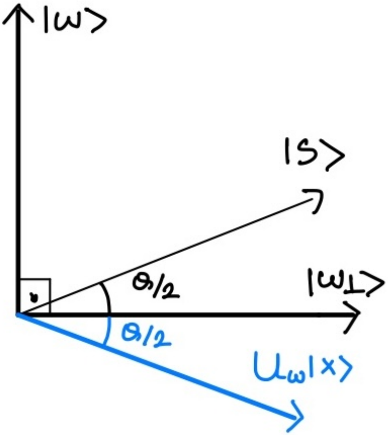
Definition

$$U_\omega = \mathbb{I} - 2|\omega\rangle\langle\omega|$$

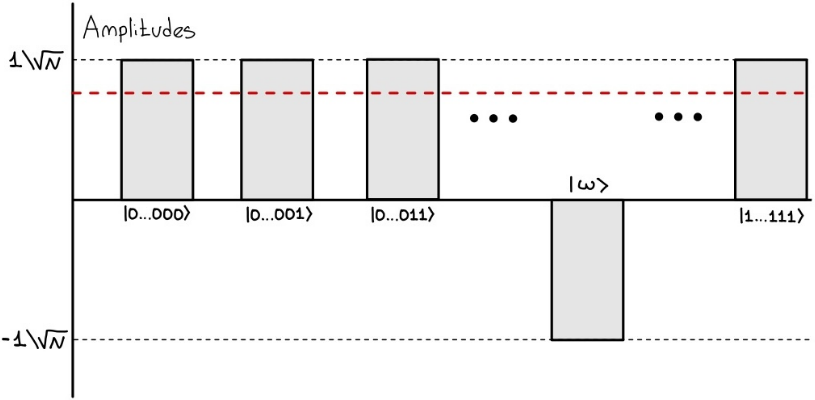
Corollary

$$U_\omega|x\rangle = (-1)^{f(x)}|x\rangle \begin{cases} |x\rangle & \textit{if } x \neq \omega \\ -|x\rangle & \textit{if } x = \omega \end{cases}$$

Iterate



Iterate



Iterate

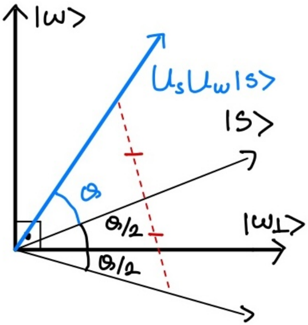
Definition

$$U_s := 2|s\rangle\langle s| - \mathbb{I}$$

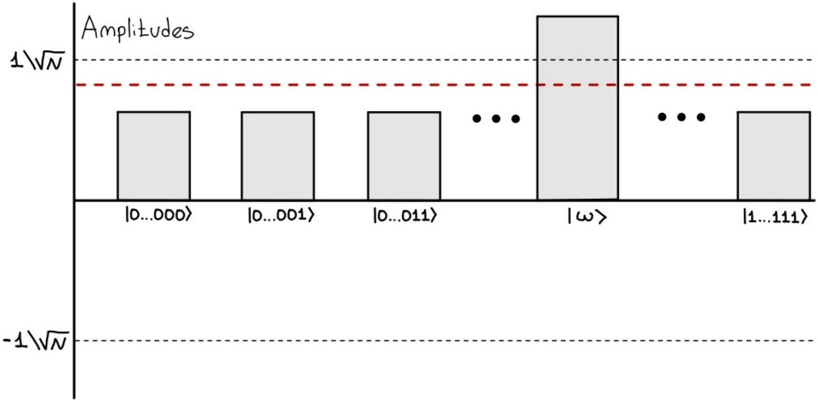
Iterate

Corollary

$$U_s|x\rangle = 2|s\rangle\langle s|x\rangle - |x\rangle$$



Iterate and Measure



(3)Measure, we measure $|\omega\rangle$ with a high probability

Observation

Observe that:

$$|s\rangle = |\omega_\perp\rangle\langle\omega_\perp|s\rangle + |\omega\rangle\langle\omega|s\rangle = \cos\frac{\theta}{2}|\omega_\perp\rangle + \sin\frac{\theta}{2}|\omega\rangle$$

$$= \sqrt{1-\frac{1}{N}}|\omega_\perp\rangle + \frac{1}{\sqrt{N}}|\omega\rangle$$

Then,

$$\langle\omega|s\rangle = \frac{1}{\sqrt{N}} = \sin\frac{\theta}{2} \approx \frac{\theta}{2}$$

from the first term of the Taylor expansion for $1 << N$.

Computational Complexity

Definition

Define the probability of measuring the desired state $|\omega\rangle$ given r iterations as,

$$P(\omega, r) := |\langle\omega|(U_s U_\omega)^r|s\rangle|^2$$

Initial Probability ($r = 0$): From Born's rule:

$$P(\omega, 0) = |\langle\omega|\phi\rangle|^2 = \frac{1}{N} = \sin^2(\theta/2),$$

from previous frame.

Computational Complexity

After r Iterations of Grover's Algorithm: The probability evolves as:

$$P(\omega, r) = \sin^2\left(\theta\left(\frac{1}{2} + r\right)\right),$$

where $(U_\omega U_s)^r$ is the r-iteration Grover operator.

To ensure $P(\omega, r) \approx 1$, the argument of the sinus function must satisfy:

$$\theta\left(\frac{1}{2} + r\right) \approx \frac{\pi}{2} \implies r \approx \frac{\pi}{2\theta} - \frac{1}{2}.$$

This condition corresponds to rotating the initial state $|s\rangle$ to align with $|\omega\rangle$ in the $|\omega\rangle$-$|\omega_\perp\rangle$ plane.

Computational Complexity

Recall that;

$$\langle \omega | s \rangle = \sin(\theta/2) = \frac{1}{\sqrt{N}} \implies \theta \approx \frac{2}{\sqrt{N}}.$$

Substitute $\theta \approx \frac{2}{\sqrt{N}}$ into the desired condition $r \approx \frac{\pi}{2\theta} - \frac{1}{2}$. yields:

$$r \approx \frac{\pi}{4}\sqrt{N}.$$

As a result, The number of iterations required to achieve a high probability of success scales as $\mathbf{O(\sqrt{N})}$